(Q4)

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function so that*

$$(1 - \cos^2 x) \leq f(x) \leq x^2$$

for all $x \in (-2022, 2022)$.
Prove that $\lim_{x \to 0} f(x)$ exists.

Proof. We observe that $1 - \cos^2 x = \sin^2 x$. We can thus rewrite the definition of $f(x)$ as:

$$\sin^2 x \leq f(x) \leq x^2$$

Taking $\lim_{x \to 0} \sin x = 0$, we can use limit laws to compute $\lim_{x \to 0} \sin^2 x$:

$$\lim_{x \to 0} \sin x = 0 \implies \lim_{x \to 0} (\sin x)(\sin x) = 0 \cdot 0 = 0$$

Using the fact that x^2 is continuous, we can also compute its limit at $x = 0$:

$$\lim_{x \to 0} x^2 = 0^2 = 0$$

By Squeeze Theorem,

$$\forall x \in (-2022, 2022),\ \sin^2 x \leq f(x) \leq x^2 \text{ and } \lim_{x \to 0} \sin^2 x = \lim_{x \to 0} x^2 = 0$$
$$\implies \lim_{x \to 0} f(x) = 0$$

Which also proves that $\lim_{x \to 0} f(x)$ exists, as required. ■