(Q7)

(a) False.

Consider the piecewise function

$$f(x) = \begin{cases} \frac{\sin x}{x} + 2021 & \text{if } x \neq 0 \\ 69 & \text{if } x = 0 \end{cases}$$

This function fulfills the following:

- It is defined for all $x \in \mathbb{R}$.
- It is bounded within $(2021, 2022)$ for all $x \neq 0$, and $(0, 2022)$ for $x = 0$, which means the entire function is bounded within $(0, 2022)$.
- $\lim\limits_{x \to 0} f(x) = 2022$.

As such, it is possible to have $0 < f(x) < 2022$ with a limit $L \geq 2022$.

(b) False.

We cannot draw conclusions given only the limit of the product of two functions, as we cannot guarantee that there exists a limit for either function.

As a counter-example, consider $f(x) = x,\ g(x) = \dfrac{1}{x}$. It follows:

$$\lim_{x \to 0} x \cdot \frac{1}{x} = 1$$
$$\lim_{x \to 0} x = 0$$
$$\lim_{x \to 0} \frac{1}{x} \text{ DNE}$$

Therefore, $\lim\limits_{x \to a} f(x) = \dfrac{1}{\lim\limits_{x \to a} g(x)}$ is not true for some functions.

(c) True.

Proof. We are given that

$$\forall \varepsilon \in (0, \tfrac{1}{100}),\ \exists\, \delta_1 > 0 \text{ s.t. } \forall x \in \mathbb{R},\ 0 < |x - 137| < \delta_1 \implies |f(x) - 2022| < \varepsilon$$

Thus we can assume that for $\varepsilon < \frac{1}{100}$, there is a δ_1 for that ε that satisfies this particular implication. We want to show that:

$$\forall \varepsilon > 0,\ \exists\, \delta > 0 \text{ s.t. } \forall x \in \mathbb{R},\ 0 < |x - 137| < \delta \implies |f(x) - 2022| < \varepsilon$$

Fix $\varepsilon \geq \frac{1}{100}$. Let $\delta = \delta_1$. It follows that:

$$\begin{aligned} &0 < |x - 137| < \delta_1 = \delta \\ \implies &0 < |x - 137| < \delta \implies |f(x) - 2022| < \tfrac{1}{100} \leq \varepsilon \\ \implies &0 < |x - 137| < \delta \implies |f(x) - 2022| < \varepsilon \end{aligned}$$

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