(Q9)

Proof. By the Fundamental Theorem of Algebra, any polynomial p over $\mathbb{C}$ can be reduced into a product of linear terms over $\mathbb{C}$, the number of terms being $\deg p$. Additionally, if a is a root of p, then $\bar{a}$ is also a root of p. Thus, roots of p come in conjugate pairs. What is left is to consider specific cases of $\deg p$.

Considering cases where $\deg p = 1$:
Suppose for the sake of contradiction, that p is reducible. Then

$$\exists a, b \in P(\mathbb{R}) : \; ab - p \implies \deg a + \deg b = \deg p$$

It is also a condition that $\deg a, \deg b \neq 0$.
However, $\deg p = 1 \implies \deg a$ or $\deg b = 0$, which is a contradiction.

Considering cases where $\deg p = 2$:
By earlier proof, if p is irreducible, then it has no real roots.

Considering cases for $\deg p$ is even and > 2:
Any polynomial of an even degree can be expressed as a product of polynomials of degree 2, so it is reducible.

Considering cases for $\deg p$ is odd and > 1:
Since roots of polynomials come in conjugate pairs, there will always be one root remaining. Since roots come in conjugate pairs, this root has to be equal to its conjugate:

$$a = \bar{a} \implies a = x + 0i$$

Since the imaginary part of the root is 0, it is wholly real.
Since p has a real root a, $\exists q \in P(\mathbb{R}) : \; p(x) = (x - a)p(x)$, thus p is reducible.

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