Another expression of the singularity coefficient for a constant right hand side

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We consider the following problem (see CC18 for all details)

$$\left|\begin{array}{l}\text{Find } u = u_r + b\zeta s,\ b \in \mathbb{C},\ u_r \in H^1(D_\rho),\ \zeta \in \mathcal{C}^\infty([0,1], D_\rho),\ s \notin H^1(D_\rho),\ \text{ s.t.:}\\ \operatorname{div}\left(\varepsilon^{-1}\nabla u\right) + k_0^2 \mu u = 0 \quad \text{in } D_\rho \\ u = f \quad \text{on } \partial D_\rho \end{array}\right. \tag{1}$$

We define the singular complement z as the solution to the homogeneous problem

$$\left|\begin{array}{l}\text{Find } z = \zeta\,(s^* + c\,s) + \tilde{z},\ c \in \mathbb{C},\ \tilde{z} \in H^1(D_\rho) \text{ such that:}\\ \operatorname{div}\left(\varepsilon^{-1}\nabla z\right) + k_0^2 \mu z = 0 \quad \text{in } D_\rho \qquad . \\ z = 0 \quad \text{on } \partial D_\rho \end{array}\right. \tag{2}$$

Defining $w := c\zeta s + \tilde{z}$, solving Problem (2) is equivalent to solving:

$$\left|\begin{array}{l}\text{Find } w \text{ such that:}\\ \operatorname{div}\left(\varepsilon^{-1}\nabla w\right) + k_0^2 \mu w = -\operatorname{div}(\varepsilon^{-1}\nabla(\zeta s^*)) - k_0^2\mu\,\zeta s^* \quad \text{in } D_\rho\ . \\ w = 0 \quad \text{on } \partial D_\rho \end{array}\right. \tag{3}$$

Using the singular complement z, one can show after integration by parts on a perforated domain at the corner (and taking the limit) that the singularity coefficient admits the expression

$$b = -\frac{\displaystyle\int_{\partial D_\rho} \varepsilon^{-1} f \partial_r \tilde{z}\, d\sigma}{\displaystyle 2\lambda \int_{-\pi}^{\pi} \varepsilon^{-1}\Phi^2\, d\theta}. \tag{4}$$

The goal of these notes is to derive another formula in the specific case where f is constant. The reasons are two-fold: it is a simple case to test and formulas can be simplified, this can avoid to compute a normal derivative numerically to compute (4).

1 Lift the solution and new expression

First let us define $\tilde{f} \in H^1(D_\rho)$, the continuous extension of f over the domain D_ρ: for example consider $\tilde{f} = cst = f$ in D_ρ for constant right-hand side. Then we solve (1) for the lifted solution $u_l = u - \tilde{f} \in H_0^1(D_\rho)$:

$$\left| \begin{array}{l} \text{Find } u_l = u - \tilde{f}, \text{ s.t.:} \\ \operatorname{div}\left(\varepsilon^{-1}\nabla u_l\right) + k_0^2 \mu u_l = -k_0^2 \mu \tilde{f} \quad \text{in } D_\rho \\ u_l = 0 \quad \text{on } \partial D_\rho \end{array} \right. \tag{5}$$

Above we have used the fact that $\operatorname{div}\left(\varepsilon^{-1}\nabla \tilde{f}\right) = 0$.

Considering $0 < \delta < \rho$, using Problems (2)-(5) we have

$$-\int_{D_\rho \setminus \overline{D_\delta}} \varepsilon^{-1}\nabla u_l \cdot \nabla z \, + k_0^2 \int_{D_\rho \setminus \overline{D_\delta}} \mu u_l \, z + \int_{\partial D_\delta} \varepsilon^{-1} \partial_n u_l \, z = -k_0^2 \int_{D_\rho \setminus \overline{D_\delta}} \mu \tilde{f} \, z$$

and

$$-\int_{D_\rho \setminus \overline{D_\delta}} \varepsilon^{-1}\nabla z \cdot \nabla u_l \, + k_0^2 \int_{D_\rho \setminus \overline{D_\delta}} \mu z \, u_l + \int_{\partial D_\delta} \varepsilon^{-1} \partial_n z \, u_l = 0,$$

leading to

$$\int_{\partial D_\delta} \varepsilon^{-1} \partial_n u_l \, z - \int_{\partial D_\delta} \varepsilon^{-1} \partial_n z \, u_l = -k_0^2 \int_{D_\rho \setminus \overline{D_\delta}} \mu \tilde{f} \, z \tag{6}$$

Using the fact that

$$\lim_{\delta \to 0} \int_{\partial D_\delta} \varepsilon^{-1} \left(\partial_r u_l \, z - \partial_r z \, u_l\right) \, d\sigma = 2b \, \lambda \int_{-\pi}^{\pi} \varepsilon^{-1} \Phi^2 \, d\theta \tag{7}$$

We obtain the alternate expression of b:

$$b = -k_0^2 \frac{\displaystyle\int_{D_\rho} \mu \tilde{f} z}{\displaystyle 2\lambda \int_{-\pi}^{\pi} \varepsilon^{-1} \Phi^2 \, d\theta}. \tag{8}$$

for which there is no normal derivative to approximate. Contrary to (4) we will need the full knowledge of z for that coefficient.

For numerical tests with a constant right-hand side we will use formula (8). Note that if we have a smooth data (meaning for any non constant chosen extension $\tilde{f} \in H_0^1(D_\rho)$), one can use the same method leading to the following expression:

$$b = \frac{\displaystyle\int_{D_\rho} \varepsilon^{-1} \nabla \tilde{f} \cdot \nabla z \, - k_0^2 \int_{D_\rho} \mu \tilde{f} z}{\displaystyle 2\lambda \int_{-\pi}^{\pi} \varepsilon^{-1} \Phi^2 \, d\theta}, \tag{9}$$

where we used the fact that $\int_{D_\rho} \operatorname{div}(\varepsilon^{-1}\nabla\tilde{f})z = -\int_{D_\rho} \varepsilon^{-1}\nabla\tilde{f}\cdot\nabla z + \int_{\partial D_\rho} \varepsilon^{-1}\partial_r\tilde{f}z$, and $z=0$ on ∂D_ρ. In that case the right-hand side in (5) is slightly modified as well.

2 Variational formulation for the regular part

Once b is computed, one can solve the problem only for the regular part $u_{l,r} = u_r - \tilde{f} = u_l - b\zeta s$:

$$\left|\begin{array}{l}\text{Find } u_{l,r} \text{ s.t.:}\\ \operatorname{div}\left(\varepsilon^{-1}\nabla u_{l,r}\right) + k_0^2\mu u_{l,r} = -k_0^2\mu\tilde{f} - b\operatorname{div}\left(\varepsilon^{-1}\nabla(\zeta s)\right) - bk_0^2\mu\zeta s \quad \text{in } D_\rho\\ u_{l,r} = 0 \quad \text{on } \partial D_\rho\end{array}\right. \tag{10}$$

Note that the right-hand side is still well defined. This leads to the following variational formulation

$$\left|\begin{array}{l}\text{Find } u_{l,r} \in H_0^1(D_\rho) \text{ s.t.:} \forall v' \in H_0^1(D_\rho)\\ -\int_{D_\rho} \varepsilon^{-1}\nabla u_{l,r}\cdot\overline{\nabla v'} + k_0^2\int_{D_\rho} \mu u_{l,r}\overline{v'} = -\int_{D_\rho} k_0^2\mu\tilde{f}\overline{v'} + b\int_{D_\rho} \varepsilon^{-1}\nabla(\zeta s)\cdot\overline{\nabla v'} - bk_0^2\int_{D_\rho} \mu\zeta s\overline{v'}\end{array}\right. \tag{11}$$

Using the expression of the cutt-off function we have (for $) < \delta < l < \rho$):

$$\left|\begin{array}{l}\text{Find } u_{l,r} \text{ s.t.:}\\ \operatorname{div}\left(\varepsilon^{-1}\nabla u_{l,r}\right) + k_0^2\mu u_{l,r} = -k_0^2\mu\tilde{f} - bk_0^2\mu s \quad \text{in } D_\delta\\ \operatorname{div}\left(\varepsilon^{-1}\nabla u_{l,r}\right) + k_0^2\mu u_{l,r} = -k_0^2\mu\tilde{f} - b\operatorname{div}\left(\varepsilon^{-1}\nabla(\eta s)\right) - bk_0^2\mu\eta s \quad \text{in } D_{l,\delta} := D_l \setminus \overline{D_\delta}\\ \operatorname{div}\left(\varepsilon^{-1}\nabla u_{l,r}\right) + k_0^2\mu u_{l,r} = -k_0^2\mu\tilde{f} \quad \text{in } D_\rho\\ u_{l,r} = 0 \quad \text{on } \partial D_\rho\end{array}\right. \tag{12}$$

leading to the variational formulation

$$\left|\begin{array}{l}\text{Find } u_{l,r} \in H_0^1(D_\rho) \text{ s.t.:} \forall v' \in H_0^1(D_\rho)\\ -\int_{D_\rho} \varepsilon^{-1}\nabla u_{l,r}\cdot\overline{\nabla v'} + k_0^2\int_{D_\rho} \mu u_{l,r}\overline{v'} = -\int_{D_\rho} k_0^2\mu\tilde{f}\overline{v'} - b\int_{D_{l,\delta}} \operatorname{div}(\varepsilon^{-1}\nabla(\zeta s))\overline{v'} - bk_0^2\int_{D_\rho} \mu\zeta s\overline{v'}\\ \Longleftrightarrow\\ -\int_{D_\rho} \varepsilon^{-1}\nabla u_{l,r}\cdot\overline{\nabla v'} + k_0^2\int_{D_\rho} \mu u_{l,r}\overline{v'} = -\int_{D_\rho} k_0^2\mu\tilde{f}\overline{v'} + b\int_{D_{l,\delta}} \varepsilon^{-1}\nabla(\zeta s)\cdot\overline{\nabla v'}\\ +\, b\int_{\partial D_\delta} \varepsilon^{-1}\partial_r s\overline{v'} - bk_0^2\int_{D_\rho} \mu\zeta s\overline{v'}\end{array}\right. \tag{13}$$

3 Numerical method

Based on CC18 we discretize terms differently depending on the the parts (singular or regular) involved. For (11), the left-hand side and the first term on the right-hand side are discretized with standard FEM, and the last tow terms in the right-hand side are discretized using Newton-Cotes quadrature rules. With the CC18 notations we have

$$\left(-\mathbb{K}_\varepsilon + k_0^2\mathbb{M}_\mu\right) U = -k_0^2\mathbb{M}_\mu F - b(\mathbb{A}_0 + \mathbb{B}_0)S$$