Q8

(a) False.

Let $f(x) = \dfrac{1}{x}$. Then

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln x\Big|_b^1\right] = \lim_{b\to\infty}\ln x = \infty$$

So $\lim\limits_{x\to\infty} f(x) = 0$, but the integral still diverges.

(b) True.

Proof. First suppose f has a horizontal asymptote $y = L$ as x tends to infinity. Then since $f(x)$ is an antiderivative of $f'(x)$, we have

$$\int_1^\infty f'(x)\,dx = \lim_{b\to\infty}[f(b) - f(1)] = L - f(1)$$

So the integral converges.

Now assume $\displaystyle\int_1^\infty f'(x)\,dx$ converges. This means $\lim\limits_{b\to\infty}[f(b) - f(1)]$ approaches a fixed value as $x \to \infty$, which by definition is an asymptote. ■

(c) True.

Let $y = e^{-x^2}$, the integral of which is bounded. We then solve for x:

$$\ln y = -x^2 = \sqrt{-\ln y}$$

which converges.