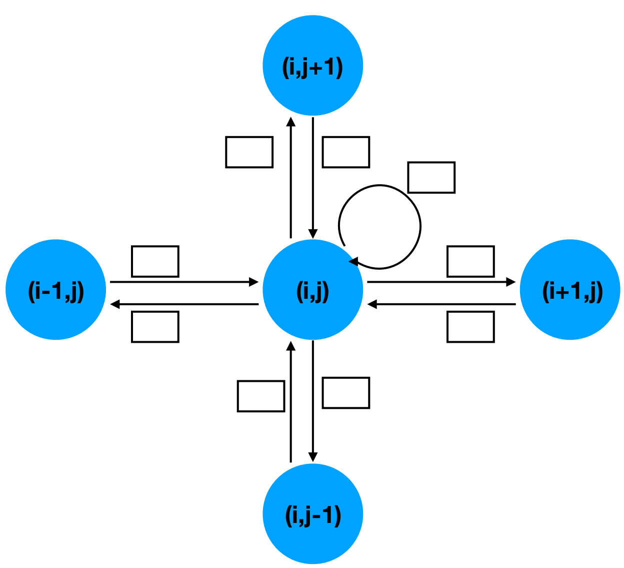
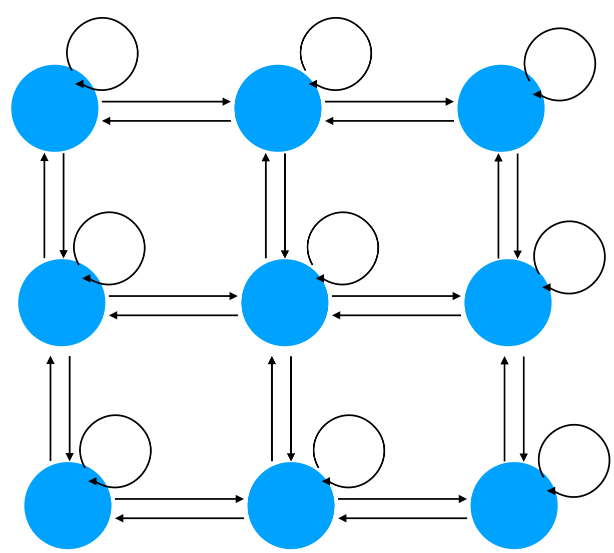
The goal of this discussion section is to get familiar with Markov chains.

Participation in discussion section counts as 5% of the grade. Completion of the worksheets counts as 20% of the grade. **Submit your worksheet work by April 21st at 2:59pm.**

Let's consider a model of population dynamics of two species in competition. We are interested to know which population will survive or if both might coexist after a long time. To model that we consider a finite birth and death chain (X_n, Y_n) in $[0, \dots, n] \times [0, \dots, n]$ with the following transitions:

- $(i, j) \to (i, j)$ with a probability m_{ij}
- $(i, j) \to (i+1, j)$ with a probability p_{ij}
- $(i, j) \to (i, j+1)$ with a probability q_{ij}
- $(i, j) \to (i-1, j)$ with a probability r_{ij}
- $(i, j) \to (i, j-1)$ with a probability s_{ij}
- $(0, 0)$ is the only absorbing state (if you're at $(0, 0)$ you remain there)
- If one coordinate is 0 or n, then there is no birth probability.





1. Complete the figure on the left.
2. For $n = 2$, try to sketch the associated Markov chain. Hint: define $A, B, C, D, E, F, G, H, I$ the 9 possible states using the figure on the right, and write the transition probability to go from one state to another.

3. Define the transition matrix $\mathcal{P}$.

$$\mathcal{P} = \begin{array}{c c}
\begin{pmatrix}
? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ?
\end{pmatrix} & \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \end{matrix} \\
\begin{matrix} A & B & C & D & E & F & G & H & I \end{matrix} &
\end{array}$$

4. Using Python, assign some values for $m_{ij}, p_{ij}, q_{ij}, r_{ij}, s_{ij}$ (be consistent with the transition matrix properties) and predict the long-term distribution. Try for 3 different cases.

5. Submit your work on Catcourses under the assignment `Worksheet 12` **as a .ipynb**.