Exercise: Model predictive control with the race car

In the following exercise, we will play around with variations of optimal control.

Tasks

1. We will start by casting a simulation task as a nonlinear optimization problem solved in Opti. For this purpose, we use a collocation scheme. With some hand-waving, this boils down to:

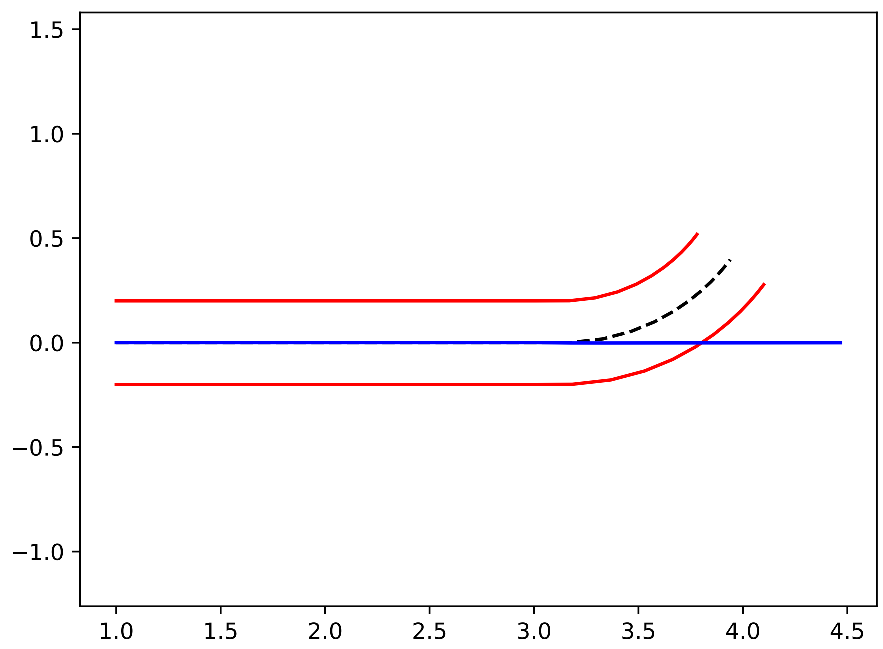
   - Subdivide the integration horizon T into N integration intervals refered to by index k.
   - Introduce state variables at the beginning of each interval: $X_k \in \mathbb{R}^{n_x}$.
   - Introduce helper variables: $X_k^c \in \mathbb{R}^{n_x \times \text{degree}}$.
   - On each interval k, consider a polynomial that exactly interpolates through X_k and X_k^c.
   - Enforce that the polynomial's slope should match the ODE, point-wise.
   - Enforce continuity of the polynomials at the interval boundaries.
   - All of the above polynomial activity can be represented by constant linear maps.

   Run the Opti example below, and verify that the car covers a distance of $4.066\,\mathrm{m}$

```
f = dae.create('f', {'x', 'u'}, {'ode'});  % System dynamics

T = 2;        % Integration horizon [s]
N = 20;       % Number of integration intervals
dt = T/N;     % Length of one interval

nx = dae.nx; % Number of states

% A helper function to get the index of a state with a particular name
indx = @(name) find(strcmp(cellstr(dae.x), name));
indy = @(name) find(strcmp(cellstr(dae.y), name));

% Numeric coefficient matrices for collocation
degree = 3;
method = 'radau';
tau = casadi.collocation_points(degree,method);
[C,D,B] = casadi.collocation_coeff(tau);

opti = casadi.Opti(); % Opti context

xk = casadi.MX(x0);

x_traj = {xk};        % Place to store the state solution trajectory
for k=1:N % Loop over integration intervals
```

```
    % Decision variables for helper states at each collocation point
    Xc = opti.variable(nx, degree);

    % Slope of polynomial at collocation points
    Z  = [xk Xc];
    Pidot = (Z * C)/dt;

    % Collocation constraints (slope matching with dynamics)
    opti.subject_to(Pidot==getfield(f('x',Xc),'ode'));

    % Continuity constraints
    xk_next = opti.variable(nx);
    opti.subject_to(Z * D==xk_next);

    % Initial guesses
    opti.set_initial(Xc, repmat(x0,1,degree));
    opti.set_initial(xk_next, x0);

    xk = xk_next;
    x_traj{end+1} = xk;
end
x_traj = [x_traj{:}];

opti.minimize(0);
options = struct;
```

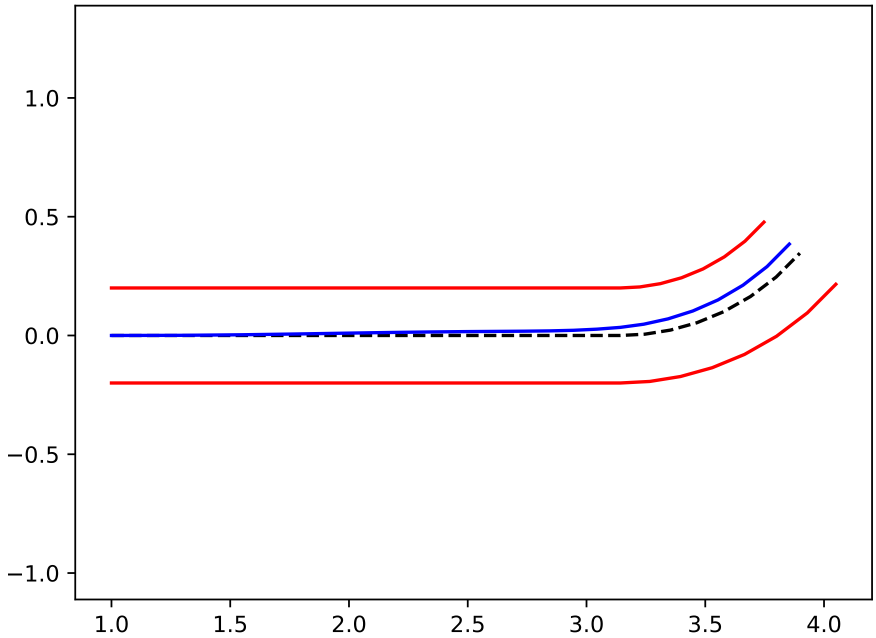
The template script also comes with plotting code that can be used to obtain:



2. Start from the solution of the previous task, but extend the problem's decision variables and constraints.

Find a formulation and a control trajectory such that the car reaches $s(T) = 4$ at the end of the horizon, whilst staying within track boundaries: $-0.2 \leq n(t) \leq 0.2$.

The result may look something like this:

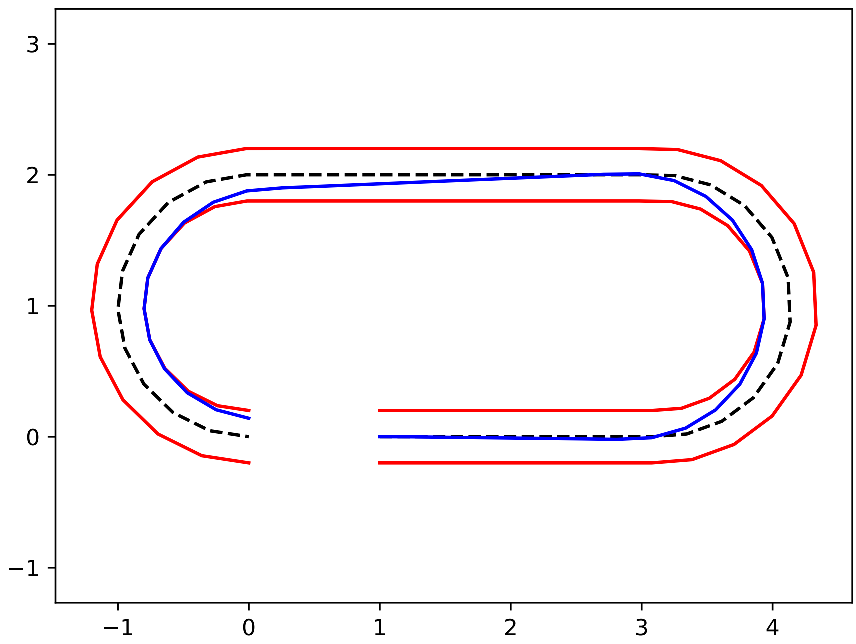


3. Optional: compute a control trajectory that minimizes lap-time. The length of the lap is 4π.

Meaningful bounds of the problem's quantities are given in the table below.

Quantity	Lower bound	Upper bound
n	-0.2	0.2
D	-1	1
$\dot{D}$	-20	20
δ	-0.8	0.8
$\dot{\delta}$	-4	4
a_{long}	-10	6
a_{lat}	-10	10

With $N = 40$, the optimal solution should look like the following picture:

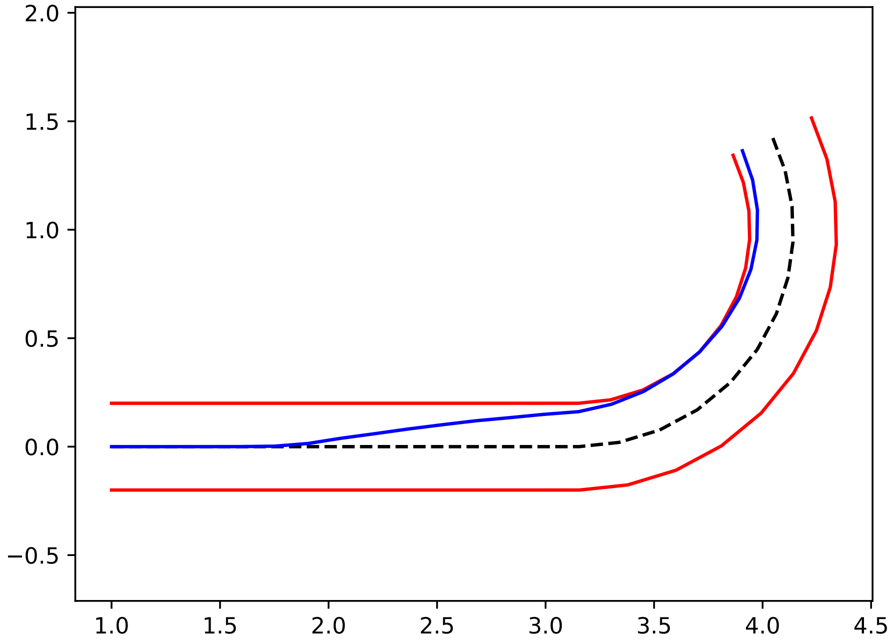


4. We will work towards model predictive control (MPC) now. Start from the solution script of Task 3.

   - Pick a prediction horizon of $T = 0.5\,\mathrm{s}$ and $N = 10$.
   - Introduce a parameter $\bar{x}_0$, i.e. a changable quantity that is not optimized over: ...
   - Introduce an extra decision variable for x_0: ...
   - Instead of a final constraint on $s(T)$, use $-s(T)$ in the objective.
   - Relax the solver tolerance with the `ipopt.tol` option set to `1e-5`.
   - Obtain a pure CasADi function out of the Opti problem: ... Here, `mpc_step` maps from the current state $\bar{x}_0$ and initial guesses for states & controls to the optimal states & controls. Verify that the car ends up at $3.13822\,\mathrm{m}$ at the end of the horizon.

   MPC is simply the computation and application of `mpc_step` in a loop: Now, MPC is about repeatedly solving this problem in a receding horizon fashion: ...

   The resulting closed-loop trajectory should look something like this:



5. Optional: Gauss-Newton and fatrop solver