

(Q9)

(a) True.

*Proof.* We observe that:

$$\begin{aligned}f + 2g &= 2g + f \\&= 2g - 2f + 3f \\&= 2(g - f) + 3f\end{aligned}$$

Since  $f + 2g$  can be expressed as a combination and transformation of two integrable functions, it is also integrable. ■

(b) False.

As a counterexample, let  $f = g = \chi_{\mathbb{Q}}$ , where

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & : x \in \mathbb{Q} \\ 0 & : x \notin \mathbb{Q} \end{cases}$$

Neither  $f$  nor  $g$  are integrable, while  $(f + g)(x) = 1$ , which is integrable.

(c) False.

Let  $\chi_{\mathbb{Q}}$  be the same as the one defined in (b). We define  $f$  as:

$$f(x) = \begin{cases} x^2 & : x \in [-2, 0] \cup [1, 6] \\ \chi_{\mathbb{Q}} & : x \in (0, 1) \end{cases}$$

$f$  is integrable on  $[-2, 0]$  and  $[1, 6]$  but not on  $(0, 1)$ , so it is not integrable on  $[-1, 4]$ .

(d) False.

Let  $f$  be defined as:

$$f(x) = \begin{cases} 1 & : x \in \mathbb{Q} \\ -1 & : x \notin \mathbb{Q} \end{cases}$$

$f^2(x) = 1$ , which is integrable, while  $f$  is not.