

LECTURE NOTES F -SINGULARITIES

JAVIER CARVAJAL-ROJAS

ABSTRACT. These are lectures notes for a course on F -singularities given at the CIMAT in the Spring Semester 2024.

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1. REGULARITY (A CRASH COURSE)

This is a course about F -singularities and in particular about singularities. In a nutshell, singularities are the absence of regularity. Before defining what a regular ring is, we need the notion of *projective and global dimensions*.

1.1. Projective resolutions and other homological algebra stuff. Let M be a module over a ring R .¹

Exercise 1.1. Prove that there is an exact sequence of R -modules

$$0 \rightarrow K_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where P_0 is free and so projective. Iterate this to obtain an exact sequence

$$0 \rightarrow K_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

where the P_i 's are free. The module K_i is referred to as *a syzygy module*.

Definition 1.1 (Resolutions). An exact sequence

$$\cdots \rightarrow P_{i+1} \rightarrow P_i \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

is called a *free (resp. projective) resolution* of M if all the P_i 's are free (resp. projective). We may denote a projective resolution as $P_\bullet \rightarrow M \rightarrow 0$.²

¹All rings are commutative with unity 1.

²Over local rings projective modules are free (Kaplansky's theorem). That is, projective modules are locally free. The converse, however, isn't true (unless the module in question is finitely generated).

Exercise 1.2. Prove that free resolutions always exist, *i.e.* the category of R -modules has “enough projectives.”

Definition 1.2 (Projective dimension). The module M is said to have *finite projective dimension* if there is a projective resolution $P_\bullet \rightarrow M \rightarrow 0$ such that $P_i = 0$ for all $i \gg 0$. In such case, the *projective dimension* of M is

$$\mathrm{pd} M = \mathrm{pd}_R M := \min\{n \in \mathbb{N} \mid \exists P_\bullet \rightarrow M \rightarrow 0 \text{ such that } P_i = 0 \forall i > n\}.$$

If M has not finite projective dimension we write $\mathrm{pd} M = \infty$.

Exercise 1.3. Prove that M is projective iff $\mathrm{pd} M = 0$.

Next lemma is key.

Lemma 1.3. Suppose that there are two exact sequences of R -modules

$$0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow$$

and

$$0 \rightarrow K'_n \rightarrow P'_{n-1} \rightarrow \cdots \rightarrow P'_0 \rightarrow M \rightarrow$$

where $1 \leq n \in \mathbb{N}$ and the P_i and P'_i are projective. Then

- (a) $K_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \cong K'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots$
- (b) K_n is projective iff so is K'_n .

Proof. Note that (b) follows from (a).³ The proof of (b) is lengthy and left as an exercise. Hint: Proceed by induction on n . Prove the case $n = 1$ first and then reduce the inductive case to this one. \square

It can be used to prove the following.

Exercise 1.4. Let

$$0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow$$

be an exact sequences where the P_i 's are projective. Prove that

- (a) $\mathrm{pd} M \leq n$ iff K_n is projective.
- (b) If $\mathrm{pd} M \geq n$ then $\mathrm{pd} K_n = \mathrm{pd} M - n$.

Exercise 1.5. Suppose that R is noetherian and that M is finitely generated. Prove that

$$\mathrm{pd}_R M = \sup\{\mathrm{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec} R\} = \sup\{\mathrm{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \mid \mathfrak{m} \text{ maximal}\}$$

Exercise 1.6. Prove that

$$\mathrm{pd}(M \oplus N) = \max\{\mathrm{pd} M, \mathrm{pd} N\}.$$

The above exercise generalizes as follows.

Exercise* 1.7. Let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be an exact sequence of R -modules. Show the following statements.

- (a) If two of the modules in the exact sequence have finite projective dimension then so does the third one.

³Observe that for this is absolutely essential to use projectiveness instead of freeness.

(b) In that case (*i.e.* the three modules have finite projective dimension), then

$$\mathrm{pd} M \leq \max\{\mathrm{pd} M', \mathrm{pd} M''\},$$

(c) and if the inequality is strict then $\mathrm{pd} M'' = \mathrm{pd} M' + 1$.

Definition 1.4 (Minimal free resolution). Let $(R, \mathfrak{m}, \mathcal{K})$ be a noetherian local ring and M a finitely generated R -module. A free resolution $P_\bullet \rightarrow M \rightarrow 0$ is said to be *minimal* if

$$\phi_i(P_{i+1}) \subset \mathfrak{m}P_i \quad \forall i \in \mathbb{N}$$

where $\phi_i: P_{i+1} \rightarrow P_i$ is the homomorphism from the free resolution.

Exercise 1.8. In the setup of Definition 1.4, let $K_i := \ker \phi_{i-1}$ for all $i \geq 1$. Prove that $\mu(P_0) = \mu(M)$ and $\mu(P_i) = \mu(K_i)$ for all $i \geq 1$. Here, we let

$$\mu(-) = \dim_{\mathcal{K}} - \otimes_R \mathcal{K}$$

denote the minimal number of generators.

Exercise 1.9. Show that minimal free resolutions exist.

Exercise 1.10. In the setup of Definition 1.4, let $P_\bullet \rightarrow M \rightarrow 0$ and $P'_\bullet \rightarrow M \rightarrow 0$ be two minimal free resolutions. Show that $\mu(P_i) = \mu(P'_i)$ for all $i \in \mathbb{N}$.

The above two exercises guarantee that the following definition makes sense.

Definition 1.5 (Betti numbers). In the setup of Definition 1.4, the i -th Betti number of M is defined as $\beta_i(M) := \mu(P_i)$ where $P_\bullet \rightarrow M \rightarrow 0$ is any minimal free resolution.

Remark 1.6. Sometimes people talk about the Betti numbers of $(R, \mathfrak{m}, \mathcal{K})$, in that case, they refer to the Betti numbers of \mathcal{K} .

Exercise 1.11. Let $P_\bullet \rightarrow M \rightarrow 0$ be a minimal free resolution. Prove that $P_i = 0$ if (and only if) $i > \mathrm{pd} M$. That is,

$$\mathrm{pd} M = \sup\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$$

Exercise 1.12. Prove that

$$\beta_i(M) = \dim_{\mathcal{K}} \mathrm{Tor}_i(\mathcal{K}, M), \quad \forall i \in \mathbb{N}.$$

and conclude that

$$\mathrm{pd} M = \sup\{i \in \mathbb{N} \mid \mathrm{Tor}_i(\mathcal{K}, M) \neq 0\} \leq \mathrm{pd} \mathcal{K}.$$

Definition 1.7 (Global dimension). The *global dimension* of a ring R is the supremum of the projective dimensions of finitely generated R -modules.

Corollary 1.8. *The global dimension of a local ring is the projective dimension of its residue field.*

Remark 1.9 (Regular sequences and depth). Recall that a *regular element* $r \in R$ on an R -module M is one for which $\cdot r: M \rightarrow M$ is injective but not surjective. A *regular sequence* $r_1, \dots, r_d \in R$ on M is defined by the following two conditions:

- (a) r_1 is regular on M , and
- (b) r_i is regular on $M/(r_1, \dots, r_{i-1})M$ for all $i = 2, \dots, d$.

Given an ideal $\mathfrak{a} \subset R$, the *depth of \mathfrak{a} on M* , denoted by $\text{depth}_R(\mathfrak{a}, M)$, is the maximal length of a regular sequence on M of elements in \mathfrak{a} . When $(R, \mathfrak{m}, \mathcal{K})$ is local, we may write $\text{depth } M = \text{depth}_R M = \text{depth}_R(\mathfrak{m}, M)$. In that case, we also have:

$$\text{depth } M = \min\{i \in \mathbb{N} \mid \text{Ext}^i(\mathcal{K}, M) \neq 0\}.$$

This formula can be proved as follows (details are left to the reader). First, prove that if $r_1, \dots, r_d \in R$ is a regular sequence on M then

$$\text{Ext}_R^i(\mathcal{K}, M) = \begin{cases} 0 & \text{if } i < d, \\ \text{Hom}_R(\mathcal{K}, M/(r_1, \dots, r_d)M) & \text{if } i = d. \end{cases}$$

This can be proved by induction on d . The base step $d = 0$ is trivial. For the inductive step, consider the exact sequence

$$0 \rightarrow M \xrightarrow{r_1} M \rightarrow M/r_1M \rightarrow 0$$

Next, apply the functor $\text{Hom}_R(\mathcal{K}, -)$ to it. Since $r_1 \in \mathfrak{m}$, it acts like 0 on \mathcal{K} and so $\text{Ext}_R^i(\mathcal{K}, \cdot r_1) = 0$. This means that the long exact sequence on Ext 's breaks down into exact sequences

$$0 \rightarrow \text{Ext}_R^i(\mathcal{K}, M) \rightarrow \text{Ext}_R^i(\mathcal{K}, M/r_1M) \rightarrow \text{Ext}_R^{i+1}(\mathcal{K}, M) \rightarrow 0$$

Since r_2, \dots, r_d is a regular sequence on M/r_1M , we may apply the inductive hypothesis and conclude.

Theorem 1.10 (Auslander–Buchsbaum formula). *In the setup of Definition 1.4, if $\text{pd } M < \infty$ then*

$$\text{pd } M + \text{depth } M = \text{depth } R.$$

In particular, if R has finite global dimension it is at most $\text{depth } R$.

Proof. We only sketch a proof and leave the details to the reader as an exercise. The proof is an induction on $\text{pd } M$. If $\text{pd } M = 0$ then M is free and so $\text{depth } M = \text{depth } R$. If $\text{pd } M = 1$ then there is an exact sequence

$$0 \rightarrow R^{\oplus m} \xrightarrow{\phi} R^{\oplus n} \rightarrow M \rightarrow 0$$

which we may assume to be minimal, *i.e.* we may assume that the entries of the $n \times m$ R -matrix $\phi: R^{\oplus m} \rightarrow R^{\oplus n}$ are in \mathfrak{m} . Consider next the long exact sequence on Ext obtained by applying the functor $\text{Hom}_R(\mathcal{K}, -)$ (write it down yourself). Observe that $\text{Ext}_R^i(\mathcal{K}, R^{\oplus k}) = \text{Ext}_R^i(\mathcal{K}, R)^{\oplus k}$ and that

$$\text{Ext}_R^i(\mathcal{K}, \phi) : \text{Ext}_R^i(\mathcal{K}, R)^{\oplus m} \rightarrow \text{Ext}_R^i(\mathcal{K}, R)^{\oplus n}$$

is given by the \mathcal{K} -matrix obtained by reducing ϕ modulo \mathfrak{m} . In particular, $\text{Ext}_R^i(\mathcal{K}, \phi) = 0$ and so there is an exact sequence

$$0 \rightarrow \text{Ext}_R^i(\mathcal{K}, R)^{\oplus n} \rightarrow \text{Ext}_R^i(\mathcal{K}, M) \rightarrow \text{Ext}_R^{i+1}(\mathcal{K}, R)^{\oplus m} \rightarrow 0$$

From this, we see that $\text{depth } M = \text{depth } R - 1$. This shows the base step of the induction. For the inductive step, suppose $\text{pd } M \geq 2$ and consider an exact sequence

$$0 \rightarrow N \rightarrow R^{\oplus m} \rightarrow M \rightarrow 0$$

where $\text{pd } N = \text{pd } M - 1$. Use the corresponding long exact sequence on Ext 's obtained by applying $\text{Hom}_R(\mathcal{K}, -)$ to find the relationship between the depths of M and N (which is $\text{depth } N = \text{depth } M + 1$). Use the inductive hypothesis to conclude. \square

Remark 1.11. It is not difficult to see (using Krull's height theorem) that every regular sequence can be extended to a system of parameters. In particular, $\text{depth } R \leq \dim R$.⁴ When this equality happens to be an equality one says that $(R, \mathfrak{m}, \mathcal{K})$ is *Cohen–Macaulay*. Thus, a local ring is Cohen–Macaulay if and only if every system of parameters⁵ is a regular sequence.

1.2. Regular local rings. Let $(R, \mathfrak{m}, \mathcal{K})$ be a noetherian local ring. Then, by Nakayama's lemma, its so-called *embedded dimension*

$$\text{edim } R := \mu(\mathfrak{m}) = \dim_{\mathcal{K}} \mathfrak{m} \otimes \mathcal{K} = \dim_{\mathcal{K}} \mathfrak{m}/\mathfrak{m}^2$$

is finite.

Exercise 1.13. Use Krull's ideal theorem to conclude that the embedded dimension is at least the Krull's dimension of the local ring. In particular, noetherian local rings have finite dimension.

Definition 1.12 (Regular local ring). A noetherian local ring $(R, \mathfrak{m}, \mathcal{K})$ is said to be *regular* if the inequality

$$\text{edim } R \geq \dim R$$

is an equality.

Exercise 1.14. Prove that if $(R, \mathfrak{m}, \mathcal{K})$ is a noetherian local ring such that \mathfrak{m} is generated by a regular sequence then it is regular.

The converse of this exercise is also true but a bit harder to prove.

Theorem 1.13. Let $(R, \mathfrak{m}, \mathcal{K})$ be a regular (noetherian) local ring. Then every set of minimal generators of \mathfrak{m} (aka regular system of parameters) is a regular sequence. In particular, $\text{pd}_R \mathcal{K} = \dim R$.⁶

This result can be seen as a consequence of the following.

Theorem 1.14. A regular local ring is an integral domain.⁷

Recall the following useful, generalized form of prime avoidance.

Lemma 1.15 (Prime avoidance). Suppose that $\mathfrak{a} \subset \mathfrak{a}_1 \cup \dots \cup \mathfrak{a}_k$ where all but up to two of the ideals \mathfrak{a}_i are prime. Then $\mathfrak{a} \subset \mathfrak{a}_i$ for some $i = 1, \dots, k$.

Lemma 1.16. Let $(R, \mathfrak{m}, \mathcal{K})$ be a local ring of positive dimension. Then R contains a regular element not in \mathfrak{m}^2 . That is, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ that avoids all minimal primes.

Proof. Use prime avoidance. □

Sketch of the proof of Theorem 1.14. Set $d = \dim R < \infty$. Let's do induction on d . If $d = 0$, the regularity of R implies that $0 = \dim_{\mathcal{K}} \mathfrak{m}/\mathfrak{m}^2$ and so $\mathfrak{m} = 0$ by Nakayama's lemma. This means that R is a field and we're done.

Assume now that $d > 0$ and that all regular local rings of dimension $< d$ are integral domains. By Lemma 1.16, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ a regular element. Observe that

⁴More generally, $\text{depth}(\mathfrak{a}, R) \leq \text{ht } \mathfrak{a}$.

⁵A system of parameters for a local ring $(R, \mathfrak{m}, \mathcal{K})$ is a collection $x_1, \dots, x_{\dim R}$ such that $\sqrt{(x_1, \dots, x_{\dim R})} = \mathfrak{m}$. System of parameters always exist.

⁶In particular, regular local rings are Cohen–Macaulay, i.e. $\text{depth } R = \dim R$.

⁷In fact UFDs and so normal.

- R/rR is a local ring whose maximal ideal is generated by $d - 1$ elements (one less than the number of generators of \mathfrak{m}), and
- the dimension of R/rR is $d - 1$.

In particular, R/rR is a regular local ring of dimension $d - 1$. By the inductive hypothesis, it is an integral domain and so $rR = (r)$ is a prime ideal. Further, observe that $(r) \subset R$ cannot be a minimal prime. Let $\mathfrak{p} \subset R$ be a minimal prime of R that is contained in (r) . We're done if we can prove that $\mathfrak{p} = 0$. Let $x \in \mathfrak{p}$, and so $x = yr$ for some $y \in R$. In fact, $y \in \mathfrak{p}$ as $r \notin \mathfrak{p}$. In other words, $\mathfrak{p} = r\mathfrak{p}$. Since $r \in \mathfrak{m}$, Nakayama's lemma yields that $\mathfrak{p} = 0$; as desired. \square

Corollary 1.17. *Let $(R, \mathfrak{m}, \mathcal{K})$ be a local ring and $r \in \mathfrak{m} \setminus \mathfrak{m}^2$.⁸ Then, R is regular if and only if r is a regular element and R/rR is regular.*

Summing up, regular local rings have finite global dimension equal to its dimension. It turns out that the converse is also true and it's a deep result due to Auslander–Buchsbaum and Serre. To prove this, we need the following observation.

Exercise 1.15. Let $(R, \mathfrak{m}, \mathcal{K})$ be a local ring and M be a finitely generated R -module. Let $r \in R$ be a regular element on R and on M . Prove that

$$\mathrm{pd}_{R/rR} M/rM = \mathrm{pd}_R M$$

Hint: Show that a minimal free resolution $P_\bullet \rightarrow M \rightarrow 0$ becomes a minimal free resolution of M/rM after base change by R/rR . Notice that this is tantamount to the vanishing

$$\mathrm{Tor}_i^R(R/rR, M) = 0, \quad \forall i > 0.$$

But this can be seen from the fact that

$$0 \rightarrow R \xrightarrow{r} R \rightarrow R/rR \rightarrow 0$$

and

$$0 \rightarrow M \xrightarrow{r} M \rightarrow M/rM \rightarrow 0$$

are both exact.

We're ready to prove the main result in this section. Please take a moment to appreciate its beauty.

Theorem 1.18 (Auslander–Buchsbaum–Serre). *Let $(R, \mathfrak{m}, \mathcal{K})$ be a local noetherian ring. Then the following statements are equivalent.*

- (a) R is regular (i.e. \mathfrak{m} is generated by a regular sequence)
- (b) The global dimension of R is $\dim R$
- (c) $\mathrm{pd}_R \mathcal{K}$ is finite.

Proof. It only remains to explain why (c) implies (a). This is an induction on $d := \dim R < \infty$. If $d = 0$, then the Auslander–Buchsbaum formula yields that $\mathrm{pd}_R \mathcal{K} = 0$ and so that \mathcal{K} is a free R -module. Hence, $R = \mathcal{K}$ and we're done.

Let's assume that $d > 0$ and that (c) implies (a) for those local rings of dimension $< d$. Since R is positive dimensional, we can find a regular element $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ and it suffices to prove that the local ring $(R/rR, \mathfrak{m}/rR, \mathcal{K})$ is regular (which has dimension $d - 1$). To that end, we can apply the inductive hypothesis and prove that $\mathrm{pd}_{R/rR} \mathcal{K}$ is finite. For this, apply Exercise 1.15. \square

⁸Note that this is to say that r is part of a minimal set of generators for \mathfrak{m} .

Exercise 1.16. Prove the following to corollaries.

Corollary 1.19. *If $(R, \mathfrak{m}, \mathbb{K})$ is a regular local ring then so is $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Corollary 1.20 (Hilbert's syzygy theorem). *Let \mathbb{K} be a field. Then, every finitely generated $\mathbb{K}[x_1, \dots, x_n]$ -module has a free resolution of length at most n .*

1.3. General regular rings. With the above in place, we can finally define regular rings beyond the local case.

Definition 1.21 (Regular rings of finite dimension). We say that a noetherian ring of finite Krull dimension $\dim R$ is regular if any of the following equivalent conditions hold:

- (a) The local ring $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$.
- (b) The global dimension of R is at most $\dim R$ (i.e. every finitely generated module has projective dimension at most $\dim R$).
- (c) R has finite global dimension.

Exercise 1.17. Prove that the above conditions are indeed equivalent.

Definition 1.22 (Regular rings). Let R be a noetherian ring. Then R is said to be regular if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$.

Exercise 1.18. Prove that if R is regular then so is $W^{-1}R$ for any multiplicative set $W \subset R$.

Exercise 1.19. Prove that for a regular ring its global dimension equals its dimension.

1.4. Complete regular rings and the Cohen structure theorems. Let $(R, \mathfrak{m}, \mathbb{K})$ be a noetherian local ring. Recall that its completion is the canonical homomorphism

$$R \rightarrow \hat{R} := \varprojlim_n R/\mathfrak{m}^n$$

It turns out that \hat{R} is a noetherian local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$, residue field \mathbb{K} , and dimension $\dim R$. Moreover, $R \rightarrow \hat{R}$ is a faithfully flat local homomorphism. In particular, R is regular if and only if so is \hat{R} .

Remark 1.23. More generally, the completion of an R -module M is the \hat{R} -module

$$\hat{M} := \varprojlim_n M/\mathfrak{m}^n M.$$

Notice that there is a canonical \hat{R} -linear map

$$\hat{R} \otimes_R M \rightarrow \hat{M}$$

but it may not be an isomorphism. However, it is an isomorphism if M is finitely generated.

Exercise 1.20. Prove that $\operatorname{depth} R = \operatorname{depth} \hat{R}$. In particular, R is Cohen–Macaulay iff so is \hat{R} .

Example 1.24. If $R = \mathbb{K}[x_1, \dots, x_n]/\mathfrak{a}$ and $\mathfrak{m} = (x_1, \dots, x_n)$, then $\hat{R}_{\mathfrak{m}} = \mathbb{K}[[x_1, \dots, x_n]]/\mathfrak{a}$.

Recall that $(R, \mathfrak{m}, \mathbb{K})$ is said to be complete if $R \rightarrow \hat{R}$ is an isomorphism. It turns out that \hat{R} is complete. In fact, every quotient of \hat{R} is a noetherian complete local ring.

Remark 1.25 (Characteristic). Recall that the characteristic of a ring R , say $\operatorname{char} R$, is the only nonnegative integer $n \in \mathbb{N}$ such that $(n) = \ker(\mathbb{Z} \rightarrow R)$. Note that if R is an integral domain (i.e. a field) then $\operatorname{char} R$ is either 0 or a prime number p .

Exercise 1.21. Prove that R contains a field as a subring if and only if $\text{char } R = \text{char } \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \text{Spec } R$. Here $\kappa(\mathfrak{p})$ denotes the residue field of R at \mathfrak{p} .

For this reason, those rings that contain a field as a subring are referred to as rings of *equi-characteristic*. If a ring does not contain a field then it is said to have *mixed-characteristic*.

If $(R, \mathfrak{m}, \mathcal{K})$ is a local ring, then it has equicharacteristic iff $\text{char } R = \text{char } \mathcal{K}$. If it is mixed characteristic then $\text{char } \mathcal{K} = p > 0$ but $0 \neq p \in R$.

Suppose that $(R, \mathfrak{m}, \mathcal{K})$ is complete. A complete local subring $(\Lambda, p\Lambda, \mathcal{K}) \subset (R, \mathfrak{m}, \mathcal{K})$ is referred to as a coefficient ring. This entails that $\mathfrak{m} \cap \Lambda = p\Lambda$ and $p = \text{char } \mathcal{K} \geq 0$. There are three cases:

- R has equi-characteristic and so Λ is a field contained in R that maps isomorphically to \mathcal{K} .
- R has mixed-characteristic and $0 \neq p \in R$ is not nilpotent. In that case, $(\Lambda, p\Lambda, \mathcal{K})$ is a complete DVR. We'll refer to these rings as *Cohen rings*.
- R has mixed-characteristic and $p \in R$ is nilpotent (*i.e.* $\text{char } R = p^n$ for some $n > 1$). In that case, $(\Lambda, p\Lambda, \mathcal{K})$ is an artinian local ring.

Theorem 1.26 (Cohen structure theorem I). *Let $(R, \mathfrak{m}, \mathcal{K})$ be a complete (noetherian) local ring. Then:*

- (a) R has a coefficient ring.
- (b) There is a surjective homomorphism $\Lambda[[x_1, \dots, x_n]] \rightarrow R$ where Λ is either a field or a Cohen ring. Moreover, Λ can be taken as a coefficient ring of R if $p \in R$ isn't nilpotent. In particular, R is a quotient of a regular complete local ring.

Remark 1.27. The most difficult part is to show the existence of a coefficient ring. If $(R, \mathfrak{m}, \mathcal{K})$ has equi-characteristic $p > 0$ and \mathcal{K} is perfect. Then it turns out that

$$\mathcal{K}_0 := \bigcap_{e \in \mathbb{N}} R^{p^e}$$

is the only coefficient field of R . Here, $R^{p^e} = \{r^{p^e} \in r \in R\}$.

Theorem 1.28 (Cohen structure theorem II). *Let $(R, \mathfrak{m}, \mathcal{K})$ be a complete regular local ring. Then:*

- If R has equi-characteristic then $R \cong \mathcal{K}[[x_1, \dots, x_n]]$.
- If R has mixed-characteristic then there is a Cohen ring Λ such that

$$R \cong \begin{cases} \Lambda[[x_1, \dots, x_n]] & \text{if } p \in R \text{ is a regular element} \\ \Lambda[[x_1, \dots, x_n]]/(p - f) \text{ for some } f \in \mathfrak{m}^2 & \text{otherwise.} \end{cases}$$

We say that R is unramified in the former case.

Theorem 1.29 (Cohen–Gabber structure theorem III). *Let $(R, \mathfrak{m}, \mathcal{K})$ be a complete local ring that either is equi-characteristic or is an integral domain. Then, there exists a subring $A \subset R$ such that:*

- (a) A is a complete local ring,
- (b) $A \subset R$ is finite induces an isomorphism on residue fields and is generically étale,
- (c) $A \cong \Lambda[[x_1, \dots, x_n]]$ where Λ is a field or a Cohen ring.

Exercise 1.22. In the setup of Theorem 1.29, show that $(R, \mathfrak{m}, \mathcal{K})$ is Cohen–Macaulay if and only if $A \subset R$ is free (i.e. R is a projective A -module). Hint: Use the Auslander–Buchsbaum formula.

Exercise 1.23. Let R be a noetherian equi-characteristic ring. Prove that R is regular iff $\hat{R}_{\mathfrak{p}} \cong \kappa(\mathfrak{p})[[x_1, \dots, x_{\text{ht } \mathfrak{p}}]]$ for all $\mathfrak{p} \in \text{Spec } R$. Recall that $\kappa(\mathfrak{p}) := \mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = \mathcal{K}(R/\mathfrak{p})$ denotes the residue field of $R_{\mathfrak{p}}$.

2. THE FROBENIUS ENDOMORPHISM AND KUNZ’S THEOREM

From now on unless otherwise stated, we are going to assume that all rings have prime characteristic p . That is, all rings are \mathbb{F}_p -algebras. We always use the shorthand notation

$$q := p^e.$$

Further, we’ll assume that all rings are noetherian. The *Frobenius endomorphism* of a ring R is the homomorphism of \mathbb{F}_p -algebras

$$F = F_R: R \rightarrow R, \quad r \mapsto r^p.$$

By iterating, we also have $F^e: r \mapsto r^{p^e}$ for all $e \in \mathbb{N}$. We let $R^q \subset R$ be the image subring of F^e .

Exercise 2.1. Prove that $F: R \rightarrow R$ is indeed a homomorphism of \mathbb{F}_p -algebras. Prove that $\text{Spec } F^e: \text{Spec } R \rightarrow \text{Spec } R$ is the identity.

Exercise 2.2. Prove that R is reduced iff F^e is injective for some/all $e \in \mathbb{N}$.

Exercise 2.3. Recall that a ring R is reduced iff its total ring of fractions $\mathcal{K}(R)$ is a product of fields $K_1 \times \dots \times K_n$. Then, we may define $\bar{\mathcal{K}}(R)$ as $\bar{K}_1 \times \dots \times \bar{K}_n$ where \bar{K}_i is an algebraic closure of K_i . Hence $r^{1/q}$ is well-defined in $\bar{\mathcal{K}}(R)$ for all $r \in \mathcal{K}(R)$. Show that

$$R^{1/q} := \{r^{1/q} \in \bar{\mathcal{K}}(R) \mid r \in R\} \subset \bar{\mathcal{K}}(R)$$

is a subring that contains R . Moreover, show that $R \subset R^{1/q}$, $F^e: R \rightarrow R$, and $R^q \rightarrow R$ are isomorphic as R -algebras.

Definition 2.1 (Frobenius powers). Let $\mathfrak{a} \subset R$ be an ideal. Then $\mathfrak{a}^{[q]}$ is the extension ideal of \mathfrak{a} along F^e , and it’s called the e -th *Frobenius power* of \mathfrak{a} .

Note that if $\theta: R \rightarrow S$ is a homomorphism of rings then there is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\theta} & S \\ F^e \downarrow & & \downarrow F^e \\ R & \xrightarrow{\theta} & S \end{array}$$

Exercise 2.4. Prove that the above diagram is cartesian for all $e \in \mathbb{N}$ if θ is a localization $R \rightarrow W^{-1}R$. Show that if $\theta: R \rightarrow R/\mathfrak{a}$ is a quotient then the diagram is cartesian iff $\mathfrak{a}^{[q]} = \mathfrak{a}$.

More generally, the following notation is going to be useful.

Notation 2.2 (Frobenius pushforward). Let M be an R -module. We let

$$F_*^e M := \{F_*^e m \mid m \in M\}$$

be the R -module defined by the rules $F_*^e m + F_*^e m' = F_*^e(m + m')$ and $rF_*^e m = F_*^e r^q m$. In other words, $F_*^e M$ is the restriction of scalars of M along F^e . Thus, $F_*^e M$ is identical to

M as an abelian group but the R -scalar action is being twisted by Frobenius. Likewise, if $M = S$ is an R -algebra then $F_*^e S$ is an R -algebra with the product $(F_*^e s)(F_*^e s') = F_*^e(ss')$. Again, $F_*^e S$ is the exact same thing as S as a ring, what changes is the R -algebra structure.

Exercise 2.5. Prove that R is reduced iff $F_*^e R$ is a faithful R -module for some/all e .

Exercise 2.6. Prove that

$$F_*^e \hat{R} = \widehat{(F_*^e R)}.$$

With the above notation in place, we see that the commutative diagram above induces a ring homomorphism

$$F_\theta^e: S \otimes_R F_*^e R \rightarrow F_*^e S, \quad s \otimes F_*^e r \mapsto F_*^e s \theta(r)$$

which is called the *relative Frobenius* of $\theta: R \rightarrow S$.

Exercise 2.7. Prove that $\text{Spec } F_\theta^e$ is a (universal) homeomorphism.

Theorem 2.3 (Kunz's theorem). *Let R be a (noetherian) ring. Then R is regular iff $F^e: R \rightarrow R$ is (faithfully) flat for some/all $e > 0$.*

Remark 2.4 (The socle). Let $(R, \mathfrak{m}, \mathcal{K})$ be a local ring and M be a finitely generated R -module. The *socle* of M is the submodule

$$\text{Soc}(M) := \{m \in M : m\mathfrak{m} = 0\} \cong \text{Hom}_R(\mathcal{K}, M) = \text{Ext}_R^0(\mathcal{K}, M).$$

In particular, $\text{depth } M = 0$ iff $\text{Soc } M \neq 0$. Since $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n M = 0$, it follows that, if $\text{depth } M = 0$, there is $n \in \mathbb{N}$ such that $\text{Soc } M \not\subset \mathfrak{m}^n M$. Let $c := \text{depth } R$ and $r_1, \dots, r_c \in R$ be a regular sequence. Set $\mathfrak{a} := (r_1, \dots, r_c)$. Observe that $\text{depth}_R R/\mathfrak{a} = 0$. Then, we may find $n \in \mathbb{N}$ such that

$$\text{Soc}_R(R/\mathfrak{a}) \not\subset \mathfrak{m}^n(R/\mathfrak{a}).$$

Lemma 2.5. *Let $(R, \mathfrak{m}, \mathcal{K})$ be a local ring of depth c . Then there is $n \in \mathbb{N}$ such that for all infinite minimal free resolutions*

$$\dots \rightarrow R^{\oplus \beta_{i+1}(M)} \xrightarrow{\phi_i} R^{\oplus \beta_i(M)} \rightarrow \dots \rightarrow R^{\beta_0(M)} \rightarrow M \rightarrow 0$$

the entries of the matrix ϕ_{c+1} are not all contained in \mathfrak{m}^n (i.e. the image of ϕ_{c+1} is not inside $\mathfrak{m}^n R^{\oplus \beta_{c+1}} = (\mathfrak{m}^n)^{\oplus \beta_{c+1}}$). Here $b_i := \beta_i(M)$.

Proof. Note that, by the Auslander–Buchsbaum formula, we have that $b_{c+1} \neq 0$ as the resolution has infinite length. This is gonna be important below.

Let $\mathfrak{a} = (r_1, \dots, r_c)$ and n be as in Remark 2.4. In particular, for $N := \text{Soc}_R(R/\mathfrak{a})$ we have that $N \not\subset \mathfrak{m}^n N$. Observe that

$$\text{pd}_R R/\mathfrak{a} = c$$

and so

$$\text{Tor}_{c+1}^R(M, R/\mathfrak{a}) = 0.$$

This implies that after base changing the given infinite minimal free resolution we obtain that

$$(R/\mathfrak{a})^{\oplus b_{c+2}} \xrightarrow{\phi_{c+1}/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_{c+1}} \xrightarrow{\phi_c/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_c}$$

is exact in the middle. In other words,

$$\ker \phi_c/\mathfrak{a} \subset \text{im } \phi_{c+1}/\mathfrak{a}$$

Now, since the given resolution is minimal, we have that the entries of ϕ_c are all in \mathfrak{m} and so

$$N^{\oplus b_{c+1}} \subset \ker \phi_c / \mathfrak{a}.$$

Thus, putting everything together, if (for the sake of contradiction) the image of ϕ_{c+1} is inside $(\mathfrak{m}^n)^{\oplus b_{c+1}}$, it would follow that

$$N^{\oplus b_{c+1}} \subset (\mathfrak{m}^n(R/\mathfrak{a}))^{\oplus b_{c+1}}.$$

But, since $b_{c+1} \neq 0$, this implies that

$$N \subset \mathfrak{m}^n(R/\mathfrak{a}),$$

which contradicts the construction of n . Isn't math just so cool? \square

Lemma 2.6. *Let $(R, \mathfrak{m}, \mathcal{K})$ be a local ring and M be an R -module. Then \hat{M} is a flat \hat{R} -module whenever $\mathrm{Tor}_1^R(M, \mathcal{K}) = 0$ and in particular whenever M is flat.⁹*

Proof. This is a particular case of [Sta23, Tag 0AGW]. \square

Lemma 2.7 ([Sta23, Tag 039V]). *Let $R \rightarrow S$ a homomorphism of rings and M be an S -module. If M is a flat R -module and a faithfully flat S -module then $R \rightarrow S$ is flat.*

Exercise 2.8. Let R be a (noetherian ring). Then, F_R^e is flat as an R -module iff $F_*^e R_{\mathfrak{p}}$ is flat as an $R_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \mathrm{Spec} R$. If R is local, then $F^e R$ is flat as an R -module iff $F^e \hat{R}$ is flat as an \hat{R} -module. Hint: Apply the two previous lemmas.

Exercise 2.9. Let $R = \mathcal{K}[x_1, \dots, x_d]$ be a polynomial ring over a field \mathcal{K} (or more generally over a ring \mathcal{K} whose Frobenius is free). Let $\{F_*^e \lambda\}_{\lambda \in \Lambda}$ be a \mathcal{K} -basis for $F_*^e \mathcal{K} = \mathcal{K}^{1/q}$ (which we may assume contains $F_*^e 1$). Prove that

$$\{F_*^e \lambda x_1^{i_1} \cdots x_d^{i_d}\}_{\lambda \in \Lambda, 0 \leq i_1, \dots, i_d \leq q-1}$$

is an R -basis for $F_*^e R$. Suppose now that Λ is finite so that $F_*^e R$ is free of finite rank. Consider the corresponding dual basis

$$\{\phi_{\lambda, i_1, \dots, i_d} := (F_*^e \lambda x_1^{i_1} \cdots x_d^{i_d})^\vee\}_{\lambda \in \Lambda, 0 \leq i_1, \dots, i_d \leq q-1}$$

for $\mathrm{Hom}_R(F_*^e R, R)$. Show that

$$F_*^e R \rightarrow \mathrm{Hom}_R(F_*^e R, R), \quad F_*^e 1 \mapsto \Phi^e := \phi_{1, q-1, \dots, q-1}$$

is an isomorphism. We will be referring to Φ^e as the e -th (power of the) Frobenius trace of R .

Exercise 2.10. Conclude that $F_*^e \mathcal{K}[[x_1, \dots, x_d]]$ is a flat $\mathcal{K}[[x_1, \dots, x_d]]$ -module. Show that it is free if $[\mathcal{K}^{1/p} : \mathcal{K}] < \infty$. What about the converse?

Proof of Kunz's theorem. We may assume that $(R, \mathfrak{m}, \mathcal{K})$ is local. Moreover, we may assume that $(R, \mathfrak{m}, \mathcal{K})$ is complete. If R is regular then $R \cong \mathcal{K}[[x_1, \dots, x_{\dim R}]]$ and we're done by Exercise 2.10.

Conversely, suppose that $F^e : R \rightarrow R$ is flat. We want to prove that $\mathrm{pd}_R \mathcal{K} < \infty$. Suppose, for the sake of contradiction that there is an infinite minimal free resolution

$$\cdots \rightarrow R^{\oplus \beta_{i+1}(\mathcal{K})} \xrightarrow{\phi_i} R^{\oplus \beta_i(\mathcal{K})} \rightarrow \cdots \rightarrow R^{\beta_0(\mathcal{K})} \rightarrow \mathcal{K} \rightarrow 0$$

⁹Be cautious, the same can't be said about freeness and hence about projectivity.

That is, $\beta_{c+1}(\mathcal{K}) \neq 0$ for $c = \text{depth } R$. Since $F^e: R \rightarrow R$ is flat for all e , we can base change this infinite minimal free resolution to obtain a minimal free resolution

$$\dots \rightarrow R^{\oplus \beta_{i+1}(\mathcal{K})} \xrightarrow{\phi_i^{[q]}} R^{\oplus \beta_i(\mathcal{K})} \rightarrow \dots \rightarrow R^{\beta_0(\mathcal{K})} \rightarrow R/\mathfrak{m}^{[q]} \rightarrow 0$$

where $\phi_i^{[q]}$ is the matrix obtained from ϕ_i by raising its entries to the q -th power. In particular, the entries of $\phi_i^{[q]}$ belong to $\mathfrak{m}^{[q]} \subset \mathfrak{m}^q$ for all i and in particular for $i = \text{depth } R + 1$. This, however, contradicts Lemma 2.5 as $\mathfrak{m}^q \subset \mathfrak{m}^n$ for all $e \gg 0$ such that $q \geq n$. \square

2.1. Relative version of Kunz's theorem. There is a relative version of Kunz's theorem that goes by the name of Radu–André's theorem. To state it, we need to recall the following definition (the relative notion of F -regularity).

Definition 2.8 (Regular algebras). Let $\theta: R \rightarrow S$ be an R -algebra (where R and S are noetherian). We say that θ is *regular* if it is flat and all its fibers are *geometrically regular*. That is, for all $\mathfrak{p} \in \text{Spec } R$ the $\kappa(\mathfrak{p})$ -algebra $S \otimes_R \kappa(\mathfrak{p})$ is noetherian and regular (and noetherian) after any base change by a finitely generated field extension $\mathcal{K}/\kappa(\mathfrak{p})$.¹⁰

Theorem 2.9 (Radu–André). *Let $\theta: R \rightarrow S$ be an R -algebra. Then, θ is regular iff F_θ^e is (faithfully) flat for all/some $e > 0$.*

On the proof. The most important step is to show that if θ is regular then $S \otimes_R F_*^e R$ is noetherian. With that in place, the result can be obtained from the absolute Kunz theorem and the critere de planitude par fibres. I hope to add more details later on. \square

2.2. Bhatt–Scholze's generalization of Kunz's theorem. The (colimit) perfection of a ring R is

$$R \rightarrow R_{\text{perf}} := \text{colim}(R \xrightarrow{F} R \xrightarrow{F} R \rightarrow \dots)$$

We say that R is perfect iff $R \rightarrow R_{\text{perf}}$ is an isomorphism, *i.e.* Frobenius is an isomorphism on R . Observe that R_{perf} is perfect. Perfect rings are rarely noetherian. In fact, a noetherian perfect ring is a finite product of perfect fields.

Exercise 2.11. Prove that $\text{Spec } R \rightarrow \text{Spec } R_{\text{perf}}$ is a homeomorphism. Conclude that the perfection of a noetherian local ring has finite dimension.

Theorem 2.10 (Bhatt–Scholze). *Let $(R, \mathfrak{m}, \mathcal{K})$ be a complete local ring (of prime characteristic p). Then its perfection is R_{perf} has finite global dimension.*

Proof. TO BE ADDED. \square

This result easily proves Kunz's theorem as follows. Recall that the substantial part of Kunz's theorem is that if $F^e: R \rightarrow R$ is flat for a complete local ring then R is regular, *i.e.* R has finite global dimension. That is, we must show that there is $n \in \mathbb{N}$ such that for all R -modules one has that

$$\text{Tor}_i^R(\mathcal{K}, M) = 0$$

for all $i \geq n$. To that end, one observes that $R \rightarrow R_{\text{perf}}$ is faithfully flat and that

$$R_{\text{perf}} \otimes_R \text{Tor}_i^R(\mathcal{K}, M) = \text{Tor}_i^{R_{\text{perf}}} (R_{\text{perf}} \otimes_R \mathcal{K}, R_{\text{perf}} \otimes_R M).$$

Then, we can take n to be the global dimension of R_{perf} , which is finite by Bhatt–Scholze's theorem.

¹⁰It suffices to ask this for all finite purely inseparable extensions $\mathcal{K}/\kappa(\mathfrak{p})$.

REFERENCES

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CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS, A.C., CALLEJÓN JALISCO S/N, 36024 COL. VALENCIANA, GUANAJUATO, GTO, MÉXICO

Email address: `javier.carvajal@cimat.mx`