

Half-plane Capacity and Conformal Radius (joint work with Steffen Rohde)

Carto Wong

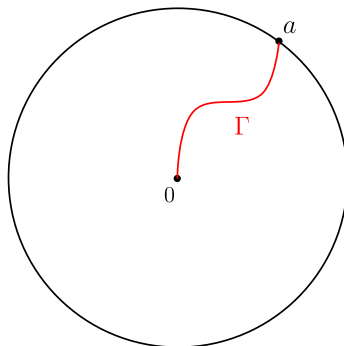
University of Washington

December 12, 2011
The University of Hong Kong

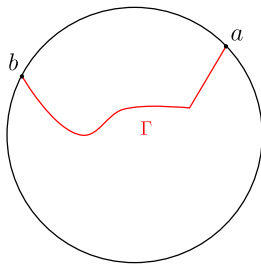
Let Γ be a slit in the unit disk \mathbb{D} going from $a \in \partial\mathbb{D}$ to 0.

Question

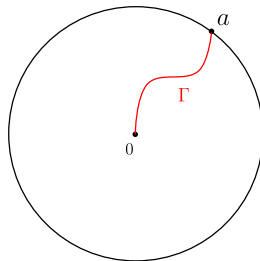
What is a “canonical” parametrization of Γ ?



Two cases



chordal case



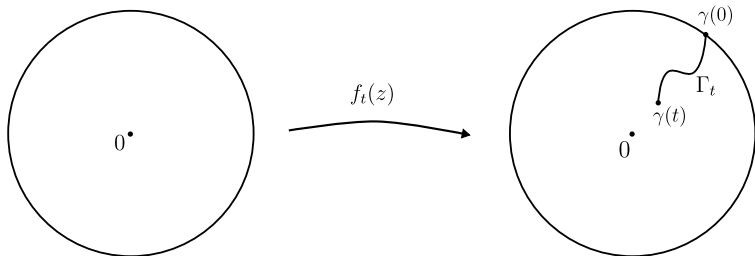
radial case

Radial case

Suppose $\gamma: (0, T) \rightarrow \mathbb{D}$ is a slit. There is a unique conformal mapping

$$f_t: \mathbb{D} \rightarrow \mathbb{D} \setminus \gamma([0, t])$$

satisfying $f_t(0) = 0$ and $f_t'(0) > 0$.



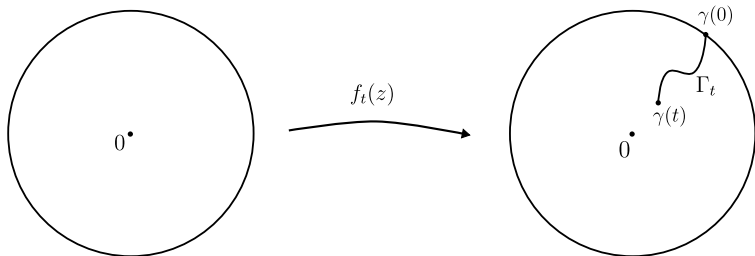
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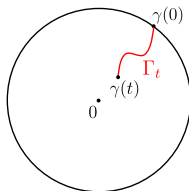
satisfying $f_t(0) = 0$ and $f'_t(0) > 0$. We define the “disk capacity”

$$\text{dcap}(\Gamma_t) := -\log f'_t(0).$$



Question

What is a “canonical” parametrization of Γ ?



Answer

Parametrize the slit so that $\text{dcap}(\Gamma_t) = t$.

Used in radial Schramm-Loewner Evolution (SLE)

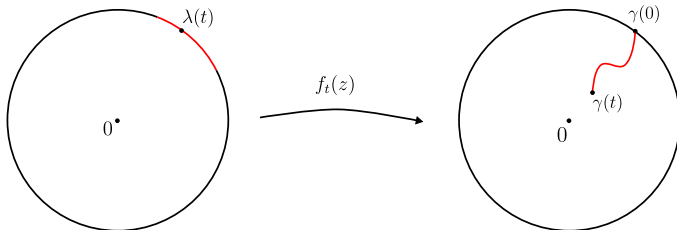
Reason

Theorem

Suppose the slit is parametrized so that $\text{dcap}(\Gamma_t) = t$. Then the conformal mapping $f_t(z)$ satisfies the (radial) Loewner differential equation

$$\partial_t f_t(z) = -f'_t(z) \frac{\lambda(t) + z}{\lambda(t) - z}$$

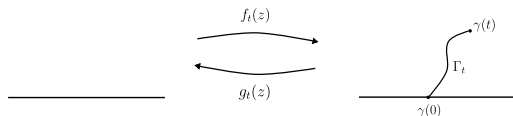
for some continuous function $\lambda(t)$ with $|\lambda(t)| \equiv 1$.



Chordal (half-plane) case

Hydrodynamic normalization:

$$g_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \dots \quad (z \rightarrow \infty)$$



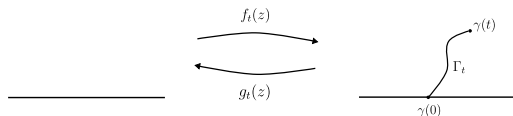
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The **half-plane capacity** of Γ_t is defined as

$$\text{hcap}(\Gamma_t) := \lim_{z \rightarrow \infty} z [g_t(z) - z] \geq 0.$$



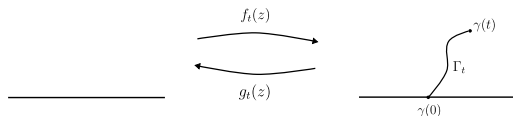
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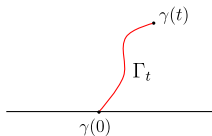


Example

$$A = [0, i], \quad g(z) = \sqrt{z^2 + 1} = z + \frac{1}{2z} + \dots \quad \text{and} \quad \text{hcap}(A) = \frac{1}{2}.$$

Question

What is a “canonical” parametrization of Γ ?



Answer

Parametrize the slit so that $\text{hcap}(\Gamma_t) = 2t$.

Used in chordal Schramm-Loewner Evolution (SLE)

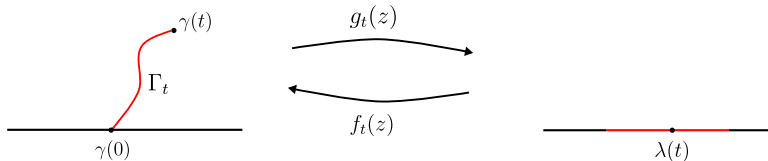
Chordal Loewner differential equation

Theorem

Suppose the slit is parametrized so that $hcap(\Gamma_t) = 2t$. Then $g_t(z)$ satisfies the (chordal) Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}$$

for some continuous real-valued function $\lambda(t)$.



Basic properties of $\text{hcap}(A)$

- monotone:

$$A \subseteq B \Rightarrow \text{hcap}(A) \leq \text{hcap}(B).$$

- invariant under horizontal translation: for any $x \in \mathbb{R}$,

$$\text{hcap}(A + x) = \text{hcap}(A).$$

- scaling property: for any $r > 0$,

$$\text{hcap}(rA) = r^2 \text{hcap}(A).$$

Question

Find a geometric quantity that is comparable to $\text{hcap}(A)$.

Scaling property: $\text{hcap}(rA) = r^2 \text{hcap}(A)$. Some candidates are

- $\text{area}(A)$
- $\text{diam}(A)^2$
- $\text{height}(A)^2$
- $\text{diam}(A) \times \text{height}(A)$

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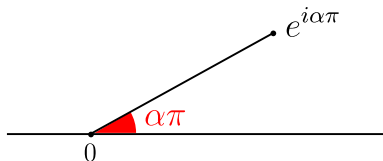
They all fail!

Counter-example

$\text{hcap}(A)$ and $\text{diam}(A)^2$ are not comparable:

$$\text{hcap}([0, e^{i\alpha\pi}]) = \frac{1}{2}\alpha^{1-2\alpha}(1-\alpha)^{2\alpha-1} \rightarrow 0$$

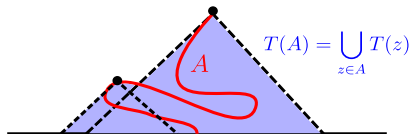
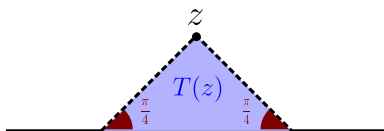
as $\alpha \rightarrow 0$.



Notation

For $z \in \mathbb{H}$, define

$T(z)$ = triangular region in the figure below

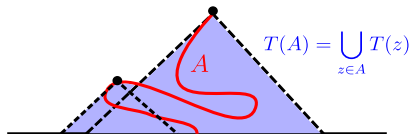
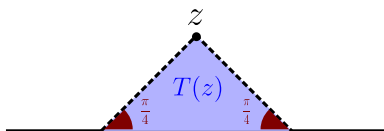


Notation

For $z \in \mathbb{H}$, define

$T(z)$ = triangular region in the figure below

$$T(A) = \bigcup_{z \in A} T(z)$$



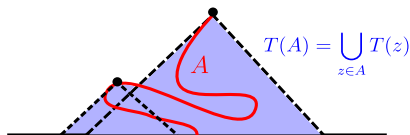
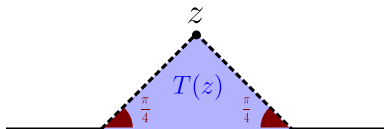
Main result

Theorem

There are constants $c_1, c_2 > 0$ such that

$$c_1 \cdot \text{area}(T(A)) \leq \text{hcap}(A) \leq c_2 \cdot \text{area}(T(A))$$

for all A .

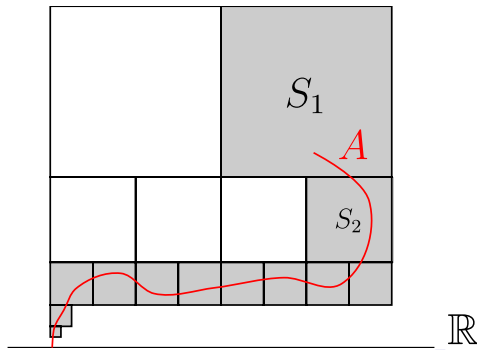


$S(A)$

Whitney squares:

$$S_{n,k} = [k2^n, (k+1)2^n] \times [2^n, 2^{n+1}] \quad (n, k \in \mathbb{Z})$$

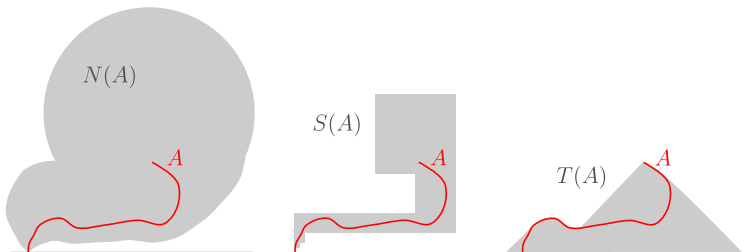
$$S(A) = \bigcup_j S_j$$



Comparable quantities

With absolute constants,

$$\text{area } N(A) \asymp \text{area } S(A) \asymp \text{area } T(A).$$



History of the problem

2001 Steffen Rohde and Michel Zinsmeister: showed a similar result for conformal radius when A is “already fat”.

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2009 Steven Lalley, Gregory Lawler, Hariharan Narayanan: proved the “ball version”. The proof also uses probabilistic argument.

2010 (with Steffen Rohde) a non-probabilistic proof, using a known result about conformal radius.

Outline of a proof for the main result

3 ingredients in our proof:

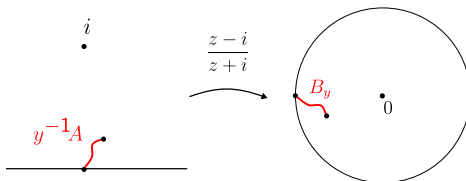
- ① an equivalent problem about dcap (conformal radius)
- ② a “fattening lemma”
- ③ a lemma for the fat case (“Variation of conformal radius”
Steffen Rohde and Michel Zinsmeister, 2001)

Half-plane capacity and conformal radius

$$\text{dcap}(B) = -\log f'(0) \asymp 1 - f'(0) \text{ for } B \subseteq \{\tfrac{1}{2} \leq |z| < 1\}.$$

Lemma

$$\lim_{y \rightarrow \infty} \frac{\text{dcap}(B_y)}{\text{hcap}(y^{-1}A)} = 2.$$



Equivalent problems

Problem

$$hcap(A) \asymp \text{area } N(A)$$

for any slit $A \subseteq \mathbb{H}$.

is equivalent to

Problem

$$dcap(B) \asymp \text{area } N(B)$$

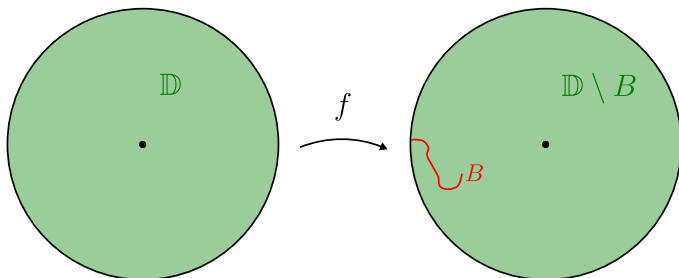
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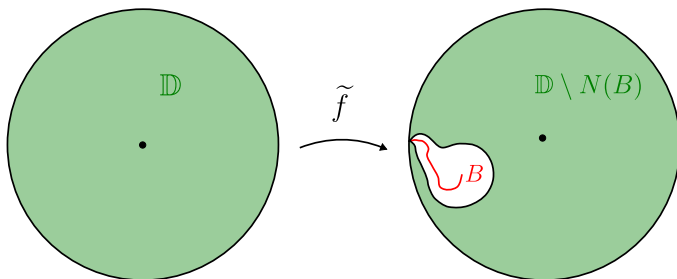
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Fattening lemma



$$\text{dcap}(B) = -\log |f'(0)| \asymp 1 - |f'(0)|$$

Fattening lemma



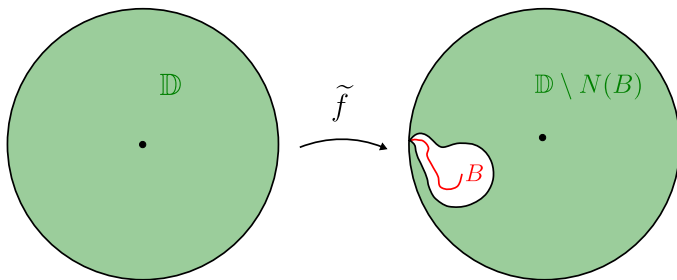
$$\text{dcap } N(B) = -\log \left| \tilde{f}'(0) \right| \asymp 1 - \left| \tilde{f}'(0) \right|$$

Fattening lemma

Lemma (Fattening lemma)

For $B \subseteq \{\frac{1}{2} \leq |z| < 1\}$,

$$d\text{cap}(B) \leq d\text{cap } N(B) \leq c \, d\text{cap}(B).$$



Lower bound for $\text{dcap}(B)$

Let $f: \mathbb{D} \rightarrow \mathbb{D} \setminus B$ be conformal with $f(0) = 0$.

$$f(z) = a_1 z + a_2 z^2 + \dots$$

$$\text{area}(\mathbb{D} \setminus B) = \pi \left(|a_1|^2 + 2|a_2|^2 + 3|a_3|^2 + \dots \right)$$

$$\geq \pi |a_1|^2$$

$$\text{area}(B) \leq \pi(1 - |a_1|^2) \asymp 1 - |a_1| \asymp \text{dcap}(B)$$

By the fattening lemma,

$$\text{dcap}(B) \gtrsim \text{dcap}N(B) \gtrsim \text{area } N(B).$$

Proof of fattening lemma (sketch)

Decompose \mathbb{D} into dyadic layers $D_n = \{\frac{1}{2^{n+1}} < 1 - |z| \leq \frac{1}{2^n}\}$.

$$\begin{aligned} \text{dcap}(B) &= -\log |f'(0)| \\ &= \frac{1}{2\pi} \int_{\partial\mathbb{D}} -\log |f(\zeta)| \, d\zeta \\ &\asymp \sum_{n=1}^{\infty} \frac{\omega_n(0)}{2^n}, \end{aligned}$$

where $\omega_n(z) = \omega(z, B_n, \mathbb{D} \setminus B_n)$ and $B_n = B \cap D_n$.

Proof of fattening lemma (sketch)

Similarly,

$$\text{dcap } N(B) \asymp \sum_{n=1}^{\infty} \frac{\tilde{\omega}_n(0)}{2^n}.$$

The fattening lemma follows if one can show that

$$\tilde{\omega}_n(z) \lesssim \omega_{n-1}(z) + \omega_n(z) + \omega_{n+1}(z)$$

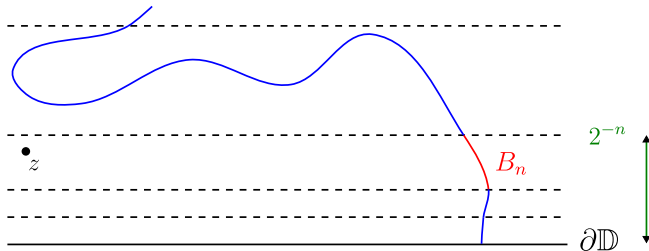
for $z \in \mathbb{D} \setminus N(B)$ and all n .

Harmonic measure estimate

It is in general not true that

$$\tilde{\omega}_n(z) \lesssim \omega_n(z)$$

for $z \in \mathbb{D} \setminus N(B)$.

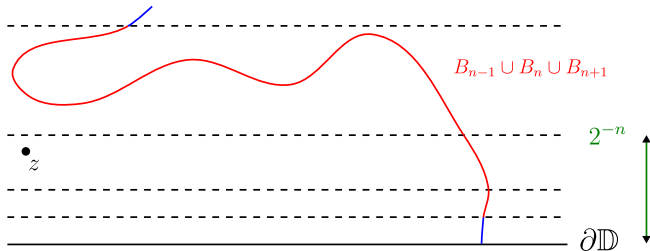


Harmonic measure estimate

By taking 3 terms, we have

$$\tilde{\omega}_n(z) \lesssim \omega_{n-1}(z) + \omega_n(z) + \omega_{n+1}(z)$$

for $z \in \mathbb{D} \setminus N(B)$.



Outline of a proof for the main result

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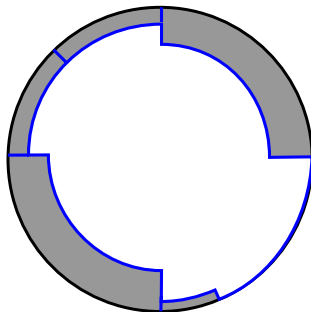
- 1 an equivalent problem about dcap (conformal radius)
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The fat case

Lemma (Rohde and Zinsmeister, 2001)

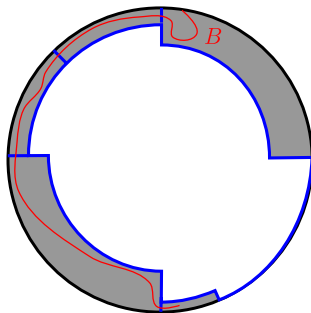
If $B = \bigcup_{j=1}^n Q_j$ is a union of disjoint dyadic squares, then

$$d\text{cap}(B) \asymp \text{area}(B).$$



Upper bound of $\text{dcap}(B)$

Cover B by dyadic squares in a minimal way. Call the larger set $Q(B) \supseteq B$.



$$\text{dcap}(B) \leq \text{dcap}Q(B) \lesssim \text{area } Q(B) \lesssim \text{area } N(B).$$

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Lemma (Rohde and Zinsmeister, 2001)

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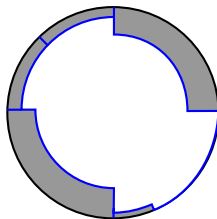
Sketch of proof. WLOG, $\text{area}(Q_1) \leq \text{area}(Q_2) \leq \dots \leq \text{area}(Q_n)$.

Using induction, it suffices to show that

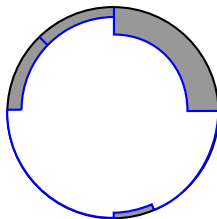
$$\text{dcap}\left(\bigcup_{j=1}^m Q_j\right) - \text{dcap}\left(\bigcup_{j=1}^{m-1} Q_j\right) \asymp \text{area}(Q_m)$$

for all m .

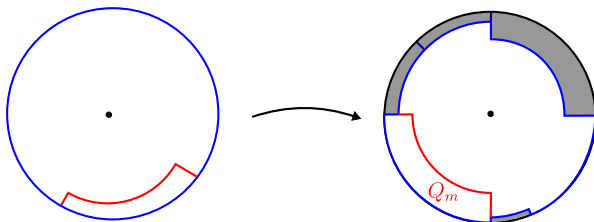
The induction step



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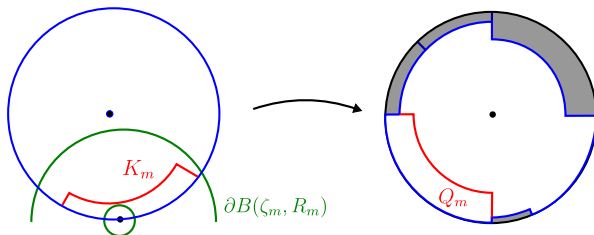
The induction step



On the right side,

$$\omega \asymp \text{diam}(Q_m) \asymp \sqrt{\text{area}(Q_m)}.$$

The induction step



On the left side, the red curve is bounded between two circles $\partial B(\zeta_m, \rho R_m)$ and $\partial B(\zeta_m, R_m)$ with center $\zeta_m \in \partial \mathbb{D}$, where $\rho \in (0, 1)$ is an absolute constant. Moreover,

$$\omega \asymp R_m \asymp \sqrt{\text{dcap}(K_m)}.$$

Thank you!

These slides are available at

<https://github.com/cartowong/halfplane-capacity>