(Q10)

Proof. We prove this by induction on n.
First, we consider $n = 1$. This gives us a matrix $A = (a_{11})$, and $C_A(x) = \det(xI_n - A) = x - a_{11}$, which has degree 1.
Next, we consider $n = 2$. This gives us a matrix

$$A = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

We then calculate $C_A(x)$ as:

$$\det \begin{pmatrix} x - b_1 & -b_2 \\ -b_3 & x - b_4 \end{pmatrix} = (x - b_1)(x - b_4) + b_2 b_3$$

Which, after expansion, becomes a polynomial of degree 2.
Now, we assume a matrix $A \in \mathcal{M}_{n \times n}$ has a $C_A(x)$ of degree n, and prove that this holds for $n + 1$.
We thus calculate $C_A(x)$ for some matrix $A \in \mathcal{M}_{n+1 \times n+1}$. This is given by $\det(xI_{n+1} - A)$.
We observe that $xI_{n+1} - A$ is given by

$$\begin{pmatrix} x - A_{11} & -A_{12} & -A_{13} & \dots & -A_{1n} \\ -A_{21} & x - A_{22} & \dots & \dots & \vdots \\ -A_{31} & \vdots & x - A_{33} & & \vdots \\ \vdots & & & \ddots & \vdots \\ -A_{n1} & \dots & \dots & \dots & x - A_{nn} \end{pmatrix}$$

So its determinant is given by

$$\sum_{j=1}^{n+1} (-1)^{1+j} (xI_{n+1} - A)_{1j} \det\left(xI_n - \tilde{A}_{1j}\right)$$

Expanding this sum, we get:

$$(x - A_{11}) \det\left(xI_n - \tilde{A}_{11}\right) - \left[(-A_{12}) \det\left(xI_n - \tilde{A}_{12}\right)\right] + \dots \\ + \left[(-A_{1(n+1)}) \det\left(xI_n - \tilde{A}_{1(n+1)}\right)\right]$$

We observe that $\det\left(xI_n - \tilde{A}_{1j}\right)$ is a polynomial of degree n, since $xI_n - \tilde{A}_{1j} \in \mathcal{M}_{n \times n}$. Thus, the first term in the sum expansion, $(x - A_{11}) \det\left(xI_n - \tilde{A}_{11}\right)$, is a polynomial of degree $n + 1$, while every other term is a polynomial of degree n multiplied by a scalar. Since the first term is not cancelled out, the entire sum evaluates to a polynomial of degree $n + 1$.
Thus, $C_A(x)$ is a polynomial of degree $n + 1$ for $A \in \mathcal{M}_{n+1 \times n+1}$, and the induction hypothesis holds. ∎