Smoothness of Loewner Slits

Carto Wong

University of Washington

Nov 9, 2010

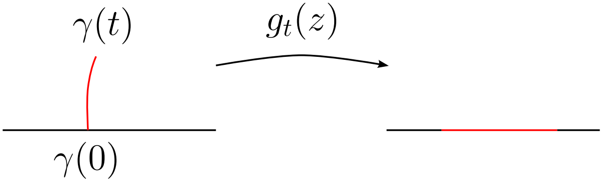
Chordal Loewner differential equation

Let $\mathbb{H} = \{\operatorname{Im} z > 0\}$ be the upper halfplane.

- A slit in $\mathbb{H}$ is a simple curve $\gamma\colon (0, T] \to \mathbb{H}$ with base $\gamma(0) \in \mathbb{R}$.
- $g_t\colon H_t \to \mathbb{H}$ is a conformal mapping from $H_t = \mathbb{H} \setminus \gamma([0, t])$ onto $\mathbb{H}$.

Hydrodynamic normalization:

$$g_t(z) = z + \frac{2t}{z} + \cdots \qquad (z \to \infty)$$



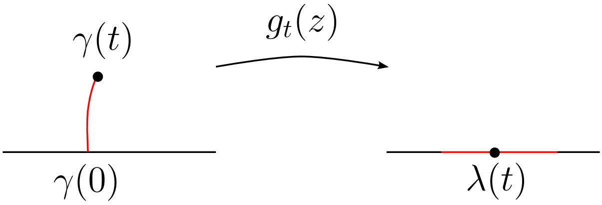
Chordal Loewner differential equation

Theorem

$g_t(z)$ satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)},$$

for all $z \in H_t = \mathbb{H} \setminus \gamma([0,t])$, where $\lambda\colon [0,T] \to \mathbb{R}$ is a continuous function. Moreover, $\lambda(t) = g_t(\gamma(t))$.



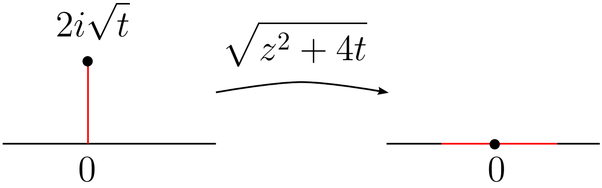
Example

vertical slit $\gamma(t) = 2i\sqrt{t}$

$$g_t(z) = \sqrt{z^2 + 4t}$$

$$\partial_t g_t(z) = \frac{2}{\sqrt{z^2 + 4t}} = \frac{2}{g_t(z)}$$

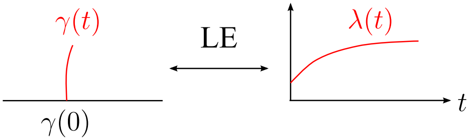
So, $\lambda(t) \equiv 0$ for the vertical slit.



We have a map

$$\left\{\begin{array}{l}\text{slits in the upper}\\ \text{halfplane } \mathbb{H} \text{ modulo}\\ \text{reparametrization}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{continuous functions}\\ \lambda\colon [0, T] \to \mathbb{R}\end{array}\right\}$$

which is 1-1, but not onto.



Kufarev gave an example (in the disc case) of a continuous $\lambda(t)$ which does not generate slit.

Question

When does $\lambda\colon [0, T] \to \mathbb{R}$ generate slit?

- (Kufarev) If λ is C^1, then it generates a slit.
- (Marshall, Rohde 2005) If $\|\lambda\|_{\mathrm{Lip}(\frac{1}{2})} < c_0$, then it generates a quasi-slit in $\mathbb{H}$.
- (Lind 2005) The best constant is $c_0 = 4$.

Question

What more can we say if λ is more regular than $\mathrm{Lip}(\frac{1}{2})$?

- (see [Ale]) If λ has bounded first derivative, then γ is C^1.

[Ale] I. A. Aleksandrov, Parametric continuations in the theory of univalent functions, Izdat. Nauka, 1976.

1st main result

Theorem (W- 2010)

Suppose $\lambda\colon [0,T] \to \mathbb{R}$ *is* $Lip(\frac{1}{2}+\delta)$ *with* $0 < \delta \le \frac{1}{2}$. *Then*

- (a) $\gamma(t^2)$ *is* $C^{1,\delta}$ *regular on* $[0,T]$.
- (b) $\gamma(t)$ *grows vertically at* $t=0$ *and*

$$\left|\gamma'(t) - \frac{i}{\sqrt{t}}\right| \le N t^{-\frac{1}{2}+\delta}$$

where $N > 0$ *is a constant.*

With an extra assumption $\|\lambda\|_{Lip(\frac{1}{2})} \le 1$, *both statements are quantitative.*

2nd main result

Theorem (W- 2010)

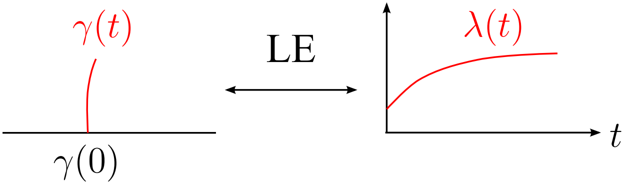
Let $\lambda\colon [0, T] \to \mathbb{R}$ *and* $0 < \delta \leq \frac{1}{2}$.

- a. *If* λ *is* $C^{1,\delta}$, *then* γ *is* $C^{1,\frac{1}{2}+\delta}$ *regular on* $(0, T]$.
- b. *If* λ *is* $C^{1,\frac{1}{2}+\delta}$, *then* γ *is* $C^{2,\delta}$ *regular on* $(0, T]$.

With an extra assumption $\|\lambda\|_{Lip(\frac{1}{2})} \leq 1$, *both statements are quantitative.*

Roughly speaking, $\lambda \in C^{n+\alpha} \Rightarrow \gamma \in C^{n+\frac{1}{2}+\alpha}$ for $n + \alpha \leq 2$.

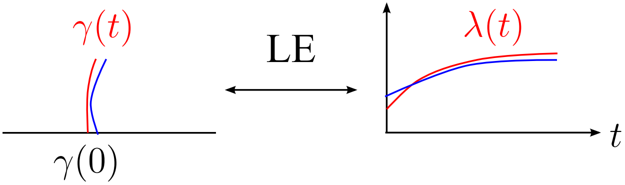
LE diagram



If we modify one picture,

- how does the other change?
- how do we quantify and estimate the change?

LE diagram



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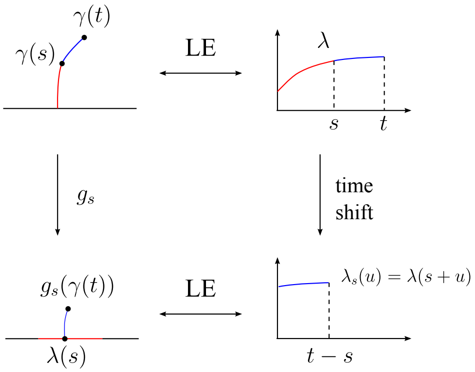
Stationary property

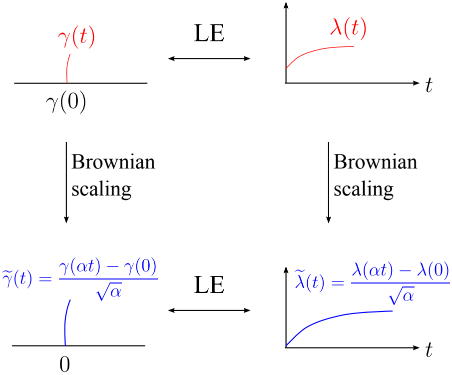


Proof.

$$\partial_u g_{s+u}(z) = \frac{2}{g_{s+u}(z) - \lambda(s+u)}.$$

$\square$

Scaling property



Proof.

$$\partial_t \left[\frac{1}{\sqrt{\alpha}} g_{\alpha t}(\sqrt{\alpha} z) \right] = \frac{2}{\frac{1}{\sqrt{\alpha}} g_{\alpha t}(\sqrt{\alpha} z) - \frac{1}{\sqrt{\alpha}} \lambda(\alpha t)}.$$

□

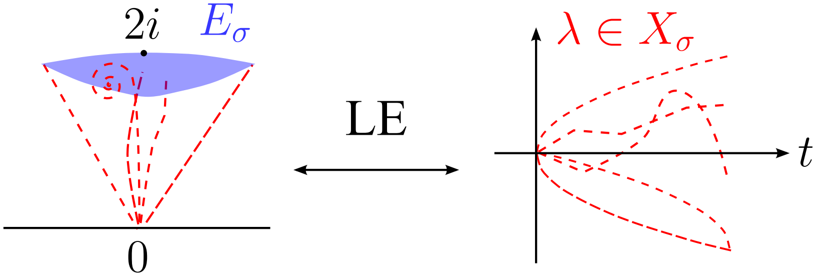
For $0 \leq \sigma < 4$, let

$$X_\sigma = \left\{ \lambda\colon [0,1] \to \mathbb{R} \text{ with } \lambda(0) = 0 \text{ and } \|\lambda\|_{\mathrm{Lip}(\frac{1}{2})} \leq \sigma \right\}$$

$(X_\sigma, \|\cdot\|_\infty)$ is a compact metric space and

$$\begin{aligned} \mathrm{Tip}\colon X_\sigma &\to \mathbb{H} \\ \lambda &\mapsto \gamma(1) \end{aligned}$$

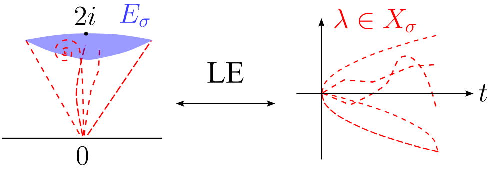
is continuous, and therefore has a compact image $E_\sigma = \mathrm{Tip}(X_\sigma)$.



Two lemmas concerning Tip: $X_\sigma \to \mathbb{H}$

Lemma (size of the image)

diam $E_\sigma = O(\sigma)$ *as* $\sigma \to 0$.



Lemma (modulus of continuity)

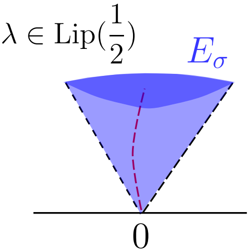
If $\|\lambda_j\|_{Lip(\frac{1}{2})} \leq 1$, *then*

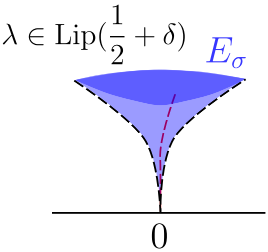
$$|Tip(\lambda_1) - Tip(\lambda_2)| \leq c\|\lambda_1 - \lambda_2\|_\infty.$$

Slit grows vertically

$$\lambda \in \text{Lip}(\frac{1}{2}) \Rightarrow \gamma(t) \in \sqrt{t}E_\sigma$$

$$\lambda \in \text{Lip}(\frac{1}{2}+\delta) \Rightarrow \gamma(t) \in \sqrt{t}E_{\sigma(t)} \text{ with } \sigma(t) = O(t^\delta)$$





1st main result

Theorem (W- 2010)

Suppose $\lambda \colon [0, T] \to \mathbb{R}$ is Lip$(\frac{1}{2} + \delta)$ with $0 < \delta \leq \frac{1}{2}$. Then

- (a) γ is $C^{1,\delta}$ regular on $(0, T]$.
- (b) $\gamma(t)$ grows vertically at $t = 0$ ✓ and

$$\left| \gamma'(t) - \frac{i}{\sqrt{t}} \right| \leq N t^{-\frac{1}{2}+\delta}$$

  where $N > 0$ is a constant.

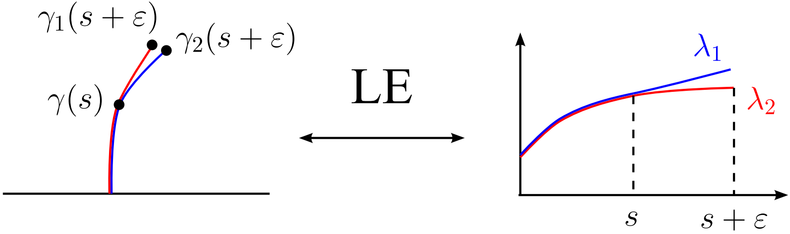
With an extra assumption $\|\lambda\|_{Lip(\frac{1}{2})} \leq 1$, both statements are quantitative.

Existence of $\gamma'(s)$

Suppose $\lambda_j\colon [0, T] \to \mathbb{R}$ ($j = 1, 2$) satisfy

$$\begin{cases} |\lambda_j(t_1) - \lambda_j(t_2)| \le M\, |t_1 - t_2|^{\frac{1}{2}+\delta} \\ \lambda_1 = \lambda_2 \text{ on } [0, s] \end{cases}$$

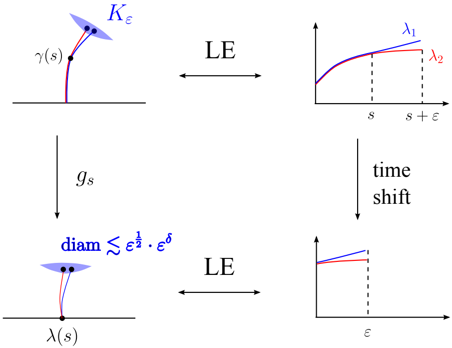
Then $|\gamma_1(s+\varepsilon) - \gamma_2(s+\varepsilon)| = O(\varepsilon^{?})$ as $\varepsilon \to 0+$.



Existence of $\gamma'(s)$

Fix $\lambda_1\colon [0,s] \to \mathbb{R}$ and let

$$K_\varepsilon = \left\{ \gamma^\lambda(s+\varepsilon) \in \mathbb{H} \;\middle|\; \begin{array}{l} \lambda\colon [0,s+\varepsilon] \to \mathbb{R} \text{ satisfying } \lambda = \lambda_1 \text{ on } [0,s] \\ \text{and } \sup\limits_{t_1 \neq t_2 \in [0,T]} \dfrac{|\lambda(t_1) - \lambda(t_2)|}{|t_1 - t_2|^{\frac{1}{2}+\delta}} \leq M \end{array} \right\}$$



Existence of $\gamma'(s)$

Lemma

Suppose $\lambda\colon [0, T] \to \mathbb{R}$ *satisfies*

- *the* (M, T, δ)*-Lip*$(\frac{1}{2} + \delta)$ *condition for some* $0 < \delta \leq \frac{1}{2}$*; and*
- $\|\lambda\|_{Lip(\frac{1}{2})} \leq 1$

then for any $0 < s < t \leq T$,

$$\frac{1}{C} \leq \sqrt{\frac{t-s}{t}}\,|g_s'(\gamma(t))| \leq C,$$

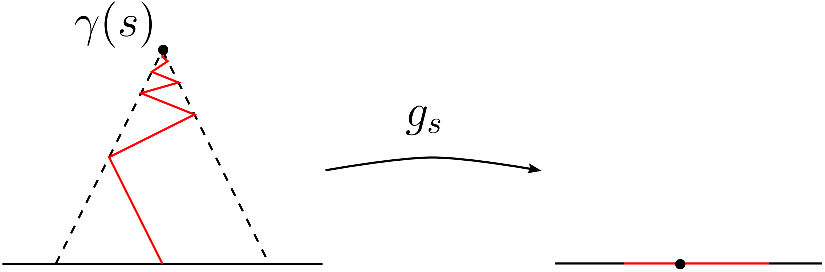
where $C = C(M, T, \delta) > 0$*. Moreover, for all* $s \in (0, T)$*, the limit*

$$\lim_{\varepsilon \to 0+} \sqrt{\varepsilon} g_s'(\gamma(s+\varepsilon)) = \sqrt{s} \exp\left[-\int_0^s \frac{1}{2u} + \frac{2}{\gamma(s-u,s)^2}\,du\right]$$

exists and is nonzero.

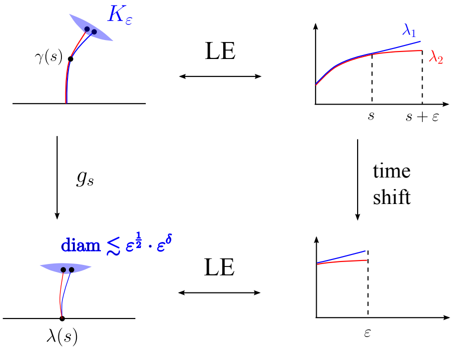
Existence of $\gamma'(s)$

For $\lambda \in Lip(\frac{1}{2} + \delta)$, the lemma says $|g_s'(\gamma(s+\varepsilon))| \asymp \varepsilon^{-\frac{1}{2}}$. Near the tip $\gamma(s)$ the slit halfplane cannot have a "corner of angle strictly less than 2π".



Existence of $\gamma'(s)$

$$\operatorname{diam} K_\varepsilon \lesssim \operatorname{diam}[g_s(K_\varepsilon)] \cdot \sup_z \left| g_s^{-1}(z) \right| \lesssim \varepsilon^{1+\delta}.$$

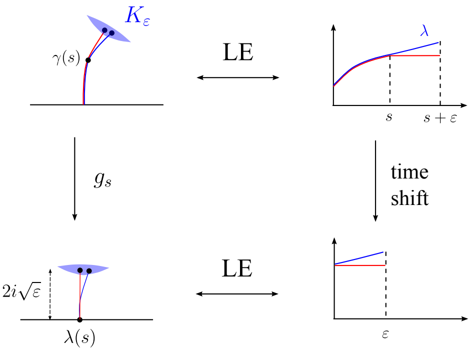


Existence of $\gamma'(s)$

Given $\lambda \in \mathrm{Lip}(\frac{1}{2}+\delta)$, modify λ so that it is constant after time s. This does not change the existence and the value of

$$\lim_{\varepsilon \to 0+} \frac{\gamma(s+\varepsilon)-\gamma(s)}{\varepsilon}.$$

Can assume $\gamma(s+\varepsilon) = f_s(\lambda(s)+2i\sqrt{\varepsilon})$, where $f_s = g_s^{-1}$.



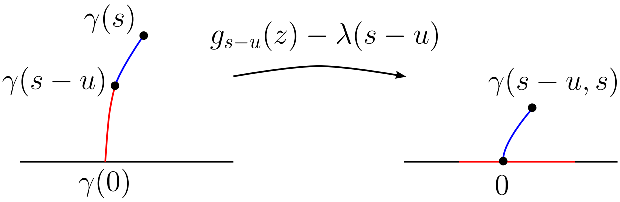
Existence of $\gamma'(s)$

Corollary

Assuming the (M, T, δ)-Lip$(\frac{1}{2} + \delta)$ condition and $\|\lambda\|_{Lip(\frac{1}{2})} \leq 1$, for any $0 < s \leq T$ we have

$$\gamma'(s) = \frac{i}{\sqrt{s}} \exp\left[\int_0^s \frac{1}{2u} + \frac{2}{\gamma(s-u,s)^2}\, du\right],$$

where $\gamma(s-u, s) = g_{s-u}(\gamma(s)) - \lambda(s-u)$.



$\gamma'(s) = \frac{i}{\sqrt{s}} e^{L(s)}$ is equally regular as the function

$$L(s) = \int_0^s \frac{1}{2u} + \frac{2}{\gamma(s-u,s)^2}\,du.$$

For $\lambda \in \mathrm{Lip}(\frac{1}{2}+\delta)$, we can show $L \in \mathrm{Lip}(\delta)$ by estimating

$$\begin{aligned} L(s+\varepsilon) - L(s) = \int_0^s & \frac{2}{\gamma(s+\varepsilon-u,s+\varepsilon)^2} - \frac{2}{\gamma(s-u,s)^2}\,du + \\ & \int_s^{s+\varepsilon} \frac{1}{2u} + \frac{2}{\gamma(s+\varepsilon-u,s+\varepsilon)^2}\,du. \end{aligned}$$

Current progress

All driving functions below are assumed to satisfy $\|\lambda\|_{\text{Lip}(\frac{1}{2})} \leq 1$.

Let $0 < \delta \leq \frac{1}{2}$.

- If $\lambda \in C^{1,\delta}$, then L is $\text{Lip}(\frac{1}{2} + \delta)$ and therefore γ is $C^{1,\frac{1}{2}+\delta}$.
- If $\lambda \in C^{1,\frac{1}{2}+\delta}$, then

$$\gamma''(s) = \frac{2\gamma'(s)}{\gamma(s)^2} - 4\gamma'(s)Q(s)$$

where $Q \in \text{Lip}(\delta)$ and is given by

$$Q(s) := \int_0^s \frac{\partial_s \gamma(s-u, s)}{\gamma(s-u,s)^3}\, du.$$

It means that γ is $C^{2,\delta}$.

Unsolved problem

- Conjecture: If $\lambda \in C^{n+\alpha}$ then $\gamma \in C^{n+\frac{1}{2}+\alpha}$.

  Proved for $n + \alpha \leq 2$.

- The converse statement: if $\gamma \in C^{n,\alpha}$, how smooth is λ?

  C. Earle, A. Epstein: if $\gamma \in C^n$ $(n \geq 2)$, then $\lambda \in C^{n-1}$.

Thank you!