(Q5)

(a) The negation of the definition of limit is:

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \varepsilon$$

(b)

$$\lim_{x \to a} f(x) \text{ DNE iff}$$
$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \in \mathbb{R} \text{ s.t.}$$
$$0 < |x - a| < d \text{ and } |f(x) - L| \geq \varepsilon$$

(c)

Proof. Formally speaking, the implication we are looking to prove is:

$$\forall L \in \mathbb{R}\, \exists \varepsilon_1 \text{ s.t. } \forall \delta > 0, \exists x \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \varepsilon_1$$
$$\iff$$
$$\forall L \in \mathbb{R}\, \exists \varepsilon \text{ s.t. } \forall \delta > 0, \exists x \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta \text{ and } |cf(x) - cL| \geq \varepsilon$$

Assuming the first implication, we can assume theat there exists a ε_1 that satisfies $|f(x) - L \geq \varepsilon_1$. We need to find an ε that satisfies $|cf(x) - cL \geq \varepsilon$.
Fix $\delta > 0$. Let $\varepsilon = \frac{\varepsilon_1}{c}$.
We observe that $|cf(x) - cL| = c|f(x) - L|$.
Then,

$$c|f(x) - L| \geq \frac{\varepsilon_1}{c} \implies |f(x) - L| \geq \varepsilon_1$$

Now, assuming the second implication, we observe that $\dfrac{\varepsilon}{c} < \varepsilon_1$. Thus,

$$|f(x) - L| \geq \varepsilon_1 > \frac{\varepsilon}{c}$$

■

(d)

Proof. By contrapositive of the given implication:

$$\lim_{x \to a} cf(x) \text{ exists iff } \lim_{x \to a} f(x) \text{ exists}$$

Considering one way of the implication, if $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} c$ exists, by limit laws we can calculate $\lim_{x \to a} cf(x)$ which also exists.
Considering the other direction of the implication, if $\lim_{x \to a} cf(x)$ exists and $\lim_{x \to a} c$ exists and is non-zero, then we can perform the following division:

$$\frac{\lim_{x \to a} cf(x)}{\lim_{x \to a} c}$$

Which would yield $\lim_{x \to a} f(x)$. ■

(e) Consider the function $\dfrac{1}{x}$.

$\lim_{x \to o} \dfrac{1}{x}$ DNE, but $\lim_{x \to 0} 0 \cdot \dfrac{1}{x} = 0$