

(Q1) The formula is:

$$f^{(n)}(x) = \frac{3^n \cdot n!}{(5 - 3x)^{n+1}}$$

*Proof.* Since we are only considering  $n \in \mathbb{N}$ , we prove this through induction. To make operations easier to see, we write  $f(x)$  as  $(5 - 3x)^{-1}$ .

*Base Case.*

Let  $n = 1$ . By application of the chain rule:

$$\begin{aligned} f^{(1)}(x) &= -1 \cdot (5 - 3x)^{-2} \cdot \frac{d}{dx}(5 - 3x) \\ &= -1 \cdot (5 - 3x)^{-2} \cdot -3 \\ &= 3(5 - 3x)^{-2} \\ &= \frac{3}{(5 - 3x)^2} = \frac{3^1 \cdot 1!}{(5 - 3x)^{1+1}} \end{aligned}$$

Thus, the base case holds.

*Induction Step.*

In this step, we aim to prove the following induction hypothesis:

$$f^{(n)}(x) = \frac{3^n(n!)}{(5 - 3x)^{n+1}} \implies f^{(n+1)}(x) = \frac{3^{n+1} \cdot (n+1)!}{(5 - 3x)^{(n+1)+1}}$$

We begin by simplifying the right hand side of the implication and rewriting it:

$$\begin{aligned} f^{(n+1)}(x) &= \frac{3^{n+1} \cdot (n+1)!}{(5 - 3x)^{(n+1)+1}} \\ &= \frac{3^n \cdot 3 \cdot n! \cdot (n+1)}{(5 - 3x)^{n+1} \cdot (5 - 3x)} \\ &= \frac{3^n(n!)}{(5 - 3x)^{n+1}} \cdot 3 \cdot (n+1) \cdot (5 - 3x)^{-1} \end{aligned}$$

We observe that  $f^{(n+1)}(x)$  is simply  $f^{(n)}(x)$  multiplied by  $[3 \cdot (n+1) \cdot (5 - 3x)^{-1}]$ . Therefore, if the derivative of  $f^{(n)}(x)$  has the same result, the induction hypothesis is proved. Differentiating  $f^{(n)}(x)$  via chain rule ( $n$  is a constant and is not affected):

$$\begin{aligned} \frac{d}{dx} 3^n(n!)(5 - 3x)^{-(n+1)} &= 3^n(n!)(5 - 3x)^{-(n+1)-1} \cdot -(n+1) \cdot -3 \\ &= 3^n(n!)(5 - 3x)^{-(n+1)-1} \cdot (n+1) \cdot 3 \text{ (minus signs cancel)} \\ &= 3^n(n!)(5 - 3x)^{-(n+1)} \cdot [(5 - 3x)^{-1} \cdot (n+1) \cdot 3] \end{aligned}$$

We observe that these additional terms are the exact same ones we multiplied to  $f^{(n)}(x)$  to obtain  $f^{(n+1)}(x)$ , thereby proving the induction hypothesis.

Since both the base case and induction hypothesis hold, we have proven the theorem, as required. ■