LANGEVIN SIMULATIONS OF CONFIGURATIONS WITH STATIC CHARGES

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By using the Langevin equation with complex action it is possible to generate (complex) configurations in the charged sector of gauge theories. We test the algorithm in the case of two-dimensional U(1) gauge theory and are able to measure the string tension with great precision and modest use of computer time for a separation of the charged particles where a conventional Wilson loop would have a value $\approx 10^{-10}$.

1. Introduction. It has always been a severe limitation of the Monte Carlo method in lattice gauge theories that one can only generate vacuum configurations. The signal from a large Wilson loop is exponentially damped when extracted from vacuum configurations. If one could generate configurations in the charged sector the situation would be very different. In the case where one has a flux string between the charged particles the signal relative to the vacuum would be constant no matter how far away one took the charges. In addition one would get for free information about the physical string between the charges, its spatial extension and the fall off of field strength. This kind of information is indeed very hard to extract using conventional Monte Carlo techniques. It requires the measurement of the correlation between a Wilson loop and a plaquette divided by the value of the Wilson loop [1]. For large Wilson loops this is almost impossible.

The euclidean action for a gauge theory with static charges is complex and hence not suitable for standard Monte Carlo techniques. In the case of U(1) this problem can be circumvented by making a Fourier transform to the dual action. The imaginary part is then absorbed into a flux-conserving δ-function. Indeed impressive results have been obtained using this dual transformation for three-dimensional U(1) [2–4]. Unfortunately this method seems to be very hard to generalize to non-abelian theories due to problems with implementing Gauss' law.

Including static charges is however quite natural in hamiltonian approaches like the projection method [5,6], or modifications of it [7]. But here one seems to have another more general obstacle; the difficulty in constructing efficient algorithms that work in the continuum region.

The purpose of this letter is to draw attention to a somewhat non-standard method of generating field configurations in the presence of charged particles which can be generalized to non-abelian gauge theories.

2. The complex Langevin equation and a toy model. The Langevin equation for a system with (euclidean) action $S(\varphi)$ is given by

$$\dot{\varphi}(t) = -\delta S(\varphi)/\delta\varphi + \eta(t), \tag{2.1}$$

where $\eta(t)$ is a gaussian random noise, appropriately normalized. To each Langevin equation there is a corresponding Fokker–Planck equation

$$\dot{P}(\varphi, t) = \frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\varphi}P(\varphi, t) - P(\varphi, t)u(\varphi, t)\right), \tag{2.2}$$

where the drift velocity u in the case of a

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stationary distribution $\exp[-S(\varphi)]$ is

$$u(\varphi) = -\delta S(\varphi)/\delta\varphi. \tag{2.3}$$

From (2.2) it can be shown that $P(\varphi, t)$ approaches the equilibrium distribution exponentially in time, the rate being determined by the smallest non-zero eigenvalue of the Fokker–Planck hamiltonian defined by

$$H_{\mathrm{FP}}(f(\varphi)) = -\left[\frac{\partial^2}{\partial\varphi^2}f(\varphi) + \frac{\partial}{\partial\varphi}\left(\frac{\delta S}{\delta\varphi}f(\varphi)\right)\right]. \tag{2.4}$$

H_{FP} is a hermitean operator with respect to the measure $\mathrm{e}^{S(\varphi)}\mathrm{d}\varphi$ and we may expand the initial distribution $P_0(\varphi)$ in terms of the eigenfunctions $\psi_n(\varphi)$ of H_{FP},

$$P_0(\varphi) = \sum_n a_n\psi_n(\varphi), \tag{2.5}$$

$$P(\varphi, t) = \sum_n a_n \mathrm{e}^{-\lambda_n t}\psi_n(\varphi), \tag{2.6}$$

where λ_n are the eigenvalues of H_{FP}.

If S is complex H_{FP} is no longer hermitean and not much is known about the spectrum $\{\lambda_n\}$ and the completeness of eigenfunctions $\{\psi_n\}$. However, from the pioneering work of Klauder [8–10], Parisi [11] and Gozzi [12] it is known that for a class of actions $S(\varphi)$ $\{\psi_n\}$ are complete and $\mathrm{Re}\,\lambda_n > 0$ for $n \neq 0$, such that $P(\varphi, t)$ converges to $\exp[-S(\varphi)]$. It might be difficult though, especially for multidimensional cases, to settle the question of convergence with complex actions.

Let us first examine a simple toy model, relevant for lattice gauge theories:

$$S(\varphi) = -(\beta\cos\varphi + \mathrm{i}\varphi). \tag{2.7}$$

We are interested in the expectation value of $\cos\varphi$ when the "probability" amplitude is given by

$$P(\varphi) = \mathrm{e}^{-S(\varphi)}, \tag{2.8}$$

$$\langle\cos\varphi\rangle_P \equiv \int_0^{2\pi}\mathrm{d}\varphi\, P(\varphi)\cos\varphi \Big/ \int_0^{2\pi}\mathrm{d}\varphi\, P(\varphi)$$

$$= I_0(\beta)/I_1(\beta) - 1/\beta. \tag{2.9}$$

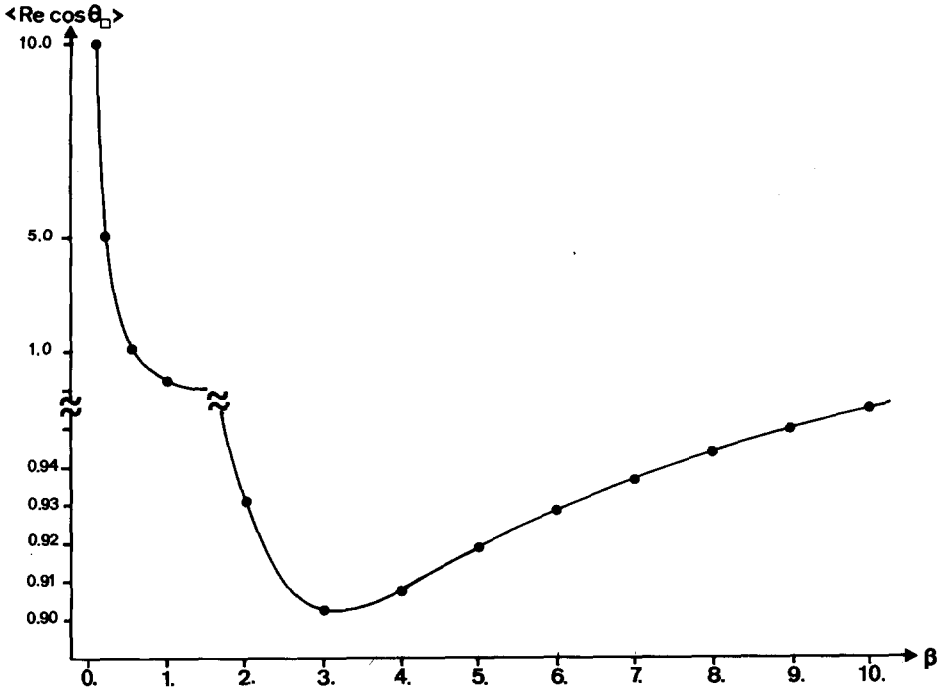

Fig. 1. Measured values of $\langle\cos\varphi\rangle_\beta$ using weight $\exp[-S(\varphi)]$, $S = \beta\cos\varphi + \mathrm{i}\varphi$, the total time $T = n\Delta t$ used at each measurement is 10^3. The error bars are smaller than the size of the dots (see table 1 for details).

That $\langle \cos\varphi \rangle_P$ goes to infinity for $\beta \to 0$ as $1/\beta$ is made possible by $P(\varphi)$ not being positive definite.

The lowest order (in Δt) discretized version of the Langevin equation (2.1) with S given by (2.7) is

$$\varphi(t_{n+1}) = \varphi(t_n) + \left[-\beta \sin\varphi(t_n) + \mathrm{i}\right] \Delta t + \eta(t_n)\sqrt{2\Delta t}\,. \quad (2.10)$$

In fig. 1 and table 1 we have shown the results from 10 independent runs of 10^5 steps for $\Delta t = 10^{-3}$ at various values of β. The measurements were performed at every fifth step for time $t = n\Delta t > 20$. The agreement with (2.9) is seen to be excellent, especially taking into account that the first order formula (2.10) does not satisfy detailed balance and is only expected to converge to $P(\varphi)$ given by (2.8) up to $O(\Delta t)$ [13].

As discovered by Klauder run-away solutions to the complex Langevin equation may be a potential problem. The reason is simple: allowing the variables to be complex makes a well behaved potential unbounded from below in certain directions in the complex φ space. The Langevin equation corresponding to (2.1) and (2.7) is

$$\dot{\varphi}(t) = -\beta \sin\varphi(t) + \mathrm{i} + \eta(t). \quad (2.11)$$

Table 1

Measured values of $\langle \cos\varphi \rangle$ using weight $\exp(\beta \cos\varphi + \mathrm{i}\varphi)$ for various β values compared to the exact values. The time steps used are $\Delta t = 10^{-3}$ and for each β ten measurements of time $T = N\Delta t = 10^2$ were made. Measurements were made at each 5th step for $T > 10$ except for the smallest and largest value of β for which $T > 20$.

	$\langle \cos\varphi \rangle_{\text{measured}}$	$I_0(\beta)/I_1(\beta) - 1/\beta$
10.0	0.953(1) + 0.0005i(30)	0.9542
9.0	0.948(1) + 0.002i(2)	0.9497
8.0	0.944(1) + 0.001i(2)	0.9443
7.0	0.937(1) + 0.0002i(20)	0.9376
6.0	0.929(1) + 0.001i(3)	0.9294
5.0	0.920(2) − 0.001i(3)	0.9193
4.0	0.911(3) + 0.004i(3)	0.9081
3.0	0.898(5) + 0.003i(10)	0.9013
2.0	0.933(5) + 0.04i(3)	0.9331
1.0	1.237(5) + 0.05i(8)	1.240
0.5	2.107(10) − 0.01i(10)	2.124
0.2	5.05(2) + 0.2i(2)	5.050
0.1	9.98(4) + 0.4i(8)	10.02

The stationary points ($\dot{\varphi}(t) = 0$, $\eta(t) = 0$) are purely imaginary:

$$\varphi_n = 2\pi n + \mathrm{i}\xi, \quad (2.12)$$

$$\sinh\xi = 1/\beta, \quad (2.13)$$

and for the fluctuations $\delta\dot{\varphi}$ around these points we have

$$\delta\dot{\varphi} = -\beta \cosh\varphi_n \delta\varphi = -(1+\beta^2)^{1/2}\delta\varphi. \quad (2.14)$$

The fixpoints are stable and for $\Delta t < 5 \times 10^{-2}$ we have never seen instabilities in our simulations, only occasionally jumps from one minimum to a neighbouring one. For $t = 0.1$ we have observed numerical instabilities though.

3. Simulation of two-dimensional U(1) lattice gauge theory with external charges. Encouraged by these results we test the method in a more realistic case where everything still can be calculated analytically: U(1) in two dimensions. The partition function $Z(\beta, L_1, L_2)$ is

$$Z(\beta, L_1, L_2) = \int \prod_{n;\mu} \mathrm{d}\theta_{n;\mu} \exp\left(\beta \sum_{\square} \cos\theta_{\square}\right) \times \exp\left(\mathrm{i} \sum_{n \in L_1} \theta_{n;2}\right) \exp\left(-\mathrm{i} \sum_{n \in L_2} \theta_{n;2}\right), \quad (3.1)$$

where

$$\theta_{\square_n} \equiv \theta_{n;1} - \theta_{n;2} - \theta_{(n_1, n_2+1);1} + \theta_{(n_1+1, n_2);2}. \quad (3.2)$$

and the external charges are included as two Wilson lines in the 2-direction (time direction) separated by a spatial distance $L = L_2 - L_1$ (see fig. 2).

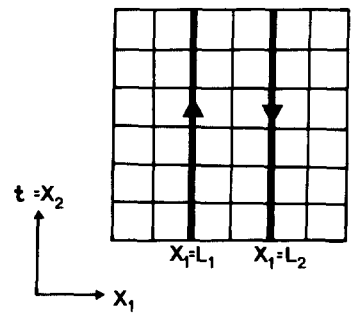

Fig. 2. Wilson lines in x_2 (time direction) located at $x_1 = L_1$ and $x_1 = L_2$.

In the infinite volume limit one can transform two plaquette variables and the expectation value of $\cos\theta_\square$ is given by:

$$\langle\cos\theta_\square\rangle_{\text{outside charges}} = I_1(\beta)/I_0(\beta), \tag{3.3}$$

$$\langle\cos\theta_\square\rangle_{\text{inside charges}} = I_0(\beta)/I_1(\beta) - 1/\beta. \tag{3.4}$$

Large β corresponds to the continuum limit ($1/\beta = e^2a^2$, a = lattice spacing) and here:

$$\beta(1-\cos\theta_\square) \approx \tfrac{1}{2}E^2(x)a^2, \tag{3.5}$$

where $E(x)$ denotes the electric field. The electric energy density in the presence of external charges can now be defined as

$$\mathscr{E}(x) \equiv -\left[\tfrac{1}{2}\langle E^2(x)\rangle_{e^+e^-} - \tfrac{1}{2}\langle E^2(x)\rangle_{\text{vacuum}}\right]. \tag{3.6}$$

The overall minus sign in (3.6) is due to the rotation $E \to iE$ in euclidean space. In the two-dimensional case the pure vacuum fluctuations $\langle E^2(x)\rangle_{\text{vacuum}}$ are of course given by an expression identical to (3.3) and we get for the energy density between the charges:

$$\mathscr{E}(x) = (\beta/a^2) \times \{[I_0(\beta)/I_1(\beta) - 1/\beta] - I_1(\beta)/I_0(\beta)\}, \tag{3.7}$$

$$\mathscr{E}(x) = \tfrac{1}{2}e^2[1 + O(e^2a^2)]. \tag{3.8}$$

This corresponds, as it should, to an electric field $E(x) = e$ between the charges. Outside the charges we have $\mathscr{E}(x) = 0$. If we define the string tension σ as the energy density between the charges we have

$$\sigma = \tfrac{1}{2}e^2[1 + O(e^2a^2)]. \tag{3.9}$$

This should be compared to the usual definition of σ in terms of Wilson loops. In two dimensions the Wilson loop factorizes and is just the plaquette expectation value to the power A, where A is the number of plaquettes inside the loop. σ_W is defined by:

$$\left\langle \exp\left(i \sum_{n,\mu\in\text{loop}} \theta_{n,\mu}\right)\right\rangle \equiv \exp(-\sigma_W a^2 A) = [I_1(\beta)/I_0(\beta)]^A, \tag{3.10}$$

$$\sigma_W = a^{-2}\log[I_0(\beta)/I_1(\beta)] = \tfrac{1}{2}e^2[1 + O(e^2a^2)]. \tag{3.11}$$

The two definitions of σ agree in the weak coupling limit.

We want to determine $\mathscr{E}(x)$ by including the charges in the partition function as in (3.1) and use the complex Langevin equation to generate (complex) configurations.

The Langevin equation for the link variables θ_l, $l \equiv n, \mu$, is:

$$\dot{\theta}_l = -\beta \sum_{\square \ni l} (\square|l) \sin\theta_\square + i(\delta_{l,L_1} - \delta_{l,L_2}) + \eta_l,$$

$$(\square|l) = \pm 1 \quad \text{depending on orientation.} \tag{3.12}$$

We have performed Langevin simulations on a 20×20 lattice for β equal to 6.0 and 9.0 for two different separations $L = L_1 - L_2$ using 50000 updatings and $\Delta t = 10^{-3}$ for each L. The results are presented in table 2 and fig. 3. They are quite satisfactory, taking into account that the systematic error is of order Δt. As can be seen from figs. 3a and 3b these systematic errors have the same sign and magnitude inside and outside the string. Quantities of physical interest, i.e. the string tension, which are related to the difference between inside and outside (vacuum) the string are hence more or less unaffected by this systematic error as can be seen from table 2. We have also investigated in more detail how the systematic errors depend on Δt by making high statistics runs for 5 different Δt-values. As can be seen from fig. 4 the difference between the theoretical value and the Monte Carlo results is approximately linear in Δt and a 4-digit accuracy can be obtained either by a small enough Δt or by extrapolating higher Δt-measurements to $\Delta t = 0$.

A comment should be made. Periodic boundary conditions should not be used when the linear extension L_0 of the system is not much larger than the charge separation. The physical reason for this is clear. If we are not in the infinite volume limit, and the charges are separated by $\approx L_0/2$ there is no stable ground state configuration as the flux can go either way around the torus. Indeed we have observed string flipping taking place after up to 20000 iterations. On the other hand, using bag boundary conditions (where $\theta_\square = 0$ at the spatial boundary in the case of two dimensions) allows one to take the two charges almost to the boundaries and still get the right field strength between them.

Table 2
Measured averages of $\mathrm{Re}\cos\theta_{\square}$ for two-dimensional U(1) inside (L) and outside (0) the flux-tube region on a 20×21 lattice with bag boundary conditions imposed on the 21st spatial plaquette. For comparison we give the theoretical values on an infinite lattice as calculated from eqs. (3.3) and (3.4). Averages are taken over 40 ksweep, with measurements every sweep, after allowing 10 ksweeps of thermalization. The time increment is $\Delta t = 0.001$. The imaginary parts are small and within the errors of the real parts.

	W-line interspace	$\langle \mathrm{Re}\cos\theta_{\square} \rangle_L$	$\langle \mathrm{Re}\cos\theta_{\square} \rangle_0$	Difference	Difference theory
6.0	6	0.9269(5)	0.9115(3)	0.0154(4)	0.0174
	14	0.9276(3)	0.9110(4)	0.0166(3)	
9.0	6	0.9480(3)	0.9417(2)	0.0063(2)	0.0070
	14	0.9482(2)	0.9414(2)	0.0068(2)	

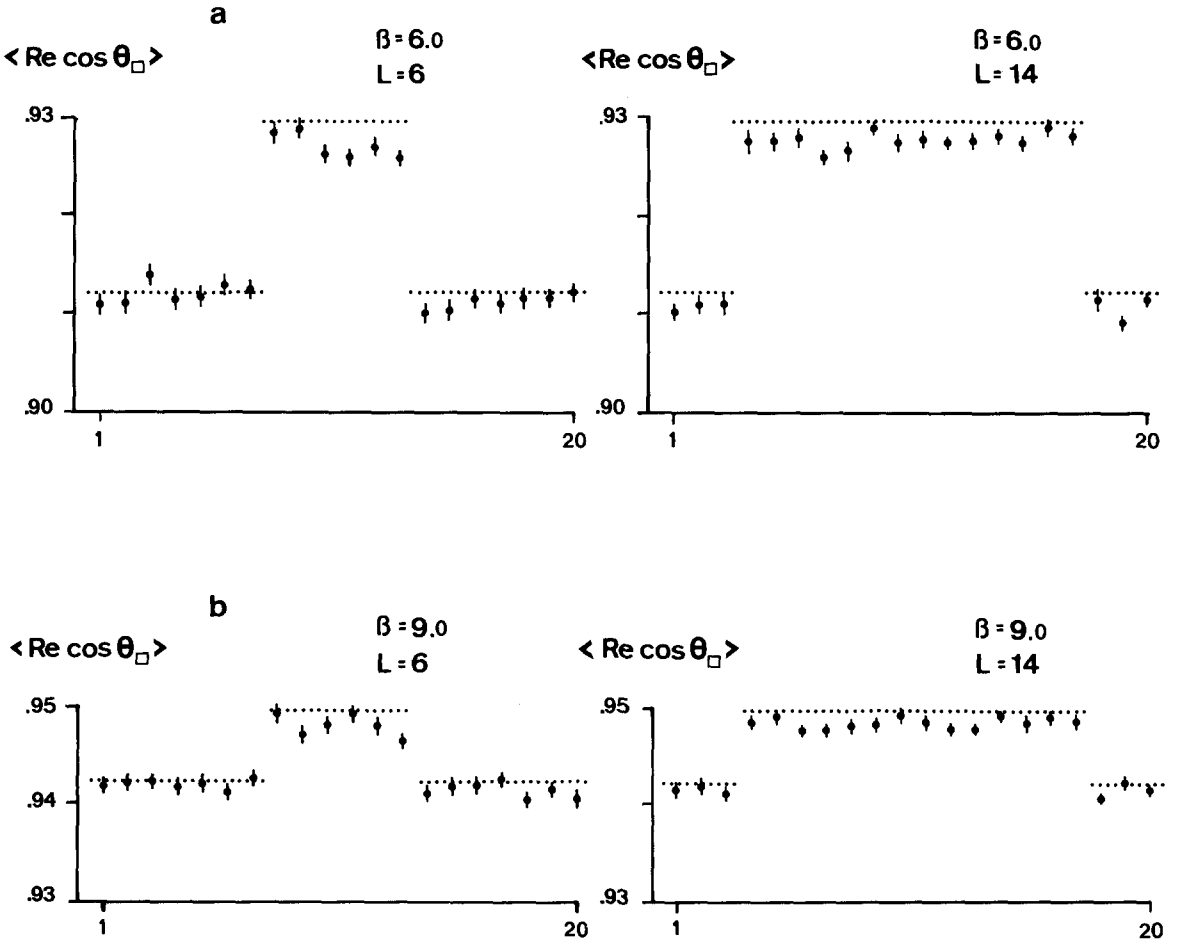

Fig. 3. Measured averages of $\mathrm{Re}\cos\theta_{\square}$ for $\beta = 6.0$ (a) and 9.0 (b) (see table 2). Dotted lines denote theoretical values calculated on an infinite lattice.

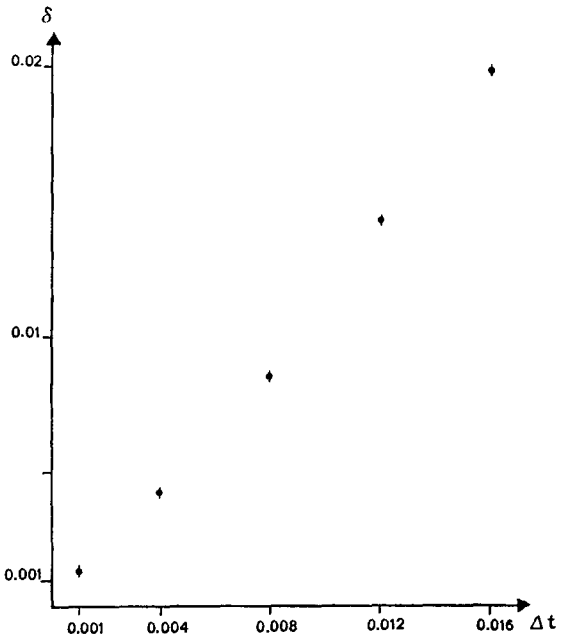

Fig. 4. Difference between the theoretical value and Monte Carlo results for $\mathrm{Re}\cos\theta_{\square}$ averaged between the charges for $\beta = 6.0$ and $L = 14$ for different time increments Δt. The number of sweeps for each Δt were chosen such that $t = n\Delta t$ is the same for all Δt; $t_{\mathrm{therm}} = 80$ and $t_{\mathrm{meas}} = 200$ respectively. (The statistics in this figure is thus roughly a factor 5 larger than in figs. 3a and 3b).

4. Discussion. The complex Langevin equation seems to be a powerful tool for generating complex "configurations" relevant to the charged sectors of gauge theories. For instance it only required the modest computer time of 1.5 VAX 11/780 hours to measure the string tension for charges separated by 14 lattice units on a 20×20 lattice at $\beta = 6.0$. Using conventional methods one would have to measure a Wilson loop with expectation value $[I_1(\beta)/I_0(\beta)]^{14\times 20} = 6 \times 10^{-12}$. In addition one gets information about field strength between the charges, which in higher dimensions can be used to investigate phenomena like string roughening in more detail [4]. It also seems as if a clever choice of boundary conditions allows one to use the lattice volume quite effectively. As mentioned we could, using bag boundary conditions, take the charge almost to the boundary.

The method still calls for a better theoretical understanding, and the high gains which seem possible when applying it to non-abelian theories should urge an extension of the pioneering work of Klauder, Parisi and Gozzi [8–12].

We are presently studying three-dimensional U(1) and non-abelian theories with these methods. The results look encouraging.

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