(Q2)

Theorem. Let $\mathbb{F} = \{a + b\sqrt{11} | a, b \in \mathbb{Q}\}$, where $+, \cdot$ are defined by:

$$
\begin{aligned}
(a + b\sqrt{11}) + (c + d\sqrt{11}) &= (a + b) + (c + d)\sqrt{11} \\
(a + b)\sqrt{11} \cdot (c + d\sqrt{11}) &= (ac + 11bd) + (ad + bc)\sqrt{11}
\end{aligned}
$$

Applying the properties of $\mathbb{Q}$, $\mathbb{F}$ is a field.

Proof. We prove $\mathbb{F}$ is a field by verifying all the field axioms one by one.

Associativity

$$
\begin{aligned}
(a + b\sqrt{11}) + \left[(c + d\sqrt{11}) + (f + g\sqrt{11})\right] &= (a + b\sqrt{11}) + \left[(c + f) + (d + g)\right]\sqrt{11} \\
&= \left[a + (c + f)\right] + \left[b + (d + g)\right]\sqrt{11} \\
&= \left[(a + c) + f\right] + \left[(b + d) + g\right]\sqrt{11} \\
&= \left[(a + c) + (b + d)\right]\sqrt{11} + (f + g\sqrt{11}) \\
&= \left[(a + b\sqrt{11}) + (c + d\sqrt{11})\right] + (f + g\sqrt{11})
\end{aligned}
$$

Commutativity

$$
\begin{aligned}
(a + b\sqrt{11}) + (c + d\sqrt{11}) &= (a + c) + (b + d)\sqrt{11} \\
&= (c + a) + (d + b)\sqrt{11} \\
&= (c + d\sqrt{11}) + (a + b\sqrt{11})
\end{aligned}
$$

Distributivity

$$
\begin{aligned}
&(a + b\sqrt{11}) \cdot \left[(c + d\sqrt{11}) + (f + g\sqrt{11})\right] \\
&= (a + b\sqrt{11}) \cdot \left[(c + f) + (d + g)\sqrt{11}\right] \\
&= \left[a(c + f) + 11b(d + g)\right] + \left[a(d + g) + b(c + f)\right]\sqrt{11} \\
&= (ac + af + 11bd + 11bg) + (ad + ag + bc + bf)\sqrt{11} \\
&= \left[(ac + 11bd) + (af + 11bg)\right] + \left[(ad + bc) + (ag + bf)\right]\sqrt{11} \\
&= \left[(cd + 11bd) + (ad + bc)\sqrt{11}\right] + \left[(af + 11bg) + (ag + bf)\sqrt{11}\right] \\
&= (a + b\sqrt{11}) \cdot (c + d\sqrt{11}) + (a + b\sqrt{11}) \cdot (f + g\sqrt{11})
\end{aligned}
$$

Identities
Let the additive identity be $0 + 0\sqrt{11}$. Then

$$(a + b\sqrt{11}) + (0 + 0\sqrt{11}) = (a + 0) + (b + 0)\sqrt{11} = a + b\sqrt{11}$$

Let the multiplicative identity be $1 + 0\sqrt{11}$. Then

$$(a + b\sqrt{11}) \cdot (1 + 0\sqrt{11}) = (a + 0) + (0 + b)\sqrt{11} = a + b\sqrt{11}$$

Inverses
Let the additive inverse be $-a - b\sqrt{11}$. Then

$$(a + b\sqrt{11}) + (-a - b\sqrt{11}) = (a - a) + (b - b)\sqrt{11} = 0 + 0\sqrt{11}$$

Let the multiplicative inverse be $\left(\dfrac{a}{a^2 - 11b^2} - \dfrac{b}{a^2 - 11b^2}\sqrt{11}\right)$. Then

$$\begin{aligned} &\left(\frac{a}{a^2 - 11b^2} - \frac{b}{a^2 - 11b^2}\sqrt{11}\right) \cdot (a + b\sqrt{11}) \\ &= \left(\frac{a^2}{a^2 - 11b^2} - \frac{11b^2}{a^2 - 11b^2}\right) + \left(\frac{ab}{a^2 - 11b^2} - \frac{ab}{a^2 - 11b^2}\right)\sqrt{11} \\ &= 1 + 0\sqrt{11} \end{aligned}$$

Since all the field axioms have been verified, $\mathbb{F}$ is a field. ■