(Q3)

Theorem 1. *Prove that* $\lim_{x \to -1} x^3 = -1$ *using the* $\delta - \varepsilon$ *definition of the limit.*

Proof. The $\delta - \varepsilon$ definition of the limit, as applied to this function, is

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R},\ 0 < |x+1| < \delta \implies |x^3 + 1| < \varepsilon$$

Fix $\varepsilon > 0$. Thus,
By the sum of cubes,

$$\begin{aligned} |x^3 + 1| &= |x+1||x^2 - x + 1| < \varepsilon \\ &\implies \delta |x^2 - x + 1| < \varepsilon \end{aligned}$$

Let $\delta = min\{1, \frac{\varepsilon}{3}\}$.
Assuming $0 < |x+1| < \delta$, $|x+1| < \frac{\varepsilon}{3}$ and $|x+1| < 1$.
Considering $\delta \leq 1$ implies:

$$\begin{aligned} & 0 < |x+1| < 1 \\ \implies & -1 < x + 1 < 1 \\ \implies & -2 < x < 0 \\ \implies & |x| < 2 \end{aligned}$$

Since $\delta \leq \frac{\varepsilon}{3}$, it follows that:

$$\begin{aligned} |x^2 - x + 1| < 3 &\implies \delta |x^2 - x + 1| < 3\delta \leq \varepsilon \\ &\implies |x^3 + 1| < \varepsilon \end{aligned}$$

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