Q(5)

(a)

Proof. We aim to prove T is injective using the definition of injectivity, that is,

$$T(x_1) = T(x_2) \implies x_1 = x_2$$

First suppose $T(x_1) = T(x_2)$ and $x_1 = x_2$. Then $T(x_1) - T(x_2) = \mathbf{0}_W$. By linearity of T, we have $T(x_1) - T(x_2) = T(x_1 - x_2) = \mathbf{0}_W$. Since $x_1 = x_2$, we have

$$T(x_1 - x_2) = T(\mathbf{0}_V) = \mathbf{0}_W$$

So $N(T) = \{\mathbf{0}_V\}$, since any other value implies $x_1 \neq x_2$.
Now suppose $N(T) = \{\mathbf{0}_W\}$, and let $T(x_1) = T(x_2)$.
Then $T(x_1) - T(x_2) = \mathbf{0}_W$. Again, by linearity of T, we have $T(x_1) - T(x_2) = T(x_1 - x_2) = \mathbf{0}_W$. Since $N(T) = \{\mathbf{0}_V\}$, $x_1 - x_2$ has to be $\mathbf{0}_V$. Then

$$x_1 - x_2 = \mathbf{0}_V \implies x_1 = x_2$$

Which satisfies the definition of injectivity. ■

(b)

Proof. We assume for the entire proof that $\dim V = \dim W = n$.
First, suppose T is injective; that is, $N(T) = \{\mathbf{0}_V\}$.
Then by the Rank-Nullity Theorem,

$$\dim V - \dim N(T) = n - 0 = n = \dim(\operatorname{im} T)$$

We have $\operatorname{im} T$ is a subspace of W, and $\dim(\operatorname{im} T) = \dim V = \dim W$, so $\operatorname{im} T = W$, and thus T is surjective.
Now suppose T is surjective, so $\operatorname{im} T = W \implies \dim(\operatorname{im} T) = \dim W = n$. Then

$$\dim(\operatorname{im} T) + \dim N(T) = n = \dim W = \dim V$$
$$\dim(\operatorname{im} T) = \dim W = n \implies \dim N(T) = 0 \implies N(T) = \{\mathbf{0}_V\}$$

So T is injective. ■