(Q2)
(a)

Proof. For $n = 1$, there is nothing to prove.
For $n = 2$, we have the matrix

$$A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

Then $\det A = ab - ab = 0$.
For the induction step, we assume this holds for n, and prove for $n + 1$.
Let $A \in \mathcal{M}_{n+1 \times n+1}$, and A has two identical rows. Using Part 2 of Theorem 2, we define $\det A$ as

$$\sum_{j=1}^{n+1} (-1)^{i+j} A_{ij} \det\left(\tilde{A}_{ij}\right)$$

Where $i \in \{1, 2, \ldots, n\}$. We choose i such that we are not choosing one of the two identical rows. Then $\tilde{A}_{ij} \in \mathcal{M}_{n \times n}$ and has two identical rows, so $\det(\tilde{A}_{ij}) = 0$.
Expanding the summation for $\det A$, we then get:

$$A_{i1} \det(\tilde{A}_{i1}) - A_{i2} \det(\tilde{A}_{i2}) + \ldots A_{i(n+1)} \det(\tilde{A}_{i(n+1)})$$

Since $\det(\tilde{A}_{ij}) = 0$, each term in this sum is 0, and thus the entire sum is 0. ∎

(b)

Proof. Let

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix}, \quad B = \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix},$$

By Theorem 57, we observe that

$$\det C = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i + a_j \\ \vdots \\ a_j + a_i \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j + a_i \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_j + a_i \\ \vdots \\ a_n \end{pmatrix}$$
$$= \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix}$$
$$= \det A + 0 + \det B + 0$$

Since C has two identical rows, $a_i + a_j$ and $a_j + a_i$, By Theorem 58 part 1, $\det C = \det A + \det B = 0$. Thus, we can conclude

$$\det A + \det B = 0 \implies \det A = -\det B$$

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