(Q6)

(a) We obtain a matrix for this linear transformation by taking $T(e_i)$ for each standard basis vector for $\mathbb{R}^4$.

$$\begin{aligned} T(1,0,0,0) &= (1,0,1,1) \\ T(0,1,0,0) &= (2,0,1,3) \\ T(0,0,1,0) &= (3,0,2,4) \\ T(0,0,0,1) &= (1,0,0,2) \end{aligned}$$

Taking each resultant vector as a row, we obtain the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 3 \\ 3 & 0 & 2 & 4 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

(b)

We find $N(T)$ by solving the matrix obtained in a for a system of homogenous equations. Applying row-reduction, we have:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 3 \\ 3 & 0 & 2 & 4 \\ 1 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $y = t$, $w = s$. Then $x = -2s$, $z = s$. So the general solution has the form

$$(-2s, t, s, s)$$

Grouping by parameter, we then obtain

$$t(0,1,0,0) + s(-2,0,1,1)$$

So a basis for $N(T)$ is $\{(0,1,0,0), (-2,0,1,1)\}$ and has dimension 2.

(c) No.

In order for T to be invertible, it has to be both injective and surjective. By proof of Q5(a), in order for T to be injective it has to have $N(T) = \{\mathbf{0}\}$. Since this is not the case, T is not injective and thus not invertible.