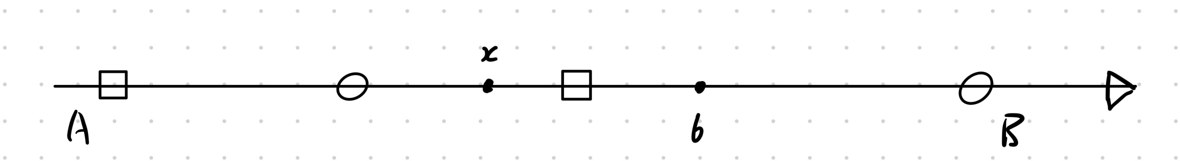
(Q9)

Theorem 1. *There exists sets of real numbers $A, B \subseteq \mathbb{R}$ satisfying* ***all*** *of the following conditions:*
(a) $A \neq \phi, B \neq \phi$
(b) There exists $b \in B$ so that $a < b$ for every $a \in A$.
(c) There exists $a \in A$ so that $b < a$ for every $b \in B$.

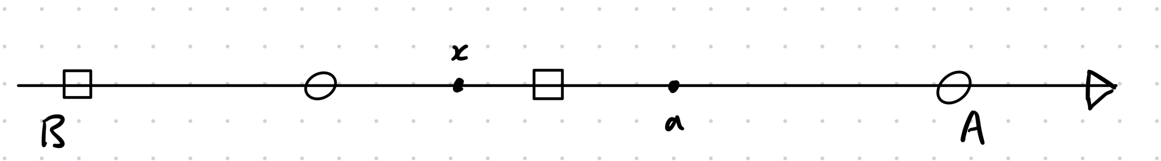
This theorem is false. We can prove this by contradiction.

Proof. We begin by assuming all three conditions are true.
By condition (a), $A \neq \phi, B \neq \phi$. Let $x \in A$ and $x \in B$. This implies $x \in A \cap B$.
Since $A, B \subseteq \mathbb{R}$, we can conceptualize them as sections on a number line.
Considering condition (b),



$$\begin{aligned} &\text{If } x \in A \cap B, \text{ then} \\ &\exists b \in B \text{ s.t. } a < b, \forall a \in A && \implies b \notin A \\ & && \implies x < b \text{ (since } x \in A) \text{ and } a < b \end{aligned}$$

Simultaneously considering condition (c),



$$\begin{aligned} &\text{If } x \in A \cap B, \text{ then} \\ &\exists a \in A \text{ s.t. } b < a, \forall b \in B && \implies a \notin B \\ & && \implies x < a \text{ (since } x \in B) \text{ and } b < a \end{aligned}$$

This leaves us with four states which must be simultaneously true:

$$\forall x \in A \cap B, \exists a \in A, b \in B \text{ such that:}$$

1. $x < b$.
2. $x < a$.
3. $a < b$.
4. $b < a$.

This implies $a < b < a \Rightarrow a < a$ which is not possible unless $A = \phi$ and $B = \phi$ via vacuous truth, which contradicts condition (a). ■