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Complex Langevin: A Numerical Method?

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There are several problems in theoretical physics which are defined by complex valued weight functions e^{-S} where S can be a complex action or classical Hamiltonian. Complex Langevin is one method to simulate integrals over such complex valued weights. The conditions under which the so called complex Langevin process can be related to the complex weight and correctly simulates such integrals are discussed.

1. Introduction

Many problems of physical interest are formulated as multidimensional integrals. Typical representatives are problems in classical statistical physics (equilibrium thermodynamics) stochastic problems (non equilibrium thermodynamics) and quantum field theory on the lattice. Such problems cannot be solved by a straightforward numerical integration. An alternative approach is to solve such problems by some stochastic procedure called Monte Carlo (MC) integration or simulation. However, a naive MC simulation would not be of much help since it would run over too many contributions of minor importance. So it would be very useful to choose a MC simulation which is sensitive to the integration measure, i.e. the simulation chooses the trials as they are weighted by the measure. This measure for example could be a typical canonical ensemble for some real variables $x_i \in \mathbb{R}$

$$d\mu(\{x\}) \sim e^{-H(\{x\})} \prod_i dx_i, \tag{1}$$

or

$$d\mu(\{x\}) \sim e^{-S(\{x\})} \prod_i dx_i, \tag{2}$$

where $H(\{x\})$ is the Hamiltonian of a classical statistical mechanics system or $S(\{x\})$ is the action of a lattice field theory and both are real. Such a proposal was made in [1] and since the trials are generated by their importance, this approach is called importance sampling. The basic idea is to generate a homogeneous Markov process by its transition probability (density) $f_t(\{x\}|\{x'\})$, with the above measures as unique invariant measures. Invariant means

$$d\mu(\{x\}) = \int_{\mathbb{R}^n} f_t(\{x\}|\{x'\}) d\mu(\{x'\}) \prod dx. \tag{3}$$

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Such an procedure of course depends strongly on the fact that $e^{-S(\{x\})}\prod_i dx_i$ is proportional to a probability measure which implies $S, H \in \mathbb{R}$. Clearly a straightforward application of standard simulation techniques like the Metropolis *et al.* algorithm [1] will fail when they are applied to problems with a complex action or Hamiltonian. In this case there is no direct probabilistic interpretation of a distribution function of the form e^{-S}.

An alternative way to generate a homogeneous Markov process is a stochastic differential equation (SDE) also called a Langevin equation with time independent diffusion and drift coefficients. Assume a one dimensional process $X_t : \Omega \to \mathbb{R}$ with its SDE

$$dX = F(X)d\tau + dW \quad , \tag{4}$$

and the drift term

$$F(x) = -\frac{1}{2}\frac{dS(x)}{dx} \quad . \tag{5}$$

W_t is a standard Wiener process with zero mean and covariance

$$E(W_{t_1}W_{t_2}) = \min(t_1, t_2). \tag{6}$$

Under certain conditions its probability density $f(x,t)$ obeys the so called Fokker-Planck (F-P) equation.

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial S(x)}{\partial x} + \frac{\partial}{\partial x}\right] f(x,t) =: \frac{1}{2}Tf(x,t). \tag{7}$$

T denotes the F-P operator. It can be seen immediately that $\exp(-S)$ is a stationary solution with zero eigenvalue, i.e. $T\exp(-S) = 0$. The zero eigenvalue is not degenerate. Let us assume further that $\exp(-S) \in L^p(\mathbb{R}, dx); p = 1, 2$. S could be an appropriate polynomial. For a general time dependent solution we have

$$f(x,t) = e^{Tt}f(x,0). \tag{8}$$

$\exp(-S)$ defines a unique invariant measure with $d\mu(x) \sim \exp(-S)dx$. The appropriate operator in $L^p(\mathbb{R}, d\mu(x)); p = 1, 2$ is given by

$$\hat{T} = e^S T e^{-S}. \tag{9}$$

For simplicity let us restrict to L^2 and make an unitary transformation by

$$-H = e^{-S/2}\hat{T}e^{S/2} \tag{10}$$

where H denotes a Schrödinger operator in $L^2(\mathbb{R}, dx)$ given by

$$H = -\frac{d^2}{dx^2} + V(x) \tag{11}$$

with

$$V(x) = \frac{1}{2}\left(\frac{1}{2}(S'(x))^2 - S''(x)\right) \tag{12}$$

This operator is obviously of the form $H = Q^*Q$ with

$$Q = \frac{d}{dx} + \frac{S'(x)}{2}. \tag{13}$$

So, we have $H \geq 0$. For simplicity let us now assume that $V(x)$ increases sufficiently fast at infinity such that the spectrum of H is purely discrete. We then have for any $f(x,t)$

$$f(x,t) = \sum_n c_n f_n(x) e^{-\lambda_n t} \tag{14}$$

$$s - \lim_{t\to\infty} f(x,t) = c_0 f_0(x), \tag{15}$$

where f_n are the eigenfunctions of H and λ_n the corresponding eigenvalues. This all together is nothing else than stochastic quantization [2].

A generalization of the above calculation leads to the following result (for details see e.g. [3,4]). Assume a homogeneous Markov process X_t with values on some manifold Ξ and measure $d\nu$. Ξ could be a Lie group. T is the corresponding F-P operator having an isolated zero eigenvalue with $T\hat{f} = 0$, $\hat{f}$ unique. Further let us assume $\hat{f} \geq 0$ *a.e.* and

$$\int_\Xi \hat{f} d\nu = 1. \tag{16}$$

$\hat{f}$ then defines an invariant measure by $d\mu = \hat{f} d\nu$. The operator $\hat{T} = \hat{f}^{-1} T \hat{f}$ is then the generator of a contraction semi group in $L^p(\Xi, d\mu)$ with $\hat{T}1 = 0,\quad 1 \in L^p(\Xi, d\mu);\quad, p = 1,2$. For any $\Phi \in L^p(\Xi, d\mu)$

$$s - \lim_{t\to\infty} e^{\hat{T}t}\Phi = c_\Phi. \tag{17}$$

c_Φ is a constant depending on Φ. From $\mu(\Xi) = 1$ follows that any distribution density f_t converges point wise to $\hat{f}$ i.e.

$$\lim_{t\to\infty} f_t = \hat{f} \ \ a.e. \tag{18}$$

On the other hand, if the zero eigenvalue of T is M-fold degenerate then there are M ergodic classes. In this case we have

$$\lim_{\tau\to\infty} f(x,y,\tau) = \sum_{i=1}^{M} c_i \hat{f}_i(x,y) \text{ a.e. } . \tag{19}$$

For the Langevin equation (4) in general S must not be restricted to real values. It could also be complex. This approach to complex actions was first proposed by John Klauder [5,6] and thereafter studied by many authors (e.g. [7]). This method is commonly denoted as complex Langevin (CL). As mentioned above there is no formal restriction to a real valued drift term. The use of the full complex action as a drift term provides CL with a genuine advantage over other methods, that it can converge directly to the desired distribution circumventing the *sign problem* of the standard algorithms. As we will see later the name complex Langevin may be slightly misleading, since the Langevin equations really describe a real diffusion process in twice as many dimensions.

Currently no complete theory of the CL method is available and unfortunately several conditions under which a real Langevin process can be shown to converge to a given distribution are not satisfied for a general CL process. Even giving a convergent numerical simulation, in some cases CL is known to converge to the wrong results [8,9]. For simple actions, this deceitful behavior can be corrected by an appropriate choice of kernel in the Langevin equation [10]. For general, more complicated, systems (e.g. lattice QCD) it is far from clear which choice of kernel is required.

2. Complex Langevin

Let us now assume $S : \mathbb{R} \to \mathbb{C}$ which leads to the following SDE.

$$dZ = F(Z)d\tau + dW \quad , \tag{20}$$

with the drift term

$$F(z) = -\frac{1}{2}\frac{dS(z)}{dz} \quad . \tag{21}$$

With $Z_t = X_t + iY_t$ this is

$$dX = G(X, Y)d\tau + dW \tag{22}$$

$$dY = H(X, Y)d\tau \tag{23}$$

where with $S(z) = u(x, y) + iv(x, y)$

$$G(x, y) = -\frac{1}{2}\frac{\partial u(x, y)}{\partial x} \; , \quad H(x, y) = \frac{1}{2}\frac{\partial u(x, y)}{\partial y} \quad . \tag{24}$$

Note that the equation for dY has a zero diffusion coefficient, which leads to a problem with a singular diffusion matrix.

The process $\{(X_t, Y_t), t \geq 0\}$ as defined by equation (20) has a distribution density $f(x, y, t)$. The first question is whether this so defined process converges in distribution at all, i.e.

$$\lim_{\tau \to \infty} f(x, y, \tau) = \hat{f}(x, y), \text{ a.e. } . \tag{25}$$

The second question is the essence of complex Langevin! It is the question whether

$$E(g(X + iY)) = \int_{\mathbb{R}^2} g(x + iy)\hat{f}(x, y)dxdy = \frac{1}{\mathcal{N}}\int_{\mathbb{R}} g(x)e^{-S(x)}dx. \tag{26}$$

or at least

$$\lim_{t \to \infty} E(g(X_t + iY_t)) = \frac{1}{\mathcal{N}}\int_{\mathbb{R}} g(x)e^{-S(x)}dx =: \langle g(x) \rangle \tag{27}$$

holds for certain functions g. This tells us that if the process as defined above has converged in some sense we might be able to calculate $\langle g(x)\rangle$ by an equivalent probabilistic system of twice as many dimensions.

As already mentioned earlier, one solution is to find a stationary solution with the required properties of the following equation.

$$\frac{\partial f(x,y,\tau)}{\partial \tau} = Tf(x,y,\tau) \tag{28}$$

where T in this case is given by

$$T = -G_x(x,y) - G(x,y)\frac{\partial}{\partial x} - H_y(x,y) - H(x,y)\frac{\partial}{\partial y} + \frac{1}{2}\frac{\partial^2}{\partial x^2} \ . \tag{29}$$

The question now is whether T has a unique stationary zero eigenvalue solution $\hat{f}(x,y)$

$$T\hat{f}(x,y) = 0 \ . \tag{30}$$

In contrary to a real Langevin process (4), a CL process defined by the real two dimensional equations (22) and (23) has a singular diffusion matrix and it is still an open question what conditions guarantee the existence of a stationary solution at all. A general statement on the existence of stationary solutions based on the form of the drift terms can not be made.

Also T, for general $S(x+iy)$, does not fall into the class operators for which a general statement on the regularity of the solutions can be made [11], thus one can not exclude the possibility that the solutions $T\hat{f} = 0$ exist only in the sense of distributions (weak solutions). If this is the case then it might be quite difficult to find (construct) the stationary density. From this one also must conclude, that in general $\hat{f}(x,y)$ can not be assumed to be a Gibbsian density, i.e. $\hat{f}$ can not be assumed to be of the form $\hat{f} = \exp(\Phi)$ with Φ some real potential. This situation seems of course acceptable since the physics of the system is described by the complex distribution $\exp(-S)$. The CL method only serves as an algorithmic trick to simulate such complex densities.

To see this, let us solve the problem $S(x) = cx^2,\ \ c \in \mathbb{R}^+$, by complex Langevin. The Langevin equation then reads:

$$dX(\tau) = -cX(\tau)d\tau + dW(\tau), \tag{31}$$

$$dY(\tau) = -cY(\tau)d\tau \tag{32}$$

The stationary density $\hat{f}(x,y)$ is then a weak solution and can be formally given as

$$\hat{f}(x,y) \sim e^{-cx^2}\delta(y). \tag{33}$$

We are now going to discuss the conditions under which a convergent CL process correctly simulates a given system with a complex action or Hamiltonian (i.e. equation (27) holds). Let us start out with an action which for simplicity is restricted one dimension $S:\mathbb{R}\to\mathbb{C}$ with S a polynomial of degree N such that $e^{-S} \in \mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ the Schwartz space of C^∞ functions of rapid decrease. With $g(x)$ a polynomial of degree M it is thus guaranteed that

$$\langle g(x)\rangle \equiv \frac{1}{\mathcal{N}}\int_{\mathbb{R}} g(x)e^{-S(x)}dx, \tag{34}$$

$$\mathcal{N} = \int_{\mathbb{R}} e^{-S(x)}dx \quad , \tag{35}$$

does exist, provided $0 < |\mathcal{N}|$.

Let us now assume:

1. $$c(k,t) \equiv E(e^{ikZ(t)}) = \int_{\mathbb{R}^2} e^{ik(x+iy)}f(x,y,t)dxdy \tag{36}$$

   the limit $t \to \infty$ exists point wise and

   $$\lim_{\tau\to\infty} c_\tau(k) \equiv c_\infty(k) \in \mathcal{S}(\mathbb{R}) \quad . \tag{37}$$

2. $$\lim_{t\to\infty} \left|E(Z^n(t)e^{ikZ(t)})\right| < \infty \text{ for all } 0 \le n \le N-1, k \in \mathbb{R} \quad . \tag{38}$$

Equation (27) then holds at least for $g(z)$ a polynomial of degree $M \le N-1$. Moreover equation (27) holds for any higher moment $E(Z^n(t))$, $n \ge N$ which exist for $t \to \infty$. This can be seen as follows.

From our assumptions we can define

$$\hat{h}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} c_\infty(k)e^{-ikx}dk \quad . \tag{39}$$

and know further that $\hat{h}(x) \in \mathcal{S}(\mathbb{R})$. We further conclude that

$$\frac{d}{dt}E(e^{ikZ(t)}) = ikE\left(e^{ikZ(t)}F(Z(t))\right) - \frac{k^2}{2}E\left(e^{ikZ(t)}\right) \tag{40}$$

exists in the limit $t \to \infty$ and equals to zero. We also conclude from the above that

$$\lim_{t\to\infty} E(Z^n(t)e^{ikZ(t)}) = \int_{\mathbb{R}} x^n e^{ikx}\hat{h}(x)dx \tag{41}$$

for $0 \le n \le N-1$ and $k \in \mathbb{R}$. All together we obtain for $\hat{h}(x)$ that

$$\frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial S(x)}{\partial x} + \frac{\partial}{\partial x}\right]\hat{h}(x) =: \mathcal{T}\hat{h}(x) = 0. \tag{42}$$

$\mathcal{T}$ has two zero eigenvalue solutions. One is

$$\hat{h}_1(x) \sim e^{-S(x)} \in \mathcal{S}(\mathbb{R}) \tag{43}$$

and as the Fourier transform of $c_\infty(k)$ it must be a Schwartz function. For the second solution

$$\hat{h}_2(x) \sim e^{-S(x)}\int_{x_0}^{x} e^{S(y)}dy \tag{44}$$

one finds $\hat{h}_2(x) \notin \mathcal{S}(\mathbb{R})$. The only possible solution thus is the one proportional to e^{-S} and

$$\lim_{t\to\infty} E(Z^n(t)e^{ikZ(t)}) = \frac{1}{\mathcal{N}}\int_{\mathbb{R}} x^n e^{ikx} e^{-S(x)} dx \tag{45}$$

for $0 \leq n \leq N-1$ and $k \in \mathbb{R}$. If further $E(Z^n(t))$, $n \geq N$ for $t \to \infty$ exist then

$$\lim_{t\to\infty} E(Z^n(t)) = \left.\frac{d^n c_\infty(k)}{dk^n}\right|_{k=0} = \frac{1}{\mathcal{N}}\int_{\mathbb{R}} x^n e^{-S(x)} dx. \tag{46}$$

The above conditions are necessary to allow a connection between the CL process Z_t and the complex Fokker Planck equation (42). These conditions also guarantee the correct convergence of certain moments of the CL process. For details see [12,13].

3. CL on Lie Groups

Since several problems like QCD at high densities lead to complex valued actions $S : G \to \mathbb{C}$, where now G is some group like $SU(n)$ or $U(1)$ we extend now these results to semisimple compact connected Lie groups and $U(1)$, .

Let us assume a unitary representation of a topological group G, $U : g \mapsto U(g)$ from G into the group of unitary operators on some Hilbert space H. Let us denote the set of equivalence classes of equivalent irreducible representations by $\hat{G}$ and use U^ρ for a member of the equivalence class ρ. The so called coordinate functions

$$u^\rho_{jk} : \left\{ \begin{array}{lll} G & \to & \mathbb{C} \\ s & \mapsto & (U^\rho_s e^\rho_k, e^\rho_j) \qquad 1 \leq j,k \leq n_\rho \ , \end{array} \right. \tag{47}$$

where $(e^\rho_j)_{1\leq j\leq n}$ is some arbitrary basis in H^ρ with complex dimension n_ρ, form an orthonormal basis for $L^2(G)$ with G a compact group . For $f \in L^2(G)$ we have

$$f = \sum_{\rho\in\hat{G}} \sum_{j,k=1}^{n_\rho} n_\rho a^\rho_{jk} u^\rho_{jk} \ , \tag{48}$$

where $a^\rho_{jk} = (f, u^\rho_{jk}) = \int_G f \overline{u}^\rho_{jk} d\mu$ [14–16]. Let us now define the character of ρ by

$$\chi^\rho : \left\{ \begin{array}{lll} G & \to & \mathbb{C} \\ g & \mapsto & TrU^\rho_g = \sum_{k=1}^{n_\rho} u^\rho_{kk} \ . \end{array} \right. \tag{49}$$

The characters also have orthogonality relations. For any function obeying $f(g^{-1}hg) = f(h) \forall\, g \in G$ one has [14–16]

$$f = \sum_{\rho\in\hat{G}} (f, \chi^\rho)\chi^\rho \ . \tag{50}$$

Let $\mathcal{T}_\rho(G)$ denote the linear space spanned by the coordinate functions u^ρ_{jk}. If G is now a real compact connected semisimple Lie group and $\Omega(\mathfrak{g})$ the Casimir element of universal enveloping algebra, then the functions of $\mathcal{T}_\rho(G)$ are eigenfunctions of $\Omega(\mathfrak{g})$ considering the Casimir element as a differential operator [14].

Let now M be some n-dimensional manifold. Let $W^i, 1 \le i \le n$ real Wiener processes and assume $n+1$ vector fields $\mathcal{A}_i, 0 \le i \le n$. Then with $f \in C_0^\infty(M)$ (for compact manifold it is sufficient to use $f \in C^\infty(M)$) one defines a stochastic differential equation by [3,17]

$$d(f \circ X) = (\mathcal{A}_0 f)(X)dt + \sum_{i=1}^{n} (\mathcal{A}_i f)(X) * dW^i \ , \tag{51}$$

where $\sum_{i=1}^{n} (\mathcal{A}_i f)(X) * dW^i$ is the stochastic differential in the sense of Stratonovich denoted by $*$. One further has

$$\frac{d}{dt} E(f \circ X_t) = E((Lf) \circ X_t) \ , \tag{52}$$

with

$$L = \mathcal{A}_0 + \frac{1}{2} \sum_{i=1}^{n} \mathcal{A}_i^2 \ . \tag{53}$$

For details see e.g. [3,17]. The adjoint operator L^* is conventionally called the Fokker-Planck operator.

To perform CL one needs to complexify G. In the context of an universal complexification of a real analytic Lie group it is just worth noting that

i) The universal complexification exists for any real connected Lie group.

ii) In the case of a compact group G the homomorphism $\gamma : G \to G_{\mathbb{C}}$ is injective and $\gamma(G)$ is a compact subgroup of $G_{\mathbb{C}}$.

iii) If $L(G)$ denotes the Lie algebra of G then

$$\mathbb{C} \otimes_{\mathbb{R}} L(G) =: L(G)_{\mathbb{C}} \simeq L(G_{\mathbb{C}}) \ . \tag{54}$$

iv) If G is semisimple then so is $G_{\mathbb{C}}$.

For details we refer e.g. to [16,18].

For the in physics important groups $SU(n)$ vice versa $U(n)$ we have the relations

$$SU(n)_{\mathbb{C}} \simeq SL(n, \mathbb{C}) \ , \ U(n)_{\mathbb{C}} \simeq GL(n, \mathbb{C}) \ . \tag{55}$$

The existence of a complexification implies that the coordinate functions u_{ij}^{ρ} have holomorphic extensions $u_{ij}^{\rho,\mathbb{C}}$. From now on G is regarded as a subgroup of $G_{\mathbb{C}}$. Assuming $(A_a)_{1 \le a \le n}$ a $\mathbb{C}$-basis of $L(G_{\mathbb{C}})$, then $((A_a)_{1 \le a \le n}, (iA_a)_{1 \le a \le n})$ forms an $\mathbb{R}$-basis of $L(G_{\mathbb{C}})$ when considered as a real Lie algebra. Let us define now the following derivations. For f a holomorphic function on $G_{\mathbb{C}}$

$$(\mathcal{A}_a f)(s) = \left. \frac{d}{dz} f(s \, exp \, z A_a) \right|_{z=0} \qquad s \in G_{\mathbb{C}} \ . \tag{56}$$

For $f \in C^\infty(G_{\mathbb{C}})$ we distinguish

$$(\mathcal{A}_a^x f)(s) = \left. \frac{d}{dt} f(s \, exp \, t A_a) \right|_{t=0} \qquad s \in G_{\mathbb{C}}, \ f \in C^\infty(G_{\mathbb{C}}) \ , \tag{57}$$

and

$$(\mathcal{A}_a^y f)(s) = \frac{d}{dt} f(s \, exp \, ti A_a)\Big|_{t=0} \qquad s \in G_{\mathbb{C}}, \ f \in C^\infty(G_{\mathbb{C}}) \ . \tag{58}$$

In the case f holomorphic on $G_{\mathbb{C}}$ we have

$$\mathcal{A}_a f = \mathcal{A}_a^x f = -i \mathcal{A}_a^y f \ . \tag{59}$$

The CL equation can now defined by

$$d(f \circ X) = -\sum_a (\theta_a(\mathcal{A}_a^x f))(X) dt - \sum_a (\phi_a(\mathcal{A}_a^y f))(X) dt + \sum_a (\mathcal{A}_a^x f)(X) * dW^a \tag{60}$$

where

$$\theta_a = \frac{1}{2}(\mathcal{A}_a^x ReS + \mathcal{A}_a^y ImS) \ , \ \ \phi_a = \frac{1}{2}(\mathcal{A}_a^x ImS - \mathcal{A}_a^y ReS) \ . \tag{61}$$

When f, S is holomorphic, then with (59) equation (60) simplifies to

$$d(f \circ X) = -\sum_a (\mathcal{A}_a S)(\mathcal{A}_a f)(X) dt + \sum_a (\mathcal{A}_a f)(X) * dW^a \tag{62}$$

and one obtains

$$\frac{d}{dt} E(f \circ X_t) = E((L^{\mathbb{C}} f) \circ X_t) \ , \tag{63}$$

where $L^{\mathbb{C}} = \frac{1}{2}\sum_a \mathcal{A}_a^2 - \sum_a (\mathcal{A}_a S)\mathcal{A}_a$.

We define now $\Lambda_{t,ij}^\rho := E(u_{ij}^{\rho,\mathbb{C}} \circ X_t)$ and let S have the expansion

$$S = \sum_{\rho \in \Theta} \sum_{l,m} c_{lm}^\rho u_{lm}^\rho \ \Rightarrow S^{\mathbb{C}} = \sum_{\rho \in \Theta} \sum_{l,m} c_{lm}^\rho u_{lm}^{\rho,\mathbb{C}} \ , \tag{64}$$

with Θ a finite subset of $\hat{G}$. If we assume that
1) the expectation values become stationary so that we may define
$\Lambda_{ij}^\rho := \lim_{t \to \infty} E(u_{ij}^{\rho,\mathbb{C}} \circ X_t)$ and

$$\sum_{\rho \in \hat{G}} \sum_{i,j} n_\rho |\Lambda_{ij}^\rho|^2 < \infty \ , \tag{65}$$

then there is a $h \in L^2(G)$ such that

$$\Lambda_{ij}^\rho = \int u_{ij}^\rho h \, d\mu \ . \tag{66}$$

h is given by

$$h = \sum_{\rho \in \hat{G}} \sum_{i,j} n_\rho \Lambda_{ij}^\rho \overline{u_{ij}^\rho} \ . \tag{67}$$

$$\lim_{t\to\infty} E((u_{lp}^{\rho,\mathbb{C}} u_{iq}^{\hat\rho,\mathbb{C}}) \circ X_t) = \int u_{lp}^{\rho} u_{iq}^{\hat\rho} h \, d\mu \quad \forall\, \rho, \hat\rho \in \hat G, \tag{68}$$

$$\forall\, 1 \le l,p \le dim\rho;\ 1 \le i,q \le dim\hat\rho \ , \tag{69}$$

then $\mathcal{T}h = 0$ in the sense of distributions with

$$\mathcal{T} := \sum_a \mathcal{A}_a^r \left(\frac{1}{2}\mathcal{A}_a^r + (\mathcal{A}_a^r S) \right) \ . \tag{70}$$

where $\mathcal{A}_a^r$ is obtained by restricting the flow to $\mathbb{R} \times \mathbb{G} \to \mathbb{G}$, thus defining the derivatives on G. Since $\mathcal{T}$ is elliptic, h is $\in C^\infty(G)$. Note that 0 is an isolated eigenvalue of $\mathcal{T}$ and the corresponding normalized eigenvector is $h = \frac{1}{\mathcal{N}} e^{-2S}$ with $\mathcal{N} = \int e^{-2S} \, d\mu$, assuming $0 < |\mathcal{N}| < \infty$. The result remains also valid for arbitrary $S \in C^\infty(G)$. For details of the proof we refer to [19].

If we restrict to $U(1)$, with $U(1)_{\mathbb{C}} = \mathbb{C}\backslash\{0\}$, the results simplify significantly. For every $f \in L^2(U(1))$ one has

$$f = \sum_q c_q e^{iqx} \quad \text{with} \quad c_q = \int_0^{2\pi} f(x) e^{-iqx} \, d\mu(x) \ , \tag{71}$$

with $d\mu(x) = dx/2\pi$. The vector fields are simply $\mathcal{A} = \frac{d}{dz}$, $\mathcal{A}^r = \frac{d}{dx}$ and equation (63) reads

$$\frac{d}{dt} E(f \circ X_t) = E\left(\left\{ \frac{1}{2}\left(\frac{d^2}{dz^2} f\right) - \left(\frac{d}{dz} S\right)\left(\frac{d}{dz} f\right) \right\} \circ X_t \right) \ . \tag{72}$$

Again S is assumed a finite series

$$S(x) = \sum_q c_q e^{iqx} \tag{73}$$

and

$$\Lambda_t^k := E(e^{ikX_t}) \ . \tag{74}$$

Assuming again the stationary expectation values

$$\Lambda^k := \lim_{t\to\infty} E\left(e^{ikX_t}\right) \tag{75}$$

and

$$\sum_{q\in\mathbb{Z}} |\Lambda^q|^2 < \infty \ , \tag{76}$$

then with $h := \sum_q \Lambda^q e^{-iqx}$ we have $\Lambda^k = \int e^{ikx} h \, d\mu$.
In the limit $t \to \infty$ one obtains with (72)

$$0 = -\frac{k^2}{2}\Lambda^k + k \sum_q q c_q \Lambda^{q+k} \tag{77}$$

$$= \frac{1}{2}\int \left(\frac{d^2}{dx^2} e^{ikx}\right) h \, d\mu - \int \left(\left(\frac{d}{dx} S\right) h \right) \left(\frac{d}{dx} e^{ikx}\right) d\mu \ . \tag{78}$$

As before one has $\mathcal{T}h = 0$ (distributional) with

$$\mathcal{T} = \frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx} + (\frac{d}{dx}S)\right) \tag{79}$$

such that $h = \frac{1}{N}e^{-2S}$, where the result also holds for arbitrary $S \in C^\infty(U(1))$.

4. Conclusions

So far there is still no closed theory of the complex Langevin method and general results are difficult to prove due to the singular structure of the F-P operator of the CL process. Nevertheless it is possible to define conditions under which a CL process simulates correctly integrals over complex valued weights. It is thus in principle possible to verify aposteriori the correctness of the process, even the conditions might be hard to prove.

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