Lab 4 : Theory

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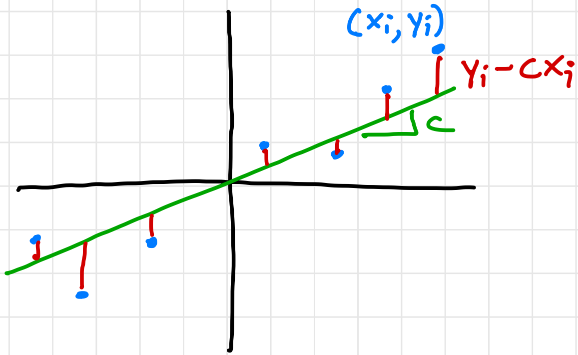
LINEAR REGRESSION (1/3)

Suppose we want to fit a linear model

$$y = c\,x$$

to pairs of scalar measurements

$$(x_1, y_1), \ldots, (x_n, y_n).$$



That is to say, suppose we want to choose a value of c for which

$$y_1 = c\,x_1 \qquad \cdots \qquad y_n = c\,x_n$$

Since measurements are never perfect, these n equations cannot all be satisfied exactly by <u>any</u> choice of c. So instead, we choose c to minimize the sum squared error:

$$\underset{c}{\text{minimize}} \quad \sum_{i=1}^{n} (y_i - c\,x_i)^2$$

LINEAR REGRESSION (2/3)

$$\underset{c}{\text{minimize}} \boxed{\sum_{i=1}^{n} (y_i - cx_i)^2}$$

call this $f(c)$ — it is a scalar function of a single variable

Apply the first derivative test to find the minimum:

$$0 = \frac{\partial f}{\partial c} = \sum_{i=1}^{n} \frac{\partial}{\partial c}\left((y_i - cx_i)^2\right)$$

$$= \sum_{i=1}^{n} 2(y_i - cx_i)(-x_i)$$

$$= -2\left(\sum_{i=1}^{n} \left((y_i x_i) - c(x_i^2)\right)\right)$$

$$= -2\left(\left(\sum_{i=1}^{n} y_i x_i\right) - c\left(\sum_{i=1}^{n} x_i^2\right)\right)$$

The solution is:

$$\boxed{c = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}}$$

LINEAR REGRESSION (3/3)

Apply the second derivative test to check if this is really a minimum:

$$f(c) = \sum_{i=1}^{n} (y_i - c x_i)^2$$

$$\frac{\partial f}{\partial c} = -2\left(\left(\sum_{i=1}^{n} y_i x_i\right) - c\left(\sum_{i=1}^{n} x_i^2\right)\right)$$

$$\frac{\partial^2 f}{\partial c^2} = 2\left(\sum_{i=1}^{n} x_i^2\right)$$

← This is a sum of squares, so it must be positive (unless all $x_i = 0$, in which case $f(c)$ does not have a unique minimum) — so, the solution to

$$0 = \frac{\partial f}{\partial c}$$

that we found is, indeed, a minimum.

FINITE DIFFERENCE APPROXIMATION (1/2)

Suppose $x(t)$ is a function of time. It's derivative with respect to time is

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

We can approximate this derivative by evaluating the right-hand side at some small Δt:

$$\dot{x}(t) \approx \frac{x(t+\Delta t) - x(t)}{\Delta t} \qquad \text{for } \Delta t \approx 0$$

This is called a finite difference approximation.

FINITE DIFFERENCE APPROXIMATION (2/2)

Suppose we have $n+1$ measurements of $x(t)$ that are sampled at a fixed rate (i.e., the difference in time between one measurement and the next is <u>always</u> Δt):

$$\underset{\substack{\uparrow\\ x_0}}{x(t_0)},\ \underset{\substack{\uparrow\\ x_1}}{x(t_0+\Delta t)},\ \underset{\substack{\uparrow\\ x_2}}{x(t_0+2\Delta t)},\ \dots,\ \underset{\substack{\uparrow\\ x_{n-1}}}{x(t_0+(n-1)\Delta t)},\ \underset{\substack{\uparrow\\ x_n}}{x(t_0+n\Delta t)}$$

We can then approximate the time derivative at each of these same times as

$$\underbrace{\left(\frac{x_1-x_0}{\Delta t}\right)}_{\approx\ \dot{x}(t_0)},\ \underbrace{\left(\frac{x_2-x_1}{\Delta t}\right)}_{\approx\ \dot{x}(t_0+\Delta t)},\ \underbrace{\left(\frac{x_3-x_2}{\Delta t}\right)}_{\approx\ \dot{x}(t_0+2\Delta t)},\ \dots,\ \underbrace{\left(\frac{x_n-x_{n-1}}{\Delta t}\right)}_{\approx\ \dot{x}(t_0+(n-1)\Delta t)}$$

Important note – we started with $n+1$ measurements of x but produced only n measurements of $\dot{x}$.

FORCE PARAMETER (1/2)

The following ODE (derived in class) describes the vertical motion of the drone:

$$\dot{v}_z = (1/m)\, f_z - g\cos\phi\cos\theta + v_x\omega_y - v_y\omega_x$$

The z-axis accelerometer measures specific force, which is defined as "the non-gravitational force per unit mass" on the drone along the body-fixed z axis. In short, this means "$\dot{v}_z$ without the gravity term":

$$a_z^{SF} = (1/m)\, f_z + v_x\omega_y - v_y\omega_x$$

If the drone is near hover — that is, if v_x, v_y, ω_x, and ω_y are all assumed small — then we can approximate this as

$$a_z^{SF} = (1/m)\, f_z$$

Plug in the expression for f_z in terms of motor power commands to get:

$$a_z^{SF} = (1/m)\, k_F (m_1 + m_2 + m_3 + m_4)$$

FORCE PARAMETER (2/2)

We found:

$$a_z^{SF} = (1/m)\, k_F (m_1 + m_2 + m_3 + m_4)$$

Rearrange this slightly:

$$m a_z^{SF} = k_F (m_1 + m_2 + m_3 + m_4)$$

Given measurements of $m_1 + m_2 + m_3 + m_4$ (the "x") and $m a_z^{SF}$ (the "y"), we can estimate k_F (the "c") by solving a linear regression problem ($y = cx$).

Here are two ways to validate your estimate:

1. Plot the raw data (y vs. x) along with your linear fit (the line described by $y = c_{est}\, x$).

2. Plot the actual accelerometer measurements (a_z^{SF} vs. t) along with your prediction of these measurements given your estimate of k_F ($k_F(m_1+m_2+m_3+m_4)/m$ vs. t).

MOMENT PARAMETER (1 / 2)

The following ODE (derived in class) describes the yaw motion of the drone:

$$\dot{\omega}_z = (1/J_z)\,\tau_z + \left(\frac{J_x - J_y}{J_z}\right)\omega_x \omega_y$$

If the drone is near hover — that is, if ω_x and ω_y are assumed small — then we can approximate this as:

$$\dot{\omega}_z = (1/J_z)\,\tau_z$$

Plug in the expression for τ_z in terms of motor power commands to get:

$$\dot{\omega}_z = (1/J_z)\,k_M(-m_1 + m_2 - m_3 + m_4)$$

Rearrange this slightly:

$$J_z \dot{\omega}_z = k_M(-m_1 + m_2 - m_3 + m_4)$$

MOMENT PARAMETER (2/2)

We found:

$$J_z \dot{\omega}_z = k_M(-m_1 + m_2 - m_3 + m_4)$$

Given measurements of $-m_1 + m_2 - m_3 + m_4$ (the "x") and $J_z \dot{\omega}_z$ (the "y"), where $\dot{\omega}_z$ is approximated using finite difference from measurements of ω_z, we can estimate k_M (the "c") by solving a linear regression problem ($y = cx$).

Here are two ways to validate your estimate:

① Plot the raw data (y vs. x) along with your linear fit (the line described by $y = c_{est}\,x$).

② Plot the estimate of $\dot{\omega}_z$ you got from finite difference ($\dot{\omega}_z$ vs t) along with your prediction of $\dot{\omega}_z$ given your estimate of k_M:

$$k_M(-m_1 + m_2 - m_3 + m_4)/J_z \quad \text{vs. } t$$