DYNAMICS

T. Bretl

AE 483

Fall 2021

EQUATIONS OF MOTION

KINEMATICS

$$\dot{o}_1^0 = R_1^0 \, v_{0,1}^1$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = N \, \omega_{0,1}^1$$

DYNAMICS

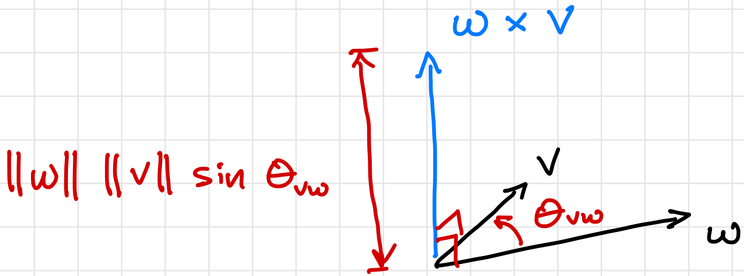
$$f^1 = m\dot{v}_{0,1}^1 + \overbrace{\widehat{\omega_{0,1}^1} \, m v_{0,1}^1}^{\text{the cross product } \omega_{0,1} \times m v_{0,1} \text{ in the coordinates of frame 1}}$$

$$\tau^1 = J^1 \dot{\omega}_{0,1}^1 + \widehat{\omega_{0,1}^1} \, J^1 \omega_{0,1}^1$$

$$J^1 = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}$$

← the moment of inertia matrix in the coordinates of frame 1, assuming the principal axes are aligned with x_1, y_1, and z_1

Cross Product



← what it is

$$w' = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad v' = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$(w \times v)' = \underbrace{\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}}_{\widehat{w'}} \underbrace{\begin{bmatrix} d \\ e \\ f \end{bmatrix}}_{v'} = \begin{bmatrix} bf - ce \\ cd - af \\ ae - bd \end{bmatrix}$$

"skew-symmetric"

← one way to compute it (with matrix multiplication)

SEE NOTEBOOK FOR OTHER WAYS TO COMPUTE IT THAT ARE EASIER IN PRACTICE

EQUATIONS OF MOTION

KINEMATICS

$$\dot{o}^0_1 = R^0_1 v^1_{0,1}$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = N \omega^1_{0,1}$$

DYNAMICS

$$f^1 = m\dot{v}^1_{0,1} + \widehat{\omega^1_{0,1}}\, m v^1_{0,1}$$

$$\tau^1 = J^1 \dot{\omega}^1_{0,1} + \widehat{\omega^1_{0,1}}\, J^1 \omega^1_{0,1}$$

$$f^1 = f^1_{gravity} + f^1_{rotors}$$

$$\tau^1 = \tau^1_{rotors}$$

ignore all other external forces and torques

FORCE OF GRAVITY

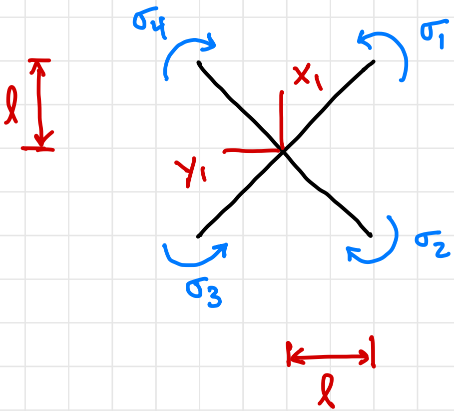
$$f^0_{gravity} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

← easy to write in frame 0

$$f^1_{gravity} = \qquad f^0_{gravity}$$

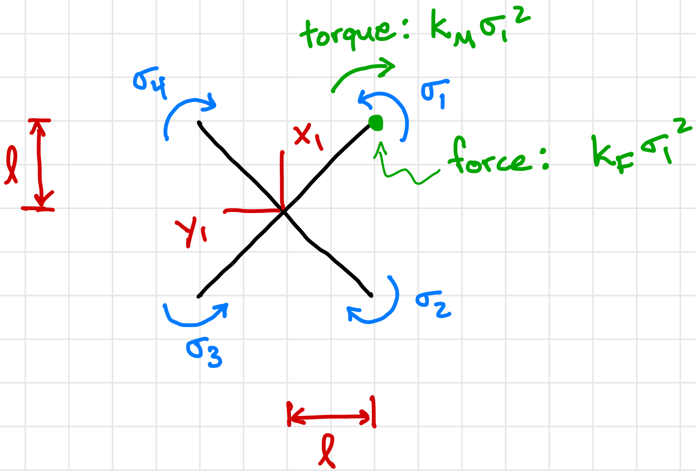
← transform to frame 1

ROTOR FORCES AND TORQUES



$$\underbrace{f_1^1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad \tau_1^1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}}^{\text{from rotor \# 1 only}}$$

FORCES AND TORQUES

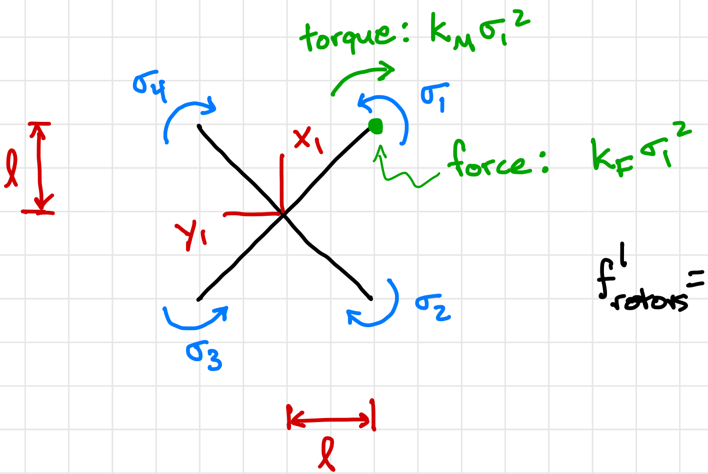


from rotor # 1 only

$$f_1^1 = \begin{bmatrix} 0 \\ 0 \\ k_F \sigma_1^2 \end{bmatrix} \qquad \tau_1^1 = \begin{bmatrix} -\ell\, k_F \sigma_1^2 \\ -\ell\, k_F \sigma_1^2 \\ -k_M \sigma_1^2 \end{bmatrix}$$

$$\begin{aligned}
\tau_1^1 &= \begin{bmatrix} 0 \\ 0 \\ -k_M \sigma_1^2 \end{bmatrix} + \widehat{p_1^1}\, f_1^1 \\
&= \begin{bmatrix} 0 \\ 0 \\ -k_M \sigma_1^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\ell \\ 0 & 0 & -\ell \\ \ell & \ell & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ k_F \sigma_1^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ -k_M \sigma_1^2 \end{bmatrix} + \begin{bmatrix} -\ell\, k_F \sigma_1^2 \\ -\ell\, k_F \sigma_1^2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\ell\, k_F \sigma_1^2 \\ -\ell\, k_F \sigma_1^2 \\ -k_M \sigma_1^2 \end{bmatrix}
\end{aligned}$$

FORCES AND TORQUES (all rotors)



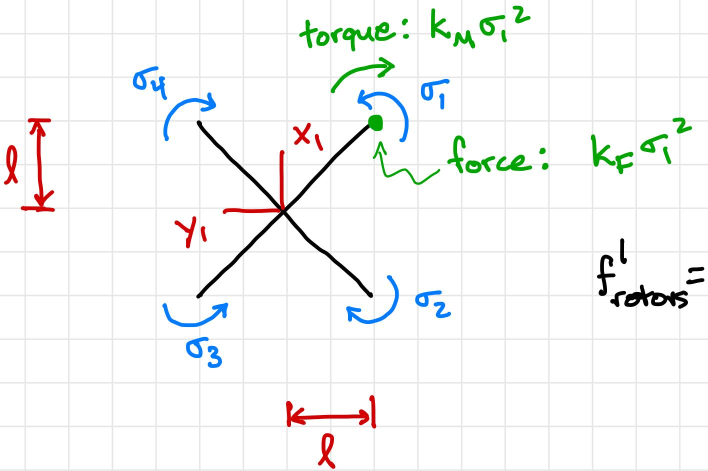
$$f_1^1 = \begin{bmatrix} 0 \\ 0 \\ k_F \sigma_1^2 \end{bmatrix} \qquad \tau_1^1 = \begin{bmatrix} -\ell k_F \sigma_1^2 \\ -\ell k_F \sigma_1^2 \\ -k_M \sigma_1^2 \end{bmatrix}$$

$$f^1_{\text{rotors}} = \begin{bmatrix} \quad \end{bmatrix} \leftarrow f_z$$

$$\tau^1_{\text{rotors}} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \begin{matrix} \leftarrow \tau_x \\ \leftarrow \tau_y \\ \leftarrow \tau_z \end{matrix}$$

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \\ \sigma_4^2 \end{bmatrix}$$

FORCES AND TORQUES (all rotors)



$$f_1^1 = \begin{bmatrix} 0 \\ 0 \\ k_F \sigma_1^2 \end{bmatrix} \qquad \tau_1^1 = \begin{bmatrix} -\ell k_F \sigma_1^2 \\ -\ell k_F \sigma_1^2 \\ -k_M \sigma_1^2 \end{bmatrix}$$

$$f^1_{\text{rotors}} = \begin{bmatrix} 0 \\ 0 \\ k_F\sigma_1^2 + k_F\sigma_2^2 + k_F\sigma_3^2 + k_F\sigma_4^2 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow f_z \end{matrix}$$

$$\tau^1_{\text{rotors}} = \begin{bmatrix} -\ell k_F\sigma_1^2 - \ell k_F\sigma_2^2 + \ell k_F\sigma_3^2 + \ell k_F\sigma_4^2 \\ -\ell k_F\sigma_1^2 + \ell k_F\sigma_2^2 + \ell k_F\sigma_3^2 - \ell k_F\sigma_4^2 \\ -k_M\sigma_1^2 + k_M\sigma_2^2 - k_M\sigma_3^2 + k_M\sigma_4^2 \end{bmatrix} \begin{matrix} \leftarrow \tau_x \\ \leftarrow \tau_y \\ \leftarrow \tau_z \end{matrix}$$

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \begin{bmatrix} -\ell k_F & -\ell k_F & \ell k_F & \ell k_F \\ -\ell k_F & \ell k_F & \ell k_F & -\ell k_F \\ -k_M & k_M & -k_M & k_M \\ k_F & k_F & k_F & k_F \end{bmatrix} \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \\ \sigma_4^2 \end{bmatrix}$$

TOTAL FORCE AND TORQUE

$$f^1 = (R_1^0)^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_z \end{bmatrix}$$

$$\tau^1 = \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}$$

$$\begin{array}{r} \text{net torque}\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ \text{net force (thrust)}\left\{\begin{array}{c} \\ \end{array}\right. \end{array} \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \overbrace{\begin{bmatrix} -\ell k_F & -\ell k_F & \ell k_F & \ell k_F \\ -\ell k_F & \ell k_F & \ell k_F & -\ell k_F \\ -k_M & k_M & -k_M & k_M \\ k_F & k_F & k_F & k_F \end{bmatrix}}^{\text{constant matrix}} \overbrace{\begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \\ \sigma_4^2 \end{bmatrix}}^{\text{squared rotor speeds}}$$

ROTOR SPEEDS VS MOTOR POWER COMMANDS

squared rotor speed $(\text{rad}/s)^2$

$$\sigma_1^2 \approx c\, m_1$$

motor power command, an integer between 0 and 65535

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \begin{bmatrix} -l k_F & -l k_F & l k_F & l k_F \\ -l k_F & l k_F & l k_F & -l k_F \\ -k_M & k_M & -k_M & k_M \\ k_F & k_F & k_F & k_F \end{bmatrix} \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \\ \sigma_4^2 \end{bmatrix}$$

ROTOR SPEEDS VS MOTOR POWER COMMANDS

squared rotor speed $(\text{rad}/s)^2$

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motor power command, an integer between 0 and 65535

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \begin{bmatrix} -l k_F & -l k_F & l k_F & l k_F \\ -l k_F & l k_F & l k_F & -l k_F \\ -k_M & k_M & -k_M & k_M \\ k_F & k_F & k_F & k_F \end{bmatrix} \begin{bmatrix} c m_1 \\ c m_2 \\ c m_3 \\ c m_4 \end{bmatrix}$$

ROTOR SPEEDS VS MOTOR POWER COMMANDS

squared rotor speed $(rad/s)^2$

$$\omega_1^2 \approx c\, m_1$$

motor power command, an integer between 0 and 65535

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \begin{bmatrix} -l(k_F c) & -l(k_F c) & l(k_F c) & l(k_F c) \\ -l(k_F c) & l(k_F c) & l(k_F c) & -l(k_F c) \\ -(k_M c) & (k_M c) & -(k_M c) & (k_M c) \\ (k_F c) & (k_F c) & (k_F c) & (k_F c) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

ROTOR SPEEDS VS MOTOR POWER COMMANDS

squared rotor speed $(\mathrm{rad/s})^2$

$$\sigma_1^2 \approx c\, m_1$$

motor power command, an integer between 0 and 65535

"k_M" really means "$k_M c$"

"k_F" really means "$k_F c$"

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \begin{bmatrix} -l k_F & -l k_F & l k_F & l k_F \\ -l k_F & l k_F & l k_F & -l k_F \\ -k_M & k_M & -k_M & k_M \\ k_F & k_F & k_F & k_F \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

TOTAL FORCE AND TORQUE

$$f^1 = (R_1^0)^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_z \end{bmatrix}$$

$$\tau^1 = \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}$$

constant matrix

motor power commands

$$\begin{array}{r l} \text{net torque} \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. & \\ \text{net force (thrust)} \left\{ \begin{matrix} \\ \end{matrix} \right. & \end{array} \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \begin{bmatrix} -l k_F & -l k_F & l k_F & l k_F \\ -l k_F & l k_F & l k_F & -l k_F \\ -k_M & k_M & -k_M & k_M \\ k_F & k_F & k_F & k_F \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

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$$f^1 = m\dot{v}_{0,1}^1 + \widehat{\omega_{0,1}^1}\, m v_{0,1}^1$$

$$\tau^1 = J^1 \dot{\omega}_{0,1}^1 + \widehat{\omega_{0,1}^1}\, J^1 \omega_{0,1}^1$$

$$f^1 = (R_1^0)^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_z \end{bmatrix}$$

$$\tau^1 = \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}$$

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix} = \begin{bmatrix} -\ell k_F & -\ell k_F & \ell k_F & \ell k_F \\ -\ell k_F & \ell k_F & \ell k_F & -\ell k_F \\ -k_M & k_M & -k_M & k_M \\ k_F & k_F & k_F & k_F \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

LIST OF PARAMETERS WE NEED TO KNOW:

???

What f_z is required to hover?

What is the max value of f_z?

What motor power commands are required to hover?

What is the min value of f_z?

$$f^1 = (R_1^0)^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_z \end{bmatrix}$$

$$\tau^1 = \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}$$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4k_F l} & -\frac{1}{4k_F l} & -\frac{1}{4k_M} & \frac{1}{4k_F} \\ -\frac{1}{4k_F l} & \frac{1}{4k_F l} & \frac{1}{4k_M} & \frac{1}{4k_F} \\ \frac{1}{4k_F l} & \frac{1}{4k_F l} & -\frac{1}{4k_M} & \frac{1}{4k_F} \\ \frac{1}{4k_F l} & -\frac{1}{4k_F l} & \frac{1}{4k_M} & \frac{1}{4k_F} \end{bmatrix} \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_z \end{bmatrix}$$