(Q7)

1. If A has a column of zeroes, then $\det A = 0$.

Proof. Let A have a column a_i where $a_{1i} = a_{2i} = \ldots a_{ni} = 0$. Then A^t has $a_{i1} = a_{i2} = \ldots a_{in} = 0$, i.e a row of zeroes, so $\det A^t = 0$ by Theorem 58 Part 1.
So $\det A^t = \det A = 0$. ∎

2. For any $i \in \{1, \ldots, n\}$, we have

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})$$

Proof. The proof for this follows from Q3. ∎

3. If A has two identical columns, then $\det A = 0$.

Proof. Let the columns of A be given by

$$\begin{pmatrix} a_1 & \ldots & a_i & \ldots & a_i \ldots & a_n \end{pmatrix}$$

Then for A^t, Each row is given by

$$[A^t]_{ij} = A_{ji}$$

So A^t has two identical rows. Thus

$$\det(A^t) = \det A = 0$$

∎

4. If B is obtained from A by swapping two columns, then $\det B = -\det A$.

Proof. Let $A = (a_{n1}, \ldots, a_{ni}, \ldots a_{nj}, \ldots, a_{nn})$, and $B = (a_{n1}, \ldots, a_{nj}, \ldots a_{ni}, \ldots, a_{nn})$
Then for each element in A^t and B^t:

$$[A^t]_{ij} = A_{ji}, [A^t]_{ij} = B_{ji}$$

So now B^t is obtained from A^t by swapping two rows. $\det A^t = -\det B^t$ by Q2(b).
Then

$$\det A^t = -\det B^t = -\det B = \det A$$

∎

5. If B is obtained from A by scaling column i by $c \in \mathbb{F}$, then $\det B = a \det A$.

Proof. Let $B = (a_{n1}, \ldots ca_{ni}, \ldots, a_{nn})$. Then $[B^t]_{ij} = B_{ji}$, and $[A^t]_{ij} = A_{ji}$, So B^t is obtained from A^t by scaling row i of A^t by c.
Then

$$\det B^t = c \det A^t = c \det A = \det B$$

∎

6. If B is obtained from A by adding a multiple of one column to another, then $\det B = \det A$.

Proof. Let

$$A = (a_{n1}, \dots, a_{ni}, \dots a_{nj}, \dots, a_{nn})$$

and

$$B = (a_{n1}, \dots, a_{ni} + a_{nj}, \dots, a_{nj}, \dots, a_{nn})$$

Then

$$A^t = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{jn} \\ \vdots \\ a_{nn} \end{pmatrix} \quad B^t = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{in} + a_{jn} \\ \vdots \\ a_{jn} \\ \vdots \\ a_{nn} \end{pmatrix}$$

Then

$$\det A^t = \det B^t = \det A = \det B$$

■