

(Q7)

(a) True.

Proof. Since f is strictly increasing, we have

$$\forall x_1, x_2 \in \mathbb{R}, x_1 < x_2 \implies f(x_1) < f(x_2)$$

For the sake of contradiction, assume f has a maximum or minimum. By definition of a local minimum or maximum, we have that $x_1 \in \mathbb{R}$ is a maximum iff

$$\exists \delta > 0 \text{ s.t. } x_2 \in (x_1 - \delta, x_1 + \delta) \implies f(x_2) \leq f(x_1)$$

Let $x_1 = c$ and $x_2 = x_1 + \frac{\delta}{2}$. We have $x_1 < x_2$, so $f(x_1) < f(x_2)$, which contradicts the assumption that f has a maximum.

This argument also holds for a minimum. ■

(b) False.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^3$, which is strictly increasing and also differentiable. $f'(x) = 3x^2$, thus $f'(0) = 0$.

(c) True.

The proof follows from proof of Q2. If $f \circ g$ is bijective, then f must be surjective and g must be injective. Thus if $f \circ f$ is bijective, then f must be both surjective and injective, and thus bijective.

(d) True.

Proof. By the definition of continuity, we have

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R}, \\ |x - p| < \delta \implies |f'(x) - f'(p)| < \varepsilon$$

Let $\varepsilon = \frac{f'(p)}{2}$. Since f' is continuous, we know that such a δ exists to satisfy this implication. Let I be $(p - \delta, p + \delta)$. Since $f'(p) > 0$,

$$|x - p| < \delta \implies |f'(x) - f'(p)| < \frac{f'(p)}{2} \implies f'(x) > 0$$

Since $\forall x \in I, f'(x) > 0$, f is strictly increasing, and thus injective. ■