Phase field approximation for Plateau's problem

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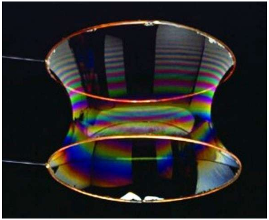
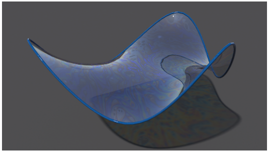
Introduction of the problem

Plateau’s Problem

Definition

Finding a set that minimizes its area and spans a given boundary.





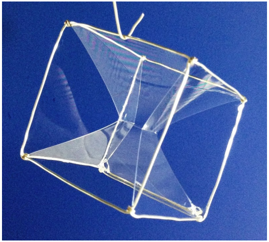
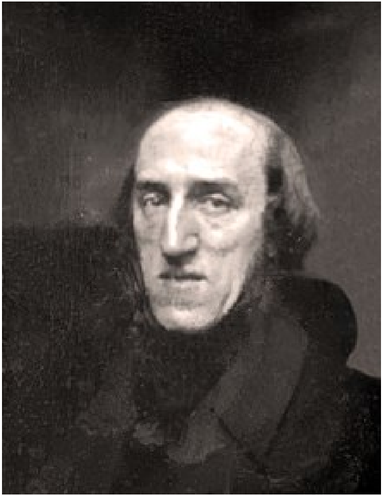


Figure: Application examples : shape of soap films

Origins



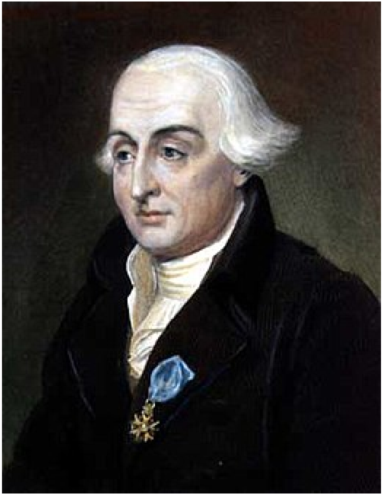


Figure: Joseph Plateau (1801-1883) and Joseph-Louis Lagrange (1736-1813)

First solutions

Figure: Jesse Douglas(1897-1965) and Tibor Rado (1895-1965)

Two major approaches

- **E.R. Reifenberg** (1928-1964) : uses Čech cohomology to define surface spanning a boundary.
- **H. Federer** (1920-2010) and **W. H. Fleming** (1928-2023) : use oriented currents to solve Plateau's oriented problem.

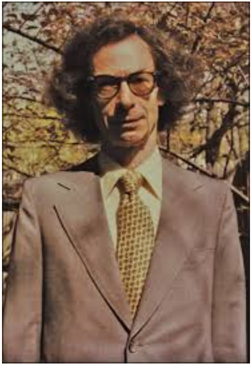


Figure: Fleming and Federer

Plateau in a cylinder

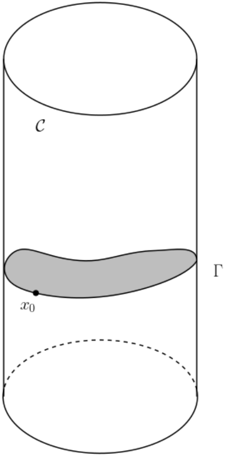


Figure: Plateau in a cylinder

C : the cylinder

Γ : graph of a Lipschitz function, defined on the boundary of the cylinder ∂C

Surfaces represented as boundary of sets

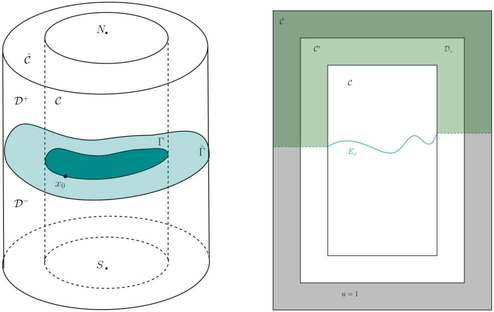


Figure: Interlocked cylinders

- $\hat{\Gamma}$: radial extension of the prescribed curve Γ
- $\mathcal{D}^+$: set above $\hat{\Gamma}$ in between cylinders
- $\mathcal{D}^-$: set below $\hat{\Gamma}$ in between cylinders

Surface spaning the curve Γ

$\downarrow$

Set containing $\mathcal{D}^+$ and not meeting $\mathcal{D}^-$

Problem definition

Definition of Plateau's problem

$$\inf \left\{ P(\Omega, \mathcal{C}') | \Omega \text{ such that } \chi_\Omega \in BV(\hat{\mathcal{C}}),\ \mathcal{D}^+ \subset \Omega \text{ and } \mathcal{D}^- \subset \Omega^c \right\} \tag{1}$$

Proposition

Plateau's problem (1) *admits solutions.*

Remark

For Ω_0 a solution of (1), the optimal surface is $\partial^*\Omega_0$.

Regularity results

Ahlfors regularity up to the boundary

Theorem (E.M.)

Let Ω_0 a minimizer of (1), then there exist two constants, $C_1, C_2 > 0$ such that for all $x \in \partial^*\Omega_0$ and $r < r_0 := d(\partial\mathcal{C}, \partial\mathcal{C}')$,

$$C_1 \leqslant \frac{P(\Omega_0, B(x,r))}{r^2} \leqslant C_2. \tag{2}$$

Ahlfors regularity up to the boundary

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Lemma (E.M.)

Let Ω_0 *a minimizer of* (1). *Then,*

$$\partial^*\Omega_0 \cap \partial\mathcal{C} = \Gamma.$$

Phase field approximation

Approximation of a rectifiable set length

Let $K \subset \Omega \subset \mathbb{R}^n$ a k-rectifiable set (for instance, if $k = 1$, a Lipschitz curve).

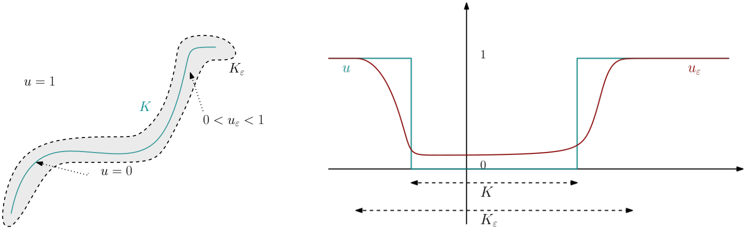

Figure: Phase field method

Ambrosio–Tortorelli energy :

$$\varepsilon \int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon} \int_\Omega (1 - u_\varepsilon)^2 \xrightarrow{\Gamma} \mathcal{H}^k(K).$$

Steiner's problem

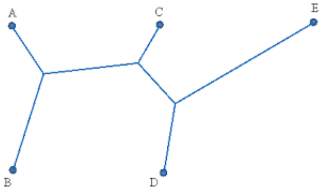

Figure: Steiner's problem with 5 points (N=4)

Approximation energy :

$$F_\varepsilon(u) := \varepsilon \int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon}\int_\Omega (1-u)^2 dx + \frac{1}{c_\varepsilon}\sum_{i=1}^{N} d_u(a_0, a_i),$$

$$d_u(a_0, a_i) = \inf_{\gamma : a_0 \to a_i} \int_\gamma |u|^2 d\mathcal{H}^1.$$

Approximation of length minimization problems among compact connected sets, M. Bonnivard, A. Lemenant, and F. Santambrogio, *SIAM Journal on Mathematical Analysis, 47(2), 1489-1529 (2015).*

Geodesic distance between closed curves

Definition

The geodesic distance between two closed curves, γ_1 and γ_2, is defined by

$$d_u(\gamma_1, \gamma_2) := \inf_{\varphi : \gamma_1 \longrightarrow \gamma_2} \int_{E_\varphi} |u|^2 d\mathcal{H}^2,$$

where, $\varphi : \gamma_1 \longrightarrow \gamma_2$ means that φ is a smooth curve in the space of closed curves connecting γ_1 and γ_2, and E_φ is the image of this curve φ.

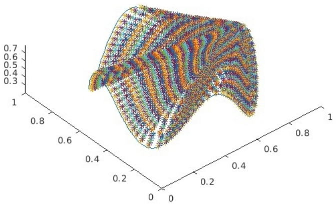

Figure: Geodesics

$x_0 \in \Gamma$: a fixed point

$\gamma_0(t) = x_0$: a constant closed curve

Approximation functional by penalisation

Definition

Let a sequence (c_ε) converging to 0 and $u \in H^1(\hat{\mathcal{C}}) \cap C(\overline{\hat{\mathcal{C}}})$ such that $0 \leqslant u \leqslant 1$ and $u = 1$ on $\overline{\hat{\mathcal{C}}} \backslash \mathcal{C}'$.
We define the approximation functional

$$F_\varepsilon(u) := \varepsilon \int_{\mathcal{C}'} |\nabla u|^2 dx + \frac{1}{4\varepsilon} \int_{\mathcal{C}'} (1-u)^2 dx + \frac{1}{c_\varepsilon} d_u(\Gamma, \gamma_0). \tag{3}$$

Γ-convergence type result

Theorem (E.M., Bonnivard, Bretin, Lemenant)

Let $\Omega \subset \hat{\mathcal{C}}$ a competitor for Plateau's problem (1). *Then, there exist a constant $C > 0$, depending only on $\hat{\Gamma}$, and $(u_\varepsilon) \in H^1(\hat{\mathcal{C}}) \cap C(\overline{\hat{\mathcal{C}}})$ such that $0 \leqslant u_\varepsilon \leqslant 1$, $u_\varepsilon = 1$ on $\overline{\hat{\mathcal{C}}} \backslash \mathcal{C}'$ and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leqslant P(\Omega, \mathcal{C}') - C.$$

Γ-convergence type result

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Let $\Omega \subset \hat{C}$ *a competitor for Plateau's problem* (1). *Then, there exist a constant* $C > 0$, *depending only on* $\hat{\Gamma}$, *and* $(u_\varepsilon) \in H^1(\hat{C}) \cap C(\overline{\hat{C}})$ *such that* $0 \leqslant u_\varepsilon \leqslant 1$, $u_\varepsilon = 1$ *on* $\overline{\hat{C}} \backslash C'$ *and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leqslant P(\Omega, C') - C.$$

Theorem (E.M., Bonnivard, Bretin, Lemenant)

Let Ω *be a solution of Plateau's problem* (1). *Then there exists a constant* $C > 0$, *depending only on* $\hat{\Gamma}$, *such that for all sequences* $u_\varepsilon \in H^1(\hat{C}) \cap C^0(\overline{\hat{C}})$ *such that* $0 \leqslant u_\varepsilon \leqslant 1$ *and* $u_\varepsilon = 1$ *on* $\overline{\hat{C}} \backslash C'$, *we have*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geqslant P(\Omega, C') - C.$$

Numerical simulations

Tilted circle

$$E_\varepsilon(u,\varphi) := \varepsilon \int_{\mathcal{C}'} |\nabla u|^2 dx + \frac{1}{4\varepsilon}\int_{\mathcal{C}'} (1-u)^2 dx + \frac{1}{c_\varepsilon}\int_{E_\varphi} |u|^2 d\mathcal{H}^2$$

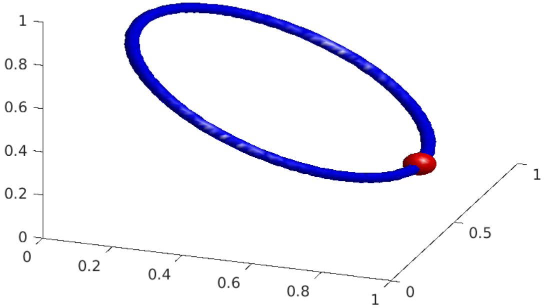

Figure: Tilted circle

Numerical approximation of the Steiner problem in dimension 2 and 3, M. Bonnivard, E. Bretin and A. Lemenant, *Mathematics of Computation, 89, 1-43 (2020)*.

Sinusoidal boundary

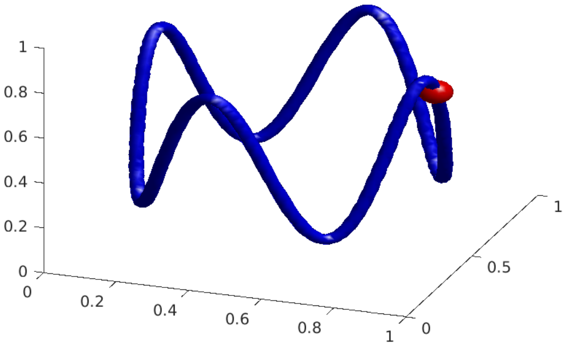

Figure: Sinusoidal boundary

2 circles

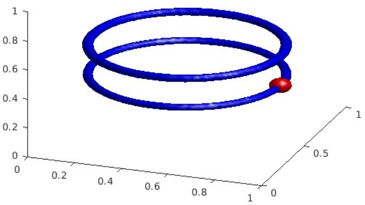

Figure: 2 close circles

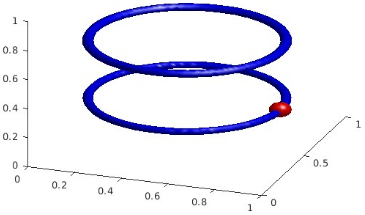

Figure: 2 less close circles

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Matthieu Bonnivard, Elie Bretin, and Antoine Lemenant.
Numerical approximation of the steiner problem in dimension 2 and 3.
Mathematics of Computation, 89(321):1–43, 2020.

Matthieu Bonnivard, Elie Bretin, Antoine Lemenant, and Eve Machefert.
Numerical phase field approximation for plateau's problem.
In preparation.

Matthieu Bonnivard, Antoine Lemenant, and Filippo Santambrogio.
Approximation of length minimization problems among compact connected sets.
SIAM Journal on Mathematical Analysis, 47(2):1489–1529, 2015.

Eve Machefert.
Ahlfors regularity up to the boundary of plateau solutions.
In preparation.

Numerical scheme for the minimization of u

We want to solve $\nabla_u E^2_\varepsilon(u,\varphi) = 0$. To that aim we decompose

$$\nabla_u E^2_\varepsilon(u,\varphi) = J_{imp}(u,\varphi) + J_{exp}(u,\varphi),$$

in which we add αu to $J_{imp}(u,\varphi)$ and deduct from $J_{exp}(u,\varphi)$. So that for α big enough, J_{exp} is concave.
Then we get a semi-implicite scheme as follows

$$J_{imp}(u^{n+1},\varphi) + J_{exp}(u^n,\varphi) = 0.$$

we deal with the implicite term with the Fourier transform.