Lab 3 : Theory

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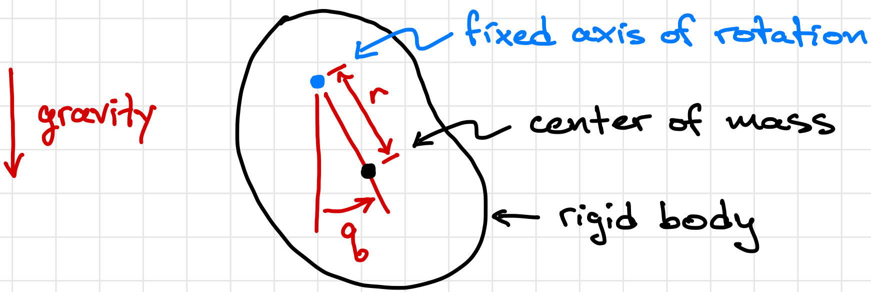
Consider a rigid body that is constrained to rotate about a fixed axis:



Assuming no friction about the fixed axis and no external forces other than gravity, the motion of the rigid body is governed by the following ODE:

$$\ddot{q} = -\left(\frac{mgr}{J+mr^2}\right)\sin q$$

m — mass

J — moment of inertia about an axis through the center of mass

r — distance between axis of rotation and center of mass

Where did the equation of motion come from? Let's apply what we know from AE352 to derive it:

$$T = \frac{1}{2} m (r \dot{q})^2 + \frac{1}{2} J \dot{q}^2$$ ← kinetic energy

$$V = - mgr \cos q$$ ← potential energy

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(J + mr^2)\, \dot{q}^2 + mgr \cos q \end{aligned}$$ ← Lagrangian

$$\begin{aligned} 0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \\ &= \frac{d}{dt}\left((J + mr^2)\, \dot{q} \right) + mgr \sin q \\ &= (J + mr^2)\, \ddot{q} + mgr \sin q \end{aligned}$$ ← Euler-Lagrange equation

Solving for $\ddot{q}$, we find:

$$\boxed{\ddot{q} = - \left(\frac{mgr}{J + mr^2} \right) \sin q}$$ ← equation of motion

So, here is the equation of motion:

$$\ddot{q} = -\left(\frac{mgr}{J+mr^2}\right)\sin q$$

equivalent to linearizing about $q=\dot{q}=0$

If we assume the angle q is small, then this EOM can be approximated by:

$$\ddot{q} \approx -\left(\frac{mgr}{J+mr^2}\right) q$$

The solution to this ODE has the form

$$q(t) = a\cos\left(\left(\frac{2\pi}{T}\right)t + b\right) \quad \leftarrow \text{a sinusoid with period } T$$

for some constants a, T, and b. Let's check this really is the solution and find the constants in the process.

If:

$$q = a\cos\left(\frac{2\pi}{T}t + b\right)$$

then:

$$\dot{q} = -a\left(\frac{2\pi}{T}\right)\sin\left(\frac{2\pi}{T}t + b\right)$$

$$\ddot{q} = -a\left(\frac{2\pi}{T}\right)^2\cos\left(\frac{2\pi}{T}t + b\right)$$

$$\boxed{\ddot{q} \approx -\left(\frac{mgr}{J+mr^2}\right)q}$$

Plug this into the ODE:

$$-a\left(\frac{2\pi}{T}\right)^2\cos\left(\frac{2\pi}{T}t + b\right) = -\left(\frac{mgr}{J+mr^2}\right)a\cos\left(\frac{2\pi}{T}t + b\right)$$

For equality to hold, it must be true that:

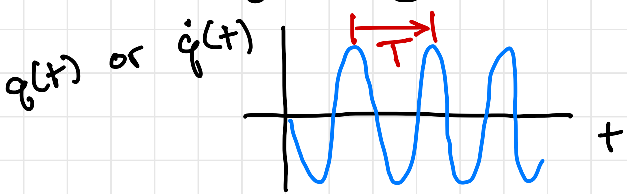
$$\boxed{\left(\frac{2\pi}{T}\right)^2 = \left(\frac{mgr}{J+mr^2}\right)}$$

The other constants come from initial conditions:

$$\left.\begin{aligned} q(0) &= a\cos(b) \\ \dot{q}(0) &= -a\left(\frac{2\pi}{T}\right)\sin(b) \end{aligned}\right\} \text{could solve for } a \text{ and } b$$

All of this gives us a way to estimate J:

1. Measure m with a scale
2. Measure r with a ruler
3. Let the rigid body swing, and estimate T by looking at either $q(t)$ or $\dot{q}(t)$



← you would probably want to find the average period over many oscillations, to make your estimate more accurate

4. Solve for J

$$\boxed{\left(\frac{2\pi}{T}\right)^2 = \left(\frac{mgr}{J+mr^2}\right)}$$