(Q5)

Theorem 1. *Prove that if f is continuous at $x = a$, and $f(a) > 0$, then there exists a $\delta > 0$, so that $f(x) > 0$ for all $x \in (a - \delta, a + \delta)$.*

Proof. Formally speaking, the implication we are trying to prove is:

$$\lim_{x \to a} f(x) = f(a) > 0 \implies \exists \delta > 0 \text{ s.t. } \forall x \in (a - \delta, a + \delta), \ f(x) > 0$$

Assuming $f(x)$ is continuous by the limit definition of continuity (rewritten slightly):

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R},$$
$$x \in (a - \delta, a + \delta) \implies |f(x) - f(a)| < \varepsilon$$

We know that there exists a δ for every ε such that the above implication holds. Therefore, all we need to do is to choose an ε such that $\varepsilon < f(a)$,
Let $\varepsilon = \frac{f(a)}{2}$. Since $f(a) > 0$, it follows $f(a) - \frac{f(a)}{2} = \frac{f(a)}{2} > 0$. By the above implication, there exists a δ such that $|f(x) - f(a)| < \frac{f(a)}{2}$ holds for all $x \in (a - \delta, a + \delta)$.
It follows:
$$\forall x \in (a - \delta, a + \delta), \ |f(x) - f(a)| < \frac{f(a)}{2} \implies f(x) > 0$$

Which proves the theorem, as required. ∎