Basic Ideas of Stochastic Quantization

Mikio NAMIKI

Department of Physics, Waseda University, Tokyo 169

We describe basic ideas of the stochastic quantization which was originally proposed by Parisi and Wu. We start from a brief survey of stochastic-dynamical approaches to quantum mechanics, as a historical background, in which one can observe important characteristics of the Parisi-Wu stochastic quantization method that are different from others. Next we give an outline of the stochastic quantization, in which a neutral scalar field is quantized as a simple example. We show that this method enables us to quantize gauge fields without resorting to the conventional gauge-fixing procedure and the Faddeev-Popov trick. Furthermore, we introduce a generalized (kerneled) Langevin equation to extend the mathematical formulation of the stochastic quantization: Its illustrative application is given by a quantization of dynamical systems with bottomless actions. Finally, we develop a general formulation of stochastic quantization within the framework of a (4+1)-dimensional field theory.

§ 1. Introduction

The Parisi-Wu stochastic quantization[1)] was designed to produce quantum mechanics from the thermal equilibrium limit of a hypothetical stochastic process with respect to a new *fictitious* time other than ordinary time. In this paper, let us use SQM as an abbreviation of the Parisi-Wu stochastic quantization method. SQM has been developing for its practical and technical merits in modern field theories, so that we can now regard it as another important way of quantization different from the conventional canonical and path-integral methods. However, we must remark that the underlying idea undoubtedly belongs to the line of thought that quantum mechanics should be replaced with a classical stochastic dynamics. We know that many physicists, including both those who agree and do not agree with the Copenhagen Interpretation, have confidentially been conceiving such an idea for a long time. For this reason, we start our discussion from a historical survey of stochastic-dynamical approaches to quantum mechanics. This point of view allows us to observe some important characteristics of SQM that are remarkably different from other approaches.

As is widely known, there exists a formal similarity between the Schrödinger equation for a free quantum-mechanical particle and the diffusion equation for a free Brownian-motion particle, because the equations are mutually transformed into each other through a substitution given by $t\leftrightarrow -it$ and $\hbar/2m\leftrightarrow\alpha$, with t, m, $\hbar$ and α being the time variable, particle mass, Planck constant and diffusion constant, respectively. The formal similarity may suggest to us a possible "hidden-variable" theory, under a naive idea that the quantum fluctuation originates in a classical random motion caused by interaction with an unknown "ether" surrounding the observed particle. In fact, we had even seen a few attempts[2] along this line of thought in the earlier period of the development of quantum mechanics. Note that they wanted to identify the quantum process with a classical stochastic one with respect to *imaginary* ordinary-time.

However, the NO-GO theorem given by von Neumann[3] had strongly prevented this kind of hidden-variable theory to develop. After some years, Bohm[4] broke through the barrier of the theorem and succeeded in making a hidden-variable theory, deriving a classical "Newton equation" from the quantum-mechanical Schrödinger equation. His "Newton equation" contains the so-called "quantum-mechanical force" to be regarded as a classical random force rooted in a hypothetical "ether" which we cannot observe. The hidden-variable theory and its modifications have been developed by himself and other authors.

While it is true that Bohm wanted to obliterate quantum mechanics with a classical stochastic-dynamics, we notice that his approach can be considered to be a sort of quantization method, if we trace back to the Schrödinger equation from the Newton equation by adding the quantum-mechanical force. Actually, Nelson[5] elegantly formulated a stochastic quantization which produces quantum mechanics within the framework of a classical Brownian-motion theory having $\hbar$ as the diffusion constant. We should remark that, contrary to the more naive theory, both Bohm's theory and Nelson's stochastic quantization are formulated in terms of *real* ordinary-time. Nelson's stochastic quantization (as well as Bohm's theory) is very interesting from the fundamental point of view on how to give a stochastic interpretation to quantum mechanics, even though it seems that his method can hardly manage such complications as many particle systems or fields. SQM, on the other hand, has so far used *imaginary* ordinary-time or *Euclidean* space-time coordinates (brought in through the Wick rotation) in order to make the theory mathematically well-posed, and is formulated on the basis of a hypothetical stochastic process with respect to a *fictitious* time other than ordinary one. From these points of view, one may feel that SQM is rather artificial in comparison with Nelson's method. Indeed, we do not know a physical meaning for the fictitious time, which is only a mathematical tool up to the present. We have no reason to name it "time". As for the first point, however, a few authors have successfully attempted to formulate SQM in terms of Minkowski space-time coordinates, keeping ordinary-time real.[6],[7] We should therefore notice that the use of *imaginary* ordinary-time is not essential to SQM. As for the second point, we shall briefly imagine a possible physical background for the fictitious time in § 6.

First of all, we should stress that SQM can start from a classical equation of motion but not directly from Hamiltonian or Lagrangian, in other words, that SQM is a theory capable of quantizing even dynamical systems without canonical formalism. For example, we can quantize some dynamical systems with non-holonomic constraints within the framework of SQM, as will be seen in the case of the stochastic gauge fixing in § 3. This is one of the most remarkable characteristics of SQM that is different from conventional theories, i.e., the canonical and path-integral ones based on Hamiltonian and/or Lagrangian. We should also point out that a wide class of stochastic processes with respect to the fictitious time can yield the same quantum mechanics. For example, we can introduce a "generalized" Langevin equation obtained by a generalization or modification of the original Langevin equation with a "kernel". We call it the "kerneled Langevin equation". Note that we are quite free to choose a kernel among many possible functional forms. In § 5, we attempt to apply the kerneled Langevin equation, with an appropriately chosen kernel, to the stochastic quantization of dynamical systems with bottomless actions. Undoubtedly, these facts show that SQM has the potential to extend the territory of quantum mechanics and of quantum field theory beyond the conventional ones.

SQM is based on a well-defined Markoffian process of the Wiener type with Gaussian white noise. The original idea of SQM is that a D-dimensional *quantum* system is equivalent to a $(D+1)$-dimensional *classical* system with random fluctuations. The idea is not quite new, because a pioneer work[8] already existed for the Feynman-Kac integral and the quantum theory of spin systems had been developed through a similar procedure based on Trotter's formula.[9]

A similar but different method named the "micro-canonical quantization" method[10] was proposed several years ago, under the rather drastic idea that a D-dimensional *quantum* system should be equivalent to a $(D+1)$-dimensional *classical* system governed by a *deterministic* dynamics with respect to a new *fictitious* time other than (imaginary) ordinary-time. This method is similar to SQM in making use of a fictitious time, but different from SQM in introducing a classical deterministic dynamics. Those who are using the method expect to obtain quantum fluctuations from classical chaotic behaviors of the system in fictitious time. Rigorously speaking, however, we should notice that equivalence of this method to quantum mechanics is not yet verified, and that the method cannot always quantize every dynamical system because some systems, such as linear systems or free fields, have no chaos. Nevertheless, this method can work only in applications to numerical simulations, because successive updates of a dynamical variable by a finite time step often make a random distribution similar to that in SQM. Therefore, this method is now often combined with SQM for the purpose of carrying out large numerical simulations in lattice QCD.

In addition to the above quantization procedures formulated in configuration or momentum space, there is a possible formulation of quantum mechanics in phase space. This kind of quantization was initiated by Wigner[11] many years ago, but we know that the Wigner distribution function is not considered to be a probability distribution due to the lack of positive definiteness. Husimi[12] proposed a positive-definite phase-space distribution function, given by averaging the Wigner distribution function over a small cell around each point in phase space. Afterward, many authors have attempted to reformulate or modify this way of quantization. However, we do not know exactly what kinds of classical stochastic processes exist behind them.

Recently, several authors have also attempted to formulate SQM in phase space.[13] We know that their theory is based on a well-defined classical Wiener-Markoffian process with Gaussian white noise so designed as to yield quantum mechanics from its thermal equilibrium limit.

As we have seen here, we have many quantum theories or quantization methods. This means that the sixty-year history of quantum mechanics would not be enough to fix its theoretical formulations and to explore its physical implications, and that we are working in an era in which quantum mechanics itself is still being examined, reformulated, and extended over its present version. We believe that SQM is one of the most powerful candidates for making progress of quantum mechanics.

In § 2, we first sketch the general prescription of SQM with its applications to simple examples. In § 3, we show that one of the most important merits of SQM is found in its application to gauge fields. In § 4, we extend the theory of SQM by introducing a kerneled Langevin equation and we apply it to the stochastic quantization of unstable dynamical systems. In § 5, we develop a mathematical formulation of SQM within the framework of (4+1)-dimensional field theory with "stochastic" actions, and then introduce its operator formalism to be useful for systematic calculations. Section 6 is devoted to concluding remarks. In the Appendix, we briefly explain some important formulas, in the theory of stochastic processes, which have been used for SQM.

§ 2. Outline of SQM

Suppose that we have a dynamical system described by the action functional $S[q]$, depending on variables $q(x)=\{q_i;\, i=1, 2, \cdots\}$, in which x stands for ordinary time for particles or 4-dimensional coordinates for fields. Later we shall denote fields by ϕ or ψ instead of q. As is well known, we can calculate the quantum-mechanical expectation value of an observable, $G(q)$, or other important quantities by means of the following path-integral formula,

$$\langle G\rangle = C\int G(q)\exp(-S[q]/\hbar)\mathcal{D}q\,, \tag{2·1}$$

or similar ones, with C being the normalization constant. Here $\int\cdots\mathcal{D}q$ means the functional integration with respect to the function $q(x)$, and we have used Euclidean space-time coordinates obtained by the Wick rotation, in order to make the formula mathematically well-posed. We first formulate a general prescription to give (2·1) by means of SQM and then apply it to a neutral scalar field as an illustrative example. Throughout this paper, we adhere to Euclidean space-time geometry for simplicity, although SQM in Minkowski space-time has already been formulated.[7)]

2.1. *General prescription*

As mentioned in § 1, the Parisi-Wu stochastic quantization method (SQM) produces quantum mechanics from the thermal equilibrium limit of a hypothetical stochastic process with respect to *fictitious time*, say t, other than *ordinary time* x for particles or *time-component* x_0 (of Euclidean coordinates x) for fields. For the purpose of making the hypothetical stochastic process, we first introduce an additional dependence of q on t, and set a Langevin equation

$$\frac{\partial q_i(x,t)}{\partial t}=-\frac{\delta S[q]}{\delta q_i(x)}\Big|_{q=q(x,t)}+\eta_i(x,t)\,, \tag{2·2}$$

to govern the hypothetical stochastic process in t, where the η_i's are Gaussian white noise subject to the statistical law

$$\langle\eta_i(x,t)\rangle=0\,, \tag{2·3a}$$

$$\langle\eta_i(x,t)\eta_j(x',t')\rangle=2\alpha\delta_{ij}\delta(x-x')\delta(t-t')\,, \tag{2·3b}$$

α being the diffusion constant. The bracket $\langle\cdots\rangle$ means an ensemble average over the η's.

Equation (2·2) accompanied by (2·3) certainly describes a typical Wiener-Markoffian stochastic process with white noise. Reading the time variable, t, as a fictitious time, we can easily transfer the theory of stochastic processes given in the Appendix into the basic formulation of SQM through the replacement of $q_i(t)$ in the former with $q_i(x,t)$ in the latter, that is, $i\to(i,x)$, $\delta_{ij}\to\delta_{ij}\delta(x-x')$, and $\Sigma_i\to\Sigma_i\int dx$, in which x is the ordinary time for particles and the space-time point for fields. Repetition of details would not be necessary.

Solving (2·2), we obtain $q(x,t)$ and then an arbitrary quantity $G(q(x,t))$ as a function or functional of η's, whose expectation values, $\langle q_i(x,t)\rangle$ and $\langle G(q(x,t))\rangle$, are given by averaging them over η's subject to (2·3). This method is the Langevin way of dealing with stochastic processes. On the other hand, we can also describe the same processes by means of the Fokker-Planck equation, instead of the Langevin equation, if we introduce the probability distribution functional, $\Phi[q,t]$, defined by

$$\int G(q)\Phi[q,t]\mathcal{D}q=\langle G(q(x,t))\rangle\,, \tag{2·4}$$

the right-hand side being the above Langevin expectation value. Following the standard procedure from (A·1) and (A·2) (or equivalently, (A·5) and (A·6) with (A·7)) to (A·11) mentioned in the Appendix), we can easily derive the Fokker-Planck equation

$$\frac{\partial}{\partial t}\Phi[q,t]=\hat{F}\Phi[q,t] \tag{2·5a}$$

with the Fokker-Planck operator

$$\hat{F}=\alpha\int dx\sum_i\frac{\delta}{\delta q_i(x)}\left\{\frac{\delta}{\delta q_i(x)}+\frac{1}{\alpha}\frac{\delta S}{\delta q_i(x)}\right\}. \tag{2·5b}$$

Equation (2·5) immediately gives the thermal equilibrium distribution

$$\Phi_{\rm eq}[q]=C\exp\left(-\frac{1}{\alpha}S[q]\right), \tag{2·6}$$

if the drift force $K_i(q,t)=-(\delta S[q]/\delta q_i)_{q=q(x,t)}$ has a damping effect. In this case, therefore, the left-hand side of (2·4) turns into the path-integral formula (2·1) in the limit $t\to\infty$, provided that we put

$$\alpha=\hbar\,. \tag{2·7}$$

This means that we have come to quantum mechanics via the above hypothetical stochastic process; that is, we have obtained a general prescription of SQM.

Consequently, we can formulate a theoretical procedure, based on the above prescription of SQM, to calculate the field-theoretical propagator in the following way:

$$\varDelta_{ij\cdots}(x_1, x_2, \cdots)=\lim_{t\to\infty}\langle q_i(x_1, t)q_j(x_2, t)\cdots\rangle\,, \tag{2·8}$$

because this gives the well-known path-integral definition

$$\varDelta_{ij}(x-x')=C\int q_i(x)q_j(x')\exp\left(-\frac{1}{\alpha}S[q]\right)\mathscr{D}q \tag{2·9}$$

for the one-particle propagator, for example.

In the theory of stochastic processes, we often use the one-particle *correlation function* defined by

$$D_{ij}(x, t; x', t')=\langle q_i(x, t)q_j(x', t')\rangle\,, \tag{2·10a}$$

which has the stationary limit

$$\begin{aligned} D_{ij}(x-x', t-t')&=\langle q_i(x, t)q_j(x', t')\rangle_{\rm st} \\ &=\lim_{t,t'\to\infty}\langle q_i(x, t)q_j(x', t')\rangle \text{ for fixed } t-t' \end{aligned} \tag{2·10b}$$

when the drift force is independent of t. The space-time uniformity of the original dynamical system is reflected in the dependence of the stationary correlation function only on $x-x'$. Thus, (2·8), (2·9) and (2·10b) give the SQM formula

$$\varDelta_{ij}(x-x')=D_{ij}(x-x', 0) \tag{2·11}$$

for the one-particle field-theoretical propagator.

A widely known fact is that $\varDelta_{ii}(x)$ informs us about the first energy gap $\varDelta E$ (or the particle mass) associated with the ith dynamical variable through the asymptotic behavior

$$\varDelta_{ii}(x)\xrightarrow{|x|\to\infty}\exp(-\varDelta E|x|) \tag{2·12}$$

apart from a gradually varying function, in which we have used the natural unit ($\hbar=1$ and $c=1$). In the next subsection, we will discuss this problem, together with another possibility of extracting the particle mass from the asymptotic behavior in the fictitious time.

Following the *stochastic* "canonical" formalism suggested in the Appendix, if we introduce the *stochastic* "Lagrangian" $\mathcal{L}$, the *stochastic* "Lagrangian density" $\bar{\mathcal{L}}$ and the *stochastic* "action" $\mathcal{S}$ defined by

$$\mathcal{L}(t)=\int d^4x\bar{\mathcal{L}}(x, t)\,; \tag{2·13a}$$

$$\bar{\mathcal{L}}(x, t)=\frac{1}{2(2\alpha)}\sum_i\left(\frac{\partial q_i(x, t)}{\partial t}+\frac{\delta S[q]}{\delta q_i(x)}\Big|_{q=q(x,t)}\right)^2, \tag{2·13b}$$

$$\mathcal{S}=\int dt\,\mathcal{L}(t)\,, \tag{2·14}$$

then we can easily derive the *stochastic* "Hamiltonian" $\mathcal{H}$ and the *stochastic* "Hamiltonian density" $\mathscr{H}$ as follows:

$$\mathcal{H}[q,p]=\int d^4x\,\mathscr{H}(x,t)\,, \tag{2·15a}$$

$$\mathscr{H}(x,t)=\sum_i p_i(x,t)\frac{\partial q_i(x,t)}{\partial t}-\mathscr{L}(x,t)$$

$$=\sum_i\left\{\alpha p_i^2(x,t)-p_i(x,t)\frac{\delta S[q]}{\delta q_i}\Big|_{q=q(x,t)}\right\} \tag{2·15b}$$

in which we have used the *stochastic* "momentum" $p_i(x,t)$ given by

$$p_i(x,t)=\frac{\partial\mathscr{L}}{\partial\{\partial q_i(x,t)/\partial t\}}$$

$$=\frac{1}{2\alpha}\left[\frac{\partial q_i(x,t)}{\partial t}+\frac{\delta S[q]}{\delta q_i(x)}\Big|_{q=q(x,t)}\right]. \tag{2·16}$$

We can put $t=0$ in all quantities in $\mathcal{H}$, because the drift force in SQM is always independent of t and hence $\mathcal{H}$ is kept constant for the stochastic process in t.

The *randomization* procedure suggested in the Appendix enables us to derive the Fokker-Planck operator from the *stochastic* "Hamiltonian" in the following way. Suppressing the t-dependence or putting t equal to 0 in all quantities obtained above, we have only to replace $p_i(x)$ with

$$\hat{\pi}_i(x)=-\frac{\delta}{\delta q_i(x)} \tag{2·17}$$

in $\mathcal{H}$, provided that $p_i(x)$ is located to the left of other quantities. Operator $\mathcal{H}[q,\hat{\pi}]$, thus obtained, is nothing other than the Fokker-Planck operator (2·5b): $\mathcal{H}[q,\hat{\pi}]=\hat{F}$. Note that $\hat{\pi}_i$ is subject to the *randomization* condition

$$[\hat{\pi}_i(x),q_j(x')]=-\delta_{ij}\delta(x-x') \tag{2·18}$$

in the case of SQM. We shall return to the *stochastic* "canonical" formalism in § 5.

Note that the Fokker-Planck operator $\hat{F}$ is not self-adjoint: $\hat{F}\neq\hat{F}^\dagger$, in which

$$\hat{F}^\dagger=\alpha\int dx\sum_i\left\{\frac{\delta}{\delta q_i(x)}-\frac{1}{\alpha}\frac{\delta S}{\delta q_i(x)}\right\}\frac{\delta}{\delta q_i(x)}\,. \tag{2·19}$$

By means of the similarity transformation with $U=e^{S[q]/2\alpha}=U^\dagger$, we can transform $\hat{F}$ and $\hat{F}$ into a self-adjoint operator $\hat{H}$ as follows,

$$\hat{H}=U\hat{F}U^{-1}=U^{-1}\hat{F}^\dagger U=-2\alpha\sum_i\int d^4x\hat{A}_i^\dagger(x)\hat{A}_i(x)\,, \tag{2·20a}$$

$$=-2\alpha\sum_i\int d^4x\left[-\frac{1}{2}\hat{\pi}_i^2(x)+\bar{V}(q)\right], \tag{2·20b}$$

where

$$\hat{A}_i(x)=\frac{1}{\sqrt{2}}\left[-\bar{\pi}_i(x)+\frac{1}{2\alpha}\frac{\delta S[q]}{\delta q_i(x)}\Big|_{q=\hat{q}}\right], \tag{2·21a}$$

$$\hat{A}_i^\dagger(x)=\frac{1}{\sqrt{2}}\left[\bar{\pi}_i(x)+\frac{1}{2\alpha}\frac{\delta S[q]}{\delta q_i(x)}\Big|_{q=\hat{q}}\right], \tag{2·21b}$$

$$\bar{V}(q)=\frac{1}{8\alpha^2}\left(\frac{\delta S}{\delta q_i}\right)^2-\frac{1}{4\alpha}\frac{\delta^2 S}{\delta q_i^2}. \tag{2·21c}$$

Equation (2·20) tells us that $\hat{H}$ is a self-adjoint operator having non-positive real eigenvalues, and that both $\hat{F}$ and $\hat{F}^\dagger$ have the same eigenvalues as those of $\hat{H}$. Assuming the existence of the eigenvalue problem of $\hat{H}$,

$$\hat{H}\bar{u}_\nu[q]=\lambda_\nu\bar{u}_\nu[q]; \tag{2·22a}$$

$$(\bar{u}_\mu, \bar{u}_\nu)=\delta_{\mu\nu}, \quad \sum_\nu \bar{u}_\nu[q]\bar{u}_\nu^*[q']=\delta[q-q'], \tag{2·22b}$$

we can obtain the eigenvalue problems of $\hat{F}$ and $\hat{F}^\dagger$ as follows:

$$\hat{F}u_\nu[q]=\lambda_\nu u_\nu[q], \qquad u_\nu[q]\equiv U^{-1}\bar{u}_\nu[q], \tag{2·23a}$$

$$\hat{F}^\dagger v_\nu[q]=\lambda_\nu v_\nu[q], \qquad v_\nu[q]\equiv U\bar{u}_\nu[q], \tag{2·23b}$$

in which we have used $\hat{F}=U^{-1}\hat{H}U$ and $\hat{F}^\dagger=U\hat{H}U^\dagger$ with $U=e^{S[q]/2\alpha}$. The complete orthonormal condition (2·22b) becomes

$$(v_\mu, u_\nu)=\delta_{\mu\nu}, \qquad \sum_\nu u_\nu[q]v_\nu^*[q']=\delta[q-q'], \tag{2·24a}$$

$$(u_\mu, v_\nu)=\delta_{\mu\nu}, \qquad \sum_\nu v_\nu[q]u_\nu^*[q']=\delta[q-q']. \tag{2·24b}$$

Note that the highest value of non-positive eigenvalue λ is *zero* and has the eigenfunction

$$u_0[q]=\frac{1}{\sqrt{C}}\Phi_{\mathrm{eq}}[q], \quad v_0[q]=\sqrt{C}, \quad \text{then} \quad (v_0, u_0)=\int\Phi_{\mathrm{eq}}[q]\mathcal{D}q=1, \tag{2·25}$$

where $\Phi_{\mathrm{eq}}[q]$ and C are given by (2·6).

Now we can discuss whether or how an arbitrary random state goes to the equilibrium state (2·6) irrespective of initial state. By making use of (2·23a) and (2·24a), we expand a random state $\Phi[q, t]$ starting from an initial state $\Phi_0[q]$ in $\{u_\nu\}$ as

$$\Phi[q, t]=\exp(\hat{F}t)\Phi_0[q], \tag{2·26a}$$

$$=\sum_\nu e^{\lambda_\nu t}u_\nu[q](v_\nu, \Phi_0). \tag{2·26b}$$

Thus we conclude, with the help of (2·25), that

$$\Phi[q, t]\xrightarrow{t\to\infty}\Phi_{\mathrm{eq}}[q] \tag{2·27}$$

irrespective of the initial state Φ_0, under the condition that the highest eigenvalue ($\lambda_0=0$) is *discrete* and *non-degenerate*.

2.2. *Neutral scalar field*

As a simple example, we apply SQM to quantization of a free neutral scalar field, $\phi(x)$, described by the Euclidean action:

$$S_0[\phi]=\frac{1}{2}\int d^4x\phi(x)(-\Box+m^2)\phi(x)\,, \tag{2·28a}$$

$$=\frac{1}{2}\int d^4k\tilde{\phi}(k)(k^2+m^2)\tilde{\phi}(k)\,, \tag{2·28b}$$

where m stands for the particle mass and $\tilde{\phi}(k)$ for the Fourier transforms defined by

$$\tilde{\phi}(k)=\frac{1}{\sqrt{(2\pi)^4}}\int d^4xe^{-ikx}\phi(x)\,. \tag{2·29}$$

According to the prescription of SQM, we introduce an additional dependence of ϕ or $\tilde{\phi}$ on *fictitious time* t and then set the Langevin equation as

$$\frac{\partial}{\partial t}\tilde{\phi}(k,t)=-(k^2+m^2)\tilde{\phi}(k,t)+\tilde{\eta}(k,t)\,; \tag{2·30a}$$

$$\langle\tilde{\eta}(k,t)\rangle=0\,,\qquad \langle\tilde{\eta}(k,t)\tilde{\eta}(k',t')\rangle=2\delta^4(k+k')\delta(t-t')\,, \tag{2·30b}$$

where $\tilde{\eta}(k,t)$ is the Fourier transform of $\eta(x,t)$ with respect to x defined by a formula similar to (2·29).

We can easily solve (2·30a) and then obtain, by making use of (2·30b), the stationary one-particle correlation function as

$$D(x-x',t-t')=\frac{1}{(2\pi)^5}\int d^4kd\omega e^{ik(x-x')-i\omega(t-t')}\tilde{D}(k,\omega) \tag{2·31}$$

with

$$\tilde{D}(k,\omega)=\frac{2}{\omega^2+(k^2+m^2)^2}\,. \tag{2·32}$$

Note that $\tilde{D}(k,\omega)$ is also given by

$$\langle\tilde{\phi}(k,\omega)\tilde{\phi}(k',\omega')\rangle_{\text{st}}=\delta(k+k')\delta(w+w')\tilde{D}(k,\omega)\,, \tag{2·33}$$

where $\tilde{\phi}(k,\omega)$ is the 5-dimensional Fourier transform defined by

$$\tilde{\phi}(k,\omega)=\frac{1}{\sqrt{(2\pi)^5}}\int d^4xdte^{-ikx+i\omega t}\phi(x,t)\,. \tag{2·34}$$

Performing the integration with respect to ω in (2·31), we obtain

$$D(x-x',t-t')=\frac{1}{(2\pi)^4}\int d^4k\frac{1}{k^2+m^2}e^{ik(x-x')-(k^2+m^2)|t-t'|}\,, \tag{2·35}$$

which immediately yields

$$D(x-x',0)=\frac{1}{(2\pi)^4}\int d^4k\frac{1}{k^2+m^2}e^{ik(x-x')}=\varDelta(x-x')\,. \tag{2·36}$$

Thus we have obtained exactly the same free propagator as given by the conventional method, via the route of SQM.

The asymptotic behaviors of $D(x, t)$ in x and t are, respectively,

$$D(x,0)\xrightarrow[|x|\to\infty]{}\frac{m^2}{16\pi^3}\left(\frac{2\pi}{m|x|}\right)^{3/2}\exp[-m|x|]\,, \tag{2·37a}$$

$$D(0,t)\xrightarrow[|t|\to\infty]{}\frac{m^2}{16\pi^2}\left(\frac{1}{m^2|t|}\right)^{2}\exp[-m^2|t|]\,. \tag{2·37b}$$

They tell us that we can extract information about the first energy gap, i.e., the particle mass, from the asymptotic behavior of the correlation function $D(x, t)$. We should note another possibility of finding the particle mass from the asymptotic behavior not only in x but also in t.

Another possibility of finding the particle mass also comes from the averaged correlation function,

$$\bar{D}(t-t')=\langle\bar{q}(t)\bar{q}(t')\rangle_{\text{st}}\,; \tag{2·38a}$$

$$\bar{q}(t)=\sqrt{\left(\int d^4x\right)^{-1}}\int d^4xq(x,t)\,, \tag{2·38b}$$

because easy calculation gives

$$\bar{D}(t-t')\equiv\frac{1}{\int d^4x}\iint d^4xd^4x'D(x-x',t-t')=\frac{1}{m^2}\exp[-m^2|t-t'|]\,. \tag{2·39}$$

The reason why we have obtained the particle mass in both the x- and t-dependence of the correlation function can naturally be understood on the basis of the dispersion formula

$$\omega^2+(k^2+m^2)^2=0\,, \tag{2·40}$$

to give the pole of the integrand of (2·31), which must completely determine the one-particle spectrum in this case.

Here we discuss, through plausible arguments, how to extract information about the first energy gap, i.e., the particle mass, from the dependence of $D(x, t)$ on not only x but also t, in the case of interacting fields. In this case, the particle mass is to be defined with one of the poles of the one-particle (*Euclidean*) propagator, $\tilde{\Delta}(k)$, on the imaginary axis of the complex k-plane — we do not know of other reliable methods. Consequently, the mass should naturally be rooted in the pole of the correlation function, $\tilde{D}(k, \omega)$, given by (2·33).

The stationary property in x and t requires $\tilde{D}(k, \omega)$ to be an even function of k and ω, so that we can put

$$\tilde{D}(k,\omega)=\frac{2}{\omega^2+(k^2+m^2)^2+g^2\Xi(k^2,\omega^2)}\,, \tag{2·41}$$

where g stands for the coupling constant and $\Xi(k^2, \omega^2)$ for the "self-energy" function in the 5-dimensional field theory in (x, t)-space (see § 5). A possible approach to Ξ

was discussed by Namiki and Yamanaka.[14] Corresponding to the presence of stable physical particles, we should have simple poles with positive residues, which are located on the imaginary axis of the complex ω-plane in a symmetrical way with respect to the origin. For simplicity, suppose that we have only one physical particle. Then the corresponding pole is given by

$$\omega^2+\Omega^4(k^2)=0\,, \tag{2·42a}$$

under the assumption that $\Omega^4(k^2)$ is only one solution of

$$\Omega^4(k^2)=(k^2+m^2)^2+g^2\Xi(k^2,\,-\Omega^4(k^2)) \tag{2·42b}$$

with positive $\Omega^2(k^2)$ depending on k^2. Thus, we can rewrite the denominator of (2·41) as $(\omega^2+\Omega^4(k^2))[1+N(k^2)+R(\omega^2+\Omega^4(k^2))]$, where $N(k^2)=(d\Xi/d\omega^2)_{\omega^2=-\Omega^4(k^2)}$ and $R(0)=0$. Note that $[1+N+R]$ has no poles. Consequently, we are led to

$$D(x,\,t)=\frac{1}{(2\pi)^4}\int d^4k\frac{1}{[1+N(k^2)]\Omega^2(k^2)}\exp[ikx-\Omega^2|t|]\,. \tag{2·43}$$

SQM gives the field-theoretical propagator by means of $\Delta(x)=D(x,0)$, so that $\Omega^2(k^2)$ must be written as

$$\Omega^2(k^2)=(k^2+m^2)+g^2\Sigma(k^2)\,, \tag{2·44}$$

where $\Sigma(k^2)$ is nothing but the self-energy part of the conventional field thery. $\Omega^2(k^2)$ should have a simple pole at the renormalized mass, m_r, corresponding to the physical particle, provided that an appropriate procedure of renormalization has been carried out. Thus, we should obtain the same asymptotic behaviors of $D(x,0)$ and $D(0,t)$ as (2·37), and the same averaged correlation function as (2·39), if we replace m with m_r and multiply by the new factor $[1+N(-m_r{}^2)]^{-1}$. The latter factor is to be absorbed into the wave function renormalization.

The above plausible arguments suggest that we can also extract information about the particle mass from the (x,t)-dependences of the correlation function even in the case of interacting fields. Concerning this kind of argument, one may refer to the numerical derivation of the first energy gap in the case of anharmonic oscillators.[15]

§ 3. Application to gauge fields

Parisi and Wu[1] presented an interesting plan to remarkably simplify the gauge fixing procedure, by means of SQM, claiming that we need neither any gauge fixing procedure nor the Faddeev-Popov ghost. Actually, this was one of the most challenging attempts in modern field theory. Before this pioneer work, we used to rely upon the Faddeev-Popov trick in order to fix the gauge within the path-integral formulation of field theory. While true that the Faddeev-Popov method turns out to reveal a rather beautiful structure of the field theory with constraints, i.e., the BRS-symmetry (see Refs. 16) and 17)), its mathematical manipulation is rather ritualistic and cannot be used beyond perturbation theory because of the Gribov problem.[18] Parisi and Wu first derived the Landau-gauge propagator of a gauge field without resorting to the usual gauge-fixing term, and conjectured that the Faddeev-Popov term to recover the gauge invariance and unitarity, in the case of a non-Abelian gauge field, could automatically be produced within the framework of SQM. We also expect to avoid the troublesome Gribov problem by means of SQM.

Namiki, Ohba, Okano and Yamanaka[19] derived the general gauge-field propagator of a free gauge field depending on an arbitrary gauge parameter by taking into account the initial distribution of the longitudinal component of the gauge field which Parisi and Wu discarded in their original paper.[1] They also showed that the Parisi-Wu conjecture on the Faddeev-Popov effect was true, at least through one loop perturbative calculations.

Zwanziger[20] introduced a new type of gauge-fixing procedure, called *stochastic* gauge-fixing, within the Fokker-Planck formalism. Nakagoshi, Namiki, Ohba and Okano[21] formulated a similar idea, by means of a Langevin equation with a dissipative-nonlinear drift force. They perturbativelly showed that the nonlinear part of the drift force actually produced the Faddeev-Popov term. Examining the effect of the drift force, Seiler, Stamatescu and Zwanziger[22] analyzed in a clear-cut way a non-perturbative feature of the non-Abelian gauge field, in particular, of the Gribov problem.

Fukai and Okano[23] showed that this kind of procedure was also useful to stochastically quantize a linearized gravitational field.

Undoubtedly, we can see many important characteristics of SQM in its application to gauge fields. In this section, we briefly discuss the gauge-fixing problem and the Faddeev-Popov effect by means of SQM.

3.1. *SQM without gauge fixing*

Let us first consider stochastic quantization of an Abelian gauge field or a free part of non-Abelian gauge fields, $A_\mu(x)$, whose Euclidean action is given by

$$S=\frac{1}{4}\int d^4x F_{\mu\nu}F_{\mu\nu}\,, \tag{3·1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,, \tag{3·2}$$

where we have suppressed color indices in the non-Abelian case.

The conventional procedure of quantizing gauge fields relies upon a "gauge-fixing" term

$$\frac{1}{2\alpha}(\partial\cdot A)^2 \tag{3·3}$$

put in the Lagrangian density, α being the gauge parameter. As is well known, this procedure destroys the gauge invariance and the unitarity, even though they are recovered by the Faddeev-Popov trick. Parisi and Wu[1] wanted to quantize gauge fields without resorting to this kind of gauge-fixing and then without the help of the Faddeev-Popov ghost.

Following the general prescription given in § 2, we introduce an additional dependence of the field on *fictitious* time t and set the basic Langevin equation for a hypothetical stochastic process with respect to t as

$$\frac{\partial}{\partial t}A_\mu(k,t)=-(k^2\delta_{\mu\nu}-k_\mu k_\nu)A_\nu(k,t)+\eta_\nu(k,t)\,; \tag{3·4a}$$

$$\langle\eta_\mu(k,t)\rangle=0\,,\quad \langle\eta_\mu(k,t)\eta_\nu(k',t')\rangle=2\delta_{\mu\nu}\delta^4(k+k')\delta(t-t') \tag{3·4b}$$

for the Fourier transform, $A_\mu(k,t)$ (see (2·24) for definition, but suppress the symbol ~ here for simplicity). Remember that we are going to quantize the gauge field which is originally described by the dynamical action (3·1) without the gauge-fixing term (3·3). Introducing the projection operators $O^T_{\mu\nu}$ and $O^L_{\mu\nu}$ defined as

$$O^T_{\mu\nu}=\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\,, \tag{3·5a}$$

$$O^L_{\mu\nu}=\frac{k_\mu k_\nu}{k^2}\,, \tag{3·5b}$$

we can decompose the Langevin equation into two components as

$$\frac{\partial}{\partial t}A_\mu{}^T(k,t)=-k^2A_\mu{}^T+\eta_\mu{}^T\,, \tag{3·6a}$$

$$\frac{\partial}{\partial t}A_\mu{}^L(k,t)=\eta_\mu{}^L\,, \tag{3·6b}$$

where $A_\mu{}^T=O^T_{\mu\nu}A_\nu$ and $A_\mu{}^L=O^L_{\mu\nu}A_\nu$ stand for the transverse and longitudinal components of A_μ, respectively, and $\eta_\mu{}^T$ and $\eta_\mu{}^L$ for the corresponding components of η_μ. Note that (3·6a) for the transverse component has a definite drift force with damping effect, while (3·6b) for the longitudinal component describes a random-walk process without drift force. Lack of the drift force in (3·6b) is a direct consequence of the gauge invariance, i.e., $\delta S/\delta A_\mu{}^L=0$. Thus the longitudinal component does not have a stationary solution. This is a general feature of a gauge theory without any gauge-fixing condition to reduce the degrees of freedom.

Solving (3·4) for an initial field $A_\mu(k,0)$ given at $t=0$, we have

$$A_\mu(k,t)=\int_0^\infty G_{\mu\nu}(k;t-t')\eta_\nu(k,t')dt'+[O^T_{\mu\nu}e^{-k^2t}+O^L_{\mu\nu}]A_\nu(k,0)\,, \tag{3·7}$$

where $G_{\mu\nu}(k;t-t')$ is the Green's function given by

$$G_{\mu\nu}(k;t-t')=\theta(t-t')[O^T_{\mu\nu}e^{-k^2(t-t')}+O^L_{\mu\nu}]\,, \tag{3·8}$$

which is a solution of the equation

$$\left[\delta_{\mu\kappa}\frac{\partial}{\partial t}+k^2O^T_{\mu\kappa}\right]G_{\kappa\nu}(k;t-t')=\delta_{\mu\nu}\delta(t-t')\,. \tag{3·9}$$

Thus we obtain

$$A_\mu{}^T(k,t)=\int_0^t e^{-k^2(t-t')}\eta_\mu{}^T(t')dt'+e^{-k^2t}A_\mu{}^T(k,0)\,, \tag{3·10a}$$

$$A_\mu{}^L(k,t)=\int_0^t \eta_\mu{}^L(k,t')dt'+A_\mu{}^L(k,0)\,. \tag{3·10b}$$

The correlation function of the gauge field is then given by

$$\begin{aligned}D^{(0)}_{\mu\nu}(k,t;k',t')&=\langle A_\mu(k,t)A_\nu(k',t')\rangle\\&=\delta^4(k+k')\Big[O^T_{\mu\nu}\frac{1}{k^2}(e^{-k^2|t-t'|}-e^{-k^2(t+t')})+2t_<O^L_{\mu\nu}\Big]\\&\quad+[O^T_{\mu\sigma}e^{-k^2t}+O^L_{\mu\sigma}][O^T_{\nu\rho}e^{-k'^2t'}+O^L_{\nu\rho}]A_\sigma(k,0)A_\rho(k',0)\,,\end{aligned} \tag{3·11}$$

where $t_<$ stands for the smaller of t and t'. The prescription of SQM requires us to take the infinite t limit of the equal-time correlation function

$$D^{(0)}_{\mu\nu}(k,t;k',t')\xrightarrow[t\to\infty]{}\delta^4(k+k')\Big[\frac{1}{k^2}O^T_{\mu\nu}+2tO^L_{\mu\nu}\Big]+O^L_{\mu\sigma}O^L_{\nu\rho}A_\sigma(k,0)A_\rho(k',0)\,, \tag{3·12}$$

which is to be equated to the field-theoretical propagator.

However, we should remark that (3·12) contains a longitudinal component which diverges as $t\to\infty$ and also retains longitudinal parts of the initial configuration in the last term which are not proportional to $\delta^4(k+k')$. The delta function $\delta^4(x-x')$ is representative of the translational invariance, so we cannot accept (3·12) itself as a proper field-theoretical propagator. Parisi and Wu[1] have set $A_\mu(k,0)=0$ by hand to erase the last term and then obtained the Landau-gauge propagator plus the diverging term. Noting that any diverging longitudinal term will not appear in the final results of all gauge invariant quantities, they stressed that SQM enabled one to quantize gauge fields without resorting to the conventional gauge-fixing procedure based on (3·3).

While true that their SQM approach without use of the conventional gauge-fixing procedure is very interesting to us, we are not satisfied with their special choice of initial configuration, $A^\mu(k,0)=0$, because a natural supposition is that $A^\mu(k,0)$ does not have such a sharp distribution, but rather a random distribution around zero. In order to take this random distribution into account, Namiki, Ohba, Okano and Yamanaka[19] put

$$A_\mu{}^L(k,0)=\frac{k_\mu}{k^2}\phi(k)\,, \tag{3·13}$$

where $\phi(k)$ is a scalar function (satisfying $\phi^*(k)=-\phi(-k)$ for the reality condition of $A_\mu(k,0)$), and then assume that $\phi(k)$ is statistically subject to a Gaussian distribution law with width α around zero. Denoting this kind of average by upperbar, thus we obtain

$$\overline{\phi(k)\phi(k')}^\phi=-\alpha\delta^4(k+k') \tag{3·14}$$

for the functional average of $\phi(k)\phi(k')$, which leads us to

$$\overline{\langle A_\mu(k,t)A_\nu(k',t)\rangle_\eta}^\phi\xrightarrow[t\to\infty]{}\delta^4(k+k')\Big[\frac{1}{k^2}\Big(\delta_{\mu\nu}-(1-\alpha)\frac{k_\mu k_\nu}{k^2}\Big)+2tO^L_{\mu\nu}\Big]\,. \tag{3·15}$$

The finite part of the r.h.s. is nothing other than the usual α-gauge propagator in the conventional theory. This implies that the choice of initial distribution just corresponds to fixing the gauge parameter. Note that the gauge parameter in this case comes from the initial distribution but not from the gauge-fixing term (3·3) in the Lagrangian. However, we wish to emphasize that SQM can give the gauge field propagator without resort to the usual gauge-fixing procedure, as was first suggested by Parisi and Wu.[1)]

On the basis of the free correlation function and the free propagator obtained above, we can develop a *stochastic* perturbation theory to show a remarkable result that the t-dependent longitudinal component yields the so-called Faddeev-Popov effect without the help of any ghost field. For details, see Ref. 19).

3.2. *SQM with stochastic gauge fixing*

In this section, we first introduce the *stochastic* gauge-fixing procedure in the non-Abelian case, following Zwanziger's idea.[20)] His theory was originally formulated within the framework of the Fokker-Planck equation, but we in this section describe the stochastic gauge-fixing procedure in terms of the Langevin equation, based on another idea.[21)]

If we start from the action (3·1) supplemented by the conventional gauge-fixing term (3·3), the general prescription of SQM should give the Langevin equation

$$\frac{\partial}{\partial t}A_\mu{}^a = -\frac{\delta S}{\delta A_\mu{}^a(k)}\Big|_{A=A(k,t)} + \frac{1}{\alpha}\partial_\mu(\partial\cdot A^a) + \eta_\mu{}^a(k,t)\,, \tag{3·16}$$

where the η's are subject to (3·4b). The second term on the r.h.s. is a constraint force generated by the gauge-fixing term (3·3), which means the presence of a sort of constraint

$$\partial_\mu(\partial\cdot A^a)=0\,. \tag{3·17}$$

We know that the constraint force destroys the gauge invariance and unitarity.

The fundamental idea of Zwanziger gauge-fixing is to modify the Langevin equation with an additional restoring force, which pulls the configuration of $A_\mu{}^a$ back to the gauge-fixing surface along the "gauge orbit". For this purpose, a simplest way is to replace (3·17) with the gauge-covariant constraint

$$D_\mu{}^{ab}(\partial\cdot A^b)=0\,, \tag{3·18}$$

in which we have used the covariant derivative

$$D_\mu{}^{ab}=\partial_\mu\delta^{ab}-gf^{abc}A_\mu{}^c\,, \tag{3·19}$$

g and f^{abc} being the coupling constant and the structure factor of the gauge field, respectively. Correspondingly, the Langevin equation becomes

$$\frac{\partial}{\partial t}A_\mu{}^a = -\frac{\delta S}{\delta A_\mu{}^a(k)}\Big|_{A=A(k,t)} + \frac{1}{\alpha}D_\mu{}^{ab}(\partial\cdot A^b) + \eta_\mu{}^a(k,t)\,. \tag{3·20}$$

The new constraint force (the second term on the r.h.s.), corresponding to the constraint (3·18), never destroys the gauge invariance and unitarity, as we will see below. The gauge parameter α introduced as a Lagrange multiplier is left undetermined because of the orthogonality between the constraint force and the main drift force $-(\delta S/\delta A_\mu{}^a)$.

Note that we cannot rewrite (3·18) as a "total differential" form; in other words, we are dealing with a *non-holonomic* system having no Lagrangian or Hamiltonian. This procedure implies that SQM offers us a possible tool for quantizing those non-holonomic systems which cannot be described within the framework of the canonical formalism. Remember that SQM can start directly from the equation of motion without resorting to Lagrangian or Hamiltonian, while the canonical theory never does. This is one of the remarkable characteristics of SQM.

In order to examine the drift motion of the gauge field derived only by the constraint force (the second term on the r.h.s. of (3·19)), we have only to consider the equation

$$\frac{\partial}{\partial t}A_\mu{}^a=\frac{1}{\alpha}D_\mu{}^{ab}(\partial\cdot A^b)\,, \tag{3·21a}$$

which gives

$$A_\mu{}^a(x,\,t+\delta t)-A_\mu{}^a(x,\,t)=\delta t D_\mu{}^{ab}\frac{1}{\alpha}(\partial\cdot A^b) \tag{3·21b}$$

for an infinitesimal development of the gauge field from t to $t+\delta t$. This means that the drift motion is produced by a gauge transformation, because we usually introduce the gauge transfomation with a continuous parameter ϵ through the following infinitesimal gauge transformation:

$$\delta A_\mu{}^a(x)=\delta\epsilon G_\omega\cdot A_\mu{}^a(x) \tag{3·22}$$

for an infinitesimal increment $\delta\epsilon$, where

$$G_\omega=\int d^4x\omega^a(x)G^a(x)\,, \tag{3·23a}$$

$$G^a(x)=-D_\mu{}^{ab}\frac{\delta}{\delta A_\mu{}^b(x)}\,. \tag{3·23b}$$

Here $\omega^a(x)$ is an arbitrary function and $G^a(x)$ is the generator of gauge transformation subject to

$$[G^a(x),\,G^b(x')]=f^{abc}G^c(x)\delta(x-x')\,. \tag{3·23c}$$

Hence (3·22) immediately becomes (3·21b), if we identify ϵ with t and put $\omega^a=\alpha^{-1}(\partial\cdot A^a)$. This does mean that the constraint force (the second term on the r.h.s. of (3·20)) never destroys the gauge invariance of the theory. We should also remark that, based on (3·21), we can analyze the Gribov problem in a non-perturbative way.[22)]

Now our perturbative calculation starts from (3·20) or its rewritten form in momentum representation as

$$\dot{A}_\mu{}^a(k,\,t)=-\left(k^2\delta_{\mu\nu}-\left(1-\frac{1}{\alpha}\right)k_\mu k_\nu\right)A_\nu{}^a(k,\,t)+X_\mu{}^a(k,\,t)\,, \tag{3·24a}$$

where

$$X_\mu{}^a(k,t)=\eta_\mu{}^a(k,t)+Y_\mu{}^a(k,t)+Z_\mu{}^a(k,t)\,, \tag{3·24b}$$

in which $Y_\mu{}^a(k,t)$ and $Z_\mu{}^a(k,t)$ are nothing but the main drift force $-(\delta S/\delta A_\mu{}^a)$ and the nonlinear part of the constraint force, respectively, in momentum representation. Note that the α-dependent linear part of the constraint force gives a damping effect to guarantee the presence of equilibrium or the stationary state except $\alpha<+\infty$. On the other hand, the nonlinear part yields the so-called Faddeev-Popov term to recover the gauge invariance and unitarity. For details, see Ref. 21).

Returning back to the original idea,[20] let us introduce the Fokker-Planck equation corresponding to the Langevin equation (3·20):

$$\frac{\partial}{\partial t}\Phi_\omega(A;t)=\hat{F}_\omega\Phi_\nu(A;t)\,,\qquad \hat{F}_\omega=\hat{F}-G_\omega{}^\dagger\,, \tag{3·25}$$

in which

$$\hat{F}=\int d^4x\frac{\delta}{\delta A_\mu{}^a(x)}\left(\frac{\delta}{\delta A_\mu{}^a(x)}+\frac{\delta S}{\delta A_\mu{}^a(x)}\right), \tag{3·26a}$$

$$G_\omega{}^\dagger=\int d^4x\frac{\delta}{\delta A_\mu{}^a(x)}(D_\mu{}^{ab}\omega^b(A;x))\,. \tag{3·26b}$$

The time-evolution operator given by (3·25) is $\exp[(\hat{F}-G_\omega{}^\dagger)t]$. The additional operator G_ω in the Fokker-Planck equation comes from the constraint force of (3·20). However, we can explicitly prove that the expectation value

$$\langle\xi[A]\rangle_\omega=\int\xi[A]\Phi_\omega(A;t)\mathcal{D}A \tag{3·27}$$

is independent of ω for a gauge invariant quantity $\xi[A]$: $G_\omega\cdot\xi[A]=0$. The conclusion holds for not only $\omega^a=\alpha^{-1}(\partial\cdot A^a)$ but also an arbitrary ω. For details, see Ref. 20).

§ 4. Kerneled Langevin equation

In § 2, we have started our SQM by setting (2·2) as a basic Langevin equation. Choice of (2·2) is one of the simplest ones but not unique, because we can find many Langevin equations capable of producing the same quantum mechanics in thermal-equilibrium limit. One of the most important generalized-Langevin equations must be a *kerneled* Langevin equation. This type of Langevin equation was proposed by Ref. 24) and applied to several problems.[25] In this section, we briefly explain SQM based on the kerneled Langevin equation, and then introduce its recent application[26] to the stochastic quantization of dynamical systems in which the action functional is bottomless. One may expect that we can quantize a gravitational field in future development of the latter work.

Suppose that we have a scalar field ϕ characterized by an action functional $S[\phi]$, then set up the following kerneled Langevin equation

$$\frac{\partial}{\partial t}\phi(x,t)=-\int dx'K(x,x';\phi)\frac{\delta S}{\delta\phi(x',t)}+\int dx'\frac{\delta K(x,x';\phi)}{\delta\phi(x',t)}$$

$$+\int dx' G(x, x'; \phi)\eta(x', t), \tag{4·1a}$$

where t is the fictitious-time variable, and the kernel $K(x, x'; \phi)$ is a real functional depending on ϕ only at fictitious time t. The Gaussian white noise $\eta(x, t)$ is characterized by the statistical properties,

$$\langle\eta(x, t)\rangle=0, \quad \langle\eta(x, t)\eta(x', t')\rangle=2\delta(x-x')\delta(t-t'). \tag{4·1b}$$

It is assumed that $K(x, x'; \phi)$ is factorizable as

$$K(x, x'; \phi)=\int dx'' G(x, x''; \phi)G(x', x''; \phi) \tag{4·1c}$$

with a real functional $G(x, x'; \phi)$. The Fokker-Planck equation corresponding to (4·1) is

$$\frac{\partial}{\partial t}\Phi[\phi, t]=\int dx\int dx' \frac{\delta}{\delta\phi(x)}K(x, x'; \phi)\left[\frac{\delta}{\delta\phi(x')}+\frac{\delta S}{\delta\phi(x')}\right]\Phi[\phi, t]. \tag{4·2}$$

When the action $S[\phi]$ has a bottom and is positive definite for large ϕ, it is easily proved that this stationary solution is realized in the equilibrium limit, and that correct path-integral expectation values (with the Feynman measure $\exp[-S]$) can be calculated based on (4·1) at $t\to\infty$. Remember that the kerneled Langevin equation still retains a wide allowance. The simplest case is given by

$$K(x, x'; \phi)=\gamma^2\delta(x-x'). \tag{4·3}$$

Constant γ represents a scale transformation of the fictitious time t. Equation (4·1) goes back to (2·2) for $\gamma=1$, but we have to retain the constant γ in order to develop a complete theory of the renormalization scheme by means of SQM.[27]

Let us consider a scalar field ϕ characterized by a bottomless action $S[\phi]$. According to the naive prescription of SQM, we set up a Langevin equation (1·1) with a kernel $K(x, x'; \phi)=\delta(x-x')$. But solutions of the Langevin equation with this naive kernel run away to plus or minus infinity, and an equilibrium state does not exist. In such a case, expectation values cannot be calculated in the limit $t\to\infty$, and the prescription of SQM fails. This is the reason why we want to quantize our bottomless system using (4·1). For simplicity, let us choose a field dependent kernel of the following form:

$$K(x, x'; \phi)=\delta(x-x')\bar{K}[\phi], \tag{4·4a}$$

where $\bar{K}[\phi]$ is assumed to be a real and positive functional of ϕ. When K has the form (4·4a), $G(x, x'; \phi)$ defined with (4·1c) is given by

$$G(x, x'; \phi)=\delta(x-x')\bar{K}^{1/2}[\phi]. \tag{4·4b}$$

Then the kerneled Langevin equation (4·1a) takes the following form:

$$\frac{\partial}{\partial t}\phi(x, t)=-\bar{K}[\phi]\frac{\delta S[\phi]}{\delta\phi(x, t)}+\frac{\delta\bar{K}[\phi]}{\delta\phi(x, t)}+\bar{K}^{1/2}[\phi]\eta(x, t). \tag{4·5}$$

Here we define a functional $S_{-}[\phi]$ as

$$S_{-}[\phi]\equiv S[\phi]-\ln\bar{K}[\phi]\,. \tag{4·6}$$

By using $S_{-}[\phi]$, we rewrite (4·5) as

$$\frac{\partial}{\partial t}\phi(x,t)=-\bar{K}[\phi]\frac{\delta S_{-}[\phi]}{\delta\phi(x,t)}+\bar{K}^{1/2}[\phi]\eta(x,t)\,. \tag{4·7}$$

Since we assume $\bar{K}[\phi]$ is positive, the drift term $-\bar{K}[\phi]\cdot\delta S_{-}[\phi]/\delta\phi(x,t)$ becomes a damping term if we *properly choose* $\bar{K}[\phi]$ so that the functional $S_{-}[\phi]$ is *positive definite* (has a *bottom*). In this case, solutions of (2·4) are confined to a finite region, and we expect that an equilibrium state exists.

As an example, we consider the following bottomless ϕ^4-theory:

$$S[\phi]=S_2[\phi]-S_4[\phi]\,, \tag{4·8a}$$

$$S_2[\phi]=\int d^dx\left[\frac{1}{2}(\partial_\mu\phi)^2+\frac{1}{2}m^2\phi^2\right], \tag{4·8b}$$

$$S_4[\phi]=\int d^dx\left[\frac{1}{4}\lambda\phi^4\right],\qquad \lambda>0\,. \tag{4·8c}$$

For the model (4·8), we choose $\bar{K}[\phi]$ as

$$\bar{K}[\phi]=\exp[-S_4[\phi]]\,. \tag{4·9}$$

Then $S_{-}[\phi]$ becomes

$$S_{-}[\phi]=S[\phi]+S_4[\phi]=S_2[\phi]\,. \tag{4·10}$$

This $S_{-}[\phi]$ is clearly positive definite, and the Langevin equation (4·7) has the form,

$$\frac{\partial}{\partial t}\phi(x,t)=-\exp[-S_4]\frac{\delta S_2}{\delta\phi(x,t)}+\exp[-S_4/2]\eta(x,t)\,. \tag{4·11}$$

The drift term of (4·11) is clearly a damping term, and the solution is confined in a finite region if a finite initial value is chosen. Note that in the limit $\lambda\to 0$, (4·11) reduces correctly to the Langevin equation for a free field in the naive SQM prescription, where $K(x,x';\phi)=\delta(x-x')$.

Next we prove that the stochastic process (4·7) has an equilibrium state if we choose a proper $\bar{K}[\phi]$ such that $S_{-}[\phi]$ is positive definite and the drift term of (4·7) becomes a damping term. (Note that the conventional arguments leading to the equilibrium state const·exp$[-S]$ (see Ref. 28), for example) cannot be applied here, since normalizability of exp$[-S]$ is assumed in these arguments.)

When the drift term of (4·7) is a damping term, the Fokker-Planck distribution $\Phi[\phi,t]$ satisfies the following boundary condition:

$$\Phi[\phi,t]\xrightarrow[|\phi|\to\infty]{}0\,. \tag{4·12}$$

Note that we need (4·12) to derive the Fokker-Planck equation equivalent to (4·7) or (4·5), since surface terms should vanish in performing the functional integration by parts in the usual way of derivation. The derived Fokker-Planck equation is

$$\frac{\partial}{\partial t}\Phi[\phi, t]=\hat{F}\Phi[\phi, t], \tag{4·13a}$$

$$\hat{F}=\int d^d x\frac{\delta}{\delta\phi(x)}\bar{K}[\phi]\left[\frac{\delta}{\delta\phi(x)}+\frac{\delta S[\phi]}{\delta\phi(x)}\right]. \tag{4·13b}$$

Using (4·4), we can easily check that this Fokker-Planck equation is nothing but a rewritten form of (4·2).

The Fokker-Planck operator $\hat{F}$ of (4·13b) is rewritten as

$$\hat{F}=\int d^d x\frac{\delta}{\delta\phi(x)}\left[\frac{\delta}{\delta\phi(x)}+\frac{\delta S_-[\phi]}{\delta\phi(x)}\right]\bar{K}[\phi], \tag{4·14}$$

using the functional $S_-[\phi]$. Here we define $\bar{\hat{F}}$ and $\bar{\Phi}$ by

$$\bar{\hat{F}}=\hat{F}\bar{K}^{-1}[\phi], \tag{4·15a}$$

$$\bar{\Phi}[\phi, \tau(t)]=\bar{K}[\phi]\Phi[\phi, t] \tag{4·15b}$$

with

$$\tau(t)\equiv\bar{K}[\phi]t. \tag{4·15c}$$

Note that $\tau(t)$ is a scale transformation of the fictitious time t since $\bar{K}[\phi]$ is positive, and that the limit $t\to\infty$ means $\tau\to\infty$. Using (4·15), the Fokker-Planck equation (4·13a) with (4·14) becomes

$$\frac{\partial}{\partial\tau}\bar{\Phi}[\phi, \tau]=\bar{\hat{F}}\bar{\Phi}[\phi, \tau], \tag{4·16a}$$

$$\bar{\hat{F}}=\int d^d x\frac{\delta}{\delta\phi(x)}\left[\frac{\delta}{\delta\phi(x)}+\frac{\delta S_-[\phi]}{\delta\phi(x)}\right]. \tag{4·16b}$$

Equation (4·16) has the same form as the Fokker-Planck equation of the naive SQM prescription, where $K(x, x'; \phi)=\delta(x-x')$. (Note that an action functional $S[\phi]$ is replaced with $S_-[\phi]$.) We know the mathematical property of (4·16) very well. When $S_-[\phi]$ is positive definite, the asymptotic behavior of $\bar{\Phi}[\phi, \tau]$ is

$$\lim_{\tau\to\infty}\bar{\Phi}[\phi, \tau]=\text{const}\cdot\exp[-S_-[\phi]]. \tag{4·17}$$

Since $\exp[-S_-]$ is a normalizable functional, the conventional arguments (see Ref. 28), for example) concerned with a Fokker-Planck equation allows us to prove (4·17) in this case. Rewriting (4·17) using (4·15), we can obtain asymptotic behavior of the Fokker-Planck distribution itself as

$$\Phi_{\text{eq}}[\phi]\equiv\lim_{t\to\infty}\Phi[\phi, t]=\text{const}\cdot\exp[-S[\phi]], \tag{4·18}$$

which is nothing but the formal solution of (4·2).

Thus we have shown that our stochastic process (4·7) has the equilibrium state expressed by the Feynman path-integral measure as far as $\bar{K}$ is regular. Note that, in principle, the equilibrium state should be supplemented with the boundary condition (4·12). For detailed discussions on numerical simulations and on the stochastic path-integral representation, see Ref. 26).

§ 5. Five-dimensional "stochastic" field theory

Here we formulate a five-dimensional "stochastic" field theory for SQM. For simplicity, we deal only with a boson field, $\phi_i(x,t)$ (i standing for intrinsic quantum numbers). Extension to the fermion field case is straightforward. In SQM the field is considered to depend on both 4-dimensional Euclidean space-time coordinates x, and a fictitious time t, i.e., on 5-dimensional coordinates $X=(x,t)$. Denote $X_a=x_\mu$ for $a=\mu=1,2,3,4$ and $X_a=t$ for $a=5$ (a runs from 1 to 5 and μ from 1 to 4). SQM starts from the Langevin equation (2·2) associated with (2·3), i.e.,

$$\dot{\phi}_i(X)=K_i(\phi(X))+\eta_i(X)\,;\quad K_i(\phi(X))=-\frac{\delta S[\phi]}{\delta\phi_i}\Big|_{\phi=\phi(X)}\,, \tag{5·1a}$$

$$\langle\eta_i(X)\rangle=0\,,\qquad \langle\eta_i(X)\eta_j(X')\rangle=2\alpha\delta_{ij}\delta^5(X-X')\,, \tag{5·1b}$$

where $\dot{\phi}_i=\partial\phi_i/\partial t=\partial_5\phi_i$. Here we have used the conventional notation $\partial_a=\partial/\partial X_a$. In the Appendix we learned that the temporal evolution of the stochastic process is described by the following "stochastic" path-integral for the kernel function:

$$T[\phi,t|\phi',t']=C\int\mathcal{D}\phi(X)\exp\Big[-\frac{1}{2\alpha}\mathcal{S}[\phi,t|\phi',t']\Big]\,; \tag{5·2a}$$

$$\mathcal{S}[\phi,t|\phi',t']=\int_{\tau=t'}^{\tau=t}d\tau\mathcal{L}[\phi(x,\tau),\dot{\phi}(x,\tau)]=\int_{X_5=t'}^{X_5=t}d^5X\mathcal{I}(\phi(X),\dot{\phi}(X))\,, \tag{5·2b}$$

$$\mathcal{L}=\int d^4x\mathcal{I} \tag{5·2c}$$

with $\phi(x,t)=\phi(X)$ and $\phi(x',t')=\phi'(X')$, where the "Lagrangian" density corresponding to (5·1a) is given by (2·13b), i.e.,

$$\mathcal{I}=\frac{1}{2}\sum_i[\dot{\phi}_i(X)-K_i(\phi(X))]^2\,. \tag{5·3}$$

In usual cases, the drift force K_i can be divided into free and interaction parts as follows:

$$K_i(\phi(X))=-\Big(\frac{\delta S}{\delta\phi_i}\Big)_{\phi=\phi(X)}=-(m_i^2-\Box)\phi_i(X)+K_i^{\rm int}(\phi(X))\,, \tag{5·4}$$

corresponding to the decomposition of the dynamical action $S[\phi]$, as

$$S=S_0+S_{\rm int}\,;\quad S_0=\frac{1}{2}\sum_i\int d^4x\phi_i(x)(m_i^2-\Box)\phi_i(x)\,, \tag{5·5}$$

where m_i stands for the particle mass associated with the i-th field. Therefore, the "stochastic" Lagrangian density is written as

$$\mathcal{I}=\frac{1}{2}\sum_i[\dot{\phi}_i(X)+(m_i^2-\Box)\phi_i(X)-K_i^{\rm int}(\phi(X))]^2\,. \tag{5·6}$$

5.1. *"Stochastic-canonical" field theory —— "classical" formalism*

The "stochastic" path in the integrand of (5·2a) fluctuates around the "classical" path given by the variational principle

$$\delta \mathcal{S} = 0 \tag{5·7}$$

to yield the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\frac{\delta \mathcal{L}}{\delta \dot{\phi}_i(X)} - \frac{\delta \mathcal{L}}{\delta \phi_i(X)} = 0\,, \tag{5·8}$$

from which we can easily derive the "classical" field equations:

$$\dot{\phi}_i(X) = -(m_i^2 - \Box)\phi_i(X) + K_i^{\text{int}}(\phi(X)) + \pi_i(X)\,, \tag{5·9a}$$

$$\dot{\pi}_i(X) = (m_i^2 - \Box)\pi_i(X) - \sum_j \pi_j(X)\frac{\partial K_j^{\text{int}}(\phi(X))}{\partial \phi_i(X)}\,. \tag{5·9b}$$

Here we have introduced the "stochastic" momentum field defined as

$$\pi_i(X) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}_i(x,t)} = \dot{\phi}_i(X) + (m_i^2 - \Box)\phi_i(X) - K_i^{\text{int}}(\phi(X))\,. \tag{5·10}$$

Noting that $\mathcal{L}$ given by (5·6) depends on the second-order derivatives of ϕ with respect to x_μ ($\mu=1, 2, 3, 4$) but only on the first-order derivative with respect to t, let us reformulate the above procedure, based on the modified "stochastic" Lagrangian density given by

$$\mathcal{L}'(\phi, \partial_a\phi, \chi_c, \partial_b\chi_c, \xi_c) = \frac{1}{2}\sum_i[\dot{\phi}_i + m_i^2\phi_i - \sum_\mu \partial_\mu\chi_{\mu i} - K_i^{\text{int}}(\phi)]^2 + \sum_{i\mu}\xi_{\mu i}(\chi_{\mu i} - \partial_\mu\phi_i)\,, \tag{5·11}$$

whose last term gives the constraint, $\chi_{\mu i} = \partial_\mu\phi_i$, $\xi_{\mu i}$ being the Lagrange multiplier. The variational principle (5·7) yields

$$\frac{\partial \mathcal{L}'}{\partial \phi_i} - \sum_a \partial_a\left(\frac{\partial \mathcal{L}'}{\partial(\partial_a\phi_i)}\right) = 0\,, \tag{5·12a}$$

$$\frac{\partial \mathcal{L}'}{\partial \chi_{ci}} - \sum_b \partial_b\left(\frac{\partial \mathcal{L}'}{\partial(\partial_b\chi_{ci})}\right) = 0\,, \tag{5·12b}$$

$$\frac{\partial \mathcal{L}'}{\partial \xi_{\mu i}} = 0\,, \tag{5·12c}$$

where the last equation is nothing but the above constraint. Needless to say, (5·12) is exactly the same equations as (5·9).

By virtue of (5·12), we can easily prove that the 5-dimensional "energy-momentum" density tensor, defined by

$$\mathcal{T}_{ab} = \sum_i \frac{\partial \mathcal{L}'}{\partial(\partial_a\phi_i)}\partial_b\phi_i + \sum_{ic}\frac{\partial \mathcal{L}'}{\partial(\partial_a\chi_{ci})}\partial_b\chi_{ci} - \delta_{ab}\mathcal{L}'\,, \tag{5·13}$$

satisfies the conservation law of "energy-momentum", i.e.,

$$\sum_a \partial_a \mathcal{T}_{ab}=0\,. \tag{5·14}$$

Hence we have the conserved "energy-momentum" flow density given by $\mathcal{T}_{5b}(x,\,t)$, i.e.,

$$\mathcal{T}_{5b}=\sum_i \frac{\partial \mathcal{L}'}{\partial \dot{\phi}_i}\partial_b\phi_i-\delta_{5b}\mathcal{L}' \tag{5·15a}$$

or

$$\mathcal{T}_{55}=\sum_i \pi_i\dot{\phi}_i-\mathcal{L}'\,, \tag{5·15b}$$

$$\mathcal{T}_{5\mu}=\sum_i \pi_i\partial_\mu\phi_i\,, \tag{5·15c}$$

which obeys

$$\frac{\partial \mathcal{T}_{5b}}{\partial t}+\sum_\mu \frac{\partial \mathcal{T}_{\mu b}}{\partial x_\mu}=0\,. \tag{5·16}$$

In (5·16), we have used $(\partial\mathcal{L}'/\partial\dot{\chi}_{ci})=0$. Thus, we know that

$$P_b\equiv\int d^4x\,\mathcal{T}_{5b}(x,\,t) \tag{5·17a}$$

or

$$P_5\equiv\int d^4x\,\mathcal{T}_{55}(x,\,t)\,, \tag{5·17b}$$

$$P_\mu\equiv\int d^4x\,\mathcal{T}_{5\mu}(x,\,t) \tag{5·17c}$$

is a *constant* of motion independent of t.

We have so far been talking about the process in "Lagrangian" language, that is, in terms of ϕ and $\dot{\phi}$. If (5·10) enables us to express $\dot{\phi}$ by π, that is, to carry out the "Legendre" transformation by means of (5·9a), then we can describe the process in "Hamiltonian" language, that is, in terms of ϕ and π. Through the "Legendre" transformation, P_5, defined by (5·17b), turns to the "Hamiltonian"

$$\mathcal{H}[\phi,\,\pi]\equiv P_5[\phi,\,\dot{\phi}(\phi,\,\pi)]\,, \tag{5·18a}$$

that is,

$$\mathcal{H}=\sum_i\int d^4x\left[\frac{1}{2}\pi_i^2+\pi_iK_i(\phi)\right]=\sum_i\int d^4x\left[\frac{1}{2}\pi_i^2-\pi_i(m_i^2-\Box)\phi_i+\pi_iK_i^{\text{int}}(\phi)\right]. \tag{5·18b}$$

Naturally it holds that

$$\{\phi_i(x,\,t),\,\phi_j(x',\,t)\}=\{\pi_i(x,\,t),\,\pi_j(x',\,t)\}=0\,, \tag{5·19a}$$

$$\{\phi_i(x,\,t),\,\pi_j(x',\,t)\}=\delta_{ij}\delta^4(x-x') \tag{5·19b}$$

for "Poisson" brackets, and then

$$\dot{\phi}_i(x,\,t)=\{\phi_i(x,\,t),\,\mathcal{H}\}\,,\quad \dot{\pi}_i(x,\,t)=\{\pi_i(x,\,t),\,\mathcal{H}\} \tag{5·20}$$

for the equation of motion, where

$$\{f, g\} \equiv \sum_i \int d^4x \left[\frac{\delta f}{\delta \phi_i(x)} \frac{\delta g}{\delta \pi_i(x)} - \frac{\delta f}{\delta \pi_i(x)} \frac{\delta g}{\delta \phi_i(x)} \right]. \tag{5·21}$$

Equation (5·20) produces exactly the same result as (5·9), by making use of (5·18) and (5·19). We can also explicitly show that

$$\{P_\mu, \mathscr{H}\} = 0 . \tag{5·22}$$

We started our theory by setting (5·3) as the Lagrangian density, in order to explicitly show the presence of the diffusion constant $\alpha = \hbar$. Of course, however, we can use another choice (2·13b), i.e.,

$$\mathcal{L} = \frac{1}{2(2\alpha)} \sum_i \left[\dot{\phi}_i^2 + \frac{\delta S[\phi]}{\delta \phi_i(x)}\bigg|_{\phi=\phi(x,t)} \right]. \tag{5·23}$$

In this case, we have to replace π_i and $\mathscr{H}$ with $2\alpha\pi_i$ and $2\alpha\mathscr{H}$ respectively in the above expressions.

5.2. *"Stochastic-canonical" field theory —— "operator" formalism*

According to the ideas mentioned in the Appendix, we can easily transfer the above "classical" theory into the abstract "operator" formalism in an abstract vector space, to perfectly describe our hypothetical stochastic process for SQM, by replacing the Poisson brackets in § 5.1 with corresponding commutators for the corresponding operators, i.e.,

$$\{f, g\} \to \frac{1}{2\alpha} [\hat{f}, \hat{g}] . \tag{5·24a}$$

Note that the replacement rule becomes

$$\{f, g\} \to [\hat{f}, \hat{g}] , \tag{5·24b}$$

when we start from (5·23), but we continue to use the original Lagrangian density (5·3). Thus, we are led to the following commutation relations

$$[\hat{\phi}_i(X), \hat{\phi}_j(X')]_{t=t'} = [\hat{\pi}_i(X), \hat{\pi}_j(X')]_{t=t'} = 0 , \tag{5·25a}$$

$$[\hat{\phi}_i(X), \hat{\pi}_j(X')]_{t=t'} = 2\alpha\delta_{ij}\delta^4(x-x') \tag{5·25b}$$

and to the "Heisenberg" equation of motion

$$\frac{\partial}{\partial t}\hat{\phi}_i(X) = [\hat{\phi}_i(X), \hat{F}] , \qquad \frac{\partial}{\partial t}\hat{\pi}_i(X) = [\hat{\pi}_i(X), \hat{F}] \tag{5·26}$$

for "Heisenberg" operators. Here we have introduced the Fokker-Planck operator given by

$$\hat{F} \equiv \frac{1}{2\alpha} \mathscr{H}(\hat{\phi}, \hat{\pi}) , \tag{5·27}$$

where $\mathscr{H}$ has the same functional form as (5·18), but note that $\hat{\pi}$ must be put on the left of other operators to give (5·31b) (to be compared with (A·69)) later. (Also note that $\hat{F}=\mathscr{H}[\hat{\phi}, \hat{\pi}]$ if we start from (5·23).) According to § 2 and the Appendix, we call (5·25) the *randomization* condition. As is easily shown, (5·26) together with (5·25) exactly yields operator equations of the same form as (5·9).

In the same way as in the Appendix, the stochastic "Heisenberg" operators are related to the time-independent "Schrödinger" operators, $\hat{\phi}_i(x)$ and $\hat{\pi}_i(x)$, through

$$\hat{\phi}_i(X)=\hat{T}^{-1}(t)\hat{\phi}_i(x)\hat{T}(t)\,, \qquad \hat{\pi}_i(X)=\hat{T}^{-1}(t)\hat{\pi}_i(x)\hat{T}(t)\,; \tag{5·28a}$$

$$\hat{T}(t)=\exp[\hat{F}t]\,. \tag{5·28b}$$

Note that $\partial\hat{F}/\partial t=0$ in SQM. The "Schrödinger" operators are also subject to the same commutation relation (5·25) and work as initial values of the "Heisenberg" operators at $t=0$.

We are now dealing with abstract operators, such as $\hat{\phi}$ and $\hat{\pi}$, in an abstract vector space. Along the same line of thought as in the Appendix, we assume the t-independent operator fields, $\{\hat{\phi}_i(x)\}$, in the "Schrödinger" picture, to be self-adjoint and to have the eigenvalue problem in the abstract vector space:

$$\hat{\phi}_i(x)|\phi\rangle=\phi_i(x)|\phi\rangle\,; \tag{5·29a}$$

$$\langle\phi|\phi'\rangle=\delta[\phi-\phi']\,, \qquad \int|\phi\rangle\mathscr{D}\phi\langle\phi|=\hat{1}\,. \tag{5·29b}$$

In the ϕ-representation, (5·25b) gives a matrix element of $\hat{\pi}$ in the form

$$\langle\phi|\hat{\pi}_i(x)|\phi'\rangle=-2\alpha\frac{\delta}{\delta\phi_i(x)}\delta[\phi-\phi']\,. \tag{5·30}$$

Remember that we have the following identities in the abstract vector space:

$$\Big\langle\int\Big|\hat{\pi}_i(x)=0 \qquad \text{and then} \qquad \Big\langle\int\Big|\hat{\pi}_i(X)=0\,, \tag{5·31a}$$

so that

$$\Big\langle\int\Big|\hat{F}=0\,, \qquad \Big\langle\int\Big|\hat{T}(t)=\Big\langle\int\Big| \tag{5·31b}$$

for Dirac's standard ket defined by

$$\Big|\int\Big\rangle=\int\mathscr{D}\phi|\phi\rangle\,. \tag{5·32}$$

With the help of (5·30), we know that (5·31a) in the ϕ-representation is nothing but the formula similar to (A·13).

Equations (5·25) and (5·30) mean that $\hat{\pi}_i$ is anti-self-adjoint, that is, an imaginary field subject to

$$\hat{\pi}_i^{\dagger}(x)=-\hat{\pi}_i(x)\,. \tag{5·33}$$

In abstract operator formalism, we describe a random state in terms of an abstract vector, say $|\Phi\rangle_t$, which is written in terms of a conventional probability-distribution functional $\Phi[\phi, t]$ as

$$|\Phi\rangle_t=\int|\phi\rangle\mathcal{D}\phi\Phi[\phi,t]; \qquad \Phi[\phi,t]=\langle\phi|\Phi\rangle_t. \tag{5·34}$$

This means that $|\Phi\rangle_t$ has $\Phi[\phi,t]=\langle\phi|\Phi\rangle_t$ as its representative. For an initial state $|\Phi\rangle_0$ with representative $\Phi_0[\phi]$ at $t=0$, we have

$$|\Phi\rangle_t=\hat{T}(t)|\Phi\rangle_0; \qquad \hat{T}(t)=e^{\hat{F}t}, \tag{5·35}$$

which obeys the abstract Fokker-Planck equation

$$\frac{d}{dt}|\Phi\rangle_t=\hat{F}|\Phi\rangle_t; \tag{5·36a}$$

$$\frac{d}{dt}\hat{T}(t)=\hat{F}\hat{T}(t). \tag{5·36b}$$

The conventional Fokker-Planck equation (2·5) in the ϕ-representation is easily derived from (5·36a) by making use of (5·29) and (5·30). Equation (5·35) also gives

$$\Phi[\phi,t]=\int\mathcal{D}\phi' T[\phi,t|\phi',t']\Phi_0[\phi']; \tag{5·37a}$$

$$T[\phi,t|\phi',t']=\langle\phi|e^{\hat{F}(t-t')}|\phi'\rangle, \tag{5·37b}$$

in the ϕ-representation. $T[\phi,t|\phi',t']$ describes the transition probability for the configuration change from ϕ' at t' to ϕ at t. We know that (5·37a) shows the Markoffian property of this stochastic process. Equation (5·31) guarantees the conservation law of probability and the normalization condition at every time, because $\langle\int|\Phi\rangle_t=\int\mathcal{D}\phi\Phi[\phi,t]=\langle\int|e^{\hat{F}}|\Phi_0\rangle=1$ and $\int\mathcal{D}\phi T[\phi,t|\phi',t']=1$.

Through the above arguments, we knew that we were dealing with random variables and stochastic "momenta" in the ϕ-representation in § 2.1. Consequently, we can translate all equations and relations, including the eigenvalue problem given in § 2.1 into the abstract ones. For example, one can easily see that

$$|\Phi\rangle_t\xrightarrow{t\to\infty}|\Phi\rangle_{\rm eq}=\sqrt{C}|u_0\rangle, \tag{5·38a}$$

$$T(\phi,t|\phi',t')\xrightarrow{t\to\infty}\Phi_{\rm eq}[\phi]. \tag{5·38b}$$

Here $|\Phi\rangle_{\rm eq}$ and $|u_0\rangle$ (or $|v_0\rangle$) are, respectively, abstract versions of $\Phi_{\rm eq}[\phi]$ and $u_0[\phi]$ (or $v_0[\phi]$), belonging to the zero eigenvalue of $\hat{F}$ (or $\hat{F}^\dagger$), given in § 2.1. Note that we have the abstract version of (2·25), i.e.,

$$|u_0\rangle=\frac{1}{\sqrt{C}}|\Phi\rangle_{\rm eq}, \qquad |v_0\rangle=\sqrt{C}\Big|\int\Big\rangle, \quad \langle v_0|u_0\rangle=1. \tag{5·39}$$

Using these formulas written in the abstract notation, we can easily describe the stationary expectation value of a random quantity, say $G=G[\phi]$, and the stationary correlation function in the form:

$$\langle G\rangle_{\rm st}=\langle v_0|G[\hat{\phi}]|u_0\rangle, \tag{5·40a}$$

$$D_{ij}(X-X')=\langle v_0|T\hat{\phi}_i(X)\hat{\phi}_j(X')|u_0\rangle. \tag{5·40b}$$

Here we have used (5·31) rewritten by $\langle v_0|$ and the abstract version of the eigenvalue equation (2·23) for $\lambda=0$, i.e.,

$$\langle v_0|\hat{F}=0\,,\qquad \hat{F}|u_0\rangle=0\,. \tag{5·41}$$

Let us introduce the X-translation operator defined by

$$\hat{F}_a\equiv\frac{1}{2\alpha}\hat{P}_a[\hat{\phi},\hat{\pi}]\,; \tag{5·42a}$$

$$[\hat{F}_a,\hat{F}_b]=0\,,\qquad (a=1,2,3,4,5) \tag{5·42b}$$

(see (5·17) for P_a), which gives

$$\partial_a\hat{\phi}_i(X)=[\hat{\phi}_i(X),\hat{F}_a]\,,\quad \partial_a\hat{\pi}_i(X)=[\hat{\pi}_i(X),\hat{F}_a] \tag{5·43a}$$

or

$$\hat{\phi}_i(X)=\exp[-\hat{F}\cdot X]\hat{\phi}_i(0)\exp[\hat{F}\cdot X]\,, \tag{5·43b}$$

where $\hat{F}\cdot X=\sum_a\hat{F}_aX_a$. Then we can rewrite the correlation function as

$$D_{ij}(X-X')=\langle v_0|\hat{\phi}_i(0)\exp[\hat{F}\cdot(X-X')]\hat{\phi}_i(0)|u_0\rangle\,, \tag{5·44}$$

where we have used

$$\langle v_0|\hat{F}_a=0\,,\qquad \hat{F}_a|u_0\rangle=0\,. \tag{5·45}$$

Equation (5·45) represents the space-time uniformity for $a=1,2,3,4$ and (5·41) for $a=5$. On the basis of (5·44), we can utilize many mathematical techniques developed in modern field theory.[27),28)]

As we have seen in this section, SQM with an additional degree of freedom coming from the fictitious time naturally requires us to formulate a 5-dimensional field theory, as its theoretical framework, in which the additional "momemtum" field, $\pi_i(X)$, necessarily appears. In the perturbative calculation of $D_{ij}(X-X')$ by means of SQM, in fact, we need to use not only the stochastic free ϕ-propagator, $D_{ij}^{(0)}(X-X')=\langle v_0^{(0)}|T\hat{\phi}_i^{(0)}(X)\hat{\phi}_j^{(0)}(X')|u_0^{(0)}\rangle$ but also the free mixed-propagator $\tilde{D}_{ij}^{(0)}(X-X')=\langle v_0^{(0)}|T\hat{\phi}_i^{(0)}(X)\hat{\pi}_j^{(0)}(X')|u_0^{(0)}\rangle$ (the superscript (0) stands for free quantities, and note that $\tilde{\tilde{D}}_{ij}^{(0)}(X-X')=\langle v_0^{(0)}|T\hat{\pi}_i^{(0)}(X)\hat{\pi}_j^{(0)}(X')|u_0^{(0)}\rangle=0$). We know that we cannot complete the SQM perturbation theory and the renormalization scheme without resorting to such an additional field, as was discussed in detail in Refs. 27) and 28). Here we do not enter into details of perturbative calculations.

Finally, we give the path-integral representation of the generating functional in "phase space" or in "Hamiltonian" language as follows:

$$\mathcal{Z}_J=C\int\mathcal{D}\phi\mathcal{D}\pi\exp\left[-\frac{1}{2\alpha}\mathcal{S}_J\right], \tag{5·46a}$$

where C, J_i^ϕ and J_i^π are an appropriate normalization constant, external sources of ϕ_i and π_i, respectively, and

$$\mathcal{S}_J=\int d^5X\sum_i\left[\pi_i\dot{\phi}_i-\pi_iK_i(\phi)-\frac{1}{2}\pi_i^2+J_i^\phi\phi_i+J_i^\pi\pi_i\right]. \tag{5·46b}$$

The integrand of (5·46b) is nothing but the "Lagrangian" density in "Hamiltonian" language. Note that $\lim_{J\to 0}\mathscr{Z}_J = T(\phi, t|\phi', t')$.

Compare this with (5·2) for the path-integral representation in the "configuration space", written in "Lagrangian language". One can prove the equivalence between (5·2) and (5·46) (in the case of vanishing J's) in a similar way, as in Ref. 29), where the corresponding proof in the ordinary path-integral field theory is given. A heuristic proof is to integrate out the π-variables in (5·46) to get (5·2). Note that the π's are imaginary fields.

§ 6. Concluding remarks

In this paper we have described the basic ideas and important characteristics of the Parisi-Wu stochastic quantization method (SQM), which is remarkably different from other stochastic-dynamical approaches to quantum mechanics. Here we stress some of the differences, recalling that SQM was originally designed to produce quantum mechanics from the thermal equilibrium limit of a hypothetical stochastic process with respect to a *fictitious* time other than *ordinary* time.

First of all, we have to mention that SQM enables us to extract the first energy gap or the relevant particle mass from asymptotic behaviors of a one-particle correlation function not only in ordinary space-time coordinates but also in the fictitious time, on the same basis of a single dispersion formula. One of the most important applications of SQM is obviously the quantization of gauge fields without resorting to the conventional gauge-fixing procedure, i.e., the Faddeev-Popov trick. On the basis of the remarkable fact that SQM can start from the equation of motion itself but not directly from Lagrangian or Hamiltonian, we have formulated a stochastic gauge-fixing procedure which never destroys the gauge invariance and unitarity. Furthermore, SQM allows us to set up a wide class of hypothetical stochastic processes and corresponding Langevin equations to give the same quantum mechanics in the equilibrium limit. As an interesting example among them, we have given a kerneled Langevin equation capable of quantizing dynamical systems with bottomless actions.

Undoubtedly, SQM offers us a powerful quantization method for enlarging the territory of quantum mechanics beyond that of the conventional theories, as far as we are concerned with practical and technical merit. In this context, we have only to look upon the "fictitious time" as a mathematical tool, but need not to find its physical meaning.

However, a fascinating task would be to seek out possible physical roots of the "fictitious time" in exotic space-time

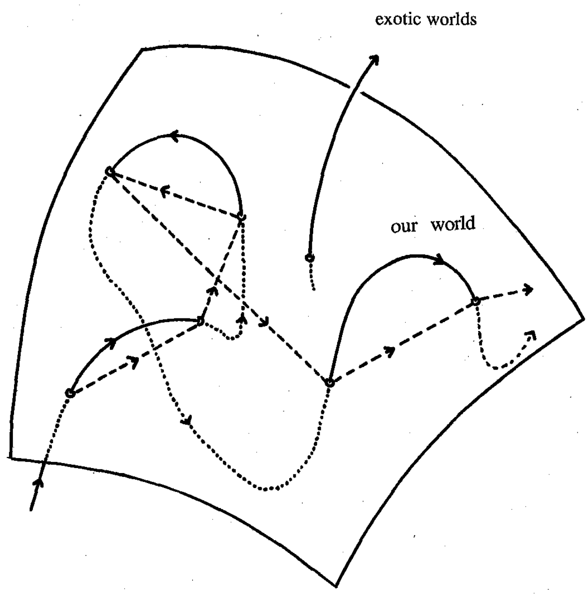

Fig. 1.

worlds surrounding ours. Recent particle cosmology suggests that the primeval universe with many space-time worlds has shrunk to the present 4-dimensional world. One may supplement this story with a new idea that the exotic worlds are still alive and fluctuating around the present one, under the control of classical dynamics in a many-dimensional curved space. According to this idea, we may imagine that even a smooth deterministic motion in the larger world makes a random trace on the present 4-dimensional world, as is suggested in Fig. 1. We may also speculate that such a classical motion yields "quantum fluctuations" through a mechanism similar to that in SQM or micro-canonical quantization, and then the motions toward the exotic worlds should be described in terms of the "fictitious time". Of course, this kind of idea is still only a dream.

The author is very much indebted to Dr. S. Tanaka for his laborious help.

Appendix
—— *Theory of Stochastic Processes* ——

In this Appendix we briefly explain important formulas in the theory of stochastic processes which are useful for our formulation of SQM, mainly based on the Ito calculus. For more details or for the Stratonovich calculus, see review articles and textbooks on stochastic quantization and the theory of stochastic processes.[30]

A.1. *The Langevin equation and the Fokker-Planck equation*

A stochastic process of the Wiener-Markoffian type is usually governed by the Langevin equation

$$\frac{dq_i(t)}{dt}-K_i(q(t))=\eta_i(t) \tag{A$\cdot$1}$$

for random variables $q(t)=\{q_1(t), q_2(t), \cdots, q_n(t)\}$, $K_i(q)$ being the drift force. The random force $\eta_i(t)$ is subject to the Gaussian statistical law

$$\langle\eta_i(t)\rangle=0\,, \tag{A$\cdot$2a}$$

$$\langle\eta_i(t)\eta_j(t')\rangle=2\alpha\delta_{ij}\delta(t-t')\,, \tag{A$\cdot$2b}$$

$$\langle\eta_i(t)\eta_j(t)\cdots\eta_k(t)\rangle=0 \quad \text{for odd number of } \eta\text{'s}\,, \tag{A$\cdot$2c}$$

$$=\sum\langle\eta_l(t)\eta_m(t)\rangle\langle\eta_p(t)\eta_q(t)\rangle\cdots \text{for even number of } \eta\text{'s}\,, \tag{A$\cdot$2d}$$

where α is the diffusion constant and the summation is taken over every possible pair of η's. From (A$\cdot$1) we obtain $q(t)$ or $G(q(t))$ as a function of the η's, whose average calculated by means of (A$\cdot$2) gives the expectation value $\langle G(q(t))\rangle$ or the correlation function $\langle q_i(t)q_j(t')\cdots q_k(t'')\rangle$. In the case of drift forces independent of time, the correlation functions usually do not depend on the time origin but only time differences in the limit $t, t', \cdots \to\infty$. We call these asymptotic limits the stationary correlation functions. In particular, the stationary two-point correlation defined by

$$D_{ij}(t-t')\equiv\langle q_i(t)q_j(t')\rangle_{\mathrm{st}}=\lim_{t,t'\to\infty}\langle q_i(t)q_j(t')\rangle \quad \text{for fixed } t-t' \tag{A$\cdot$3}$$

will play an important role in the theory of stochastic processes. We can talk about the *unit power spectrum* of q_i, $P_{q_i}(\omega)$, given from its Fourier transform $\tilde{D}_{ij}(\omega)$ as follows,

$$D_{ii}(\tau)=\int_0^\infty P_{q_i}(\omega)\cos\omega\tau d\omega \quad \text{with} \quad P_{q_i}(\omega)=\frac{1}{\pi}\tilde{D}_{ii}(\omega)\,, \tag{A·4a}$$

$$\tilde{D}_{ij}(\omega)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau D_{ij}(\tau)e^{i\omega\tau}\,. \tag{A·4b}$$

Note that $D_{ij}(\tau)=D_{ji}(-\tau)$ and $\tilde{D}^*_{ij}(\omega)=\tilde{D}_{ji}(\omega)$.

For a better mathematical formulation, we should replace the Langevin equation with the following stochastic difference-equation of the Ito-type:

$$dq_i(t)-K_i(q(t))dt=dw_i(t)\,; \qquad dq_i(t)\equiv q_i(t+dt)-q_i(t)\,, \tag{A·5}$$

in which the random force part, $dw_i(t)$, should be subject to

$$\langle dw_i(t)\rangle=0\,, \tag{A·6a}$$

$$\langle dw_i(t)dw_j(t')\rangle=\begin{cases}0 & \text{for} \quad t\neq t'\,,\\ 2\alpha\delta_{ij}dt & \text{for} \quad t=t'\,.\end{cases} \tag{A·6b}$$

Behind (A·6b) we have the distribution law

$$W_{dw}(x)dx=\prod_i\frac{1}{\sqrt{2\pi(2\alpha dt)}}\exp\left[-\frac{x_i^2}{2(2\alpha dt)}\right]dx_i \tag{A·7}$$

for the probability of finding a value of dw_i in (x_i, x_i+dx_i).

Equation (A·6) tells us that we have to estimate the order of magnitude of dw_i to be $\sqrt{dt}$, or equivalently, the order of η_i to be $(\sqrt{dt})^{-1}$. Therefore, we have to retain terms proportional to $(dq_i(t))^2$ in the Taylor expansion of $G(q(t+dt))$ with respect to $dq_i(t)$ in the following way:

$$\begin{aligned}G(q(t+dt))&=G(q(t)+dq(t))\\ &=G(q(t))+\sum_i\left(\frac{\partial G(q)}{\partial q_i}\right)_{q=q(t)}dq_i+\frac{1}{2}\sum_i\sum_j\left(\frac{\partial^2G(q)}{\partial q_i\partial q_j}\right)_{q=q(t)}dq_idq_j+\cdots\\ &=G(q(t))+\sum_i\left(\frac{\partial G(q)}{\partial q_i}\right)_{q=q(t)}[K_i(q(t))dt+dw_i(t)]\\ &\quad+\frac{1}{2}\sum_i\sum_j\left(\frac{\partial^2G(q)}{\partial q_i\partial q_j}\right)_{q=q(t)}dw_i(t)dw_j(t)+O(\sqrt{(dt)^3})\,,\end{aligned} \tag{A·8}$$

in order to keep terms of the first order in dt. We also learn by (A·5) that $q_i(t)$ is not determined by $\eta_i(t)$ but by $\eta_i(t-dt)$, in other words, that $q_i(t)$ never correlates with $\eta_i(t)$, so that

$$\langle G(q(t))\eta_i(t)\rangle=\langle G(q(t))\rangle\langle\eta_i(t)\rangle=0\,, \tag{A·9a}$$

$$\left\langle\left(\frac{\partial G(q)}{\partial q_i}\right)_{q=q(t)}\eta_j(t)\right\rangle=0,\ \cdots\,. \tag{A·9b}$$

Let us introduce the probability distribution function, $\Phi(q, t)$, defined by

$$\int G(q)\Phi(q, t)dq = \langle G(q(t))\rangle\,, \tag{A·10}$$

of finding a value of q in $(q, q+dq)$ at time t, where $dq = \prod_i dq_i$. The q's on the l.h.s. are simply integration variables, while the $q(t)$'s on the r.h.s. are random variables obeying the Langevin equation (A·1) or (A·5). Differentiating (A·10) and putting $\int G(q)[\partial\Phi(q, t)/\partial t]dq = d\langle G(q(t))\rangle/dt = [\langle G(q(t+\Delta t))\rangle - \langle G(q(t))\rangle]/(\Delta t)$, with the help of (A·8) and (A·9), we obtain

$$\begin{aligned}
&\int G(q)\left[\frac{\partial\Phi(q, t)}{\partial t}\right]dq \\
&= \left\langle \sum_i \left(\frac{\partial G(q)}{\partial q_i}\right)_{q=q(t)} K_i(q(t))\right\rangle + \alpha \left\langle \sum_i \left(\frac{\partial^2 G(q)}{\partial q_i^2}\right)_{q=q(t)}\right\rangle \\
&= \int G(q)\left[\sum_i \left\{-\frac{\partial}{\partial q_i}K_i(q) + \alpha\left(\frac{\partial^2}{\partial q_i^2}\right)\right\}\Phi(q, t)\right]dq\,,
\end{aligned}$$

where we have done integration by parts, from the second line to the last one, under an appropriate boundary condition on $\Phi(q, t)$. From the equality of the first line to the last one, we can immediately derive the Fokker-Planck equation

$$\frac{\partial}{\partial t}\Phi(q, t) = \hat{F}\Phi(q, t)\,; \tag{A·11a}$$

$$\hat{F} \equiv \sum_i [\alpha\hat{D}_i^2 - \hat{D}_i K_i(q)] \quad \text{with} \quad \hat{D}_i = \frac{\partial}{\partial q_i}\,. \tag{A·11b}$$

We call $\hat{F}$ the Fokker-Planck operator. Note that $\hat{F}$ is not self-adjoint because $\hat{F} \neq \hat{F}^\dagger$, where the adjoint operator is given by

$$\hat{F}^\dagger = \sum_i [\alpha\hat{D}_i^2 + K_i(q)\hat{D}_i]\,. \tag{A·11c}$$

Note that the Fokker-Planck equation permits us to put the normalization condition

$$\int dq\Phi(q, t) = 1 \tag{A·12}$$

independent of time, because $d/dt\int\Phi(q, t) = \int dq\hat{F}\Phi(q, t)$ is reduced to the vanishing integral of the form

$$\int dq\hat{D}_i(\cdots\Phi(q, t)) = 0 \tag{A·13}$$

under an appropriate boundary condition on Φ.

In the case of time-independent drift force, the Fokker-Planck equation (A·11) gives the equilibrium distribution, $\Phi_{\mathrm{eq}}(q)$, in a stationary state or thermal equilibrium limit by

$$\frac{\partial}{\partial t}\Phi_{\mathrm{eq}}(q) = 0 \quad \text{or equivalently} \quad \sum_i [\alpha\hat{D}_i - K_i(q)]\Phi_{\mathrm{eq}}(q) = 0\,. \tag{A·14}$$

For the case in which the drift force has a time-independent potential $V(q)$, i.e.,

$$K_i(q)=-\frac{\partial V(q)}{\partial q_i}\,, \tag{A·15}$$

(A·14) has a solution

$$\Phi_{\rm eq}(q)=C\exp\Big(-\frac{1}{\alpha}V(q)\Big)\,, \tag{A·16}$$

C being a constant determined by the normalization condition (A·12).

As a simple example, consider a 1-dimensional Brownian motion under the drift force

$$K=-\Omega^2 q \quad \text{or} \quad V(q)=-\frac{1}{2}\Omega^2 q^2 \tag{A·17}$$

for which we have the Langevin equation

$$\frac{dq}{dt}+\Omega^2 q=\eta\,. \tag{A·18}$$

Its asymptotic solution for $t\to\infty$ is obviously independent of the initial condition $q(0)=q_0$ due to the damping effect of the drift force, so that we obtain the following stationary correlation function and the unit power spectrum

$$D(t-t')=\langle q(t)q(t')\rangle_{\rm st}=\frac{\alpha}{\Omega^2}e^{-\Omega^2|t-t'|}\,, \tag{A·19a}$$

$$P_q(\omega)=\frac{2\alpha}{\pi}\frac{1}{\Omega^4+\omega^2}\,. \tag{A·19b}$$

$D(0)=\langle q^2\rangle_{\rm st}=(\alpha/\Omega^2)$ is nothing but the equi-partition law in thermal equilibrium, which can also be derived from (A·16).

Under the assumption that $\partial K_i/\partial t=0$, i.e., $\partial\hat{F}/\partial t=0$ and $\partial\hat{F}^\dagger/\partial t=0$, we study the eigenvalue problem of the Fokker-Planck operator $\hat{F}$ and $\hat{F}^\dagger$. For this purpose, the similarity transformation with

$$U=e^{1/(2\alpha)V(q)} \tag{A·20}$$

is convenient to use, because we have $Uq_iU^{-1}=q_i$ and $U\hat{D}_iU^{-1}=\hat{D}_i-(1/2\alpha)\times(\partial V(q)/\partial q_i)$ and then transform $\hat{F}$ into

$$\hat{H}=U\hat{F}U^{-1}=-2\alpha\sigma\sum_i\hat{A}_i{}^\dagger\hat{A}_i\,, \tag{A·21a}$$

$$=-2\alpha\sum_i[-D_i{}^2+\bar{V}(q)]\,, \tag{A·21b}$$

in which σ is an arbitrary constant to be fixed later and

$$\hat{A}_i=\frac{1}{\sqrt{2\sigma}}\Big[\hat{D}_i+\frac{1}{2\alpha}\frac{\partial V}{\partial q_i}\Big]\,,\qquad \hat{A}_i{}^\dagger=\frac{1}{\sqrt{2\sigma}}\Big[-\hat{D}_i+\frac{1}{2\alpha}\frac{\partial V}{\partial q_i}\Big]\,, \tag{A·22a}$$

$$\bar{V}(q)=\frac{1}{2}\Big[\Big(\frac{1}{2\alpha}\Big)^2\Big(\frac{\partial V(q)}{\partial q_i}\Big)^2-\frac{1}{2\alpha}\frac{\partial^2 V(q)}{\partial q_i{}^2}\Big]\,. \tag{A·22b}$$

Equation (A·21b) shows that operator $(-\hat{H})$ has the same form as the Hamiltonian of a standard non-relativistic Schrödinger equation with potential $\bar{V}(q)$. On the basis of (A·21) and (A·22), we can easily prove that $\hat{H}$ is a self-adjoint operator with non-positive eigenvalues and its eigenvalue problem is written as

$$\hat{H}\bar{u}_\nu(q)=(-\lambda_\nu)\bar{u}_\nu(q)\,, \tag{A·23a}$$

$$(\bar{u}_\mu, \bar{u}_\nu)=\delta_{\mu\nu}\,, \qquad \sum_\nu \bar{u}_\nu(q)\bar{u}_\nu{}^*(q')=\delta(q-q')\,, \tag{A·23b}$$

where λ_ν takes a non-negative value in $[0, +\infty)$. Here we have assumed $\{\bar{u}_\nu\}$ to make a complete orthogonal set. Note that $\bar{u}_0$ belonging to the highest eigenvalue ($\lambda_0=0$), i.e., subject to the equation $\hat{A}_i\bar{u}_0=0$, is given by

$$\bar{u}_0(q)=\sqrt{C}\exp\left(-\frac{1}{2\alpha}V(q)\right) \tag{A·24}$$

with the same C as in (A·16).

Using the easily proved relations

$$\hat{F}=U^{-1}\hat{H}U\,, \qquad \hat{F}^\dagger=U\hat{H}U^{-1}\,, \tag{A·25}$$

we can immediately transform (A·23) into the eigenvalue problems of $\hat{F}$ and $\hat{F}^\dagger$ as

$$\hat{F}u_\nu=(-\lambda_\nu)u_\nu \quad \text{for} \quad u_\nu(q)=U^{-1}\bar{u}_\nu(q)\,, \tag{A·26a}$$

$$\hat{F}^\dagger v_\nu=(-\lambda_\nu)v_\nu \quad \text{for} \quad v_\nu(q)=U\bar{u}_\nu(q)\,, \tag{A·26b}$$

$$(v_\mu, u_\nu)=(u_\mu, v_\nu)=\delta_{\mu\nu}\,, \quad \sum_\nu u_\nu(q)v_\nu{}^*(q')=\sum_\nu v_\nu(q)u_\nu{}^*(q')=\delta(q-q') \tag{A·26c}$$

with the same λ_ν and $\bar{u}_\nu$. Hence we know that $\{u_\nu\}$ and $\{v_\nu\}$ make a reciprocal set to each other, and that

$$u_0=\frac{1}{\sqrt{C}}\Phi_{\rm eq}(q)\,, \quad v_0=\sqrt{C}\,. \tag{A·27}$$

Equation (A·26c) enables us to expand the formal solution

$$\Phi(q, t)=e^{\hat{F}t}\Phi_0(q) \tag{A·28}$$

for an initial distribution $\Phi_0(q)$ at $t=0$, in the following series:

$$\Phi(q, t)=\sum_\nu e^{-\lambda_\nu t}(v_\nu, \Phi_0)u_\nu(q)\,. \tag{A·29}$$

Thus, with help of (A·27), we know that $\Phi(q, t)$ tends to $\Phi_{\rm eq}$, i.e.,

$$\lim_{t\to\infty}\Phi(q, t)=\Phi_{\rm eq}(q) \tag{A·30}$$

irrespective of the initial condition Φ_0, if the highest eigenvalue ($\lambda_0=0$) is discrete and non-degenerate. Note that, under this condition, the uniqueness of the equilibrium solution (A·16) is confirmed.

Putting $\sigma=(\Omega^2/2\alpha)$ for the simple example of (A·17), we obtain

$$\hat{H}=-\Omega^2\hat{A}^\dagger\hat{A} \tag{A·31}$$

with the canonical creation and annihilation operators $\hat{A}$ and $\hat{A}^\dagger$ given by

$$[\hat{A}, \hat{A}^\dagger]=1\,; \quad \hat{A}=\sqrt{\frac{\alpha}{\Omega^2}}\left[\hat{D}+\frac{1}{2\alpha}\Omega^2 q\right], \quad \hat{A}^\dagger=\sqrt{\frac{\alpha}{\Omega^2}}\left[-\hat{D}+\frac{1}{2\alpha}\Omega^2 q\right]. \qquad (\text{A}\cdot 32)$$

Operator $(-\hat{H}+\Omega^2/2)$ is, therefore, just the same Schrödinger operator as a quantum-mechanical harmonic oscillator with mass $(2\alpha)^{-1}$, frequency Ω^2 and $\hbar=1$. Thus, we know that

$$-\lambda_\nu=-\Omega^2\nu\,; \quad \nu=0, 1, 2, \cdots, \qquad (\text{A}\cdot 33)$$

in which the highest eigenvalue ($\lambda_0=0$) is discrete and non-degenerate for $\Omega^2\neq 0$.

A.2. *Path-integral formalism and the randomization condition*

The Fokker-Planck equation is a partial differential equation with first-order time derivative and linear operator $\hat{F}$, so that its solution for finite time-evolution can be uniquely determined by giving its initial distribution. This is one of the most important characteristics of the Markoffian process. Thus, we can put

$$\Phi(q, t)=\int T(q, t|q', t')\Phi(q', t')dq'\,; \qquad (\text{A}\cdot 34\text{a})$$

$$T(q, t|q', t')=e^{-\hat{F}(t-t')}\delta(q-q') \qquad (\text{A}\cdot 34\text{b})$$

for the initial distribution $\Phi(q', t')$ at initial time $t'(<t)$. For simplicity we have assumed that $\partial\hat{F}/\partial t=0$. $T(q, t|q', t')$ represents the transition probability of finding the dynamical system in configuration q at time t for an initial configuration q' at initial time t', in other words, for the initial condition $T(q, t'|q', t')=\delta(q-q')$. We can also easily see that

$$\int T(q, t|q', t')dq=1\,, \quad \lim_{t\to\infty} T(q, t|q', t')=\Phi_{\text{eq}}(q)\,. \qquad (\text{A}\cdot 35)$$

Using (A·34a) again to write $\Phi(q', t')$ in terms of $\Phi(q'', t'')$ and $T(q', t'|q'', t'')$ with $t''(<t')$, we obtain

$$T(q, t|q'', t'')=\int T(q, t|q', t')\,T(q', t'|q'', t'')dq'\,. \qquad (\text{A}\cdot 36)$$

Successive use of (A·36), describing the Markoffian property of the process, leads us to

$$T(q, t|q', t')=\int\cdots\int\cdots\int T(q, t|q^{N-1}, t^{N-1})\cdots T(q^k, t^k|q^{k-1}, t^{k-1})$$
$$\cdots\times T(q^1, t^1|q', t')dq^{N-1}\cdots dq^k\cdots dq^1 \qquad (\text{A}\cdot 37)$$

for N subintervals of the time interval (t, t') i.e., $(t=t^N, t^{N-1}), \cdots, (t^k, t^{k-1}), \cdots, (t^1, t^0=t')$. Each time interval in (A·37) becomes very short for very large N. For an infinitesimal time interval, (t^k, t^{k-1}), with $\Delta t^k=t^k-t^{k-1}=O((t-t')/N)$, we can easily derive $T(q^k, t^k|q^{k-1}, t^{k-1})$ from (A·5). The kernel function for (t^k, t^{k-1}) represents the probability that the system passes through two fixed gates, (q^k, t^k) and (q^{k-1}, t^{k-1}), in which $q_i(t^k)$ and $q_i(t^{k-1})$ are fixed equal to q_i^k and q_i^{k-1} respectively. Rewrite

(A·5) for (t^k, t^{k-1}) as

$$q_i^k - q_i^{k-1} - K_i(q^{k-1})\Delta t^k = dw_i(t^{k-1})\,; \tag{A·38}$$

then we know that the kernel function with fixed gates is equal to the probability that $dw_i(t^{k-1})$ just takes the value given by the left-hand side of (A·38). Thus the probability distribution of the dw's, (A·7), immediately becomes

$$T(q^k, t^k|q^{k-1}, t^{k-1}) = \exp\left[-\sum_i \frac{\{q_i^k - q_i^{k-1} - K_i(q^{k-1})\Delta t^k\}^2}{2(2\alpha\Delta t^k)}\right]\prod_i \frac{1}{\sqrt{2\pi(2\alpha\Delta t^k)}}\,. \tag{A·39}$$

Inserting (A·39) into (A·37), we obtain

$$T(q, t|q', t') = \lim_{N\to\infty}\int \exp\left[-\frac{1}{4\alpha}\sum_i \sum_{k=1}^{k=N}\left\{\frac{q_i^k - q_i^{k-1}}{\Delta t^k} - K_i(q^{k-1})\right\}^2 \Delta t^k\right]$$
$$\times \prod_i \prod_{k=1}^{K=N-1} \frac{dq_i^k}{\sqrt{2\pi(2\alpha\Delta t^k)}}\,, \tag{A·40}$$

which is the path-integral representation of the kernel function. Equation (A·40) is symbolically written as

$$T(q, t|q', t') = C\int \mathcal{D}q(\tau)\exp\left[-\frac{1}{2\alpha}\mathcal{S}(q, t|q', t')\right], \tag{A·41}$$

where

$$\mathcal{S}(q, t|q', t') = \int_{t'}^{t} \mathcal{L}(q(\tau), \dot{q}(\tau))d\tau\,, \tag{A·42a}$$

$$\mathcal{L} = \frac{1}{2}\sum_i \{\dot{q}_i(\tau) - K_i(q(\tau))\}^2 \tag{A·42b}$$

with $q_i(t) = q_i$ and $q_i(t') = q_i'$. In (A·41), C is a normalization constant to be determined by $\int T(q, t|q', t')dq = 1$. In the above procedure to (A·41), note that the Jacobian is equal to unity for the transformation of integration variables from the dw's to the q's, i.e.,

$$\det\left(\frac{\partial(dw_i)}{\partial q_j}\right) = 1\,, \tag{A·43}$$

because (A·38) tells us that only q_i^k depends on $dw_i(t^{k-1})$, but q_i^{k-1} does not. This is one of the most important characteristics of the Ito calculus.

Here we should briefly remark on another procedure, based on conventional differential calculus, for formulating the path-integral representation of the kernel function, which starts from

$$\tilde{W}[\eta] = C\exp\left[-\frac{1}{4\alpha}\sum_i \int d\tau \eta_i^2(\tau)\right] \tag{A·44}$$

for the distribution of η, instead of (A·2) or (A·6).[31] The fact that (A·44) is formally equivalent to (A·2) is easy to show. Inserting (A·1) into (A·44), the authors cited in

Ref. 31) obtained

$$T(q,t|q',t')=C\int \mathcal{D}q(\tau)\exp\left[-\frac{1}{2\alpha}\mathcal{S}\right]\det\left(\frac{\delta\eta}{\delta q}\right). \tag{A·45}$$

For the case in which $K_i(q)$ is given by (A·15), the determinant factor is calculated as

$$\det\left(\frac{\delta\eta}{\delta q}\right)=\exp\left[\frac{1}{2}\sum_i\int d\tau\left(\frac{\partial^2 V}{\partial q_i^2}\right)_{q=q(\tau)}\right], \tag{A·46}$$

where the convention $\theta(0)=1/2$ has been used for the Heaviside step function defined with $\theta(t)=1$ for $t>0$ and $\theta(t)=0$ for $t<0$. The above approach based on the Ito calculus is apparently different from theirs by lack of the determinant. However, we know that the same term is automatically included in $\mathcal{S}$ as given by (A·42), as is easily shown in the following way.[32] $\mathcal{L}$ given by (A·42b) is equal to

$$\sum_i\frac{1}{2}\left\{\left(\frac{dq_i}{d\tau}\right)^2+\left(\frac{\partial V}{\partial q_i}\right)^2\right\}+\sum_i\frac{\partial V}{\partial q_i}\frac{dq_i}{d\tau}. \tag{A·47a}$$

Discretizing the last term of (A·47a) based on the Ito calculus, we can rewrite it as

$$\sum_i\frac{1}{2}\left\{\left(\frac{\partial V}{\partial q_i}\right)_{q=q(\tau+\Delta\tau)}+\left(\frac{\partial V}{\partial q_i}\right)_{q=q(\tau)}\right\}\frac{q_i(\tau+\Delta\tau)-q_i(\tau)}{\Delta\tau}-\sum_i\frac{1}{2}(2\alpha)\left(\frac{\partial^2 V}{\partial q_i^2}\right)_{q=q(\tau)}. \tag{A·47b}$$

The last term exactly gives the exponent of (A·46).

Returning to (A·41), we compare it with the quantum-mechanical Feynman path-integral formula

$$T(q,t|q',t')=C\int \mathcal{D}q(\tau)\exp\left[\frac{i}{\hbar}S(q,t|q',t')\right]; \tag{A·48a}$$

$$S(q,t|q',t')=\int_{t'}^{t}L(q(\tau),\dot q(\tau))d\tau \tag{A·48b}$$

for the probability amplitude of transition from (q',t') to (q,t), where S and L are respectively the *dynamical* action and Lagrangian of the dynamical system. That there should exist a formal similarity between the theory of stochastic processes and quantum mechanics is easily observed under the following correspondence:

$$-\frac{1}{2\alpha}\leftrightarrow\frac{i}{\hbar},\qquad \mathcal{L}\leftrightarrow L. \tag{A·49}$$

Because of this similarity, we call $\mathcal{S}$ and $\mathcal{L}$ respectively the *stochastic* "action" and "Lagrangian" within the *stochastic* "canonical" formalism.

This similarity also suggests us to follow the same theoretical procedure as in quantum mechanics, from the above path-integral representation (A·41) to a possible canonical operator formalism, proposed by Saito and Namiki many years ago[33] and which is briefly described for SQM in § 5. For this purpose, it is convenient to define a *stochastic* "momentum", say p_i, and *stochastic* "Hamiltonian," say $\mathcal{H}$, with

$$p_i=\frac{\partial\mathcal{L}}{\partial\dot q_i}=\frac{\partial\mathcal{S}}{\partial q_i}, \tag{A·50a}$$

$$\mathcal{H}(q, p)=\sum_i p_i \dot{q}_i - \mathcal{L} \,. \tag{A·50b}$$

Partially differentiating (A·41) with respect to t, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}T(q, t|q', t') &= -\frac{1}{2\alpha}\int \mathcal{D}q(\tau)\left[\frac{d\mathcal{S}}{dt}-\sum_i \frac{\partial \mathcal{S}}{\partial q_i}\dot{q}_i\right]\exp\left[-\frac{1}{2\alpha}\mathcal{S}\right] \\ &= \frac{1}{2\alpha}\int \mathcal{D}q(\tau)\mathcal{H}(q, p)\exp\left[-\frac{1}{2\alpha}\mathcal{S}\right],\end{aligned} \tag{A·51}$$

where we have used $d\mathcal{S}/dt=\mathcal{L}$ and (A·50). In (A·51) p_i can be replaced with

$$\hat{\pi}_i = -2\alpha\frac{\partial}{\partial q_i} \tag{A·52}$$

because $p_i\exp[-(1/2\alpha)\mathcal{S}]=-2\alpha(\partial/\partial q_i)\exp[-(1/2\alpha)\mathcal{S}]$, so that (A·51) becomes

$$\frac{\partial}{\partial t}T(q, t|q', t')=\hat{F}T(q, t|q', t')\,; \quad \hat{F}=\frac{1}{2\alpha}\mathcal{H}(q, \hat{\pi})\,, \tag{A·53}$$

which is nothing other than the Fokker-Planck equation.

In the case of (A·42b) for the stochastic "Lagrangian", we can explicitly write down the stochastic "momentum" and "Hamiltonian" as

$$p_i = \dot{q}_i - K_i\,, \tag{A·54a}$$

$$\mathcal{H}(q, p)=\sum_i\left[\frac{1}{2}p_i^2+p_iK_i\right]. \tag{A·54b}$$

The above replacement of the p's with the π's immediately gives

$$\hat{F}=\frac{1}{2\alpha}\mathcal{H}(q, \hat{\pi})=\sum_i\left[\alpha\frac{\partial^2}{\partial q_i^2}-\frac{\partial}{\partial q_i}K_i\right], \tag{A·55}$$

which is exactly equal to the Fokker-Planck operator given by (A·11b). Note that we have to put $\hat{\pi}_i$ in the left-hand side of K_i in order to keep the conservation law of probability and the normalization condition at every time, because we have

$$\frac{d}{dt}\int dq\Phi(q, t)=\int dq\sum_i\frac{\partial}{\partial q_i}\left[\alpha\frac{\partial}{\partial q_i}-K_i\right]\Phi(q, t)=0$$

under an appropriate boundary condition on $\Phi(q, t)$. In quantum mechanics the ordering of operators in the Hamiltonian is so arranged as to keep its hermiticity. The difference comes from the physical meanings of the probability distribution in the theory of stochastic processes and the probability amplitude in quantum mechanics. Mathematically, the above ordering is a result of the stochastic difference equation of the Ito-type, (A·5).

Here we should remark that the stochasticity has been represented by the operator nature of the stochastic "momentum", i.e., the replacement of the p's with the $\hat{\pi}$'s given by (A·52), or equivalently, by the following commutation relation:

$$[\hat{\pi}_i, q_j]=-2\alpha\delta_{ij}\,, \tag{A·56}$$

which is to be compared with the *quantum* or *quantization* condition responsible for quantum fluctuations in quantum mechanics. The diffusion constant α gives us a measure of random fluctuations in our stochastic processes, while the Planck constant $\hbar$ measures quantum fluctuations. The commutation condition, (A·56), is worthy of being named the *random* or *randomization* condition. Actually, the $\hat{\pi}$'s tend to be commutable c-numbers if α goes to zero. This means that the deterministic dynamics will be revived in the small α limit. The presence of α in (A·56) is in this way physically important. For the sake of mathematical simplicity, however, a more convenient method is to put α into the *stochastic* "Lagrangian" itself from the outset as

$$\mathcal{L}=\frac{1}{2(2\alpha)}\sum_i[\dot{q}_i-K_i]^2\,, \tag{A·57}$$

which yields the *stochastic* "canonical quantities" as follows:

$$p_i=\frac{1}{2\alpha}\{\dot{q}_i-K_i\}\,, \tag{A·58a}$$

$$\mathcal{H}(q,p)=\sum_i[\alpha p_i^2+p_iK_i]\,. \tag{A·58b}$$

Correspondingly, we should set the *replacement* rule and *randomization* condition, respectively, in the following ways:

$$p_i\to\hat{\pi}_i=-\frac{\partial}{\partial q_i}\,, \tag{A·59a}$$

$$[\hat{\pi}_i,q_j]=-\delta_{ij}\,. \tag{A·59b}$$

Consequently, the *stochastic* "Hamiltonian" operator itself, i.e., $\mathcal{H}(q,\hat{\pi})$, is just equal to the Fokker-Planck operator, (A·11b).

A.3. *Operator formalism*

We have so far described our stochastic "canonical" formalism by means of the conventional representation, in which the q's are c-number variables and the $\hat{\pi}$'s are differential operators with respect to the q's. On the analogy of quantum mechanics, we can introduce an abstract representation of the theory of stochastic processes in the following way.[33] Starting from the *stochastic* "Lagrangian" (A·57), we replace the *stochastic* "canonical" variables, q and p, with abstract operators, $\hat{q}$ and $\hat{p}$, subject to the abstract randomization condition of the same form as (A·59b), i.e.,

$$[\hat{p}_i,\hat{q}_j]=-\delta_{ij}\,. \tag{A·60}$$

Here both $\hat{q}$'s and $\hat{p}$'s are operators on an abstract vector space.

Assuming the existence of the eigenvalue problem of $\hat{q}_i$ with eigenvalue q_i'

$$\hat{q}_i|q'\rangle=q_i'|q'\rangle\,; \tag{A·61a}$$

$$\langle q'|q''\rangle=\delta(q'-q'')\,,\qquad\int dq|q\rangle\langle q|=\hat{1}\,, \tag{A·61b}$$

$\hat{1}$ being the identity operator, then we use $\{|q'\rangle\}$ as the basic system of the q-

representation. Equations (A·60) and (A·61) immediately give the following expression of the stochastic "momentum" in the q-representation:

$$\langle q'|\hat{p}_i|q''\rangle=-\frac{\partial}{\partial q_i'}\delta(q'-q'')\,,\qquad \langle q'|G(\hat{q},\hat{p})|q''\rangle=G\Big(q',-\frac{\partial}{\partial q'}\Big)\delta(q'-q'')\,. \tag{A·62}$$

Thus we know that the differential operator $\hat{\pi}_i$ in (A·59a) is nothing other than $\hat{p}_i$ in the q-representation.

In the abstract representation, we consider an abstract vector, say $|\Phi\rangle_t$, to represent a stochastic state described by a distribution function $\Phi(q,t)$. Using the eigenvalue equation (A·61), we can give the relationship between the abstract vector $|\Phi\rangle_t$ and its representative $\Phi(q,t)$ as follows:

$$|\Phi\rangle_t=\int dq'|q'\rangle\langle q'|\Phi\rangle_t\,;\quad \Phi(q',t)=\langle q'|\Phi\rangle_t\,. \tag{A·63}$$

This also means that the eigenvector $|q'\rangle$ represents the state in which one of the random variables, say q_i, takes a definite value q_i'. Note that the normalization condition is now written as

$$\Big\langle\int\Big|\Phi\Big\rangle_t=1\,, \tag{A·64}$$

where we have used Dirac's standard ket defined by

$$\Big|\int\Big\rangle\equiv\int|q'\rangle dq'=\frac{1}{\sqrt{C}}|v_0\rangle\,, \tag{A·65}$$

in which $|v_0\rangle$ is the abstract vector corresponding to the eigenfunction $v_0(q)$ given by (A·27).

As is easily shown, the abstract version of the Fokker-Planck equation

$$\frac{\partial}{\partial t}|\Phi\rangle_t=\hat{F}|\Phi\rangle_t\,;\quad \hat{F}=\mathcal{H}(\hat{q},\hat{p}) \tag{A·66}$$

immediately gives (A·11) as its representaive, and then

$$T(q',t'|q'',t'')=\langle q'|e^{\hat{F}(t'-t'')}|q''\rangle\,. \tag{A·67}$$

We can also rewrite the expectation value of a quantity (say, $G(q)$) and the correlation function in the stationary state, respectively, as

$$\langle G\rangle_{\rm st}=\langle v_0|G(\hat{q}(t))|u_0\rangle\,, \tag{A·68a}$$

$$D_{ij}(t-t')=\langle v_0|T\hat{q}_i(t)\hat{q}_j(t')|u_0\rangle\,, \tag{A·68b}$$

where T stands for the time-ordering symbol and $|u_0\rangle$ for the abstract vector corresponding to the eigenfunction $u_0(q)$ given by (A·27). We have put $\hat{q}_i(t)=e^{-\hat{F}t}\hat{q}_ie^{\hat{F}t}$, which is regarded as the *stochastic* "Heisenberg" operator given by the evolution operator $\hat{F}$ and the time-independent "Schrödinger" operator $\hat{q}_i$. In the derivation of (A·68), we have also used

$$\langle v_0|\hat{F}=0 \quad \text{or} \quad \hat{F}|u_0\rangle=0 \tag{A·69}$$

corresponding to (A·13) and (A·26).

Within the operator formalism given by the abstract representation, we can utilize many techniques developed in quantum mechanics and quantum field theory.[26),27)]

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