# Exercise: Model predictive control with the race car

In the following exercise, we will play around with variations of optimal control.

## Tasks

1. We will start by casting a simulation task as a nonlinear optimization problem solved in Opti. For this purpose, we use a collocation scheme. With some hand-waving, this boils down to:

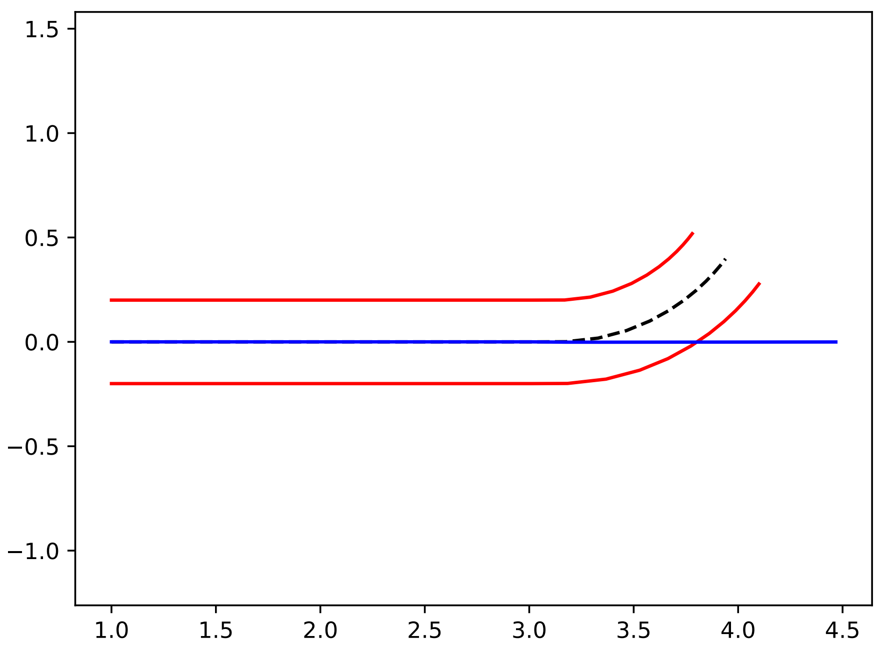
   - Subdivide the integration horizon $T$ into $N$ integration intervals refered to by index $k$.
   - Introduce state variables at the beginning of each interval: $X_k \in \mathbb{R}^{n_x}$.
   - Introduce helper variables: $X_k^c \in \mathbb{R}^{n_x \times \text{degree}}$.
   - On each interval $k$, consider a polynomial that exactly interpolates through $X_k$ and $X_k^c$.
   - Enforce that the polynomial's slope should match the ODE, point-wise.
   - Enforce continuity of the polynomials at the interval boundaries.
   - All of the above polynomial activity can be represented by constant linear maps.

   Run the Opti example below, and verify that the car covers a distance of $4.066\,\mathrm{m}$

```python
x0 = dae.start(dae.x())                      # Initial state
f = dae.create('f', ['x', 'u'], ['ode']) # System dynamics

T = 2       # Integration horizon [s]
N = 20      # Number of integration intervals
dt = T/N    # Length of one interval

nx = dae.nx() # Number of states

xvar = dae.x()

# Numeric coefficient matrices for collocation
degree = 3
method = 'radau'
tau = ca.collocation_points(degree,method)
[C,D,B] = ca.collocation_coeff(tau)

opti = ca.Opti() # Opti context

xk = ca.MX(x0)

x_traj = [xk]       # Place to store the state solution trajectory
for k in range(N): # Loop over integration intervals
```

```python
        # Decision variables for helper states at each collocation point
        Xc = opti.variable(nx, degree)

        # Slope of polynomial at collocation points
        Z  = ca.horzcat(xk,Xc)
        Pidot = (Z @ C)/dt

        # Collocation constraints (slope matching with dynamics)
        opti.subject_to(Pidot==f(x=Xc)["ode"])

        # Continuity constraints
        xk_next = opti.variable(nx)
        opti.subject_to(Z @ D==xk_next)

        # Initial guesses
        opti.set_initial(Xc, ca.repmat(x0,1,degree))
        opti.set_initial(xk_next, x0)

        xk = xk_next
        x_traj.append(xk)
    x_traj = ca.hcat(x_traj)

    opti.minimize(0)
    options = {'ipopt.hessian_approximation':'limited-memory'}
```

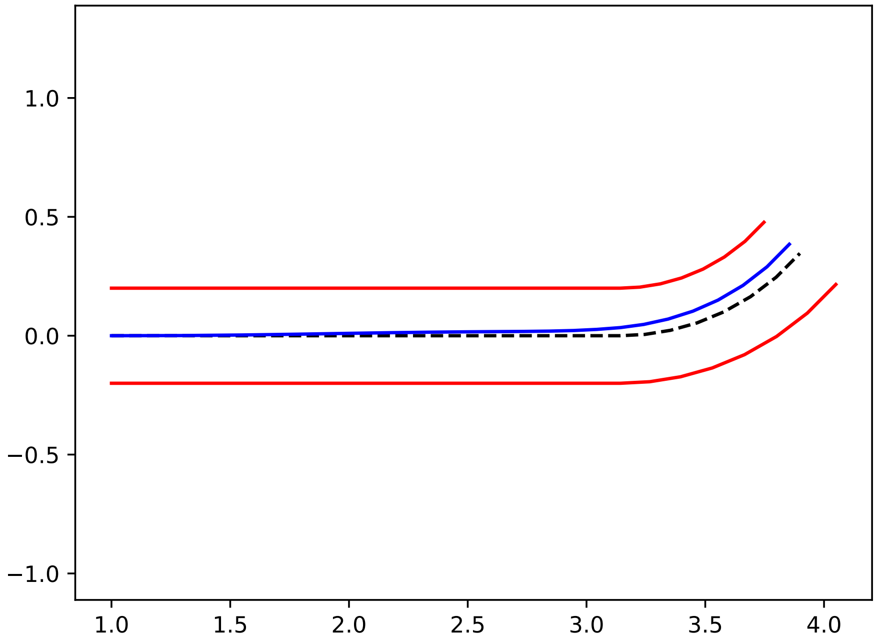
The template script also comes with plotting code that can be used to obtain:




2. Start from the solution of the previous task, but extend the problem's decision variables and constraints.

   Find a formulation and a control trajectory such that the car reaches $s(T) = 4$ at the end of the horizon, whilst staying within track boundaries: $-0.2 \leq n(t) \leq 0.2$.

The result may look something like this:

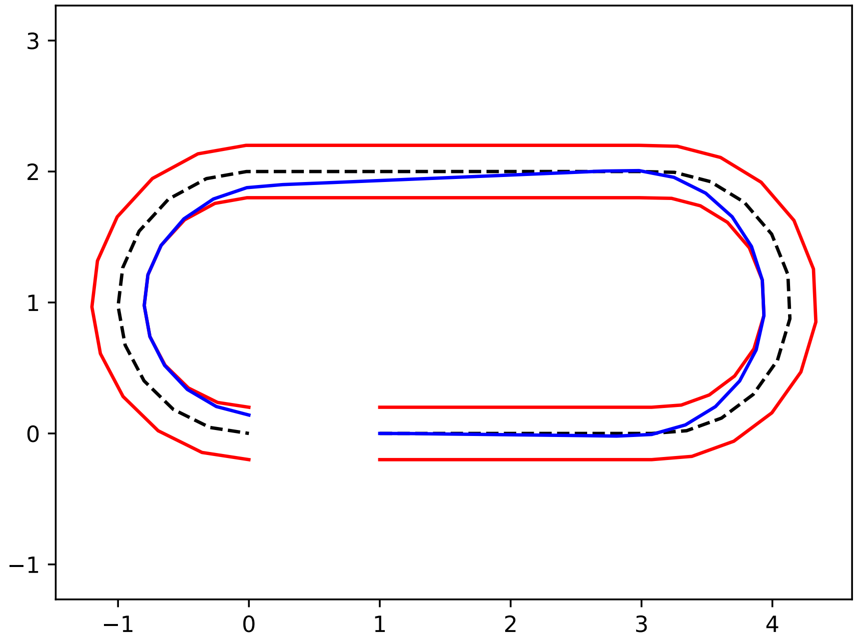



3. Optional: compute a control trajectory that minimizes lap-time. The length of the lap is $4\pi$.

   Meaningful bounds of the problem's quantities are given in the table below.

| Quantity | Lower bound | Upper bound |
|---|---|---|
| $n$ | $-0.2$ | $0.2$ |
| $D$ | $-1$ | $1$ |
| $\dot{D}$ | $-20$ | $20$ |
| $\delta$ | $-0.8$ | $0.8$ |
| $\dot{\delta}$ | $-4$ | $4$ |
| $a_{\text{long}}$ | $-10$ | $6$ |
| $a_{\text{lat}}$ | $-10$ | $10$ |

   With $N = 40$, the optimal solution should look like the following picture:




4. We will work towards model predictive control (MPC) now. Start from the solution script of Task 3.

   - Pick a prediction horizon of $T = 0.5\,\mathrm{s}$ and $N = 10$.
   - Introduce a parameter $\bar{x}_0$, i.e. a changable quantity that is not optimized over:

     ```
     x0_bar = opti.parameter(nx)
     opti.set_value(x0_bar, x0)
     ```

   - Introduce an extra decision variable for $x_0$:

     ```
     xk = opti.variable(nx)
     opti.subject_to(xk==x0_bar)
     ```

   - Instead of a final constraint on $s(T)$, use $-s(T)$ in the objective.
   - Relax the solver tolerance with the `ipopt.tol` option set to `1e-5`.
   - Obtain a pure CasADi function out of the Opti problem:

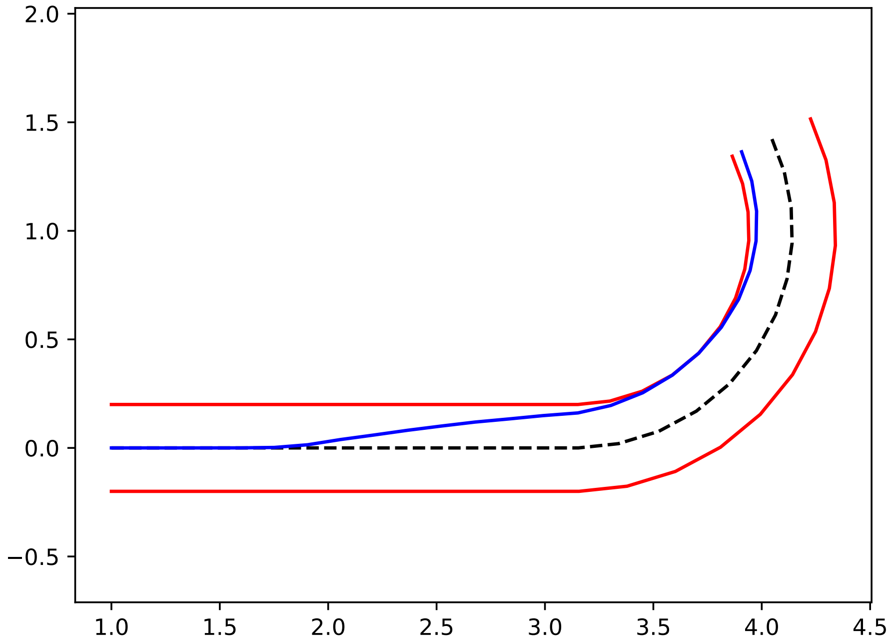
     ```
     mpc_step = opti.to_function('mpc_step', [x0_bar, x_traj, u_traj],
         [x_traj, u_traj])
     [x_opt,u_opt] = mpc_step(x0,0,0)
     print("x_opt(T): ", x_opt[:,-1])
     ```

     Here, `mpc_step` maps from the current state $\bar{x}_0$ and initial guesses for states & controls to the optimal states & controls. Verify that the car ends up at $3.13822\,\mathrm{m}$ at the end of the horizon.

MPC is simply the computation and application of `mpc_step` in a loop: Now, MPC is about repeatedly solving this problem in a receding horizon fashion:

```python
simulator = ca.integrator('simulator', 'cvodes', dae.create(), 0, dt)

y_traj = [H(x0)]
for k in range(25):
  # Compute optimal trajectories
  [x_opt,u_opt] = mpc_step(x0, x_opt, u_opt)

  # What part of the trajectory to apply?
  u_apply = u_opt[:,0]

  # Plant model
  x0 = simulator(x0=x0,u=u_apply)["xf"]

  y_traj.append(H(x0))
y_opt = ca.hcat(y_traj)
```

The resulting closed-loop trajectory should look something like this:




5. Optional: Gauss-Newton and fatrop solver