KINEMATICS

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AE 483

Fall 2021

EQUATIONS OF MOTION

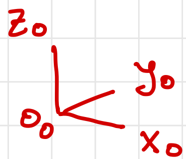
$$\text{KINEMATICS}\left\{\begin{aligned} \dot{o}_1^0 &= R_1^0 \, v_{0,1}^1 \\ \begin{bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} &= N \, \omega_{0,1}^1 \end{aligned}\right.$$

$$\text{DYNAMICS}\left\{\begin{aligned} f^1 &= m\dot{v}_{0,1}^1 + \widehat{\omega_{0,1}^1} \, m v_{0,1}^1 \\ \tau^1 &= J^1 \dot{\omega}_{0,1}^1 + \widehat{\omega_{0,1}^1} \, J^1 \omega_{0,1}^1 \end{aligned}\right.$$

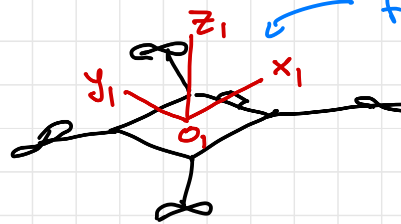
our goal is to understand what these mean and where they come from

Assume the drone is a rigid body.

The position and orientation of a rigid body is the same as the position and orientation of a frame attached to the rigid body.



frame 0 (world frame)



frame 1 (body frame)

Position is the location of the point o_1 written in the coordinates of frame 0 $\Big\}$ o_1^0 $\leftarrow$ 3×1 matrix

Orientation is the direction of the unit vectors x_1, y_1, z_1 written in the coordinates of frame 0 $\Big\}$ $R_1^0 = \underbrace{\begin{bmatrix} \underset{\downarrow}{\overset{3\times 1}{}}\; x_1^0 & \underset{\downarrow}{\overset{3\times 1}{}}\; y_1^0 & \underset{\downarrow}{\overset{3\times 1}{}}\; z_1^0 \end{bmatrix}}_{3\times 3 \text{ matrix}}$

PROPERTIES OF A ROTATION MATRIX

$$(R_1^0)^T = (R_1^0)^{-1} = R_0^1 \qquad \det(R_1^0) = 1$$

COORDINATE TRANSFORMATION (VECTORS)

$$v^0 = R_1^0 v^1$$

COORDINATE TRANSFORMATION (POINTS)

$$p^0 = o_1^0 + R_1^0 p^1$$

SEQUENTIAL TRANSFORMATION

$$R_2^0 = R_1^0 R_2^1$$

$$o_2^0 = o_1^0 + R_1^0 o_2^1$$

CONSTRAINTS ON ~~PROPERTIES OF~~ A ROTATION MATRIX

$$(R_1^0)^T = (R_1^0)^{-1} = R_0^1$$

columns must describe orthogonal unit vectors

$$\det(R_1^0) = 1$$

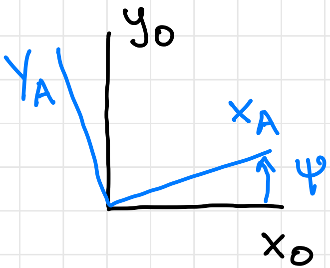
these unit vectors must satisfy the right-hand rule

R_1^0 has 9 numbers, but only 3 of these numbers can be chosen independently

EULER ANGLES (YAW, PITCH, ROLL)

$O \to A$

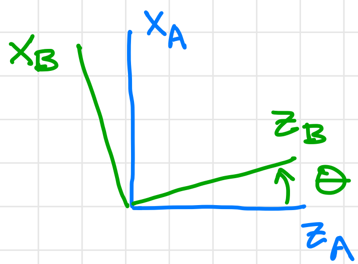
rotate by ψ (psi)
about z_O



$$R_A^O = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A \to B$

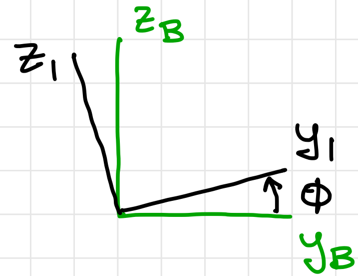
rotate by θ (theta)
about y_A



$$R_B^A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$B \to 1$

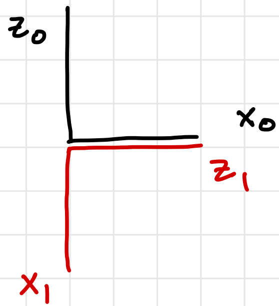
rotate by ϕ (phi)
about x_B



$$R_1^B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$R_1^O = R_A^O R_B^A R_1^B = R_z(\psi) R_y(\theta) R_x(\phi)$$

REVIEW



$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \leftarrow$$

① First, do this

Find two <u>different</u> sets of yaw, pitch, and roll angles that would have produced this rotation matrix

② In the time remaining, answer these questions

Does <u>any</u> choice of ψ, θ, ϕ produce a valid R_1^0?

Do different ψ, θ, ϕ always produce different R_1^0?

Does there always exist <u>some</u> ψ, θ, ϕ that produces a given R_1^0?

LINEAR VELOCITY

a vector whose direction is the axis along which frame 1 is translating at this instant and whose magnitude is the speed

$$\dot{o}_1^0 = v_{0,1}^0$$

the time derivative of o_1^0 (3×1 matrix)

linear velocity

of frame 1

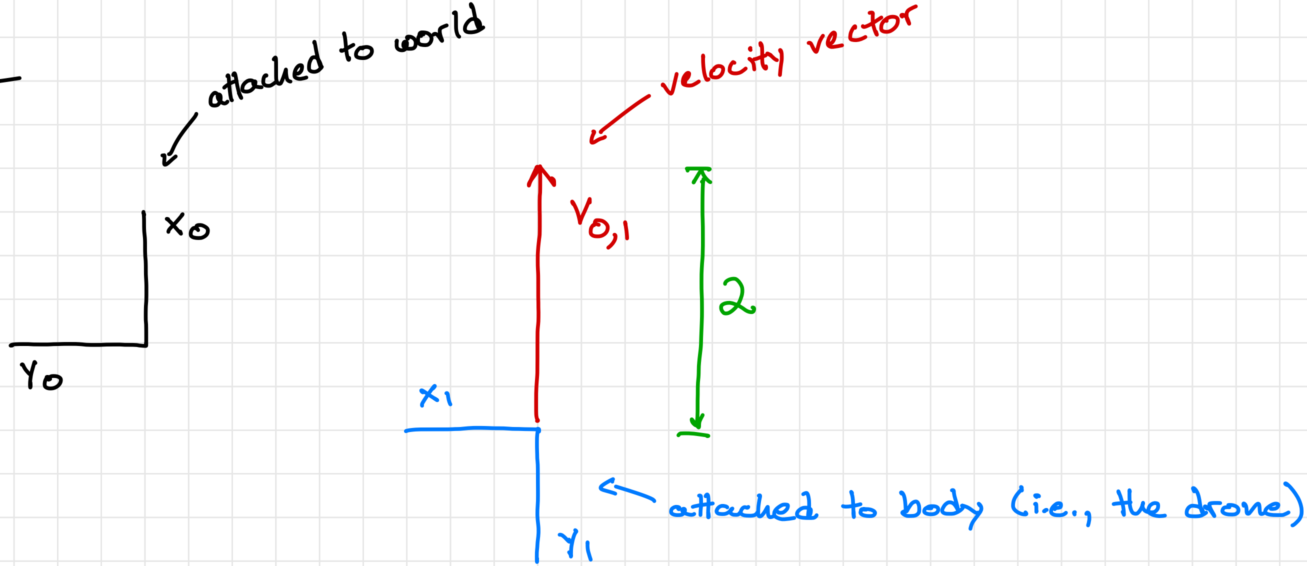
with respect to frame 0

written in the coordinates of frame 0

$$= R_1^0 \underbrace{v_{0,1}^1}$$

the same thing written in the coordinates of frame 1

EXAMPLE



$$v^0_{0,1} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

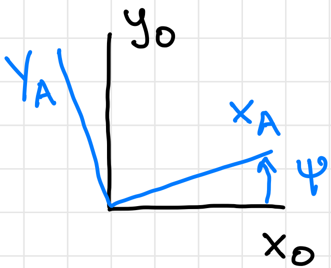
$$R^0_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

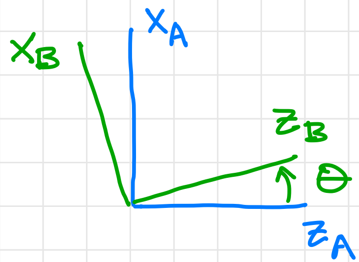
$$v^1_{0,1} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

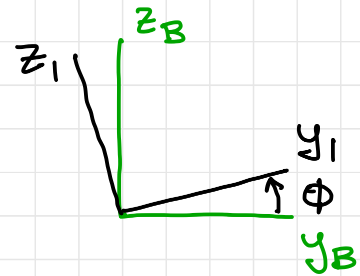
ANGULAR VELOCITY

a vector whose direction is the axis about which frame 1 is rotating at this instant and whose magnitude is the rate of rotation

break the rotation into pieces:







$$\omega^{0}_{0,A} = \omega^{A}_{0,A} = \begin{bmatrix}0\\0\\1\end{bmatrix}\dot{\psi} \qquad \omega^{A}_{A,B} = \omega^{B}_{A,B} = \begin{bmatrix}0\\1\\0\end{bmatrix}\dot{\theta} \qquad \omega^{B}_{B,1} = \omega^{1}_{B,1} = \begin{bmatrix}1\\0\\0\end{bmatrix}\dot{\phi}$$

vectors add: $\omega_{0,1} = \omega_{0,A} + \omega_{A,B} + \omega_{B,1}$

FROM ANGULAR VELOCITY TO ANGULAR RATES

$$\omega_{0,1} = \omega_{0,A} + \omega_{A,B} + \omega_{B,1}$$

← chop into pieces

$$\underbrace{\boxed{\omega^1_{0,1}}}_{\text{angular velocity}} = \omega^1_{0,A} + \omega^1_{A,B} + \omega^1_{B,1}$$

← write in coordinates

$$= R^1_A\, \omega^A_{0,A} + R^1_B\, \omega^B_{A,B} + \omega^1_{B,1}$$

$$= (R^A_1)^T \omega^A_{0,A} + (R^B_1)^T \omega^B_{A,B} + \omega^1_{B,1}$$

apply coordinate transform and inverse transform to rewrite in terms of what we know

$$= \left(R^A_B R^B_1\right)^T \begin{bmatrix}0\\0\\1\end{bmatrix}\dot{\psi} + (R^B_1)^T \begin{bmatrix}0\\1\\0\end{bmatrix}\dot{\theta} + \begin{bmatrix}1\\0\\0\end{bmatrix}\dot{\phi}$$

← plug in what we know

$$= \begin{bmatrix} \underbrace{(R^A_B R^B_1)^T \begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{first column}} & \underbrace{(R^B_1)^T \begin{bmatrix}0\\1\\0\end{bmatrix}}_{\text{second column}} & \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\text{third column}} \end{bmatrix} \underbrace{\boxed{\begin{bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix}}}_{\text{angular rates}}$$

call this N^{-1}

← collect terms and write as matrix multiplication

REVIEW

① First, do this

Suppose frames 0 and 1 are initially aligned and that both linear and angular velocity are constant when written in the coordinates of frame 1:

$$v^1_{0,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad w^1_{0,1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

What does the drone do? Draw the path it follows.

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② Then, in the time remaining, answer these questions

If the angular velocity is <u>constant</u>, are the angular rates also constant?

Is it always possible to go back and forth between angular velocity and angular rates?

EQUATIONS OF MOTION

KINEMATICS

$$\dot{o}_1^0 = R_1^0 v_{0,1}^1$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = N \omega_{0,1}^1$$

← We understand kinematics now

DYNAMICS

$$f^1 = m\dot{v}_{0,1}^1 + \widehat{\omega_{0,1}^1}\, m v_{0,1}^1$$

$$\tau^1 = J^1 \dot{\omega}_{0,1}^1 + \widehat{\omega_{0,1}^1}\, J^1 \omega_{0,1}^1$$

← Next, we will look at dynamics