Lab 5 : Theory

T. Bretl

AE 483

Fall 2021

This is the so-called linear quadratic regulator (LQR) problem:

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \int_0^\infty \left(x^T Q x + u^T R u \right) dt \\ \text{subject to} \quad & \dot{x} = Ax + Bu, \quad x(0) = x_0. \end{aligned}$$

The optimal solution is just linear state feedback

$$u = -Kx$$

for a special choice of K that is easily computed (see the helper function lqr in your analysis notebook.

This optimal choice of K depends on the weights Q and R, which "penalize" (or impose a quadratic "cost" on) non-zero states x and inputs u.

So, to get a good controller, you need to make a good choice of the weights.

What is a good choice of weights? "Bryson's rule" is often the right place to start. This rule says

$$Q = \operatorname{diag}\left(\left(\frac{1}{x_{1,max}}\right)^2, \ldots, \left(\frac{1}{x_{n,max}}\right)^2 \right) \quad \leftarrow Q \text{ is diagonal}$$

$$R = \operatorname{diag}\left(\left(\frac{1}{u_{1,max}}\right)^2, \ldots, \left(\frac{1}{u_{m,max}}\right)^2 \right) \quad \leftarrow R \text{ is diagonal}$$

where

$x_{i,max}$ is largest value of x_i (the ith element of the state) that you would ever want to see

and

$u_{i,max}$ is the largest value of u_i (the ith element of the input) that you would ever want to see.

To get started, let's ignore Q (take it as idenity) and focus on R. Recall that:

$$u = \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z - mg \end{bmatrix}$$

How might we determine the maximum value of each element of u?

Remember that

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = P \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

for the matrix P that is derived in your analysis notebook.

Do the following:

1. Find the motor commands that correspond to $u = 0$ — that is, to $\tau_x = \tau_y = \tau_z = 0$ and $f_z = mg$.

2. Find the maximum change to each motor command that is possible without exceeding limits, i.e., while staying in the range $[0, 65535]$.

   For example, if you found that $m_1 = m_2 = m_3 = m_4 = 45000$, then the max change would be 20535.

3. Since all four inputs $(\tau_x, \tau_y, \tau_z, f_z)$ might need to change at once, and since it might be neccessary to change all motor commands to achieve a change in any one of these four inputs, divide the "max possible change" to each motor command that you found in step (2) by four to approximate the max this motor command can change in response to changing just one input.

   Continuing our example, this max would be $20535/4 \approx 5000$.

(4) Find the max value of each input (i.e., the max change in τ_x, τ_y, τ_z, and f_z that can be achieved by changing each motor command the amount you found in step (3).

Continuing our example, let's look at τ_x. Notice that

$$\tau_x = -(\ell k_F)m_1 - (\ell k_F)m_2 + (\ell k_F)m_3 + (\ell k_F)m_4 \quad \leftarrow \text{from the derivation of } P$$

$$= \ell k_F\,(\underbrace{-m_1 - m_2}_{\text{negative}} \underbrace{+ m_3 + m_4}_{\text{positive}})$$

If $m_1, m_2, m_3, m_4 \in [-5000, 5000]$, then τ_x is maximized by:

$$\underbrace{m_1 = -5000 \quad m_2 = -5000}_{\text{negative}} \quad \underbrace{m_3 = 5000 \quad m_4 = 5000}_{\text{positive}}$$

In particular:

$$u_{1,max} = \ell k_F(-(-5000) - (-5000) + 5000 + 5000)$$
$$= 4\ell k_F(5000)$$
$$= 20000\,\ell k_F$$

cont. →

So, as the first diagonal entry r_1 of R, I would choose:

$$r_1 = \left(\frac{1}{u_{1,max}}\right)^2 = \left(\frac{1}{20000\, \ell k_F}\right)^2$$

I would do a similar calculation to find r_2, r_3, and r_4.

You should find that this rule tells us to use <u>really</u> big numbers for the diagonal entries of R! This is why the default choice of

$$R = I \quad \leftarrow \text{identity matrix}$$

does <u>not</u> produce a working controller. (Most likely.)