Half-plane Capacity and Conformal Radius (joint work with Steffen Rohde)

Carto Wong

University of Washington

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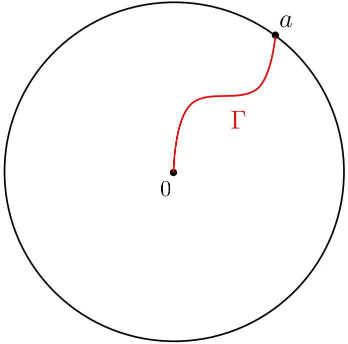
The Chinese University of Hong Kong

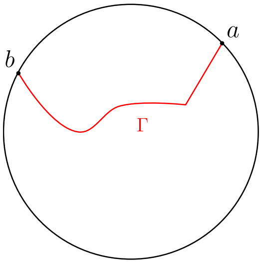
Let Γ be a slit in the unit disk $\mathbb{D}$ going from $a \in \partial\mathbb{D}$ to 0.

Question

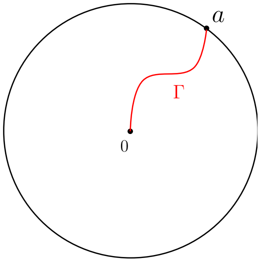
What is a "canonical" parametrization of Γ?



Two cases



chordal case



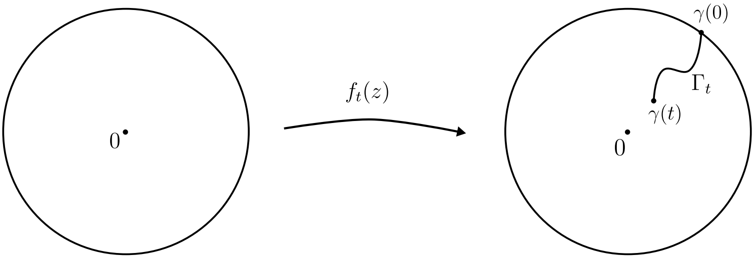
radial case

Radial case

Suppose $\gamma\colon (0, T) \to \mathbb{D}$ is a slit. There is a unique conformal mapping

$$f_t\colon \mathbb{D} \to \mathbb{D} \setminus \gamma([0, t])$$

satisfying $f_t(0) = 0$ and $f_t'(0) > 0$.



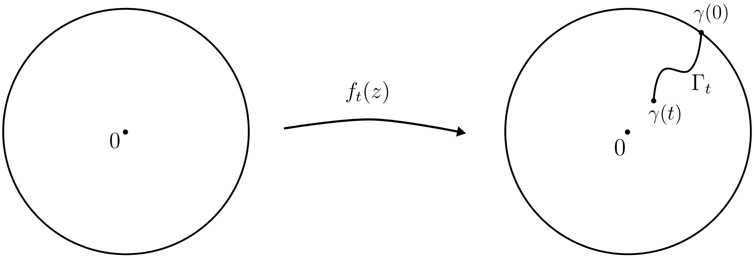
Radial case

Suppose $\gamma\colon (0, T) \to \mathbb{D}$ is a slit. There is a unique conformal mapping

$$f_t\colon \mathbb{D} \to \mathbb{D} \setminus \gamma([0, t])$$

satisfying $f_t(0) = 0$ and $f_t'(0) > 0$. We define the "disk capacity"

$$\mathrm{dcap}(\Gamma_t) := -\log f_t'(0).$$



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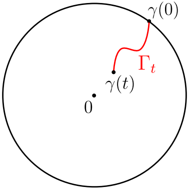


Answer

Parametrize the slit so that $\mathrm{dcap}(\Gamma_t) = t$.

Used in radial Schramn-Loewner Evolution (SLE)

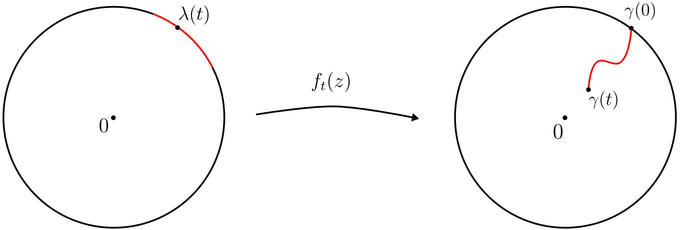
Reason

Theorem

Suppose the slit is parametrized so that $dcap(\Gamma_t) = t$. *Then the conformal mapping* $f_t(z)$ *satisfies the (radial) Loewner differential equation*

$$\partial_t f_t(z) = -f_t'(z)\,\frac{\lambda(t)+z}{\lambda(t)-z}$$

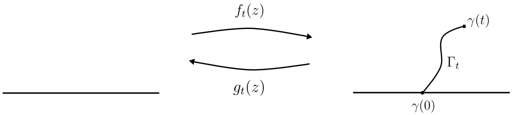
for some continuous function $\lambda(t)$ *with* $|\lambda(t)| \equiv 1$.



Chordal (half-plane) case

Hydrodynamic normalization:

$$g_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \cdots \qquad (z \to \infty)$$



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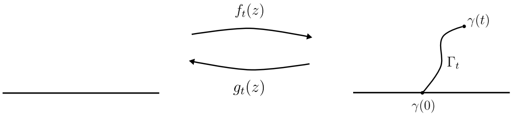
$$g_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \cdots \qquad (z \to \infty)$$

The half-plane capacity of Γ_t is defined as

$$\mathrm{hcap}(\Gamma_t) := \lim_{z\to\infty} z\,[g_t(z) - z] \geq 0.$$



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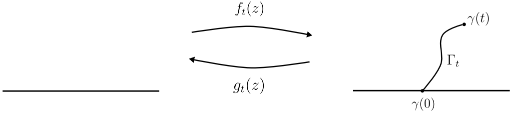

Example

$A = [0, i]$, $g(z) = \sqrt{z^2+1} = z + \frac{1}{2z} + \cdots$ and $\mathrm{hcap}(A) = \frac{1}{2}$.

Question

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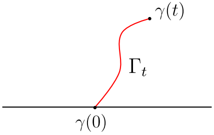


Answer

Parametrize the slit so that $\text{hcap}(\Gamma_t) = 2t$.

Used in chordal Schramm-Loewner Evolution (SLE)

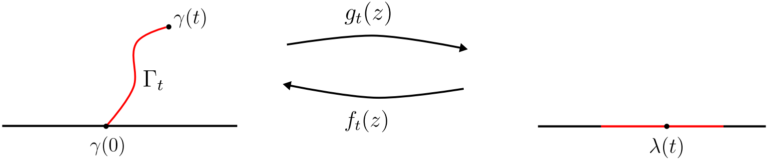
Chordal Loewner differential equation

Theorem

Suppose the slit is parametrized so that $hcap(\Gamma_t) = 2t$*. Then* $g_t(z)$ *satisfies the (chordal) Loewner differential equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}$$

for some continuous real-valued function $\lambda(t)$*.*



Basic properties of hcap(A)

- monotone:

$$A \subseteq B \Rightarrow \mathrm{hcap}(A) \leq \mathrm{hcap}(B).$$

- invariant under horizontal translation: for any $x \in \mathbb{R}$,

$$\mathrm{hcap}(A + x) = \mathrm{hcap}(A).$$

- scaling property: for any $r > 0$,

$$\mathrm{hcap}(rA) = r^2 \mathrm{hcap}(A).$$

Question

Find a geometric quantity that is comparable to $\mathrm{hcap}(A)$.

Scaling property: $\mathrm{hcap}(rA) = r^2\mathrm{hcap}(A)$. Some candidates are

- $\mathrm{area}\,(A)$
- $\mathrm{diam}\,(A)^2$
- $\mathrm{height}(A)^2$
- $\mathrm{diam}\,(A) \times \mathrm{height}(A)$

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- area (A)
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- diam $(A) \times \mathrm{height}(A)$

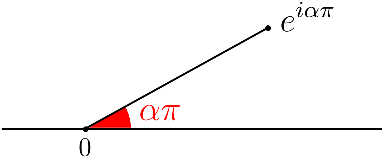
They all fail!

Counter-example

hcap(A) and diam $(A)^2$ are not comparable:

$$\text{hcap}([0, e^{i\alpha\pi}]) = \frac{1}{2}\alpha^{1-2\alpha}(1-\alpha)^{2\alpha-1} \to 0$$

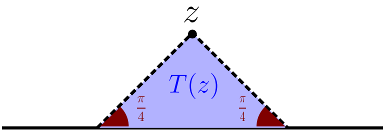
as $\alpha \to 0$.

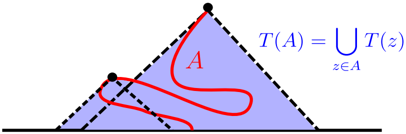


Notation

For $z \in \mathbb{H}$, define

$$T(z) = \text{triangular region in the figure below}$$

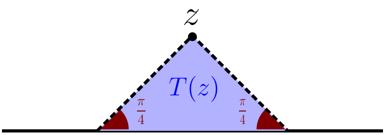


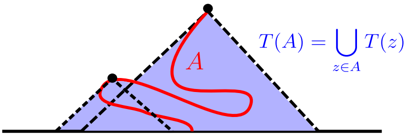


Notation

For $z \in \mathbb{H}$, define

$$T(z) = \text{triangular region in the figure below}$$
$$T(A) = \bigcup_{z \in A} T(z)$$





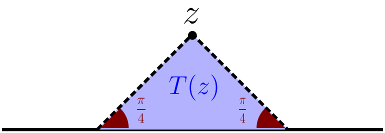
Main result

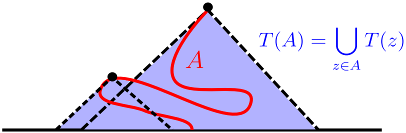
Theorem

There are constants c_1, $c_2 > 0$ such that

$$c_1 \cdot area\,(T(A)) \leq hcap(A) \leq c_2 \cdot area\,(T(A))$$

for all A.



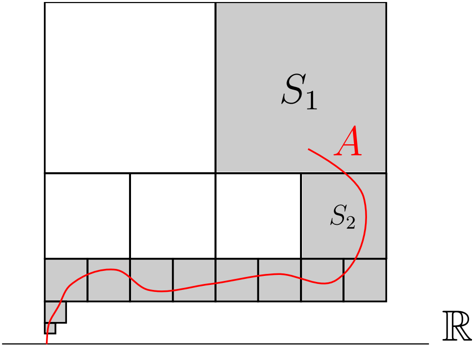


S(A)

Whitney squares:

$$S_{n,k} = [k2^n, (k+1)2^n] \times [2^n, 2^{n+1}] \qquad (n, k \in \mathbb{Z})$$

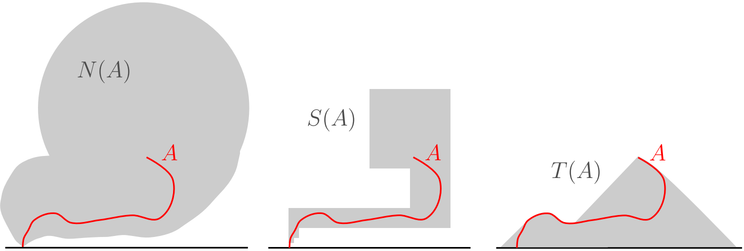
$$S(A) = \bigcup_j S_j$$



Comparable quantities

With absolute constants,

$$\text{area } N(A) \asymp \text{area } S(A) \asymp \text{area } T(A).$$



History of the problem

2001 Steffen Rohde and Michel Zinsmeister: showed a similar result for conformal radius when A is “already fat”.

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2009 Steven Lalley, Gregory Lawler, Hariharan Narayanan: proved the "ball version". The proof also uses probabilistic argument.

2010 (with Steffen Rohde) a non-probabilistic proof, using a known result about conformal radius.

Outline of a proof for the main result

3 ingredients in our proof:

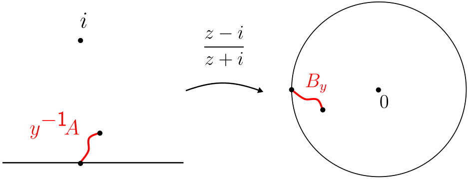
1. an equivalent problem about dcap (conformal radius)
2. a "fattening lemma"
3. a lemma for the fat case ("Variation of conformal radius" Steffen Rohde and Michel Zinsmeister, 2001)

Half-plane capacity and conformal radius

$$\mathrm{dcap}(B) = -\log f'(0) \asymp 1 - f'(0) \text{ for } B \subseteq \{\tfrac{1}{2} \le |z| < 1\}.$$

Lemma

$$\lim_{y \to \infty} \frac{dcap(B_y)}{hcap(y^{-1}A)} = 2.$$



Equivalent problems

Problem

$$hcap(A) \asymp area\, N(A)$$

for any slit $A \subseteq \mathbb{H}$.

is equivalent to

Problem

$$dcap(B) \asymp area\, N(B)$$

for any slit $B \subseteq \left\{\frac{1}{2} \leq |z| < 1\right\}$.

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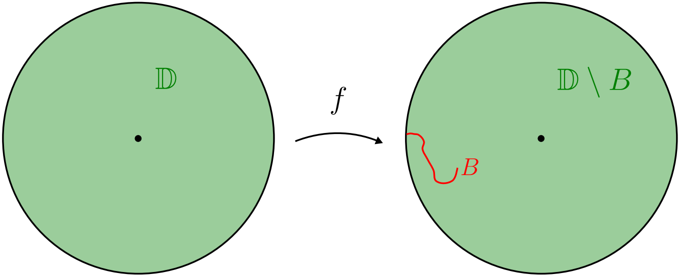
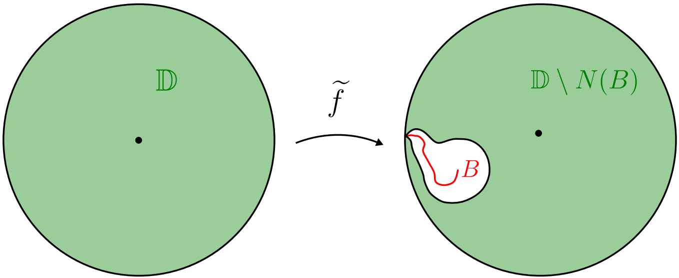
Fattening lemma



$$\mathrm{dcap}(B) = -\log\left|f'(0)\right| \asymp 1 - \left|f'(0)\right|$$

Fattening lemma



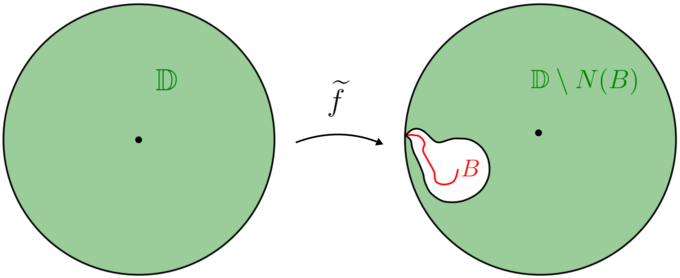
$$\operatorname{dcap} N(B) = -\log \left| \widetilde{f}'(0) \right| \asymp 1 - \left| \widetilde{f}'(0) \right|$$

Fattening lemma

Lemma (Fattening lemma)

For $B \subseteq \{\frac{1}{2} \leq |z| < 1\}$,

$$dcap(B) \leq dcap\, N(B) \leq c\, dcap(B).$$



Lower bound for dcap(B)

Let $f : \mathbb{D} \to \mathbb{D} \setminus B$ be conformal with $f(0) = 0$.

$$\begin{aligned} f(z) &= a_1 z + a_2 z^2 + \cdots \\ \text{area}\,(\mathbb{D} \setminus B) &= \pi \left(|a_1|^2 + 2\,|a_2|^2 + 3\,|a_3|^2 + \cdots \right) \\ &\geq \pi\,|a_1|^2 \\ \text{area}\,(B) &\leq \pi(1 - |a_1|^2) \asymp 1 - |a_1| \asymp \text{dcap}(B) \end{aligned}$$

By the fattening lemma,

$$\text{dcap}(B) \gtrsim \text{dcap}N(B) \gtrsim \text{area}\,N(B).$$

Proof of fattening lemma (sketch)

Decompose $\mathbb{D}$ into dyadic layers $D_n = \{\frac{1}{2^{n+1}} < 1 - |z| \leq \frac{1}{2^n}\}$.

$$\begin{aligned} \operatorname{dcap}(B) &= -\log\left|f'(0)\right| \\ &= \frac{1}{2\pi}\int_{\partial\mathbb{D}} -\log|f(\zeta)|\; d\zeta \\ &\asymp \sum_{n=1}^{\infty} \frac{\omega_n(0)}{2^n}, \end{aligned}$$

where $\omega_n(z) = \omega(z, B_n, \mathbb{D} \setminus B_n)$ and $B_n = B \cap D_n$.

Proof of fattening lemma (sketch)

Similarly,

$$\operatorname{dcap} N(B) \asymp \sum_{n=1}^{\infty} \frac{\widetilde{\omega}_n(0)}{2^n}.$$

The fattening lemma follows if one can show that

$$\widetilde{\omega}_n(z) \lesssim \omega_{n-1}(z) + \omega_n(z) + \omega_{n+1}(z)$$

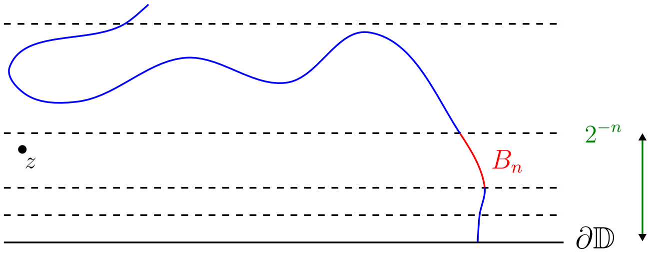
for $z \in \mathbb{D} \setminus N(B)$ and all n.

Harmonic measure estimate

It is in general not true that

$$\widetilde{\omega}_n(z) \lesssim \omega_n(z)$$

for $z \in \mathbb{D} \setminus N(B)$.

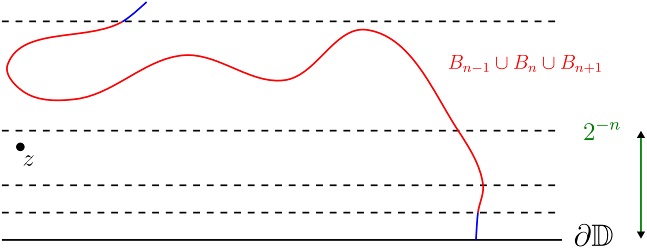


Harmonic measure estimate

By taking 3 terms, we have

$$\widetilde{\omega}_n(z) \lesssim \omega_{n-1}(z) + \omega_n(z) + \omega_{n+1}(z)$$

for $z \in \mathbb{D} \setminus N(B)$.



Outline of a proof for the main result

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1. an equivalent problem about dcap (conformal radius)
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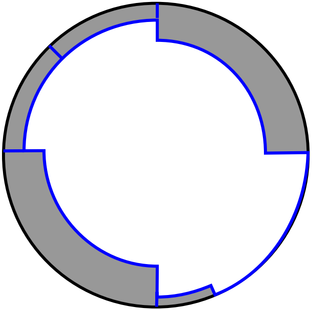
3. a lemma for the fat case ("Variation of conformal radius" Steffen Rohde and Michel Zinsmeister, 2001)

The fat case

Lemma (Rohde and Zinsmeister, 2001)

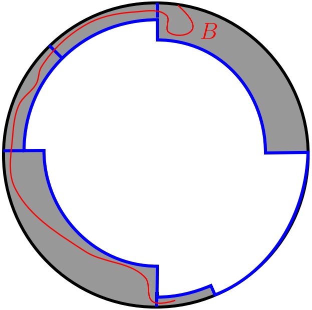
If $B = \bigcup_{j=1}^{n} Q_j$ *is a union of disjoint dyadic squares, then*

$$dcap(B) \asymp area\,(B).$$



Upper bound of dcap(B)

Cover B by dyadic squares in a minimal way. Call the larger set $Q(B) \supseteq B$.



$$\mathrm{dcap}(B) \leq \mathrm{dcap}\, Q(B) \lesssim \text{area } Q(B) \lesssim \text{area } N(B).$$

The fat case

Lemma (Rohde and Zinsmeister, 2001)

If $B = \bigcup_{j=1}^{n} Q_j$ is a union of disjoint dyadic squares, then

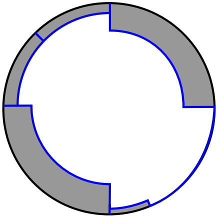
$$dcap(B) \asymp area\,(B).$$

Sketch of proof. WLOG, $\text{area}\,(Q_1) \leq \text{area}\,(Q_2) \leq \cdots \leq \text{area}\,(Q_n)$. Using induction, it suffices to show that

$$\text{dcap}\left(\bigcup_{j=1}^{m} Q_j\right) - \text{dcap}\left(\bigcup_{j=1}^{m-1} Q_j\right) \asymp \text{area}\,(Q_m)$$

for all m.

The induction step



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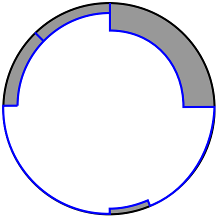

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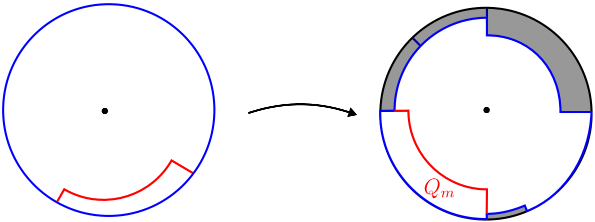
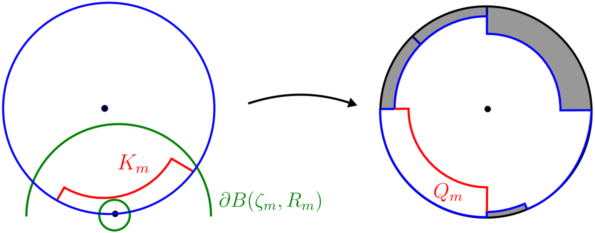


On the right side,

$$\omega \asymp \mathrm{diam}\,(Q_m) \asymp \sqrt{\mathrm{area}\,(Q_m)}.$$

The induction step



On the left side, the red curve is bounded between two circles $\partial B(\zeta_m, \rho R_m)$ and $\partial B(\zeta_m, R_m)$ with center $\zeta_m \in \partial\mathbb{D}$, where $\rho \in (0,1)$ is an absolute constant. Moreoever,

$$\omega \asymp R_m \asymp \sqrt{\mathrm{dcap}(K_m)}.$$

Thank you!

These slides are available at
https://github.com/cartowong/halfplane-capacity