(Q6)

Theorem 1. $|\sin x| \leq |x|$ *for all* $x \neq 0$.

Proof. Fix x as arbitrary.
Case 1: $x > 0$. We define a closed interval $[0, x]$. By MVT applied on $\sin x$:

$$\begin{aligned} \forall x > 0,\ \exists c \in (0, x) \text{ s.t. } \frac{\sin x - \sin 0}{x - 0} &= \cos c \\ \implies \frac{\sin x}{x} &= \cos c \end{aligned}$$

Case 2: $x < 0$. We define a closed interval $[x, 0]$. By MVT applied on $\sin x$:

$$\begin{aligned} \forall x < 0,\ \exists c \in (x, 0) \text{ s.t. } \frac{\sin 0 - \sin x}{0 - x} &= \cos c \\ \implies \frac{-\sin x}{-x} &= \cos c \\ \implies \frac{\sin x}{x} &= \cos c \end{aligned}$$

In both cases, we have $\dfrac{\sin x}{x} = \cos c$. Then,

$$\begin{aligned} \frac{\sin x}{x} = \cos c &\implies \left|\frac{\sin x}{x}\right| = |\cos c| \leq 1 \\ &\implies |\sin x| = |\cos c||x| \leq |x| \end{aligned}$$

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