(Q7)

Theorem 1. *For any bounded function f, g and partition P of $[a, b]$, we have*

$$U_P(f+g) \geq U_P(f) + U_P(g)$$

Proof. For convenience, let $I = [a, b]$ and $\Delta_a = b - a$. We observe that

$$\sup\{(f+g)(I)\} \geq \sup\{f(I)\} + \sup\{g(I)\}$$
$$\Downarrow$$
$$\sup\{(f+g)(I)\}\Delta_a \geq \sup\{f(I)\}\Delta_a + \sup\{g(I)\}\Delta_a \quad (1)$$

Therefore, for any partition $P = \{x_0, x_1, \ldots x_n\}$ of $[a, b]$:

$$U_P(f) = \sum_{i=1}^{n} \sup f([x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$
$$U_P(g) = \sum_{i=1}^{n} \sup g([x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$
$$U_P(f+g) = \sum_{i=1}^{n} \sup(f+g)([x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

Since (1) holds for every term in the summation, we can conclude that:

$$U_P(f+g) \geq U_P(f) + U_P(g)$$

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