ON COMPLEX PROBABILITIES

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We show how the formalism of the Langevin equation may be used for evaluating the averages of quantities which have a complex valued distribution.

It is fashionable to use stochastic techniques for computing the expectation values in statistical mechanics, especially if the number of degrees of freedom is very high: for example if a configuration of the system is denoted by φ, we may be interested to compute

$$\int \mathrm{d}\mu[\varphi]\, f(\varphi) \equiv \langle f \rangle ,$$
$$\mathrm{d}\mu[\varphi] = \mathrm{d}[\varphi] \exp\{H[\varphi]\}/Z ,$$
$$Z = \int \mathrm{d}[\varphi] \exp\{-H[\varphi]\}, \quad \int \mathrm{d}\mu[\varphi] = 1 . \tag{1}$$

A very efficient method is the Monte Carlo method [‡1] in which we use explicitly the fact that $\mathrm{d}\mu[\varphi]$ is a probability measure: generally speaking one constructs a recursive random algorithm which generates the configurations of the system according to the probability measure $\mathrm{d}\mu[\varphi]$; more precisely we have that:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\varphi_n) = \langle f \rangle , \tag{2}$$

where the φ_n are the configurations generated according to the algorithm.

An other method for continuous systems is based on the Langevin equation:

$$\dot{\varphi} = -\partial H/\partial\varphi + \eta(t), \quad \overline{\eta(t)\eta(t')} = 2\delta(t-t') . \tag{3}$$

Standard mathematical manipulations shows that the probability distribution for the field φ evolves according to the Fokker–Planck equation [2]:

$$\dot{P}(\varphi, t) = (\partial/\partial\varphi)(-p\partial H/\partial\varphi + \partial p/\partial\varphi) . \tag{4}$$

The reduced probability distribution $p(\varphi, t)$ will evolve according to the equation:

$$p(\varphi, t) = P(\varphi, t)u(\varphi), \quad u(\varphi) = \exp[\tfrac{1}{2}H(\varphi)] ,$$
$$\dot{p}(p, t) = -\mathscr{H}p \equiv \tfrac{1}{4}(\partial H/\partial\varphi)^2 - \tfrac{1}{2}\partial^2 H/\partial\varphi^2 , \tag{5}$$

which is a Schrödinger type equation. If the number of degrees of freedom is finite and the hamiltonian H is not singular, the associated Fokker–Planck hamiltonian $\mathscr{H}$ is a self adjoint positive operator (provided that $Z < \infty$) which has only one zero eigenvalue $u(\varphi)$. These properties imply that in the large time limit $\exp(-t\mathscr{H})$ becomes the projection over $u(\varphi)$ and

$$p(\varphi, t) \xrightarrow[t\to\infty]{} \exp\{-H[\varphi]\}/Z ,$$

$$\frac{1}{t}\int_0^t f(\varphi(t'))\, \mathrm{d}t' \xrightarrow[t\to\infty]{} \langle f \rangle , \tag{6}$$

Up to now everything is well known.

The problem we are interested here is to study what happens if H becomes complex valued [‡2]. We argue that, while the Monte Carlo method cannot be used in this situation, nothing

[‡1] For a review see ref. [1].

[‡2] While I was writing this paper I received a very interesting paper [3] in which a similar suggestion is put forward; he shows that the existence of logarithmic cuts for the function $H(\varphi)$ does not destroy the existence of solutions of the Langevin equation. In view of this result the case in which $\partial H/\partial\varphi$ is a meromorfic function of φ with only single poles and for large $|\varphi| \partial H/\partial\varphi \propto \mathrm{A}\cdot\varphi$ where A is a positive definite operator, seems the easiest case for obtaining rigorous results.

forbids to write a Langevin equation also for complex H. In this case (let us assume that H is an analytic function) eq. (3) gives an evolution law for φ in the complex plane; a real valued probability distribution on the complex plane can be defined $p(\varphi_R, \varphi_I, t)$ ($\varphi = \varphi_R + i\varphi_I$) in the usual way; it would satisfy a generalized Fokker–Planck equation:

$$\dot{P}(\varphi_R, \varphi_I, t) = (\partial^2/\partial\varphi_R^2)p + (\partial/\partial\varphi_R)[V_R(\varphi)p] + (\partial/\partial\varphi_I)[V_I(\varphi)p], \tag{7}$$

whose solution is no more as simple as before. Our main observation is that we can define a complex valued function on the real axis $P(\varphi_R, t)$, such that

$$\int d[\varphi_R] P(\varphi_R, t)\varphi_R^n = \int d[\varphi_R]\, d[\varphi_I]\, P(\varphi_R, \varphi_I, t)(\varphi_R + i\varphi_I)^n . \tag{8}$$

$P(\varphi_R, t)$ satisfies the Fokker–Planck equations (4), (5). In other words a complex probability distribution may be emulated by taking the expectation values of complex quantities [‡3].

It is thus natural to ask the following questions:

(a) Does eq. (3) have a solution for all times (with probability one)?

(b) Do the probabilities $P(\varphi_R, t)$, $P(\varphi_R, \varphi_I, t)$ (if they exist) have a limit when t goes to infinity?

(c) Do the averages $t^{-1}\int_0^t \varphi(t')^n\, dt'$ exist (in some sense, e.g. as distributions in time) for finite t and do they have a limit when t goes to infinity?

(d) Finally, does the following relation hold?

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi^n(t')\, dt' = \lim_{t\to\infty} \int d\varphi_R P(\varphi_R, t)\varphi_R^n = \langle\varphi^n\rangle . \tag{9}$$

[‡3] The argument is formal and it is based on the implicit assumption that the moment problem (8) has a solution. If this happens the usual Ito differential calculus relation $d/dt\langle f\rangle = \langle \partial f/\partial\varphi\ \partial H/\partial\varphi + \partial^2 f/\partial\varphi^2\rangle$, $f(\varphi)$, being a generic function implies that $\int d\varphi P(\varphi, t)\,[\partial f/\partial t - \partial f/\partial\varphi\ \partial H/\partial\varphi - \partial^2 f/\partial\varphi^2] = 0$. By integration by part we obtain the Fokker–Planck equation as stated in text.

The answer to these questions is not evident, indeed something must go wrong for those complex hamiltonian such that $Z = 0$; in this case $\langle\varphi^n\rangle$ would be infinite. Moreover for a generic class of potentials there will be choices of the noise η such that the trajectory $\varphi(t)$ arrives to infinity in a finite time; we can only hope (it is likely to be true, provided that certain conditions are satisfied) that a solution of the differential equation exists with probability one. To be definite let us consider the case in which the hamiltonian $\mathscr{H}$ is an analytic function of a parameter g. The eigenvalues and the eigenvectors of $\mathscr{H}$ (which is supposed to be a positive selfadjoint operator for real non-negative g) are analytic functions of g which may be continued at complex g. A necessary condition for the existence of the limit at large time of $P(\varphi_R, t)$ is that all the eigenvalues of $\mathscr{H}$ have a non-negative real part. If we close our eyes on the possibility that $\int_0^t \varphi^n(t')\, dt'$ may be not defined because the field $\varphi(t)$ has a finite probability to be as large as we want, it is easy to see that if the lhs of eq. (9) exists, eq. (9) must be satisfied, as soon as the conditions on the eigenvalues of $\mathscr{H}$ are satisfied.

It is obvious that if we study the problem by assuming that the imaginary part of H is small and by using perturbation theory all points (a)–(d) are satisfied, so that there is nothing obvious against the conjecture that points (a)–(d) hold in the region of positive eigenvalues of $\mathscr{H}$: however it is quite likely that the properties of H for complex φ will play a relevant role in a rigorous proof.

In order to obtain some ideas on the possible relevance of thise proposal I have done some numerical experiments in the very simple case $H = g\varphi^4$. The solutions of the Langevin equation have been simulated on the computer for different values of g ($|g| = 1$); roughly speaking in the region where the real part of g were positive, I have checked that both

$$\frac{1}{t}\int_0^t dt'\, \varphi^2(t'), \quad \frac{1}{t}\int_0^t dt\, \varphi^4(t'), \tag{10}$$

seemed to have a limit when the time was going to infinity, at least with a precision of a few percent; within the same precision they where equal, respectively, to:

$$\int \mathrm{d}\varphi \, \exp(-g\varphi^4)\varphi^2/Z, \quad \int \mathrm{d}\varphi \, \exp(-g\varphi^4)\varphi^4/Z. \tag{11}$$

For purely imaginary g the situation was rather confused and no evidence for the existence of the limit at large time was found. Quite remarkably in the region of the nonzero imaginary part but negative real part of g, the limits of (10) at large time seem to exist, however the integrals in eq. (11) are divergent in this region and the limits estimated for (10) did not coincide with the analytic continuation of the integrals in (11) from their convergence region.

We see that the Langevin equation may be a useful tool for studying a system with complex hamiltonians H in the region where the imaginary part of H is small (certainly far from the region of zeros of the partition function Z). It is unclear if this method may be used to study the Feynman path integral for nearly real times, not at imaginary times as it is done nowadays.

References

[1] K. Binder, ed., Monte Carlo methods (Springer, Berlin, 1979).

[2] See for example; I. Guikhman and A. Skorokhod, Introduction à la Théorie des Processus Aléatoires (Mir, Moscow, 1980);
G. Parisi and Wu Yong-shi, Sci. Sinica 24 (1981) 483.

[3] J.R. Klauder, Bell Lab. preprint, Lectures given XXIIth Schladming school).