LECTURE NOTES F-SINGULARITIES

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Abstract. These are lectures notes for a course on F-singularities given at the CIMAT in the Spring Semester 2024.

1. Regularity (a crash course)

This is a course about F-singularities and in particular about singularities. In a nutshell, singularities are the absence of regularity. Before defining what a regular ring is, we need the notion of *projective and global dimensions*.

1.1. **Projective resolutions and other homological algebra stuff.** Let M be a module over a ring R.[1]

Exercise 1.1. Prove that there is an exact sequence of R-modules

$$0 \to K_1 \to P_0 \to M \to 0$$

where P_0 is free and so projective. Iterate this to obtain an exact sequence

$$0 \to K_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

where the P_i's are free. The module K_i is referred to as *a syzygy module*.

Definition 1.1 (Resolutions)**.** An exact sequence

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

is called a *free (resp. projective) resolution* of M if all the P_i's are free (resp. projective). We may denote a projective resolution as $P_\bullet \to M \to 0$.[2]

Exercise 1.2. Prove that free resolutions always exist, *i.e.* the category of R-modules has "enough projectives."

Definition 1.2 (Projective dimension)**.** The module M is said to have *finite projective dimension* if there is a projective resolution $P_\bullet \to M \to 0$ such that $P_i = 0$ for all $i \gg 0$. In such case, the *projective dimension* of M is

$$\operatorname{pd} M = \operatorname{pd}_R M \coloneqq \min\{n \in \mathbb{N} \mid \exists P_\bullet \to M \to 0 \text{ such that } P_i = 0 \forall i > n\}.$$

If M has not finite projective dimension we write $\operatorname{pd} M = \infty$.

Exercise 1.3. Prove that M is projective iff $\operatorname{pd} M = 0$.

Next lemma is key.

[1] All rings are commutative with unity 1.

[2] Over local rings projective modules are free.

Lemma 1.3. *Suppose that there are two exact sequences of R-modules*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow$$

and

$$0 \longrightarrow K'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow M \longrightarrow$$

where $1 \leq n \in \mathbb{N}$ and the P_i and P'_i are projective. Then

(a) $K_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \cong K'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots$

(b) K_n *is projective iff so is* K'_n.

Proof. Note that (b) follows from (a).[3] The proof of (b) is lengthy and left as an exercise. Hint: Proceed by induction on n. Prove the case $n = 1$ first and then reduce the inductive case to this one. □

It can be used to prove the following.

Exercise 1.4. Let

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow$$

be an exact sequences where thee P_i's are projective. Prove that

(a) $\operatorname{pd} M \leq n$ iff K_n is projective.

(b) If $\operatorname{pd} M \geq n$ then $\operatorname{pd} K_n = \operatorname{pd} M - n$.

Exercise 1.5. Suppose that R is noetherian and that M is finitely generated. Prove that

$$\operatorname{pd}_R M = \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup\{\operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \text{ maximal}\}$$

Exercise 1.6. Prove that

$$\operatorname{pd}(M \oplus N) = \max\{\operatorname{pd} M, \operatorname{pd} N\}.$$

The above exercise generalizes as follows.

Exercise* 1.7. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R-modules. Show the following statements.

(a) If two of the modules in the exact sequence have finite projective dimension then so does the third one.

(b) In that case (*i.e.* the three modules have finite projective dimension), then

$$\operatorname{pd} M \leq \max\{\operatorname{pd} M', \operatorname{pd} M''\},$$

(c) and if the inequality is strict then $\operatorname{pd} M'' = \operatorname{pd} M' + 1$.

Definition 1.4 (Minimal free resolution)**.** Let $(R, \mathfrak{m}, \mathcal{k})$ be a noetherian local ring and M a finitely generated R-module. A free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ is said to be *minimal* if

$$\phi_i(P_{i+1}) \subset \mathfrak{m} P_i \quad \forall i \in \mathbb{N}$$

where $\phi_i \colon P_{i+1} \longrightarrow P_i$ is the homomorphism from the free resolution.

Exercise 1.8. In the setup of Definition 1.4, let $K_i := \ker \phi_{i-1}$ for all $i \geq 1$. Prove that $\mu(P_0) = \mu(M)$ and $\mu(P_i) = \mu(K_i)$ for all $i \geq 1$. Here, we let

$$\mu(-) = \dim_{\mathcal{k}} - \otimes_R \mathcal{k}$$

denote the minimal number of generators.

[3] Observe that for this is absolutely essential to use projectiveness instead of freeness.

Exercise 1.9. Show that minimal free resolutions exist.

Exercise 1.10. In the setup of Definition 1.4, let $P_\bullet \longrightarrow M \longrightarrow 0$ and $P'_\bullet \longrightarrow M \longrightarrow 0$ be two minimal free resolutions. Show that $\mu(P_i) = \mu(P'_i)$ for all $i \in \mathbb{N}$.

The above two exercises guarantee that the following definition makes sense.

Definition 1.5 (Betti numbers)**.** In the setup of Definition 1.4, the *i-th Betti number* of M is defined as $\beta_i(M) \coloneqq \mu(P_i)$ where $P_\bullet \longrightarrow M \longrightarrow 0$ is any minimal free resolution.

Remark 1.6. Sometimes people talk about the Betti numbers of $(R, \mathfrak{m}, \mathscr{k})$, in that case, they refer to the Betti numbers of $\mathscr{k}$.

Exercise 1.11. Let $P_\bullet \longrightarrow M \longrightarrow 0$ be a minimal free resolution. Prove that $P_i = 0$ if (and only if) $i > \operatorname{pd} M$. That is,

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$$

Exercise 1.12. Prove that

$$\beta_i(M) = \dim_{\mathscr{k}} \operatorname{Tor}_i(\mathscr{k}, M), \quad \forall i \in \mathbb{N}.$$

and conclude that

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i(\mathscr{k}, M) \neq 0\} \leq \operatorname{pd} \mathscr{k}.$$

Definition 1.7 (Global dimension)**.** The *global dimension* of a ring R is the supremum of the projective dimensions of finitely generated R-modules.

Corollary 1.8. *The global dimension of a local ring is the projective dimension of its residue field.*

Remark 1.9 (Regular sequences and depth). Recall that a *regular element* $r \in R$ on an R-module M is one for which $\cdot r \colon M \longrightarrow M$ is injective but not surjective. A *regular sequence* $r_1, \ldots, r_d \in R$ on M is defined by the following two conditions:

(a) r_1 is regular on M, and
(b) r_i is regular on $M/(r_1, \ldots, r_{i-1})M$ for all $i = 2, \ldots d$.

Given an ideal $\mathfrak{a} \subset R$, the *depth of* $\mathfrak{a}$ *on* M, denoted by $\operatorname{depth}_R(\mathfrak{a}, M)$, is the maximal length of a regular sequence on M of elements in $\mathfrak{a}$. When $(R, \mathfrak{m}, \mathscr{k})$ is local, we may write $\operatorname{depth} M = \operatorname{depth}_R M = \operatorname{depth}_R(\mathfrak{m}, M)$. In that case, we also have:

$$\operatorname{depth} M = \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i(\mathscr{k}, M) \neq 0\}.$$

This formula can be proved as follows (details are left to the reader). First, prove that if $r_1, \ldots, r_d \in R$ is a regular sequence on M then

$$\operatorname{Ext}^i_R(\mathscr{k}, M) = \begin{cases} 0 & \text{if } i < d, \\ \operatorname{Hom}_R(\mathscr{k}, M/(r_1, \ldots, r_d)M) & \text{if } i = d. \end{cases}$$

This can be proved by induction on d. The base step $d = 0$ is trivial. For the inductive step, consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot r_1} M \longrightarrow M/r_1 M \longrightarrow 0$$

Next, apply the functor $\operatorname{Hom}_R(\mathscr{k}, -)$ to it. Since $r_1 \in \mathfrak{m}$, it acts like 0 on $\mathscr{k}$ and so $\operatorname{Ext}^i_R(\mathscr{k}, \cdot r_1) = 0$. This means that the long exact sequence on Ext's breaks down into exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M) \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M/r_1 M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathscr{k}, M) \longrightarrow 0$$

Since $r_2, \dots, r_d$ is a regular sequence on M/r_1M, we may apply the inductive hypothesis and conclude.

Theorem 1.10 (Auslander–Buchsbaum formula)**.** *In the setup of Definition 1.4, if* $\operatorname{pd} M < \infty$ *then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R.$$

In particular, if R has finite global dimension it is at most $\operatorname{depth} R$*.*

Proof. We only sketch a proof and leave the details to the reader as an exercise. The proof is an induction on $\operatorname{pd} M$. If $\operatorname{pd} M = 0$ then M is free and so $\operatorname{depth} M = \operatorname{depth} R$. If $\operatorname{pd} M = 1$ then there is an exact sequence

$$0 \longrightarrow R^{\oplus m} \xrightarrow{\phi} R^{\oplus n} \longrightarrow M \longrightarrow 0$$

which we may assume to be minimal, *i.e.* we may assume that the entries of the $n \times m$ R-matrix $\phi \colon R^{\oplus m} \longrightarrow R^{\oplus n}$ are in $\mathfrak{m}$. Consider next the long exact sequence on Ext obtained by applying the functor $\operatorname{Hom}_R(\mathscr{k}, -)$ (write it down yourself). Observe that $\operatorname{Ext}^i_R(\mathscr{k}, R^{\oplus k}) = \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus k}$ and that

$$\operatorname{Ext}^i_R(\mathscr{k}, \phi) : \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus m} \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus n}$$

is given by the $\mathscr{k}$-matrix obtained by reducing ϕ modulo $\mathfrak{m}$. In particular, $\operatorname{Ext}^i_R(\mathscr{k}, \phi) = 0$ and so there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, R)^{\oplus n} \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathscr{k}, R)^{\oplus m} \longrightarrow 0$$

From this, we see that $\operatorname{depth} M = \operatorname{depth} R - 1$. This shows the base step of the induction. For the inductive step, suppose $\operatorname{pd} M \geq 2$ and consider an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0$$

where $\operatorname{pd} N = \operatorname{pd} M - 1$. Use the corresponding long exact sequence on Ext's obtained by applying $\operatorname{Hom}_R(\mathscr{k}, -)$ to find the relationship between the depths of M and N (which is $\operatorname{depth} N = \operatorname{depth} M + 1$). Use the inductive hypothesis to conclude. □

Remark 1.11. It is not difficult to see (using Krull's height theorem) that every regular sequence can be extended to a system of parameters. In particular, $\operatorname{depth} R \leq \dim R$.[4] When this equality happens to be an equality one says that $(R, \mathfrak{m}, \mathscr{k})$ is *Cohen–Macaulay*. Thus, a local ring is Cohen–Macaualay if and only if every system of parameters[5] is a regular sequence.

1.2. **Regular local rings.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring. Then, by Nakayama's lemma, its so-called *embedded dimension*

$$\operatorname{edim} R \coloneqq \mu(\mathfrak{m}) = \dim_{\mathscr{k}} \mathfrak{m} \otimes \mathscr{k} = \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2$$

is finite.

Exercise 1.13. Use Krull's ideal theorem to conclude that the embedded dimension is at least the Krull's dimension of the local ring. In particular, noetherian local rings have finite dimension.

[4]More generally, $\operatorname{depth}(\mathfrak{a}, R) \leq \operatorname{ht} \mathfrak{a}$.

[5]A system of parameters for a local ring $(R, \mathfrak{m}, \mathscr{k})$ is a collection $x_1, \dots, x_{\dim R}$ such that $\sqrt{(x_1, \dots, x_{\dim R})} = \mathfrak{m}$. System of parameters always exist.

Definition 1.12 (Regular local ring)**.** A noetherian local ring $(R, \mathfrak{m}, \mathscr{k})$ is said to be *regular* if the inequality

$$\operatorname{edim} R \geq \dim R$$

is an equality.

Exercise 1.14. Prove that if $(R, \mathfrak{m}, \mathscr{k})$ is a noetherian local ring such that $\mathfrak{m}$ is generated by a regular sequence then it is regular.

The converse of this exercise is also true but a bit harder to prove.

Theorem 1.13. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a regular (noetherian) local ring. Then every set of minimal generators of $\mathfrak{m}$ (aka regular system of parameters) is a regular sequence. In particular,* $\operatorname{pd}_R \mathscr{k} = \dim R$.[6]

This result can be seen as a consequence of the following.

Theorem 1.14. *A regular local ring is an integral domain.*[7]

Recall the following useful, generalized form of prime avoidance.

Lemma 1.15 (Prime avoidance)**.** *Suppose that $\mathfrak{a} \subset \mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_k$ where all but up to two of the ideals $\mathfrak{a}_i$ are prime. Then $\mathfrak{a} \subset \mathfrak{a}_i$ for some $i = 1, \ldots, k$.*

Lemma 1.16. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring of positive dimension. Then R contains a regular element not in $\mathfrak{m}^2$. That is, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ that avoids all minimal primes.*

Proof. Use prime avoidance. □

Sketch of the proof of Theorem 1.14. Set $d = \dim R < \infty$. Let's do induction on d. If $d = 0$, the regularity of R implies that $0 = \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2$ and so $\mathfrak{m} = 0$ by Nakayama's lemma. This means that R is a field and we're done.

Assume now that $d > 0$ and that all regular local rings of dimension $< d$ are integral domains. By Lemma 1.16, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ a regular element. Observe that

- R/rR is a local ring whose maximal ideal is generated by $d-1$ elements (one less than the number of generators of $\mathfrak{m}$), and
- the dimension of R/rR is $d-1$.

In particular, R/rR is a regular local ring of dimension $d-1$. By the inductive hypothesis, it is an integral domain and so $rR = (r)$ is a prime ideal. Further, observe that $(r) \subset R$ cannot be a minimal prime. Let $\mathfrak{p} \subset R$ be a minimal prime of R that is contained in (r). We're done if we can prove that $\mathfrak{p} = 0$. Let $x \in \mathfrak{p}$, and so $x = yr$ for some $y \in R$. In fact, $y \in \mathfrak{p}$ as $r \notin \mathfrak{p}$. In other words, $\mathfrak{p} = r\mathfrak{p}$. Since $r \in \mathfrak{m}$, Nakayama's lemma yields that $\mathfrak{p} = 0$; as desired. □

Corollary 1.17. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring and $r \in \mathfrak{m} \setminus \mathfrak{m}^2$.*[8] *Then, R is regular if and only if r is a regular element and R/rR is regular.*

Summing up, regular local rings have finite global dimension equal to its dimension. It turns out that the converse is also true and it's a deep result due to Auslander–Buchsbaum and Serre. To prove this, we need the following observation.

[6]In particular, regular local rings are Cohen–Macaulay, *i.e.* $\operatorname{depth} R = \dim R$.

[7]In fact UFDs and so normal.

[8]Note that this is to say that r is part of a minimal set of generators for $\mathfrak{m}$.

Exercise 1.15. Let $(R, \mathfrak{m}, \Bbbk)$ be a local ring and M be a finitely generated R-module. Let $r \in R$ be a regular element on R and on M. Prove that

$$\operatorname{pd}_{R/rR} M/rM = \operatorname{pd}_R M$$

Hint: Show that a minimal free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ becomes a minimal free resolution of M/rM after base change by R/rR. Notice that this is tantamount to the vanishing

$$\operatorname{Tor}_i^R(R/rR, M) = 0, \quad \forall i > 0.$$

But this can be seen from the fact that

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/rR \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\cdot r} M \longrightarrow M/rM \longrightarrow 0$$

are both exact.

We're ready to prove the main result in this section. Please take a moment to appreciate its beauty.

Theorem 1.18 (Auslander–Buchsbaum–Serre)**.** *Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring. Then the following statements are equivalent.*

(a) R is regular (i.e. $\mathfrak{m}$ is generated by a regular sequence)
(b) The global dimension of R is $\dim R$
(c) $\operatorname{pd}_R \Bbbk$ is finite.

Proof. It only remains to explain why (c) implies (a). This is an induction on $d \coloneqq \dim R < \infty$. If $d = 0$, then the Auslander–Buchsbaum formula yields that $\operatorname{pd}_R \Bbbk = 0$ and so that $\Bbbk$ is a free R-module. Hence, $R = \Bbbk$ and we're done.

Let's assume that $d > 0$ and that (c) implies (a) for those local rings of dimension $< d$. Since R is positive dimensional, we can find a regular element $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ and it suffices to prove that the local ring $(R/rR, \mathfrak{m}/rR, \Bbbk)$ is regular (which has dimension $d-1$). To that end, we can apply the inductive hypothesis and prove that $\operatorname{pd}_{R/rR} \Bbbk$ is finite. For this, apply Exercise 1.15. ∎

Exercise 1.16. Prove the following to corollaries.

Corollary 1.19. *If $(R, \mathfrak{m}, \Bbbk)$ is a regular local ring then so is $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Corollary 1.20 (Hilbert's syzygy theorem)**.** *Let $\Bbbk$ be a field. Then, every finitely generated $\Bbbk[x_1, \ldots, x_n]$-module has a free resolution of length at most n.*

1.3. **General regular rings.** With the above in place, we can finally define regular rings beyond the local case.

Definition 1.21 (Regular rings)**.** We say that a noetherian ring of finite Krull dimension $\dim R$ is regular if any of the following equivalente conditions hold:

(a) The local ring $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$.
(b) The global dimension of R is at most $\dim R$.
(c) R has finite global dimension.

Exercise 1.17. Prove that the above conditions are indeed equivalent.

Exercise 1.18. Prove that if R is regular then so is $W^{-1}R$ for any multiplicative set $W \subset R$.

1.4. **Complete regular rings and the Cohen structure theorems.** Let $(R, \mathfrak{m}, \Bbbk)$ be a noetherian local ring. Recall that its completion is the canonical homomorphism

$$R \longrightarrow \hat{R} \coloneqq \varprojlim_n R/\mathfrak{m}^n$$

It turns out that $\hat{R}$ is a noetherian local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$, residue field $\Bbbk$, and dimension $\dim R$. Moreover, $R \longrightarrow \hat{R}$ is a faithfully flat local homomorphism. In particular, R is regular if and only if so is $\hat{R}$.

Exercise 1.19. Prove that $\operatorname{depth} R = \operatorname{depth} \hat{R}$. In particular, R is Cohen–Macaulay iff so is $\hat{R}$.

Example 1.22. If $R = \Bbbk[x_1, \ldots, x_n]/\mathfrak{a}$ and $\mathfrak{m} = (x_1, \ldots, x_n)$, then $\hat{R}_{\mathfrak{m}} = \Bbbk[\![x_1, \ldots, x_n]\!]/\mathfrak{a}$.

Recall that $(R, \mathfrak{m}, \Bbbk)$ is said to be complete if $R \longrightarrow \hat{R}$ is an isomorphism. It turns out that $\hat{R}$ is complete. In fact, every quotient of $\hat{R}$ is a noetherian complete local ring.

Remark 1.23 (Characteristic). Recall that the characteristic of a ring R, say $\operatorname{char} R$, is the only nonnegative integer $n \in \mathbb{N}$ such that $(n) = \ker(\mathbb{Z} \longrightarrow R)$. Note that if R is an integral domain (*i.e.* a field) then $\operatorname{char} R$ is either 0 or a prime number p.

Exercise 1.20. Prove that R contains a field as a subring if and only if $\operatorname{char} R = \operatorname{char} \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Here $\kappa(\mathfrak{p})$ denotes the residue field of R at $\mathfrak{p}$.

For this reason, those rings that contain a field as a subring are referred to as rings of *equi-characteristic*. If a ring does not contain a field then it is said to have *mixed-characteristic*.

If $(R, \mathfrak{m}, \Bbbk)$ is a local ring, then it has equicharacteristic iff $\operatorname{char} R = \operatorname{char} \Bbbk$. If it is mixed characteristic then $\operatorname{char} \Bbbk = p > 0$ but $0 \neq p \in R$.

Suppose that $(R, \mathfrak{m}, \Bbbk)$ is complete. A complete local subring $(\Lambda, p\Lambda, \Bbbk) \subset (R, \mathfrak{m}, \Bbbk)$ is referred to as a coefficient ring. This entails that $\mathfrak{m} \cap \Lambda = p\Lambda$ and $p = \operatorname{char} \Bbbk \geq 0$. There are three cases:

- R has equi-characteristic and so Λ is a field contained in R that maps isomorphically to $\Bbbk$.
- R has mixed-caracteristic and $0 \neq p \in R$ is not nilpotent. In that case, $(\Lambda, p\Lambda, \Bbbk)$ is a complete DVR. We'll referred to this rings as *Cohen rings*.
- R has mixed-caracteristic and $p \in R$ is nilpotent (*i.e.* $\operatorname{char} R = p^n$ for some $n > 1$). In that case, $(\Lambda, p\Lambda, \Bbbk)$ is an artinian local ring.

Theorem 1.24 (Cohen structure theorem I)**.** *Let $(R, \mathfrak{m}, \Bbbk)$ be a noetherian local ring. Then:*

(a) R has a coefficient ring.
(b) There is a surjective homomorphism $\Lambda[\![x_1, \ldots, x_n]\!] \longrightarrow R$ where Λ is either a field or a Cohen ring. Moreover, Λ can be taken as a coefficient ring of R if $p \in R$ isn't nilpotent. In particular, R is a quotient of a regular complete local ring.

Theorem 1.25 (Cohen structure theorem II)**.** *Let $(R, \mathfrak{m}, \Bbbk)$ be a complete regular local ring. Then:*

- *If R has equi-characteristic then $R \cong \Bbbk[\![x_1, \ldots, x_n]\!]$.*

- *If R has mixed-characteristic then*

$$R \cong \begin{cases} \Lambda[\![x_1, \dots, x_n]\!] & \textit{if } p \in R \textit{ is a regular element} \\ \Lambda[\![x_1, \dots, x_n]\!]/(p-f) \textit{ for some } f \in \mathfrak{m}^2 & \textit{otherwise.} \end{cases}$$

We say that R is unramified *in the former case.*

Theorem 1.26 (Cohen–Gabber structure theorem III)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local integral domain. Then, there exists a subring $A \subset R$ such that:*

(a) A is a complete local ring,
(b) $A \subset R$ is finite induces an isomorphism on residue fields and a separable extension on fraction fields,
(c) $A \cong \Lambda[\![x_1, \dots, x_n]\!]$ where Λ is a field or a Cohen ring.

Exercise 1.21. In the setup of Theorem 1.26, show that $(R, \mathfrak{m}, \mathscr{k})$ is Cohen–Macaulay if and only if $A \subset R$ is free (*i.e.* R is a projective A-module). Hint: Use the Auslander–Buchsbaum formula.

References

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