(Q8)

(a) We find $C_A(x)$ by computing $\det(xI_n - A)$:

$$B = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} x-1 & 0 & -1 & -1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-3 & -2 \\ 0 & 0 & -1 & x \end{pmatrix}$$

$$\det B = (x-1)^2\,(x(x-3)+2) = (x-1)^3(x-2)$$

Since $C_A(x)$ splits over $\mathbb{R}$, A has eigenvalues 1 and 2.

(b) For each eigenvalue λ, we solve for is eigenspace $E_\lambda = N(\lambda I_n - A)$ and find a basis for E_λ.

For E_1:

$$I_4 - A = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solving the general solution, we get the basis

$$\beta_{E_1} = \{(1,0,0,0), (0,1,0,0), (0,0,-1,1)\}$$

We then do the same for E_2:

$$2I_4 - A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solving the general solution, we get the basis

$$\beta_{E_2} = \{(-1,0,-2,1)\}$$

(c) The union $\beta_1 \cup \beta_2$ is given by:

$$\beta = \{(1,0,0,0), (0,1,0,0), (0,0,-1,1)(-1,0,-2,1)\}$$

β spans $\mathbb{R}^4$, and is linearly independent. Thus, β is a basis for $\mathbb{R}^4$.

(d) Since $\mathbb{R}^4$ is a direct sum of the eigenspaces of A, A is diagonalizable. To do so, we arrange the vectors of β to the columns of a matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

We then solve for P^{-1}:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R3\cdot -1} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R4-R3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R4\cdot -1} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{array}\right)$$

$$\xrightarrow{R1+R4, R3-2R4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{array}\right)$$

We then solve for $D = P^{-1}AP$:

$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Which yields the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$