

(Q7)

(a) False.

The contrapositive of the given statement is that if $\dim V \geq \dim W$, then there exists at least one injective linear map. This means that there exists one linear map T such that $N(T) = \{\mathbf{0}\}$.

By the Rank-Nullity Theorem:

$$\begin{aligned}\dim(\operatorname{im} T) + \dim N(T) &= \dim V \implies \dim(\operatorname{im} T) = \dim V - \dim N(T) \\ &\implies \dim W = \dim V - \dim N(T)\end{aligned}$$

For cases where $\dim V = \dim W$, $\dim N(T)$ can clearly be 0, so injective linear maps exist for that case. However, if $\dim V > \dim W$, then $\dim N(T)$ can never be 0, so $N(T) \neq \{\mathbf{0}\}$.

(b) False.

As a counterexample, let $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, and $T : V \rightarrow W$ be given by $T(x, y, z) = (x, y)$.

Then there are infinite choices for z for which $T(x, y, z)$ will map to the same (x, y) . For example, $(1, 2, 3)$ and $(1, 2, 4)$ both map to $(1, 2)$.

(c) False.

Let T be the linear map defined in (b). Since it maps from a vector space of dimension 3, the vector space should have basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. However the vector space it maps to is of dimension 2, so it has basis $\{\mathbf{x}_1, \mathbf{x}_2\}$, which is of a different cardinality to the basis of T 's domain.