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ON THE COMPLEX LANGEVIN EQUATION*

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We discuss the Langevin equation for a complex Boltzmann distribution, allowing for modifications of the process. The relation between the two different time development operators involved is analyzed, with emphasis on their spectra.

1. Introduction

Sometimes one has to deal with actions that are complex. This is the case in euclidean lattice gauge theory with static charges [1], or in theories with Wess-Zumino terms in the action [2]. The corresponding Boltzmann distribution cannot then be regarded as a probability distribution, and some of the standard Monte Carlo methods (Metropolis, heat bath etc.) cannot be applied.

The Langevin method [3] for generating a given distribution is based on a stochastic process in an artificial time. For a real action, and under very general conditions, the statistical distribution in configurations will converge in artificial time to the desired Boltzmann distribution.

When this method is applied to a system with a complex action, supposedly real variables will diffuse out into the complex plane, and accordingly the resulting distribution will not be confined to real values on the variables [4]. Under certain conditions, however, expectation values of analytic functions of the dynamical variables will converge to the correct values, as obtained from the correct complex-valued Boltzmann distribution.

In this paper we will consider the Langevin method with complex analytic actions in a configuration space R^d. The mechanism responsible for the applicability of the method will be thoroughly analyzed, and the effects of various modifications of the method investigated.

The paper is organized as follows. In sect. 2 we briefly analyze the Langevin method for a real action. The corresponding analysis for the complex case is done in sect. 3, together with a discussion of the mechanism responsible for making the

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method work under certain conditions. Effects of a finite time-step are also considered. In sect. 4, we investigate what kind of distributions in the complex plane will simulate a given complex Boltzmann distribution on the real line. Some particular modifications of the Langevin process will be considered in sect. 5, as well as an alternative approach. In sect. 6, finally, we present an application to a simple example, a complex gaussian, illustrating some of the ideas of the previous sections.

2. The Langevin equation with a real action

Consider a system of real variables x_i, with a desired Boltzmann distribution $\rho_0(x) = \exp(-S(x))$. The standard Langevin method for generating this distribution is based on a stochastic process in a fictitious time t:

$$\dot{x}_i = -\partial_i S(x) + \dot{W}_i(t), \tag{2.1}$$

where W_i is a gaussian white noise with the normalization

$$\langle \dot{W}_i(t) \cdot \dot{W}_j(t') \rangle = 2\delta_{ij}\delta(t-t'). \tag{2.2}$$

On a computer, one has to use the discretized version

$$\Delta x_i = -\varepsilon\, \partial_i S + \sqrt{\varepsilon}\,\eta_i, \tag{2.3}$$

where ε is the time step, and η_i a gaussian noise with

$$\langle \eta_i \eta_j \rangle = 2\,\delta_{ij}. \tag{2.4}$$

In the continuum limit, the real probability distribution $\rho(x)$ will evolve according to

$$\dot{\rho} = -\partial_i J_i, \tag{2.5}$$

with the probability current J_i given by

$$J_i = -\left(\hat{\partial}_i + \partial_i S\right)\rho. \tag{2.6}$$

We thus obtain the diffusion equation for ρ,

$$\dot{\rho} = \hat{\partial}_i\left(\hat{\partial}_i + \partial_i S\right)\rho = -\hat{K}\rho. \tag{2.7}$$

By a rescaling with $\exp(\frac{1}{2}S)$, the evolution operator $\hat{K}$ is transformed into the self-adjoint Fokker-Planck hamiltonian:

$$\hat{H}_{\mathrm{FP}} = \left(-\hat{\partial}_i + \tfrac{1}{2}\,\partial_i S\right)\left(\hat{\partial}_i + \tfrac{1}{2}\,\partial_i S\right) = \left(\hat{\partial}_i + \tfrac{1}{2}\,\partial_i S\right)^{\dagger}\left(\hat{\partial}_i + \tfrac{1}{2}\,\partial_i S\right). \tag{2.8}$$

This is a positive semi-definite operator, with the unique ground state

$$\psi_0 = \exp\left(-\tfrac{1}{2}S\right), \tag{2.9}$$

corresponding to the equilibrium distribution

$$\rho_0 = \exp(-S). \tag{2.10}$$

This guarantees the convergence of ρ to ρ_0. We note that for ρ_0, detailed balance is satisfied, since the current in eq. (2.6) vanishes itself, and not only its divergence. The Langevin eq. (2.3) is in no way unique. It can be modified to the more general process

$$\Delta x_i = \varepsilon v_i(x) + \sqrt{\varepsilon} A_i^{(k)}(x)\eta_k \tag{2.11}$$

where the drift term v_i must satisfy the constraint

$$v_i = \partial_j g_{ij} - \partial_j S \cdot g_{ij}. \tag{2.12}$$

Here, the noise correlation g_{ij} is defined as

$$g_{ij} \equiv A_i^{(k)} A_j^{(k)}. \tag{2.13}$$

The noise amplitudes $A_i^{(k)}$ are arbitrary real functions of x, and should be chosen so as to avoid stability problems due to a finite time step. This type of problem can sometimes be avoided by a dynamical choice of time step, ensuring that $|\Delta x|$ is always small [5]. The corresponding modified Fokker-Planck hamiltonian is given by

$$\hat{H}_{\mathrm{FP}} = \left(-\hat{\partial}_i + \tfrac{1}{2}\partial_i S\right) g_{ij} \left(\hat{\partial}_j + \tfrac{1}{2}\partial_j S\right), \tag{2.14}$$

or, differently expressed,

$$\hat{H}_{\mathrm{FP}} = \left[A_i^{(k)}\left(\hat{\partial}_i + \tfrac{1}{2}\partial_i S\right)\right]^{\dagger} \cdot \left[A_j^{(k)}\left(\hat{\partial}_j + \tfrac{1}{2}\partial_j S\right)\right], \tag{2.15}$$

so that convergence is guaranteed, provided the noise amplitude matrix $A_i^{(k)}$ is non-degenerate.

3. The Langevin equation with a complex action

When $S(x)$ is complex, the Langevin process (2.11) might still be considered. The drift term (or the noise) will then prevent the supposedly real variables from staying on the real axis. We will therefore denote the variables z_i instead of x_i, and the corresponding complex conjugate z_i^*. The corresponding probability density will be denoted $P(z, z^*)$. We will assume that the action is an analytic function $S(z)$.

The process (2.11) will then look as follows.

$$\Delta z_i = \varepsilon v_i(z) + \sqrt{\varepsilon} A_i^{(k)}(z) \cdot \eta_k$$

$$\Rightarrow \Delta z_i^* = \varepsilon v_i^*(z^*) + \sqrt{\varepsilon} A_i^{(k)*}(z^*) \eta_k . \tag{3.1}$$

We define the complex generalization of the noise correlation

$$g_{ij}(z) \equiv A_i^{(k)} A_j^{(k)} ,$$

$$h_{ij}(z, z^*) \equiv A_i^{(k)} A_j^{(k)*} = h_{ji}^* ,$$

$$g_{ij}^*(z^*) \equiv A_i^{(k)*} A_j^{(k)*} . \tag{3.2}$$

With v_i still given by (2.12), we obtain the probability currents

$$J_i(z, z^*) = \left(-\hat{\partial}_j g_{ji} - \hat{\partial}_j^* h_{ji} + v_i \right) P(z, z^*) ,$$

$$J_i^* = \left(-\hat{\partial}_j^* g_{ji}^* - \hat{\partial}_j h_{ji}^* + v_i^* \right) P . \tag{3.3}$$

By the conservation of probability, the evolution equation for P follows:

$$\dot{P} = -\partial_i J_i - \partial_i^* J_i^* = \hat{\partial}_i \left(\hat{\partial}_j g_{ji} + \hat{\partial}_j^* h_{ji} - v_i \right) P$$

$$+ \hat{\partial}_i^* \left(\hat{\partial}_j^* g_{ji}^* + \hat{\partial}_j h_{ij} - v_i^* \right) P = -\hat{K}_c P . \tag{3.4}$$

This defines the time evolution operator $\hat{K}_c$. Now, consider the time evolution of the expectation value of an analytic variable $f(z)$. One readily obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f(z) \rangle = -\langle \hat{K}_c^{\mathrm{T}} f \rangle$$

$$\equiv \left\langle \left(g_{ij} \hat{\partial}_i + h_{ij} \hat{\partial}_i^* + v_i \right) \hat{\partial}_j f + \left(g_{ij}^* \hat{\partial}_i^* + h_{ij}^* \hat{\partial}_i + v_i^* \right) \hat{\partial}_j^* f \right\rangle , \tag{3.5}$$

where the superscript T stands for transpose. By the analyticity of f, this will reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f(z) \rangle = \left\langle \left(g_{ij} \hat{\partial}_i + v_i \right) \hat{\partial}_j f \right\rangle = -\left\langle \hat{K}^{\mathrm{T}} f \right\rangle . \tag{3.6}$$

Hence, with $A_i^{(k)}$ analytic (and thus also v_i and g_{ij}), the non-analytic part of $\hat{K}_c$ is irrelevant, and only its "analytic projection" $\hat{K}$ has to be considered. Obviously, the spectrum of $\hat{K}$ will be a subset of the complete spectrum of $\hat{K}_c$. The time evolution

(3.6) simulates the result for $\langle f(x)\rangle$ from a distribution $\rho(x)$ on the real line, with a time evolution

$$\dot{\rho}(x) = \hat{\partial}_j\big(\hat{\partial}_i g_{ij} - v_j\big)\rho(x) = -\hat{K}\rho\,. \tag{3.7}$$

As in the real case, we define the Fokker-Planck hamiltonian $\hat{H}_{\mathrm{FP}}$ as (cf. eq. (2.14))

$$\begin{aligned}\hat{H}_{\mathrm{FP}} &\equiv \mathrm{e}^{S/2}\hat{K}\,\mathrm{e}^{-S/2} = \big(-\hat{\partial}_i + \tfrac{1}{2}\,\partial_i S\big)g_{ij}\big(\hat{\partial}_j + \tfrac{1}{2}\,\partial_j S\big)\\ &= \big(\hat{\partial}_i + \tfrac{1}{2}\,\partial_i S\big)^{\mathrm{T}} g_{ij}\big(\hat{\partial}_j + \tfrac{1}{2}\,\partial_j S\big)\,.\end{aligned} \tag{3.8}$$

Although $\hat{H}_{\mathrm{FP}}$ is not self-adjoint for a complex S, it is still symmetric. Thus, the eigenfunctions will form a complete, orthogonal set in R^d, provided the scalar product is defined without complex conjugation. The spectrum will however be complex, and convergence might be destroyed by an eigenvalue with negative real part, at least if $\mathrm{Im}\,S$ is not small. The situation might however depend crucially on the choice of g_{ij}.

The updating algorithm (3.1) with a finite time step ε gives rise to a transfer operator $\hat{T}_{\mathrm{c}}(\varepsilon)$, which is supposed to be a good approximation to the continuum version $\exp(-\varepsilon\hat{K}_{\mathrm{c}})$.

With a gaussian noise, one can actually obtain an exact formal expression for $\hat{T}_{\mathrm{c}}(\varepsilon)$. With $\hat{K}_{\mathrm{c}}$ expressed as in (3.4), $\hat{T}_{\mathrm{c}}(\varepsilon)$ is given by

$$\hat{T}_{\mathrm{c}}(\varepsilon) = :\exp\big(-\varepsilon\hat{K}_{\mathrm{c}}\big):\,, \tag{3.9}$$

where : : stands for the normal ordering prescription; derivatives to the left and functions to the right. Again, when considering analytic expectation values, it suffices to use the reduced version

$$\hat{T}(\varepsilon) = :\exp(-\varepsilon\hat{K}):\,. \tag{3.10}$$

This expression can be used, e.g. by making an expansion in ε, to construct improved versions of the discrete Langevin updating algorithm. In what follows, we will assume that $\hat{T}(\varepsilon)$ approximates $\exp(-\varepsilon\hat{K})$ well for a small enough ε. For an analytic expectation value, we thus obtain

$$\langle f(z)\rangle_{t+\varepsilon} = \langle \hat{T}^{\mathrm{T}} f\rangle_t\,. \tag{3.11}$$

For a Monte Carlo error analysis, the reduced spectrum does not quite suffice. Suppose that after thermalization an analytic observable $f(z)$ is measured at N consecutive times, and an experimental average is formed:

$$\bar{f} = \frac{1}{N}\sum_{k=1}^{N} f\big(z(t_0 + k\varepsilon)\big) \equiv \frac{1}{N}\sum_{1}^{N} f_k\,. \tag{3.12}$$

With a real action, and $f(x)$ real-valued, we would define the ensemble variance of the average as

$$\sigma_{\bar{f}} = \langle\langle \bar{f} \cdot \bar{f} \rangle\rangle_{\mathrm{C}}, \tag{3.13}$$

with $\langle\langle\ \rangle\rangle_{\mathrm{C}}$ standing for connected ensemble average. This would satisfy the inequality

$$\frac{1}{N}\langle f^2\rangle_{\mathrm{C}} \leqslant \sigma_{\bar{f}} \leqslant \frac{1}{N}\coth\tfrac{1}{2}\varepsilon\omega_1\langle f^2\rangle_{\mathrm{C}}, \tag{3.14}$$

with ω_1 the eigenvalue (assumed non-zero) of the first excited state of $\hat{H}_{\mathrm{FP}}$, and $\langle f^2\rangle_{\mathrm{C}}$ the ground state variance of f.

If the measurements are separated by a number of time steps m, large enough for the auto-correlation to die out, i.e. if

$$\varepsilon m\omega_1 \ll 1, \tag{3.15}$$

then the ensemble average will be well approximated by the lower limit:

$$\sigma_{\bar{f}} \approx \frac{1}{N}\langle f^2\rangle_{\mathrm{C}}. \tag{3.16}$$

With a complex action, this analysis is not relevant, and the full spectrum of $\hat{K}_{\mathrm{c}}$ would in principle be required. With a small enough Im S, though, eq. (3.16) could be a good approximation.

4. Faking complex probabilities

The complex Langevin algorithm (3.1) will, provided the complete spectrum of $\hat{K}_{\mathrm{c}}$ consists entirely of eigenvalues with positive real parts, lead to a stable, real distribution P in the complex variables z_i and z_i^*. This distribution simulates the Boltzmann distribution $\rho = \exp(-S(x))$ in the sense

$$\int \mathrm{d}^d \operatorname{Re} z_i\, \mathrm{d}^d \operatorname{Im} z_i P(z, z^*) f(z) = \int \mathrm{d}^d x_i\, \rho(x) f(x) \qquad \text{for } f \text{ analytic}. \tag{4.1}$$

Obviously, P will depend on the particular version of updating being used.

As has been suggested by Ambjørn et al. [4], if the explicit form of P were known, it could be generated in some alternative way, e.g. by a Langevin process in terms of the real variables Re z_i, Im z_i, using $-\log P$ as an action. This would be a way of circumventing the stability problems often encountered when using the complex Langevin method. In this section, we will investigate what kind of options one has in the choice of P, given ρ. For simplicity we will work with only a single variable x.

Thus we want a transformation that takes ρ into an admissible P, i.e. one such that (4.1) is satisfied. Let us begin by considering a related problem: What are the transformations that conserves admissibility? Consider for simplicity only linear transformations,

$$P_2(z, z^*) = \int \mathrm{d}^2\omega\, K(z, z^*; \omega, \omega^*) P_1(\omega, \omega^*). \tag{4.2}$$

Conservation of analytic moments then leads to the condition

$$\int \mathrm{d}^2 z\, z^k K(z, z^*; \omega, \omega^*) = \omega^k. \tag{4.3}$$

The simplest solution is a spherically symmetric convolution, i.e.

$$K(z, z^*; \omega, \omega^*) = f(|z-\omega|^2), \tag{4.4}$$

with f properly normalized, but otherwise arbitrary. Proof:

$$\int \mathrm{d}^2 z f(|z-\omega|^2) z^k = \int \mathrm{d}^2 z f(z^* z)(z+\omega)^k = \omega^k \int \mathrm{d}^2 z f(z^* z). \tag{4.5}$$

A more general solution is obtained by modifying f with an explicit analytic dependence on $z-\omega$, and an arbitrary dependence on ω and ω^*,

$$K = f(|z-\omega|^2, z-\omega, \omega, \omega^*), \tag{4.6}$$

provided f satisfies

$$\int \mathrm{d}^2 z f(z^* z, z, \omega, \omega^*) = 1 \tag{4.7}$$

independently of ω, ω^*.

This kind of a linear transformation can now be applied to ρ, considered as a distribution in the complex plane, and will lead to an admissible P,

$$P(z, z^*) = \int_{\mathbf{R}} \mathrm{d}x\, K(z, z^*; x, x) \rho(x). \tag{4.8}$$

Since ρ is assumed analytic, another way of transforming ρ is by simply changing the path of integration from the real line. Thus, if ρ on the real line corresponds to

$$P_0(z, z^*) = \rho(z) \cdot \delta\left(\frac{z-z^*}{2i}\right), \tag{4.9}$$

it will, on another path γ, correspond to

$$P_1(z, z^*) = 2i\rho(z) \cdot \delta(F_\gamma) \cdot \partial^* F_\gamma(z, z^*), \tag{4.10}$$

where F_γ is a real-valued function defining γ by the condition $F_\gamma = 0$. In some cases it is possible to find a particular path γ_0, such that eq. (4.8) defines a real, positive P_1. This means that γ_0 should be chosen in such a way that the integrand in the integral

$$\int_{\gamma_0} \mathrm{d}z\, \rho(z) \qquad (=1) \tag{4.11}$$

is everywhere real. This condition defines a direction field

$$\mathrm{d}z \left\| \frac{1}{\rho(z)} \right. . \tag{4.12}$$

Thus, the "real path" γ_0 should everywhere be tangent to this direction field, it should be simply connected, and it should connect $-\infty$ with $+\infty$. The corresponding P_1 could then be transformed according to eq. (4.2), with a real, positive kernel of the type (4.4) without the explicit $(z-\omega)$-dependence (in order for the kernel to be real).

To find γ_0 we define a real parametrization σ from

$$\rho(z) = \frac{\mathrm{d}\sigma}{\mathrm{d}z}. \tag{4.13}$$

Defining a particular primitive function λ of ρ,

$$\lambda(z) = \int_{-\infty}^{z} \rho(z')\, \mathrm{d}z', \tag{4.14}$$

the real path γ_0 will be defined in terms of λ as

$$\lambda \text{ real}, \qquad 0 < \lambda < 1. \tag{4.15}$$

For cases in which this defines a simply connected path, it turns out that the resulting distribution,

$$P_{\gamma_0}(z, z^*) = -|\rho(z)|^2 \delta(\operatorname{Im} \lambda(z)) \tag{4.16}$$

might be possible to generate with a quite general method, to be considered in the next section.

5. Some particular modifications of the Langevin approach

In this section we will consider the problem of how to modify the standard Langevin approach as to get rid of instabilities. We will also discuss a few interesting particular modifications. For simplicity the discussion will be concentrated to the case of one variable. The modified Langevin equation then reads

$$\Delta z = \varepsilon v(z) + \sqrt{\varepsilon} A(z)\eta, \tag{5.1a}$$

$$\Delta z^* = \varepsilon \cdot v^*(z^*) + \sqrt{\varepsilon} A^*(z^*)\eta. \tag{5.1b}$$

Instability problems might occur whenever $|v|$ or $|A|$ is large, especially if the drift term v is repulsive. Since both terms are analytic, at least one of them will be unbounded for realistic cases. For a complex gaussian, $S = \frac{1}{2}\sigma z^2$, $\mathrm{Re}\,\sigma > 0$, the standard Langevin equation will be

$$\Delta z = -\varepsilon\sigma z + \eta\sqrt{\varepsilon}\,. \tag{5.2}$$

The drift term will always point more or less to the origin, leading to a stable situation. In more complicated cases though, this will not be true. Consider e.g. the anharmonic "oscillator"

$$S = \tfrac{1}{2}\sigma z^2 + \tfrac{1}{4}z^4, \tag{5.3}$$

with a standard drift term being dominantly cubic for large $|z|$, attractive (repulsive) along the real (imaginary) axis.

In some cases it might help introducing a formally complex ε (for z, ε^* for z^*). This will have the effect of a rotation of the spectrum of the Fokker-Planck hamiltonian, which could improve the convergence. One could also imagine having two independent noise terms with different phases. This will have no effect on the Fokker-Planck spectrum though, that can't be obtained with only one noise term. The main effect would be a smearing of the equilibrium distribution. A stable situation would seem to result, if one could arrange for a drift term asymptotically being of the form

$$v \approx -z \qquad \text{for } |z| \to \infty\,. \tag{5.4}$$

This would require $g = A^2$ to have the asymptotic behaviour

$$g \approx \mathrm{e}^{S(z)} \int_{z_0}^{z} z' \mathrm{e}^{-S(z')}\,\mathrm{d}z', \tag{5.5}$$

which obviously is possible for a gaussian with power corrections.

An interesting fact is, that one can always arrange for a vanishing drift term (also in higher dimensions) by choosing

$$g = c^2 \mathrm{e}^{S}, \qquad c = \text{complex constant}. \tag{5.6}$$

The updating algorithm will then only have a noise term

$$\Delta z = \sqrt{\varepsilon}\, c \exp\left(\tfrac{1}{2}S\right) \cdot \eta . \tag{5.7}$$

With $\exp(-S(x))$ normalizable on the real axis though, the noise amplitude would grow large for large $|x|$, leading to instability, unless some kind of cutoff is introduced. With a real action (and a real c) however, the method will work perfectly well on any compact manifold of any dimension. The mechanism is analogous to what happens when one pours sand on a vibrating membrane: the sand will rapidly concentrate where the vibration amplitude is small.

Another intriguing choice of g is

$$g = c^2 \mathrm{e}^{2S}, \tag{5.8}$$

leading to the Langevin equation

$$\Delta z = c\sqrt{\varepsilon}\ \mathrm{e}^{S}\eta + c^2 \varepsilon S' \mathrm{e}^{2S}. \tag{5.9}$$

The evolution equation for P will be

$$\dot{P} = \left(c\,\hat{\partial}\,\mathrm{e}^{S} + c^*\,\hat{\partial}^*\,\mathrm{e}^{S^*}\right)^2 P. \tag{5.10}$$

Using the transformed variable λ (familiar from sect. 4),

$$\lambda = \int_{-\infty}^{z} \mathrm{e}^{-S(z')}\,\mathrm{d}z' \tag{5.11}$$

and the correspondingly transformed distribution Q,

$$Q = \mathrm{e}^{S}\mathrm{e}^{S^*}P, \tag{5.12}$$

eq. (5.10) becomes the simple heat equation,

$$\dot{Q} = \left(c\,\partial_\lambda + c^*\,\partial_{\lambda^*}\right)^2 Q, \tag{5.13}$$

with the obvious solution

$$Q(\lambda) = f\left(\mathrm{Im}\,\frac{1}{c}\lambda\right). \tag{5.14}$$

With a real c, and starting with a real λ (which often could be obtained from symmetry considerations), f would be a δ-function. This is of course nothing but the real path of the preceding section. Obviously there will be instabilities for large real z, though these might be controlled by a cutoff for large $|z|$.

A completely different approach can be used in cases where one is interested only in averages of the kind

$$\mathrm{Re}\langle f(z)\rangle, \tag{5.15}$$

with f real on the real line. Then only the real part of the Boltzmann distribution will contribute, since

$$\mathrm{Re}\langle f(x)\rangle_\rho = \langle f(x)\rangle_{\mathrm{Re}\,\rho}. \tag{5.16}$$

If $\mathrm{Re}\,\rho$ is positive on the entire real line, one can generate it using any of the standard methods, and there is no need at all to consider the complex problem. Further, if the real line can be divided into a finite number of domains, within which the sign of $\mathrm{Re}\,\rho$ is fixed, one can still generate $|\mathrm{Re}\,\rho|$ separately in the different domains, and add the results of measurements together in the end with proper weights, obtained e.g. numerically. This is an approach that would apply also to systems with several variables, like e.g. euclidean U(1) lattice gauge theory with static charges.

6. Application to a simple example

In this section, we will apply some of the ideas of the earlier sections to a simple example: the complex gaussian in one variable, given by

$$S = \tfrac{1}{2}\sigma z^2, \qquad \mathrm{Re}\,\sigma > 0. \tag{6.1}$$

The Langevin equation (5.2) modified with a phase, becomes

$$\Delta z = -\varepsilon e^{i\varphi}\sigma z + \sqrt{\varepsilon e^{i\varphi}}\,\eta, \tag{6.2}$$

leading to the evolution for P

$$\dot{P} = \left[e^{i\varphi}\hat{\partial}\left(\hat{\partial} + \sigma z\right) + 2\,\hat{\partial}\,\hat{\partial}^* + e^{-i\varphi}\hat{\partial}^*\left(\hat{\partial}^* + \sigma^* z^*\right)\right]P = -\hat{K}_c P. \tag{6.3}$$

The operator $\hat{K}_c$ is recognized as a two-dimensional complex harmonic oscillator, with the spectrum

$$\omega_{k,l} = k\sigma e^{i\varphi} + l\sigma^* e^{-i\varphi}, \qquad k, l \geqslant 0. \tag{6.4}$$

This certifies stability of the ground state

$$P_0 = \exp\left[\frac{\mathrm{Re}(e^{i\varphi}\sigma)}{\left(\mathrm{Im}(e^{i\varphi}\sigma)\right)^2}\left(\tfrac{1}{2}\,\mathrm{Re}(e^{i\varphi}\sigma)\,\sigma z^2 - \sigma\sigma^* z z^* + \tfrac{1}{2}\,\mathrm{Re}(e^{i\varphi}\sigma)\cdot\sigma^* z^{*2}\right)\right],$$

$$\text{for } \mathrm{Re}(e^{i\varphi}\sigma) > 0. \tag{6.5}$$

The reduced operator $\hat{K}$ of eq. (3.7) becomes

$$\hat{K} = e^{i\varphi}\hat{\partial}_x\left(\hat{\partial}_x + \sigma x\right), \tag{6.6}$$

with the reduced spectrum

$$\omega_k = k\sigma e^{i\varphi}, \qquad k = 0, 1, \ldots \tag{6.7}$$

and the ground state (as expected):

$$\rho_0 = \exp\left(-\tfrac{1}{2}\sigma x^2\right) = \exp(-S(x)). \tag{6.8}$$

The effects of a finite ε in this case will only cause a complex renormalization of the parameters, illustrated by the true ground state

$$\rho_0 = \exp\left(-\tfrac{1}{2}\sigma\left(1 - \tfrac{1}{2}\varepsilon\sigma e^{i\varphi}\right)x^2\right). \tag{6.9}$$

The distribution (6.5) can be parametrized in a simpler way using the real, positive parameter

$$a = \left(\frac{\mathrm{Re}(e^{i\varphi}\sigma)}{\mathrm{Im}(e^{i\varphi}\sigma)}\right)^2. \tag{6.10}$$

We thus obtain

$$P_0 = \exp\left(\tfrac{1}{2}a\sigma z^2 - \sqrt{a(a+1)}\,|\sigma|\,|z|^2 + \tfrac{1}{2}a\sigma^* z^{*2}\right). \tag{6.11}$$

That this distribution really simulates the Boltzmann distribution (6.8), can be understood by formally integrating over z^*, keeping z fixed.

It is interesting to note, that for $a \to \infty$, P_0 shrinks to a line distribution, of course corresponding to the familiar real path, in this case a straight line. Another interesting point, outside the allowed region, is $a = -1$, corresponding to the factorizable distribution $\exp(-(S + S^*))$.

It is straightforward to generate P_0 from ρ_0, following the ideas of sect. 4, with a convolution

$$P_0(z, z^*) \propto \int \mathrm{d}x\, e^{-\sigma x^2/2} f\left(|z - x|^2, z - x\right), \tag{6.12}$$

with f given by

$$f(z^* z, z) = \exp\left[-\frac{a\sigma^*\sigma}{\left(|\sigma|\sqrt{a(a+1)} - \sigma^* a\right)^2}\right.$$
$$\left.\times\left(-ia\,\mathrm{Im}\,\sigma \cdot z^2 + \left(|\sigma|\sqrt{a(a+1)} - \sigma^* a\right)z^* z\right)\right], \tag{6.13}$$

in accordance with eq. (4.4), with a trivial (ω, ω^*) dependence.

7. Conclusions

The above analysis of the mechanisms behind the complex Langevin equation has hopefully provided the reader with some understanding of its advantages, as well as the mathematical problems connected to it. A complete solution to these problems is not given, but a number of possible lines of approach are suggested, one of which might be fruitful.

However, to gain full control over the method, a better understanding of the relation between the full time evolution operator $\hat{K}_c$ and its analytic projection $\hat{K}$, and the connection to the relation between the corresponding eigenstates $P(z, z^*)$ and $\rho(z)$, would be required.

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References

[1] C. Peterson and L. Sköld, Nucl. Phys. B255 (1985) 365
[2] G. Bhanot, E. Rabinovici, N. Seiberg and P. Woit, Nucl. Phys. B230 (1984) 291
[3] G. Parisi and Wu Y.-S., Sci. Sin. 24 (1981) 483
[4] G. Parisi, Phys. Lett. 131B (1983) 393;
J. R. Klauder, Phys. Rev. A29 (1984) 2036;
J. Ambjørn, M. Flensburg and C. Peterson, Phys. Lett. 159B (1985);
P. H. Damgaard and H. Hüffel, Phys. Reports 152 (1987) 227–398
[5] J. Ambjørn, M. Flensburg and C. Peterson, Nucl. Phys. B275{FS17} (1986) 375