(Q1)
(a)

Proof. Based on the definition of limits at infinity, the implication we are trying to prove is:

$$\forall \varepsilon > 0,\ \exists M \in \mathbb{R} \text{ s.t. } x > M \implies \left|\frac{1}{x}\right| < \varepsilon$$

Fix $\varepsilon > 0$. Let $M = \frac{1}{\varepsilon}$. If $x > M$, then $x > \frac{1}{\varepsilon} > 0$. Assuming $x > M$, it follows:

$$\begin{aligned} x > M &\implies x > \frac{1}{\varepsilon} \\ &\implies \varepsilon x > 1 \\ &\implies \frac{1}{x} < \varepsilon \\ &\implies \left|\frac{1}{x}\right| < \varepsilon \quad (\text{since } \tfrac{1}{x} > 0) \end{aligned}$$

■

(b)

Proof. In order to prove that $\frac{1}{x^k} < \frac{1}{x}$ for all integers $x > 1$ and $k \geq 2$, we first have to prove that $x < x^k$ for all $k \geq 2$. This can be done with a proof by induction.
Base Case. The base case we have to prove is $x^2 > x$. Since $x > 1$, it follows:

$$x > 1 \implies x^2 > x$$

Induction Step. Following this, we need to prove for any natural $k \geq 2$:

$$x^k > x \implies x^{k+1} > x$$

We observe that $x^{k+1} = x^k \cdot x$. Thus, since $x > 1$:

$$\begin{aligned} x^k \cdot x > x^k > x &\implies x^k \cdot x > x \\ &\implies x^{k+1} > x \end{aligned}$$

Thus, we prove that $x < x^k$ for all $k \geq 2$ and in turn $\frac{1}{x^k} < \frac{1}{x}$.
Since $x > 0, k > 0, x^k > 0 \implies \frac{1}{x^k} > 0$. Therefore,

$$0 < \frac{1}{x^k} < \frac{1}{x} \text{ for all integers } x > 1, k \geq 2$$

■

(c)

Proof. We know from the Squeeze Theorem that for any three functions f, g, h:
If:

- For x close to any point a but not a, $h(x) \leq g(x) \leq f(x)$;
- $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$;

then $\lim_{x \to a} g(x) = L$.
Let $h(x) = 0, g(x) = \frac{1}{x^k}, f(x) = \frac{1}{x}$.
By *(b)* we know that

$$0 < \frac{1}{x^k} < \frac{1}{x} \text{ for all integers } x > 1, k \geq 2$$

Therefore by the Squeeze Theorem,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = 0 \implies \lim_{x \to \infty} g(x) = 0$$

■