# LECTURE NOTES $F$-SINGULARITIES

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Abstract. These are lectures notes for a course on $F$-singularities given at the CIMAT in the Spring Semester 2024.

Contents



## 1. Regularity (a crash course)

This is a course about $F$-singularities and in particular about singularities. In a nutshell, singularities are the absence of regularity. Before defining what a regular ring is, we need the notion of *projective and global dimensions*.

### 1.1. **Projective resolutions and other homological algebra stuff.** Let $M$ be a module over a ring $R$.[1]

**Exercise 1.1.** Prove that there is an exact sequence of $R$-modules

$$0 \to K_1 \to P_0 \to M \to 0$$

where $P_0$ is free and so projective. Iterate this to obtain an exact sequence

$$0 \to K_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

where the $P_i$'s are free. The module $K_i$ is referred to as *a syzygy module*.

**Definition 1.1** (Resolutions)**.** An exact sequence

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

[1] All rings are commutative with unity 1.

is called a *free (resp. projective) resolution* of $M$ if all the $P_i$'s are free (resp. projective). We may denote a projective resolution as $P_\bullet \to M \to 0$.[2]

**Exercise 1.2.** Prove that free resolutions always exist, *i.e.* the category of $R$-modules has "enough projectives."

**Definition 1.2** (Projective dimension)**.** The module $M$ is said to have *finite projective dimension* if there is a projective resolution $P_\bullet \to M \to 0$ such that $P_i = 0$ for all $i \gg 0$. In such case, the *projective dimension* of $M$ is

$$\operatorname{pd} M = \operatorname{pd}_R M \coloneqq \min\{n \in \mathbb{N} \mid \exists P_\bullet \to M \to 0 \text{ such that } P_i = 0 \forall i > n\}.$$

If $M$ has not finite projective dimension we write $\operatorname{pd} M = \infty$.

**Exercise 1.3.** Prove that $M$ is projective iff $\operatorname{pd} M = 0$.

Next lemma is key.

**Lemma 1.3.** *Suppose that there are two exact sequences of $R$-modules*

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

*and*

$$0 \to K'_n \to P'_{n-1} \to \cdots \to P'_0 \to M \to 0$$

*where $1 \leq n \in \mathbb{N}$ and the $P_i$ and $P'_i$ are projective. Then*

*(a) $K_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \cong K'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots$*
*(b) $K_n$ is projective iff so is $K'_n$.*

*Proof.* Note that (b) follows from (a).[3] The proof of (b) is lengthy and left as an exercise. Hint: Proceed by induction on $n$. Prove the case $n = 1$ first and then reduce the inductive case to this one. □

It can be used to prove the following.

**Exercise 1.4.** Let

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

be an exact sequences where thee $P_i$'s are projective. Prove that

(a) $\operatorname{pd} M \leq n$ iff $K_n$ is projective.
(b) If $\operatorname{pd} M \geq n$ then $\operatorname{pd} K_n = \operatorname{pd} M - n$.

**Exercise 1.5.** Suppose that $R$ is noetherian and that $M$ is finitely generated. Prove that

$$\operatorname{pd}_R M = \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup\{\operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \text{ maximal}\}$$

**Exercise 1.6.** Prove that

$$\operatorname{pd}(M \oplus N) = \max\{\operatorname{pd} M, \operatorname{pd} N\}.$$

The above exercise generalizes as follows.

**Exercise* 1.7.** Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of $R$-modules. Show the following statements.

[2]Over local rings projective modules are free (Kaplansky's theorem). That is, projective modules are locally free. The converse, however, isn't true (unless the module in question is finitely generated).
[3]Observe that for this is absolutely essential to use projectiveness instead of freeness.

(a) If two of the modules in the exact sequence have finite projective dimension then so does the third one.
(b) In that case (*i.e.* the three modules have finite projective dimension), then
$$\operatorname{pd} M \leq \max\{\operatorname{pd} M', \operatorname{pd} M''\},$$
(c) and if the inequality is strict then $\operatorname{pd} M'' = \operatorname{pd} M' + 1$.

**Definition 1.4** (Minimal free resolution)**.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring and $M$ a finitely generated $R$-module. A free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ is said to be *minimal* if
$$\phi_i(P_{i+1}) \subset \mathfrak{m} P_i \quad \forall i \in \mathbb{N}$$
where $\phi_i \colon P_{i+1} \longrightarrow P_i$ is the homomorphism from the free resolution.

**Exercise 1.8.** In the setup of Definition 1.4, let $K_i := \ker \phi_{i-1}$ for all $i \geq 1$. Prove that $\mu(P_0) = \mu(M)$ and $\mu(P_i) = \mu(K_i)$ for all $i \geq 1$. Here, we let
$$\mu(-) = \dim_{\mathscr{k}} - \otimes_R \mathscr{k}$$
denote the minimal number of generators.

**Exercise 1.9.** Show that minimal free resolutions exist.

**Exercise 1.10.** In the setup of Definition 1.4, let $P_\bullet \longrightarrow M \longrightarrow 0$ and $P'_\bullet \longrightarrow M \longrightarrow 0$ be two minimal free resolutions. Show that $\mu(P_i) = \mu(P'_i)$ for all $i \in \mathbb{N}$.

The above two exercises guarantee that the following definition makes sense.

**Definition 1.5** (Betti numbers)**.** In the setup of Definition 1.4, the *$i$-th Betti number* of $M$ is defined as $\beta_i(M) := \mu(P_i)$ where $P_\bullet \longrightarrow M \longrightarrow 0$ is any minimal free resolution.

*Remark* 1.6. Sometimes people talk about the Betti numbers of $(R, \mathfrak{m}, \mathscr{k})$, in that case, they refer to the Betti numbers of $\mathscr{k}$.

**Exercise 1.11.** Let $P_\bullet \longrightarrow M \longrightarrow 0$ be a minimal free resolution. Prove that $P_i = 0$ if (and only if) $i > \operatorname{pd} M$. That is,
$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$$

**Exercise 1.12.** Prove that
$$\beta_i(M) = \dim_{\mathscr{k}} \operatorname{Tor}_i(\mathscr{k}, M), \quad \forall i \in \mathbb{N}.$$
and conclude that
$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i(\mathscr{k}, M) \neq 0\} \leq \operatorname{pd} \mathscr{k}.$$

**Definition 1.7** (Global dimension)**.** The *global dimension* of a ring $R$ is the supremum of the projective dimensions of finitely generated $R$-modules.

**Corollary 1.8.** *The global dimension of a local ring is the projective dimension of its residue field.*

*Remark* 1.9 (Regular sequences and depth). Recall that a *regular element* $r \in R$ on an $R$-module $M$ is one for which $\cdot r \colon M \longrightarrow M$ is injective but not surjective. A *regular sequence* $r_1, \ldots, r_d \in R$ on $M$ is defined by the following two conditions:
(a) $r_1$ is regular on $M$, and
(b) $r_i$ is regular on $M/(r_1, \ldots, r_{i-1})M$ for all $i = 2, \ldots d$.

Given an ideal $\mathfrak{a} \subset R$, the *depth of* $\mathfrak{a}$ *on* $M$, denoted by $\operatorname{depth}_R(\mathfrak{a}, M)$, is the maximal length of a regular sequence on $M$ of elements in $\mathfrak{a}$. When $(R, \mathfrak{m}, \mathcal{k})$ is local, we may write $\operatorname{depth} M = \operatorname{depth}_R M = \operatorname{depth}_R(\mathfrak{m}, M)$. In that case, we also have:

$$\operatorname{depth} M = \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i(\mathcal{k}, M) \neq 0\}.$$

This formula can be proved as follows (details are left to the reader). First, prove that if $r_1, \ldots, r_d \in R$ is a regular sequence on $M$ then

$$\operatorname{Ext}^i_R(\mathcal{k}, M) = \begin{cases} 0 & \text{if } i < d, \\ \operatorname{Hom}_R(\mathcal{k}, M/(r_1, \ldots, r_d)M) & \text{if } i = d. \end{cases}$$

This can be proved by induction on $d$. The base step $d = 0$ is trivial. For the inductive step, consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot r_1} M \longrightarrow M/r_1M \longrightarrow 0$$

Next, apply the functor $\operatorname{Hom}_R(\mathcal{k}, -)$ to it. Since $r_1 \in \mathfrak{m}$, it acts like 0 on $\mathcal{k}$ and so $\operatorname{Ext}^i_R(\mathcal{k}, \cdot r_1) = 0$. This means that the long exact sequence on Ext's breaks down into exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, M) \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, M/r_1M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathcal{k}, M) \longrightarrow 0$$

Since $r_2, \ldots, r_d$ is a regular sequence on $M/r_1M$, we may apply the inductive hypothesis and conclude.

**Theorem 1.10** (Auslander–Buchsbaum formula)**.** *In the setup of Definition 1.4, if* $\operatorname{pd} M < \infty$ *then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R.$$

*In particular, if $R$ has finite global dimension it is at most* $\operatorname{depth} R$.

*Proof.* We only sketch a proof and leave the details to the reader as an exercise. The proof is an induction on $\operatorname{pd} M$. If $\operatorname{pd} M = 0$ then $M$ is free and so $\operatorname{depth} M = \operatorname{depth} R$. If $\operatorname{pd} M = 1$ then there is an exact sequence

$$0 \longrightarrow R^{\oplus m} \xrightarrow{\phi} R^{\oplus n} \longrightarrow M \longrightarrow 0$$

which we may assume to be minimal, *i.e.* we may assume that the entries of the $n \times m$ $R$-matrix $\phi \colon R^{\oplus m} \longrightarrow R^{\oplus n}$ are in $\mathfrak{m}$. Consider next the long exact sequence on Ext obtained by applying the functor $\operatorname{Hom}_R(\mathcal{k}, -)$ (write it down yourself). Observe that $\operatorname{Ext}^i_R(\mathcal{k}, R^{\oplus k}) = \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus k}$ and that

$$\operatorname{Ext}^i_R(\mathcal{k}, \phi) : \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus m} \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus n}$$

is given by the $\mathcal{k}$-matrix obtained by reducing $\phi$ modulo $\mathfrak{m}$. In particular, $\operatorname{Ext}^i_R(\mathcal{k}, \phi) = 0$ and so there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus n} \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathcal{k}, R)^{\oplus m} \longrightarrow 0$$

From this, we see that $\operatorname{depth} M = \operatorname{depth} R - 1$. This shows the base step of the induction. For the inductive step, suppose $\operatorname{pd} M \geq 2$ and consider an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0$$

where $\operatorname{pd} N = \operatorname{pd} M - 1$. Use the corresponding long exact sequence on Ext's obtained by applying $\operatorname{Hom}_R(\mathcal{k}, -)$ to find the relationship between the depths of $M$ and $N$ (which is $\operatorname{depth} N = \operatorname{depth} M + 1$). Use the inductive hypothesis to conclude. $\square$

*Remark* 1.11. It is not difficult to see (using Krull's height theorem and prime avoidance) that every regular sequence can be extended to a system of parameters.[4] In particular, $\operatorname{depth} R \leq \dim R$.[5] When this equality happens to be an equality one says that $(R, \mathfrak{m}, \mathscr{k})$ is *Cohen–Macaulay*. Thus, a local ring is Cohen–Macaualay if and only if every system of parameters[6] is a regular sequence.

1.2. **Regular local rings.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring. Then, by Nakayama's lemma, its so-called *embedded dimension*

$$\operatorname{edim} R \coloneqq \mu(\mathfrak{m}) = \dim_{\mathscr{k}} \mathfrak{m} \otimes \mathscr{k} = \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2$$

is finite.

**Exercise 1.13.** Use Krull's ideal theorem to conclude that the embedded dimension is at least the Krull's dimension of the local ring. In particular, noetherian local rings have finite dimension.

**Definition 1.12** (Regular local ring)**.** A noetherian local ring $(R, \mathfrak{m}, \mathscr{k})$ is said to be *regular* if the inequality

$$\operatorname{edim} R \geq \dim R$$

is an equality.

**Exercise 1.14.** Prove that if $(R, \mathfrak{m}, \mathscr{k})$ is a noetherian local ring such that $\mathfrak{m}$ is generated by a regular sequence then it is regular.

The converse of this exercise is also true but a bit harder to prove.

**Theorem 1.13.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a regular (noetherian) local ring. Then every set of minimal generators of $\mathfrak{m}$ (aka regular system of parameters) is a regular sequence. In particular, $\operatorname{pd}_R \mathscr{k} = \dim R$.*[7]

This result can be seen as a consequence of the following.

**Theorem 1.14.** *A regular local ring is an integral domain.*[8]

Recall the following useful, generalized form of prime avoidance.

**Lemma 1.15** (Prime avoidance)**.** *Suppose that $\mathfrak{a} \subset \mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_k$ where all but up to two of the ideals $\mathfrak{a}_i$ are prime. Then $\mathfrak{a} \subset \mathfrak{a}_i$ for some $i = 1, \ldots, k$.*

**Lemma 1.16.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring of positive dimension. Then $R$ contains a regular element not in $\mathfrak{m}^2$. That is, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ that avoids all associated primes.*

*Proof.* Use prime avoidance. □

[4]Indeed, Krull's height theorem let us see that if $r_1, \ldots, r_n \in R$ is a regular sequence then $(r_1, \ldots, r_n)$ has height $n$. On the other hand, prime avoidance can be used to see that an ideal $(r_1, \ldots, r_n)$ that has height $n$ can be extended to a system of parameters.

[5]More generally, $\operatorname{depth}(\mathfrak{a}, R) \leq \operatorname{ht} \mathfrak{a}$.

[6]A system of parameters for a local ring $(R, \mathfrak{m}, \mathscr{k})$ is a collection $x_1, \ldots, x_{\dim R}$ such that $\sqrt{(x_1, \ldots, x_{\dim R})} = \mathfrak{m}$. System of parameters always exist.

[7]In particular, regular local rings are Cohen–Macaulay, *i.e.* $\operatorname{depth} R = \dim R$.

[8]In fact, they are UFDs and so normal integral domains.

*Sketch of the proof of Theorem 1.14.* Set $d = \dim R < \infty$. Let's do induction on $d$. If $d = 0$, the regularity of $R$ implies that $0 = \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2$ and so $\mathfrak{m} = 0$ by Nakayama's lemma. This means that $R$ is a field and we're done.

Assume now that $d > 0$ and that all regular local rings of dimension $< d$ are integral domains. By Lemma 1.16, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ a regular element. Observe that

- $R/rR$ is a local ring whose maximal ideal is generated by $d-1$ elements (one less than the number of generators of $\mathfrak{m}$), and
- the dimension of $R/rR$ is $d-1$.

In particular, $R/rR$ is a regular local ring of dimension $d-1$. By the inductive hypothesis, it is an integral domain and so $rR = (r)$ is a prime ideal. Further, observe that $(r) \subset R$ cannot be a minimal prime. Let $\mathfrak{p} \subset R$ be a minimal prime of $R$ that is contained in $(r)$. We're done if we can prove that $\mathfrak{p} = 0$. Let $x \in \mathfrak{p}$, and so $x = yr$ for some $y \in R$. In fact, $y \in \mathfrak{p}$ as $r \notin \mathfrak{p}$. In other words, $\mathfrak{p} = r\mathfrak{p}$. Since $r \in \mathfrak{m}$, Nakayama's lemma yields that $\mathfrak{p} = 0$; as desired. $\square$

**Corollary 1.17.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring and $r \in \mathfrak{m} \setminus \mathfrak{m}^2$.*[9] *Then, $R$ is regular if and only if $r$ is a regular element and $R/rR$ is regular.*

Summing up, regular local rings have finite global dimension equal to its dimension. It turns out that the converse is also true and it's a deep result due to Auslander–Buchsbaum and Serre. To prove this, we need the following observation.

**Exercise 1.15.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring and $M$ be a finitely generated $R$-module. Let $r \in R$ be a regular element on $R$ and on $M$. Prove that

$$\operatorname{pd}_{R/rR} M/rM = \operatorname{pd}_R M$$

Hint: Show that a minimal free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ becomes a minimal free resolution of $M/rM$ after base change by $R/rR$. Notice that this is tantamount to the vanishing

$$\operatorname{Tor}_i^R(R/rR, M) = 0, \quad \forall i > 0.$$

But this can be seen from the fact that

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/rR \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\cdot r} M \longrightarrow M/rM \longrightarrow 0$$

are both exact.

We're ready to prove the main result in this section. Please take a moment to appreciate its beauty.

**Theorem 1.18** (Auslander–Buchsbaum–Serre)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local noetherian ring. Then, the following statements are equivalent.*

*(a) $R$ is regular (i.e. $\mathfrak{m}$ is generated by a regular sequence)*
*(b) The global dimension of $R$ is $\dim R$*
*(c) $\operatorname{pd}_R \mathscr{k}$ is finite.*

[9]Note that this is to say that $r$ is part of a minimal set of generators for $\mathfrak{m}$.

*Proof.* It only remains to explain why (c) implies (a). This is an induction on $d \coloneqq \dim R < \infty$. If $d = 0$, then the Auslander–Buchsbaum formula yields that $\mathrm{pd}_R \mathscr{k} = 0$ and so that $\mathscr{k}$ is a free $R$-module. Hence, $R = \mathscr{k}$ and we're done.

Let's assume that $d > 0$ and that (c) implies (a) for those local rings of dimension $< d$. Since $R$ is positive dimensional, we can find a regular element $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ and it suffices to prove that the local ring $(R/rR, \mathfrak{m}/rR, \mathscr{k})$ is regular (which has dimension $d-1$). To that end, we can apply the inductive hypothesis and prove that $\mathrm{pd}_{R/rR} \mathscr{k}$ is finite. For this, apply Exercise 1.15. $\square$

**Exercise 1.16.** Prove the following tWo corollaries.

**Corollary 1.19.** *If $(R, \mathfrak{m}, \mathscr{k})$ is a regular local ring then so is $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

**Corollary 1.20** (Hilbert's syzygy theorem)**.** *Let $\mathscr{k}$ be a field. Then, every finitely generated $\mathscr{k}[x_1, \ldots, x_n]$-module has a free resolution of length at most $n$.*

1.3. **General regular rings.** With the above in place, we can finally define regular rings beyond the local case.

**Definition 1.21** (Regular rings of finite dimension)**.** We say that a noetherian ring of finite Krull dimension $\dim R$ is regular if any of the following equivalent conditions hold:

(a) The local ring $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$.
(b) The global dimension of $R$ is at most $\dim R$ (*i.e.* every finitely generated module has projective dimension at most $\dim R$).
(c) $R$ has finite global dimension.

**Exercise 1.17.** Prove that the above conditions are indeed equivalent.

**Definition 1.22** (Regular rings)**.** Let $R$ be a noetherian ring. Then $R$ is said to be regular if $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$.

**Exercise 1.18.** Prove that if $R$ is regular then so is $W^{-1}R$ for any multiplicative set $W \subset R$.

**Exercise 1.19.** Prove that for a regular ring its global dimension equals its dimension.

1.4. **Complete regular rings and the Cohen structure theorems.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring. Recall that its completion is the canonical homomorphism

$$R \longrightarrow \hat{R} \coloneqq \varprojlim_n R/\mathfrak{m}^n$$

It turns out that $\hat{R}$ is a noetherian local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$, residue field $\mathscr{k}$, and dimension $\dim R$. Moreover, $R \longrightarrow \hat{R}$ is a faithfully flat local homomorphism. In particular, $R$ is regular if and only if so is $\hat{R}$.

*Remark* 1.23. More generally, the completion of an $R$-module $M$ is the $\hat{R}$-module

$$\hat{M} \coloneqq \varprojlim_n M/\mathfrak{m}^n M.$$

Notice that there is a canonical $\hat{R}$-linear map

$$\hat{R} \otimes_R M \longrightarrow \hat{M}$$

but it may not be an isomorphism. However, it is an isomorphism if $M$ is finitely generated.

**Exercise 1.20.** Prove that $\operatorname{depth} R = \operatorname{depth} \hat{R}$. In particular, $R$ is Cohen–Macaulay iff so is $\hat{R}$.

**Example 1.24.** If $R = \mathcal{k}[x_1, \ldots, x_n]/\mathfrak{a}$ and $\mathfrak{m} = (x_1, \ldots, x_n)$, then $\hat{R}_{\mathfrak{m}} = \mathcal{k}[\![x_1, \ldots, x_n]\!]/\mathfrak{a}$.

Recall that $(R, \mathfrak{m}, \mathcal{k})$ is said to be complete if $R \longrightarrow \hat{R}$ is an isomorphism. It turns out that $\hat{R}$ is complete. In fact, every quotient of $\hat{R}$ is a noetherian complete local ring.

*Remark* 1.25 (Characteristic). Recall that the characteristic of a ring $R$, say $\operatorname{char} R$, is the only nonnegative integer $n \in \mathbb{N}$ such that $(n) = \ker(\mathbb{Z} \longrightarrow R)$. Note that if $R$ is an integral domain (*i.e.* a field) then $\operatorname{char} R$ is either 0 or a prime number $p$.

**Exercise 1.21.** Prove that $R$ contains a field as a subring if and only if $\operatorname{char} R = \operatorname{char} \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Here $\kappa(\mathfrak{p})$ denotes the residue field of $R$ at $\mathfrak{p}$.

For this reason, those rings that contain a field as a subring are referred to as rings of *equi-characteristic*. If a ring does not contain a field then it is said to have *mixed-characteristic*.
If $(R, \mathfrak{m}, \mathcal{k})$ is a local ring, then it has equicharacteristic iff $\operatorname{char} R = \operatorname{char} \mathcal{k}$. If it is mixed characteristic then $\operatorname{char} \mathcal{k} = p > 0$ but $0 \neq p \in R$.

Suppose that $(R, \mathfrak{m}, \mathcal{k})$ is complete. A complete local subring $(\Lambda, p\Lambda, \mathcal{k}) \subset (R, \mathfrak{m}, \mathcal{k})$ is referred to as a coefficient ring. This entails that $\mathfrak{m} \cap \Lambda = p\Lambda$ and $p = \operatorname{char} \mathcal{k} \geq 0$. There are three cases:

- $R$ has equi-characteristic and so $\Lambda$ is a field contained in $R$ that maps isomorphically to $\mathcal{k}$.
- $R$ has mixed-caracteristic and $0 \neq p \in R$ is not nilpotent. In that case, $(\Lambda, p\Lambda, \mathcal{k})$ is a complete DVR. We'll referred to this rings as *Cohen rings*.
- $R$ has mixed-caracteristic and $p \in R$ is nilpotent (*i.e.* $\operatorname{char} R = p^n$ for some $n > 1$). In that case, $(\Lambda, p\Lambda, \mathcal{k})$ is an artinian local ring.

**Theorem 1.26** (Cohen structure theorem I)**.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a complete (noetherian) local ring. Then:*

*(a) $R$ has a coefficient ring.*
*(b) There is a surjective homomorphism $\Lambda[\![x_1, \ldots, x_n]\!] \longrightarrow R$ where $\Lambda$ is either a field or a Cohen ring. Moreover, $\Lambda$ can be taken as a coefficient ring of $R$ if $p \in R$ isn't nilpotent. In particular, $R$ is a quotient of a regular complete local ring.*

*Remark* 1.27. The most difficult part is to show the existence of a coefficient ring. If $(R, \mathfrak{m}, \mathcal{k})$ has equi-characteristic $p > 0$ and $\mathcal{k}$ is perfect. Then it turns out that

$$\mathcal{k}_0 \coloneqq \bigcap_{e \in \mathbb{N}} R^{p^e}$$

is the only coefficient field of $R$. Here, $R^{p^e} = \{r^{p^e} \in r \in R\}$.

**Theorem 1.28** (Cohen structure theorem II)**.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a complete regular local ring. Then:*

- *If $R$ has equi-characteristic then $R \cong \mathcal{k}[\![x_1, \ldots, x_n]\!]$.*
- *If $R$ has mixed-characteristic then there is a Cohen ring $\Lambda$ such that*

$$R \cong \begin{cases} \Lambda[\![x_1, \ldots, x_n]\!] & \text{if } p \in R \text{ is a regular element} \\ \Lambda[\![x_1, \ldots, x_n]\!]/(p - f) \text{ for some } f \in \mathfrak{m}^2 & \text{otherwise.} \end{cases}$$

*We say that $R$ is* unramified *in the former case.*

**Theorem 1.29** (Cohen–Gabber structure theorem III)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local ring that either is equi-characteristic or is an integral domain. Then, there exists a subring $A \subset R$ such that:*

*(a) $A$ is a complete local ring,*
*(b) $A \subset R$ is finite induces an isomorphism on residue fields and is generically étale,*
*(c) $A \cong \Lambda[\![x_1, \ldots, x_n]\!]$ where $\Lambda$ is a field or a Cohen ring.*

**Exercise 1.22.** In the setup of Theorem 1.29, show that $(R, \mathfrak{m}, \mathscr{k})$ is Cohen–Macaulay if and only if $A \subset R$ is free (*i.e.* $R$ is a projective $A$-module). Hint: Use the Auslander–Buchsbaum formula.

**Exercise 1.23.** Let $R$ be a noetherian equi-characteristic ring. Prove that $R$ is regular iff $\hat{R}_\mathfrak{p} \cong \kappa(\mathfrak{p})[\![x_1, \ldots, x_{\operatorname{ht}\mathfrak{p}}]\!]$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Recall that $\kappa(\mathfrak{p}) \coloneqq \mathfrak{p}R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = \mathscr{K}(R/\mathfrak{p})$ denotes the residue field of $R_\mathfrak{p}$.

## 2. The Frobenius Endomorphism and Kunz's Theorem

From now on unless otherwise stated, we are going to assume that all rings have prime characteristic $p$. That is, all rings are $\mathbb{F}_p$-algebras. We always use the shorthand notation

$$q \coloneqq p^e.$$

Further, we'll assume that all rings are noetherian. The *Frobenius endomorphism* of a ring $R$ is the homomorphism of $\mathbb{F}_p$-algebras

$$F = F_R \colon R \longrightarrow R, \quad r \mapsto r^p.$$

By iterating, we also have $F^e \colon r \mapsto r^q$ for all $e \in \mathbb{N}$. We let $R^q \subset R$ be the image subring of $F^e$.

**Exercise 2.1.** Prove that $F \colon R \longrightarrow R$ is indeed a homomorphism of $\mathbb{F}_p$-algebras. Prove that $\operatorname{Spec} F^e \colon \operatorname{Spec} R \longrightarrow \operatorname{Spec} R$ is the identity.

**Exercise 2.2.** Prove that $R$ is reduced iff $F^e$ is injective for some/all $e \in \mathbb{N}$.

**Exercise 2.3.** Recall that a ring $R$ is reduce iff its total ring of fractions $\mathscr{K}(R)$ is a product of fields $K_1 \times \cdots \times K_n$. Then, we may define $\bar{\mathscr{K}}(R)$ as $\bar{K}_1 \times \cdots \times \bar{K}_n$ where $\bar{K}_i$ is an algebraic closure of $K_i$. Hence $r^{1/q}$ is well-defined in $\bar{\mathscr{K}}(R)$ for all $r \in \mathscr{K}(R)$. Show that

$$R^{1/q} \coloneqq \{r^{1/q} \in \bar{\mathscr{K}}(R) \mid r \in R\} \subset \bar{\mathscr{K}}(R)$$

is a subring that contains $R$. Moreover, show that $R \subset R^{1/q}$, $F^e \colon R \longrightarrow R$, and $R^q \longrightarrow R$ are isomorphic as $R$-algebras.

**Definition 2.1** (Frobenius powers)**.** Let $\mathfrak{a} \subset R$ be an ideal. Then $\mathfrak{a}^{[q]}$ is the extension ideal of $\mathfrak{a}$ along $F^e$, and it's called the *$e$-th Frobenius power of* $\mathfrak{a}$.

Note that if $\theta \colon R \longrightarrow S$ is a homomorphism of rings then there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, "F^e"'] & S \arrow[d, "F^e"] \\ R \arrow[r, "\theta"] & S \end{tikzcd}$$

**Exercise 2.4.** Prove that the above diagram is cartesian for all $e \in \mathbb{N}$ if $\theta$ is a localization $R \longrightarrow W^{-1}R$. Show that if $\theta\colon R \longrightarrow R/\mathfrak{a}$ is a quotient then the diagram is cartesian iff $\mathfrak{a}^{[q]} = \mathfrak{a}$.

More generally, the following notation is going to be useful.

**Notation 2.2** (Frobenius pushforward)**.** Let $M$ be an $R$-module. We let

$$F^e_* M \coloneqq \{F^e_* m \mid m \in M\}$$

be the $R$-module defined by the rules $F^e_* m + F^e_* m' = F^e_*(m + m')$ and $rF^e_* m = F^e_* r^q m$. In other words, $F^e_* M$ is the restriction of scalars of $M$ along $F^e$. Thus, $F^e_* M$ is identical to $M$ as an abelian group but the $R$-scalar action is being twisted by Frobenius. Likewise, if $M = S$ is an $R$-algebra then $F^e_* S$ is an $R$-algebra with the product $(F^e_* s)(F^e_* s') = F^e_*(ss')$. Again, $F^e_* S$ is the exact same thing as $S$ as a ring, what changes is the $R$-algebra structure.

**Exercise 2.5.** Prove that $R$ is reduced iff $F^e_* R$ is a faithful $R$-module for some/all $e$.

**Exercise 2.6.** Prove that

$$F^e_* \hat{R} = \widehat{(F^e_* R)}.$$

With the above notation in place, we see that the commutative diagram above induces a ring homomorphism

$$F^e_\theta \colon S \otimes_R F^e_* R \longrightarrow F^e_* S, \quad s \otimes F^e_* r \mapsto F^e_* s^q \theta(r)$$

which is called the *relative Frobenius* of $\theta\colon R \longrightarrow S$.

**Exercise 2.7.** Prove that $\operatorname{Spec} F^e_\theta$ is a (universal) homeomorphism.

**Theorem 2.3** (Kunz's theorem)**.** *Let $R$ be a (noetherian) ring. Then $R$ is regular iff $F^e\colon R \longrightarrow R$ is (faithfully) flat for some/all $e > 0$.*

*Remark* 2.4 (The socle). Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring and $M$ be a finitely generated $R$-module. The *socle* of $M$ is the submodule

$$\operatorname{Soc}(M) \coloneqq \{m \in M : m\mathfrak{m} = 0\} \cong \operatorname{Hom}_R(\mathscr{k}, M) = \operatorname{Ext}^0_R(\mathscr{k}, M).$$

In particular, $\operatorname{depth} M = 0$ iff $\operatorname{Soc} M \neq 0$. Since $\bigcap_{n\in\mathbb{N}} \mathfrak{m}^n M = 0$, it follows that, if $\operatorname{depth} M = 0$, there is $n \in \mathbb{N}$ such that $\operatorname{Soc} M \not\subset \mathfrak{m}^n M$. Let $c \coloneqq \operatorname{depth} R$ and $r_1, \ldots, r_c \in R$ be a regular sequence. Set $\mathfrak{a} \coloneqq (r_1, \ldots, r_c)$. Observe that $\operatorname{depth}_R R/\mathfrak{a} = 0$. Then, we may find $n \in \mathbb{N}$ such that

$$\operatorname{Soc}_R(R/\mathfrak{a}) \not\subset \mathfrak{m}^n(R/\mathfrak{a}).$$

**Lemma 2.5.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring of depth $c$. Then there is $n \in \mathbb{N}$ such that for all infinite minimal free resolutions*

$$\cdots \longrightarrow R^{\oplus\beta_{i+1}(M)} \xrightarrow{\phi_i} R^{\oplus\beta_i(M)} \longrightarrow \cdots \longrightarrow R^{\beta_0(M)} \longrightarrow M \longrightarrow 0$$

*the entries of the matrix $\phi_{c+1}$ are not all contained in $\mathfrak{m}^n$ (i.e. the image of $\phi_{c+1}$ is not inside $\mathfrak{m}^n R^{\oplus b_{c+1}} = (\mathfrak{m}^n)^{\oplus b_{c+1}}$). Here $b_i \coloneqq \beta_i(M)$.*

*Proof.* Note that, by the Auslander–Buchsbaum formula, we have that $b_{c+1} \neq 0$ as the resolution has infinite length. This is gonna be important below.

Let $\mathfrak{a} = (r_1, \ldots, r_c)$ and $n$ be as in Remark 2.4. In particular, for $N \coloneqq \operatorname{Soc}_R(R/\mathfrak{a})$ we have that $N \not\subset \mathfrak{m}^n N$. Observe that

$$\operatorname{pd}_R R/\mathfrak{a} = c$$

and so

$$\operatorname{Tor}_{c+1}^R(M, R/\mathfrak{a}) = 0.$$

This implies that after base changing the given infinite minimal free resolution we obtain that

$$(R/\mathfrak{a})^{\oplus b_{c+2}} \xrightarrow{\phi_{c+1}/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_{c+1}} \xrightarrow{\phi_c/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_c}$$

is exact in the middle. In other words,

$$\ker \phi_c/\mathfrak{a} \subset \operatorname{im} \phi_{c+1}/\mathfrak{a}$$

Now, since the given resolution is minimal, we have that the entries of $\phi_c$ are all in $\mathfrak{m}$ and so

$$N^{\oplus b_{c+1}} \subset \ker \phi_c/\mathfrak{a}.$$

Thus, putting everything together, if (for the sake of contradiction) the image of $\phi_{c+1}$ is inside $(\mathfrak{m}^n)^{\oplus b_{c+1}}$, it would follow that

$$N^{\oplus b_{c+1}} \subset \big(\mathfrak{m}^n (R/\mathfrak{a})\big)^{\oplus b_{c+1}}.$$

But, since $b_{c+1} \neq 0$, this implies that

$$N \subset \mathfrak{m}^n(R/\mathfrak{a}),$$

which contradicts the construction of $n$. Isn't math just so cool? $\square$

**Lemma 2.6.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and $M$ be an $R$-module. Then $\hat{M}$ is a flat $\hat{R}$-module whenever $\operatorname{Tor}_1^R(M, \mathcal{k}) = 0$ and in particular whenever $M$ is flat.*[10]

*Proof.* This is a particular case of [Sta23, Tag 0AGW]. $\square$

**Lemma 2.7** ([Sta23, Tag 039V])**.** *Let $R \longrightarrow S$ a homomorphism of rings and $M$ be an $S$-module. If $M$ is a flat $R$-module and a faithfully flat $S$-module then $R \longrightarrow S$ is flat.*

**Exercise 2.8.** Let $R$ be a (noetherian ring). Then, $F^e_R$ is flat as an $R$-module iff $F^e_* R_{\mathfrak{p}}$ is flat as an $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$. If $R$ is local, then $F^e R$ is flat as an $R$-module iff $F^e \hat{R}$ is flat as an $\hat{R}$-module. Hint: Apply the two previous lemmas.

**Exercise 2.9.** Let $R = \mathcal{k}[x_1, \ldots, x_d]$ be a polynomial ring over a field $\mathcal{k}$ (or more generally over a ring $\mathcal{k}$ whose Frobenius is free). Let $\{F^e_* \lambda\}_{\lambda \in \Lambda}$ be a $\mathcal{k}$-basis for $F^e_* \mathcal{k} = \mathcal{k}^{1/q}$ (which we may assume contains $F^e_* 1$). Prove that

$$\{F^e_* \lambda x_1^{i_1} \cdots x_d^{i_d}\}_{\lambda \in \Lambda, 0 \leq i_1, \ldots, i_d \leq q-1}$$

is an $R$-basis for $F^e_* R$. Suppose now that $\Lambda$ is finite so that $F^e_* R$ is free of finite rank. Consider the corresponding dual basis

$$\{\phi_{\lambda, i_1, \ldots, i_d} := (F^e_* \lambda x_1^{i_1} \cdots x_d^{i_d})^\vee\}_{\lambda \in \Lambda, 0 \leq i_1, \ldots, i_d \leq q-1}$$

for $\operatorname{Hom}_R(F^e_* R, R)$. Show that

$$F^e_* R \longrightarrow \operatorname{Hom}_R(F^e_* R, R), \quad F^e_* 1 \mapsto \Phi^e := \phi_{1, q-1, \ldots, q-1}$$

is an isomorphism. We will be referreing to $\Phi^e$ as the *$e$-th (power of the) Frobenius trace* of $R$.

**Exercise 2.10.** Conclude that $F^e_* \mathcal{k}[\![x_1, \ldots, x_d]\!]$ is a flat $\mathcal{k}[\![x_1, \ldots, x_d]\!]$-module. Show that it is free if $[\mathcal{k}^{1/p} : \mathcal{k}] < \infty$. What about the converse?

[10]Be cautious, the same can't be said about freenes and hence about projectivity.

*Proof of Kunz's theorem.* We may assume that $(R, \mathfrak{m}, \mathscr{k})$ is local. Moreover, we may assume that $(R, \mathfrak{m}, \mathscr{k})$ is complete. If $R$ is regular then $R \cong \mathscr{k}[\![x_1, \ldots, x_{\dim R}]\!]$ and we're done by Exercise 2.10.

Conversely, suppose that $F^e \colon R \longrightarrow R$ is flat. We want to prove that $\mathrm{pd}_R\, \mathscr{k} < \infty$. Suppose, for the sake of contradiction that there is an infinite minimal free resolution

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(\mathscr{k})} \xrightarrow{\phi_i} R^{\oplus \beta_i(\mathscr{k})} \longrightarrow \cdots \longrightarrow R^{\beta_0(\mathscr{k})} \longrightarrow \mathscr{k} \longrightarrow 0$$

That is, $\beta_{c+1}(\mathscr{k}) \neq 0$ for $c = \operatorname{depth} R$. Since $F^e \colon R \longrightarrow R$ is flat for all $e$, we can base chang this inifinite minimal free resolution to obtain a minimal free resolution

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(\mathscr{k})} \xrightarrow{\phi_i^{[q]}} R^{\oplus \beta_i(\mathscr{k})} \longrightarrow \cdots \longrightarrow R^{\beta_0(\mathscr{k})} \longrightarrow R/\mathfrak{m}^{[q]} \longrightarrow 0$$

where $\phi_i^{[q]}$ is the matrix obtained from $\phi_i$ by raising its entries to the $q$-th power. In particular, the entries of $\phi_i^{[q]}$ belong to $\mathfrak{m}^{[q]} \subset \mathfrak{m}^q$ for all $i$ and in particular for $i = \operatorname{depth} R + 1$. This, however, contradicts Lemma 2.5 as $\mathfrak{m}^q \subset \mathfrak{m}^n$ for all $e \gg 0$ such that $q \geq n$. ☐

2.1. **Relative version of Kunz's theorem.** There is a relative version of Kunz's theorem that goes by the name of Radu–André's theorem. To state it, we need to recall the following definition (the relative notion of $F$-regularity).

**Definition 2.8** (Regular algebras)**.** Let $\theta \colon R \longrightarrow S$ be an $R$-algebra (where $R$ and $S$ are noetherian). We say that $\theta$ is *regular* if it is flat and all its fibers are *geometrically regular*. That is, for all $\mathfrak{p} \in \operatorname{Spec} R$ the $\kappa(\mathfrak{p})$-algebra $S \otimes_R \kappa(\mathfrak{p})$ is noetherian and regular (and noetherian) after any base change by a finitely generated field extension $\mathscr{k}/\kappa(\mathfrak{p})$.[11]

**Theorem 2.9** (Radu–André)**.** *Let $\theta \colon R \longrightarrow S$ be an $R$-algebra. Then, $\theta$ is regular iff $F^e_\theta$ is (faithfully) flat for all/some $e > 0$.*

*On the proof.* The most important step is to show that if $\theta$ is regular than $S \otimes_R F^e_* R$ is noetherian. With that in place, the result can be obtained from the absolute Kunz theorem and the critere de planitude par fibres. I hope to add more details later on. ☐

2.2. **Bhatt–Scholze's generalization of Kunz's theorem.** The *(colimit) perfection* of a ring $R$ is

$$R \longrightarrow R_{\mathrm{perf}} \coloneqq \operatorname{colim}(R \xrightarrow{F} R \xrightarrow{F} R \longrightarrow \cdots)$$

We say that $R$ is perfect iff $R \longrightarrow R_{\mathrm{perf}}$ is an isomorphism, *i.e.* Frobenius is an isomorphism on $R$. Observe that $R_{\mathrm{perf}}$ is perfect. Perfect rings are rarely noetherian. In fact, a noetherian perfect ring is a finite product of perfect fields.

**Exercise 2.11.** Prove that $\operatorname{Spec} R \longrightarrow \operatorname{Spec} R_{\mathrm{perf}}$ is a homeomorphism. Conclude that the perfection of a noetherian local ring has finite dimension.

**Theorem 2.10** (Bhatt–Scholze)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local ring (of prime characteristic $p$). Then its perfection is $R_{\mathrm{perf}}$ has finite global dimension.*

*Proof.* TO BE ADDED. ☐

[11]It suffices to ask this for all finite purely inseparable extensions $\mathscr{k}/\kappa(\mathfrak{p})$.

This result easily proves Kunz's theorem as follows. Recall that the substantial part of Kunz's theorem is that if $F^e\colon R \longrightarrow R$ is flat for a complete local ring then $R$ is regular, *i.e.* $R$ has finite global dimension. That is, we must show that there is $n \in \mathfrak{n}$ such that for all $R$-modules one has that

$$\operatorname{Tor}_i^R(\mathscr{k}, M) = 0$$

for all $i \geq n$. To that end, one observes that $R \longrightarrow R_{\text{perf}}$ is faithfully flat and that

$$R_{\text{perf}} \otimes_R \operatorname{Tor}_i^R(\mathscr{k}, M) = \operatorname{Tor}_i^{R_{\text{perf}}}(R_{\text{perf}} \otimes_R \mathscr{k}, R_{\text{perf}} \otimes_R M).$$

Then, we can take $n$ to be the global dimension of $R_{\text{perf}}$, which is finite by Bhatt–Scholze's theorem.

**Exercise 2.12.** Let $R \longrightarrow S$ be faithfully flat. Show that the global dimension of $R$ is no more than the global dimension of $S$.

## 3. $F$-finiteness and Gabber's Theorem

In studying regularity and therefore singularities one imposes noetherianity as a basic finiteness condition. In studying singularities, one imposes one additional condition. Namely,

**Definition 3.1.** An $\mathbb{F}_p$-algebra $R$ is *$F$-finite* if $F^e\colon R \longrightarrow R$ is finite for some/all $e > 0$ (*i.e.* $F^e_* R$ is a finitely generated $R$-module for all $e$).

**Exercise 3.1.** Let $R$ be $F$-finite. Show that so are its localizations, quotients, and polynomial extensions $R[x_1, \ldots, x_n]$. Prove that a field $\mathscr{k}$ is $F$-finite iff $[\mathscr{k}^{1/q} : \mathscr{k}] < \infty$. Conclude that in such case $\mathscr{k}$-algebras that are either essentially of finite type or complete are $F$-finite.

**Exercise 3.2.** $F$-finiteness has nothing to do with noetherianity. Show that there are noetherian rings that aren't $F$-finite and vice-versa.

*Remark* 3.2 ($F$-finiteness equals kählerianity over $\mathbb{F}_p$). According to Fogarty, $R/\mathbb{F}_{\mathfrak{p}}$ is $F$-finite iff its $R$-module of Kähler differentials $\Omega_{R/\mathbb{F}_p}$ is finitely generated, in which case $R/\mathbb{F}_p$ is referred to as kählerian. See [Fog80]. The forward implication is rather trivial and can be left as an exercise for those familiar with Kähler differentials. Although this equivalence is conceptually satisfying, we won't use it in the sequel.

Kunz's theorem takes a much simpler form in that case.

**Theorem 3.3** (Kunz's theorem in the $F$-finite case)**.** *Let $R$ be an $F$-finite (and noetherian) $\mathbb{F}_p$-algebra. Then, $R$ is regular if and only if $F^e_* R$ is a projective (i.e. locally free of finite rank) $R$-module. If $R$ is further local, it is regular iff $F^e_* R$ is locally free of finite rank.*

**Exercise 3.3.** Show that if $R$ is $F$-finite then its regular locus is (Zariski-)open.

**Exercise 3.4.** Let $(R, \mathfrak{m}, \mathscr{k})$ be an $F$-finite local ring. Show that its completion $R \longrightarrow \hat{R}$ is regular. Hint: Show that $F_{\hat{R}/R}$ is an isomorphism and then conclude using Radu–André's theorem.

**Exercise 3.5.** Suppose that $R$ is a regular $F$-finite ring and $\mathfrak{p} \in \operatorname{Spec} R$. Show that the following inclusion of ideals

$$\{r \in R \mid \phi(F^e_* r) \in \mathfrak{p}, \forall \phi \in \operatorname{Hom}_R(F^e_* R, R)\} \supset \mathfrak{p}^{[q]}$$

is an equality.

**Definition 3.4** ($p$-basis)**.** A *(regular) $p$-basis* for a regular $F$-finite $R/\mathbb{F}_p$ is a set $x_1, \dots, x_n$ such that
$$F^e_* R = \bigoplus_{0 \leq i_1, \dots, i_n \leq q-1} R F^e_* x_1^{i_1} \cdots x_n^{i_n}.$$
In particular, the rank of $F^e_* R$ is $q^n$.

*Remark* 3.5. According to Tyc, a $p$-basis is the same thing as a *differential basis* (*i.e.* $\Omega_{R/\mathbb{F}_p} = \bigoplus_{i=1}^n R dx_i$). See [Tyc88]. In particular, $F$-finite fields always admit a $p$-basis.

**Exercise 3.6.** Let $R := \mathbb{F}_p[\![x, y]\!]/(x^2 + y^2 - 1)$. Prove that $R$ is regular iff $p \neq 2$. However, $R$ admits a $p$-basis iff $p \equiv 1 \bmod 4$.

**Example 3.6.** Let $\Bbbk$ be an $F$-finite field. Note that $\Bbbk[x_1, \dots, x_n]$ and $\Bbbk[\![x_1, \dots, x_n]\!]$ both admit a $p$-basis.

*Remark* 3.7 (On restriction, extension, and co-extension of scalars)*.* Let $\theta \colon R \longrightarrow S$ be an $R$-algebra, and say $f := \operatorname{Spec} \theta \colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. This induces three covariant functors $f_*, f^*, f^!$; respectively known as *restriction, extension, and co-extension of scalars.* The restriction of scalars functor $f_*$ goes from the category of $S$-modules to the one of $R$-modules. If we have a morphism of $S$-modules $N \longrightarrow N'$, we can think of it as a morphism of $R$-modules by restricting scalars along $\theta \colon R \longrightarrow S$ , which we denote by $f_* N \longrightarrow f_* N'$. On the other hand, the functor of extension of scalars (aka base change) $f^*$ goes from the category of $R$-modules to the one of $S$-modules and it's defined by base change. Namely, if $\phi \colon M \longrightarrow M'$ is a morphism of $R$-modules then its extension of scalars is the morphism of $S$-modules
$$f^* M := S \otimes_R M \xrightarrow{S \otimes_R \phi \colon s \otimes m \mapsto s \otimes \phi(m)} f^* M' := S \otimes_R M'.$$
Finally, the functor $f^!$ of co-extension of scalars goes from $R$-modules to $S$-modules and is defined as follows. If $\phi \colon M \longrightarrow M'$ is a morphism of $R$-modules then $f^! \phi$ is the following morphism of $S$-modules:
$$f^! M := \operatorname{Hom}_R(S, M) \longrightarrow f^! M' := \operatorname{Hom}_R(S, M')$$
$$\mu \mapsto \phi \circ \mu$$
It is important to notice that $\operatorname{Hom}_R(S, M)$ is indeed an $S$-module, where the scalar action of $S$ is given by
$$s\mu := \mu \circ (\cdot s) \colon s' \mapsto \mu(ss').$$
Thus, it may be better to denote this as an right action, *i.e.* we may write $\mu s$ instead of $s\mu$. Note that $\operatorname{Hom}_R(S, R)$ is also an $R$-module where $r\mu = (\cdot r) \circ \mu \colon s \mapsto r\mu(s)$. Nevertheless, these two linear structures are related as follows:
$$r\mu = \mu\theta(r),$$
from which one may say that the $S$-module structure determines the $R$-module one (by restriction of scalars).

These three functors are related by the adjointness:
$$f^* \dashv f_* \dashv f^!$$
Indeed, the co-unit $\epsilon \colon f^* f_* \longrightarrow \mathrm{id}$ is given by
$$\epsilon_N S \otimes_R N \xrightarrow{s \otimes n \mapsto sn} N$$

whereas the unit $\eta\colon \operatorname{id} \longrightarrow f_* f^*$ is given by

$$\eta_M\colon M \xrightarrow{m \mapsto 1\otimes m} S \otimes_R M.$$

Likewise, the co-unit $\operatorname{Tr}\colon f_* f^! \longrightarrow \operatorname{id}$ for the adjointness $f_* \dashv f^!$ is known as *the trace* and is defined as

$$\operatorname{Tr}_M\colon \operatorname{Hom}_R(S, M) \xrightarrow{\mu \mapsto \mu(1)} M$$

whereas its unit $\nu\colon \operatorname{id} \longrightarrow f^! f_*$ is the natural transformation

$$\begin{aligned} \nu_N\colon N &\longrightarrow \operatorname{Hom}_R(S, N) \\ n &\mapsto (s \mapsto sn). \end{aligned}$$

**Exercise 3.7.** Show that the above pairs of units and co-units define a par of adjointness relations $f^* \dashv f_* \dashv f^!$. That is, show that there are commutative diagrams of natural transformations

$$\begin{tikzcd} f^* \arrow[r, "f^*\eta"] \arrow[dr, "\mathrm{id}"'] & f^* f_* f^* \arrow[d, "\epsilon f^*"] \\ & f^* \end{tikzcd} \qquad \begin{tikzcd} f_* \arrow[r, "\eta f_*"] \arrow[dr, "\mathrm{id}"'] & f_* f^* f_* \arrow[d, "f_*\epsilon"] \\ & f_* \end{tikzcd}$$

defining $f^* \dashv f_*$. Likewise, for $f_* \dashv f^!$, show that we have commutative diagrams of natural transformations

$$\begin{tikzcd} f_* \arrow[r, "f_*\nu"] \arrow[dr, "\mathrm{id}"'] & f_* f^! f_* \arrow[d, "\operatorname{Tr} f_*"] \\ & f_* \end{tikzcd} \qquad \begin{tikzcd} f^! \arrow[r, "\nu f^!"] \arrow[dr, "\mathrm{id}"'] & f^! f_* f^! \arrow[d, "f^! \operatorname{Tr}"] \\ & f^! \end{tikzcd} .$$

The above means that the natural maps

$$\operatorname{Hom}_S(f^*M, N) \xrightarrow{\psi \mapsto f_*\psi \circ \eta_M} \operatorname{Hom}_R(M, f_*N) \text{ and } \operatorname{Hom}_R(M, f_*N) \xrightarrow{\phi \mapsto \epsilon_N \circ f^*\phi} \operatorname{Hom}_S(f^*M, N)$$

are inverse to each other. Similarly, the natural maps

$$\operatorname{Hom}_S(N, f^!M) \xrightarrow{\psi \mapsto \operatorname{Tr}_M \circ f_*\psi} \operatorname{Hom}_R(f_*N, M) \text{ y } \operatorname{Hom}_R(f_*N, M) \xrightarrow{\phi \mapsto f^!\phi \circ \nu_N} \operatorname{Hom}_S(N, f^!M)$$

are mutually inverse.

**Exercise 3.8.** Notice that $f_*$ is exact and so that $f^*$ is right-exact whereas $f^!$ is left exact. Observe that $f^*$ is exact iff $f_*S$ is flat but $f^!$ is exact iff $f_*S$ is proyective.

**Exercise 3.9.** Show that the mapping

$$\operatorname{Hom}(f^*, f^!) \longrightarrow f^!R \coloneqq \operatorname{Hom}_R(S, R), \quad \xi \mapsto \xi_R(1)$$

is a bijection, what's its inverse?

This finishes our general observations on restriction, extension, and co-extension of scalars. How does all this apply to $F^e$?

**Exercise 3.10.** Suppose that $R/\mathbb{F}_p$ admits a $p$-basis (and so it is in particular regular and $F$-finite), say $x_1, \ldots, x_n$. Let

$$\{\phi_{i_1,\ldots,i_d} \coloneqq (F^e_* x_1^{i_1} \cdots x_d^{i_d})^\vee\}_{0 \le i_1,\ldots,i_d \le q-1}$$

be the corresponding dual basis for $\operatorname{Hom}_R(F^e_* R, R)$. Show that:

(a) The $F^e_*R$-linear mapping
$$F^e_*R \longrightarrow \mathrm{Hom}_R(F^e_*R, R), \quad F^e_*1 \mapsto \Phi^e \coloneqq \phi_{q-1,\dots,q-1}$$
is an isomorphism. We will be referreing to $\Phi^e$ as the *$e$-th (power of the) Frobenius trace* of $R$.

(b) The equalities
$$\Phi^{e-1} \circ F^{e-1}_*\Phi^1 = \Phi^e = \Phi^1 \circ F_*\Phi^{e-1}.$$
hold, which justifies to say that $\Phi^e$ is the $e$-th power of $\Phi \coloneqq \Phi^1$. In fact, $\Phi^e = \Phi^a \circ F^a_*\Phi^b$ whenever $e = a + b$.

(c) For all $r \in R$ and $\mathfrak{a}, \mathfrak{b} \subset R$ ideals,
$$(\Phi^e r)(F^e_*\mathfrak{a}) \subset \mathfrak{b} \iff r \in \mathfrak{b}^{[q]} : \mathfrak{a}.$$

(d) For every ideal $\mathfrak{a} \subset R$ with quotient $R \longrightarrow A \coloneqq R/\mathfrak{a}$, there is an exact sequence of $F^e_*R$-modules
$$0 \longrightarrow \mathfrak{a}F^e_*R = F^e_*\mathfrak{a}^{[q]} \longrightarrow F^e_*(\mathfrak{a}^{[q]} : \mathfrak{a}) \xrightarrow{F^e_*r \mapsto (\Phi^e r)/\mathfrak{a}} \mathrm{Hom}_A(F^e_*A, A) \longrightarrow 0$$
which induces an isomorphism of $F^e_*A$-modules
$$F^e_*\left(\frac{\mathfrak{a}^{[q]} : \mathfrak{a}}{\mathfrak{a}^{[q]}}\right) \xrightarrow{\cong} \mathrm{Hom}_A(F^e_*A, A).$$

(e) If $x_n \in R$ is not a unit then $x_1, \dots, x_{n-1}$ yields a $p$-basis on $R/x_nR$. Furthermore, if $I \subset \{1, \dots, n\}$ is such that $(x_i \mid i \in I) \neq R$ then $\{x_i\}_{i\in I}$ is a regular sequence on $R$.

(f) If all the $x_1, \dots, x_n$ are units then $\dim R = 0$.

(g) More generally, $\dim R \leq n$.

(h) Relabel if necessary so that $x_1, \cdots, x_m$ is such that $S/(x_1, \dots, x_m)$ is zero dimensional. Show that the canonical map
$$R_{\mathrm{perf}} \coloneqq \bigcap_e R^q \xrightarrow{\subseteq} S \longrightarrow S/(x_1, \dots, x_m)$$
is injective.

(i) Conclude that $R_{\mathrm{perf}}$ is noetherian and so a product of perfect fields.

**Theorem 3.8** (Gabber [Gab04])**.** *Let $R/\mathbb{F}_p$ be $F$-finite (and noetherian). Then, there is an $F$-finite regular ring $S$ admiting a $p$-basis (and so having) finite dimension such that $R$ is a homomorphic image of $S$, i.e. there is a quotient $S \twoheadrightarrow R$.*

*Main idea of the proof.* The proof is constructive. Let $F_*r_1, \dots, F_*r_n \in F_*R$ be $R$-generators of $F_*R$. Equivalently, $r_1, \dots, r_n \in R$ are generators of $R$ as an $R^p$-module. Consider the $R$-algebra
$$S_e \coloneqq R[x_1, \dots, x_n]/(x_1^q - r_n, \dots, x_n^q - r_n)$$
Observe that its $e$-th Frobenius factors as follows
$$\begin{tikzcd} S_e \arrow[r, "F^e"] \arrow[dr, "\phi_e"'] & S_e \\ & R \arrow[u, hook] \end{tikzcd}$$

Moreover, the map $\phi_e$ further factors as

$$\begin{tikzcd} S_e \arrow[rr, twoheadrightarrow, "\sigma_e"] \arrow[dr, "\phi_e"'] & & S_{e-1} \arrow[dl, "\phi_{e-1}"] \\ & R & \end{tikzcd}$$

Where $\sigma_e$ acts like Frobenius on $R$ and as the identity on the $x$'s. Therefore, we may take the limit over this inverse system to obtain

$$S \coloneqq \varprojlim_{e \in \mathbb{N}} S_e$$

Concretely, recall that an element $s \in S$ can be thought of as a sequence $(s_0, s_1, s_2, \ldots)$ where $s_e \in S_e$ and $\sigma_e(s_e) = s_{e-1}$. In particular, we may define (with a slight abuse of notation)

$$x_i \coloneqq (r_i, x_i, x_i, \ldots) \in S$$

as the constant sequence.[12] Now, note that $S_e$ is an $S_{e-1}$-algebra and its Frobenius factors as

$$\begin{tikzcd} S_e \arrow[r, "F"] \arrow[dr, "\varphi_e"'] & S_e \\ & S_{e-1} \arrow[u, hook, "\vartheta_e"'] \end{tikzcd}$$

And, moreover, such factorization is compatible with the structural maps of the inverse system defined $S$. More precisely, we have the following commutative diagram

$$\begin{tikzcd} & S_{e+1} \arrow[rr, "F"] \arrow[dl, twoheadrightarrow, bend right, "\sigma_{e+1}"'] \arrow[drr, "\varphi_{e+1}"] & & S_{e+1} \arrow[dll, twoheadrightarrow, "\sigma_{e+1}"] \\ S_e \arrow[r, "F"] \arrow[dr, "\varphi_e"'] & S_e & & S_e \arrow[u, hook, "\vartheta_{e+1}"'] \arrow[dll, twoheadrightarrow, bend left, "\sigma_e"] \\ & S_{e-1} \arrow[u, hook, "\vartheta_e"'] & & \end{tikzcd}$$

Therefore, by taking the inverse limit, we obtain a commutative diagram

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[dr, "\varphi"'] & S \\ & S \arrow[u, hook, "\vartheta"'] \end{tikzcd}$$

of rings. One readily sees that $\varphi$ is injective and therefore $S$ is reduced. On the other hand,

$$\{x_1^{i_1} \cdots x_n^{i_n}\}_{0 \leq i_1, \ldots, i_n \leq p-1}$$

is a basis for $S$ as an $S$-module by restriction of scalars along $\theta$. Thus, putting everything together, we see that

$$F_* S = \bigoplus_{0 \leq i_1, \ldots, i_n \leq p-1} S F_* x_1^{i_1} \cdots x_n^{i_n}.$$

Thus, we're done if $S$ is noetherian, which is the content of the theorem. This is actually an involved proof. Those interested, can try themselves or read a proof in [MP]. $\square$

[12] Caution, not every element of $R$ can be lifted to $S$. In fact, $S$ isn't in any meaningful way an $R$-algebra.

**Corollary 3.9.** *$F$-finite rings have finite dimension.*

*Remark* 3.10. Corollary 3.9 was originally obtained by Kunz [Kun76]. However, I understand his proof is flawed due to some equi-dimensionality issues. Nonetheless, there are proofs of Corollary 3.9 that are independent of Gabber's result. I think we'll see one later on.

*Question* 3.11 (Noether normalization of $F$-finite rings). Is an $F$-finite ring a finite (separable) extension of an $F$-finite regular ring that admits a $p$-basis?

**Corollary 3.12.** *$F$-finite rings admit a canonical module.*[13] *Namely,*

$$\omega_R \coloneqq \operatorname{Ext}_S^{\dim S - \dim R}(R, S),$$

*where $S \twoheadrightarrow R$ is as in Theorem 3.8.*

*Remark* 3.13. By the way, not all excellent rings admit a canonical module, see [**?**]. This is an aspect in which $F$-finite rings beat excellent ones.

**Exercise* 3.11.** Prove that $R$ is Cohen–Macaulay iff $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \neq \dim S - \dim R$. So far, we've only defined local Cohen–Macaulay rings. Take the definition of general Cohen–Macaulay as being Cohen–Macaulay at all localizations at prime ideals (so you may reduce to the local case).

I hope the above convinces the reader that $F$-finite rings are pretty awesome. There's yet another reason why this is the case. Those rings that are pretty awesome for algebraic geometry have already been axiomatized and named, namely *excellent rings*.[14] Their definition is a bit of a mouthful though.

**Definition 3.14.** A noethering ring is said to be *excellent* if

(a) the completion homomorphism $R_{\mathfrak{p}} \longrightarrow \hat{R}_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$,
(b) all $R$-algebras of finite type have open regular loci, and
(c) all $R$-algebras of finite type are catenary (aka universally catenary).

**Theorem 3.15** (Kunz [Kun76])**.** *$F$-finite rings are excellent. Conversely, a local ring $(R, \mathfrak{m}, \mathscr{k})$ is $F$-finite if (and only if) it is excellent and $\mathscr{k}$ is $F$-finite.*

*Remark* 3.16 (On the proof). The proof is too lengthy to be worthwhile doing here. However, the reader should be able to prove already as an exercise that $F$-finite rings satisfy the first two properties of excellence; which are referred to as *quasi-excellent*, using Radu–André's theorem for (a). Furthermore, the point is the $F$-finite property is already a notion of excellence in positive characteristics that is much better to deal with than excellence itself. So for instance, there will be many properties excellent rings have and we'll need that can be obtained directly from $F$-finiteness. So that's the approach we'll take. A very nice detailed proof can be found in [MP].

## 4. $F$-purity, $F$-splitness, and $F$-splittings

## References

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[13] Don't worry at all if you don't know what this means. It's a little bit of a mess but we'll get back to it later when we need it. But it's a really important thing worth noting right away.

[14] Feel free to read their Wikipedia entry to glimpse at why.

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