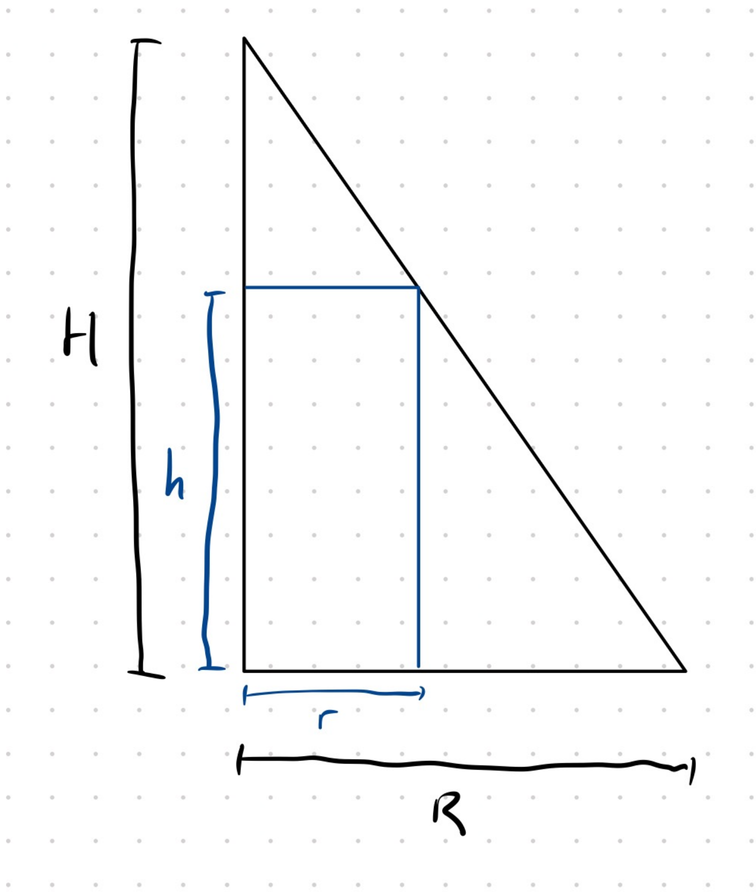
(Q1)
For clarity, we redefine the height and radius of the cone as H and R respectively, and the height and radius of the inscribed cylinder to be h and r.



From the above figure, we observe from similar triangles that:

$$\frac{H}{R} = \frac{H-h}{r} \tag{1}$$
$$= \frac{h}{R-r} \tag{2}$$

We will use (2) for this solution.
From the formula for volume of a cylinder, we define the volume V_c of a given cylinder inscribed within the cone with:

$$V_c = \pi r^2 h$$

Since $h = \frac{H(R-r)}{R}$, we have

$$V_c = \frac{\pi}{R} r^2 H(R-r)$$

We aim to find the maximum for V_c, so we need to find r for which this function is at a maximum.

Differentiating it and then solving for zero, we have

$$\begin{aligned}
\frac{2\pi}{R} rH(R-r) + \frac{\pi}{R} r^2(-H) &= 0 \\
2Rr - 3r^2 &= 0 \\
r(2R-3r) &= 0 \\
r = 0, \quad r &= \frac{2R}{3}
\end{aligned}$$

Since r cannot be 0, $\frac{2R}{3}$ is the only possible answer. Substituting this value of r into V_c, we have:

$$\begin{aligned}
V_c &= \frac{\pi}{R}\left(\frac{2R}{3}\right)^2 \cdot H\left(R - \frac{2R}{3}\right) \\
&= \frac{4\pi R}{9} \cdot \frac{HR}{3} \\
&= \frac{4}{3}\pi R^2 H
\end{aligned}$$