LECTURE NOTES F-SINGULARITIES

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Abstract. These are lectures notes for a course on F-singularities given at the CIMAT in the Spring Semester 2024.

Contents

1. Regularity (a crash course)

This is a course about F-singularities and in particular about singularities. In a nutshell, singularities are the absence of regularity. Before defining what a regular ring is, we need the notion of *projective and global dimensions*.

1.1. **Projective resolutions and other homological algebra stuff.** Let M be a module over a ring R.[1]

[1]All rings are commutative with unity 1.

Exercise 1.1. Prove that there is an exact sequence of R-modules

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_0 is free and so projective. Iterate this to obtain an exact sequence

$$0 \longrightarrow K_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where the P_i's are free. The module K_i is referred to as *a syzygy module.*

Definition 1.1 (Resolutions)**.** An exact sequence

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is called a *free (resp. projective) resolution* of M if all the P_i's are free (resp. projective). We may denote a projective resolution as $P_\bullet \longrightarrow M \longrightarrow 0$.[2]

Exercise 1.2. Prove that free resolutions always exist, *i.e.* the category of R-modules has "enough projectives."

Definition 1.2 (Projective dimension)**.** The module M is said to have *finite projective dimension* if there is a projective resolution $P_\bullet \longrightarrow M \longrightarrow 0$ such that $P_i = 0$ for all $i \gg 0$. In such case, the *projective dimension* of M is

$$\operatorname{pd} M = \operatorname{pd}_R M \coloneqq \min\{n \in \mathbb{N} \mid \exists P_\bullet \longrightarrow M \longrightarrow 0 \text{ such that } P_i = 0 \forall i > n\}.$$

If M has not finite projective dimension we write $\operatorname{pd} M = \infty$.

Exercise 1.3. Prove that M is projective iff $\operatorname{pd} M = 0$.

Next lemma is key.

Lemma 1.3. *Suppose that there are two exact sequences of R-modules*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow K'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow M \longrightarrow 0$$

where $1 \leq n \in \mathbb{N}$ and the P_i and P'_i are projective. Then

(a) $K_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \cong K'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots$
(b) K_n is projective iff so is K'_n.

Proof. Note that (b) follows from (a).[3] The proof of (b) is lengthy and left as an exercise. Hint: Proceed by induction on n. Prove the case $n = 1$ first and then reduce the inductive case to this one. □

It can be used to prove the following.

Exercise 1.4. Let

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an exact sequences where thee P_i's are projective. Prove that

(a) $\operatorname{pd} M \leq n$ iff K_n is projective.
(b) If $\operatorname{pd} M \geq n$ then $\operatorname{pd} K_n = \operatorname{pd} M - n$.

[2]Over local rings projective modules are free (Kaplansky's theorem). That is, projective modules are locally free. The converse, however, isn't true (unless the module in question is finitely generated).
[3]Observe that for this is absolutely essential to use projectiveness instead of freeness.

Exercise 1.5. Suppose that R is noetherian and that M is finitely generated. Prove that

$$\operatorname{pd}_R M = \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup\{\operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \text{ maximal}\}$$

Exercise 1.6. Prove that

$$\operatorname{pd}(M \oplus N) = \max\{\operatorname{pd} M, \operatorname{pd} N\}.$$

The above exercise generalizes as follows.

Exercise* 1.7. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R-modules. Show the following statements.

(a) If two of the modules in the exact sequence have finite projective dimension then so does the third one.

(b) In that case (*i.e.* the three modules have finite projective dimension), then

$$\operatorname{pd} M \leq \max\{\operatorname{pd} M', \operatorname{pd} M''\},$$

(c) and if the inequality is strict then $\operatorname{pd} M'' = \operatorname{pd} M' + 1$.

Definition 1.4 (Minimal free resolution)**.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring and M a finitely generated R-module. A free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ is said to be *minimal* if

$$\phi_i(P_{i+1}) \subset \mathfrak{m} P_i \quad \forall i \in \mathbb{N}$$

where $\phi_i \colon P_{i+1} \longrightarrow P_i$ is the homomorphism from the free resolution.

Exercise 1.8. In the setup of Definition 1.4, let $K_i := \ker \phi_{i-1}$ for all $i \geq 1$. Prove that $\mu(P_0) = \mu(M)$ and $\mu(P_i) = \mu(K_i)$ for all $i \geq 1$. Here, we let

$$\mu(-) = \dim_{\mathscr{k}} - \otimes_R \mathscr{k}$$

denote the minimal number of generators.

Exercise 1.9. Show that minimal free resolutions exist.

Exercise 1.10. In the setup of Definition 1.4, let $P_\bullet \longrightarrow M \longrightarrow 0$ and $P'_\bullet \longrightarrow M \longrightarrow 0$ be two minimal free resolutions. Show that $\mu(P_i) = \mu(P'_i)$ for all $i \in \mathbb{N}$.

The above two exercises guarantee that the following definition makes sense.

Definition 1.5 (Betti numbers)**.** In the setup of Definition 1.4, the *i-th Betti number* of M is defined as $\beta_i(M) := \mu(P_i)$ where $P_\bullet \longrightarrow M \longrightarrow 0$ is any minimal free resolution.

Remark 1.6. Sometimes people talk about the Betti numbers of $(R, \mathfrak{m}, \mathscr{k})$, in that case, they refer to the Betti numbers of $\mathscr{k}$.

Exercise 1.11. Let $P_\bullet \longrightarrow M \longrightarrow 0$ be a minimal free resolution. Prove that $P_i = 0$ if (and only if) $i > \operatorname{pd} M$. That is,

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$$

Exercise 1.12. Prove that

$$\beta_i(M) = \dim_{\mathscr{k}} \operatorname{Tor}_i(\mathscr{k}, M), \quad \forall i \in \mathbb{N}.$$

and conclude that

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i(\mathscr{k}, M) \neq 0\} \leq \operatorname{pd} \mathscr{k}.$$

Definition 1.7 (Global dimension)**.** The *global dimension* of a ring R is the supremum of the projective dimensions of finitely generated R-modules.

Corollary 1.8. *The global dimension of a local ring is the projective dimension of its residue field.*

Remark 1.9 (Regular sequences and depth). Recall that a *regular element* $r \in R$ on an R-module M is one for which $\cdot r \colon M \longrightarrow M$ is injective but not surjective. A *regular sequence* $r_1, \dots, r_d \in R$ on M is defined by the following two conditions:

(a) r_1 is regular on M, and
(b) r_i is regular on $M/(r_1, \dots, r_{i-1})M$ for all $i = 2, \dots, d$.

Given an ideal $\mathfrak{a} \subset R$, the *depth of* $\mathfrak{a}$ *on* M, denoted by $\operatorname{depth}_R(\mathfrak{a}, M)$, is the maximal length of a regular sequence on M of elements in $\mathfrak{a}$. When $(R, \mathfrak{m}, \mathscr{k})$ is local, we may write $\operatorname{depth} M = \operatorname{depth}_R M = \operatorname{depth}_R(\mathfrak{m}, M)$. In that case, if $M \neq 0$, we also have:

$$\operatorname{depth} M = \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i(\mathscr{k}, M) \neq 0\}.$$

This formula can be proved as follows (details are left to the reader). First, prove that if $r_1, \dots, r_d \in R$ is a regular sequence on M then

$$\operatorname{Ext}^i_R(\mathscr{k}, M) = \begin{cases} 0 & \text{if } i < d, \\ \operatorname{Hom}_R(\mathscr{k}, M/(r_1, \dots, r_d)M) & \text{if } i = d. \end{cases}$$

This can be proved by induction on d. The base step $d = 0$ is trivial. For the inductive step, consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot r_1} M \longrightarrow M/r_1M \longrightarrow 0$$

Next, apply the functor $\operatorname{Hom}_R(\mathscr{k}, -)$ to it. Since $r_1 \in \mathfrak{m}$, it acts like 0 on $\mathscr{k}$ and so $\operatorname{Ext}^i_R(\mathscr{k}, \cdot r_1) = 0$. This means that the long exact sequence on Ext's breaks down into exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M) \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M/r_1M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathscr{k}, M) \longrightarrow 0$$

Since $r_2, \dots, r_d$ is a regular sequence on M/r_1M, we may apply the inductive hypothesis and conclude.

More generally, if $\mathfrak{a}M \neq M$ then

$$\operatorname{depth}_R(\mathfrak{a}, M) = \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M)\}$$

Exercise 1.13. Here's an important trick to know (particularly, when dealing with Frobenius). Prove that if $r_1, \dots, r_n$ is a regular sequence on M then so is $r_1^{e_1}, \dots, r_d^{e_d}$ for any sequence of exponents $e_1, \dots, e_n \in \mathbb{N}$.

Exercise 1.14. Prove that if r, s is a regular sequence then so is s, r. Find an example of a regular sequence of three elements which is no longer regular after swapping two elements.

Remark 1.10 (Permutations of regular sequences). In general, it turns out that the notion of regular sequence (for more than two elements) is susceptible to the order of the elements. However, this is not the case if we work over a local ring (or in a graded setup). Indeed, given a local ring $(R, \mathfrak{m}, \mathscr{k})$, if $r_1, \dots, r_d$ is an (M-)regular sequence then so is any permutation of them. Naturally, it suffices to prove this for transpositions of elements. This is something you may try to prove yourself in an elementary fashion (although not so easy). There's, however, a neat (but not so elementary) proof based on a much more general principle. To any sequence of elements we may attach its *Koszul complex*. It turns out that if the sequence is regular its Koszul complex is acyclic. The converse is true over local rings assuming the elements of the sequence belong to the maximal ideal (and in the morally local graded setup). Since the Koszul complex is independent of the permutation of elements in the sequence, then so is the notion of regularity over local rings.

Theorem 1.11 (Auslander–Buchsbaum formula)**.** *In the setup of Definition 1.4, if* $\operatorname{pd} M < \infty$ *then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R.$$

In particular, if R has finite global dimension it is at most $\operatorname{depth} R$.

Proof. We only sketch a proof and leave the details to the reader as an exercise. The proof is an induction on $\operatorname{pd} M$. If $\operatorname{pd} M = 0$ then M is free and so $\operatorname{depth} M = \operatorname{depth} R$. If $\operatorname{pd} M = 1$ then there is an exact sequence

$$0 \longrightarrow R^{\oplus m} \xrightarrow{\phi} R^{\oplus n} \longrightarrow M \longrightarrow 0$$

which we may assume to be minimal, *i.e.* we may assume that the entries of the $n \times m$ R-matrix $\phi \colon R^{\oplus m} \longrightarrow R^{\oplus n}$ are in $\mathfrak{m}$. Consider next the long exact sequence on Ext obtained by applying the functor $\operatorname{Hom}_R(\mathscr{k}, -)$ (write it down yourself). Observe that $\operatorname{Ext}^i_R(\mathscr{k}, R^{\oplus k}) = \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus k}$ and that

$$\operatorname{Ext}^i_R(\mathscr{k}, \phi) : \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus m} \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus n}$$

is given by the $\mathscr{k}$-matrix obtained by reducing ϕ modulo $\mathfrak{m}$. In particular, $\operatorname{Ext}^i_R(\mathscr{k}, \phi) = 0$ and so there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus n} \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathscr{k}, R)^{\oplus m} \longrightarrow 0$$

From this, we see that $\operatorname{depth} M = \operatorname{depth} R - 1$. This shows the base step of the induction. For the inductive step, suppose $\operatorname{pd} M \geq 2$ and consider an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0$$

where $\operatorname{pd} N = \operatorname{pd} M - 1$. Use the corresponding long exact sequence on Ext's obtained by applying $\operatorname{Hom}_R(\mathscr{k}, -)$ to find the relationship between the depths of M and N (which is $\operatorname{depth} N = \operatorname{depth} M + 1$). Use the inductive hypothesis to conclude. □

Remark 1.12. It is not difficult to see (using Krull's height theorem and prime avoidance) that every regular sequence can be extended to a system of parameters.[4] In particular, $\operatorname{depth} R \leq \dim R$.[5] When this equality happens to be an equality one says that $(R, \mathfrak{m}, \mathscr{k})$ is *Cohen–Macaulay*. Thus, a local ring is Cohen–Macaualay if and only if every system of parameters[6] is a regular sequence.

[4]Indeed, Krull's height theorem let us see that if $r_1, \ldots, r_n \in R$ is a regular sequence then $(r_1, \ldots, r_n)$ has height n. On the other hand, prime avoidance can be used to see that an ideal $(r_1, \ldots, r_n)$ that has height n can be extended to a system of parameters.

[5]More generally, $\operatorname{depth}(\mathfrak{a}, R) \leq \operatorname{ht} \mathfrak{a}$.

[6]A system of parameters for a local ring $(R, \mathfrak{m}, \mathscr{k})$ is a collection $x_1, \ldots, x_{\dim R}$ such that $\sqrt{(x_1, \ldots, x_{\dim R})} = \mathfrak{m}$. System of parameters always exist.

1.2. **Regular local rings.** Let $(R, \mathfrak{m}, \Bbbk)$ be a noetherian local ring. Then, by Nakayama's lemma, its so-called *embedded dimension*

$$\operatorname{edim} R \coloneqq \mu(\mathfrak{m}) = \dim_{\Bbbk} \mathfrak{m} \otimes \Bbbk = \dim_{\Bbbk} \mathfrak{m}/\mathfrak{m}^2$$

is finite.

Exercise 1.15. Use Krull's ideal theorem to conclude that the embedded dimension is at least the Krull's dimension of the local ring. In particular, noetherian local rings have finite dimension.

Definition 1.13 (Regular local ring)**.** A noetherian local ring $(R, \mathfrak{m}, \Bbbk)$ is said to be *regular* if the inequality

$$\operatorname{edim} R \geq \dim R$$

is an equality.

Exercise 1.16. Prove that if $(R, \mathfrak{m}, \Bbbk)$ is a noetherian local ring such that $\mathfrak{m}$ is generated by a regular sequence then it is regular.

The converse of this exercise is also true but a bit harder to prove.

Theorem 1.14. *Let $(R, \mathfrak{m}, \Bbbk)$ be a regular (noetherian) local ring. Then every set of minimal generators of $\mathfrak{m}$ (aka regular system of parameters) is a regular sequence. In particular, $\operatorname{pd}_R \Bbbk = \dim R$.*[7]

This result can be seen as a consequence of the following.

Theorem 1.15. *A regular local ring is an integral domain.*[8]

Recall the following useful, generalized form of prime avoidance.

Lemma 1.16 (Prime avoidance)**.** *Suppose that $\mathfrak{a} \subset \mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_k$ where all but up to two of the ideals $\mathfrak{a}_i$ are prime. Then $\mathfrak{a} \subset \mathfrak{a}_i$ for some $i = 1, \ldots, k$.*

Lemma 1.17. *Let $(R, \mathfrak{m}, \Bbbk)$ be a local ring of positive dimension. Then R contains a regular element not in $\mathfrak{m}^2$. That is, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ that avoids all associated primes.*

Proof. Use prime avoidance. □

Sketch of the proof of Theorem 1.15. Set $d = \dim R < \infty$. Let's do induction on d. If $d = 0$, the regularity of R implies that $0 = \dim_{\Bbbk} \mathfrak{m}/\mathfrak{m}^2$ and so $\mathfrak{m} = 0$ by Nakayama's lemma. This means that R is a field and we're done.

Assume now that $d > 0$ and that all regular local rings of dimension $< d$ are integral domains. By Lemma 1.17, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ a regular element. Observe that

- R/rR is a local ring whose maximal ideal is generated by $d - 1$ elements (one less than the number of generators of $\mathfrak{m}$), and
- the dimension of R/rR is $d - 1$.

In particular, R/rR is a regular local ring of dimension $d-1$. By the inductive hypothesis, it is an integral domain and so $rR = (r)$ is a prime ideal. Further, observe that $(r) \subset R$ cannot be a minimal prime. Let $\mathfrak{p} \subset R$ be a minimal prime of R that is contained in (r). We're done if we can prove that $\mathfrak{p} = 0$. Let $x \in \mathfrak{p}$, and so $x = yr$ for some $y \in R$. In fact, $y \in \mathfrak{p}$ as $r \notin \mathfrak{p}$. In other words, $\mathfrak{p} = r\mathfrak{p}$. Since $r \in \mathfrak{m}$, Nakayama's lemma yields that $\mathfrak{p} = 0$; as desired. □

[7]In particular, regular local rings are Cohen–Macaulay, *i.e.* $\operatorname{depth} R = \dim R$.

[8]In fact, they are UFDs and so normal integral domains.

Corollary 1.18. *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and $r \in \mathfrak{m} \setminus \mathfrak{m}^2$.*[9] *Then, R is regular if and only if r is a regular element and R/rR is regular.*

Summing up, regular local rings have finite global dimension equal to its dimension. It turns out that the converse is also true and it's a deep result due to Auslander–Buchsbaum and Serre. To prove this, we need the following observation.

Exercise 1.17. Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and M be a finitely generated R-module. Let $r \in R$ be a regular element on R and on M. Prove that

$$\operatorname{pd}_{R/rR} M/rM = \operatorname{pd}_R M$$

Hint: Show that a minimal free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ becomes a minimal free resolution of M/rM after base change by R/rR. Notice that this is tantamount to the vanishing

$$\operatorname{Tor}_i^R(R/rR, M) = 0, \quad \forall i > 0.$$

But this can be seen from the fact that

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/rR \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\cdot r} M \longrightarrow M/rM \longrightarrow 0$$

are both exact.

We're ready to prove the main result in this section. Please take a moment to appreciate its beauty.

Theorem 1.19 (Auslander–Buchsbaum–Serre)**.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local noetherian ring. Then, the following statements are equivalent.*

(a) R is regular (i.e. $\mathfrak{m}$ is generated by a regular sequence)
(b) The global dimension of R is $\dim R$
(c) $\operatorname{pd}_R \mathcal{k}$ is finite.

Proof. It only remains to explain why (c) implies (a). This is an induction on $d := \dim R < \infty$. If $d = 0$, then the Auslander–Buchsbaum formula yields that $\operatorname{pd}_R \mathcal{k} = 0$ and so that $\mathcal{k}$ is a free R-module. Hence, $R = \mathcal{k}$ and we're done.

Let's assume that $d > 0$ and that (c) implies (a) for those local rings of dimension $< d$. Since R is positive dimensional, we can find a regular element $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ and it suffices to prove that the local ring $(R/rR, \mathfrak{m}/rR, \mathcal{k})$ is regular (which has dimension $d - 1$). To that end, we can apply the inductive hypothesis and prove that $\operatorname{pd}_{R/rR} \mathcal{k}$ is finite. For this, apply Exercise 1.17. $\square$

Exercise 1.18. Prove the following tWo corollaries.

Corollary 1.20. *If $(R, \mathfrak{m}, \mathcal{k})$ is a regular local ring then so is $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Corollary 1.21 (Hilbert's syzygy theorem)**.** *Let $\mathcal{k}$ be a field. Then, every finitely generated $\mathcal{k}[x_1, \ldots, x_n]$-module has a free resolution of length at most n.*

[9]Note that this is to say that r is part of a minimal set of generators for $\mathfrak{m}$.

1.3. **General regular rings.** With the above in place, we can finally define regular rings beyond the local case.

Definition 1.22 (Regular rings of finite dimension)**.** We say that a noetherian ring of finite Krull dimension $\dim R$ is regular if any of the following equivalent conditions hold:

(a) The local ring $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$.
(b) The global dimension of R is at most $\dim R$ (*i.e.* every finitely generated module has projective dimension at most $\dim R$).
(c) R has finite global dimension.

Exercise 1.19. Prove that the above conditions are indeed equivalent.

Definition 1.23 (Regular rings)**.** Let R be a noetherian ring. Then R is said to be regular if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$.

Exercise 1.20. Prove that if R is regular then so is $W^{-1}R$ for any multiplicative set $W \subset R$.

Exercise 1.21. Prove that for a regular ring its global dimension equals its dimension.

1.4. **Complete regular rings and the Cohen structure theorems.** Let $(R, \mathfrak{m}, \mathcal{k})$ be a noetherian local ring. Recall that its completion is the canonical homomorphism

$$R \longrightarrow \hat{R} \coloneqq \varprojlim_n R/\mathfrak{m}^n$$

It turns out that $\hat{R}$ is a noetherian local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$, residue field $\mathcal{k}$, and dimension $\dim R$. Moreover, $R \longrightarrow \hat{R}$ is a faithfully flat local homomorphism. In particular, R is regular if and only if so is $\hat{R}$.

Remark 1.24. More generally, the completion of an R-module M is the $\hat{R}$-module

$$\hat{M} \coloneqq \varprojlim_n M/\mathfrak{m}^n M.$$

Notice that there is a canonical $\hat{R}$-linear map

$$\hat{R} \otimes_R M \longrightarrow \hat{M}$$

but it may not be an isomorphism. However, it is an isomorphism if M is finitely generated.

Exercise 1.22. Prove that $\operatorname{depth} R = \operatorname{depth} \hat{R}$. In particular, R is Cohen–Macaulay iff so is $\hat{R}$.

Example 1.25. If $R = \mathcal{k}[x_1, \ldots, x_n]/\mathfrak{a}$ and $\mathfrak{m} = (x_1, \ldots, x_n)$, then $\hat{R}_{\mathfrak{m}} = \mathcal{k}[\![x_1, \ldots, x_n]\!]/\mathfrak{a}$.

Recall that $(R, \mathfrak{m}, \mathcal{k})$ is said to be complete if $R \longrightarrow \hat{R}$ is an isomorphism. It turns out that $\hat{R}$ is complete. In fact, every quotient of $\hat{R}$ is a noetherian complete local ring.

Remark 1.26 (Characteristic). Recall that the characteristic of a ring R, say $\operatorname{char} R$, is the only nonnegative integer $n \in \mathbb{N}$ such that $(n) = \ker(\mathbb{Z} \longrightarrow R)$. Note that if R is an integral domain (*i.e.* a field) then $\operatorname{char} R$ is either 0 or a prime number p.

Exercise 1.23. Prove that R contains a field as a subring if and only if $\operatorname{char} R = \operatorname{char} \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Here $\kappa(\mathfrak{p})$ denotes the residue field of R at $\mathfrak{p}$.

For this reason, those rings that contain a field as a subring are referred to as rings of *equi-characteristic*. If a ring does not contain a field then it is said to have *mixed-characteristic*.

If $(R, \mathfrak{m}, \mathscr{k})$ is a local ring, then it has equicharacteristic iff $\operatorname{char} R = \operatorname{char} \mathscr{k}$. If it is mixed characteristic then $\operatorname{char} \mathscr{k} = p > 0$ but $0 \neq p \in R$.

Suppose that $(R, \mathfrak{m}, \mathscr{k})$ is complete. A complete local subring $(\Lambda, p\Lambda, \mathscr{k}) \subset (R, \mathfrak{m}, \mathscr{k})$ is referred to as a coefficient ring. This entails that $\mathfrak{m} \cap \Lambda = p\Lambda$ and $p = \operatorname{char} \mathscr{k} \geq 0$. There are three cases:

- R has equi-characteristic and so Λ is a field contained in R that maps isomorphically to $\mathscr{k}$.
- R has mixed-caracteristic and $0 \neq p \in R$ is not nilpotent. In that case, $(\Lambda, p\Lambda, \mathscr{k})$ is a complete DVR. We'll referred to this rings as *Cohen rings*.
- R has mixed-caracteristic and $p \in R$ is nilpotent (*i.e.* $\operatorname{char} R = p^n$ for some $n > 1$). In that case, $(\Lambda, p\Lambda, \mathscr{k})$ is an artinian local ring.

Theorem 1.27 (Cohen structure theorem I)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete (noetherian) local ring. Then:*

(a) R has a coefficient ring.
(b) There is a surjective homomorphism $\Lambda[\![x_1, \ldots, x_n]\!] \longrightarrow R$ where Λ is either a field or a Cohen ring. Moreover, Λ can be taken as a coefficient ring of R if $p \in R$ isn't nilpotent. In particular, R is a quotient of a regular complete local ring.

Remark 1.28. The most difficult part is to show the existence of a coefficient ring. If $(R, \mathfrak{m}, \mathscr{k})$ has equi-characteristic $p > 0$ and $\mathscr{k}$ is perfect. Then it turns out that

$$\mathscr{k}_0 := \bigcap_{e \in \mathbb{N}} R^{p^e}$$

is the only coefficient field of R. Here, $R^{p^e} = \{r^{p^e} \in r \in R\}$.

Theorem 1.29 (Cohen structure theorem II)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete regular local ring. Then:*

- *If R has equi-characteristic then $R \cong \mathscr{k}[\![x_1, \ldots, x_n]\!]$.*
- *If R has mixed-characteristic then there is a Cohen ring Λ such that*

$$R \cong \begin{cases} \Lambda[\![x_1, \ldots, x_n]\!] & \text{if } p \in R \text{ is a regular element} \\ \Lambda[\![x_1, \ldots, x_n]\!]/(p - f) \text{ for some } f \in \mathfrak{m}^2 & \text{otherwise.} \end{cases}$$

We say that R is unramified *in the former case.*

Theorem 1.30 (Cohen–Gabber structure theorem III)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local ring that either is equi-characteristic or is an integral domain. Then, there exists a subring $A \subset R$ such that:*

(a) A is a complete local ring,
(b) $A \subset R$ is finite induces an isomorphism on residue fields and is generically étale,
(c) $A \cong \Lambda[\![x_1, \ldots, x_n]\!]$ where Λ is a field or a Cohen ring.

Exercise 1.24. In the setup of Theorem 1.30, show that $(R, \mathfrak{m}, \mathscr{k})$ is Cohen–Macaulay if and only if $A \subset R$ is free (*i.e.* R is a projective A-module). Hint: Use the Auslander–Buchsbaum formula.

Exercise 1.25. Let R be a noetherian equi-characteristic ring. Prove that R is regular iff $\hat{R}_{\mathfrak{p}} \cong \kappa(\mathfrak{p})[\![x_1,\ldots,x_{\operatorname{ht}\mathfrak{p}}]\!]$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Recall that $\kappa(\mathfrak{p}) \coloneqq \mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = \mathscr{K}(R/\mathfrak{p})$ denotes the residue field of $R_{\mathfrak{p}}$.

2. The Frobenius Endomorphism and Kunz's Theorem

From now on unless otherwise stated, we are going to assume that all rings have prime characteristic p. That is, all rings are $\mathbb{F}_p$-algebras. We always use the shorthand notation

$$q \coloneqq p^e.$$

Further, we'll assume that all rings are noetherian. The *Frobenius endomorphism* of a ring R is the homomorphism of $\mathbb{F}_p$-algebras

$$F = F_R \colon R \longrightarrow R, \quad r \mapsto r^p.$$

By iterating, we also have $F^e \colon r \mapsto r^q$ for all $e \in \mathbb{N}$. We let $R^q \subset R$ be the image subring of F^e.

Exercise 2.1. Prove that $F \colon R \longrightarrow R$ is indeed a homomorphism of $\mathbb{F}_p$-algebras. Prove that $\operatorname{Spec} F^e \colon \operatorname{Spec} R \longrightarrow \operatorname{Spec} R$ is the identity.

Exercise 2.2. Prove that R is reduced iff F^e is injective for some/all $e \in \mathbb{N}$.

Exercise 2.3. Recall that a ring R is reduce iff its total ring of fractions $\mathscr{K}(R)$ is a product of fields $K_1 \times \cdots \times K_n$. Then, we may define $\bar{\mathscr{K}}(R)$ as $\bar{K}_1 \times \cdots \times \bar{K}_n$ where $\bar{K}_i$ is an algebraic closure of K_i. Hence $r^{1/q}$ is well-defined in $\bar{\mathscr{K}}(R)$ for all $r \in \mathscr{K}(R)$. Show that

$$R^{1/q} \coloneqq \{r^{1/q} \in \bar{\mathscr{K}}(R) \mid r \in R\} \subset \bar{\mathscr{K}}(R)$$

is a subring that contains R. Moreover, show that $R \subset R^{1/q}$, $F^e \colon R \longrightarrow R$, and $R^q \longrightarrow R$ are isomorphic as R-algebras.

Definition 2.1 (Frobenius powers)**.** Let $\mathfrak{a} \subset R$ be an ideal. Then $\mathfrak{a}^{[q]}$ is the extension ideal of $\mathfrak{a}$ along F^e, and it's called the *e-th Frobenius power of* $\mathfrak{a}$.

Note that if $\theta \colon R \longrightarrow S$ is a homomorphism of rings then there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, "F^e"'] & S \arrow[d, "F^e"] \\ R \arrow[r, "\theta"] & S \end{tikzcd}$$

Exercise 2.4. Prove that the above diagram is cartesian for all $e \in \mathbb{N}$ if θ is a localization $R \longrightarrow W^{-1}R$. Show that if $\theta \colon R \longrightarrow R/\mathfrak{a}$ is a quotient then the diagram is cartesian iff $\mathfrak{a}^{[q]} = \mathfrak{a}$.

More generally, the following notation is going to be useful.

Notation 2.2 (Frobenius pushforward)**.** Let M be an R-module. We let

$$F^e_* M \coloneqq \{F^e_* m \mid m \in M\}$$

be the R-module defined by the rules $F^e_* m + F^e_* m' = F^e_*(m + m')$ and $rF^e_* m = F^e_* r^q m$. In other words, $F^e_* M$ is the restriction of scalars of M along F^e. Thus, $F^e_* M$ is identical to M as an abelian group but the R-scalar action is being twisted by Frobenius. Likewise, if $M = S$ is an R-algebra then $F^e_* S$ is an R-algebra with the product $(F^e_* s)(F^e_* s') = F^e_*(ss')$. Again, $F^e_* S$ is the exact same thing as S as a ring, what changes is the R-algebra structure.

Exercise 2.5. Prove that R is reduced iff $F^e_* R$ is a faithful R-module for some/all e.

Exercise 2.6. Prove that
$$F^e_* \hat{R} = \widehat{(F^e_* R)}.$$

With the above notation in place, we see that the commutative diagram above induces a ring homomorphism
$$F^e_\theta \colon S \otimes_R F^e_* R \longrightarrow F^e_* S, \quad s \otimes F^e_* r \mapsto F^e_* s^q \theta(r)$$
which is called the *relative Frobenius* of $\theta \colon R \longrightarrow S$.

Exercise 2.7. Prove that $\operatorname{Spec} F^e_\theta$ is a (universal) homeomorphism.

Theorem 2.3 (Kunz's theorem)**.** *Let R be a (noetherian) ring. Then R is regular iff $F^e \colon R \longrightarrow R$ is (faithfully) flat for some/all $e > 0$.*

Remark 2.4 (The socle). Let $(R, \mathfrak{m}, \Bbbk)$ be a local ring and M be a finitely generated R-module. The *socle* of M is the submodule
$$\operatorname{Soc}(M) \coloneqq \{m \in M : m\mathfrak{m} = 0\} \cong \operatorname{Hom}_R(\Bbbk, M) = \operatorname{Ext}^0_R(\Bbbk, M).$$
In particular, $\operatorname{depth} M = 0$ iff $\operatorname{Soc} M \neq 0$. Since $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n M = 0$, it follows that, if $\operatorname{depth} M = 0$, there is $n \in \mathbb{N}$ such that $\operatorname{Soc} M \not\subset \mathfrak{m}^n M$. Let $c \coloneqq \operatorname{depth} R$ and $r_1, \ldots, r_c \in R$ be a regular sequence. Set $\mathfrak{a} \coloneqq (r_1, \ldots, r_c)$. Observe that $\operatorname{depth}_R R/\mathfrak{a} = 0$. Then, we may find $n \in \mathbb{N}$ such that
$$\operatorname{Soc}_R(R/\mathfrak{a}) \not\subset \mathfrak{m}^n (R/\mathfrak{a}).$$

Lemma 2.5. *Let $(R, \mathfrak{m}, \Bbbk)$ be a local ring of depth c. Then there is $n \in \mathbb{N}$ such that for all infinite minimal free resolutions*
$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(M)} \xrightarrow{\phi_i} R^{\oplus \beta_i(M)} \longrightarrow \cdots \longrightarrow R^{\beta_0(M)} \longrightarrow M \longrightarrow 0$$
the entries of the matrix ϕ_{c+1} are not all contained in $\mathfrak{m}^n$ (i.e. the image of ϕ_{c+1} is not inside $\mathfrak{m}^n R^{\oplus b_{c+1}} = (\mathfrak{m}^n)^{\oplus b_{c+1}}$). Here $b_i \coloneqq \beta_i(M)$.

Proof. Note that, by the Auslander–Buchsbaum formula, we have that $b_{c+1} \neq 0$ as the resolution has infinite length. This is gonna be important below.

Let $\mathfrak{a} = (r_1, \ldots, r_c)$ and n be as in Remark 2.4. In particular, for $N \coloneqq \operatorname{Soc}_R(R/\mathfrak{a})$ we have that $N \not\subset \mathfrak{m}^n N$. Observe that
$$\operatorname{pd}_R R/\mathfrak{a} = c$$
and so
$$\operatorname{Tor}^R_{c+1}(M, R/\mathfrak{a}) = 0.$$
This implies that after base changing the given infinite minimal free resolution we obtain that
$$(R/\mathfrak{a})^{\oplus b_{c+2}} \xrightarrow{\phi_{c+1}/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_{c+1}} \xrightarrow{\phi_c/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_c}$$
is exact in the middle. In other words,
$$\ker \phi_c/\mathfrak{a} \subset \operatorname{im} \phi_{c+1}/\mathfrak{a}$$

Now, since the given resolution is minimal, we have that the entries of ϕ_c are all in $\mathfrak{m}$ and so
$$N^{\oplus b_{c+1}} \subset \ker \phi_c/\mathfrak{a}.$$

Thus, putting everything together, if (for the sake of contradiction) the image of ϕ_{c+1} is inside $(\mathfrak{m}^n)^{\oplus b_{c+1}}$, it would follow that

$$N^{\oplus b_{c+1}} \subset \big(\mathfrak{m}^n(R/\mathfrak{a})\big)^{\oplus b_{c+1}}.$$

But, since $b_{c+1} \neq 0$, this implies that

$$N \subset \mathfrak{m}^n(R/\mathfrak{a}),$$

which contradicts the construction of n. Isn't math just so cool? □

Lemma 2.6. *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and M be an R-module. Then $\hat{M}$ is a flat $\hat{R}$-module whenever $\operatorname{Tor}_1^R(M, \mathcal{k}) = 0$ and in particular whenever M is flat.*[10]

Proof. This is a particular case of [Sta23, Tag 0AGW]. □

Lemma 2.7 ([Sta23, Tag 039V])**.** *Let $R \longrightarrow S$ a homomorphism of rings and M be an S-module. If M is a flat R-module and a faithfully flat S-module then $R \longrightarrow S$ is flat.*

Exercise 2.8. Let R be a (noetherian ring). Then, F^e_R is flat as an R-module iff $F^e_* R_{\mathfrak{p}}$ is flat as an $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$. If R is local, then $F^e R$ is flat as an R-module iff $F^e \hat{R}$ is flat as an $\hat{R}$-module. Hint: Apply the two previous lemmas.

Exercise 2.9. Let $R = \mathcal{k}[x_1, \dots, x_d]$ be a polynomial ring over a field $\mathcal{k}$ (or more generally over a ring $\mathcal{k}$ whose Frobenius is free). Let $\{F^e_* \lambda\}_{\lambda \in \Lambda}$ be a $\mathcal{k}$-basis for $F^e_* \mathcal{k} = \mathcal{k}^{1/q}$ (which we may assume contains $F^e_* 1$). Prove that

$$\{F^e_* \lambda x_1^{i_1} \cdots x_d^{i_d}\}_{\lambda \in \Lambda, 0 \leq i_1, \dots, i_d \leq q-1}$$

is an R-basis for $F^e_* R$. Suppose now that Λ is finite so that $F^e_* R$ is free of finite rank. Consider the corresponding dual basis

$$\{\phi_{\lambda, i_1, \dots, i_d} \coloneqq (F^e_* \lambda x_1^{i_1} \cdots x_d^{i_d})^\vee\}_{\lambda \in \Lambda, 0 \leq i_1, \dots, i_d \leq q-1}$$

for $\operatorname{Hom}_R(F^e_* R, R)$. Show that

$$F^e_* R \longrightarrow \operatorname{Hom}_R(F^e_* R, R), \quad F^e_* 1 \mapsto \Phi^e \coloneqq \phi_{1, q-1, \dots, q-1}$$

is an isomorphism. We will be referreing to Φ^e as the *e-th (power of the) Frobenius trace* of R.

Exercise 2.10. Conclude that $F^e_* \mathcal{k}[\![x_1, \dots, x_d]\!]$ is a flat $\mathcal{k}[\![x_1, \dots, x_d]\!]$-module. Show that it is free if $[\mathcal{k}^{1/p} : \mathcal{k}] < \infty$. What about the converse?

Proof of Kunz's theorem. We may assume that $(R, \mathfrak{m}, \mathcal{k})$ is local. Moreover, we may assume that $(R, \mathfrak{m}, \mathcal{k})$ is complete. If R is regular then $R \cong \mathcal{k}[\![x_1, \dots, x_{\dim R}]\!]$ and we're done by Exercise 2.10.

Conversely, suppose that $F^e \colon R \longrightarrow R$ is flat. We want to prove that $\operatorname{pd}_R \mathcal{k} < \infty$. Suppose, for the sake of contradiction that there is an infinite minimal free resolution

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(\mathcal{k})} \xrightarrow{\phi_i} R^{\oplus \beta_i(\mathcal{k})} \longrightarrow \cdots \longrightarrow R^{\beta_0(\mathcal{k})} \longrightarrow \mathcal{k} \longrightarrow 0$$

That is, $\beta_{c+1}(\mathcal{k}) \neq 0$ for $c = \operatorname{depth} R$. Since $F^e \colon R \longrightarrow R$ is flat for all e, we can base chang this inifinite minimal free resolution to obtain a minimal free resolution

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(\mathcal{k})} \xrightarrow{\phi_i^{[q]}} R^{\oplus \beta_i(\mathcal{k})} \longrightarrow \cdots \longrightarrow R^{\beta_0(\mathcal{k})} \longrightarrow R/\mathfrak{m}^{[q]} \longrightarrow 0$$

[10]Be cautious, the same can't be said about freenes and hence about projectivity.

where $\phi_i^{[q]}$ is the matrix obtained from ϕ_i by raising its entries to the q-th power. In particular, the entries of $\phi_i^{[q]}$ belong to $\mathfrak{m}^{[q]} \subset \mathfrak{m}^q$ for all i and in particular for $i = \operatorname{depth} R + 1$. This, however, contradicts Lemma 2.5 as $\mathfrak{m}^q \subset \mathfrak{m}^n$ for all $e \gg 0$ such that $q \geq n$. □

2.1. **Relative version of Kunz's theorem.** There is a relative version of Kunz's theorem that goes by the name of Radu–André's theorem. To state it, we need to recall the following definition (the relative notion of F-regularity).

Definition 2.8 (Regular algebras)**.** Let $\theta\colon R \longrightarrow S$ be an R-algebra (where R and S are noetherian). We say that θ is *regular* if it is flat and all its fibers are *geometrically regular*. That is, for all $\mathfrak{p} \in \operatorname{Spec} R$ the $\kappa(\mathfrak{p})$-algebra $S \otimes_R \kappa(\mathfrak{p})$ is noetherian and regular (and noetherian) after any base change by a finitely generated field extension $\Bbbk/\kappa(\mathfrak{p})$.[11]

Theorem 2.9 (Radu–André)**.** *Let $\theta\colon R \longrightarrow S$ be an R-algebra. Then, θ is regular iff F^e_θ is (faithfully) flat for all/some $e > 0$.*

On the proof. The most important step is to show that if θ is regular than $S \otimes_R F^e_* R$ is noetherian. With that in place, the result can be obtained from the absolute Kunz theorem and the critere de platitude par fibres. I hope to add more details later on. □

2.2. **Bhatt–Scholze's generalization of Kunz's theorem.** The *(colimit) perfection* of a ring R is

$$R \longrightarrow R_{\mathrm{perf}} \coloneqq \operatorname{colim}(R \xrightarrow{F} R \xrightarrow{F} R \longrightarrow \cdots)$$

We say that R is perfect iff $R \longrightarrow R_{\mathrm{perf}}$ is an isomorphism, *i.e.* Frobenius is an isomorphism on R. Observe that R_{perf} is perfect. Perfect rings are rarely noetherian. In fact, a noetherian perfect ring is a finite product of perfect fields.

Exercise 2.11. Prove that $\operatorname{Spec} R \longrightarrow \operatorname{Spec} R_{\mathrm{perf}}$ is a homeomorphism. Conclude that the perfection of a noetherian local ring has finite dimension.

Theorem 2.10 (Bhatt–Scholze)**.** *Let $(R, \mathfrak{m}, \Bbbk)$ be a complete local ring (of prime characteristic p). Then its perfection is R_{perf} has finite global dimension.*

Proof. TO BE ADDED. □

This result easily proves Kunz's theorem as follows. Recall that the substantial part of Kunz's theorem is that if $F^e\colon R \longrightarrow R$ is flat for a complete local ring then R is regular, *i.e.* R has finite global dimension. That is, we must show that there is $n \in \mathfrak{n}$ such that for all R-modules one has that

$$\operatorname{Tor}_i^R(\Bbbk, M) = 0$$

for all $i \geq n$. To that end, one observes that $R \longrightarrow R_{\mathrm{perf}}$ is faithfully flat and that

$$R_{\mathrm{perf}} \otimes_R \operatorname{Tor}_i^R(\Bbbk, M) = \operatorname{Tor}_i^{R_{\mathrm{perf}}}(R_{\mathrm{perf}} \otimes_R \Bbbk, R_{\mathrm{perf}} \otimes_R M).$$

Then, we can take n to be the global dimension of R_{perf}, which is finite by Bhatt–Scholze's theorem.

Exercise 2.12. Let $R \longrightarrow S$ be faithfully flat. Show that the global dimension of R is no more than the global dimension of S.

[11]It suffices to ask this for all finite purely inseparable extensions $\Bbbk/\kappa(\mathfrak{p})$.

3. F-finiteness and Gabber's Theorem

In studying regularity and therefore singularities, one imposes noetherianity as a basic finiteness condition. In studying F-singularities, one imposes one additional condition. Namely,

Definition 3.1. An $\mathbb{F}_p$-algebra R is *F-finite* if $F^e\colon R \longrightarrow R$ is finite for some/all $e > 0$ (*i.e.* $F^e_* R$ is a finitely generated R-module for all e).

Exercise 3.1. Let R be F-finite. Show that so are its localizations, quotients, and polynomial extensions $R[x_1, \ldots, x_n]$. Prove that a field $\mathcal{k}$ is F-finite iff $[\mathcal{k}^{1/q} : \mathcal{k}] < \infty$. Conclude that in such case $\mathcal{k}$-algebras that are either essentially of finite type or complete are F-finite.

Exercise 3.2. F-finiteness has nothing to do with noetherianity. Show that there are noetherian rings that aren't F-finite and vice-versa.

Remark 3.2 (F-finiteness equals kählerianity over $\mathbb{F}_p$)*.* According to Fogarty, $R/\mathbb{F}_\mathfrak{p}$ is F-finite iff its R-module of Kähler differentials $\Omega_{R/\mathbb{F}_p}$ is finitely generated, in which case $R/\mathbb{F}_p$ is referred to as kählerian. See [Fog80]. The forward implication is rather trivial and can be left as an exercise for those familiar with Kähler differentials. Although this equivalence is conceptually satisfying, we won't use it in the sequel.

Kunz's theorem takes a much simpler form in that case.

Theorem 3.3 (Kunz's theorem in the F-finite case)**.** *Let R be an F-finite (and noetherian) $\mathbb{F}_p$-algebra. Then, R is regular if and only if $F^e_* R$ is a projective (i.e. locally free of finite rank) R-module. If R is further local, it is regular iff $F^e_* R$ is free of finite rank.*

Exercise 3.3. Show that if R is F-finite then its regular locus is (Zariski-)open.

Exercise 3.4. Let $(R, \mathfrak{m}, \mathcal{k})$ be an F-finite local ring. Show that its completion $R \longrightarrow \hat{R}$ is regular. Hint: Show that $F_{\hat{R}/R}$ is an isomorphism and then conclude using Radu–André's theorem.

Exercise 3.5. Suppose that R is a regular F-finite ring and $\mathfrak{p} \in \operatorname{Spec} R$. Show that the following inclusion of ideals

$$\{r \in R \mid \phi(F^e_* r) \in \mathfrak{p}, \forall \phi \in \operatorname{Hom}_R(F^e_* R, R)\} \supset \mathfrak{p}^{[q]}$$

is an equality.

Definition 3.4 (p-basis)**.** A *(regular) p-basis* for a regular F-finite $R/\mathbb{F}_p$ is a set $x_1, \ldots, x_n$ such that

$$F^e_* R = \bigoplus_{0 \leq i_1, \ldots, i_n \leq q-1} R F^e_* x_1^{i_1} \cdots x_n^{i_n}.$$

In particular, the rank of $F^e_* R$ is q^n.

Remark 3.5. According to Tyc, a p-basis is the same thing as a *differential basis* (*i.e.* $\Omega_{R/\mathbb{F}_p} = \bigoplus_{i=1}^n R dx_i$). See [Tyc88]. In particular, F-finite fields always admit a p-basis.

Exercise 3.6. Let $R \coloneqq \mathbb{F}_p[x, y]/(x^2 + y^2 - 1)$. Prove that R is regular iff $p \neq 2$. However, R admits a p-basis iff $p \equiv 1 \bmod 4$. Hint: the point is that $-1 = p - 1 \in \mathbb{F}_p$ has a square if and only $p \equiv 1 \bmod 4$.

Example 3.6. Let $\Bbbk$ be an F-finite field. Note that $\Bbbk[x_1, \ldots, x_n]$ and $\Bbbk[\![x_1, \ldots, x_n]\!]$ both admit a p-basis.

Example 3.7. More generally, a local kählerian regular algebra admits a differentials by a result of Matsumura [**?**]. Then, by Tyc's result, a local F-finite regular ring admits a p-basis. I hope to ellaborate more on this later on.

Remark 3.8 (On restriction, extension, and co-extension of scalars). Let $\theta\colon R \longrightarrow S$ be an R-algebra, and say $f \coloneqq \operatorname{Spec}\theta\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. This induces three covariant functors $f_*, f^*, f^!$; respectively known as *restriction, extension, and co-extension of scalars*. The restriction of scalars functor f_* goes from the category of S-modules to the one of R-modules. If we have a morphism of S-modules $N \longrightarrow N'$, we can think of it as a morphism of R-modules by restricting scalars along $\theta\colon R \longrightarrow S$, which we denote by $f_*N \longrightarrow f_*N'$. On the other hand, the functor of extension of scalars (aka base change) f^* goes from the category of R-modules to the one of S-modules and it's defined by base change. Namely, if $\phi\colon M \longrightarrow M'$ is a morphism of R-modules then its extension of scalars is the morphism of S-modules

$$f^*M \coloneqq S \otimes_R M \xrightarrow{S\otimes_R\phi\colon s\otimes m\mapsto s\otimes\phi(m)} f^*M' \coloneqq S \otimes_R M'.$$

Finally, the functor $f^!$ of co-extension of scalars goes from R-modules to S-modules and is defined as follows. If $\phi\colon M \longrightarrow M'$ is a morphism of R-modules then $f^!\phi$ is the following morphism of S-modules:

$$\begin{gathered} f^!M \coloneqq \operatorname{Hom}_R(S, M) \longrightarrow f^!M' \coloneqq \operatorname{Hom}_R(S, M') \\ \mu \mapsto \phi \circ \mu \end{gathered}$$

It is important to notice that $\operatorname{Hom}_R(S, M)$ is indeed an S-module, where the scalar action of S is given by

$$s\mu \coloneqq \mu \circ (\cdot s)\colon s' \mapsto \mu(ss').$$

Thus, it may be better to denote this as an right action, *i.e.* we may write μs instead of $s\mu$. Note that $\operatorname{Hom}_R(S, R)$ is also an R-module where $r\mu = (\cdot r) \circ \mu\colon s \mapsto r\mu(s)$. Nevertheless, these two linear structures are related as follows:

$$r\mu = \mu\theta(r),$$

from which one may say that the S-module structure determines the R-module one (by restriction of scalars).

These three functors are related by the adjointness:

$$f^* \dashv f_* \dashv f^!$$

Indeed, the co-unit $\epsilon\colon f^*f_* \longrightarrow \mathrm{id}$ is given by

$$\epsilon_N S \otimes_R N \xrightarrow{s\otimes n\mapsto sn} N$$

whereas the unit $\eta\colon \mathrm{id} \longrightarrow f_*f^*$ is given by

$$\eta_M\colon M \xrightarrow{m\mapsto 1\otimes m} S \otimes_R M.$$

Likewise, the co-unit $\operatorname{Tr}\colon f_*f^! \longrightarrow \mathrm{id}$ for the adjointness $f_* \dashv f^!$ is known as *the trace* and is defined as

$$\operatorname{Tr}_M\colon \operatorname{Hom}_R(S, M) \xrightarrow{\mu\mapsto\mu(1)} M$$

whereas its unit $\nu\colon \mathrm{id} \longrightarrow f^! f_*$ is the natural transformation

$$\begin{aligned} \nu_N\colon N &\longrightarrow \mathrm{Hom}_R(S,N) \\ n &\mapsto (s \mapsto sn). \end{aligned}$$

Exercise 3.7. Show that the above pairs of units and co-units define a par of adjointness relations $f^* \dashv f_* \dashv f^!$. That is, show that there are commutative diagrams of natural transformations

$$\begin{tikzcd} f^* \arrow[r, "f^*\eta"] \arrow[dr, "\mathrm{id}"'] & f^*f_*f^* \arrow[d, "\epsilon f^*"] \\ & f^* \end{tikzcd} \qquad \begin{tikzcd} f_* \arrow[r, "\eta f_*"] \arrow[dr, "\mathrm{id}"'] & f_*f^*f_* \arrow[d, "f_*\epsilon"] \\ & f_* \end{tikzcd}$$

defining $f^* \dashv f_*$. Likewise, for $f_* \dashv f^!$, show that we have commutative diagrams of natural transformations

$$\begin{tikzcd} f_* \arrow[r, "f_*\nu"] \arrow[dr, "\mathrm{id}"'] & f_*f^!f_* \arrow[d, "\mathrm{Tr}\, f_*"] \\ & f_* \end{tikzcd} \qquad \begin{tikzcd} f^! \arrow[r, "\nu f^!"] \arrow[dr, "\mathrm{id}"'] & f^!f_*f^! \arrow[d, "f^!\,\mathrm{Tr}"] \\ & f^! \end{tikzcd} .$$

The above means that the natural maps

$$\mathrm{Hom}_S(f^*M,N) \xrightarrow{\psi\mapsto f_*\psi\circ\eta_M} \mathrm{Hom}_R(M,f_*N) \text{ and } \mathrm{Hom}_R(M,f_*N) \xrightarrow{\phi\mapsto\epsilon_N\circ f^*\phi} \mathrm{Hom}_S(f^*M,N)$$

are inverse to each other. Similarly, the natural maps

$$\mathrm{Hom}_S(N,f^!M) \xrightarrow{\psi\mapsto \mathrm{Tr}_M\circ f_*\psi} \mathrm{Hom}_R(f_*N,M) \text{ y } \mathrm{Hom}_R(f_*N,M) \xrightarrow{\phi\mapsto f^!\phi\circ\nu_N} \mathrm{Hom}_S(N,f^!M)$$

are mutually inverse.

Exercise 3.8. Notice that f_* is exact and so that f^* is right-exact whereas $f^!$ is left exact. Observe that f^* is exact iff f_*S is flat but $f^!$ is exact iff f_*S is proyective.

Exercise 3.9. Show that the mapping

$$\mathrm{Hom}(f^*,f^!) \longrightarrow f^!R \coloneqq \mathrm{Hom}_R(S,R), \quad \xi \mapsto \xi_R(1)$$

is a bijection, what's its inverse?

This finishes our general observations on restriction, extension, and co-extension of scalars. How does all this apply to F^e?

Exercise 3.10. Suppose that $R/\mathbb{F}_p$ admits a p-basis (and so it is in particular regular and F-finite), say $x_1,\dots,x_n$. Let

$$\{\phi_{i_1,\dots,i_d} \coloneqq (F^e_* x_1^{i_1}\cdots x_d^{i_d})^\vee\}_{0\le i_1,\dots,i_d\le q-1}$$

be the corresponding dual basis for $\mathrm{Hom}_R(F^e_*R,R)$. Show that:

(a) The F^e_*R-linear mapping

$$F^e_*R \longrightarrow \mathrm{Hom}_R(F^e_*R,R), \quad F^e_*1 \mapsto \Phi^e \coloneqq \phi_{q-1,\dots,q-1}$$

is an isomorphism. We will be referreing to Φ^e as the *e-th (power of the) Frobenius trace* of R.

(b) The equalities
$$\Phi^{e-1} \circ F_*^{e-1}\Phi^1 = \Phi^e = \Phi^1 \circ F_*\Phi^{e-1}.$$
hold, which justifies to say that Φ^e is the e-th power of $\Phi := \Phi^1$. In fact, $\Phi^e = \Phi^a \circ F_*^a \Phi^b$ whenever $e = a + b$.

(c) For all $r \in R$ and $\mathfrak{a}, \mathfrak{b} \subset R$ ideals,
$$(\Phi^e r)(F_*^e \mathfrak{a}) \subset \mathfrak{b} \Longleftrightarrow r \in \mathfrak{b}^{[q]} : \mathfrak{a}.$$

(d) For every ideal $\mathfrak{a} \subset R$ with quotient $R \to A := R/\mathfrak{a}$, there is an exact sequence of $F_*^e R$-modules
$$0 \longrightarrow \mathfrak{a} F_*^e R = F_*^e \mathfrak{a}^{[q]} \longrightarrow F_*^e(\mathfrak{a}^{[q]} : \mathfrak{a}) \xrightarrow{F_*^e r \mapsto (\Phi^e r)/\mathfrak{a}} \operatorname{Hom}_A(F_*^e A, A) \longrightarrow 0$$
which induces an isomorphism of $F_*^e A$-modules
$$F_*^e \left(\frac{\mathfrak{a}^{[q]} : \mathfrak{a}}{\mathfrak{a}^{[q]}} \right) \xrightarrow{\cong} \operatorname{Hom}_A(F_*^e A, A).$$

(e) If $x_n \in R$ is not a unit then $x_1, \ldots, x_{n-1}$ yields a p-basis on $R/x_n R$. Furthermore, if $I \subset \{1, \ldots, n\}$ is such that $(x_i \mid i \in I) \neq R$ then $\{x_i\}_{i \in I}$ is a regular sequence on R.

(f) If all the $x_1, \ldots, x_n$ are units then $\dim R = 0$.

(g) More generally, $\dim R \leq n$.

(h) Relabel if necessary so that $x_1, \cdots, x_m$ is such that $S/(x_1, \ldots, x_m)$ is zero dimensional. Show that the canonical map
$$R^{\mathrm{perf}} := \bigcap_e R^q \xrightarrow{\subset} S \longrightarrow S/(x_1, \ldots, x_m)$$
is injective.

(i) Conclude that R^{perf} is noetherian and so a product of perfect fields.

Theorem 3.9 (Gabber [Gab04]). *Let $R/\mathbb{F}_p$ be F-finite (and noetherian). Then, there is an F-finite regular ring S admiting a p-basis (and so having) finite dimension such that R is a homomorphic image of S, i.e. there is a quotient $S \twoheadrightarrow R$.*

Main idea of the proof. The proof is constructive. Let $F_*r_1, \ldots, F_*r_n \in F_*R$ be R-generators of F_*R. Equivalently, $r_1, \ldots, r_n \in R$ are generators of R as an R^p-module. Consider the R-algebra
$$S_e := R[x_1, \ldots, x_n]/(x_1^q - r_n, \ldots, x_n^q - r_n)$$
Observe that its e-th Frobenius factors as follows
$$\begin{tikzcd} S_e \arrow[r, "F^e"] \arrow[dr, twoheadrightarrow, "\phi_e"'] & S_e \\ & R \arrow[u, hook] \end{tikzcd}$$

Moreover, the map ϕ_e further factors as
$$\begin{tikzcd} S_e \arrow[rr, twoheadrightarrow, "\sigma_e"] \arrow[dr, twoheadrightarrow, "\phi_e"'] & & S_{e-1} \arrow[dl, twoheadrightarrow, "\phi_{e-1}"] \\ & R & \end{tikzcd}$$

Where σ_e acts like Frobenius on R and as the identity on the x's. Therefore, we may take the limit over this inverse system to obtain

$$S \coloneqq \varprojlim_{e \in \mathbb{N}} S_e$$

Concretely, recall that an element $s \in S$ can be thought of as a sequence $(s_0, s_1, s_2, \ldots)$ where $s_e \in S_e$ and $\sigma_e(s_e) = s_{e-1}$. In particular, we may define (with a slight abuse of notation)

$$x_i \coloneqq (r_i, x_i, x_i, \ldots) \in S$$

as the constant sequence.[12] Now, note that S_e is an S_{e-1}-algebra and its Frobenius factors as

$$\begin{tikzcd}
S_e \arrow[r, "F"] \arrow[dr, two heads, "\varphi_e"'] & S_e \\
 & S_{e-1} \arrow[u, hook, "\vartheta_e"']
\end{tikzcd}$$

And, moreover, such factorization is compatible with the structural maps of the inverse system defined S. More precisely, we have the following commutative diagram

$$\begin{tikzcd}
 & S_{e+1} \arrow[r, "F"] \arrow[dl, two heads, "\sigma_{e+1}"'] \arrow[dr, two heads, "\varphi_{e+1}"] & S_{e+1} \arrow[dl, two heads, "\sigma_{e+1}"] \\
S_e \arrow[r, "F"] \arrow[dr, two heads, "\varphi_e"'] & S_e & S_e \arrow[u, hook, "\vartheta_{e+1}"'] \arrow[dl, two heads, "\sigma_e"] \\
 & S_{e-1} \arrow[u, hook, "\vartheta_e"'] &
\end{tikzcd}$$

Therefore, by taking the inverse limit, we obtain a commutative diagram

$$\begin{tikzcd}
S \arrow[r, "F"] \arrow[dr, two heads, "\varphi"'] & S \\
 & S \arrow[u, hook, "\vartheta"']
\end{tikzcd}$$

of rings. One readily sees that φ is injective and therefore S is reduced. On the other hand,

$$\{x_1^{i_1} \cdots x_n^{i_n}\}_{0 \leq i_1, \ldots, i_n \leq p-1}$$

is a basis for S as an S-module by restriction of scalars along θ. Thus, putting everything together, we see that

$$F_* S = \bigoplus_{0 \leq i_1, \ldots, i_n \leq p-1} S F_* x_1^{i_1} \cdots x_n^{i_n}.$$

Thus, we're done if S is noetherian, which is the content of the theorem. This is actually an involved proof. Those interested, can try themselves or read a proof in [MP]. $\square$

Corollary 3.10. *F-finite rings have finite dimension.*

Remark 3.11. Corollary 3.10 was originally obtained by Kunz [Kun76]. However, I understand his proof is flawed due to some equi-dimensionality issues. Nonetheless, there are proofs of Corollary 3.10 that are independent of Gabber's result. I think we'll see one later on.

[12]Caution, not every element of R can be lifted to S. In fact, S isn't in any meaningful way an R-algebra.

Question 3.12 (Noether normalization of F-finite rings). Is an F-finite ring a finite (separable) extension of an F-finite regular ring that admits a p-basis?

Corollary 3.13. *F-finite rings admit a canonical module.*[13] *Namely,*

$$\omega_R := \operatorname{Ext}_S^{\dim S - \dim R}(R, S),$$

where $S \twoheadrightarrow R$ is as in Theorem 3.9.

Remark 3.14. By the way, not all excellent rings admit a canonical module, see [**?**]. This is an aspect in which F-finite rings beat excellent ones.

Exercise* 3.11. Prove that R is Cohen–Macaulay iff $\operatorname{Ext}_S^i(R, S) = 0$ for all $i \neq \dim S - \dim R$. So far, we've only defined local Cohen–Macaulay rings. Take the definition of general Cohen–Macaulay as being Cohen–Macaulay at all localizations at prime ideals (so you may reduce to the local case).

I hope the above convinces the reader that F-finite rings are pretty awesome. There's yet another reason why this is the case. Those rings that are pretty awesome for algebraic geometry have already been axiomatized and named, namely *excellent rings.*[14] Their definition is a bit of a mouthful though.

Definition 3.15. A noethering ring is said to be *excellent* if

(a) the completion homomorphism $R_{\mathfrak{p}} \longrightarrow \hat{R}_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$,
(b) all R-algebras of finite type have open regular loci, and
(c) all R-algebras of finite type are catenary (aka universally catenary).

Theorem 3.16 (Kunz [Kun76])**.** *F-finite rings are excellent. Conversely, a local ring $(R, \mathfrak{m}, \mathscr{k})$ is F-finite if (and only if) it is excellent and $\mathscr{k}$ is F-finite.*

Remark 3.17 (On the proof). The proof is too lengthy to be worthwhile doing here. However, the reader should be able to prove already as an exercise that F-finite rings satisfy the first two properties of excellence; which are referred to as *quasi-excellent*, using Radu–André's theorem for (a). Furthermore, the point is the F-finite property is already a notion of excellence in positive characteristics that is much better to deal with than excellence itself. So for instance, there will be many properties excellent rings have and we'll need that can be obtained directly from F-finiteness. So that's the approach we'll take. A very nice detailed proof can be found in [MP].

4. F-purity, F-splittings, and Fedder's criterion

Ok, here we are, we're ready to introduce our first notion of F-singularity. Let $\theta \colon R \longrightarrow S$ be an R-algebra and $f := \operatorname{Spec} \theta \colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. Recall that θ is flat if f^* is an exact functor and it is faithfully flat if it further satisfies any and so all of the following equivalent conditions (for a flat morphism):

(a) For every R-module M, if $M \neq 0$ then $f^*M \neq 0$.
(b) For every sequence $M' \longrightarrow M \longrightarrow M''$ of R-modules, if $f^*M' \longrightarrow f^*M \longrightarrow f^*M''$ is exact then is exact $M' \longrightarrow M \longrightarrow M''$.
(c) The map f is surjective.

[13] Don't worry at all if you don't know what this means. It's a little bit of a mess but we'll get back to it later when we need it. But it's a really important thing worth noting right away.

[14] Feel free to read their Wikipedia entry to glimpse at why.

(d) For every maximal ideal $\mathfrak{m} \in \operatorname{Spec} S$, we have $\mathfrak{m}S \neq S$.

Recall that, since $\operatorname{Spec} F = \mathrm{id}$, flatness and faithfull flatness are the same thing for the Frobenius map. And by Kunz's theorem, they're the same as the regularity of the ring. So what's a natural weakening for faithfull flatness?

Definition 4.1 (Purity and splitness)**.** We say that $\theta \colon R \longrightarrow S$ is *pure* if

$$\eta_M \colon M \longrightarrow f_* f^* M$$

is injective for all R-modules M. We say that $\theta \colon R \longrightarrow S$ is *split* if

$$\operatorname{Tr}_R \colon f_* f^! R \longrightarrow R$$

is surjective.

Remark 4.2. Observe that $\eta_M = \eta_R \otimes M$, and so the purity of $\theta \colon R \longrightarrow S$ means that $\eta_R = \theta \colon R \longrightarrow S$ remains injective after tensoring with any R-module M. On the other hand, note that $\theta \colon R \longrightarrow S$ is split iff there is $\phi \in \operatorname{Hom}_R(S, R)$ such that $\phi(1) = 1$, *i.e.* such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\eta_R = \theta"] \arrow[dr, "\mathrm{id}"'] & S \arrow[d, "\phi"] \\ & R \end{tikzcd}$$

We may refer to any such ϕ as a θ-splitting. In particular, one can tensor this diagram by M to get that $\phi \otimes_R M \colon f_* f^* M \longrightarrow M$ is a splitting of η_M, which forces η_M to be injective. In other words,

$$\text{SPLITNESS} \Longrightarrow \text{PURITY}$$

what about the converse?

Exercise 4.1. Show that in the definition of purity we may have restricted to finitely generated R-modules M only.

Proposition 4.3 (Faithfull flatness implies purity)**.** *If $\theta \colon R \longrightarrow S$ is faitfully flat then it is pure.*

Proof. Let M be an R-module. Then, $\eta_M \colon M \longrightarrow f_* f^* M$ is injective if and only if so is

$$f^* \eta_M \colon f^* M \longrightarrow f^* f_* f^* M$$

However, according to Exercise 3.7, the map

$$\epsilon_{f^* M} \colon f^* f_* f^* M \longrightarrow f^* M$$

is a splitting of $f^* \eta_M$. And so we're done. ∎

Exercise 4.2. Let $\mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}_p$ the canonical homomorphism, *i.e.* p-adic completion. It is faithfully flat and so pure. Show that it is not split, so that purity doesn't imply splitness.

Exercise 4.3. Show that $\theta \colon R \longrightarrow S$ is pure (resp. faithfully flat) iff so is $\theta_{\mathfrak{p}} \colon R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Prove that if $\theta \colon R \longrightarrow S$ is split then so are $\theta_{\mathfrak{p}} \colon R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. What about the converse?

One reason why splitness doesn't have a behavior as nice as the one exhibit by purity is that Hom doesn't commute with flat base change. However,

Exercise 4.4. Let M, N be R-modules and consider a flat R-algebra T. Consider the canonical homomorphism of T-modules

$$T \otimes_R \operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_T(T \otimes_R M, T \otimes_R N), \quad 1 \otimes \phi \mapsto T \otimes_R \phi.$$

Prove that it is an isomorphism if M is finitely presented Hint: Use a presentation of M to reduce to the free case where it is trivial. Conclude that if $\theta\colon R \longrightarrow S$ is a finite R-algebra such that $\theta_{\mathfrak{p}}\colon R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{p}}$ is split for all $\mathfrak{p} \in \operatorname{Spec} R$ then θ is split.

The next ingredient to elucidate when splitness is the same as purity is the so-called *Matlis Duality.*

4.1. **A quick overview of Matlis duality.** I strongly recommend/urge you to go read the appendix of [ILL+07] if you haven't heard about Matlis Duality and all the stuff around it. But, in a nutshell, here's the deal [Sta23, Tag 08Z9]. Let $(R, \mathfrak{m}, \mathcal{k})$ be an equi-characteristic complete local ring.[15] We consider an *injective hull*

$$E = E(\mathcal{k}) = E_R(\mathcal{k})$$

of $\mathcal{k}$ (as an R-module). By the definition, $E \supset \mathcal{k}$ is an injective R-module such that every non-zero submodule of it intersect $\mathcal{k}$ non-trivially. As a matter of fact, E exists and is unique up non-unique isomorphism.[16] For notation ease, let $\mathscr{N}$ be the (full) subcategory of noetherian R-modules (*i.e.* f.g. R-modules) and $\mathscr{A}$ the one of artinian R-modules. Then, $\mathscr{D} \coloneqq \operatorname{Hom}_R(-, E)$ induces a faithful exact (contravariant) functor

$$\mathscr{D}\colon \mathscr{N} \longrightarrow \mathscr{A}$$

and at the same time

$$\mathscr{D}\colon \mathscr{A} \longrightarrow \mathscr{N}$$

and, moreover, there are natural isomorphisms

$$\mathrm{id}_{\mathscr{N}} \cong \mathscr{D} \circ \mathscr{D} \quad \mathrm{id}_{\mathscr{A}} = \mathscr{D} \circ \mathscr{D}.$$

In other words, $\mathscr{D}$ induces an anti-equivalence between the category of noetherian R-modules and the one of artinian R-modules! That's pretty wild if you ask me. Now, we might come back to it later on with the details. For now, this is all I need you to know. For example, the above relies on the canonical map

$$R \xrightarrow{r \mapsto \cdot r} \operatorname{Hom}_R(E, E)$$

being an isomorphism (telling you that R and E are Matlis dual to one another). Use this to show the following.

Exercise 4.5. Show that, for every finitely generated R-module M, the (finitely generated R-module) $\operatorname{Hom}_R(M, R)$ is Matlis dual to the artinian R-module $E \otimes_R M$.

Later on, we'll see the so-called local duality theorem, which is (at the very least) an extremely powerful tool to compute Matlis duals. For instance, it tells you that the Matlis dual of $\operatorname{Hom}_R(M, \omega_R)$ (which is noetherian) is the *local cohomology* module $H^{\dim R}_{\mathfrak{m}}(M)$ (which is artinian).[17] In particular, ω_R is the Matlis dual of $H^{\dim R}_{\mathfrak{m}}(R)$, and $E = H^{\dim R}_{\mathfrak{m}}(\omega_R)$.

[15] That is, it is a $\mathcal{k}$-algebra.

[16] This is a very general thing—it has nothing to do with $\mathcal{k}$ being the residue field nor R being a complete local ring.

[17] Here, we use that R is complete to say that it has a canonical moodule ω_R, say $\omega_R = \operatorname{Hom}_A(R, A)$ where $A \subset R$ is any noether normalization.

We started our remark assuming that $(R, \mathfrak{m}, \mathscr{k})$ is complete. If we drop this hypothesis, we obtain instead a natural isomorphism

$$\hat{R} \otimes_R - \xrightarrow{\cong} \mathscr{D} \circ \mathscr{D}$$

on $\mathscr{N}$. In particular, we obtain a canonical isomorphism

$$\hat{R} \xrightarrow{\cong} \operatorname{Hom}_R(E, E)$$

which realizes E as an $\hat{R}$-module! In fact, E is an injective hull of $\mathscr{k}$ as an $\hat{R}$-module. Let that sink in. Thus, an injective hull of $\mathscr{k}$ as an $\hat{R}$-module is the same as one as an R-module. This is gonna come up below. Here's an example worth having in mind that illustrates this.

Example 4.4 (Explicit description of injective hulls of residue fields)**.** The injective hull of the residue field, although rather mysterious looking at first, it's often in practice a very explicit object. Let's consider, for example, $R = \mathscr{k}[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$. It turns out that we may take E to be the $\mathscr{k}$-vector space

$$E = E_R(\mathscr{k}) \coloneqq \bigoplus_{0<i_1,\ldots,i_n \in \mathbb{N}} \mathscr{k} \cdot \frac{1}{x_1^{i_1} \cdots x_n^{i_n}} \xleftarrow{u \coloneqq \frac{1}{x_1 \cdots x_n} \leftarrow\!\shortmid 1} \mathscr{k}$$

where the R-linear action is what you expect:

$$x_j \cdot \frac{1}{x_1^{i_1} \cdots x_n^{i_n}} = \begin{cases} \frac{1}{x_1^{i_1} \cdots x_j^{i_j - 1} \cdots x_n^{i_n}} & \text{if } i_j > 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, a fix $1/(x_1^{i_1} \cdots x_n^{i_n})$ is killed by all monomials of sufficiently large degree. This is why the above action extends to an $\hat{R} = \mathscr{k}[\![x_1, \ldots, x_n]\!]$-linear! That is, E is also and $\hat{R}$-module. In fact, it equals $E_{\hat{R}}(\mathscr{k})$. Check this:

Exercise 4.6. Verify that the above E is an injective hull of $\mathscr{k}$ over both R and $\hat{R}$.

The element

$$u \coloneqq \frac{1}{x_1 \cdots x_n} \in E$$

is particularly distinguished gentleman. It's called the *socle element* of E. To see why, do the following exercise.

Exercise 4.7. Prove that $\operatorname{Soc} E = \mathscr{k} \cdot u$, where $\mathscr{k} \cdot u$ is the copy of $\mathscr{k}$ inside E.

It's also customary to give E a $\mathbb{Z}$-grading by declaring

$$\deg \frac{1}{x_1^{i_1} \cdots x_n^{i_n}} \coloneqq -(i_1 + \cdots + i_n)$$

Let E_d the direct summand of E of degree d (*e.g.* $E_d = 0$ for all $d > -n$). Letting S_d denote the $\mathscr{k}$-module of polynomials of degree d, the above induces a perfect pairing of $\mathscr{k}$-modules

$$S_d \otimes_{\mathscr{k}} E_{-n-d} \longrightarrow E_{-n} = \mathscr{k} \cdot u \cong \mathscr{k}.$$

The above tell us how to write down an injective hull of $\mathscr{k}$ over $\mathscr{k}[\![x_1, \ldots, x_n]\!]$. But what about other complete $\mathscr{k}$-algebras, *i.e.* quotients of $\mathscr{k}[\![x_1, \ldots, x_n]\!]$? Let $A \coloneqq \mathscr{k}[\![x_1, \ldots, x_n]\!]/\mathfrak{a}$ be a complete local $\mathscr{k}$-algebra. Then, we may take its injective hull of the residue field as

$$E_A(\mathscr{k}) \coloneqq \{\varepsilon \in E = E_{\hat{R}}(\mathscr{k}) \mid \varepsilon\mathfrak{a} = 0\} = \operatorname{Hom}_{\hat{R}}(A, E),$$

which is an (injective) A-module by definition—it is annihilated by $\mathfrak{a}$.

Exercise 4.8. Verify that this is indeed an injective hull of $\Bbbk$ over A and show that its socle is $\Bbbk \cdot u$. In particular, $u \in E_A(\Bbbk)$ and we refer to it as the *socle element* of $E_A(\Bbbk)$.[18]

This is in fact just a particular case of the following.

Theorem 4.5 ([ILL$^+$07, Theorem A.25])**.** *Let $\theta\colon (R, \mathfrak{m}, \Bbbk) \longrightarrow (S, \mathfrak{n}, \ell)$ be a local finite homomorphsim. Then, $E_S(\ell) = \operatorname{Hom}_R(S, E_R(\Bbbk))$.*

The above explicit description of injective hulls has the virtue of telling us why the following holds.

Exercise 4.9. Let $\ell/\Bbbk$ be an arbitrary extension of fields and consider the canonical local and flat homomorphism

$$R_{\Bbbk} \coloneqq \Bbbk[\![x_1, \ldots, x_n]\!] \longrightarrow R_{\ell} \coloneqq \ell[\![x_1, \ldots, x_n]\!]$$

obtained as the $(x_1, \ldots, x_n)$-adic completion of the flat canonical homomorphism

$$\Bbbk[x_1, \ldots, x_n] \longrightarrow \ell[x_1, \ldots, x_n] = \ell \otimes_{\Bbbk} \Bbbk[x_1, \ldots, x_n].$$

Show that

$$E_{R_\ell}(\ell) = R_\ell \otimes_{R_\Bbbk} E_{R_\Bbbk}(\Bbbk),$$

where the socle element of $E_{R_\ell}(\ell)$ corresponds to 1 tensor the socle element of $E_{R_\Bbbk}(\Bbbk)$. More generally, use this to show that, if $\mathfrak{a} \subset R_\Bbbk$ is an ideal, then

$$E_{R_\ell/\mathfrak{a}R_\ell}(\ell) = R_\ell/\mathfrak{a}R_\ell \otimes_{R_\Bbbk/\mathfrak{a}} E_{R_\Bbbk/\mathfrak{a}}(\Bbbk)$$

and that likewise the socle element base changes to the socle element. Observe that $R_\Bbbk/\mathfrak{a} \longrightarrow R_\ell/\mathfrak{a}R_\ell$ is a (faithfully) flat local homomorphism and so pure. In particular, we obtain a canonical commutative diagram

$$\begin{tikzcd}
E_{R_\Bbbk/\mathfrak{a}}(\Bbbk) \arrow[r, hook] & E_{R_\ell/\mathfrak{a}R_\ell}(\ell) \\
\Bbbk \arrow[u, hook, "1\mapsto u"] \arrow[r, hook] & \ell \arrow[u, hook, "1\mapsto u"']
\end{tikzcd}$$

where the diagonal arrows are the socle maps.

Exercise 4.10. Show that a map $\phi \in \operatorname{Hom}_R(E, M)$ is injective if (and only if) $\phi(u) \neq 0$ (where $u \in E$ is the socle element).

4.2. **Back to purity vs splitness.** Using Matlis duality we can see the following.

Proposition 4.6. *Let $\theta\colon (R, \mathfrak{m}, \Bbbk) \longrightarrow S$ be an R-algebra such that $(R, \mathfrak{m}, \Bbbk)$ is complete (and equi-characteristic). Let E the injective hull of $\Bbbk$ and $u \in E$ be a socle element. Then, the following statements are equivalent*

(a) θ is pure.
(b) $\eta_E\colon E \longrightarrow S \otimes_R E$ is injective.
(c) $0 \neq 1 \otimes u \in S \otimes_R E$.
(d) θ is split.

[18]Probably we should be saying *a* socle element. A socle element is any generator of the socle and they differ up to multplication by units of $\Bbbk$. I guess it's ok to say the socle when it has been chosen in such a specific way.

Proof. We only need to explain why (b) implies (d) (see Exercise 4.10). That is, we gotta show that the trace

$$\operatorname{Tr}_R\colon \operatorname{Hom}_R(S,R) \longrightarrow R$$

is surjective assuming that

$$\eta_E\colon E \longrightarrow S\otimes_R E$$

is injective. Applying the exact contravariant functor $\mathscr{D}(-)=\operatorname{Hom}_R(-,E)$,[19], we obtain that $\mathscr{D}(\eta_E)$ is surjective. However, we have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_R(S,R) \arrow[rr, "\operatorname{Tr}_R"] \arrow[d, "\sigma\mapsto\sigma\otimes E"'] & & R \arrow[d, "\cong\colon 1\mapsto \mathrm{id}_E"] \\
\operatorname{Hom}_R(S\otimes_R E,E) \arrow[rr, "\mathscr{D}(\eta_E)"] & & \operatorname{Hom}_R(E,E)
\end{tikzcd}$$

where the right-hand vertical arrow is an isomorphism by Matlis duality. We claim that the left-hand vertical arrow is an isomorphism too. Indeed, recall that by $\otimes$-Hom adjointness

$$\operatorname{Hom}_R(S\otimes_R E,E) \xleftrightarrow[(\psi(s)(\varepsilon)\leftarrow\!\shortmid s\otimes\varepsilon)\leftarrow\!\shortmid\psi]{\phi\mapsto(s\mapsto\phi\circ E\otimes(s\colon R\longrightarrow S))} \operatorname{Hom}_R(S,\operatorname{Hom}_R(E,E))$$

is an isomorphism. Moreover, the composition

$$\operatorname{Hom}_R(S,R)\xrightarrow{\sigma\mapsto\sigma\otimes E}\operatorname{Hom}_R(S\otimes_R E,E)\xrightarrow{\phi\mapsto(s\mapsto\phi\circ E\otimes(s\colon R\longrightarrow S))}\operatorname{Hom}_R(S,\operatorname{Hom}_R(E,E))$$

is exactly

$$f^!(R\xrightarrow{\cong\colon 1\mapsto\mathrm{id}_E}\operatorname{Hom}_R(E,E))$$

and so an isomorphism. This proves the proposition. Isn't all this nonsense just pretty? □

Exercise 4.11. In the setup of Proposition 4.6, let M be an R-module. Prove that $\mathscr{D}$ of $\eta_{\mathscr{D}(M)}$ is (up to canonical isomorphism) Tr_M. Hint: in Proposition 4.6 we did the case $M=R$.

Scholium 4.7. *Let $\theta\colon(R,\mathfrak{m},\mathscr{k})\longrightarrow S$ be a finite R algebra such that R is complete. Then, η_E and Tr_R are Matlis dual to one another. More generally, $\eta_{\mathscr{D}(M)}$ and Tr_M are Matlis dual to one another for all finitely generated R-modules M.*

Corollary 4.8. *Let $\theta\colon R\longrightarrow S$ be a finite R-algebra. If θ is pure then it is split.*

Proof. Since θ is finite, we may assume that R is local (see Exercise 4.4). Since $R\longrightarrow\hat{R}$ is faithfully flat, we assume that R is further complete (see Exercise 4.4). The result then follows from Proposition 4.6. □

Corollary 4.9. *Let $\theta\colon(R,\mathfrak{m},\mathscr{k})\longrightarrow S$ be an R-algebra and E be an injective hull of $\mathscr{k}$ with socle element $u\in E$. The following statements are equivalent:*

(a) $\theta\colon R\longrightarrow S$ is pure.
(b) $\eta_E\colon E\longrightarrow S\otimes_R E$ is injective (i.e. $0\neq 1\otimes u\in S\otimes_R E$).
(c) $\hat{\theta}\colon\hat{R}\longrightarrow\hat{S}$ is pure.

[19]Whose exactness simply means that E is an injective R-module

Proof. Let's start showing the equivalence between (b) and (c). For this, we use that E is automatically an injective hull of $\mathscr{k}$ as an $\hat{R}$-module. Observe that $\eta_E\colon E \longrightarrow \hat{S} \otimes_{\hat{R}} E$ factors as

$$E \xrightarrow{\eta_E} S \otimes_R E \to \hat{S} \otimes_R E = \hat{S} \otimes_{\hat{R}} E$$

where the second map is E tensor the canonical faithfully flat (and so pure) homomorphism $S \longrightarrow \hat{S}$, whence it is injective. This means that $E \longrightarrow \hat{S} \otimes_{\hat{R}} E$ is injective iff so is $E \longrightarrow S \otimes_R E$.

It remains to explain why (c) implies (a). Assume that $\hat{\theta}$ is pure and so split. Let $\phi \in \operatorname{Hom}_{\hat{R}}(\hat{S}, \hat{R})$ be a splitting. Now, let M be an R-module. Since $R \longrightarrow \hat{R}$ is faithfully flat, to show that $\eta_M\colon M \longrightarrow S \otimes_R M$ is injective, we may do it after base changing it to $\hat{R}$. But $\hat{R} \otimes_R \eta_M$ is injective because it is split by the composition

$$\hat{R} \otimes_R S \otimes_R M \longrightarrow \hat{S} \otimes_R M \xrightarrow{\phi \otimes M} \hat{R} \otimes_R M,$$

where the first map is M tensor the canonical map $\hat{R} \otimes_R S \longrightarrow \hat{S}$, $\square$

4.3. **The purity and splitness of Frobenius.** We made it, here's the key concept in the theory.

Definition 4.10 (F-purity and F-splitness)**.** Let $R/\mathbb{F}_p$ be a noetherian $\mathbb{F}_p$-algebra. We say that R is *F-pure* (resp. *F-split*) if $F^e\colon R \longrightarrow R$ is pure (resp. split) for some/all $0 \neq e \in \mathbb{N}$. A map $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* 1) = 1$ is referred to as an *F^e-splitting.*

Corollary 4.11. *Both regular rings and F-split rings are F-pure. Further, F-pure rings are reduced.*

Corollary 4.12. *A ring R is F-pure if and only if $R_{\mathfrak{p}}$ is F-pure for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Remark 4.13. The same can't be said about F-splitness. In fact, there are regular rings that aren't F-split; see [**?**]. This is why I don't think you can think of F-splitness as an honest notion of singularity, unless it matches F-purity by some good external reason.

Corollary 4.14. *Suppose that R is either a complete local ring or F-finite. Then, it is F-pure iff it is F-split.*

Corollary 4.15. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring. Then it is F-pure if and only if $\hat{R}$ is F-split.*

Exercise 4.12. Let $\theta\colon R \longrightarrow S$ be a pure (resp. split) R-algebra. Prove that if S is F-pure (resp. F-split) then so is R. Conclude that "direct summands" of regular rings are F-pure.

Exercise 4.13. Working in the setup of Exercise 4.9, prove that $R_{\mathscr{k}}/\mathfrak{a}$ is F-pure (*i.e.* F-split) if and only if so is $R_\ell/\mathfrak{a}R_\ell$. Hint: The forward implication "$\Rightarrow$" follows at once from Exercise 4.12 as $A_{\mathscr{k}} \coloneqq R_{\mathscr{k}}/\mathfrak{a} \longrightarrow A_\ell \coloneqq R_\ell/\mathfrak{a}R_\ell$ is faithfully flat and so pure. The converse is the more interesting part. To prove it, use Exercise 4.9 to chase the socle element along the following commutative diagram

$$\begin{tikzcd} E_{A_{\mathscr{k}}}(\mathscr{k}) \arrow[r, "\eta"] \arrow[d, hook] & F^e_* A_{\mathscr{k}} \otimes_{A_{\mathscr{k}}} E_{A_{\mathscr{k}}}(\mathscr{k}) \arrow[d, hook, "\theta"] \\ E_{A_\ell}(\ell) \arrow[r, "\eta"] & F^e_* A_\ell \otimes_{A_\ell} E_{A_\ell}(\ell) \end{tikzcd}$$

where the map θ is injective as it corresponds to the canonical map

$$F^e_* A_{\mathscr{k}} \otimes_{A_{\mathscr{k}}} E_{A_{\mathscr{k}}}(\mathscr{k}) \longrightarrow F^e_* A_\ell \otimes_{F^e_* A_{\mathscr{k}}} (F^e_* A_{\mathscr{k}} \otimes_{A_{\mathscr{k}}} E_{A_{\mathscr{k}}}(\mathscr{k})),$$

which is injective as $A_{\mathscr{k}} \longrightarrow A_\ell$ and so $F^e_* A_{\mathscr{k}} \longrightarrow F^e_* A_\ell$ are pure.

4.4. **Fedder's criterion.** How to know when a ring is F-pure you may wonder? Fedder's criterion is the answer. Let's start with a warm up and let's express Fedder's criterion for rings that admit a p-basis.

Exercise 4.14 (Fedder's criterion for p-bases)**.** Let S be a regular F-finite ring that admits a p-basis; see Exercise 3.10. Let $\mathfrak{a} \subset S$ be an ideal with quotient $R \coloneqq S/\mathfrak{a}$. Let $\mathfrak{p} \subset \mathfrak{a}$ be a prime ideal, which we may think of as a prime ideal of R. In the context of the exact sequence

$$0 \longrightarrow F^e_* \mathfrak{a}^{[q]} \longrightarrow F^e_*(\mathfrak{a}^{[q]} : \mathfrak{a}) \longrightarrow \operatorname{Hom}_R(F^e_* R, R) \longrightarrow 0$$

Show that $\operatorname{Hom}_R(F^e_* R, \mathfrak{p})$ corresponds to

$$F^e_* \left(\frac{(\mathfrak{a}^{[q]} : \mathfrak{a}) \cap \mathfrak{p}^{[q]}}{\mathfrak{a}^{[q]}} \right)$$

Conclude that $R_{\mathfrak{p}}$ is F-pure if and only if $\mathfrak{a}^{[q]} : \mathfrak{a} \not\subset \mathfrak{p}^{[q]}$.

Of course, Fedder's criterion is not very useful unless we can compute $\mathfrak{a}^{[q]} : \mathfrak{a}$, which is in general no joke. There's, however, an important case that isn't terribly hard. Namely,

Lemma 4.16. *Let $\mathfrak{a} \subset S$ be generated by a regular sequence $f_1, \ldots, f_n$ (here, we don't assume that S is regular) that remains regular after any permutation.*[20] *Then, the inclusion*

$$\mathfrak{a}^{[q]} : \mathfrak{a} \supset (f_1^q, \ldots, f_n^q, (f_1 \cdots f_n)^{q-1})$$

is an equality.

Proof. We proceed by induction on n. If $n = 1$, we readily see that $(f)^{[q]} : (f) = (f^{q-1})$ as soon as f is a regular element on S. Indeed, if $sf = hf^q$ then $s = hf^{q-1}$ as f is not a zero-divisor. Assume the result for all $1, \ldots, n-1$. Let $s \in S$ be such that

$$sf_i = \sum_{j=1}^{n} s_{ij} f_j^q, \quad \forall i = 1, \ldots, n$$

for some matrix $s_{ij} \in S$. We want to show that s is divisible bt $(f_1 \ldots f_n)^{q-1}$ module $(f_1^q, \ldots, f_n^q)$. Now, we use Exercise 1.13 to say that $f_1^q, \ldots, f_{n-1}^q, f_n$ is a regular sequence. So, if we look at the n-th equation above, this tells us that $s - s_{nn} f_n^{q-1} \in (f_1^q, \ldots, f_{n-1}^q)$ and so we may assume that s is is divisible by f_n^{q-1}. That is, we're free to replace s by; say, $s' f_n^{q-1}$. Let's look then at the equations

$$s' f_n^{q-1} f_i = \sum_{j=1}^{n} s_{ij} f_j^q, \quad \forall i = 1, \ldots, n-1.$$

Since $f_1^q, \ldots, f_{n-1}^q, f_n^{q-1}$ is a regular sequence, we conclude that $s' f_i - s_{in} f_n \in (f_1^q, \ldots, f_{n-1}^q)$ for all $i = 1, \ldots, n-1$. In particular, $s' f_i \in (f_1^q, \ldots, f_{n-1}^q, f_n)$ for all $i = 1, \ldots, n-1$. Now, here's the thing, let's use that $f_n, f_1^q, \ldots, f_{n-1}^q$ is a regular sequence[21] together with the inductive hypothesis to conclude that

$$s' \in (f_1^q, \ldots, f_{n-1}^q, (f_1 \cdots f_{n-1})^{q-1}, f_n)$$

[20]For instance, we may assume that $n \leq 2$ or well that S is either local or that it is a polynomial ring over a field and the f_i's are homogeneous. See Remark 1.10.

[21]And here's where we need that we need to be able to permute the elements in the regular sequence!

and so
$$s = s' f_n^{q-1} \in (f_1^q, \dots, f_{n-1}^q, (f_1 \cdots f_n)^{q-1});$$
as required. $\square$

Remark 4.17. I don't know whether Lemma 4.16 still works if we drop the permutability assumption. I think I might've heard people claim it without it but I'm not sure. Certainly, I don't see how to prove it without this hypothesis. Do you? If so, please let me know!

Corollary 4.18 (Fedder's criterion of complete intersections)**.** *Let S be an F-finite regular ring such that $\operatorname{Hom}_S(F^e_* S, S)$ is principally generated by a (Frobenius trace) Φ^e as an $F^e_* S$-module.*[22] *If $\mathfrak{a} \subset S$ is generated by a permutable regular sequence, then $\operatorname{Hom}_R(F^e_* R, R)$ is principally generated by the restriction of $\Phi^e(f_1 \cdots f_n)^{q-1}$ as an $F^e_* R$-module. Moreover, if S admits a p-basis, then R is F-pure at $\mathfrak{p} \subset \mathfrak{a}$ if and only if $(f_1 \cdots f_n)^{q-1} \notin \mathfrak{p}^{[q]}$.*[23]

Exercise 4.15. Let $f \coloneqq x_0^n + \cdots + x_d^n \in S \coloneqq \mathbb{F}_p[x_0, \dots, x_d]$. Use Fedder's criterion to characterize those triples (p, n, d) for which $R \coloneqq S/f$ is F-pure. Good luck and have fun!

Lemma 4.19 (Colon ideals and flat base change)**.** *Let $\theta \colon R \longrightarrow S$ be a flat algebra and $\mathfrak{a}, \mathfrak{b} \subset R$ be ideals. Then, the inclusion of ideals*
$$(\mathfrak{b} :_R \mathfrak{a})S \subset \mathfrak{b}S :_S \mathfrak{a}S$$
is an equality.

Proof. Since θ is flat, the canonical surjective map $S \otimes_R \mathfrak{a} \longrightarrow \mathfrak{a}S$ is an isomorphism for all ideals $\mathfrak{a} \subset S$. On the other hand, writing $\mathfrak{a} = (a_1, \dots, a_n)$ (which uses noetherianity) yields an exact sequence
$$0 \longrightarrow \mathfrak{b} \colon \mathfrak{a} \xrightarrow{\subset} R \xrightarrow{r \mapsto (ra_1, \dots, ra_n)} (R/\mathfrak{b})^{\times n}$$
We may FLAT base change it to obtain an exact sequence
$$0 \longrightarrow S \otimes_R (\mathfrak{b} : \mathfrak{a}) \longrightarrow S \xrightarrow{s \mapsto (sa_1, \dots, sa_n)} (S/\mathfrak{b}S)^{\times n}$$
which implies what we want. $\square$

Exercise 4.16. Let $\theta \colon (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a local homomorphism and $\mathfrak{a} \subset R$ be an ideal. Show that $\mathfrak{a}^{[q]} : \mathfrak{a} \subset \mathfrak{m}^{[q]}$ if $(\mathfrak{a}S)^{[q]} : \mathfrak{a}S \subset \mathfrak{n}^{[q]}$. Show that the converse holds if θ is flat (and so faithfully flat). Hint: use the previous Lemma 4.19.

Theorem 4.20 (Local Fedder's criterion)**.** *Let $(S, \mathfrak{n}, \mathcal{k})$ be a regular local ring and $\mathfrak{a} \subset \mathfrak{n} \subset S$ be an ideal. Then, the local ring $(R \coloneqq S/\mathfrak{a}, \mathfrak{m} \coloneqq \mathfrak{n}/\mathfrak{a}, \mathcal{k})$ is F-pure if and only if $\mathfrak{a}^{[q]} : \mathfrak{a} \not\subset \mathfrak{n}^{[q]}$.*

Proof. Notice that we may assume that S and so R are complete. Indeed, F-purity is preserved and can be checked after completion (see Corollary 4.15). Furthermore, $\hat{R} = \hat{S}/\mathfrak{a}\hat{S}$, and $\mathfrak{a}^{[q]} : \mathfrak{a} \not\subset \mathfrak{n}^{[q]}$ if and only if $\hat{\mathfrak{a}}^{[q]} : \hat{\mathfrak{a}} \not\subset \hat{\mathfrak{n}}^{[q]}$ (see Exercise 4.16), where $\hat{\mathfrak{a}} \coloneqq \mathfrak{a}\hat{S}$ and $\hat{\mathfrak{n}} = \mathfrak{n}\hat{S}$.

So it suffices to show the case $S = \mathcal{k}[\![x_1, \dots, x_d]\!]$. If $\mathcal{k}$ and so S were F-finite then we're done as we'd have a p-basis (see Exercise 4.14). To reduce to that case, we consider $S' \coloneqq \mathcal{k}_{\mathrm{perf}}[\![x_1, \dots, x_n]\!]$ where $\mathcal{k}_{\mathrm{perf}} \supset \mathcal{k}$ is the perfection of $\mathcal{k}$ (which is a perfect and so F-finite field). We can then apply Exercise 4.13 and Exercise 4.16 to $S \longrightarrow S'$ to conclude. $\square$

[22]For instance, if either S admits a p-basis or it is local. It turns out that this is a general abstract property local Gorenstein rings enjoy as we'll see later on. And, as you may expect, regular local rings are Gorenstein.
[23]This last part only needs that $f_1, \dots, f_n$ is a regular sequence on $R_{\mathfrak{p}}$.

5. Canonical Modules and a First Encounter with F-injectivity

Here's what we want. Let R be a (noetherian as always) ring. We want a finitely generated R-module ω_R such that

$$\operatorname{Hom}_{R_\mathfrak{p}}\big((\omega_R)_\mathfrak{p}, E_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))\big) = H^{\dim R_\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. That is, we want a module that is locally at every point Matlis dual to the top local cohomology module; which is Artinian. When such module ω_R exists, we praise it and refer to it as *a canonical module* over R. If a canonical module exists, it's far from unique. In fact, if ω_R is a canonical module then so is $\omega_R \otimes_R L$ for all invertible[24] R-modules L. However, this is the worst that can happen. For example, over a *normal ring*,[25] we'll see that what is unique is the divisor class associated to ω_R.

As for the existence of canonical modules, this is intimately related to *Gorenstein singularities.* A *Gorenstein ring* can be defined as a Cohen–Macaulay ring where R itself is a canonical module. As a matter of fact, a (noetherian) ring R admits a canonical module if and only if it is the quotient of a finite dimensional Gorenstein ring. As an example, regular rings are Gorenstein.

To be more precise, what one is looking for is a *(normalized) dualizing complex* $\omega_R^\bullet$ which lives in certain derived category of R-modules. The canonical module is just its cohomology modulo in degree $-\dim R$, which needs to be finite so that the canonical module exists. The existence of dualizing complexes is equivalent to being finite over a Gorenstein ring of finite dimension. The Cohen–Macaulay property is engineered exactly so that the dualizing complex collapses into the canonical module, which then becomes a *dualizing module.* That's really it in a nutshell. That's how Cohen–Macaulay and Gorenstein singularities fit into the general landscape of algebraic geometry. Moreover, this is done so that something called *Serre duality* holds—it is the algebro-geomtric analog of Poincaré duality and so it's of paramount importance. We'll see its local version aka local duality.

You see? It's a whole bunch of difficult stuff to grasp. That's life sometimes. Let's give it a shot and let's try to understand this a bit more as it lies at the heart of F-singularity theory too. Ah! And the reason is the following. If R is further F-finite and is *e.g.* local or essentially of finite type over field, then it turns out that it comes equipped with a so-called *Cartier operator*

$$\kappa_R^e \colon F^e_* \omega_R \longrightarrow \omega_R$$

which is arguably the most important object in the whole theory! For example,

Theorem 5.1. *With notation as above, suppose that R is Cohen–Macaulay and that R is either complete local or normal. Then, R is F-pure if and only if $\kappa_R^e \colon F^e_*\omega_R \longrightarrow \omega_R$ is split.*

So, there's an important weakening to that condition (at least when R is Cohen–Macaulay).

Definition 5.2 (F-injective Cohen–Macaulay rings)**.** With notation as above, R is said to be *F-injective* if $\kappa_R^e \colon F^e_*\omega_R \longrightarrow \omega_R$ is surjective.

Ok. Let's try to understand a bit more of all that. The first thing to know about is local cohomology, which was in part invented by Grothendieck (ofc) for that specific purpose.

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[24]Meaning locally free of rank 1.

[25]Which we haven't define yet and I don't know if the reader knows well.

5.1. **A real quick overview on local cohomology.** Let R be a noetherian ring. We've been using the *functor of sections* $\Gamma \coloneqq \operatorname{Hom}_R(R, -)$ quite a bit, which is quite boring as the canonical map $M \longrightarrow \operatorname{Hom}_R(R, M)$ is an isomorphism. In particular, $\operatorname{Hom}_R(R, M)$ is exact. To make this more interesting cohomologically speaking, we need to look at *sections with support* on some closed subset of $\operatorname{Spec} R$. To this end, let $\mathfrak{a} \subset R$ be an ideal. All we're gonna do depend on $V(\mathfrak{a}) \subset \operatorname{Spec} R$ so we may take $\mathfrak{a}$ to be radical. Anyways, we consider the so-called *functor of sections with support in* $V(\mathfrak{a})$

$$\Gamma_{\mathfrak{a}}(-) \coloneqq \varinjlim_{n \in \mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n, -).$$

This is also known with the more algebraic name of $\mathfrak{a}$-*torsion functor*, and the reason is that

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \in \mathbb{N}} \{m \in M \mid m\mathfrak{a}^n = 0\} = \{m \in M \mid \operatorname{supp} m \subset V(\mathfrak{a})\} \subset M.$$

Exercise 5.1. Prove that $\Gamma_{\mathfrak{a}}$ is a left exact (covariant) functor but not necessarily exact.

Thus, one defines the i-th *local cohomology functor with support on* $\mathfrak{a}$; denoted by $H^i_{\mathfrak{a}}$, as the i-th derived functor of $\Gamma_{\mathfrak{a}}$. In practice, this means that $H^i_{\mathfrak{a}}(M)$ is the i-th cohomology of the complex $\Gamma_{\mathfrak{a}}(0 \longrightarrow E^\bullet)$ where $0 \longrightarrow M \longrightarrow E^\bullet$ is any injective resolution.

Exercise 5.2. Prove that

$$H^i_{\mathfrak{a}}(-) = \varinjlim_{n \in \mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, -).$$

Remark 5.3. Observe that $H^i_{\mathfrak{a}}(-) = H^i_{\sqrt{\mathfrak{a}}}(-)$ and moreover

$$\Gamma_I(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M).$$

Also, given a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we obtain a long exact sequence on local cohomology

$$\cdots \longrightarrow H^{i-1}_{\mathfrak{a}}(M'') \xrightarrow{\delta} H^i_{\mathfrak{a}}(M') \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M'') \xrightarrow{\delta} H^i_{\mathfrak{a}}(M') \longrightarrow \cdots$$

Example 5.4. Using that

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is an injective resolution, we readily see that $H^1_{(p)}(\mathbb{Z})$ is zero unless $i = 1$ in which case it equals $\mathbb{Z}[p^{-1}]/\mathbb{Z}$.

Sometimes one may want/need compute directly a local cohomology module. In that case, the Cech complex is pretty useful. Let $r \in R$ (with R noetherian), it's (extended) Cech complex is

$$C(r; R) \colon 0 \longrightarrow R \longrightarrow R_r \longrightarrow 0$$

If we have a sequence $r_1, \ldots, r_n \in R$, then its Cech complex is

$$C(r_1, \ldots, r_n; R) \coloneqq \bigotimes_{i=1}^{n} C(r_i; R).$$

If M is an R-module, we also define the Complex

$$C(r_1, \ldots, r_n; M) \coloneqq C(r_1, \ldots, r_n; R) \otimes_R M.$$

The proof of the following result is beyond the scope of this course.

Theorem 5.5. *The i-th cohomology of the Cech complex $C(r_1, \dots, r_n; M)$ turns out to be $H^i_{(r_1,\dots,r_n)}(M)$.*

With the above, we can do the following computation.

Exercise 5.3. Prove that if $R = \mathcal{k}[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)$ then

$$H^i_{\mathfrak{m}}(R) = \begin{cases} 0 & \text{if } i \neq d, \\ E_{R_{\mathfrak{m}}}(\mathcal{k}) & \text{otherwise.} \end{cases}$$

NB this makes the polynomial and power series rings pretty awesome and we'd like all rings to be like this. We'll end up defining Gorensteing rings after this.

The following properties are useful and left as an exercise.

Exercise 5.4. Let $\mathfrak{a} \subset R$ be an ideal in a noetherian ring and M be an R-module. Let $\theta \colon R \longrightarrow S$ be an algebra and N be an S-module. Prove that following:

(a) If θ is flat then the canonical morphism $H^i_{\mathfrak{a}}(M) \otimes_R S \longrightarrow H^i_{\mathfrak{a}S}(M \otimes_R S)$ is an isomorphism. That is, $f^* H^i_{\mathfrak{a}}(M) = H^i_{f^*\mathfrak{a}}(f^*M)$ if f is flat.
(b) $H^i_{\mathfrak{a}}(f_*N) = f_* H^i_{\mathfrak{a}S}(N)$.
(c) If R is further local and M is finitely generated then $H^i_{\mathfrak{m}}(M) = H^i_{\hat{\mathfrak{m}}}(\hat{R})$.

5.2. **Cohen–Macaulay and Gorenstein singularities.** With the basics of local cohomology down, we can see what depth and Cohen–Macaulay means in these terms.

Exercise 5.5. Prove that if $\mathfrak{a}M \neq M$ then

$$\operatorname{depth}_R(\mathfrak{a}, M) = \min\{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M) \neq 0\}$$

if M is finitely generated. Conclude that a ring R is Cohen–Macaulay if and only if for all $\mathfrak{p} \in \operatorname{Spec} R$ it follows that

$$H^i_{\mathfrak{p}}(R) = 0, \quad \forall i < \operatorname{ht} \mathfrak{p}.$$

Equivalently, if this happens at all maximal ideals only. In particular, a local ring $(R, \mathfrak{m}, \mathcal{k})$ is Cohen–Macaulay if and only if

$$H^i_{\mathfrak{m}}(R) = 0, \quad \forall i < \dim R.$$

Remark 5.6. Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring. We'll see later that $H^i_{\mathfrak{m}}(R) = 0$ for all $i > \dim R$ and that $H \dim R_{\mathfrak{m}}(R)$ is never zero. This means that a Cohen–Macaulay local ring is one in which all local cohomology groups $H^i_{\mathfrak{m}}(R)$ vanish except $H^{\dim R}_{\mathfrak{m}}(R)$.

Definition 5.7 (Gorenstein rings)**.** A Cohen–Macaulay ring R is said to be *Gorenstein* if $H^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{p}}(R)$ is an injective hull of $\kappa(\mathfrak{p})$ as an $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$.[26] In other words, $\mathscr{D}(R_{\mathfrak{p}}) = H^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{p}}(R)$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

Example 5.8. From Exercise 5.3, one readily sees that regular rings are Gorenstein.

What are Gorenstein rings good for you may ask? We'll, let's see that next.

[26]Note that Gorenstein rings are first and foremost Cohen–Macaulay (and noetherian!).

5.3. **Local duality.** Local duality is roughly at the very least an amazing way to understand/calculate Matlis duality. Let's see how grandiose it is for Gorenstein rings. Local duality is pretty much the Poincaré duality of singularities. It's proof, for now, is beyond the scope of this course.

Theorem 5.9 (Local duality over Gorenstein singularities)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a Gorenstein local ring of dimension d. Then, for all $0 \leq i \leq d$, the functors $H^i_{\mathfrak{m}}$ and $\mathscr{D} \circ \operatorname{Ext}^i_R(-, R)$ are naturally isomorphic on the category of finitely generated modules.*[27]*. In particular, if R is complete and M is a finitely generated R-module, the Matlis dual of $\operatorname{Ext}^i_R(M, R)$ is naturally isomorphic to $H^{d-i}_{\mathfrak{m}}(M)$ (for all $0 \leq i \leq d$).*

As a first corollary, one obtains the following.

Exercise 5.6. Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, \mathscr{k})$. Show that $H^i_{\mathfrak{m}}(M)$ is an artinian R-module. Hint: reduce to the complete case. Then, use the Cohen structure theorem to reduce to the regular and so Gorenstein case. From here, use local and Matlis duality.

Now, let G be a Gorenstein ring of finite dimension (which is equidimensional as it is Cohen–Macaulay) and suppose that R is a finite G-algebra. In particular, we can write down a factorization

$$G \twoheadrightarrow A \hookrightarrow R$$

so that R is a finite extension of A, which is itself a quotient of G. By the Cohen structure theorems, this can always be setup if R is complete. Likewise, by Gabber's theorem, also if R is F-finite. Suppose that R and so A are equidimensional (*e.g.* Cohen–Macaulay).

First, we set

$$\omega_G \coloneqq G$$

and then

$$\omega_A \coloneqq \operatorname{Ext}^{\dim G - \dim A}_G(A, \omega_G)$$

which is an A-module. Finally, set

$$\omega_R \coloneqq \operatorname{Hom}_A(R, \omega_A).$$

This definition may look quite strange at first. However, it makes total sense under the light of *Grothendieck's duality*. Unfortunately, this is a topic that we have no time to cover as it'd imply to delve into derived categories and so on. This is the right language to make all this very natural and satisfying. As an application of Grothendieck duality, one could see the following.

Proposition 5.10. *With notation as above, let $\mathfrak{p} \in \operatorname{Spec} R$. Then, there is a natural isomorphism of functors*

$$\mathscr{D} \circ \operatorname{Hom}_{R_\mathfrak{p}}(-, (\omega_R)_\mathfrak{p}) \cong H^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}$$

on finitely generated $R_\mathfrak{p}$-modules. In particular, $\omega_{\hat{R}_\mathfrak{p}} \coloneqq \hat{R}_\mathfrak{p} \otimes \omega_R$ is naturally Matlis dual to $H^{\dim R_\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$.

In other words, we've succeded in constructing *canonical modules* as mentioned in the introduction, at least for equidimensional finite algebras over Gorentein rings. If R is further Cohen–Macaulay, a canonical module becomes a *dualizing module*:

[27]Reall that $\mathscr{D} \coloneqq \operatorname{Hom}_R(-, E)$ where $E = E_R(\mathscr{k})$

Theorem 5.11 (Local Duality over Cohen–Macaulay singularities)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a Cohen–Macaulay local ring of dimension d that is a finite algebra over a Gorenstein ring (e.g. complete or F-finite). Let ω_R be the corresponding canonical module. Then, for all $0 \leq i \leq d$, there is a natural isomorphism of functors*

$$\mathscr{D} \circ \operatorname{Ext}^i_R(-, \omega_R) \cong H^{d-i}_{\mathfrak{m}}$$

over finitely generated R-modules. In particular, if R is complete and M is a finitely generated R-module, then $\operatorname{Ext}^i_R(M, \omega_R)$ and $H^{d-i}_{\mathfrak{m}}(M)$ are naturally Matlis dual to one another.

To recap and what you need to take home, when R happens to be an equi-dimensional finite algebra over a Gorenstein ring of finite dimension (*e.g.* complete or F-finite), we define its canonical module as before. The first cool thing about it (thanks to Grothendieck duality and local duality for Gorenstein singularities) is that there are local natural isomorphisms at every $\mathfrak{p} \in \operatorname{Spec} R$

$$\mathscr{D} \circ \operatorname{Hom}_{R_{\mathfrak{p}}}(-, (\omega_R)_{\mathfrak{p}}) \cong H^{\operatorname{ht} \mathfrak{p}}_{\mathfrak{p} R_{\mathfrak{p}}}$$

In particular, $\omega_{\hat{R}_{\mathfrak{p}}} \coloneqq \hat{R}_{\mathfrak{p}} \otimes \omega_R$ is naturally Matlis dual to $H^{\dim R_{\mathfrak{p}}}_{\mathfrak{p} R_{\mathfrak{p}}}(R_{\mathfrak{p}})$. So, in the complete case, ω_R is the Matlis dual of the top local cohomology module. If R is local then all its canonical modules are isomorphic, so in that case we may talk about of the canonical module.

Moreover, if R is further Cohen–Macaulay, then the canonical module acquires new powers known as local duality: there is a natural isomorphism

$$\mathscr{D} \circ \operatorname{Ext}^i_{R_{\mathfrak{p}}}(-, (\omega_R)_{\mathfrak{p}}) \cong H^{\operatorname{ht} \mathfrak{p} - i}_{\mathfrak{p} R_{\mathfrak{p}}}$$

on finitely generated $R_{\mathfrak{p}}$-modules for all $\mathfrak{p} \in \operatorname{Spec} R$.

5.4. **The Cartier operator and F-injectivity.** Let's look up at the functoriality of canonical modules under finite extensions. Let $R \subset S$ be a finite extension such that R admits a canonical module as before. Let $f\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ be the induced map as usual. By this Grothendieck duality abstract nonsense we've been using shamelessly (I may add the details at some point but don't hold your breath), it follows that $f^! \omega_R$ is a canonical module over S. But what if we already had defined a canonical module ω_S over S? At the very least, we can say that

$$f^! \omega_R \cong \omega_S \otimes_S L$$

for some invertible S-module L. In particular, if $(S, \mathfrak{n})$ and so $(R, \mathfrak{m})$ are local, we can say that

$$f^! \omega_R \cong \omega_S$$

In particular, in this local case, the trace map can be written as

$$\operatorname{Tr}_{\omega_R}\colon f_* \omega_S \longrightarrow \omega_R$$

It further has the property that the S-linear map

$$S \xrightarrow{1 \mapsto \operatorname{Tr}_{\omega_R}} \operatorname{Hom}_S(f_* \omega_S, \omega_R)$$

is an isomorphism; and this is again a consequence of Grothendieck duality.

As the reader may verify themselves, applying $\mathscr{D}$ to $\operatorname{Tr}_{\omega_R}$ yields the map:

$$H^d_{\mathfrak{m}}(f)\colon H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(S) = f_* H^d_{\mathfrak{n}}(S)$$

where d is the common dimension of R and S. Thus, the completion of $\operatorname{Tr}_{\omega_R}$ is the Matlis dual of the canonical map $H^d_{\mathfrak{m}}(R) \longrightarrow f_* H^d_{\mathfrak{n}}(S)$.

There is one more non-local case where it is possible to construct a trace $\mathrm{Tr}_{\omega_R}\colon f_*\omega_S \longrightarrow \omega_R$. Namely, if R and S are (essentially) of finite type over some field, say $\mathcal{k} \subset \ell$; respectively, and $\ell/\mathcal{k}$ is finite. This is Grothendieck duality again. In that case, it must localize (up to isomorphism) to the local traces we had above.

We want to apply the above to the Frobenius map $F^e\colon R \longrightarrow R$, assuming R is further reduced and F-finite.[28] As before, we want the following to hold:

$$F^{e,!}\omega_R \cong \omega_R$$

This can be guaranteed either if R is local or essentially of finite type over an F-finite field. But, whenever that's the case, we get the following map

$$\kappa_R^e \coloneqq \mathrm{Tr}_{\omega_R}\colon F_*^e\omega_R \longrightarrow \omega_R$$

which we'll referred to as the *Cartier operator* of R. It follows that

$$F_*^e R \xrightarrow{1\mapsto\kappa_R^e} \mathrm{Hom}_R(F_*^e\omega_R, \omega_R)$$

is an isomorphism.

Exercise 5.7. Prove that $\kappa_R^e = \kappa_R^a \circ F_*^a\kappa_R^b$ whenever $a+b=e$. Writting $\kappa_R \coloneqq \kappa_R^1$, this justifies the notation of κ_R^e as the e-th power of κ_R.

Exercise 5.8. With notation as above, suppose that R is further Gorenstein. Show that $\mathrm{Hom}_R(F_*^eR, R)$ is a free F_*^eR module of rank 1. A generator is offer referred to as a Frobenius trace.

We know what κ_R^e has to be formally locally around any point. Indeed, $\hat{R}_\mathfrak{p} \otimes \kappa_R^e = \kappa_{\hat{R}_\mathfrak{p}}^e$ is Matlis dual to the canonical map

$$H_\mathfrak{p}^{\mathrm{ht}\,\mathfrak{p}}(F^{e,\#})\colon H_\mathfrak{p}^{\mathrm{ht}\,\mathfrak{p}}(R) \longrightarrow F_*^e H_\mathfrak{p}^{\mathrm{ht}\,\mathfrak{p}}(R)$$

at every point $\mathfrak{p} \in \operatorname{Spec} R$.

Exercise 5.9. Let $R = \mathcal{k}[\![x_1, \ldots, x_n]\!]$ where $\mathcal{k}$ is an F-finite field. Show that κ_R^e is (up to isomorphism) the map Φ^e obtained from the p-basis $x_1, \ldots, x_n$. See Exercise 3.10.

Definition 5.12 (F-injective ring)**.** A ring R is said to be *F-injective* if the canonical map

$$H_\mathfrak{p}^i(F^{e,\#})\colon H_\mathfrak{p}^i(R) \longrightarrow F_*^e H_\mathfrak{p}^i(R)$$

is injective for all $\mathfrak{p} \in \operatorname{Spec} R$ and all i.

Exercise* 5.10. Show that R is F-injective if and only if so is $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

Exercise 5.11. Prove that an F-pure ring is F-injective.

Exercise 5.12. Assume that R admits a canonical module and further a Cartier operator $\kappa_R^e\colon F_*^e\omega_R \longrightarrow \omega_R$; as before. Prove that if R is F-injective then κ_R^e is surjective. Show that converse holds if R is further Cohen–Macaulay. Conclude that if R is further Gorenstein then F-purity and F-injectivity are equivalent notions.

Exercise* 5.13. Let $(R, \mathfrak{m}, \mathcal{k})$ be a local Cohen–Macaulay ring with Cartier operator $\kappa_R^e\colon F_*^e\omega_R \longrightarrow \omega_R$. Let $r \in R$ be a regular element. Prove that if R/r is F-injective then so is R.

[28] This is arguably the main reason why F-finiteness matters.

5.5. Normal rings and divisors.

6. Cartier Algebras and Schwede's Correspondence

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