Mean Field Games with jumps

Annette Dumas
Joint work with Filippo Santambrogio

Institut Camille Jordan - Université Claude Bernard Lyon 1

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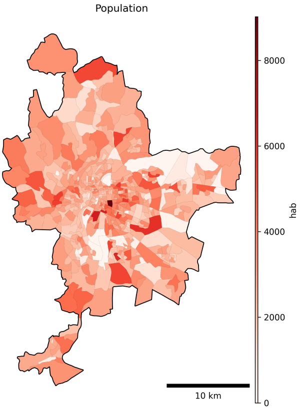

Figure – Carte de la population dans la métropole de Lyon.

[1] *Source : https ://datalyon.denoux.eu/*

Nash equilibrium with N players

- $\alpha_i \in \mathcal{S}_i$ = strategy of the player i,
- $J_i(\alpha_1, \ldots, \alpha_N)$ = cost of the player i to minimize

Definition

A *Nash equilibrium* is N-tuple $(\bar{\alpha}_1, \ldots, \bar{\alpha}_N)$ such that for all i,

$$\forall \alpha_i, J_i(\bar{\alpha}_1, \ldots, \bar{\alpha}_N) \leq J_i(\bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1}, \alpha_i, \bar{\alpha}_{i+1}, \cdots, \bar{\alpha}_N)$$

Theorem (Nash, 1951)

There exists at least one Nash equilibrium with mixed strategies.

$N \to +\infty$

- $\alpha_i \in \Omega \subset \mathbb{R}^d$ = strategy of the player i,
- $J\left(\alpha_i, \frac{1}{N-1}\sum_{j\neq i}\delta_{\alpha_j}\right)$ = cost to minimize
- $\rho_N = \frac{1}{N-1}\sum_{j\neq i}\delta_{\alpha_j} \in \mathcal{P}(\Omega)$ = probability measure over Ω

Theorem (Notes on Mean Field Games by Cardaliaguet, 2012)

If $(\bar\alpha_1, \dots, \bar\alpha_N)$ is a Nash equilibrium of the game, then up to a subsequence, $\bar\rho_N = \frac{1}{N-1}\sum_{j\neq i}\delta_{\bar\alpha_j}$ converges towards a measure $\bar\rho$ such that

$$\int_\Omega J(x,\bar\rho)d\bar\rho(x) = \inf_{\rho\in\mathcal{P}(\Omega)} \int_\Omega J(x,\bar\rho)d\rho(x)$$

$$i.e., \ \forall\rho\in\mathcal{P}(\Omega), \ \int_\Omega J(x,\bar\rho)d\bar\rho(x) \le \int_\Omega J(x,\bar\rho)d\rho(x)$$

Nash equilibrium with a continuum of players

- $\Omega \subset \mathbb{R}^d$ = city
- $\gamma \colon [0, T] \to \Omega$ = strategy of a player and $\gamma \in \mathcal{C}$
- $Q \in \mathcal{P}(\mathcal{C})$ = probability measure over the curves
- $J_Q(\gamma)$ = cost to minimize

Definition

A Nash equilibrium for Mean Field Games is a measure $\bar{Q}$ such that

$$\forall Q \in \mathcal{P}(\mathcal{C}), \ \int_{\mathcal{P}(\mathcal{C})} J_{\bar{Q}}(\gamma) d\bar{Q}(\gamma) \leq \int_{\mathcal{P}(\mathcal{C})} J_{\bar{Q}}(\gamma) dQ(\gamma)$$

Our model

$$\min_{\substack{\gamma \in BV([0,T],\Omega) \\ \gamma(0)=x_0}} J_Q(\gamma) := \underbrace{S(\gamma)}_{\text{number of jumps}} + \int_0^T \underbrace{\frac{dI}{dm}(e_t \sharp Q)(\gamma(t))}_{I \text{ admits a first variation}} \, dt$$

$$+ \int_0^T \underbrace{F(\gamma(t), e_t \sharp Q)}_{\in C(\Omega, \mathcal{P}(\Omega))} dt + \underbrace{\varphi_T(\gamma(T))}_{\text{penalization at time } T}$$

Our model

$$\min_{\substack{\gamma\in BV([0,T],\Omega)\\ \gamma(0)=x_0}} S(\gamma)+\int_0^T \frac{dI}{dm}(e_t\sharp Q)(\gamma(t))dt+\int_0^T F(\gamma(t),e_t\sharp Q)dt+\varphi_T(\gamma(T))$$

- $S(\gamma)=\begin{cases}\inf\limits_{\bar\gamma \overset{L^1}{=}\gamma}\sharp\{\text{discontinuity points of }\bar\gamma\}, & \text{if }\gamma\text{ is piecewise constant},\\ +\infty, & \text{otherwise.}\end{cases}$
- I admits a *first variation* in m, i.e. $\frac{dI}{dm}$ verifies

$$\frac{d}{d\varepsilon}I(m+\varepsilon\chi)_{|\varepsilon=0}=\int\frac{dI}{dm}(m)d\chi$$

- $e_t\sharp Q$ is the *push-forward measure* of Q by the function e_t where $e_t(\gamma)=\gamma(t)$ and

$$\forall\varphi\in C_b(\Omega),\ \int_\Omega\varphi(x)d(e_t\sharp Q)(x)=\int_{\mathcal{C}}\varphi(\gamma(t))dQ(\gamma)$$

Our model

A Nash equilibrium $\bar{Q}$ verifies

$$\forall Q \in \mathcal{P}(\mathcal{C}),\ \int_{\mathcal{P}(\mathcal{C})} J_{\bar{Q}}(\gamma) d\bar{Q}(\gamma) \leq \int_{\mathcal{P}(\mathcal{C})} J_{\bar{Q}}(\gamma) dQ(\gamma),$$

namely, $\forall Q \in \mathcal{P}(\mathcal{C})$,

$$\int_{\mathcal{C}} \left[S(\gamma) + \int_0^T \frac{dI}{dm}(e_t \sharp \bar{Q})(\gamma(t)) + F(\gamma(t), e_t \sharp \bar{Q}) dt + \varphi(\gamma(T)) \right] d\bar{Q}(\gamma)$$

$$\leq \int_{\mathcal{C}} \left[S(\gamma) + \int_0^T \underbrace{\frac{dI}{dm}(e_t \sharp \bar{Q})(\gamma(t))}_{\text{variational}} + \underbrace{F(\gamma(t), e_t \sharp \bar{Q})}_{\text{non-variational}} dt + \varphi(\gamma(T)) \right] dQ(\gamma)$$

Formulation of the problem

$$\int_{\mathcal{C}} \left[S(\gamma) + \int_0^T \frac{dI}{dm}(e_t \sharp \bar{Q})(\gamma(t)) + F(\gamma(t), e_t \sharp \bar{Q}) dt + \varphi(\gamma(T)) \right] d\bar{Q}(\gamma)$$
$$\leq \int_{\mathcal{C}} \left[S(\gamma) + \int_0^T \frac{dI}{dm}(e_t \sharp \bar{Q})(\gamma(t)) + F(\gamma(t), e_t \sharp \bar{Q}) dt + \varphi(\gamma(T)) \right] dQ(\gamma)$$

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$$\leq \int_{\mathcal{C}} \left[S(\gamma) + \int_0^T \frac{dI}{dm}(e_t\sharp\bar{Q})(\gamma(t)) + F(\gamma(t), e_t\sharp \tilde{Q})dt + \varphi(\gamma(T)) \right] dQ(\gamma)$$

so we define

$$\mathcal{U}_{\tilde{Q}}(Q) := \int S(\gamma)dQ(\gamma) + \int I(e_t\sharp Q)dt + \iint F(\gamma(t), e_t\sharp \tilde{Q})dt \; dQ(\gamma)$$

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$$\int_{\mathcal{C}} \left[S(\gamma) + \int_0^T \frac{dI}{dm}(e_t\sharp\bar{Q})(\gamma(t)) + F(\gamma(t), e_t\sharp\, \tilde{Q}\,)dt + \varphi(\gamma(T)) \right] d\bar{Q}(\gamma)$$
$$\leq \int_{\mathcal{C}} \left[S(\gamma) + \int_0^T \frac{dI}{dm}(e_t\sharp\bar{Q})(\gamma(t)) + F(\gamma(t), e_t\sharp\, \tilde{Q}\,)dt + \varphi(\gamma(T)) \right] dQ(\gamma)$$

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and we look for $\bar{Q}$ such that

$$\bar{Q} \in \mathrm{argmin}_Q\, \mathcal{U}_{\bar{Q}}(Q)$$

Formulation of the problem

Indeed, if $\bar{Q} \in \text{argmin}_Q \mathcal{U}_{\bar{Q}}(Q)$, then we define

$$\forall \varepsilon \in [0,1], \forall Q, \ \ Q_\varepsilon := \bar{Q} + \varepsilon(Q - \bar{Q})$$

$$\text{and } u(\varepsilon) := \mathcal{U}_{\bar{Q}}(Q_\varepsilon) = \int S(\gamma) dQ_\varepsilon(\gamma) + \int I(e_t \sharp Q_\varepsilon) dt$$

$$+ \iint F(\gamma(t), e_t \sharp \bar{Q}) dt \ dQ_\varepsilon(\gamma)$$

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$$\text{so that } 0 \leq u'(0) = \int S(\gamma) d(Q - \bar{Q})(\gamma) + \int \frac{dI}{dm}(e_t \sharp \bar{Q}) d(Q - \bar{Q})(\gamma)$$

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$$+ \iint F(\gamma(t), e_t \sharp \bar{Q}) dt \; d(Q - \bar{Q})(\gamma)$$

Beware !

This is true if $\iint I(e_t \sharp Q) dx dt < \infty$.

Existence of a Nash equilibrium

We look for a fixed point of the multivalued function

$$\begin{aligned} \mathcal{F}\colon \mathcal{P}_{m_0}(\mathrm{BV}) &\longrightarrow P(\mathcal{P}_{m_0}(\mathrm{BV})) \\ \tilde{Q} &\longmapsto \underset{Q,\, e_0\sharp Q = m_0}{\operatorname{argmin}} \ \mathcal{U}_{\tilde{Q}}(Q) \end{aligned}$$

where $\mathcal{U}_{\tilde{Q}}(Q) = \int S(\gamma)dQ(\gamma) + \int I(e_t\sharp Q)dt + \iint F(\gamma(t), e_t\sharp \tilde{Q})dt\ dQ(\gamma)$

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where $\mathcal{U}_{\tilde{Q}}(Q) = \int S(\gamma) dQ(\gamma) + \int I(e_t \sharp Q) dt + \iint F(\gamma(t), e_t \sharp \tilde{Q}) dt \; dQ(\gamma)$

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and $\Gamma = \{Q; \int S(\gamma)dQ(\gamma) \leq C\}$

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where $\mathcal{U}_{\tilde{Q}}(Q) = \int S(\gamma) dQ(\gamma) + \int I(e_t \sharp Q) dt + \iint F(\gamma(t), e_t \sharp \tilde{Q}) dt \; dQ(\gamma)$
and $\Gamma = \{Q; \int S(\gamma) dQ(\gamma) \leq C\}$
$\longrightarrow$ by Kakutani's theorem, a fixed point $\bar{Q}$ exists.

From Lagrangian to Eulerian

Lagrangian : $Q \in \mathcal{P}(\mathrm{BV}([0,T],\Omega))$

Eulerian : for each $t \in [0,T]$, $\rho(t) := e_t \sharp Q \in \mathcal{P}(\Omega)$, so $\rho\colon [0,T] \to \mathcal{P}(\Omega)$

Reminder of an optimal transport result

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{\Omega \times \Omega} \mathbb{1}_{x \neq y} \, d\pi(x,y) = \frac{1}{2} \left\| \mu - \nu \right\|_{TV}$$

Reminder of an optimal transport result

$$\inf_{\pi\in\Pi(\mu,\nu)} \int_{\Omega\times\Omega} \mathbb{1}_{x\neq y}\, d\pi(x,y) = \frac{1}{2}\,\|\mu-\nu\|_{TV}$$

This implies **formally** that

$$\begin{aligned}
\int_{\mathrm{BV}} S(\gamma)dQ(\gamma) &= \int_{\mathrm{BV}} \sup_{(t_i)_i} \sum_i \mathbb{1}_{\gamma(t_i)\neq\gamma(t_{i+1})} dQ(\gamma) \\
&= \sup_{(t_i)_i} \sum_i \int_{\mathrm{BV}} \mathbb{1}_{\gamma(t_i)\neq\gamma(t_{i+1})} dQ(\gamma) \\
&= \sup_{(t_i)_i} \sum_i \frac{1}{2} \|\rho(t_{i+1})-\rho(t_i)\|_{TV} \\
&= \frac{1}{2}\,\mathrm{length}(\rho) = \frac{1}{2}\int_0^T |\dot\rho(t)|dt
\end{aligned}$$

From Lagrangian to Eulerian

Let us take $F = 0$ and $I(e_t\sharp Q) = I(\rho) = V \cdot \rho + f(\rho)$, where $V\colon [0, T] \times \Omega \to \mathbb{R}$ is a given function.

Theorem

$$\min_Q \int S(\gamma)dQ(\gamma) + \int_0^T \int_\Omega [V \cdot (e_t\sharp Q) + f(e_t\sharp Q)] + \int \varphi_T(\rho(T))dQ(\gamma) \quad \text{(L)}$$

$$= \min_{\substack{\rho\in E, \rho\geq 0 \\ \forall t\in[0,T], \int_\Omega \rho(t,x)dx=1}} \int_0^T \int_\Omega \left[\frac{1}{2}|\dot\rho| + V\rho + f(\rho)\right] + \psi_0(\rho(0)) + \psi_T(\rho(T)) \quad \text{(E)}$$

In addition, if $\bar\rho$ minimizes (E), then there exists $\bar Q$ such that $e_t\sharp\bar Q = \bar\rho(t)$ and $\bar Q$ minimizes (L).

Lipschitz regularity in time

$$\min_{\substack{\rho \in E, \rho \geq 0 \\ \forall t \in [0,T], \int_\Omega \rho(t,x)dx = 1}} \int_0^T \int_\Omega |\dot{\rho}| + V\rho + f(\rho) dx dt + \psi_0(\rho(0)) + \psi_T(\rho(T)), \quad \text{(PB)}$$

where $E = \mathrm{BV}([0, T], L^1(\Omega)) \cap L^2([0, T] \times \Omega)$.

Lipschitz regularity in time

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where $E = \mathrm{BV}([0,T], L^1(\Omega)) \cap L^2([0,T]\times\Omega)$.

Theorem

- If $f : \mathbb{R} \to \mathbb{R}$ is c_0-convex, i.e $f'' \geq c_0$
- $V : [0,T]\times\Omega \to \mathbb{R}$ is such that $\sup_{t\in[0,T]} \|V'(t,\cdot)\|_{L^2(\Omega)} < \infty$
- ψ_0 and ψ_T are 1-Lipschitz on $(L^1(\Omega), \|\cdot\|_{L^1(\Omega)})$ and weakly l.s.c.

then there exists a unique minimizer ρ of (PB) and it verifies

$$\sup_{t\in[0,T]} \int_\Omega |\dot\rho(t,x)|^2 dx \leq C$$

where $C = \frac{C_0^2}{c_0^2}$ with $C_0^2 = \sup_{t\in[0,T]} \|V'(t,\cdot)\|^2_{L^2(\Omega)}$.

Lipschitz regularity in time

On the hypothesis of ψ_0 and ψ_T :

- The Dirichlet condition $\rho(0) = \rho_0$ can be replaced by a penalization $\psi_0(\rho(0^+)) = \|\rho(0^+) - \rho_0\|_{L^1}$.

Lipschitz regularity in time

On the hypothesis of ψ_0 and ψ_T :

- The Dirichlet condition $\rho(0) = \rho_0$ can be replaced by a penalization $\psi_0(\rho(0^+)) = \|\rho(0^+) - \rho_0\|_{L^1}$.
- The penalization $\int \varphi_T d\rho_T$ can be replaced by $\psi_T(\rho(T)) = \inf_{\mu \in \mathcal{P}(\Omega)} \|\mu - \rho(T)\| + \int \varphi_T d\rho(T)$.

Lipschitz regularity in time in infinite horizon

$$\min_{\substack{\rho\in E,\rho\geq 0\\ \forall t\in[0,+\infty[,\int_\Omega \rho(t,x)dx=1}} \int_0^{+\infty}\int_\Omega e^{-rt}(|\dot\rho|+V\rho+f(\rho))dxdt+\psi_0(\rho(0))+\cancel{\psi_T(\rho(T))}, \tag{PB}$$

where $E=\mathrm{BV}([0,T],L^1(\Omega))\cap L^2([0,T]\times\Omega)$.

Theorem

- If ψ_0 is 1-Lipschitz on $(L^1(\Omega),\|\cdot\|_{L^1(\Omega)})$ and weakly l.s.c.
- $V:[0,+\infty[\times\Omega\to\mathbb{R}$ is such that $\sup_{t\in[0,+\infty[}\|V'(t,\cdot)\|_{L^2(\Omega)}<\infty$
- $f:\mathbb{R}\to\mathbb{R}$ is c_0-convex, i.e $f''\geq c_0$

then there exists a unique minimizer ρ of (PB) and it verifies

$$\sup_{t\in[0,+\infty[}\int_\Omega|\dot\rho(t,x)|^2dx\leq C$$

where $C>0$ depends on V et c_0.

Regularity in space

Let us consider

$$\min_{\substack{\rho\in E,\rho\geq 0\\ \rho(0)=a,\rho(T)=b\\ \forall t\in[0,T],\int_\Omega \rho(t,x)dx=1}} \int_0^T \int_\Omega |\dot{\rho}| + V\rho + f(\rho)dxdt \qquad \text{(PB)}$$

Theorem

If for all x,x',

- $a(x) - a(x') \leq \omega(x,x')$,
- $b(x) - b(x') \leq \omega(x,x')$,
- $\sup_t V(t,x) - V(t,x') \leq \omega(x,x')$,

then the solution ρ of (PB) verifies

$$\text{for a.e } t,x,x',\ \rho(t,x) - \rho(t,x') \leq \omega(x,x').$$

Simulations

$$\min_{\substack{\rho\in E,\rho\geq 0\\ \rho(0)=a,\rho(T)=b\\ \forall t\in[0,T],\int_\Omega \rho(t,x)dx=1}} \int_0^T\int_\Omega \left(\lambda|\dot{\rho}|+V\rho+\frac{\rho^2}{2}\right)+\psi_0(\rho(0))+\psi_T(\rho(T))$$

Simulations

$$\min_{\substack{\rho\in E,\rho\geq 0\\ \rho(0)=a,\,\rho(T)=b\\ \forall t\in[0,T],\int_\Omega \rho(t,x)dx=1}} \int_0^T\int_\Omega \left(\lambda|\dot{\rho}| + V\rho + \frac{\rho^2}{2}\right) + \psi_0(\rho(0)) + \psi_T(\rho(T))$$

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Simulations

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$$\longrightarrow \min_\rho g(\mathcal{A}\rho) + f(\rho)$$

where $\mathcal{A}\rho = (A\rho, \rho, \rho)$ and $g(\rho_1,\rho_2,\rho_3) = \iint |\dot\rho_1| + \delta_{\geq 0}(\rho_2) + \delta_{\int \rho=1}(\rho_3)$,

where $\delta_C(x) = \begin{cases} 0 & \text{if } x\in C, \\ +\infty & \text{otherwise.}\end{cases}$

We use the dual proximal gradient method[1].

[1] *Ref. : Beck, First-Order Methods in Optimization. 2017*

On the proximal gradient method

At step k :

- Gradient descent method :
  $\rightarrow x_{k+1} = x_k - t_k \nabla f(x_k)$.

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On the proximal gradient method

At step k :

- Gradient descent method :
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- Projected gradient descent method :
  $\rightarrow x_{k+1} = P_C(x_k - t_k \nabla f(x_k))$.

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- Proximal gradient method :
  $\rightarrow x_{k+1} = \mathrm{prox}_{t_k g}(x_k - t_k \nabla f(x_k))$,
  where $\mathrm{prox}_g(x) = \mathrm{argmin}_u \{g(u) + \frac{1}{2}\|u - x\|^2\}$.

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On the proximal gradient method

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  where $\mathrm{prox}_g(x) = \mathrm{argmin}_u \{g(u) + \frac{1}{2}\|u - x\|^2\}$.
- Dual Proximal Gradient method :
  $\rightarrow x_{k+1} = \mathrm{prox}_{\frac{1}{L} G}(x_k - \frac{1}{L} \nabla F(x_k))$
  where $F(y) = f^*(\mathcal{A}^T y)$ and $G(y) = g^*(-y)$.

[1] *Ref. : Beck, First-Order Methods in Optimization. 2017*

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Example : $V(t,x) = a_0 \cos(\frac{2\pi}{T}(t-x))$, then $\bar{\rho} = \rho(t-x)$.

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By the change of variable $y = t - x$, we have

$$\min_{\substack{\rho \geq 0 \\ \int \rho = 1}} \int_0^T \left(|\lambda \dot{\rho}(y)| + V(y)\rho(y) + \frac{\rho(y)^2}{2} \right) dy$$

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The Euler-Lagrange equation is

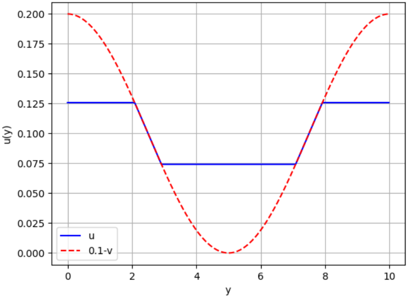
$$z' = V + \rho - c,$$

where $z(y) \in \partial(\lambda|\cdot|)(\dot{\rho}(y))$ and c is a Lagrange multiplier due to the mass constraint. In this example, $c = \frac{1}{T}$.

Solution of

$$\min_{\substack{\rho \geq 0 \\ \int \rho = 1}} \int_0^T \left(|\lambda \dot{\rho}(y)| + V(y)\rho(y) + \frac{\rho(y)^2}{2} \right) dy$$

with $V(y) = 0.1 \times \cos(\frac{2\pi}{T}(y))$:



Other examples of solutions

$$\min_{\substack{\rho\in E,\rho\geq 0\\ \rho(0)=a,\rho(T)=b\\ \forall t\in[0,T],\int_\Omega \rho(t,x)dx=1}} \int_0^T \int_\Omega \left(\lambda|\dot{\rho}| + V\rho + \frac{\rho^2}{2}\right) + \psi_0(\rho(0)) + \psi_T(\rho(T))$$

Thank you for your attention !