LECTURE NOTES F-SINGULARITIES

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ABSTRACT. These are lectures notes for a course on F-singularities given at the CIMAT in the Spring Semester 2024.

1. REGULARITY

Before defining what a regular ring is, we need the notion of *projective and global dimensions*.

1.1. **Projective resolutions and other homological algebra stuff.** Let M be a module over a ring R.[1]

Exercise 1.1. Prove that there is an exact sequence of R-modules

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_0 is free and so projective. Iterate this to obtain an exact

$$0 \longrightarrow K_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where the P_i's are free. The module K_i is referred to as *a syzygy module.*

Definition 1.1 (Resolutions)**.** An exact sequence

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is called a *free (resp. projective) resolution* of M if all the P_i's are free (resp. projective). We may denote a projective resolution as $P_\bullet \longrightarrow M \longrightarrow 0$. [2]

Exercise 1.2. Prove that free resolutions always exist and so that the category of R-modules has "enough projectives."

Definition 1.2 (Projective dimension)**.** The module M is said to have *finite projective dimension* if there is a projective resolution $P_\bullet \longrightarrow M \longrightarrow 0$ such that $P_i = 0$ for all $i \gg 0$. In such case, the *projective dimension* of M is

$$\operatorname{pd} M = \operatorname{pd}_R M \coloneqq \min\{n \in \mathbb{N} \mid \exists P_\bullet \longrightarrow M \longrightarrow 0 \text{ such that } P_i = 0 \forall i > n\}.$$

If M has not finite projective dimension we write $\operatorname{pd} M = \infty$.

Exercise 1.3. Prove that M is projective iff $\operatorname{pd} M = 0$.

Next lemma is key.

[1] All rings are commutative with unity 1.

[2] Over local rings projective modules are free.

Lemma 1.3. *Suppose that there are two exact sequences of R-modules*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow$$

and

$$0 \longrightarrow K'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow M \longrightarrow$$

where $1 \leq n \in \mathbb{N}$ and the P_i and P'_i are projective. Then

(a) $K_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \cong K'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots$

(b) K_n *is projective iff so is* K'_n.

Proof. Note that (b) follows from (a).[3] The proof of (b) is lengthy and left as an exercise. Hint: Proceed by induction on n. Prove the case $n = 1$ first and then reduce the inductive case to this one. □

It can be used to prove the following.

Exercise 1.4. Let

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow$$

be an exact sequences where thee P_i's are projective. Prove that

(a) $\operatorname{pd} M \leq n$ iff K_n is projective.

(b) If $\operatorname{pd} M \geq n$ then $\operatorname{pd} K_n = \operatorname{pd} M - n$.

Exercise 1.5. Suppose that R is noetherian and that M is finitely generated. Prove that

$$\operatorname{pd}_R M = \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup\{\operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \text{ maximal}\}$$

Exercise 1.6. Prove that

$$\operatorname{pd}(M \oplus N) = \max\{\operatorname{pd} M, \operatorname{pd} N\}.$$

The above exercise generalizes as follows.

Exercise* 1.7. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R-modules. Show the following statements.

(a) If two of the modules in the exact sequence have finite projective dimension so does the third one.

(b) In that case (*i.e.* the three modules have finite projective dimension) then

$$\operatorname{pd} M \leq \max\{\operatorname{pd} M', \operatorname{pd} M''\},$$

(c) and if the inequality is strict then $\operatorname{pd} M'' = \operatorname{pd} M' + 1$.

Definition 1.4 (Minimal free resolution)**.** Let $(R, \mathfrak{m}, \mathcal{k})$ be a noetherian local ring and M a finitely generated R-module. A free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ is said to be *minimal* if

$$\phi_i(P_{i+1}) \subset \mathfrak{m} P_i \quad \forall i \in \mathbb{N}$$

where $\phi_i \colon P_{i+1} \longrightarrow P_i$ is the homomorphism from the free resolution.

Exercise 1.8. In the setup of Definition 1.4, let $K_i \coloneqq \ker \phi_{i-1}$ for all $i \geq 1$. Prove that $\mu(P_0) = \mu(M)$ and $\mu(P_i) = \mu(K_i)$ for all $i \geq 1$. Here, we let $\mu(-) = \dim_{\mathcal{k}} - \otimes_R \mathcal{k}$ denote the minimal number of generators.

[3] Observe that for this is absolutely essential to use projectiveness instead of freeness.

Exercise 1.9. Show that minimal free resolutions exist.

Exercise 1.10. In the setup of Definition 1.4, let $P_\bullet \longrightarrow M \longrightarrow 0$ and $P'_\bullet \longrightarrow M \longrightarrow 0$ be two minimal free resolutions. Show that $\mu(P_i) = \mu(P'_i)$ for all $i \in \mathbb{N}$.

The above two exercises guarantee that the following definition makes sense.

Definition 1.5 (Betti numbers)**.** In the setup of Definition 1.4, the *i-th Betti number* of M is defined as $\beta_i(M) \coloneqq \mu(P_i)$ where $P_\bullet \longrightarrow M \longrightarrow 0$ is any minimal free resolution.

Remark 1.6. Sometimes people talk about the Betti numbers of $(R, \mathfrak{m}, \Bbbk)$, in that case they refer to the Betti numbers of $\Bbbk$.

Exercise 1.11. Let $P_\bullet \longrightarrow M \longrightarrow 0$ be a minimal free resolution. Prove that $P_i = 0$ if (and only if) $i > \operatorname{pd} M$. That is,

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$$

Exercise 1.12. Prove that

$$\beta_i(M) = \dim_{\Bbbk} \operatorname{Tor}_i(\Bbbk, M), \quad \forall i \in \mathbb{N}.$$

and conclude that

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i(\Bbbk, M) \neq 0\} \leq \operatorname{pd} \Bbbk.$$

Definition 1.7 (Global dimension)**.** The *global dimension* of a ring R is the supremum of the projective dimensions of finitely generated R-modules.

Corollary 1.8. *The global dimension of a local ring is the projective dimension of its residue field.*

Remark 1.9 (Regular sequences and depth). Recall that a *regular element* $r \in R$ on an R-module M is one for which $\cdot r \colon M \longrightarrow M$ is injective but not surjective. A *regular sequence* $r_1, \ldots, r_d \in R$ on M is defined by the following two conditions:

(a) r_1 is regular on M, and
(b) r_i is regular on $M/(r_1, \ldots, r_{i-1})M$ for all $i = 2, \ldots d$.

Given an ideal $\mathfrak{a} \subset R$, the *depth of* $\mathfrak{a}$ *on* M, denoted by $\operatorname{depth}_R(\mathfrak{a}, M)$, is the maximal length of a regular sequence on M of elements in $\mathfrak{a}$. When $(R, \mathfrak{m}, \Bbbk)$ is local, we may write $\operatorname{depth} M = \operatorname{depth}_R M = \operatorname{depth}_R(\mathfrak{m}, M)$. In that case, we also have:

$$\operatorname{depth} M = \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i(\Bbbk, M) \neq 0\}.$$

This formula can be proved as follows (details are left to the reader). First, prove that if $r_1, \ldots, r_d \in R$ is a regular sequence on M then

$$\operatorname{Ext}^i_R(\Bbbk, M) = \begin{cases} 0 & \text{if } i < d, \\ \operatorname{Hom}_R(\Bbbk, M/(r_1, \ldots, r_d)M) & \text{if } i = d. \end{cases}$$

This can be proved by induction on d. The base step $d = 0$ is trivial. For the inductive step, consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot r_1} M \longrightarrow M/r_1M \longrightarrow 0$$

Next, apply the functor $\operatorname{Hom}_R(\Bbbk, -)$ to it. Since $r_1 \in \mathfrak{m}$, it acts like 0 on $\Bbbk$ and so $\operatorname{Ext}^i_R(\Bbbk, \cdot r_1) = 0$. This means that the long exact sequence on Ext's breaks down into exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_R(\Bbbk, M) \longrightarrow \operatorname{Ext}^i_R(\Bbbk, M/r_1M) \longrightarrow \operatorname{Ext}^{i+1}_R(\Bbbk, M) \longrightarrow 0$$

Since $r_2, \dots, r_d$ is a regular sequence on M/r_1M, we may apply the inductive hypothesis and conclude.

Theorem 1.10 (Auslander–Buchsbaum formula)**.** *In the setup of Definition 1.4, if* $\operatorname{pd} M < \infty$ *then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R.$$

In particular, if R has finite global dimension it is at most $\operatorname{depth} R$.

Proof. We only sketch a proof and leave the details to the reader as an exercise. The proof is an induction on $\operatorname{pd} M$. If $\operatorname{pd} M = 0$ then M is free and so $\operatorname{depth} M = \operatorname{depth} R$. If $\operatorname{pd} M = 1$ then there is an exact sequence

$$0 \longrightarrow R^{\oplus m} \xrightarrow{\phi} R^{\oplus n} \longrightarrow M \longrightarrow 0$$

which we may assume to be minimal, *i.e.* we may assume that the entries of the $n \times m$ R-matrix $\phi\colon R^{\oplus m} \longrightarrow R^{\oplus n}$ are in $\mathfrak{m}$. Consider next the long exact sequence on Ext obtained by applying the functor $\operatorname{Hom}_R(\mathscr{k}, -)$ (write it down yourself). Observe that $\operatorname{Ext}^i_R(\mathscr{k}, R^{\oplus k}) = \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus k}$ and that

$$\operatorname{Ext}^i_R(\mathscr{k}, \phi) : \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus m} \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus n}$$

is given by the $\mathscr{k}$-matrix obtained by reducing ϕ modulo $\mathfrak{m}$. In particular, $\operatorname{Ext}^i_R(\mathscr{k}, \phi) = 0$ and so there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus n} \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathscr{k}, R)^{\oplus m} \longrightarrow 0$$

From this, we see that $\operatorname{depth} M = \operatorname{depth} R - 1$. This shows the base step of the induction. For the inductive step, suppose $\operatorname{pd} M \geq 2$ and consider an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0$$

where $\operatorname{pd} N = \operatorname{pd} M - 1$. Use the corresponding long exact sequence on Ext's obtained by applying $\operatorname{Hom}_R(\mathscr{k}, -)$ to find the relationship between the depths of M and N (which is $\operatorname{depth} N = \operatorname{depth} M + 1$). Use the inductive hypothesis to conclude. $\square$

Remark 1.11. It is not difficult to see (using Krull's height theorem) that every regular sequence can be extended to a system of parameters. In particular, $\operatorname{depth} R \leq \dim R$.[4] When this equality happens to be an equality one says that $(R, \mathfrak{m}, \mathscr{k})$ is *Cohen–Macaulay.* That is, every system of parameters is a regular sequence.

1.2. **Regular local rings.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring. Then, by Nakayama's lemma its so-called *embedded dimension*

$$\operatorname{edim} R \coloneqq \mu(\mathfrak{m}) = \dim_{\mathscr{k}} \mathfrak{m} \otimes \mathscr{k} = \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2$$

is finite.

Exercise 1.13. Use Krull's Hauptidealsatz to conclude that the embedded dimension is at least the Krull's dimension of the local ring.

Definition 1.12 (Regular local ring)**.** A noetherian local ring $(R, \mathfrak{m}, \mathscr{k})$ is said to be *regular* if the inequality

$$\operatorname{edim} R \geq \dim R$$

is an equality.

[4]More generally, $\operatorname{depth}(\mathfrak{a}, R) \leq \operatorname{ht} \mathfrak{a}$.

Exercise 1.14. Prove that if $(R, \mathfrak{m}, \mathscr{k})$ is a noetherian local ring such that $\mathfrak{m}$ is generated by a regular sequence then it is regular.

The converse of this exercise is also true but a bit harder to prove.

Theorem 1.13. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a regular (noetherian) local ring. Then every set of minimal generators of $\mathfrak{m}$ (aka regular system of parameters) is a regular sequence. In particular, $\operatorname{pd}_R \mathscr{k} = \dim R$.*[5]

This result can be seen as a consequence of the following.

Theorem 1.14. *A regular local ring is an integral domain.*[6]

Proof. TO BE ADDED. □

Corollary 1.15. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a regular local ring and $r \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then R/rR is regular and the converse holds if r is not in any minimal prime.*

Summing up, regular local rings have finite global dimension equal to its dimension. It turns out that the converse is also true and it's a deep result due to Auslander–Buchsbaum and Serre. To prove this, we need the following observation.

Exercise 1.15. Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring and M be a finitely generated R-module. Let $r \in R$ be a regular element on M. Prove that

$$\operatorname{pd}_{R/xR} M/xM = \operatorname{pd}_R M$$

Theorem 1.16 (Auslander–Buchsbaum–Serre)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local noetherian ring. Then the following statements are equivalent.*

(a) R is regular (i.e. $\mathfrak{m}$ is generated by a regular sequence)
(b) The global dimension of R is $\dim R$
(c) $\operatorname{pd}_R \mathscr{k}$ is finite.

Definition 1.17 (Regular rings)**.** We say that a noetherian ring of finite Krull dimension $\dim R$ is regular if any of the following equivalente conditions hold:

(a) The local ring $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$.
(b) The global dimension of R is at most $\dim R$.
(c) R has finite global dimension.

References

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[5]In particular, regular local rings are Cohen–Macaulay, *i.e.* $\operatorname{depth} R = \dim R$.
[6]In fact normal.