(Q11)

Proof. Let S be a basis for $W_1 \cap W_2$, where $S = \{u_1, u_2, \dots u_r\}$ and $\dim(W_1 \cap W_2) = r$. Let B_1 be a basis of W_1 and B_2 be a basis of W_2. By the Replacement Theorem, since $(W_1 \cap W_2) \subseteq W_1$ and $\subseteq W_2$, we can express B_1 and B_2 as follows:

$$B_1 = \{u_1, u_2, \dots, v_1, \dots v_s\} \implies \dim W_1 = r + s$$
$$B_2 = \{u_1, u_2, \dots, w_1, \dots w_t\} \implies \dim W_2 = r + t$$

Let $B = B_1 \cup B_2$. Then

$$B = \{u_1, u_2, \dots v_1, \dots v_s, w_1, \dots w_t\}$$

Since $|B| = (r + s) + (r + t) - r = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_@)$, it now remains to prove that B is a basis for V.
To show linear independence, let

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j + \sum_{k=1}^{t} c_k w_k = 0$$

Since u_i and v_j form B_1 we can conclude that $a, b = 0 \ \ \forall i, j$.
Similarly, since u_i, w_k for B_2, we can conclude the same for $b, c \ \ \forall i, j$.
We now prove that B spans $W_1 + W_2$. Let $w \in (W_1 + W_2)$. Then $w = w_1 + w_2$ where $w_1 \in W_1, w_2 \in W_2$.
Then

$$w_1 = \sum_{i=1}^{r} p_i u_i + \sum_{j=1}^{s} q_j v_j$$
$$w_2 = \sum_{i=1}^{r} g_i u_i + \sum_{k=1}^{t} h_k w_k$$
$$w = \sum_{i=1}^{r} (p_i + g_i) u_i + \sum_{j=1}^{s} q_j v_j + \sum_{k=1}^{t} h_k w_k$$

Which is in $\operatorname{span} B$.

$\blacksquare$