(Q7)

Theorem 1. *Suppose that f is bounded on $\mathbb{R}$. Then for all $\varepsilon > 0$ the function $g(x) = |x|^{1+\varepsilon} f(x)$ is differentiable at 0.*

Proof. We are given that f is bounded. This means:

$$\forall x \in \mathbb{R},\ \exists M > 0 \text{ s.t. } |f(x)| \leq M$$

Since we aim to prove that $g(x)$ is differentiable at 0, it suffices to prove that $g'(0)$ exists. By the limit definition of the derivative, we have:

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} \frac{g(x)}{x} \text{ since } g(0) = 0^{1+\varepsilon} f(x) = 0$$

Then,

$$g'(0) = \lim_{x \to 0} \frac{|x|^{1+\varepsilon} f(x)}{x} = \lim_{x \to 0} \frac{|x|^{1+\varepsilon}}{x} f(x)$$

Since f is bounded on both positive and negative:

$$\begin{aligned} -M \leq f(x) \leq M &\implies -M\frac{|x|^{1+\varepsilon}}{x} \leq \frac{|x|^{1+\varepsilon}}{x} f(x) \leq M\frac{|x|^{1+\varepsilon}}{x} \\ &\implies -M|x|^{\varepsilon} \leq \frac{|x|^{1+\varepsilon}}{x} f(x) \leq M|x|^{\varepsilon} \end{aligned}$$

We observe by known limits that:

$$\lim_{x \to 0} M|x|^{\varepsilon} = 0 = \lim_{x \to 0} -M|x|^{\varepsilon}$$

Therefore by Squeeze Theorem:

$$\lim_{x \to 0} \frac{|x|^{1+\varepsilon}}{x} f(x) = 0 = g'(0)$$

Since $g'(0)$ exists, we have proven the theorem, as required. ■