(Q5)

Theorem 1. $f(x) = x^2 - 1$ *is integrable on* $[-1, 1]$.

Proof. We use the ε-characterisation of integrability for this proof:

$$\forall \varepsilon > 0, \ \exists \text{ a partition } P \text{ of } [a, b]\colon \quad U_P(f) - L_P(f) < \varepsilon$$

We evaluate $U_P(f) - L_P(f)$ by cases for odd and even n.
For even n:

$$U_P(f) - L_P(f) = \frac{4}{n}$$

For odd n:

$$U_P(f) - L_P(f) = \frac{4}{n} - \frac{2}{n^3}$$

Ultimately, we aim to prove:

$$\forall \varepsilon > 0, \exists n \in \mathbb{N}\colon \quad U_{P_n}(f) - L_{P_n}(f) < \varepsilon$$

Fix $\varepsilon > 0$. For even n, let n be a natural such that $n > \frac{4}{\varepsilon}$. Then

$$\frac{4}{n} < \frac{4}{\frac{4}{\varepsilon}} \implies \frac{4}{n} < 4 \cdot \frac{\varepsilon}{4} \implies \frac{4}{n} < \varepsilon$$

For odd n, let n be a natural such that $n > \max\{\frac{8}{\varepsilon}, \left(\frac{4}{\varepsilon}\right)^{\frac{1}{3}}\}$. Then

$$\frac{4}{n} - \frac{2}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since we have proven that this implication holds for all n, f is integrable on $[-1, 1]$. ■