(Q6)

Theorem 1. *$U_P(f) \geq L_Q(f)$ for any bounded function f and partitions P, Q of $[a, b]$.*

Proof. Let R be a common refinement of P and Q such that $R = P \cup Q$.
We have $P \subseteq R$ and $Q \subseteq R$. Then by earlier proof:

$$\begin{aligned} P \subseteq R &\implies U_P(f) \geq U_R(f) \\ Q \subseteq R &\implies L_R(f) \geq L_Q(f) \end{aligned}$$

We also have $U_R(f) \geq L_R(f)$ by earlier proof. Thus:

$$\begin{gathered} U_P(f) \geq U_R(f) \geq L_R(f) \geq L_Q(f) \\ \Downarrow \\ U_P(f) \geq L_Q(f) \end{gathered}$$

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