(Q3)

(a)

$$\begin{pmatrix} 5 & 6 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 7 & 3 & 2 \\ 0 & 0 & 4 & 5 & 8 \end{pmatrix}$$

(b)

Proof. Breaking down the two block-diagonal matrices that we are multiplying, are of the form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} & 0_{1(k+1)} & \dots & 0_{1n} \\ a_{21} & \ddots & & & \vdots & & \\ \dots & & \ddots & & \vdots & & \\ a_{k1} & \dots & & a_{kk} & 0_{k(k+1)} & \dots & 0_{kn} \\ 0_{(k+1)1} & 0_{(k+1)2} & \dots & 0_{(k+1)k} & b_{(k+1)(k+1)} & \dots & b_{(k+1)n} \\ \vdots & & & & \vdots & & \\ \dots & & & & \vdots & \ddots & \\ 0_{n1} & \dots & \dots & 0_{nk} & b_{n(k+1)} & \dots & b_{nn} \end{bmatrix}$$

For the other multiplicand, the a and b entries are replaced with c and d respectively. From the defintiion of matrix multiplication, we have that for any entry ij in the resultant matrix, it is equivalent to

$$(MN)_{ij} = \sum_{l=1}^{n} M_{il}N_{lj}$$

Considering both $l \leq k$ for both matrices, then for some l we have

$$(MN)_{il} = \sum_{q=1}^{l} a_{iq}c_{ql}$$

and

$$(MN)_{lj} = \sum_{q=1}^{l} a_{qi}c_{jq}$$

If there exists one $l > k$, then that entry in one matrix will be multiplied by a zero entry in the other matrix and so the entry in the resulting matrix will be zero.
However, if both $l > k$, then for some l we have

$$(MN)_{il} = \sum_{q=1}^{n-k} b_{iq}d_{ql}$$

and

$$(MN)_{lj} = \sum_{q=1}^{n-k} b_{qi}d_{jq}$$

and a similar result if there is one $l \leq k$.

Considering all these cases, we have the following matrix:

$$\begin{bmatrix}
a_{11}c_{11} & a_{12}c_{12} & \dots & a_{1k}c_{1k} & 0_{1(k+1)} & \dots & 0_{1n} \\
a_{21}c_{21} & \ddots & & & \vdots & & \\
\dots & & \ddots & & \vdots & & \\
a_{k1}c_{k1} & \dots & & a_{kk}c_{kk} & 0_{k(k+1)} & \dots & 0_{kn} \\
0_{(k+1)1} & 0_{(k+1)2} & \dots & 0_{(k+1)k} & bd_{(k+1)(k+1)} & \dots & bd_{(k+1)n} \\
\vdots & & & & \vdots & & \\
\dots & & & & \vdots & \ddots & \\
0_{n1} & \dots & \dots & 0_{nk} & bd_{n(k+1)} & \dots & bd_{nn}
\end{bmatrix}$$

which takes the form $\begin{pmatrix} AC & O_{k,(n-k)} \\ O_{(n-k),k} & BD \end{pmatrix}$. ∎

(c)

Proof. First assume M is invertible, so $\exists N \in \mathcal{M}_{n\times n} : MN = NM$.
Let $N = \begin{pmatrix} C & O_{k,(n-k)} \\ O_{(n-k),k} & D \end{pmatrix}$. N is block diagonal.
Then

$$MN = \begin{pmatrix} AC & O_{k,(n-k)} \\ O_{(n-k),k} & BD \end{pmatrix}, \quad NM = \begin{pmatrix} CA & O_{k,(n-k)} \\ O_{(n-k),k} & DB \end{pmatrix}$$

Since $MN = NM$, $AC = CA$ and $BD = DB$ and thus A and B are invertible.
Now assume A and B are invertible, so $\exists C, D : AC - CA$ and $BD = DB$. Then let N be defined as above.
Then MN, NM are defined by the above equations. Since $AC = CA$ and $BD - DB$, $MN = NM$ and thus M is invertible. ∎