(Q6)

Proof. First, suppose $T(\beta)$ is a basis for W. Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$. Then $T(\beta) = \{T(v_1), T(v_2), \ldots T(v_n)\}$. Since T is linear and $|\beta| = |T(\beta)| \implies \dim V = \dim W$, T is an isomorphism.
Then suppose T is an isomorphism. Thus T is injective.
Since T is injective, every element in $T(\beta)$ is distinct, and W spans $T(\beta)$ Then to prove linear independence, we express the elements of $T(\beta)$ as a linear combination equating to 0:

$$a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n) = 0$$

Then by linearity and the properties of the null space:

$$\begin{aligned} a_1T(v_1) + \ldots + a_nT(v_n) &= T(a_1v_1) + \ldots + T(a_nv_n) \\ &= T(a_1v_1 + \ldots + a_nv_n) \\ &= 0_W \end{aligned}$$

Since T is injective, $N(T) = \{0_V\}$ and thus

$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0_V$$

Since β is a basis, it is linearly independent and thus all the coefficients are also 0, thus $T(\beta)$ is also linearly independent. ■