Disentangling pulse-coupled oscillators through the pseudo-inverse in a dilated timescale

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Pulse-coupled oscillators

Peskin (1975) proposed the following model. Consider oscillators of voltage $V_1, \dots, V_n \in [0, V_F]$ following the dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} V_i(t) = f(V_i(t)) + \frac{\varepsilon}{N} \sum_{j=1, j\neq i}^{n} \sum_{k=0}^{+\infty} \delta(t - t_k^j),$$

where t_k^j is the time such that $V_i(t^-) = V_F$. Then $V_i(t^+) = 0$.
A cascade is possible where multiple firing events happen at the same t.

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- Many applications : neuroscience, fireflies, pacemeaker, ...
- Many methods were developped for this problem at the particle level
- Seminal paper by Mirollo Strogatz in 1990.

Pulse-coupled oscillators in phase variable

Phase variable :

$$\phi = \omega \int_0^v \frac{1}{f(s)} ds$$

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- Oscillators $X_1, \ldots, X_n \in [0, \Phi_F]$.
- They move at constant speed $X_i'(t) = \omega > 0$.
- When $X_i(t^-) = \Phi_F$, all X_j receive kick $K(X_j)$. Reset at $\phi = 0$ and cascade mechanism.

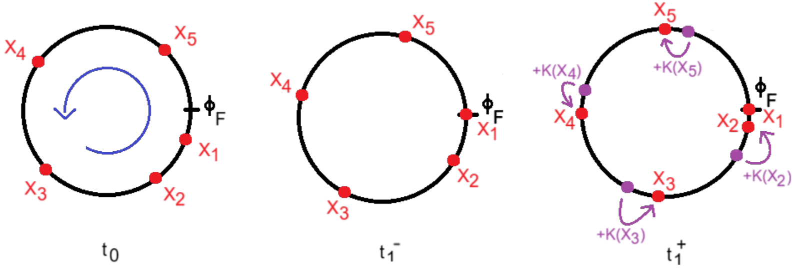

Figure – *Bad paint drawing*

Mean-field system

In the limit $n \to +\infty$, the following mean-field formulation is used :
$t > 0,\ \phi \in [0, \Phi_F]$,

$$\begin{aligned}
&\partial_t \rho + \partial_\phi([1 + K(\phi)N(t)]\rho) = 0,\\
&N(t) = [1 + K(\Phi_F)N(t)]\rho(t, \Phi_F),\\
&[1 + K(0)N(t)]\rho(t, 0) = [1 + K(\Phi_F)N(t)]\rho(t, \Phi_F) = N(t),\\
&\rho(t = 0, \phi) = \rho_{\mathrm{init}}(\phi).
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We can deduce that

$$N(t) = \frac{\rho(t, \Phi_F)}{1 - K(\Phi_F)\rho(t, \Phi_F)}.$$

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- Studied by Mauroy and Sepulchre (2013)
- They show that $K' < 0$ enforces convergence to a steady state (desynchronisation) and $K' > 0$ finite-time blow-up into a Dirac mass (synchronisation).
- Their proof is not fully rigorous and requires technical assumptions.

A new framework : the pseudo-inverse of the c.d.f. in a dilated timescale

Let $F(t, \phi)$ be the cumulative density function

$$F(t, \phi) = \int_0^{\phi} \rho(t, \tilde{\phi}) d\tilde{\phi}. \tag{1}$$

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We define the pseudo-inverse $Q(t,\eta) : \mathbb{R}_+ \times [0,1] \to [0,\Phi_F]$

$$Q(t,\eta) = \inf\{\phi \in [0,\Phi_F] \mid F(t,\phi) \geq \eta\}. \tag{2}$$

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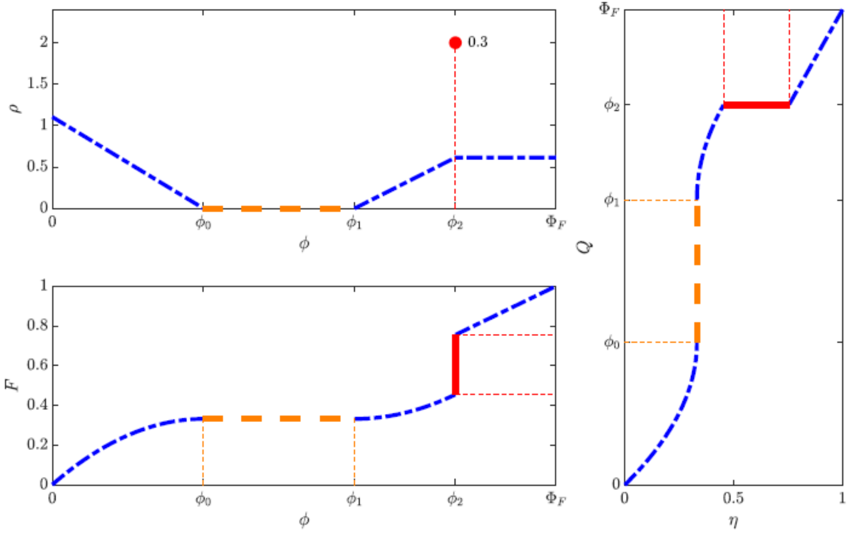
$\longrightarrow$ Note that, provided enough regularity

$$\frac{\partial Q}{\partial \eta}(\tau,\eta) = \frac{1}{\rho(Q(\tau,\eta))}, \qquad \text{and} \qquad F(t,Q(t,\eta)) = \eta.$$

Pseudo-inverse of the cummulative distribution



Formulation in the dilated timescale

$$\begin{cases} \partial_\tau Q + \partial_\eta Q = \dfrac{1}{N(\tau)} + K(Q), & \tau > 0,\ \eta \in (0,1), \\ Q(\tau, 0) = 0, & \tau > 0, \\ \dfrac{1}{N(\tau)} = \partial_\eta Q(\tau, 1) - K(\Phi_F), & \tau > 0, \\ Q(\tau = 0, \eta) = Q_{\mathrm{init}}(\eta), & \eta \in [0,1]. \end{cases}$$

with the constraint to ensure that $0 < N(\tau) < +\infty$

$$K(\Phi_F) < \partial_\eta Q(\tau, 1) < +\infty.$$

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Well-posedness

Theorem (Carrillo, Dou, R., Zhou)

For compatible initial data, there exists a classical solution either global-in-time or defined up to time τ^* such that

$$\limsup_{\tau \to \tau^*} N(\tau) = +\infty.$$

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Idea of the proof

- Mild solution with characteristics for $\partial_\tau Q + \partial_\eta Q = \tilde{N}(\tau) + K(Q)$ with given $\tilde{N}$.
- Do a fixed point on $\tilde{N}$ to find $\tilde{N} = \partial_\tau Q - K(Q)$.
- Prove well-posedness and regularity to get back to original problem.

You shall eat the fruit of the labor of your hands

Theorem (Carrillo, Dou, R., Zhou)

Suppose $K \in C^2[0, \Phi_F]$ is either convex or concave on $[0, \Phi_F]$. Let Q_1, Q_2 be two C^2 solutions, such that $\partial_\eta Q_1, \partial_\eta Q_2$ are C^1. Then,

$$e^{k_{\min}\tau} \|\partial_\eta Q_1(0) - \partial_\eta Q_2(0)\|_{L^1(0,1)} \leqslant \|\partial_\eta Q_1(\tau) - \partial_\eta Q_2(\tau)\|_{L^1(0,1)} \leqslant e^{k_{\max}\tau} \|\partial_\eta Q_1(0) - \partial_\eta Q_2(0)\|_{L^1(0,1)},$$

where,

$$k_{\min} = \min_{\phi \in [0, \Phi_F]} K'(\phi), \qquad k_{\max} = \max_{\phi \in [0, \Phi_F]} K'(\phi).$$

Main ideas of the proof

Since

$$\partial_\tau Q_i + \partial_\eta Q_i = \frac{1}{N_i(\tau)} + K(Q_i),$$

taking the difference would give $\frac{1}{N_1} - \frac{1}{N_2}$. However,

$$(\partial_\tau + \partial_\eta)(\partial_\eta Q_i) = K'(Q_i)\partial_\eta Q_i, \tag{3}$$

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$\longrightarrow$ Denote $I(t) = \|\partial_\eta Q_1(\tau) - \partial_\eta Q_2(\tau)\|_{L^1(0,1)}$.
We can prove

$$k_{\min} I \leqslant \frac{dI}{d\tau} \leqslant k_{\max} I.$$

Main ideas of the proof

Using the equation for $\partial_\eta Q$, we can prove

$$\frac{dI}{d\tau} = \int_0^1 \partial_\eta \Big(K(Q_1) - K(Q_2) \Big) \operatorname{sign}\left(\partial_\eta Q_1(\tau,\eta) - \partial_\eta Q_2(\tau,\eta) \right) d\eta$$
$$- \underbrace{\int_0^1 \partial_\eta \Big(\partial_\eta Q_1 - \partial_\eta Q_2 \Big) \operatorname{sign}\left(\partial_\eta Q_1(\tau,\eta) - \partial_\eta Q_2(\tau,\eta) \right) d\eta}_{=0 \text{ because Boundary Conditions}} .$$

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Then, we exploit

$$K'(Q_i(\eta)) = K'(0) + \int_0^{Q_i(\eta)} K''(\phi) d\phi$$

and the convexity/concavity of K.

Result in original timescale

The problem is that the timescale is different for Q_1 and Q_2 :

$$\tau_1 = \int_0^t N_1(s)ds, \qquad \tau_2 = \int_0^t N_2(s)ds$$

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$$\|\partial_\eta Q_1(\tau_1^{-1}(\tau)) - \partial_\eta Q_2(\tau_2^{-1}(\tau))\|_{L^1(0,1)} \leq e^{k_{\max}\tau} \|\partial_\eta Q_1(0) - \partial_\eta Q_2(0)\|_{L^1(0,1)}.$$

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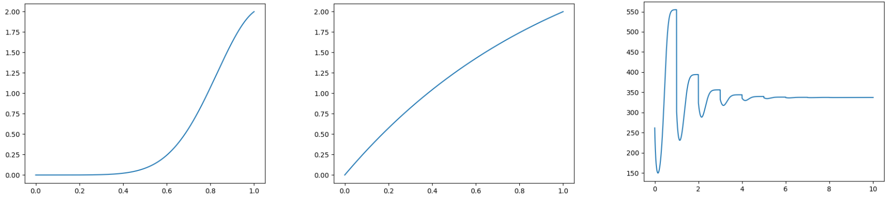
If we substitute $\tau_1^{-1}(\tau) = t$ and $\tau = \tau_1(t)$,

$$\|\partial_\eta Q_1(t) - \partial_\eta Q_2(\tau_2^{-1}(\tau_1(t)))\|_{L^1(0,1)} \leq e^{k_{\max}\tau_1(t)} \|\partial_\eta Q_1(0) - \partial_\eta Q_2(0)\|_{L^1(0,1)}.$$

Numerical confirmation



tau = 0 | tau = 10 | N(tau)

Figure – $K(Q) = -Q + 0.2$

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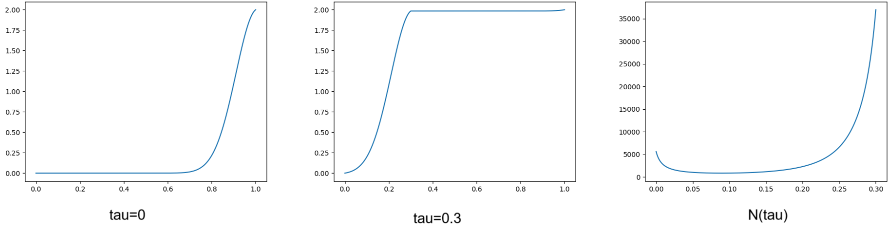

Figure – $K(Q) = 0.01\ Q + 0.2$

L^2 estimates

Another way to kill the constant term $1/N(\tau)$ is to take the average : consider

$$P := \mathrm{Id} - \int_0^1 d\eta. \qquad (4)$$

Theorem (Carrillo, Dou, R., Zhou)

Suppose $K \in C[0, \Phi_F]$. Let Q_1, Q_2 be two solutions. Then,

$$\frac{d}{d\tau} \int_0^1 (P(Q_1) - P(Q_2))^2 d\eta = 2 \int_0^1 P(Q_1 - Q_2) P(K(Q_1) - K(Q_2)) d\eta.$$

In particular, if $K(Q) = kQ + b$ is affine,

$$\|PQ_1(\tau) - PQ_2(\tau)\|_{L^2(0,1)} = e^{k\tau} \|PQ_1(0) - PQ_2(0)\|_{L^2(0,1)}.$$

Killing projector

In the L^1 and L^2 result, the idea is the same : to eliminate the challenging term $1/N(\tau)$.
To remove it, we can apply any operator P, whose kernel contains constants :

- For the L^1 result, $P = \partial_\eta$.
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For any suitable linear operator and any C^1 function f,

$$\frac{d}{d\tau}\int_0^1 f(P(Q_1 - Q_2))d\eta = \int_0^1 f'(P(Q_1 - Q_2))P(K(Q_1) - K(Q_2))d\eta.$$

Explicit example of blowup

Choose $K(\phi) \equiv 1$, $\Phi_F = 1$; then

$$\partial_t \rho + \partial_\phi ((1 + N(t))\rho) = 0, \tag{5}$$

with

$$N(t) = \frac{\rho(t,1)}{1-\rho(t,1)}, \quad t > 0. \tag{6}$$

Choose initial datum

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With $d\tau = (1 + N(t))dt$ we obtain a constant speed transport

$$\partial_\tau \rho + \partial_\phi \rho = 0, . \tag{8}$$

Explicit example of blowup

Blow up at $\tau^* = 1$ and we can solve exactly

$$\begin{aligned} N(\tau) =& \frac{\rho(\tau,1)}{1-\rho(\tau,1)} = \frac{\rho_{\text{init}}(1-\tau)}{1-\rho_{\text{init}}(1-\tau)} = \frac{\rho_{\text{init}}(\tau^*-\tau)}{1-\rho_{\text{init}}(0,\tau^*-\tau)} \\ =& \frac{1-(\tau^*-\tau)^k}{(\tau^*-\tau)^k} = \frac{1}{(\tau^*-\tau)^k} - 1. \end{aligned}$$

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Hence the blow-up time in the original timescale is given by

$$T^* = \int_0^{\tau^*} \frac{1}{N+1} d\tau = \int_0^{\tau^*} (\tau^*-\tau)^k d\tau = \frac{1}{k+1}.$$

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In general

$$t = \int_0^{\tau} (\tau^* - \tilde{\tau})^k d\tilde{\tau}.$$

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In general

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Then we conclude

$$N(t) = \frac{1}{(k+1)^{\frac{k}{k+1}}} \frac{1}{(T^*-t)^{\frac{k}{k+1}}} - 1, \quad 0 \le t < T^* = \frac{1}{k+1}.$$

Towards generalized solutions

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with the constraint to ensure that $0 < N(\tau) < +\infty$

$$K(\Phi_F) < \partial_\eta Q(\tau,1) < +\infty.$$

Future investigations :

1. Generalized solutions past the first blow-up time.
2. Unification between the particle system and the mean-field system.
3. Rigorous mean-field limit.
4. Better study of the blow-up phenomenon
5. Rigorous link to the voltage-conductance model for neurons.

Thank you !