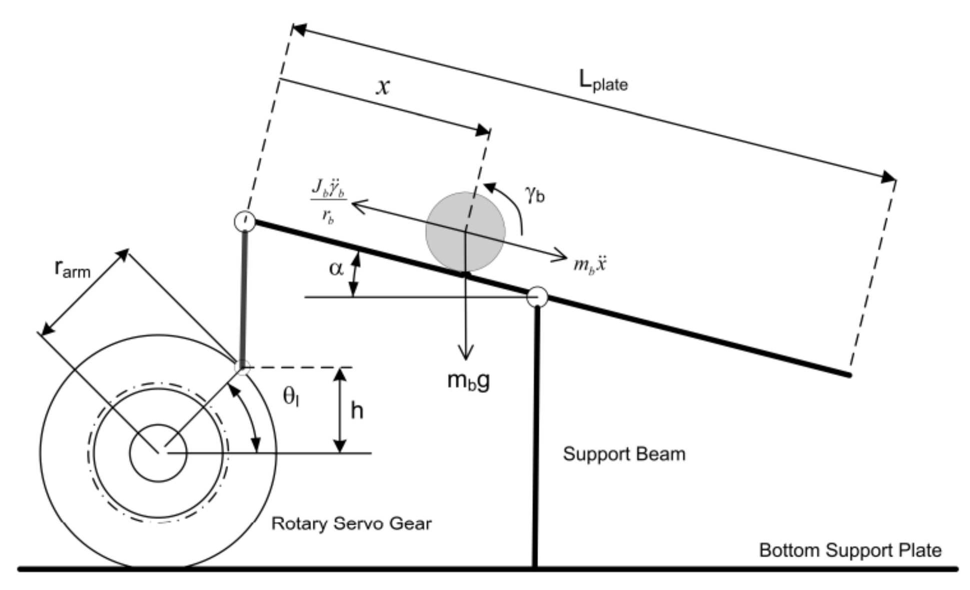
Mathematical Model

Tuesday, December 7, 2021 7:13 PM

first law of motion $F = ma$

$$m_b \ddot{x}(t) = \sum F = F_{xt} - F_{xr}$$



$$F_{xt} = m_b g \sin\alpha(t)$$

$$F_{xr} = \frac{\tau_b}{r_b}$$

$$\tau_b = J_b \ddot{\gamma}_b(t) \qquad \text{torque}$$

$$F_{xr} = \frac{J_b \ddot{x}(t)}{r_b^2}$$

$$m_b \ddot{x}(t) = m_b g \sin\alpha(t) - \frac{J_b \ddot{x}(t)}{r_b^2}$$

$$\ddot{x}(t) = \frac{m_b g \sin\alpha(t)\, r_b^2}{m_b r_b^2 + J_b} \qquad \text{linear acceleration}$$

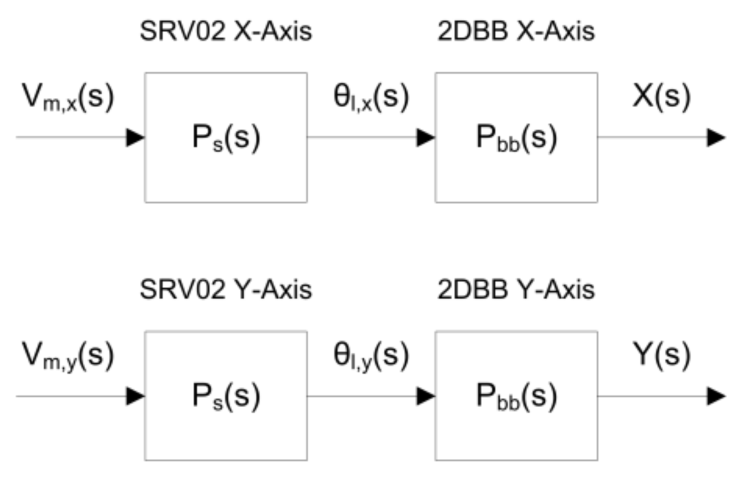
$$\sin\alpha(t) = \frac{2h}{L_{plate}}$$

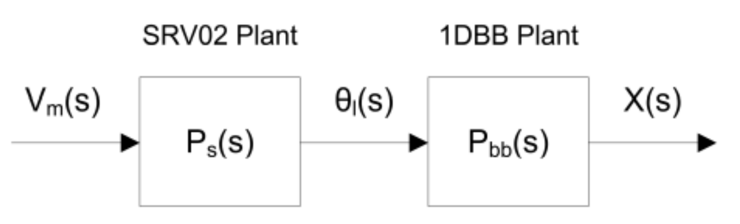
$$\sin\theta_l(t) = \frac{h}{r_{arm}}$$

$$\sin\alpha(t) = \frac{2 r_{arm} \sin\theta_l(t)}{L_{plate}} \qquad \text{relationship between plate and angle}$$

$$\ddot{x}(t) = \frac{2 m_b g r_{arm} r_b^2}{L_{plate}(m_b r_b^2 + J_b)} \sin\theta_l(t)$$

$$\ddot{x}(t) = \frac{2 m_b g r_{arm} r_b^2}{L_{plate}(m_b r_b^2 + J_b)} \theta_l(t)$$





obtaining the transfer function

$$P(s) = P_{bb}(s)\, P_s(s)$$

$$P_{bb}(s) = \frac{X(s)}{\Theta_l(s)} \qquad \text{angle to ball position}$$

$$P_s(s) = \frac{\Theta_l(s)}{V_m(s)} \qquad \text{voltage to angle}$$

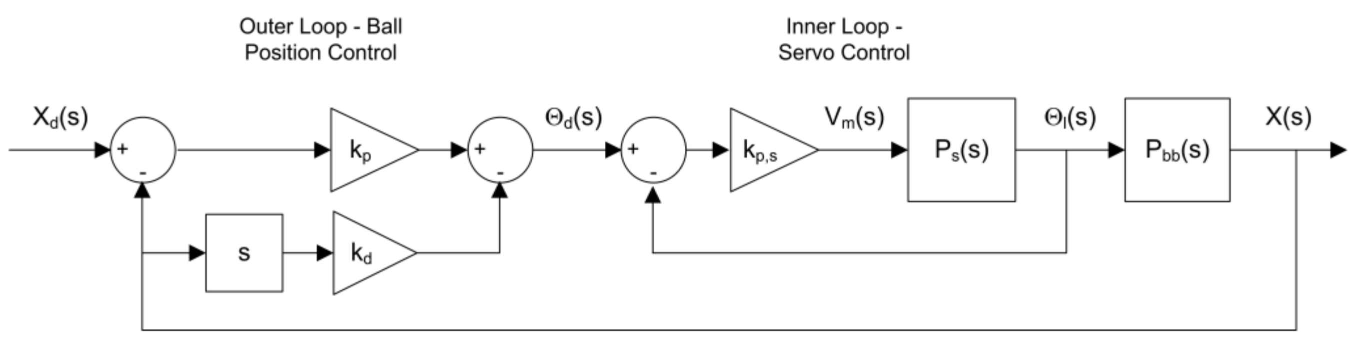
$$P_s(s) = \frac{K}{s(\tau s + 1)}$$

$$K = 1.53 \ \frac{\text{rad}}{Vs}$$

$$\tau = .0248 \text{ sec}$$

$$P_{bb}(s) = \frac{X(s)}{\Theta_l(s)} = \frac{K_{bb}}{s^2} \qquad \text{Laplace}$$

$$P(s) = \frac{X(s)}{V_m(s)} = \frac{K_{bb} K}{s^3(\tau s + 1)} \qquad \text{voltage to ball displacement}$$



$$\Theta_l(t) = \Theta_d(t)$$

$$\Theta_d(s) = K_p(X_d(s) - X(s)) - K_d s X(s)$$

$$\frac{X(s)}{X_d(s)} = \frac{K_{bb} K_p}{s^2 + K_{bb} K_d s + K_{bb} K_p} \qquad \text{transfer function}$$

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s\, \omega_n^2}$$

$$t_s = \frac{-\ln\left(c_{ts}\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

percent overshoot

ω_n = natural frequency

ζ = damping ratio

$$PO = 100 \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

state space transfer function

$$\underset{A}{\begin{bmatrix} -K_{bb}K_d & -K_{bb}K_p \\ 1 & 0 \end{bmatrix}} \quad \underset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \quad \underset{C}{\begin{bmatrix} 0 \\ K_{bb}K_p \end{bmatrix}} \quad D$$