# LECTURE NOTES $F$-SINGULARITIES

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ABSTRACT. These are lectures notes for a course on $F$-singularities given at the CIMAT in the Spring Semester 2024.

CONTENTS



## 1. REGULARITY (A CRASH COURSE)

This is a course about $F$-singularities and in particular about singularities. In a nutshell, singularities are the absence of regularity. Before defining what a regular ring is, we need the notion of *projective and global dimensions*.

### 1.1. **Projective resolutions and other homological algebra stuff.** Let $M$ be a module over a ring $R$.[1]

**Exercise 1.1.** Prove that there is an exact sequence

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $P_0$ is free and so projective. Iterate this to obtain an exact sequence

$$0 \longrightarrow K_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where the $P_i$'s are free. The module $K_i$ is referred to as *a syzygy module*.

**Definition 1.1** (Resolutions)**.** An exact sequence

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is called a *free (resp. projective) resolution* of $M$ if all the $P_i$'s are free (resp. projective). We may denote a projective resolution as $P_\bullet \longrightarrow M \longrightarrow 0$.[2]

[1]All rings are commutative with unity 1.
[2]Over local rings projective modules are free.

**Exercise 1.2.** Prove that free resolutions always exist, *i.e.* the category of $R$-modules has "enough projectives."

**Definition 1.2** (Projective dimension)**.** The module $M$ is said to have *finite projective dimension* if there is a projective resolution $P_\bullet \longrightarrow M \longrightarrow 0$ such that $P_i = 0$ for all $i \gg 0$. In such case, the *projective dimension* of $M$ is

$$\operatorname{pd} M = \operatorname{pd}_R M \coloneqq \min\{n \in \mathbb{N} \mid \exists P_\bullet \longrightarrow M \longrightarrow 0 \text{ such that } P_i = 0 \forall i > n\}.$$

If $M$ has not finite projective dimension we write $\operatorname{pd} M = \infty$.

**Exercise 1.3.** Prove that $M$ is projective iff $\operatorname{pd} M = 0$.

Next lemma is key.

**Lemma 1.3.** *Suppose that there are two exact sequences of $R$-modules*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow$$

*and*

$$0 \longrightarrow K'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow M \longrightarrow$$

*where $1 \leq n \in \mathbb{N}$ and the $P_i$ and $P'_i$ are projective. Then*

*(a)* $K_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \cong K'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots$

*(b) $K_n$ is projective iff so is $K'_n$.*

*Proof.* Note that (b) follows from (a).[3] The proof of (b) is lengthy and left as an exercise. Hint: Proceed by induction on $n$. Prove the case $n = 1$ first and then reduce the inductive case to this one. ☐

It can be used to prove the following.

**Exercise 1.4.** Let

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow$$

be an exact sequences where thee $P_i$'s are projective. Prove that

(a) $\operatorname{pd} M \leq n$ iff $K_n$ is projective.

(b) If $\operatorname{pd} M \geq n$ then $\operatorname{pd} K_n = \operatorname{pd} M - n$.

**Exercise 1.5.** Suppose that $R$ is noetherian and that $M$ is finitely generated. Prove that

$$\operatorname{pd}_R M = \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup\{\operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \text{ maximal}\}$$

**Exercise 1.6.** Prove that

$$\operatorname{pd}(M \oplus N) = \max\{\operatorname{pd} M, \operatorname{pd} N\}.$$

The above exercise generalizes as follows.

**Exercise* 1.7.** Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of $R$-modules. Show the following statements.

(a) If two of the modules in the exact sequence have finite projective dimension then so does the third one.

[3]Observe that for this is absolutely essential to use projectiveness instead of freeness.

(b) In that case (*i.e.* the three modules have finite projective dimension), then

$$\operatorname{pd} M \leq \max\{\operatorname{pd} M', \operatorname{pd} M''\},$$

(c) and if the inequality is strict then $\operatorname{pd} M'' = \operatorname{pd} M' + 1$.

**Definition 1.4** (Minimal free resolution)**.** Let $(R, \mathfrak{m}, \mathcal{k})$ be a noetherian local ring and $M$ a finitely generated $R$-module. A free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ is said to be *minimal* if

$$\phi_i(P_{i+1}) \subset \mathfrak{m} P_i \quad \forall i \in \mathbb{N}$$

where $\phi_i \colon P_{i+1} \longrightarrow P_i$ is the homomorphism from the free resolution.

**Exercise 1.8.** In the setup of Definition 1.4, let $K_i := \ker \phi_{i-1}$ for all $i \geq 1$. Prove that $\mu(P_0) = \mu(M)$ and $\mu(P_i) = \mu(K_i)$ for all $i \geq 1$. Here, we let

$$\mu(-) = \dim_{\mathcal{k}} - \otimes_R \mathcal{k}$$

denote the minimal number of generators.

**Exercise 1.9.** Show that minimal free resolutions exist.

**Exercise 1.10.** In the setup of Definition 1.4, let $P_\bullet \longrightarrow M \longrightarrow 0$ and $P'_\bullet \longrightarrow M \longrightarrow 0$ be two minimal free resolutions. Show that $\mu(P_i) = \mu(P'_i)$ for all $i \in \mathbb{N}$.

The above two exercises guarantee that the following definition makes sense.

**Definition 1.5** (Betti numbers)**.** In the setup of Definition 1.4, the *i-th Betti number* of $M$ is defined as $\beta_i(M) := \mu(P_i)$ where $P_\bullet \longrightarrow M \longrightarrow 0$ is any minimal free resolution.

*Remark* 1.6. Sometimes people talk about the Betti numbers of $(R, \mathfrak{m}, \mathcal{k})$, in that case, they refer to the Betti numbers of $\mathcal{k}$.

**Exercise 1.11.** Let $P_\bullet \longrightarrow M \longrightarrow 0$ be a minimal free resolution. Prove that $P_i = 0$ if (and only if) $i > \operatorname{pd} M$. That is,

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$$

**Exercise 1.12.** Prove that

$$\beta_i(M) = \dim_{\mathcal{k}} \operatorname{Tor}_i(\mathcal{k}, M), \quad \forall i \in \mathbb{N}.$$

and conclude that

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i(\mathcal{k}, M) \neq 0\} \leq \operatorname{pd} \mathcal{k}.$$

**Definition 1.7** (Global dimension)**.** The *global dimension* of a ring $R$ is the supremum of the projective dimensions of finitely generated $R$-modules.

**Corollary 1.8.** *The global dimension of a local ring is the projective dimension of its residue field.*

*Remark* 1.9 (Regular sequences and depth). Recall that a *regular element* $r \in R$ on an $R$-module $M$ is one for which $\cdot r \colon M \longrightarrow M$ is injective but not surjective. A *regular sequence* $r_1, \ldots, r_d \in R$ on $M$ is defined by the following two conditions:

(a) $r_1$ is regular on $M$, and
(b) $r_i$ is regular on $M/(r_1, \ldots, r_{i-1})M$ for all $i = 2, \ldots d$.

Given an ideal $\mathfrak{a} \subset R$, the *depth of* $\mathfrak{a}$ *on* $M$, denoted by $\operatorname{depth}_R(\mathfrak{a}, M)$, is the maximal length of a regular sequence on $M$ of elements in $\mathfrak{a}$. When $(R, \mathfrak{m}, \mathcal{k})$ is local, we may write $\operatorname{depth} M = \operatorname{depth}_R M = \operatorname{depth}_R(\mathfrak{m}, M)$. In that case, we also have:

$$\operatorname{depth} M = \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i(\mathcal{k}, M) \neq 0\}.$$

This formula can be proved as follows (details are left to the reader). First, prove that if $r_1, \ldots, r_d \in R$ is a regular sequence on $M$ then

$$\operatorname{Ext}^i_R(\mathcal{k}, M) = \begin{cases} 0 & \text{if } i < d, \\ \operatorname{Hom}_R(\mathcal{k}, M/(r_1, \ldots, r_d)M) & \text{if } i = d. \end{cases}$$

This can be proved by induction on $d$. The base step $d = 0$ is trivial. For the inductive step, consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot r_1} M \longrightarrow M/r_1M \longrightarrow 0$$

Next, apply the functor $\operatorname{Hom}_R(\mathcal{k}, -)$ to it. Since $r_1 \in \mathfrak{m}$, it acts like 0 on $\mathcal{k}$ and so $\operatorname{Ext}^i_R(\mathcal{k}, \cdot r_1) = 0$. This means that the long exact sequence on Ext's breaks down into exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, M) \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, M/r_1M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathcal{k}, M) \longrightarrow 0$$

Since $r_2, \ldots, r_d$ is a regular sequence on $M/r_1M$, we may apply the inductive hypothesis and conclude.

**Theorem 1.10** (Auslander–Buchsbaum formula)**.** *In the setup of Definition 1.4, if* $\operatorname{pd} M < \infty$ *then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R.$$

*In particular, if $R$ has finite global dimension it is at most* $\operatorname{depth} R$*.*

*Proof.* We only sketch a proof and leave the details to the reader as an exercise. The proof is an induction on $\operatorname{pd} M$. If $\operatorname{pd} M = 0$ then $M$ is free and so $\operatorname{depth} M = \operatorname{depth} R$. If $\operatorname{pd} M = 1$ then there is an exact sequence

$$0 \longrightarrow R^{\oplus m} \xrightarrow{\phi} R^{\oplus n} \longrightarrow M \longrightarrow 0$$

which we may assume to be minimal, *i.e.* we may assume that the entries of the $n \times m$ $R$-matrix $\phi \colon R^{\oplus m} \longrightarrow R^{\oplus n}$ are in $\mathfrak{m}$. Consider next the long exact sequence on Ext obtained by applying the functor $\operatorname{Hom}_R(\mathcal{k}, -)$ (write it down yourself). Observe that $\operatorname{Ext}^i_R(\mathcal{k}, R^{\oplus k}) = \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus k}$ and that

$$\operatorname{Ext}^i_R(\mathcal{k}, \phi) : \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus m} \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus n}$$

is given by the $\mathcal{k}$-matrix obtained by reducing $\phi$ modulo $\mathfrak{m}$. In particular, $\operatorname{Ext}^i_R(\mathcal{k}, \phi) = 0$ and so there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus n} \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathcal{k}, R)^{\oplus m} \longrightarrow 0$$

From this, we see that $\operatorname{depth} M = \operatorname{depth} R - 1$. This shows the base step of the induction. For the inductive step, suppose $\operatorname{pd} M \geq 2$ and consider an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0$$

where $\operatorname{pd} N = \operatorname{pd} M - 1$. Use the corresponding long exact sequence on Ext's obtained by applying $\operatorname{Hom}_R(\mathcal{k}, -)$ to find the relationship between the depths of $M$ and $N$ (which is $\operatorname{depth} N = \operatorname{depth} M + 1$). Use the inductive hypothesis to conclude. $\square$

*Remark* 1.11. It is not difficult to see (using Krull's height theorem) that every regular sequence can be extended to a system of parameters. In particular, $\operatorname{depth} R \leq \dim R$.[4] When this equality happens to be an equality one says that $(R, \mathfrak{m}, \mathcal{k})$ is *Cohen–Macaulay.* Thus, a local ring is Cohen–Macaualay if and only if every system of parameters[5] is a regular sequence.

1.2. **Regular local rings.** Let $(R, \mathfrak{m}, \mathcal{k})$ be a noetherian local ring. Then, by Nakayama's lemma, its so-called *embedded dimension*

$$\operatorname{edim} R \coloneqq \mu(\mathfrak{m}) = \dim_{\mathcal{k}} \mathfrak{m} \otimes \mathcal{k} = \dim_{\mathcal{k}} \mathfrak{m}/\mathfrak{m}^2$$

is finite.

**Exercise 1.13.** Use Krull's ideal theorem to conclude that the embedded dimension is at least the Krull's dimension of the local ring. In particular, noetherian local rings have finite dimension.

**Definition 1.12** (Regular local ring)**.** A noetherian local ring $(R, \mathfrak{m}, \mathcal{k})$ is said to be *regular* if the inequality

$$\operatorname{edim} R \geq \dim R$$

is an equality.

**Exercise 1.14.** Prove that if $(R, \mathfrak{m}, \mathcal{k})$ is a noetherian local ring such that $\mathfrak{m}$ is generated by a regular sequence then it is regular.

The converse of this exercise is also true but a bit harder to prove.

**Theorem 1.13.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a regular (noetherian) local ring. Then every set of minimal generators of $\mathfrak{m}$ (aka regular system of parameters) is a regular sequence. In particular, $\operatorname{pd}_R \mathcal{k} = \dim R$.*[6]

This result can be seen as a consequence of the following.

**Theorem 1.14.** *A regular local ring is an integral domain.*[7]

Recall the following useful, generalized form of prime avoidance.

**Lemma 1.15** (Prime avoidance)**.** *Suppose that $\mathfrak{a} \subset \mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_k$ where all but up to two of the ideals $\mathfrak{a}_i$ are prime. Then $\mathfrak{a} \subset \mathfrak{a}_i$ for some $i = 1, \ldots, k$.*

**Lemma 1.16.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring of positive dimension. Then $R$ contains a regular element not in $\mathfrak{m}^2$. That is, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ that avoids all minimal primes.*

*Proof.* Use prime avoidance. □

*Sketch of the proof of Theorem 1.14.* Set $d = \dim R < \infty$. Let's do induction on $d$. If $d = 0$, the regularity of $R$ implies that $0 = \dim_{\mathcal{k}} \mathfrak{m}/\mathfrak{m}^2$ and so $\mathfrak{m} = 0$ by Nakayama's lemma. This means that $R$ is a field and we're done.

Assume now that $d > 0$ and that all regular local rings of dimension $< d$ are integral domains. By Lemma 1.16, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ a regular element. Observe that

[4]More generally, $\operatorname{depth}(\mathfrak{a}, R) \leq \operatorname{ht} \mathfrak{a}$.

[5]A system of parameters for a local ring $(R, \mathfrak{m}, \mathcal{k})$ is a collection $x_1, \ldots, x_{\dim R}$ such that $\sqrt{(x_1, \ldots, x_{\dim R})} = \mathfrak{m}$. System of parameters always exist.

[6]In particular, regular local rings are Cohen–Macaulay, *i.e.* $\operatorname{depth} R = \dim R$.

[7]In fact UFDs and so normal.

- $R/rR$ is a local ring whose maximal ideal is generated by $d-1$ elements (one less than the number of generators of $\mathfrak{m}$), and
- the dimension of $R/rR$ is $d-1$.

In particular, $R/rR$ is a regular local ring of dimension $d-1$. By the inductive hypothesis, it is an integral domain and so $rR = (r)$ is a prime ideal. Further, observe that $(r) \subset R$ cannot be a minimal prime. Let $\mathfrak{p} \subset R$ be a minimal prime of $R$ that is contained in $(r)$. We're done if we can prove that $\mathfrak{p} = 0$. Let $x \in \mathfrak{p}$, and so $x = yr$ for some $y \in R$. In fact, $y \in \mathfrak{p}$ as $r \notin \mathfrak{p}$. In other words, $\mathfrak{p} = r\mathfrak{p}$. Since $r \in \mathfrak{m}$, Nakayama's lemma yields that $\mathfrak{p} = 0$; as desired. $\square$

**Corollary 1.17.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and $r \in \mathfrak{m} \setminus \mathfrak{m}^2$.[8] Then, $R$ is regular if and only if $r$ is a regular element and $R/rR$ is regular.*

Summing up, regular local rings have finite global dimension equal to its dimension. It turns out that the converse is also true and it's a deep result due to Auslander–Buchsbaum and Serre. To prove this, we need the following observation.

**Exercise 1.15.** Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and $M$ be a finitely generated $R$-module. Let $r \in R$ be a regular element on $R$ and on $M$. Prove that

$$\operatorname{pd}_{R/rR} M/rM = \operatorname{pd}_R M$$

Hint: Show that a minimal free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ becomes a minimal free resolution of $M/rM$ after base change by $R/rR$. Notice that this is tantamount to the vanishing

$$\operatorname{Tor}_i^R(R/rR, M) = 0, \quad \forall i > 0.$$

But this can be seen from the fact that

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/rR \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\cdot r} M \longrightarrow M/rM \longrightarrow 0$$

are both exact.

We're ready to prove the main result in this section. Please take a moment to appreciate its beauty.

**Theorem 1.18** (Auslander–Buchsbaum–Serre)**.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local noetherian ring. Then the following statements are equivalent.*

*(a) $R$ is regular (i.e. $\mathfrak{m}$ is generated by a regular sequence)*
*(b) The global dimension of $R$ is $\dim R$*
*(c) $\operatorname{pd}_R \mathcal{k}$ is finite.*

*Proof.* It only remains to explain why (c) implies (a). This is an induction on $d \coloneqq \dim R < \infty$. If $d = 0$, then the Auslander–Buchsbaum formula yields that $\operatorname{pd}_R \mathcal{k} = 0$ and so that $\mathcal{k}$ is a free $R$-module. Hence, $R = \mathcal{k}$ and we're done.

Let's assume that $d > 0$ and that (c) implies (a) for those local rings of dimension $< d$. Since $R$ is positive dimensional, we can find a regular element $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ and it suffices to prove that the local ring $(R/rR, \mathfrak{m}/rR, \mathcal{k})$ is regular (which has dimension $d-1$). To that end, we can apply the inductive hypothesis and prove that $\operatorname{pd}_{R/rR} \mathcal{k}$ is finite. For this, apply Exercise 1.15. $\square$

[8]Note that this is to say that $r$ is part of a minimal set of generators for $\mathfrak{m}$.

**Exercise 1.16.** Prove the following to corollaries.

**Corollary 1.19.** *If $(R, \mathfrak{m}, \mathscr{k})$ is a regular local ring then so is $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

**Corollary 1.20** (Hilbert's syzygy theorem)**.** *Let $\mathscr{k}$ be a field. Then, every finitely generated $\mathscr{k}[x_1, \ldots, x_n]$-module has a free resolution of length at most $n$.*

1.3. **General regular rings.** With the above in place, we can finally define regular rings beyond the local case.

**Definition 1.21** (Regular rings of finite dimension)**.** We say that a noetherian ring of finite Krull dimension $\dim R$ is regular if any of the following equivalent conditions hold:

(a) The local ring $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$.
(b) The global dimension of $R$ is at most $\dim R$ (*i.e.* every finitely generated module has projective dimension at most $\dim R$).
(c) $R$ has finite global dimension.

**Exercise 1.17.** Prove that the above conditions are indeed equivalent.

**Definition 1.22** (Regular rings)**.** Let $R$ be a noetherian ring. Then $R$ is said to be regular if $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$.

**Exercise 1.18.** Prove that if $R$ is regular then so is $W^{-1}R$ for any multiplicative set $W \subset R$.

**Exercise 1.19.** Prove that for a regular ring its global dimension equals its dimension.

1.4. **Complete regular rings and the Cohen structure theorems.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring. Recall that its completion is the canonical homomorphism

$$R \longrightarrow \hat{R} \coloneqq \varprojlim_n R/\mathfrak{m}^n$$

It turns out that $\hat{R}$ is a noetherian local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$, residue field $\mathscr{k}$, and dimension $\dim R$. Moreover, $R \longrightarrow \hat{R}$ is a faithfully flat local homomorphism. In particular, $R$ is regular if and only if so is $\hat{R}$.

**Exercise 1.20.** Prove that $\operatorname{depth} R = \operatorname{depth} \hat{R}$. In particular, $R$ is Cohen–Macaulay iff so is $\hat{R}$.

**Example 1.23.** If $R = \mathscr{k}[x_1, \ldots, x_n]/\mathfrak{a}$ and $\mathfrak{m} = (x_1, \ldots, x_n)$, then $\hat{R}_\mathfrak{m} = \mathscr{k}[\![x_1, \ldots, x_n]\!]/\mathfrak{a}$.

Recall that $(R, \mathfrak{m}, \mathscr{k})$ is said to be complete if $R \longrightarrow \hat{R}$ is an isomorphism. It turns out that $\hat{R}$ is complete. In fact, every quotient of $\hat{R}$ is a noetherian complete local ring.

*Remark* 1.24 (Characteristic)*.* Recall that the characteristic of a ring $R$, say $\operatorname{char} R$, is the only nonnegative integer $n \in \mathbb{N}$ such that $(n) = \ker(\mathbb{Z} \longrightarrow R)$. Note that if $R$ is an integral domain (*i.e.* a field) then $\operatorname{char} R$ is either $0$ or a prime number $p$.

**Exercise 1.21.** Prove that $R$ contains a field as a subring if and only if $\operatorname{char} R = \operatorname{char} \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Here $\kappa(\mathfrak{p})$ denotes the residue field of $R$ at $\mathfrak{p}$.

For this reason, those rings that contain a field as a subring are referred to as rings of *equi-characteristic*. If a ring does not contain a field then it is said to have *mixed-characteristic*.

If $(R, \mathfrak{m}, \mathscr{k})$ is a local ring, then it has equicharacteristic iff $\operatorname{char} R = \operatorname{char} \mathscr{k}$. If it is mixed characteristic then $\operatorname{char} \mathscr{k} = p > 0$ but $0 \neq p \in R$.

Suppose that $(R, \mathfrak{m}, \mathscr{k})$ is complete. A complete local subring $(\Lambda, p\Lambda, \mathscr{k}) \subset (R, \mathfrak{m}, \mathscr{k})$ is referred to as a coefficient ring. This entails that $\mathfrak{m} \cap \Lambda = p\Lambda$ and $p = \operatorname{char} \mathscr{k} \geq 0$. There are three cases:

- ◦ $R$ has equi-characteristic and so $\Lambda$ is a field contained in $R$ that maps isomorphically to $\mathscr{k}$.
- ◦ $R$ has mixed-caracteristic and $0 \neq p \in R$ is not nilpotent. In that case, $(\Lambda, p\Lambda, \mathscr{k})$ is a complete DVR. We'll referred to this rings as *Cohen rings*.
- ◦ $R$ has mixed-caracteristic and $p \in R$ is nilpotent (*i.e.* $\operatorname{char} R = p^n$ for some $n > 1$). In that case, $(\Lambda, p\Lambda, \mathscr{k})$ is an artinian local ring.

**Theorem 1.25** (Cohen structure theorem I)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring. Then:*

*(a) $R$ has a coefficient ring.*
*(b) There is a surjective homomorphism $\Lambda[\![x_1, \ldots, x_n]\!] \longrightarrow R$ where $\Lambda$ is either a field or a Cohen ring. Moreover, $\Lambda$ can be taken as a coefficient ring of $R$ if $p \in R$ isn't nilpotent. In particular, $R$ is a quotient of a regular complete local ring.*

*Remark* 1.26. The most difficult part is to show the existence of a coefficient ring. If $(R, \mathfrak{m}, \mathscr{k})$ has equi-characteristic $p > 0$ and $\mathscr{k}$ is perfect. Then it turns out that

$$\mathscr{k}_0 \coloneqq \bigcap_{e \in \mathbb{N}} R^{p^e}$$

is the only coefficient field of $R$. Here, $R^{p^e} = \{r^{p^e} \in r \in R\}$.

**Theorem 1.27** (Cohen structure theorem II)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete regular local ring. Then:*

- ◦ *If $R$ has equi-characteristic then $R \cong \mathscr{k}[\![x_1, \ldots, x_n]\!]$.*
- ◦ *If $R$ has mixed-characteristic then there is a Cohen ring $\Lambda$ such that*

$$R \cong \begin{cases} \Lambda[\![x_1, \ldots, x_n]\!] & \text{if } p \in R \text{ is a regular element} \\ \Lambda[\![x_1, \ldots, x_n]\!]/(p - f) \text{ for some } f \in \mathfrak{m}^2 & \text{otherwise.} \end{cases}$$

*We say that $R$ is* unramified *in the former case.*

**Theorem 1.28** (Cohen–Gabber structure theorem III)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local ring that either is equi-characteristic or is an integral domain. Then, there exists a subring $A \subset R$ such that:*

*(a) $A$ is a complete local ring,*
*(b) $A \subset R$ is finite induces an isomorphism on residue fields and a separable extension on fraction fields,*
*(c) $A \cong \Lambda[\![x_1, \ldots, x_n]\!]$ where $\Lambda$ is a field or a Cohen ring.*

**Exercise 1.22.** In the setup of Theorem 1.28, show that $(R, \mathfrak{m}, \mathscr{k})$ is Cohen–Macaulay if and only if $A \subset R$ is free (*i.e.* $R$ is a projective $A$-module). Hint: Use the Auslander–Buchsbaum formula.

**Exercise 1.23.** Let $R$ be a noetherian equi-characteristic ring. Prove that $R$ is regular iff $\hat{R}_{\mathfrak{p}} \cong \kappa(\mathfrak{p})[\![x_1, \ldots, x_{\operatorname{ht} \mathfrak{p}}]\!]$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

## 2. The Frobenius Endomorphism and Kunz's Theorem

From now on unless otherwise stated, we are going to assume that all rings have prime characteristic $p$. That is, all rings are $\mathbb{F}_p$-algebras. We always use the shorthand notation

$$q := p^e.$$

Further, we'll assume that all rings are noetherian. The *Frobenius endomorphism* of a ring $R$ is the homomorphism of $\mathbb{F}_p$-algebras

$$F = F_R \colon R \longrightarrow R, \quad r \mapsto r^p.$$

By iterating, we also have $F^e \colon r \mapsto r^q$ for all $e \in \mathbb{N}$. We let $R^q \subset R$ be the image subring of $F^e$.

**Exercise 2.1.** Prove that $F \colon R \longrightarrow R$ is indeed a homomorphism of $\mathbb{F}_p$-algebras. Prove that $\operatorname{Spec} F^e \colon \operatorname{Spec} R \longrightarrow \operatorname{Spec} R$ is the identity.

**Exercise 2.2.** Prove that $R$ is reduced iff $F^e$ is injective for some/all $e \in \mathbb{N}$.

**Exercise 2.3.** Recall that a ring $R$ is reduce iff its total ring of fractions $\mathscr{K}(R)$ is a product of fields $K_1 \times \cdots \times K_n$. Then, we may define $\bar{\mathscr{K}}(R)$ as $\bar{K}_1 \times \cdots \times \bar{K}_n$ where $\bar{K}_i$ is an algebraic closure of $K_i$. Hence $r^{1/q}$ is well-defined in $\bar{\mathscr{K}}(R)$ for all $r \in \mathscr{K}(R)$. Show that

$$R^{1/q} := \{r^{1/q} \in \bar{\mathscr{K}}(R) \mid r \in R\} \subset \bar{\mathscr{K}}(R)$$

is a subring that contains $R$. Moreover, show that $R \subset R^{1/q}$, $F^e \colon R \longrightarrow R$, and $R^q \longrightarrow R$ are isomorphic as $R$-algebras.

**Definition 2.1** (Frobenius powers)**.** Let $\mathfrak{a} \subset R$ be an ideal. Then $\mathfrak{a}^{[q]}$ is the extension ideal of $\mathfrak{a}$ along $F^e$, and it's called the *$e$-th Frobenius power of* $\mathfrak{a}$.

Note that if $\theta \colon R \longrightarrow S$ is a homomorphism of rings then there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, "F^e"'] & S \arrow[d, "F^e"] \\ R \arrow[r, "\theta"] & S \end{tikzcd}$$

**Exercise 2.4.** Prove that the above diagram is cartesian for all $e \in \mathbb{N}$ if $\theta$ is either a localization $R \longrightarrow W^{-1}R$ or a local completion $R \longrightarrow \hat{R}$. Show that if $\theta \colon R \longrightarrow R/\mathfrak{a}$ is a quotient then the diagram is cartesian iff $\mathfrak{a}^{[q]} = \mathfrak{a}$.

More generally, the following notation is going to be useful.

**Notation 2.2** (Frobenius pushforward)**.** Let $M$ be an $R$-module. We let

$$F^e_* M := \{F^e_* m \mid m \in M\}$$

be the $R$-module defined by the rules $F^e_* m + F^e_* m' = F^e_*(m + m')$ and $rF^e_* m = F^e_* r^q m$. In other words, $F^e_* M$ is the restriction of scalars of $M$ along $F^e$. Thus, $F^e_* M$ is identical to $M$ as an abelian group but the $R$-scalar action is being twisted by Frobenius. Likewise, if $M = S$ is an $R$-algebra then $F^e_* S$ is an $R$-algebra with the product $(F^e_* s)(F^e_* s') = F^e_*(ss')$. Again, $F^e_* S$ is the exact same thing as $S$ as a ring, what changes is the $R$-algebra structure.

With the above notation in place, we see that the commutative diagram above induces a ring homomorphism

$$F^e_\theta \colon S \otimes_R F^e_* R \longrightarrow F^e_* S, \quad s \otimes F^e_* r \mapsto F^e_* s^q \theta(r)$$

which is called the *relative Frobenius* of $\theta \colon R \longrightarrow S$.

**Exercise 2.5.** Prove that $\operatorname{Spec} F^e_\theta$ is a (universal) homeomorphism.

**Theorem 2.3** (Kunz's theorem). *Let $R$ be a (noetherian) ring. Then $R$ is regular iff $F^e \colon R \longrightarrow R$ is (faithfully) flat for some/all $e > 0$.*

*Remark* 2.4 (The socle). Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and $M$ be a finitely generated $R$-module. The *socle* of $M$ is the submodule

$$\operatorname{Soc}(M) \coloneqq \{m \in M : m\mathfrak{m} = 0\} \cong \operatorname{Hom}_R(\mathcal{k}, M) = \operatorname{Ext}^0_R(\mathcal{k}, M).$$

In particular, $\operatorname{depth} M = 0$ iff $\operatorname{Soc} M \neq 0$. Since $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n M = 0$, it follows that, if $\operatorname{depth} M = 0$, there is $n \in \mathbb{N}$ such that $\operatorname{Soc} M \not\subset \mathfrak{m}^n M$. Let $c \coloneqq \operatorname{depth} R$ and $r_1, \ldots, r_c \in R$ be a regular sequence. Set $\mathfrak{a} \coloneqq (r_1, \ldots, r_c)$. Observe that $\operatorname{depth}_R R/\mathfrak{a} = 0$. Then, we may find $n \in \mathbb{N}$ such that

$$\operatorname{Soc}_R(R/\mathfrak{a}) \not\subset \mathfrak{m}^n (R/\mathfrak{a}).$$

**Lemma 2.5.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring of depth $c$. Then there is $n \in \mathbb{N}$ such that for all infinite minimal free resolutions*

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(M)} \xrightarrow{\phi_i} R^{\oplus \beta_i(M)} \longrightarrow \cdots \longrightarrow R^{\beta_0(M)} \longrightarrow M \longrightarrow 0$$

*the entries of the matrix $\phi_{c+1}$ are not all contained in $\mathfrak{m}^n$ (i.e. the image of $\phi_{c+1}$ is not inside $\mathfrak{m}^n R^{\oplus b_{c+1}} = (\mathfrak{m}^n)^{\oplus b_{c+1}}$). Here $b_i \coloneqq \beta_i(M)$.*

*Proof.* Note that, by the Auslander–Buchsbaum formula, we have that $b_{c+1} \neq 0$ as the resolution has infinite length. This is gonna be important below.

Let $\mathfrak{a} = (r_1, \ldots, r_c)$ and $n$ be as in Remark 2.4. In particular, for $N \coloneqq \operatorname{Soc}_R(R/\mathfrak{a})$ we have that $N \not\subset \mathfrak{m}^n N$. Observe that

$$\operatorname{pd}_R R/\mathfrak{a} = c$$

and so

$$\operatorname{Tor}^R_{c+1}(M, R/\mathfrak{a}) = 0.$$

This implies that after base changing the given infinite minimal free resolution we obtain that

$$(R/\mathfrak{a})^{\oplus b_{c+2}} \xrightarrow{\phi_{c+1}/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_{c+1}} \xrightarrow{\phi_c/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_c}$$

is exact in the middle. In other words,

$$\ker \phi_c/\mathfrak{a} \subset \operatorname{im} \phi_{c+1}/\mathfrak{a}$$

Now, since the given resolution is minimal, we have that the entries of $\phi_c$ are all in $\mathfrak{m}$ and so

$$N^{\oplus b_{c+1}} \subset \ker \phi_c/\mathfrak{a}.$$

Thus, putting everything together, if (for the sake of contradiction) the image of $\phi_{c+1}$ is inside $(\mathfrak{m}^n)^{\oplus b_{c+1}}$, it would follow that

$$N^{\oplus b_{c+1}} \subset \big(\mathfrak{m}^n (R/\mathfrak{a})\big)^{\oplus b_{c+1}}.$$

But, since $b_{c+1} \neq 0$, this implies that

$$N \subset \mathfrak{m}^n(R/\mathfrak{a}),$$

which contradicts the construction of $n$. Isn't math just so cool? $\square$

*Proof of Kunz's theorem.* We may assume that $(R, \mathfrak{m}, \mathscr{k})$ is local. Moreover, we may assume that $(R, \mathfrak{m}, \mathscr{k})$ is complete. If $R$ is regular then $R \cong \mathscr{k}[\![x_1, \ldots, x_{\dim R}]\!]$. From which we see that $F^e_* R$ is a free $R$-module with a basis given by

$$\{F^e_* \lambda_i x_1^{i_1} \ldots x_d^{i_d}\}_{i \in I, 0 \leq i_1, \ldots, i_d \leq q-1}$$

where $\{F^e_* \lambda_i\}_{i \in I}$ is a $\mathscr{k}$-basis for $F^e_* \mathscr{k}$.

Conversely, suppose that $F^e \colon R \longrightarrow R$ is flat. We want to prove that $\mathrm{pd}_R\, \mathscr{k} < \infty$. Suppose, for the sake of contradiction that there is an infinite minimal free resolution

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(\mathscr{k})} \xrightarrow{\phi_i} R^{\oplus \beta_i(\mathscr{k})} \longrightarrow \cdots \longrightarrow R^{\beta_0(\mathscr{k})} \longrightarrow \mathscr{k} \longrightarrow 0$$

That is, $\beta_{c+1}(\mathscr{k}) \neq 0$ for $c = \operatorname{depth} R$. Since $F^e \colon R \longrightarrow R$ is flat for all $e$, we can base chang this inifinite minimal free resolution to obtain a minimal free resolution

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(\mathscr{k})} \xrightarrow{\phi_i^{[q]}} R^{\oplus \beta_i(\mathscr{k})} \longrightarrow \cdots \longrightarrow R^{\beta_0(\mathscr{k})} \longrightarrow R/\mathfrak{m}^{[q]} \longrightarrow 0$$

where $\phi_i^{[q]}$ is the matrix obtained from $\phi_i$ by raising its entries to the $q$-th power. In particular, the entries of $\phi_i^{[q]}$ belong to $\mathfrak{m}^{[q]} \subset \mathfrak{m}^q$ for all $i$ and in particular for $i = \operatorname{depth} R + 1$. This, however, contradicts Lemma 2.5 as $\mathfrak{m}^q \subset \mathfrak{m}^n$ for all $e \gg 0$ such that $q \geq n$. $\square$

2.1. **Relative version of Kunz's theorem.** There is a relative version of Kunz's theorem that goes by the name of Radu–André's theorem. To state it, we need to recall the following definition (the relative notion of $F$-regularity).

**Definition 2.6** (Regular algebras)**.** Let $\theta \colon R \longrightarrow S$ be an $R$-algebra (where $R$ and $S$ are noetherian). We say that $\theta$ is *regular* if it is flat and all its fibers are *geometrically regular*. That is, for all $\mathfrak{p} \in \operatorname{Spec} R$ the $\kappa(\mathfrak{p})$-algebra $S \otimes_R \kappa(\mathfrak{p})$ is noetherian and regular (and noetherian) after any base change by a finitely generated field extension $\mathscr{k}/\kappa(\mathfrak{p})$.[9]

**Theorem 2.7** (Radu–André)**.** *Let $\theta \colon R \longrightarrow S$ be an $R$-algebra. Then, $\theta$ is regular iff $F^e_\theta$ is (faithfully) flat for all/some $e > 0$.*

*On the proof.* The most important step is to show that if $\theta$ is regular than $S \otimes_R F^e_* R$ is noetherian. With that in place, the result can be obtained from the absolute Kunz theorem and the critere de planitude par fibres. I hope to add more details later on. $\square$

2.2. **Bhatt–Scholze's generalization of Kunz's theorem.** The *(colimit) perfection* of a ring $R$ is

$$R \longrightarrow R_{\mathrm{perf}} \coloneqq \operatorname{colim}(R \xrightarrow{F} R \xrightarrow{F} R \longrightarrow \cdots)$$

We say that $R$ is perfect iff $R \longrightarrow R_{\mathrm{perf}}$ is an isomorphism, *i.e.* Frobenius is an isomorphism on $R$. Observe that $R_{\mathrm{perf}}$ is perfect. Perfect rings are rarely noetherian. In fact, a noetherian perfect ring is a finite product of perfect fields.

**Exercise 2.6.** Prove that $\operatorname{Spec} R \longrightarrow \operatorname{Spec} R_{\mathrm{perf}}$ is a homeomorphism. Conclude that the perfection of a noetherian local ring has finite dimension.

[9]It suffices to ask this for all finite purely inseparable extensions $\mathscr{k}/\kappa(\mathfrak{p})$.

**Theorem 2.8** (Bhatt–Scholze)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local ring (of prime characteristic $p$). Then its perfection is $R_{\mathrm{perf}}$ has finite global dimension.*

*Proof.* TO BE ADDED. $\square$

This result easily proves Kunz's theorem as follows. Recall that the substantial part of Kunz's theorem is that if $F^e \colon R \longrightarrow R$ is flat for a complete local ring then $R$ is regular, *i.e.* $R$ has finite global dimension. That is, we must show that there is $n \in \mathfrak{n}$ such that for all $R$-modules one has that

$$\mathrm{Tor}_i^R(\mathscr{k}, M) = 0$$

for all $i \geq n$. To that end, one observes that $R \longrightarrow R_{\mathrm{perf}}$ is faithfully flat and that

$$R_{\mathrm{perf}} \otimes_R \mathrm{Tor}_i^R(\mathscr{k}, M) = \mathrm{Tor}_i^{R_{\mathrm{perf}}}(R_{\mathrm{perf}} \otimes_R \mathscr{k}, R_{\mathrm{perf}} \otimes_R M).$$

Then, we can take $n$ to be the global dimension of $R_{\mathrm{perf}}$, which is finite by Bhatt–Scholze's theorem.

## References

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