Implementation of Transient Heat Conduction Equation in Finite Element Methods: Theory and Implementation Details

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Contents

1 Introduction

In this Document, I will try to give a brief introduction to:

- The derivation of transient heat conduction equation in a physical view;
- Variational format of transient heat conduction equation (the weak form) ;
- Corresponding FEM discretization and implementation details on regular quadrilateral grid

If anyone has any question or recommendation, just feel free to email me: `ctj19990927@mail.ustc.edu.cn`

2 Physical Model of Transient Heat Conduction Equation

We focus on an enough small piece of square aligned to the 2D Cartesian coordinate:

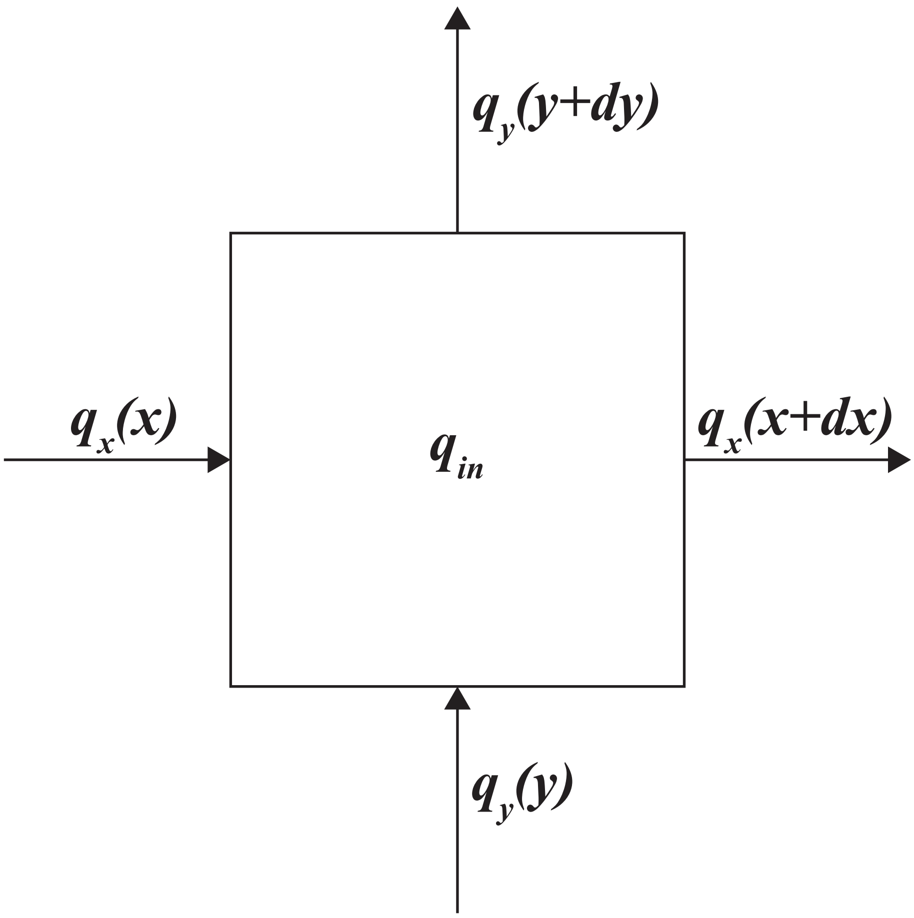

Figure 1: Physical model of heat conduction in an infinitesimal unit.

The infinitesimal unit (the square) is assumed to have length dx and dy in x-axis and y-axis direction correspondingly. In x-axis direction, the flowing-through heat flux has different tensities on two boundaries: $q_x(x)$, $q_x(x + dx)$ (Here q_x

means the x-component of the heat flux $\boldsymbol{q}$. q_y in the following paragraph has similar meaning). Similarly, in y-axis direction, the flowing-through heat flux has different tensities on two boundaries: $q_y(y)$, $q_y(y+dy)$. Assuming additionally that the unit square itself generates heat with a tensity of q_{in} (Unit: W/m^2 in 2D). According to the Heat-Balance Equation, the change of unit's internal energy ΔU in a given timestep Δt equals to the energy input (heat flowing inwards E_{in} and heat generated by itself E_s) minus the energy output (heat flowing outwards E_{out}). That is:

$$\Delta U = E_{in} + E_s - E_{out} \tag{1}$$

- $E_{in} = \{q_x(x)dy + q_y(y)dx\}\Delta t$
- $E_{out} = \{q_x(x+dx)dy + q_y(y+dy)dx\}\Delta t$
- $E_s = q_{in}dxdy\Delta t$
- Assume that the unit square's density is ρ, and its heat capacity is c. In timestep Δt, its temperature change is ΔT. Then its change in internal energy is: $\Delta U = \rho c dxdy\Delta T$.

Combining the above terms all together into equation (1), we obtain:

$$\rho c dxdy\Delta T = \{q_x(x)dy+q_y(y)dx\}\Delta t-\{q_x(x+dx)dy+q_y(y+dy)dx\}\Delta t+q_{in}dxdy\Delta t \tag{2}$$

Reorganizing the equation above will lead us to the following:

$$\rho c\frac{\Delta T}{\Delta t} = -\frac{q_x(x+dx)-q_x(x)}{dx} - \frac{q_y(y+dy)-q_y(y)}{dy} + q_{in} \tag{3}$$

Using the Taylor expansion and taking the limit $dx \to 0, dy \to 0, \Delta t \to 0$:

$$\rho c\frac{\partial T}{\partial t} = -\frac{\partial q_x(x)}{\partial x} - \frac{\partial q_y(y)}{\partial y} + q_{in} \tag{4}$$

According to Fourier's Law, heat flux is propotional to the negative of temperature gradient:

$$\boldsymbol{q} = -\boldsymbol{\kappa}\nabla T \tag{5}$$

where $\boldsymbol{\kappa}$ is the unit's thermal conductivity tensor. Substituting (5) to (4) will produce the so called transient heat conduction equation:

$$\rho c\frac{\partial T}{\partial t} = \nabla\cdot(\boldsymbol{\kappa}\nabla T) + q_{in} \tag{6}$$

3 Variation of transient heat conduction equation-Weak form

Considering heat conduction in region Ω, governed by the following equation (7) with specific initial and boundary conditions:

$$\begin{aligned}
\rho c\frac{\partial T}{\partial t} &= \nabla\cdot(\boldsymbol{\kappa}\nabla T) + q_{in} \quad in\ \Omega\\
T &= T_0 \quad on\ \partial\Omega_D\\
-(\boldsymbol{\kappa}\nabla T)\cdot\boldsymbol{n} &= q_0 \quad on\ \partial\Omega_N\\
-(\boldsymbol{\kappa}\nabla T)\cdot\boldsymbol{n} &= h(T-T^{ref}) \quad on\ \partial\Omega_C\\
T|_{t=0} &= T_{init}
\end{aligned} \tag{7}$$

Here $\partial\Omega_D$, $\partial\Omega_N$, $\partial\Omega_C$ refer to part of boundary $\partial\Omega$ with Dirichlet, Neumann and Convection boundary conditions respectively. $\boldsymbol{n}$ is the unit normal vector pointing outwards the boundary $\partial\Omega$. It is also depicted in Figure 2 as demonstrated below:

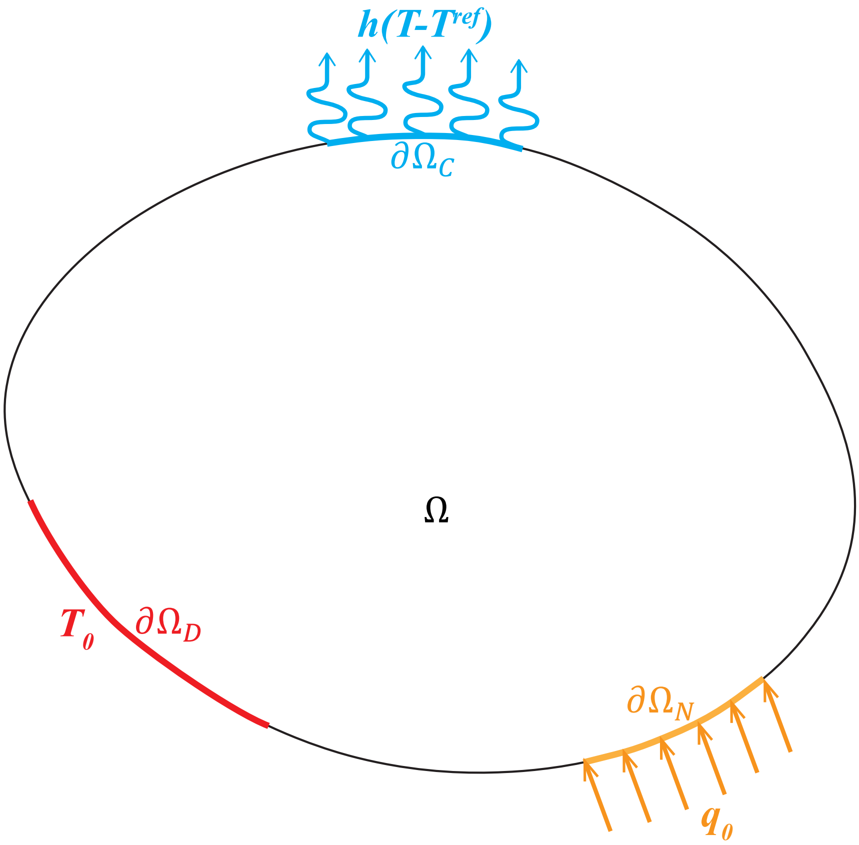

Figure 2: Simulation model of heat conduction with different boundary conditions.

Now we will multiply a test function ϕ to the heat conduction equation (6) and integrate both sides in Ω: (Here it's assumed that $\phi_{\Omega_D} \equiv 0$)

$$\iint_\Omega \rho c \frac{\partial T}{\partial t} \phi d\Omega = \iint_\Omega \nabla \cdot (\boldsymbol{\kappa} \nabla T) \phi d\Omega + \iint_\Omega q_{in} \phi d\Omega \tag{8}$$

According to the Gauss's theorem, term $\iint_\Omega \nabla \cdot (\boldsymbol{\kappa} \nabla T) \phi d\Omega$ in the right side can be rewritten as the following form:

$$\begin{aligned} \iint_\Omega \nabla \cdot (\boldsymbol{\kappa} \nabla T) \phi d\Omega &= -\iint_\Omega \nabla \phi \cdot (\boldsymbol{\kappa} \nabla T) d\Omega + \int_{\partial\Omega} \phi (\boldsymbol{\kappa} \nabla T) \cdot \boldsymbol{n} ds \\ &= -\iint_\Omega \nabla \phi \cdot (\boldsymbol{\kappa} \nabla T) d\Omega \\ &\quad - \int_{\partial\Omega_N} \phi q_0 d\partial\Omega_N - \int_{\partial\Omega_C} \phi h (T - T^{ref}) d\partial\Omega_C \end{aligned} \tag{9}$$

Substituting (9) to (8) and resort the formulation:

$$\begin{aligned} &\iint_\Omega \rho c \frac{\partial T}{\partial t} \phi d\Omega + \iint_\Omega \nabla \phi \cdot (\boldsymbol{\kappa} \nabla T) d\Omega \\ &= \iint_\Omega q_{in} \phi d\Omega - \int_{\partial\Omega_N} \phi q_0 d\partial\Omega_N - \int_{\partial\Omega_C} \phi h (T - T^{ref}) d\partial\Omega_C \end{aligned} \tag{10}$$

Equation (10) is the variational form of transient heat conduction equation we wanted. (i.e. the weak form)

4 FEM Discretization of heat conduction equation

Consider a finite element in Ω with m nodes (named Ω_e). On each node we will assign a nodal basis function $\{N_i\}_{i=0}^{m-1}$.Suggest that each node corresponds to a nodal temperature $\{T_i\}_{i=0}^{m-1}$, then the temperature inside the finite element can be expressed as $T=\sum_{i=0}^{m-1} T_i N_i$. Putting it into equation (10) and restrict the integrating area to Ω_e, we have:

$$\begin{aligned}&\iint_{\Omega_e}\rho c(\sum_{i=0}^{m-1}\frac{\partial T_i}{\partial t}N_i)\phi d\Omega+\iint_{\Omega_e}\sum_{i=0}^{m-1}(\nabla\phi\cdot(\boldsymbol{\kappa}\nabla N_i))T_i d\Omega\\&=\iint_{\Omega_e}q_{in}\phi d\Omega-\int_{\partial\Omega_N\cap\Omega_e}\phi q_0 d\partial\Omega_N-\int_{\partial\Omega_C\cap\Omega_e}\phi h(\sum_{i=0}^{m-1}T_iN_i-T^{ref})d\partial\Omega_C\end{aligned}\tag{11}$$

let $\phi=N_j,\ j=0,1,\cdots,m-1$:

$$\begin{aligned}&\iint_{\Omega_e}\rho c(\sum_{i=0}^{m-1}\frac{\partial T_i}{\partial t}N_i)N_j d\Omega+\iint_{\Omega_e}\sum_{i=0}^{m-1}(\nabla N_j\cdot(\boldsymbol{\kappa}\nabla N_i))T_i d\Omega\\&=\iint_{\Omega_e}q_{in}N_j d\Omega-\int_{\partial\Omega_N\cap\Omega_e}N_j q_0 d\partial\Omega_N-\int_{\partial\Omega_C\cap\Omega_e}N_j h(\sum_{i=0}^{m-1}T_iN_i-T^{ref})d\partial\Omega_C\end{aligned}\tag{12}$$

Listing $N_j(j=0,1,\cdots,m-1)$, we can obtain a finite element discretization of heat conduction equation in Ω_e:

$$\mathrm{M}_e\frac{\partial\boldsymbol{T}_e}{\partial t}+\mathrm{K}_e\boldsymbol{T}_e=\boldsymbol{Q}_e\tag{13}$$

where:

$$\mathrm{M}_e=[M_{ij}]_{i,j=0}^{m-1}=[\iint_{\Omega_e}\rho c N_i N_j d\Omega]_{i,j=0}^{m-1}\tag{14}$$

$$\mathrm{K}_e=[K_{ij}]_{i,j=0}^{m-1}=[\iint_{\Omega_e}(\nabla N_j\cdot(\boldsymbol{\kappa}\nabla N_i))d\Omega]_{i,j=0}^{m-1}\tag{15}$$

$$\begin{aligned}&\boldsymbol{Q}_e=[Q_j]_{j=0}^{m-1}\\&=[\iint_{\Omega_e}q_{in}N_j d\Omega-\int_{\partial\Omega_N\cap\Omega_e}N_j q_0 d\partial\Omega_N-\int_{\partial\Omega_C\cap\Omega_e}N_j h(\sum_{i=0}^{m-1}T_iN_i-T^{ref})d\partial\Omega_C]_{j=0}^{m-1}\end{aligned}\tag{16}$$

Note that when convection boundary condition exists, further adjustments should be made (like moving some terms relative to $\boldsymbol{T}_e$ in the right side to the left side).

5 Detailed Finite Element Methods on regular quadrilateral grid

Here we restrict our discussion on regular quadrilateral grid as shown in Figure.3, and there is assumed to be no convection boundary conditions for simplify.

(Furthermore, the assembly of the right side term of the FEM format (13) is neglected and is defined as a constant $\boldsymbol{Q}_e$ on the element Ω_e in the following parts)

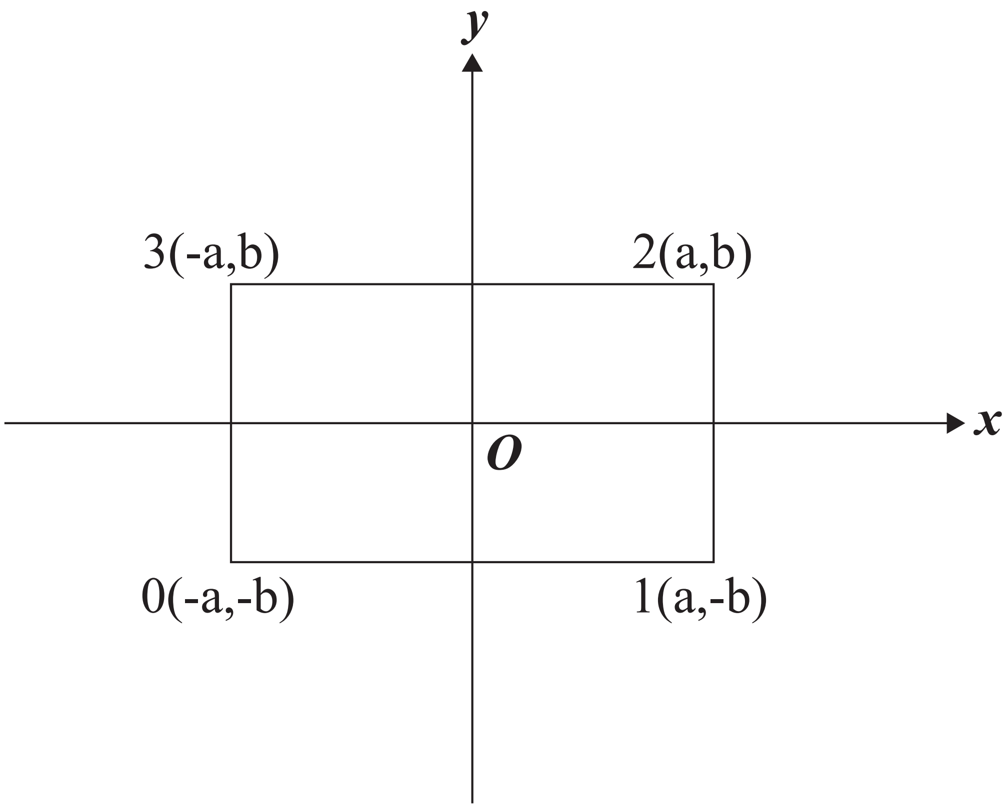

Figure 3: Axis-Aligned quadrilateral finite element we will use for discussion in this document.

the heat capacity, thermal conductivity tensor and density are considered constant in the whole element Ω_e, i.e., $\rho \equiv \rho_e,\ c \equiv c_e,\ \boldsymbol{\kappa} \equiv \boldsymbol{\kappa}_e$.

5.1 Assembly of Conduction Matrices

In Figure.3, nodal basis functions can be defined as:

$$\begin{aligned} N_0(x,y) &= \frac{1}{4}(1-\frac{x}{a})(1-\frac{y}{b}) \\ N_1(x,y) &= \frac{1}{4}(1+\frac{x}{a})(1-\frac{y}{b}) \\ N_2(x,y) &= \frac{1}{4}(1+\frac{x}{a})(1+\frac{y}{b}) \\ N_3(x,y) &= \frac{1}{4}(1-\frac{x}{a})(1+\frac{y}{b}) \end{aligned} \tag{17}$$

Substituting (17) to equation (14) and (15), that will be:

$$\mathrm{M}_e = \frac{ab}{9}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \tag{18}$$

$$
\mathrm{K}_e = \boldsymbol{\kappa}_{11}\begin{bmatrix} \frac{b}{3a} & -\frac{b}{3a} & -\frac{b}{6a} & \frac{b}{6a} \\ -\frac{b}{3a} & \frac{b}{3a} & \frac{b}{6a} & -\frac{b}{6a} \\ -\frac{b}{6a} & \frac{b}{6a} & \frac{b}{3a} & -\frac{b}{3a} \\ \frac{b}{6a} & -\frac{b}{6a} & -\frac{b}{3a} & \frac{b}{3a} \end{bmatrix} + \boldsymbol{\kappa}_{12}\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} + \boldsymbol{\kappa}_{22}\begin{bmatrix} \frac{a}{3b} & \frac{a}{6b} & -\frac{a}{6b} & -\frac{a}{3b} \\ \frac{a}{6b} & \frac{a}{3b} & -\frac{a}{3b} & -\frac{a}{6b} \\ -\frac{a}{6b} & -\frac{a}{3b} & \frac{a}{3b} & \frac{a}{6b} \\ -\frac{a}{3b} & -\frac{a}{6b} & \frac{a}{6b} & \frac{a}{3b} \end{bmatrix} \tag{19}
$$

Here $\boldsymbol{\kappa}_e = \begin{bmatrix} \boldsymbol{\kappa}_{11} & \boldsymbol{\kappa}_{12} \\ \boldsymbol{\kappa}_{12} & \boldsymbol{\kappa}_{22} \end{bmatrix}$. Assembling M_e, K_e and $\boldsymbol{Q}_e$ across the whole region Ω will bring us to the global finite element discretization:

$$
\mathrm{M}\frac{\partial \boldsymbol{T}}{\partial t} + \mathrm{K}\boldsymbol{T} = \boldsymbol{Q} \tag{20}
$$

5.2 Finite Difference Method for Transient Heat Conduction

To solve equation (20), we need to step forward along the time t from a given initial temperature field $\boldsymbol{T}_{init}$. Using the basic finite difference method, from time t_n to t_{n+1} (t_n corresponds to temperature field $\boldsymbol{T}_n$), the relationship can be depicted as:

$$
\begin{aligned} &\mathrm{M}\frac{\boldsymbol{T}_{n+1} - \boldsymbol{T}_n}{\Delta t} + \mathrm{K}\boldsymbol{T}_{n+1} = \boldsymbol{Q} \\ \Rightarrow &(\mathrm{M} + \Delta t\mathrm{K})\boldsymbol{T}_{n+1} = \Delta t\boldsymbol{Q} + \mathrm{M}\boldsymbol{T}_n \end{aligned} \tag{21}
$$

Here we adopt implicit finite difference method for better stability. (And for the same reason, $\Delta t < 1$ should be guaranteed)

6 Numerical Example and Link to a Demostration MATLAB Code

Based on the discussion above, I've built a simple MATLAB code adopted from the well-known top88.m to simulate transient heat conduction on single 2D rectangular plates. Please just feel free to have a try:
`https://github.com/cas-ustc-meow/2DTransientHeatConductionDemo`

Also, if anyone has any question or recommendation, just feel free to contact me:
`ctj19990927@mail.ustc.edu.cn`