(Q4)
(a)

Proof. By earlier proof, $\gcd(a, n) = 1 \implies \exists x \in \mathbb{Z} : [a] \cdot [x] = [1]$ in $\mathbb{Z}_n$.
We now seek to prove the converse:

$$\exists x \in \mathbb{Z}_n : [a] \cdot [x] = [1] \implies \gcd(a, n) = 1$$

Let $[a] \cdot [x] = [1]$ in $\mathbb{Z}_n$. Then by definition of congruence modulo n, we have

$$ax \equiv 1 \pmod{n} \implies n \mid 1 - ax \implies \exists y \in \mathbb{Z} : yn = 1 - ax$$

Then by Bézout's Identity:

$$\exists x, y \in \mathbb{Z} : ax + yn = 1 \implies \gcd(a, n) = 1$$

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Since both directions have been proven, this statement is if and only if.
(b) Since $\forall n \in \mathbb{N}, \mathbb{Z}_n$ fulfills all the field axioms except for that of the multiplicative identity, if every nonzero element of $\mathbb{Z}_n$ has a multiplicative identity, $\mathbb{Z}_n$ is a field. This occurs iff n is prime.