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LATTICE QCD WITH FERMIONS AT STRONG COUPLING: A DIMER SYSTEM

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Lattice QCD with Susskind fermions and gauge group U(N) is shown to be equivalent to a generalized dimer system, when the gauge coupling is infinite. In that formulation chiral symmetry breaking is investigated numerically. Results are confronted with large-N mean field calculations, where the first $1/N$ correction to the chiral order parameter is computed. It is argued that the PCAC relation between the pion mass and the quark mass can be considered as a mean field scaling relation, and that for this system the mean field approximation is valid in dimension 4.

1. Introduction

Lattice QCD at strong coupling has been useful for understanding qualitatively the long-distance structure of the physical continuum theory that is presumably embodied in the nonperturbative (e.g. $\propto e^{-c/g^2}$) content of the weak coupling limit. The pure gauge theory at strong coupling produces almost trivially a model of confinement with Wilson loops decaying exponentially with the enclosed minimal area. In the absence of a phase transition for some family of actions interpolating from strong to weak coupling this feature should persist in the continuum limit. Of course, to really derive confinement for the continuum theory, the correct renormalization group scaling has to be observed. Unfortunately, this question has so far been accessible to numerical analysis only.

Recently [1,2] it has been found, that a similarly appealing picture holds for the realization of chiral symmetry at strong coupling, once fermionic fields are included. In the extreme strong coupling limit (no Wilson plaquette term) with staggered Kogut-Susskind fermions the exact chiral U(1) invariance present in the lattice theory breaks spontaneously. The appearance of a Goldstone boson – tentatively

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called pion – can be demonstrated. More precisely, we consider the euclidean action

$$S(\bar\psi_a,\psi_a,U)=\tfrac{1}{2}\sum_{\substack{\ell\\ x\to x'}}\Gamma_\ell\left[\bar\psi(x)U_\ell\psi(x')-\bar\psi(x')U_\ell^\dagger\psi(x)\right]+m\sum_x\bar\psi(x)\psi(x),$$

$$a=1,\dots,N. \quad (1.1)$$

Here the first sum runs over all links that lead in the positive direction from site x to x'. U_ℓ are compact U(N) or SU(N) gauge fields and Γ_ℓ are the Kogut-Susskind phase factors derived from the Dirac matrices. A popular representation is

$$\Gamma_\ell=(-)^{x_1+\cdots+x_{\mu-1}}\ \text{for the link}\ x'=x+\hat\mu. \quad (1.2)$$

The chiral U(1) transformation (d = euclidean dimension)

$$\left.\begin{array}{l}\psi(x)\to e^{i\alpha(-)^x}\psi(x)\\ \bar\psi(x)\to e^{i\alpha(-)^x}\bar\psi(x)\end{array}\right\}\qquad (-)^x=(-)^{x_1+\cdots+x_d} \quad (1.3)$$

is a symmetry of the action (1.1) for $m=0$ (only odd sites coupled to even sites). $\langle\bar\psi\psi(x)\rangle$ is an order parameter for (1.3), and chiral symmetry is broken if

$$\lim_{m\to 0_+}\lim_{V\to\infty}\langle\bar\psi\psi(x)\rangle\neq 0, \quad (1.4)$$

where V is the volume (number of sites), and

$$\langle\bar\psi\psi(x)\rangle=\frac{1}{Z(m)}\int D\psi D\bar\psi DU e^{S(\bar\psi,\psi,U)}\bar\psi(x)\psi(x). \quad (1.5)$$

$\int DU$ means an integration with the Haar measure for every link, $Z(m)$ is fixed by $\langle 1\rangle=1$ and $\int D\psi D\bar\psi$ are Grassmann integrals. The situation resembles somewhat a magnet or spin system with m analogous to a magnetic field and $\bar\psi\psi$ to a magnetization. However the statistical mechanics of euclidean fermionic bilinears is rather different. While a strong magnetic field leads to a complete polarization of spins (for Ising systems typically $\langle\sigma\rangle=\tanh H$), in (1.4) the mass term alone would give

$$\frac{1}{N}\langle\bar\psi\psi\rangle=\frac{1}{m}. \quad (1.6)$$

Thus the interaction rather than generating coherence between spins leading to symmetry breaking has first to regularize the divergence in (1.6). With ordered $U_\ell\equiv 1$ (free fermions) $\langle\bar\psi\psi\rangle\to_{m\to 0}0$ results, while randomizing the links with the

Haar measure gives $\langle\bar{\psi}\psi\rangle \to_{m\to 0}$finite. This is another aspect of the intuitive picture [3] that confinement (disordered U_ℓ's) and chiral symmetry breakdown are intimately correlated.

$\langle\bar{\psi}\psi\rangle \neq 0$ has so far been demonstrated analytically by graphical [1] and by mean field [2] methods. In all cases some approximation beyond infinite coupling is required, either $N \to \infty$ or $d \to \infty$. Both lead automatically to quenched averaging too, where closed fermionic loops do not manifest themselves to leading order. In this article we show that for finite d, N strong coupling QCD with fermions is equivalent to a generalization of what is known as a dimer [4, 5] system in statistical mechanics (for SU(N) there are additional barionic (see below) variables). Apart from yielding an interesting point of view for a fermionic model, this formulation allows a computer simulation that provides high precision data with a numerical effort, that is negligible as compared to dynamical fermion algorithms simulating the determinant. The dimer formulation is explained in sect. 2. Sect. 3 contains some analytic mean field results for large N, including the first $1/N$ correction for $\langle\bar{\psi}\psi\rangle$. In sect. 4 we summarize our numerical data, confront them with analytic results, and offer some conclusions.

2. Generalized dimers alias mesons

In the Boltzmann factor corresponding to action (1.1) we can first carry out the U integration on every link. This is accomplished by the exact one link integral in grassmannian sources derived for SU(N) and U(N) in the appendix. The U(N) result reads (A.2)

$$H(\bar{\chi}\chi\bar{\varphi}\varphi) = \int DU \mathrm{e}^{\bar{\chi}U\varphi - \bar{\varphi}U^\dagger\chi} = \sum_{k=0}^{N} \alpha_k (\bar{\chi}\chi\bar{\varphi}\varphi)^k \tag{2.1}$$

$$\text{with } \alpha_k = \frac{(N-k)!}{k!N!}. \tag{2.2}$$

Strong coupling U(N) QCD is thus converted into a self-coupled fermionic theory:

$$Z(m) = \int D\psi D\bar{\psi}\, \mathrm{e}^{m\Sigma_x \bar{\psi}\psi(x)} \prod_{\substack{\ell \\ x\to x'}} H\left(\tfrac{1}{4}\bar{\psi}\psi(x)\bar{\psi}\psi(x')\right). \tag{2.3}$$

This effective action depends only on local bilinears $\bar{\psi}\psi$. Thus at strong coupling the fermionic nature of the original system still manifests itself through the Pauli exclusion principle ($(\bar{\psi}\psi)^{N+1} = 0$) but not through antisymmetry (anticommutativity). For every link ℓ pointing from site x to x' H represents a sum over $N+1$ terms, which, when multiplied out, form a sum of statistical proportions. Every term in this sum can be labeled by integers $k_\ell \in \{0, 1, \ldots, N\}$ for every link. For a given term in the sum, the $\bar{\psi}, \psi$ integrations are trivial and can be carried out. On every site we have to do the integral

$$\int \mathrm{d}\psi(x)\, \mathrm{d}\bar{\psi}(x)\, \mathrm{e}^{m\bar{\psi}\psi(x)} (\bar{\psi}\psi(x))^{\sigma_x} = \frac{N!}{(N-\sigma_x)!} m^{N-\sigma_x}. \tag{2.4}$$

σ_x is an integer "divergence" of the k_ℓ configurations:

$$\sigma_x = \sum_{\ell, x \in \partial\ell} k_\ell, \tag{2.5}$$

where we sum over the star of all links touching x. (2.4) is understood to vanish for $N - \sigma_x < 0$. Now $Z(m)$ is given as a dimer partition function:

$$Z(m) = \sum_{\{k_\ell\}} \prod_\ell \alpha_{k_\ell} \left(\tfrac{1}{4}\right)^{k_\ell} \prod_x \frac{N!}{(N-\sigma_x)!} m^{N-\sigma_x}. \tag{2.6}$$

Thus we have to occupy the links in our lattice with between 0 and N dimers. No more than a total of N dimers are allowed to touch at any given vertex. If less of them are touching the missing $\bar{\psi}\psi$'s to saturate the integration are "filled up" by the mass term, which acts as a monomer term in this context. The Monte Carlo simulation of this dimer model will be discussed in sect. 4. Samuel in a series of papers [5] has emphasized before that dimer models can be formulated efficiently with the help of Grassmann variables. As we have seen our dimers obey a kind of parastatistics of degree N for gauge group U(N). This is reminiscent of the original motivation for introducing color before it acquired a dynamical rôle.

For SU(N) in (2.3) every link factor H is replaced by (see the appendix)

$$H \to H\left(\tfrac{1}{4}\bar{\psi}\psi(x)\bar{\psi}\psi(x')\right) + \frac{\Gamma_\ell^N}{N!}\left[\left(\tfrac{1}{2}\bar{\psi}(x)\psi(x')\right)^N + (-)^N\left(\tfrac{1}{2}\bar{\psi}(x')\psi(x)\right)^N\right]. \tag{2.7}$$

Clearly the new term corresponds to a baryon consisting of N quarks moving along the link in either direction. Such moves are possible only on dimer-free links ("all ψ variables are used") and they have to form closed non-intersecting loops. Their weight is most easily worked out by considering the prototype loop of fig. 1:

$$\begin{aligned} \text{weight} &= \frac{1}{2^{4N} N!^4} \int \mathrm{d}\psi_1\, \mathrm{d}\bar{\psi}_1 \ldots \mathrm{d}\psi_4\, \mathrm{d}\bar{\psi}_4\, \Gamma_{12}^N \Gamma_{23}^N (-\Gamma_{43})^N (-\Gamma_{14})^N \\ &\quad \times (\bar{\psi}_1\psi_2)^N (\bar{\psi}_2\psi_3)^N (\bar{\psi}_3\psi_4)^N (\bar{\psi}_4\psi_1)^N \\ &= 2^{-PN} (-)^N \prod_{\text{links}} (\text{oriented } \Gamma_\ell)^N. \end{aligned} \tag{2.8}$$

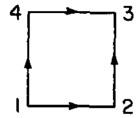

Fig. 1. Simplest baryonic closed path.

This formula, where P is the perimeter, is already the generalization for arbitrary loops. The $(-)^N$ is the fermionic sign from N closed quark loops. For N even (bosonic baryons), (2.8) is simply 2^{-PN}, and a Monte Carlo sampling of the positive integrand seems quite feasible, given an algorithm to enumerate the closed paths [6]. For N odd the last factor in (2.8), which is representation independent, can have either sign. For the elementary plaquette loop the total weight is positive, since $\Pi_{\square}\Gamma_{\ell} = -1$. Here the difficult problem of Monte Carlo sampling strongly oscillating fermionic partition functions reappears. If one believes in a smooth N dependence independent of baryon statistics (odd versus even N), one could try to interpolate between the even N values. Correlation functions of powers of $\bar{\psi}\psi(x)$ can be obtained from the elementary variables of the dimer problem. Promoting m to a source $m(x)$ for a moment we can easily show that the transcription is given by

$$(\bar{\psi}\psi(x))^{\ell} \leftrightarrow m^{-\ell} \frac{(N-\sigma_x)!}{(N-\sigma_x-\ell)!}, \tag{2.9}$$

where the right-hand side again has to be replaced by zero whenever $N-\sigma_x-\ell<0$. The chiral order parameter reads

$$\frac{1}{N}\langle\bar{\psi}\psi\rangle = \frac{1}{mN}\langle N-\sigma_x\rangle^{\text{dimer}}. \tag{2.10}$$

Clearly, as $m \to 0_+$ we shall find $\langle N-\sigma_x\rangle \to 0$ since all allowed configurations in a Monte Carlo have $\sigma_x \equiv N$ (no more monomers). The question of chiral symmetry breaking is decided by the rate at which $\langle\sigma_x\rangle = N$ is approached as the mass (monomer strength) is turned off. Finite $\langle\bar{\psi}\psi\rangle$ means linear decay while a symmetric situation would require a more rapid decay.

3. Mean field calculation

3.1. THE METHOD FOR FERMIONS

For analytic calculations the dimer formulation with constrained integer variables does not appear advantageous. Rather we shall follow a strategy similar to [2] and rewrite the grassmannian system as a C-number integral that can be evaluated approximately by a saddle-point expansion, where $1/N$ is the loop counting parameter. It should be noted how similar this formulation for $\bar{\psi}\psi$ type variables is to mean field saddle-point expansions in spin models [7,8]. The peculiarities of Grassmann systems alluded to in the introduction shows up in the solution to the mean field equations.

Let us develop the mean field method first for a single-site variable $\Sigma_a \bar{\psi}_a \psi_a$, $a = 1,\dots, N$. Any function of $\bar{\psi}\psi$ is given as

$$f(\bar{\psi}\psi) = \sum_{k\geq 0} f_k (\bar{\psi}\psi)^k. \tag{3.1}$$

Using

$$\lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} \mathrm{d}\sigma \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\alpha}{2\pi i} \mathrm{e}^{-\alpha\sigma+\varepsilon(\alpha^2-\sigma^2)} \alpha^k \sigma^\ell = k!\delta_{k\ell}, \tag{3.2}$$

we can obviously rewrite the expansion (3.1) as

$$f(\bar{\psi}\psi) = \int_{-\infty}^{\infty} \mathrm{d}\sigma \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\alpha}{2\pi i} \mathrm{e}^{-\alpha(\sigma-\bar{\psi}\psi)} f(\sigma). \tag{3.3}$$

Here, as in the future, the convergence factor $\varepsilon(\alpha^2 - \sigma^2)$ has been suppressed. (3.3) shows that formally "$\sigma = \bar{\psi}\psi$" due to a delta function; the rigorous justification is however the coinciding expansion in the Grassmann variables. Integrating over $\bar{\psi}$ and ψ we find

$$\int \mathrm{d}\psi\, \mathrm{d}\bar{\psi}\, f(\bar{\psi}\psi) = \int_{-\infty}^{\infty} \mathrm{d}\sigma \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\alpha}{2\pi i} \mathrm{e}^{N\log\alpha - \alpha\sigma} f(\sigma). \tag{3.4}$$

Here we have succeeded in replacing $\bar{\psi}\psi$ by a real variable σ coupled to an imaginary random source α. The source is averaged over with a weight given by the action of the source $N\log\alpha$. The same steps carried out for an Ising variable [8] produce $\log\cosh\alpha$ instead as the only but crucial difference. Another remark is in order: while in (3.1) coefficients f_k, $k > n$ are undefined (the expansion truncates) this is no more the case in the integrand in (3.3). However from our derivation it is clear that the integral (3.4) should be independent of an arbitrary choice for the higher coefficients as long as the asymptotic σ behaviour does not change too drastically to allow using (3.2) order by order.

The whole problem has to do with the fact that using the nilpotency property $(\bar{\psi}\psi)^{N+1} = 0$ and the introduction of mean field variables $\bar{\psi}\psi \to \sigma$ do not commute. The same problem arises with compact spins and equations like $(\sigma_{\text{Ising}})^2 = 1$ (see [9] for more remarks on this point). A one-variable prototype integral is given by

$f(\bar{\psi}\psi) = \exp(m\bar{\psi}\psi)$ yielding

$$Z(m) = \int \mathrm{d}\psi\, \mathrm{d}\bar{\psi}\, \mathrm{e}^{m\bar{\psi}\psi} = m^N .$$

The mean field version after changing $\sigma \to N\sigma$ reads

$$Z(m) = N\int_{-\infty}^{\infty} \mathrm{d}\sigma \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\alpha}{2\pi i}\, \mathrm{e}^{N[\log\alpha - \alpha\sigma + m\sigma]} . \tag{3.5}$$

Thinking of N being large we attempt a saddle-point evaluation. The extremum of the exponent is at

$$\alpha_0 = m , \qquad \sigma_0 = \frac{1}{\alpha_0} = \frac{1}{m} . \tag{3.6}$$

The path of steepest descent through (α_0, σ_0) is parameterized by real (τ, μ):

$$\begin{aligned} \sigma &= \sigma_0(1 + i\mu) , \\ \alpha &= \alpha_0(1 + \tau - i\mu) , \end{aligned} \tag{3.7}$$

and gives an expansion of Z in Feynman graphs:

$$Z = \mathrm{e}^{N\log m} N \int \frac{\mathrm{d}\mu\, \mathrm{d}\tau}{2\pi} \exp\left\{ N\left[-\tfrac{1}{2}\tau^2 - \tfrac{1}{2}\mu^2 + \tfrac{1}{3}(\tau - i\mu)^3 - \cdots \right] \right\} . \tag{3.8}$$

Actually it is more convenient to leave the integral in the original variables and perform it by analytic continuation. Putting

$$\begin{aligned} \sigma &= \sigma_0(1 + \tilde{\sigma}) , \\ \alpha &= \alpha_0(1 + \tilde{\alpha}) , \end{aligned} \tag{3.9}$$

we have

$$Z = \mathrm{e}^{N\log m} N \int \frac{\mathrm{d}\tilde{\sigma}\, \mathrm{d}\tilde{\alpha}}{2\pi i} \exp\left\{ N\left[-\tfrac{1}{2}\tilde{\alpha} - \tilde{\alpha}\tilde{\sigma} + \tfrac{1}{3}\tilde{\alpha}^3 - \tfrac{1}{4}\tilde{\alpha}^4 + \cdots \right] \right\} . \tag{3.10}$$

In this form it is obvious that the $\tilde{\alpha}$ vertices from expanding the log are irrelevant, since inverting the quadratic form in $(\tilde{\alpha}, \tilde{\sigma})$, we find that $\tilde{\alpha}$ does not propagate (or rather with factors ε if the damping term is included). Thus the $1/N$ series truncates as $\hbar$ expansions do in simple problems (oscillator, hydrogen), and we recover the exact result as one should. The same is, of course, true for (3.8), where all contributions to higher orders cancel.

3.2. STRONG COUPLING QCD

In the mean field transcription of QCD at strong coupling (2.3) we introduce an $\alpha(x)$ and $\sigma(x)$ on every site and derive from (2.3)

$$Z(m)=N^V\int\prod_x\frac{\mathrm{d}\sigma(x)\,\mathrm{d}\alpha(x)}{2\pi i}\exp\left\{N\left[\sum_x(\log\alpha(x)+m\sigma(x)-\alpha(x)\sigma(x))\right.\right.$$

$$\left.\left.+\sum_{\substack{\ell\\ x\to x'}}W(\sigma(x)\sigma(x'))\right]\right\},\tag{3.11}$$

using

$$H\left(\tfrac{1}{4}N^2\sigma(x)\sigma(x')\right)=\mathrm{e}^{NW(\sigma(x)\sigma(x'))}\tag{3.12}$$

from (A.4) in the appendix. The exponent reaches its extremum at

$$\alpha_0(x)=m+\sum_{y(x)}W'(\sigma_0(x)\sigma_0(y))\sigma_0(y),$$

$$\sigma_0(x)=\frac{1}{\alpha_0(x)},\tag{3.13}$$

with $y(x)$ the nearest neighbours of x. For a constant saddle-point we combine (3.13) into

$$1=m\sigma_0+2dW'\left(\sigma_0^2\right)\sigma_0^2.\tag{3.14}$$

The asymptotic N behaviour of $Z(m)$ is given by the integrand at the saddle-point with W replaced by its large-N limit W_0. In this way we derive the known [1,2] results:

$$\sigma_0=\frac{d\sqrt{m^2+2d-1}-(d-1)m}{d^2+m^2},\tag{3.15}$$

$$Z_{\mathrm{SP}}(m)=\exp\left\{NV\left[-\log\sigma_0+m\sigma_0-1+dW_0\left(\sigma_0^2\right)\right]\right\},\tag{3.16}$$

$$\frac{1}{N}\langle\bar\psi\psi\rangle=\sigma_0+\mathrm{O}(1/N).\tag{3.17}$$

$\lim_{m\to 0_+}\sigma_0=\sqrt{2d-1}/d$ does not vanish and chiral symmetry breaking has thus been found.

We wish to compute the $1/N$ correction to $\langle\bar{\psi}\psi\rangle/N$. To that end we shift and rescale (σ, α):

$$\begin{aligned}\sigma &= \sigma_0(1+\tilde{\sigma}),\\ \alpha &= \alpha_0(1+\tilde{\alpha}),\end{aligned} \tag{3.18}$$

and the relation with $\bar{\psi}\psi$ is now

$$\frac{1}{N}\langle\bar{\psi}\psi\rangle = \sigma_0(1+\langle\tilde{\sigma}\rangle). \tag{3.19}$$

Dropping the factor Z_{SP} the action to compute $\langle\tilde{\sigma}\rangle$ is the one appearing in

$$\begin{aligned} Z(m) \propto \int \mathrm{d}\tilde{\sigma}\,\mathrm{d}\tilde{\alpha}\exp\Big\{ & N\sum_x\left(-\tfrac{1}{2}\tilde{\alpha}^2(x)+\tfrac{1}{3}\tilde{\alpha}^3(x)-\tilde{\alpha}(x)\tilde{\sigma}(x)\right)\\ & +2dW_1^{(1)}\sum_x\tilde{\sigma}(x)+N\left(W_0^{(1)}+W_0^{(2)}\right)\sum_{\substack{\ell\\ x\to x'}}\tilde{\sigma}(x)\tilde{\sigma}(x')\\ & +NdW_0^{(2)}\sum_x\tilde{\sigma}^2(x)+N\tfrac{1}{3}dW_0^{(3)}\sum_x\tilde{\sigma}^3(x)+N\left(W_0^{(2)}+\tfrac{1}{2}W_0^{(3)}\right)\\ & \times\sum_{\substack{\ell\\ x\to x'}}\left(\tilde{\sigma}(x)\tilde{\sigma}(x')^2+\tilde{\sigma}(x)^2\tilde{\sigma}(x')\right)+\mathrm{O}\left(\tilde{\sigma}^4,\tilde{\alpha}^4,1/N\right)\Big\}. \end{aligned} \tag{3.20}$$

Here we exhibit explicitly those terms in the expansion needed to compute $\langle\tilde{\sigma}\rangle$ to order $1/N$ and introduce the notation

$$W^{(k)} = \sigma_0^{2k}\left.\frac{\partial^k W}{\partial\lambda^k}\right|_{\lambda=\sigma_0^2}. \tag{3.21}$$

Denoting σ lines by —— and α lines by $\sim\!\sim$ we read off the propagators:

$$\begin{pmatrix} \text{——} & \text{—}\!\sim \\ \sim\!\text{—} & \sim\!\sim \end{pmatrix} = \frac{1}{N}(T-1)^{-1}\begin{pmatrix} 1 & -1\\ -1 & T\end{pmatrix}, \tag{3.22}$$

with

$$T = 2dW_0^{(1)} - \left(W_0^{(1)}+W_0^{(2)}\right)\bar{\Delta}, \tag{3.23}$$

where $\bar{\Delta}$ is the antiferromagnetic laplacian (opposite central term):

$$(\bar{\Delta}\sigma)(x)=\sum_{n.n.y(x)}\sigma(y)+2d\sigma(x). \tag{3.24}$$

For momenta $p=(\pi,\dots,\pi)+q$ with small q we have

$$— = -\frac{1}{N}\left[\left(W_0^{(1)}+W_0^{(2)}\right)q^2+m\sigma_0\right]^{-1}. \tag{3.25}$$

The staggered σ ("pion") field is the Goldstone boson associated with the broken symmetry (1.3). Its correlation length diverges $\sim m^{-1/2}$ for small m. This is the PCAC relation of current algebra. Here it is reminiscent of mean field scaling behaviour: a correlation length diverges as the inverse square root of a parameter in the action

$$\xi(\beta)\sim|\beta-\beta_c|^{-1/2},$$

a deviation from the critical inverse temperature typically. For spin models it is known that the critical exponent $\frac{1}{2}$ is correct only above the upper critical dimension 4. One way to see this as a consistency requirement within the saddle-point mean field calculation is the degree of infrared divergence of self-energy corrections given by the diagram in fig. 2 (in the broken phase with a triple vertex). It gives a small correction to the small-p behaviour only for $d>4$ where $\int d^dq/q^4$ is infrared-convergent. For staggered fields the situation is different. The long-range correlations occur for $p_\mu\sim\pi$. Due to momentum conservation at the vertices the internal lines carry momenta differing by π in all components and thus do not blow up simultaneously. We thus conjecture that the upper critical dimension of this fermion derived system is below 4.

Finally we turn to the actual corrections to $\bar{\psi}\psi$. They are the sum of the diagrams in fig. 3, where the first term is coming from W_1, the $1/N$ correction to the one-link integral. Using translation invariance and $— = - \sim$ we have

$$\langle\tilde{\sigma}\rangle=\sum_x(\circ\!-\!\!-\!\times)(\,| + O - O\,). \tag{3.26}$$

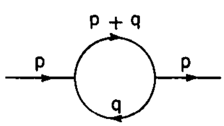

Fig. 2. Self-energy correction.

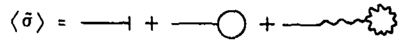

Fig. 3. Diagrams contributing to the $1/N$ correction of $\langle\bar{\psi}\psi\rangle/N$.

For the zero momentum propagator we have

$$\sum_x(\circ - x) = -\frac{1}{N}\left(1 + 2dW_0^{(1)} + 4dW_0^{(2)}\right)^{-1}. \tag{3.27}$$

Furthermore ($\circ - \circ$) is the propagator at the origin)

$$| = 2dW_1^{(1)} = 2d\frac{W_0^{(2)}}{1 - 2W_0^{(1)}}, \tag{3.28}$$

$$\begin{aligned}\bigcirc = & \left[2d\left(W_0^{(2)} + W_0^{(3)}\right)\right]N \circ - \circ + 4d\left(W_0^{(2)} + \tfrac{1}{2}W_0^{(3)}\right) \\ & \times N(\circ - \circ + \hat{\mu}),\end{aligned} \tag{3.29}$$

$$\text{✿} = N(\circ\!\sim\!\circ). \tag{3.30}$$

The σ propagator fulfils the equation

$$(T-1)N - = \delta_{xy} = \left[2dW_0^{(1)} - 1 - \left(W_0^{(1)} + W_0^{(2)}\right)\bar{\Delta}\right]N - , \tag{3.31}$$

which allows us to relate (using 90 degree isotropy)

$$1 = \left(-1 - 2dW_0^{(2)}\right)N \circ - \circ - 2d\left(W_0^{(1)} + W_0^{(2)}\right)N(\circ - \circ + \hat{\mu}), \tag{3.32}$$

$$N \circ - \circ = \frac{-1}{V}\sum_p\left\{2dW_0^{(2)} + 1 + 2\left(W_0^{(1)} + W_0^{(2)}\right)\sum_\mu \cos p_\mu\right\}^{-1}, \tag{3.33}$$

$$N \circ\!\sim\!\circ = 1 + N \circ - \circ. \tag{3.34}$$

Here we refer to the appendix for $W_0^{(1)}, W_0^{(2)}, W_0^{(3)}$ as functions of m. All contributions to the $1/N$ correction of $\bar{\psi}\psi$ are explicit now, and their numerical values for some range of masses are displayed in fig. 4 and plotted with the Monte Carlo data in sect. 4. The correction coefficient for infinite volume is very small throughout the entire range of masses with its maximal value of zero mass: $\langle\tilde{\sigma}\rangle = -0.0297$. At finite volume (momenta in the Brillouin zone summed instead of integrated) the coefficient rises sharply at small mass. This happens of course where the correlation length as estimated from the pion mass in (3.25) reaches the order of the volume. In that range the negative correction starts to cancel the lowest order mean field value

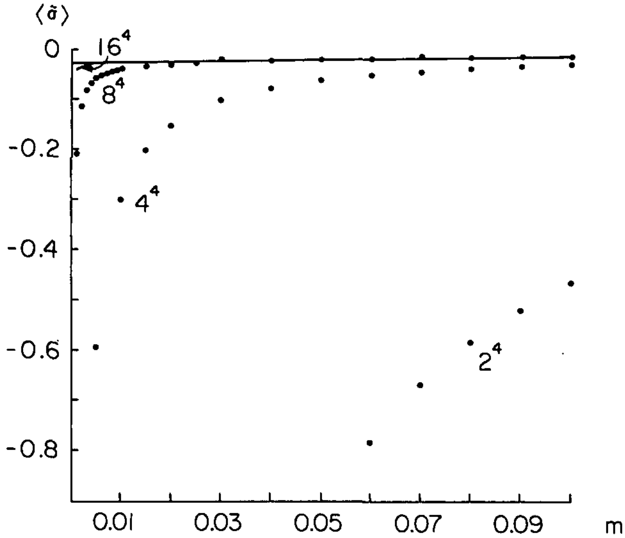

Fig. 4. The $1/N$ correction to the chiral order parameter $\langle\tilde{\sigma}\rangle$ appearing in $\langle\bar{\psi}\psi\rangle/N = \sigma_0(1 + \langle\tilde{\sigma}\rangle/N)$.

which for a translation invariant saddle-point is insensitive to the volume. Since symmetries do not break in finite volume, $\langle\bar{\psi}\psi\rangle$ eventually has to bend down if one sends m to zero at finite volume. Our $1/N$ correction shows the onset of this happening, although, once the first order in $1/N$ is comparable to the zeroth order, a calculation to this order is not reliable any more.

4. Monte Carlo simulation and conclusions

On the basis of the dimer form of QCD at infinite coupling (2.6) a Monte Carlo sampling procedure can be designed. Our elementary updating steps take place on links that are systematically swept through. With probability $\frac{1}{2}$ we envisage either increasing or decreasing the dimer occupation number on a link by one unit. After that decision is taken the ratio of the Boltzmann factors of the potential new configuration and the present one is computed. It is local in the dimer variables. Then the move is accepted or rejected à la Metropolis. For U(1) runs we used heatbath updating for the two possible occupation numbers to increase the efficiency of the program. We found it convenient to store and update the in principle redundant variables σ_x. Due to the limited memory available at our Cyber 170/370 ($2^{17}-1=131$k 60 bit words) we packed the σ_x of a given site and the four links emerging from it in positive directions into one word. This enabled us to simulate lattices up to volume 16^4 with 12 bits for every integer variable allowing for U(N) with $N \leqslant 2^{12}-1=4095$. The torus geometry was handled by Fortran 5 bit shifting

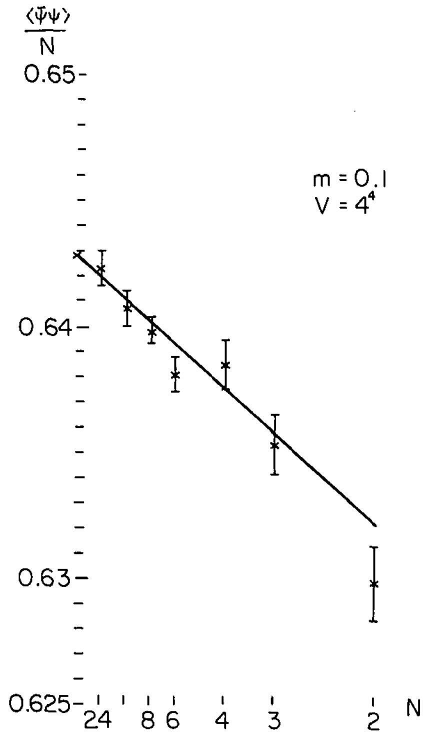

Fig. 5. N dependence of the chiral order parameter at $m = 0.1$ on a 4^4 lattice and the analytic result up to order $1/N$ (computed for finite volume).

operations restricting the side lengths of our four-dimensional cube to powers of two.

In figs. 5 and 6 we plot the N dependence of $\langle\bar{\psi}\psi\rangle/N$ at a fixed mass of 0.1 on a 4^4 and an 8^4 lattice. The abscissae are linear in $1/N$. At this quark mass the pion mass in the $N\to\infty$ limit is 0.69 or the correlation length 1.45. We see that the large-N limit with the first $1/N$ correction describes the order parameter well all the way down to $N=1$. Its whole variation from $N=\infty$ to $N=1$ is within 2%. Our data are gathered with typically 1000 sweeps per point and show an accuracy of $\frac{1}{5}$%. Errors are estimated from root mean square deviations with the data blocked into different bunches of $1,2,4,\dots,2^8$ sweeps to see that autocorrelation effects die out. One sweep on the 8^4 lattice takes 0.9 sec on the Cyber 170/370.

Figs. 7 and 8 display the chiral order parameter as a function of the mass. The agreement between data and large-N analytic results is excellent. The onset of the depression of the order parameter in finite volume as expected on general grounds and predicted by the volume dependence of the $1/N$ coefficient is consistent with our data. Measurements in this range become increasingly time consuming since the

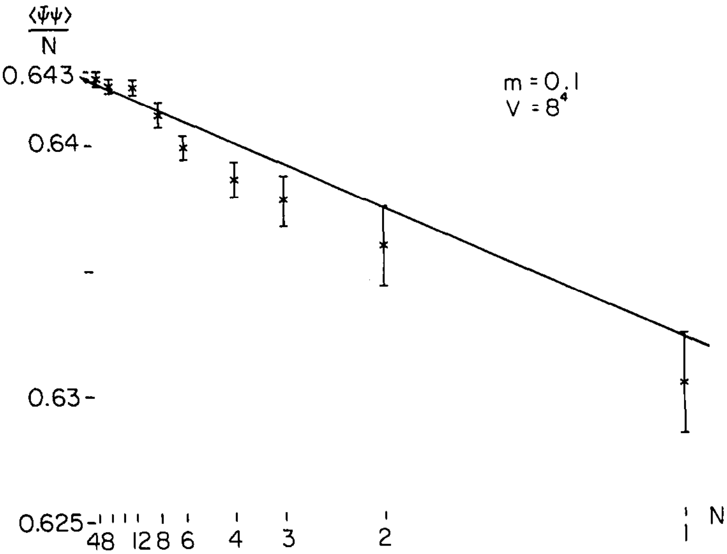

Fig. 6. Same as fig. 5 but for an 8^4 lattice.

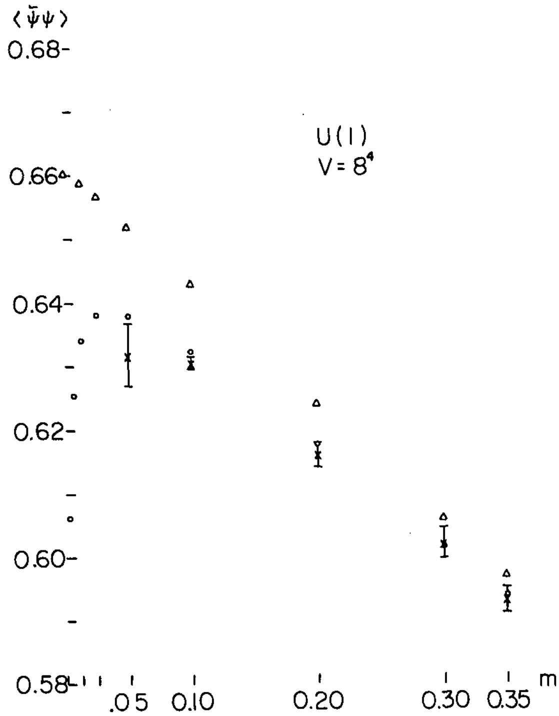

Fig. 7. $\langle\bar\psi\psi\rangle$ as a function of the mass for group U(1) on an 8^4 lattice, $\times$ are Monte Carlo points, $\triangle$ the ($N=\infty$) mean field results and $\circ$ includes the first $1/N$ correction.

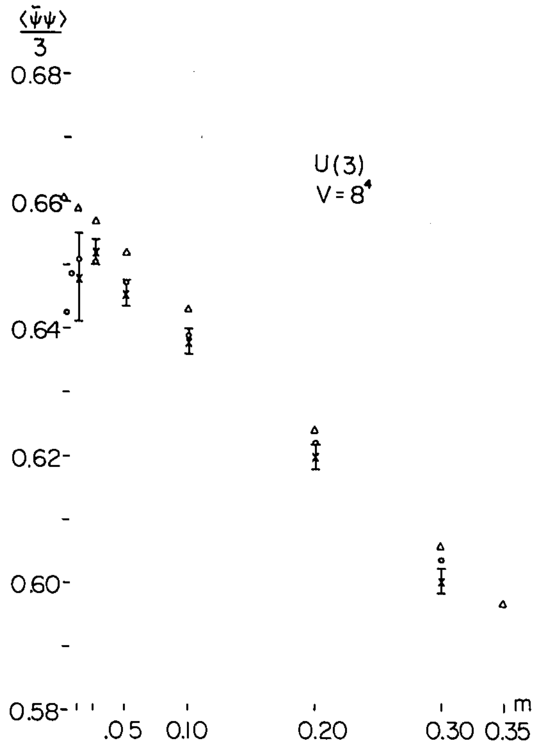

Fig. 8. Same as fig. 7 but for U(3).

physical correlation length (inverse pion mass) increases; for a bare quark mass of 0.0125 in the $N \to \infty$ limit, it is about six lattice units. The variation of the order parameter with m is small too. An extrapolation of the 8^4 data to $m = 0$ represents no problem and agrees well with the $1/N$ result when the infinite volume correction is taken at $m = 0$. We also ran the program at very large N, $N = 2047$ in our case. We found that $\bar{\psi}\psi/N$ is easily measurable for masses in the range as displayed in figs. 7 and 8. Our precision (and agreement with the saddle-point value σ_0) was pushed to 10^{-5}. Finite size effects are suppressed by $1/N$ as expected, and invisible.

Concluding, we wish to emphasize that our data prove that strong coupling QCD (gauge group U(N)) with fermions is very well described in the large-N approximation, that is a saddle-point mean field calculation for this system, for all values of N. Possibly the real expansion parameter is $1/N$ times a small parameter arising dynamically from the saddle-point equation, as it happens occasionally in saddle-point mean field expansions ("expansion without expansion parameter"). We found that the quantities $W^{(k)}$ accompanying higher-order vertices decrease rapidly, but could not manifestly demonstrate the form of the effective expansion parameter. The dimer Monte Carlo algorithm produces accurate data incorporating the full fermion determinant at strong coupling with very little numerical effort. This is due to the fact that we found a representation of the fermionic system as a positive spin model type statistical sum instead of an alternating one. Unfortunately this seems to be possible only in the unphysical $\beta=0$ infinite-coupling limit, as is the straightforward demonstration of confinement. More general algorithms to simulate dynamical fermions away from $\beta=0$ could be compared with our data (or large-N results) for $\beta=0$ as a test.

Appendix

THE ONE-LINK INTEGRAL

We shall evaluate the integral over the U(N) or SU(N) Haar measure with N component Grassmann sources $\varphi, \bar{\varphi}, \chi, \bar{\chi}$:

$$H=\int \mathrm{d}U \mathrm{e}^{\bar{\chi}U\varphi - \bar{\varphi}U^{\dagger}\chi}. \tag{A.1}$$

The more general case with the general matrix source has been extensively studied for large N [10].

The integration over the center of the group in (A.1) tells us that the only terms in the expansion of the exponential that contribute are those with equal powers of U and $U^{\dagger}$ modulo N. Combining this information with the separate invariance under $\chi\to W\chi$, $\bar{\chi}\to\bar{\chi}W^{\dagger}$ and $\varphi\to V\varphi, \bar{\varphi}\to\bar{\varphi}V^{\dagger}$ (right and left invariance of the Haar measure) and the nilpotency and antisymmetry of the Grassmann variables implies that H has the form

$$H=\sum_{k=0}^{N}\alpha_k(\bar{\chi}\chi\bar{\varphi}\varphi)^k+\begin{cases}0 & \text{for U}(N)\\ \dfrac{1}{N!}\left[(\bar{\chi}\varphi)^N+(-)^N(\bar{\varphi}\chi)^N\right] & \text{for SU}(N).\end{cases} \tag{A.2}$$

To determine the coefficients α_k we integrate (A.1) as well as (A.2) over $\bar{\chi}, \chi$ with an extra factor $\mathrm{e}^{\bar{\chi}\chi}$:

$$\int \mathrm{d}\chi\,\mathrm{d}\bar{\chi}\int \mathrm{d}U \mathrm{e}^{\bar{\chi}\chi+\bar{\chi}U\varphi-\bar{\varphi}U^{+}\chi}=\mathrm{e}^{\bar{\varphi}\varphi},$$

$$\int \mathrm{d}\chi\,\mathrm{d}\bar{\chi}\,\mathrm{e}^{\bar{\chi}\chi}\sum_{k=0}^{N}\alpha_k(\bar{\chi}\chi\bar{\varphi}\varphi)^k=\sum_{k=0}^{N}\alpha_k\frac{N!}{(N-K)!}(\bar{\varphi}\varphi)^k.$$

Comparing coefficients of $\bar{\varphi}\varphi$ we find

$$\alpha_k=\frac{(N-k)!}{N!k!} \tag{A.3}$$

We mention in passing that a completely analogous result can be obtained for bosonic sources X, Y. The U(N) result is

$$\int \mathrm{d}U \mathrm{e}^{\overline{X}UY+\overline{Y}U^{\dagger}X} = \sum_{k=0}^{\infty} \frac{(N-1)!k!}{(N+k-1)!} (\overline{X}X\overline{Y}Y)^k .$$

We shall need the large-N limit of H with $\overline{\varphi}\varphi\overline{\chi}\chi$ replaced by $\frac{1}{4}\lambda N^2$ with λ fixed as $N \to \infty$. There are two strategies giving identical results for the first two terms in the $1/N$ expansion for U(N)

$$H\left(\tfrac{1}{4}\lambda N^2\right) = \mathrm{e}^{NW(\lambda)} = \mathrm{e}^{N(W_0(\lambda)+(1/N)W_1(\lambda)+\cdots)} . \tag{A.4}$$

The first is to approximate all factorials by the Stirling formula, make the summation index $k = 0, \ldots, N$ continuous ($t \in [0,1]$ for fermions, $[0, \infty)$ for bosons), and evaluate the resulting integral by the saddle-point method. The second strategy similar to the methods used in [10, 2] starts from a differential equation for H. It is easily obtained by applying the operator $\Sigma_a(\partial/\partial\varphi_a)(\partial/\partial\overline{\varphi}_a)$ on (A.1):

$$\lambda W' - \lambda^2 W'^2 - \frac{1}{N}\lambda^2 W'' - \tfrac{1}{4}\lambda = 0 . \tag{A.5}$$

For finite N (A.2) is of course a solution to all orders in λ provided one takes into account $\lambda^{N+1} = 0$. For $N \to \infty$ we neglect the nilpotency of λ and solve for successive powers in $1/N$:

$$W_0' - \lambda W_0'^2 - \tfrac{1}{4} = 0 , \tag{A.6}$$

$$W_1' - 2\lambda W_0' W_1' - \lambda W_0'' = 0 , \tag{A.7}$$

with solutions (boundary condition: $W(0) = 0$)

$$W_0(\lambda) = 1 - \sqrt{1-\lambda} + \log\left[\tfrac{1}{2}(1+\sqrt{1-\lambda})\right] , \tag{A.8}$$

$$W_1(\lambda) = \log\left[\tfrac{1}{2}(1+\sqrt{1-\lambda})\right] - \tfrac{1}{4}\log(1-\lambda) . \tag{A.9}$$

The quantities used in sect. 3 are summarized by

$$W^{(k)} = \lambda^k \frac{\partial W}{\partial \lambda^k} , \tag{A.10}$$

$$W_0^{(1)} = \tfrac{1}{2}(1 - \sqrt{1-\lambda}) , \tag{A.11}$$

$$W_0^{(1)} + W_0^{(2)} = \frac{1}{4} \frac{\lambda}{\sqrt{1-\lambda}} , \tag{A.12}$$

$$2W_0^{(2)} + W_0^{(3)} = \tfrac{1}{8}\lambda^2 (1-\lambda)^{-3/2} ; \tag{A.13}$$

$$W_1^{(1)} = \frac{W_0^{(2)}}{1 - 2W_0^{(1)}} \tag{A.14}$$

follows directly from (A.5) and is the only form in which W_1 enters into our calculation.

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