Outstanding questions

- [ ] Justification for treating V_{trap}, L_z, and V_{int} as gauge interactions → what is the benefit over a naive discretization as we go to the continuum limit?

- [ ] Can we study the Maxwell relations?

$$\frac{\partial^2(\beta\Omega)}{\partial(\beta\mu)(\partial\beta\omega_z)} = \frac{-\partial^2 \ln \mathcal{Z}}{\partial(\beta\mu)\partial(\beta\omega_z)} = \frac{-\partial\langle\hat{N}\rangle}{\partial(\beta\omega_z)} = -\frac{\partial\langle\hat{L}_z\rangle}{\partial(\beta\mu)}$$

- [ ] Going to AMR

→ a will change. how will this affect our parameters?

$$S = \sum_{\tilde{x},t} a^d \left(\frac{\omega_z}{a}(\tilde{x}a) + \quad \ldots \quad \left(\frac{\omega_{tr}}{a^2}\right)^2 (\tilde{r}_\perp^2 a^2)\right)$$

$$a = dx\_cell$$

$$\omega_z \rightarrow \omega_z/(d\tau\, a)$$

- [ ] Is catastrophic cancellation occurring?
  - subtracting two good approximations may yield a very bad approximation of the difference
  - amplification of error introduced by previous calculations
  - is there any way to identify this?

To do

- [ ] Add new observables to notes
  → need to complexify

- [ ] Start naming analysis code w/date in name so you know which data files it will work on?

- [ ] hack out HDF5 storage w/ Don
- [ ] investigate w/Don using plotfiles again
- [ ] re-read Busch et al 2 cold atoms paper
- [ ] Talk through factors of dx-cell, dy-cell in x-center, y-center, r^2 with Don

- [ ] Talk through Bridges & ERCAP work
  - start dates
  - what should we run on each?
  - what testing do we need to do?

Completed To-Dos

- [x] merge hdfs branch w/main branch
- [x] Casey will add $\hat{T}$, $\hat{V}_{int}$, $\hat{V}_{tr}$ to Observables_kernel.H
- [x] check new version of S (compiled from observables) against old version currently in the code.
- [x] work w/ Don on reductions for observables
  - [x] Question → addition of a_x in line 377 of Observables_Kernel.H (in local_observables)
  - [x] Best way to test $L_z = S\omega/\bar{\omega}_z$? cout? write to file?
  - [x] Adding new observables to observables[obs:: __]
    ↳ I think this is in observables.cpp but don't want to get it wrong
  - [x] Add new observables to logfile.
    - all components of S + new observables to logfile
- [x] multiply by a^d in S, Ka^R, and Ka^{Im}
- [x] Add to notes: this code can really only do $d=2$. If we want to do $d=1$ or $d=3$, we need to change a lot.

Deriving the action for the lattice

Continuum action:

$r_\perp^2 = (x^2+y^2)$

$$S[\phi] = \int d^dx\, d\tau \left[\phi^*\left(\partial_\tau - \mu - \frac{\nabla^2}{2m} - \frac{m}{2}\omega_{tr}^2 r_\perp^2 - i\omega_z(x\partial_y - y\partial_x)\right)\phi + \lambda(\phi^*\phi)^2\right]$$

To discretize:

$$\int d^dx\, d\tau \to \sum_{\vec{x},\tau} a^d d\tau \quad , \quad \alpha\phi^*\phi \to \alpha\phi_r^*\phi_{r-\hat{\tau}} \text{ (see appendix)}$$

Discrete action:

$$S[\phi] = \sum_{\vec{x},\tau} a^d d\tau \left[\phi_r^*\left(\partial_\tau - \frac{\nabla^2}{2m}\right)\phi_r - \phi_r^*\left(\mu + \frac{m}{2}\omega_{tr}^2 r_\perp^2 + i\omega_z(x\partial_y - y\partial_x)\right)\phi_{r-\hat{\tau}} + \lambda(\phi_r^*\phi_{r-\hat{\tau}})^2\right]$$

Discrete derivatives:

$$\partial_j\phi = \frac{1}{a}(\phi_r - \phi_{r-\hat{j}}) \quad \text{and} \quad \nabla^2\phi = \sum_{j=1}^{d}\frac{1}{a^2}(\phi_{r-\hat{j}} - 2\phi_r + \phi_{r+\hat{j}})$$

$$\phi_r^*\partial_\tau\phi_r = \frac{1}{d\tau}(\phi_r^*\phi_{r-\hat{\tau}} - \phi_r^*\phi_r)$$

$$\phi_r^*\nabla^2\phi_r = \frac{1}{a^2}\sum_{j=1}^{d}(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}})$$

$$\phi_r^*\partial_x\phi_{r-\hat{\tau}} = \frac{1}{a}(\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{x}}), \quad \phi_r^*\partial_y\phi_{r-\hat{\tau}} = \frac{1}{a}(\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{y}})$$

So our discrete (lattice) action is:

$$S = \sum_{\vec{x},\tau}\Big[a^d\phi_r^*\phi_r - a^d\phi_r^*\phi_{r-\hat{\tau}} - a^d d\tau\,\mu\,\phi_r^*\phi_{r-\hat{\tau}} - \frac{a^{d-2}d\tau}{2m}\sum_{j=1}^{d}(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}})$$
$$- \frac{ma^d d\tau}{2}\omega_{tr}^2 r_\perp^2\phi_r^*\phi_{r-\hat{\tau}} + i\omega_z a^{d-1}d\tau\,(x\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - x\phi_r^*\phi_{r-\hat{\tau}} - y\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}$$
$$+ y\phi_r^*\phi_{r-\hat{\tau}}) + a^d d\tau\,\lambda(\phi_r^*\phi_{r-\hat{\tau}})^2\Big]$$

Let's update our parameters to incorporate $d\tau$ and a

$$x = a\tilde{x} \quad y = a\tilde{y} \quad r^2 = a^2\tilde{r}^2 \quad \mu = \frac{\tilde{\mu}}{d\tau} \quad m = \frac{\tilde{m}\,d\tau}{a^2} \quad \omega_{tr} = \frac{\tilde{\omega}_{tr}}{d\tau}$$

$$\omega_z = \frac{\tilde{\omega}_z}{d\tau} \qquad \lambda = \frac{\tilde{\lambda}}{d\tau}$$

$$S=\sum_{\vec{x},\tau}\Big[a^d\phi^*_r\phi_r-a^d\phi^*_r\phi_{r-\hat{\tau}}-a^d\tilde{\mu}\,\phi^*_r\phi_{r-\tau}-\frac{a^d}{2\tilde{m}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}}-2\phi^*_r\phi_r+\phi^*_r\phi_{r+\hat{j}}\right)$$

$$-\frac{\tilde{m}a^d}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau}+i\tilde{\omega}_z a^d\left(\tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}}-\tilde{x}\phi^*_r\phi_{r-\hat{\tau}}-\tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}+\tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)$$

$$+a^d\tilde{\lambda}\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2\Big]$$

We can pull out a constant factor of a^d now:

$$S=\sum_{\vec{x},\tau}a^d\Big[\phi^*_r\phi_r-\phi^*_r\phi_{r-\hat{\tau}}-\tilde{\mu}\,\phi^*_r\phi_{r-\tau}-\frac{1}{2\tilde{m}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}}-2\phi^*_r\phi_r+\phi^*_r\phi_{r+\hat{j}}\right)$$

$$-\frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau}+i\tilde{\omega}_z\left(\tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}}-\tilde{x}\phi^*_r\phi_{r-\hat{\tau}}-\tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}+\tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)+\tilde{\lambda}\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2\Big]$$

$\tilde{x}=n_x=x/a \rightarrow x=\tilde{x}a$

Noting that, to second order in $d\tau$, $(1+d\tau\mu)=e^{d\tau\mu}$, we further simplify:

$$S=\sum_{\vec{x},\tau}a^d\Big[\underbrace{\phi^*_r\phi_r-e^{\tilde{\mu}}\phi^*_r\phi_{r-\tau}}_{S_\mu}\underbrace{-\frac{1}{2\tilde{m}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}}-2\phi^*_r\phi_r+\phi^*_r\phi_{r+\hat{j}}\right)}_{S_\nabla}$$

$$-\underbrace{\frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau}}_{S_{tr}}+\underbrace{i\tilde{\omega}_z\left(\tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}}-\tilde{x}\phi^*_r\phi_{r-\hat{\tau}}-\tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}+\tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)}_{-S_w}+\underbrace{\tilde{\lambda}\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2}_{S_{int}}\Big]$$

We can make our work easier by writing this out as:

$$S=\sum_{\vec{x},\tau}a^d\left[S_\mu+S_\nabla-S_{trap}-S_w+S_{int}\right]$$

☐ rewrite each component to include $\sum_{\vec{x},\tau}a^d$

$[\phi]=L^{-d/2}$??

Writing our complex fields in terms of two real components

The next step is to note that $\phi = \phi_1 + i\phi_2$ and apply this coordinate transformation to the lattice.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \text{ and } \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \longrightarrow$$

$$S_\mu = \phi^*_r\phi_r - e^{\tilde{\mu}}\phi^*_r\phi_{r-\tau} \longrightarrow \frac{1}{2}\Big[\phi_{1,r}^2 + \phi_{2,r}^2 - e^{\mu}\big(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)\Big]$$

$$S_\nabla = -\frac{1}{2\tilde{M}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}} - 2\phi^*_r\phi_r + \phi^*_r\phi_{r+\hat{j}}\right)$$

$$\rightarrow \frac{1}{4\tilde{m}}\sum_{j=1}^{d}\Big[2\phi_{1,r}^2 + 2\phi_{2,r}^2 - \big(\phi_{1,r}\phi_{1,r-j} - i\phi_{2,r}\phi_{1,r-j} + i\phi_{1,r}\phi_{2,r-j} + \phi_{2,r}\phi_{2,r-j}$$
$$+ \phi_{1,r}\phi_{1,r+j} - i\phi_{2,r}\phi_{1,r+j} + i\phi_{1,r}\phi_{2,r+j} + \phi_{2,r}\phi_{2,r+j}\big)\Big]$$

$$S_{trap} = \frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau} \longrightarrow \frac{\tilde{m}}{4}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\big(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)$$

$$S_\omega = i\tilde{\omega}_z\left(\tilde{x}\phi^*_r\phi_{r-\tau} - \tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}} + \tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}} - \tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)$$

$$\rightarrow i\frac{\tilde{\omega}_z}{2}\Big[\tilde{x}\big(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} + i\phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}}$$
$$- i\phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big) + \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}}$$
$$+ i\phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{\tau}} + i\phi_{2,r}\phi_{1,r-\tau} - i\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)\Big]$$

$$= i\frac{\tilde{\omega}_z}{2}\Big[\tilde{x}\big(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big)$$
$$+ \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)$$
$$+ \frac{\tilde{\omega}_z}{2}\Big[\tilde{x}\big(\phi_{2,r}\phi_{1,r-\tau} - \phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}} + \phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}}\big)$$
$$+ \tilde{y}\big(\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\tau} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\big)\Big]$$

$$= i\frac{\tilde{\omega}_z}{2}\Big[(\tilde{x}-\tilde{y})\big(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\big) - \tilde{x}\big(\phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big) + \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}}\big)\Big]$$
$$+ \frac{\tilde{\omega}_z}{2}\Big[(\tilde{x}-\tilde{y})\big(\phi_{2,r}\phi_{1,r-\tau} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\big) - \tilde{x}\big(\phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}}\big) + \tilde{y}\big(\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}}\big)\Big]$$

$$S_{int} = \tilde{\lambda}(\phi^*_r \phi_{r-\hat{\tau}})^2 \rightarrow \frac{\hat{\tilde{\lambda}}}{4}\left(\phi_{1,r}\phi_{1,r-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right)^2$$

$$\begin{aligned}
= \frac{\hat{\tilde{\lambda}}}{4}\Big(& \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - i\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + i\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
& - i\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
& + i\phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
& + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - i\phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + i\phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}}\Big)
\end{aligned}$$

$$\begin{aligned}
= \frac{\hat{\tilde{\lambda}}}{4}\Big(& \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} \\
& + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}}\Big) \\
+ i\frac{\hat{\tilde{\lambda}}}{4}\Big(& -\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
& + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}}\Big)
\end{aligned}$$

Now, we can derive the drift functions by taking a complex derivative of the action, S

$$K_a = \frac{\delta}{\delta\phi_a} S[\phi]$$

not sure we even need this second sum...

When taking derivatives on the lattice, recall that $\partial_i \phi_r = \frac{1}{a}(\phi_r - \phi_{r-\hat{i}})$

Therefore: $$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{a=1}^{2}\sum_{q=1}^{N}\phi_{a,q}\,\phi_{a,q+\hat{i}}\right) = \sum_{a=1}^{2}\sum_{q=1}^{N}\left(\phi_{a,q}\frac{\delta}{\delta\phi_{c,r}}\phi_{a,q+\hat{i}} + \frac{\delta}{\delta\phi_{c,r}}\phi_{a,q}\,\phi_{a,q+\hat{i}}\right)$$

$$= \sum_{a=1}^{2}\sum_{q=1}^{N}\left(\phi_{a,q}\,\delta_{c,a}\,\delta_{r,q+\hat{i}} + \delta_{c,a}\,\delta_{r,q}\,\phi_{a,q+\hat{i}}\right)$$

the sums collapse, due to the Kronecker deltas

$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{a=1}^{2}\sum_{q=1}^{N}\phi_{a,q}\,\phi_{a,q+\hat{i}}\right) = \phi_{c,r-\hat{i}}\ \text{if}(r>0) + \phi_{c,r+\hat{i}}\ \text{if}\left(r+\hat{i}<N\right)$$

similarly $$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2}\phi_{a,q}\,\phi_{a,q-\hat{i}}\right) = \phi_{c,r+\hat{i}}\ \text{if}(r+\hat{i}<N) + \phi_{c,r-\hat{i}}\ \text{if}(r>0)$$

and $$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N}\sum_{a=1}^{2}\sum_{b=1}^{2}\varepsilon_{ab}\,\phi_{a,q}\,\phi_{b,q+\hat{i}}\right) = \sum_{b=1}^{2}\varepsilon_{cb}\left(\phi_{b,r-\hat{i}} + \phi_{b,r+\hat{i}}\right)$$

and $$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N}\sum_{a=1}^{2}\sum_{b=1}^{2}\varepsilon_{ab}\,\phi_{a,q}\,\phi_{b,q-\hat{i}}\right) = \sum_{b=1}^{2}\varepsilon_{cb}\left(\phi_{b,r+\hat{i}} + \phi_{b,r-\hat{i}}\right)$$

Question: don't our boundary conditions already satisfy the constraints?
If $r=N$, $\phi_{r+\hat{i}}=0$ automatically

$$K_a = \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau}S_{\mu} + \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau}S_{\nabla} - \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau}S_{trap} - \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau}S_{w} + \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau}S_{int}$$

Let's take this derivative for each part of the action individuall

$$K_{a,\mu,r} = \frac{\delta}{\delta\phi_{ar}}S_{\mu r} = \frac{1}{2}\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\left[\phi_{c,r}^{2} - e^{\tilde{\mu}}\phi_{c,r}\,\phi_{c,r-\hat{\tau}} - i e^{\hat{\mu}}\sum_{b=1}^{2}\varepsilon_{cb}\,\phi_{c,r}\,\phi_{b,r-\hat{\tau}}\right]$$

$$= \frac{1}{2}\left[2\phi_{a,r} - e^{\tilde{\mu}}\left(\phi_{a,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\right) - i e^{\hat{\mu}}\sum_{b=1}^{2}\varepsilon_{ab}\left(\phi_{b,r+\hat{\tau}} + \phi_{b,r-\hat{\tau}}\right)\right]$$

$$K_{a,\mu,r} = \phi_{a,r} - \frac{1}{2}e^{\tilde{\mu}}\left(\phi_{a,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\right) - \frac{i}{2}e^{\tilde{\mu}}\sum_{b=1}^{2}\varepsilon_{ab}\left(\phi_{b,r+\hat{\tau}} + \phi_{b,r-\hat{\tau}}\right)$$

$$K_{a,\nabla,r} = \frac{\delta}{\delta\Phi_{a,r}} S_{\nabla,r} = \frac{\delta}{\delta\Phi_{a,r}} \frac{1}{4\tilde{m}} \sum_{j=1}^{d} \sum_{c=1}^{2} \Big[2\Phi_{c,r}^2 - \left(\Phi_{c,r}\Phi_{c,r+\hat{j}} + \Phi_{c,r}\Phi_{c,r-\hat{j}}\right)$$

$$- i \sum_{b=1}^{2} \varepsilon_{cb} \left(\Phi_{c,r}\Phi_{b,r-\hat{j}} + \Phi_{c,r}\Phi_{b,r+\hat{j}}\right) \Big]$$

$$= \frac{1}{4\tilde{m}} \sum_{j=1}^{d} \Big[4\Phi_{a,r} - \left(\Phi_{a,r+\hat{j}} + \Phi_{a,r-\hat{j}} + \Phi_{a,r+\hat{j}} + \Phi_{a,r-\hat{j}}\right)$$

$$- i \sum_{b=1}^{2} \varepsilon_{ab} \left(\Phi_{b,r-\hat{j}} + \Phi_{b,r+\hat{j}} + \Phi_{b,r+\hat{j}} + \Phi_{b,r-\hat{j}}\right) \Big]$$

$$= \frac{1}{4\tilde{m}} \sum_{j=1}^{d} \Big[4\Phi_{a,r} - 2\Phi_{a,r+\hat{j}} - 2\Phi_{a,r-\hat{j}} - 2i \sum_{b=1}^{2} \varepsilon_{ab} \left(\Phi_{b,r+\hat{j}} + \Phi_{b,r-\hat{j}}\right) \Big]$$

$$= \frac{1}{2\tilde{m}} \sum_{j=1}^{d} \Big[2\Phi_{a,r} - \left(\Phi_{a,r+\hat{j}} + \Phi_{a,r-\hat{j}}\right) - i \sum_{b=1}^{2} \varepsilon_{ab} \left(\Phi_{b,r+\hat{j}} + \Phi_{b,r-\hat{j}}\right) \Big]$$

$$K_{a,tr,r} = \frac{\delta}{\delta\Phi_{a,r}} S_{tr,r} = \frac{\delta}{\delta\Phi_{a,r}} \frac{\tilde{m}\tilde{\omega}_{tr}^2 \tilde{r}_\perp^2}{4} \sum_{c=1}^{2} \Big[\Phi_{c,r}\Phi_{c,r-\hat{\tau}} + i \sum_{b=1}^{2} \varepsilon_{cb} \Phi_{c,r}\Phi_{b,r-\hat{\tau}} \Big]$$

$$= \frac{\tilde{m}\tilde{\omega}_{tr}^2 \hat{r}_\perp^2}{4} \Big[\Phi_{a,r-\hat{\tau}} + \Phi_{a,r+\hat{\tau}} + i \sum_{b=1}^{2} \varepsilon_{ab} \left(\Phi_{b,r-\hat{\tau}} + \Phi_{b,r+\hat{\tau}}\right) \Big]$$

$$K_{a,\omega,r} = \frac{d}{\delta\Phi_{a,r}} S_{\omega,r} = \frac{\delta}{\delta\Phi_{a,r}} \frac{\tilde{\omega}_z}{2} \sum_{c=1}^{2} \Big[\sum_{b=1}^{2} \varepsilon_{cb} \Big((\tilde{y}-\tilde{x}) \Phi_{c,r}\Phi_{b,r-\hat{\tau}} + \tilde{x}\Phi_{c,r}\Phi_{b,r-\hat{y}-\hat{\tau}}$$

$$- \tilde{y}\Phi_{c,r}\Phi_{b,r-\hat{x}-\hat{\tau}} \Big) + i \left((\tilde{x}-\tilde{y})\Phi_{c,r}\Phi_{c,r-\hat{\tau}} - \tilde{x}\Phi_{c,r}\Phi_{c,r-\hat{y}-\hat{\tau}} + \tilde{y}\Phi_{c,r}\Phi_{c,r-\hat{x}-\hat{\tau}} \right) \Big]$$

$$= \frac{\tilde{\omega}_z}{2} \Big[\sum_{b=1}^{2} \varepsilon_{ab} \Big((\tilde{y}-\tilde{x})\left(\Phi_{b,r+\hat{\tau}} + \Phi_{b,r-\hat{\tau}}\right) + \tilde{x}\left(\Phi_{b,r-\hat{y}-\hat{\tau}} + \Phi_{b,r+\hat{y}+\hat{\tau}}\right)$$

$$- \tilde{y}\left(\Phi_{b,r-\hat{x}-\hat{\tau}} + \Phi_{b,r+\hat{x}+\hat{\tau}}\right) \Big) + i \Big((\tilde{x}-\tilde{y})\left(\Phi_{a,r-\hat{\tau}} + \Phi_{a,r+\hat{\tau}}\right)$$

$$- \tilde{x}\left(\Phi_{a,r-\hat{y}-\hat{\tau}} + \Phi_{a,r+\hat{y}+\hat{\tau}}\right) + \tilde{y}\left(\Phi_{a,r-\hat{x}-\hat{\tau}} + \Phi_{a,r+\hat{x}+\hat{\tau}}\right) \Big) \Big]$$

$$K_{a,\lambda,r} = \frac{\delta}{\delta\Phi_{a,r}} S_{\lambda,r} = \frac{\delta}{\delta\Phi_{a,r}} \frac{\bar{\lambda}}{4} \sum_{c=1}^{2} \sum_{b=1}^{2} \Big[2\Phi_{c,r}\Phi_{c,r-\hat{\tau}}\Phi_{b,r}\Phi_{b,r-\hat{\tau}} - \Phi_{c,r}^2 \Phi_{b,r-\hat{\tau}}^2$$

$$+ 2i\varepsilon_{cb} \left(\Phi_{c,r}^2 \Phi_{c,r-\hat{\tau}}\Phi_{b,r-\hat{\tau}} - \Phi_{c,r}\Phi_{b,r}\Phi_{b,r-\hat{\tau}}^2 \right) \Big]$$

$$= \frac{\bar{\lambda}}{4} \frac{\delta}{\delta\phi_{a,r}} \sum_{c=1}^{2}\sum_{b=1}^{2}\Big[\underbrace{2\phi_{c,r}\phi_{c,r-\hat{\tau}}\phi_{b,r}\phi_{b,r-\hat{\tau}}}_{A} - \underbrace{\phi_{c,r}^{2}\phi_{b,r-\hat{\tau}}^{2}}_{B}$$

$$+2i\varepsilon_{cb}\Big(\underbrace{\phi_{c,r}^{2}\phi_{c,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}}_{C} - \underbrace{\phi_{c,r}\phi_{b,r}\phi_{b,r-\hat{\tau}}^{2}}_{D}\Big)\Big]$$

A:

$$\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\sum_{b=1}^{2}\left(2\phi_{c,r'}\phi_{c,r'-\hat{\tau}}\phi_{b,r'}\phi_{b,r'-\tau}\right)$$

$$=2\sum_{b,c=1}^{2}\left[\phi_{c,r'}\phi_{b,r'}\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\tau}\right)+\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'}\phi_{b,r'}\right)\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}\right]$$

$$=2\sum_{b,c=1}^{2}\Big[\phi_{c,r'}\phi_{b,r'}\phi_{c,r'-\hat{\tau}}\frac{\delta\phi_{b,r'-\tau}}{\delta\phi_{a,r}}+\phi_{c,r'}\phi_{b,r'}\frac{\delta\phi_{c,r'-\hat{\tau}}}{\delta\phi_{a,r}}\phi_{b,r'-\hat{\tau}}$$

$$+\phi_{c,r'}\frac{\delta\phi_{b,r'}}{\delta\phi_{a,r}}\phi_{c,r'-\tau}\phi_{b,r'-\tau}+\frac{\delta\phi_{c,r'}}{\delta\phi_{a,r}}\phi_{b,r'}\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\tau}\Big]$$

$$=2\sum_{b,c=1}^{2}\Big[\phi_{c,r'}\phi_{b,r'}\phi_{c,r'-\hat{\tau}}\delta_{r,r'-\tau}\delta_{ab}+\phi_{c,r'}\phi_{b,r'}\delta_{r,r'-\hat{\tau}}\delta_{ac}\phi_{b,r'-\hat{\tau}}$$

$$+\phi_{c,r'}\delta_{rr'}\delta_{ab}\phi_{c,r'-\tau}\phi_{b,r'-\tau}+\delta_{rr'}\delta_{ac}\phi_{b,r'}\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\tau}\Big]$$

$$=2\sum_{b,c=1}^{2}\Big(\phi_{c,r+\hat{\tau}}\phi_{b,r+\hat{\tau}}\phi_{c,r}\delta_{ab}+\phi_{c,r+\hat{\tau}}\phi_{b,r+\hat{\tau}}\phi_{b,r}\delta_{ac}$$

$$+\phi_{c,r}\phi_{c,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\delta_{ab}+\phi_{b,r}\phi_{c,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\delta_{ac}\Big)$$

$$=2\Big(\sum_{c=1}^{2}\phi_{c,r+\hat{\tau}}\phi_{a,r+\hat{\tau}}\phi_{c,r}+\sum_{b=1}^{2}\phi_{a,r+\hat{\tau}}\phi_{b,r+\hat{\tau}}\phi_{b,r}$$

$$+\sum_{c=1}^{2}\phi_{c,r}\phi_{c,r-\hat{\tau}}\phi_{a,r-\hat{\tau}}+\sum_{b=1}^{2}\phi_{b,r}\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\Big)$$

Since independent sums have independent indices, we can now change our sums over c to be sums over b

$$=2\Big(2\sum_{b=1}^{2}\phi_{b,r}\phi_{b,r+\hat{\tau}}\phi_{a,r+\hat{\tau}}+2\sum_{b=1}^{2}\phi_{b,r}\phi_{b,r-\hat{\tau}}\phi_{a,r-\hat{\tau}}\Big)$$

$$=4\sum_{b=1}^{2}\phi_{b,r}\left(\phi_{b,r+\hat{\tau}}\phi_{a,r+\hat{\tau}}+\phi_{b,r-\hat{\tau}}\phi_{a,r-\hat{\tau}}\right) \quad A$$

B: $$-\frac{\delta}{\delta\phi_{a,r}}\sum_{b=1}^{2}\left(\phi_{c,r'}^{2}\phi_{b,r-\hat{\tau}}^{2}\right)$$

$$=-\sum_{b=1}^{2}\sum_{c=1}^{2}\left[\phi_{c,r'}^{2}\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{b,r'-\hat{\tau}}^{2}\right)+\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'}^{2}\right)\phi_{b,r'-\hat{\tau}}^{2}\right]$$

$$=-\sum_{b=1}^{2}\sum_{c=1}^{2}\left[2\phi_{c,r'}^{2}\phi_{b,r'-\hat{\tau}}\delta_{a,b}\delta_{r,r'-\hat{\tau}}+2\phi_{c,r'}\delta_{ac}\delta_{rr'}\phi_{b,r'-\hat{\tau}}^{2}\right]$$

$$=-2\left[\sum_{c=1}^{2}\phi_{c,r+\hat{\tau}}^{2}\phi_{a,r}+\sum_{b=1}^{2}\phi_{a,r}\phi_{b,r-\hat{\tau}}^{2}\right]$$

sum over $c\rightarrow$ sum over b

$$=-2\sum_{b=1}^{2}\phi_{a,r}\left(\phi_{b,r+\hat{\tau}}^{2}+\phi_{b,r-\hat{\tau}}^{2}\right)$$

B

C: $$\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\sum_{b=1}^{2}2i\varepsilon_{cb}\,\phi_{c,r'}^{2}\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}$$

$$=\sum_{c=1}^{2}\sum_{b=1}^{2}2i\varepsilon_{cb}\left[\phi_{c,r'}^{2}\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}\right)+\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'}^{2}\right)\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}\right]$$

$$=\sum_{c=1}^{2}\sum_{b=1}^{2}2i\varepsilon_{cb}\Big[\phi_{c,r'}^{2}\phi_{c,r'-\hat{\tau}}\delta_{ab}\delta_{r,r'-\hat{\tau}}+\phi_{c,r'}^{2}\delta_{ac}\delta_{r,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}$$

$$+2\phi_{c,r'}\delta_{ac}\delta_{rr'}\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}\Big]$$

$$=2i\left[\sum_{c=1}^{2}\varepsilon_{ca}\phi_{c,r+\hat{\tau}}^{2}\phi_{c,r}+\sum_{b=1}^{2}\varepsilon_{ab}\phi_{a,r+\hat{\tau}}^{2}\phi_{b,r}+2\sum_{b=1}^{2}\varepsilon_{ab}\phi_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\right]$$

sum over $c\rightarrow$ sum over b

C

$$=2i\sum_{b=1}^{2}\varepsilon_{ab}\left[\phi_{b,r}\phi_{a,r+\hat{\tau}}^{2}-\phi_{b,r}\phi_{b,r+\hat{\tau}}^{2}+2\phi_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\right]$$

NOTE: minus sign here is because $\varepsilon_{ab}=-\varepsilon_{ba}$!

$$D: -\frac{\delta}{\delta\Phi_{a,r}}\sum_{c=1}^{2}\sum_{b=1}^{2}\left[2i\,\varepsilon_{cb}\,\Phi_{c,r'}\Phi_{b,r'}\Phi_{b,r'-\hat{\tau}}^{2}\right]$$

$$= -2i\sum_{c=1}^{2}\sum_{b=1}^{2}\varepsilon_{cb}\left[\Phi_{c,r'}\Phi_{b,r'}\frac{\delta}{\delta\Phi_{a,r}}\left(\Phi_{b,r'-\hat{\tau}}^{2}\right)+\frac{\delta}{\delta\Phi_{a,r}}\left(\Phi_{c,r'}\Phi_{b,r'}\right)\Phi_{b,r'-\hat{\tau}}^{2}\right]$$

$$= -2i\sum_{c=1}^{2}\sum_{b=1}^{2}\varepsilon_{cb}\left[2\Phi_{c,r'}\Phi_{b,r'}\delta_{ab}\delta_{r,r'-\hat{\tau}}\Phi_{b,r'-\hat{\tau}}+\Phi_{c,r'}\delta_{ab}\delta_{rr'}\Phi_{b,r'-\hat{\tau}}^{2}\right.$$
$$\left.+\delta_{ac}\delta_{rr'}\Phi_{b,r'}\Phi_{b,r'-\hat{\tau}}^{2}\right]$$

$$= -2i\left[\sum_{c=1}^{2}\varepsilon_{ca}\,2\Phi_{c,r+\hat{\tau}}\Phi_{a,r+\hat{\tau}}\Phi_{a,r}+\sum_{c=1}^{2}\varepsilon_{ca}\Phi_{cr}\Phi_{a,r-\hat{\tau}}^{2}\right.$$
$$\left.+\sum_{b=1}^{2}\varepsilon_{ab}\Phi_{b,r}\Phi_{b,r-\hat{\tau}}^{2}\right]$$

sum over $c \to$ sum over b

$$D \qquad = -2i\sum_{b=1}^{2}\varepsilon_{ab}\left[\Phi_{b,r}\Phi_{b,r-\hat{\tau}}^{2}-\Phi_{b,r}\Phi_{a,r-\hat{\tau}}^{2}-2\Phi_{a,r}\Phi_{a,r+\hat{\tau}}\Phi_{b,r+\hat{\tau}}\right]$$

NOTE: minus signs again come from $\varepsilon_{ab}=-\varepsilon_{ba}$

CHECK:

$$(A+B)_{a\to 1}=\frac{\delta}{\delta\Phi_{1,r}}\left(\overbrace{\Phi_{1,r}^{2}\Phi_{1,r-\hat{\tau}}^{2}+\Phi_{2,r}^{2}\Phi_{2,r-\hat{\tau}}^{2}+4\Phi_{1,r}\Phi_{2,r}\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}}-\Phi_{1,r}^{2}\Phi_{2,r-\hat{\tau}}^{2}-\Phi_{1,r-\hat{\tau}}^{2}\Phi_{2,r}^{2}}^{\mathrm{Re}[S_\lambda]/(\pi/4)}\right)$$

$$(A+B)_{a\to 1}=4\Phi_{1,r}\left(\Phi_{1,r+\hat{\tau}}^{2}+\Phi_{1,r-\hat{\tau}}^{2}\right)+4\Phi_{2,r}\left(\Phi_{2,r+\hat{\tau}}\Phi_{1,r+\hat{\tau}}+\Phi_{2,r-\hat{\tau}}\Phi_{1,r-\hat{\tau}}\right)$$
$$-2\Phi_{1,r}\left(\Phi_{1,r+\hat{\tau}}^{2}+\Phi_{1,r-\hat{\tau}}^{2}\right)-2\Phi_{1,r}\left(\Phi_{2,r+\hat{\tau}}^{2}+\Phi_{2,r-\hat{\tau}}^{2}\right)$$

$$=2\Phi_{1,r}\left(\Phi_{1,r+\hat{\tau}}^{2}+\Phi_{1,r-\hat{\tau}}^{2}\right)+4\Phi_{2,r}\left(\Phi_{2,r+\hat{\tau}}\Phi_{1,r+\hat{\tau}}+\Phi_{2,r-\hat{\tau}}\Phi_{1,r-\hat{\tau}}\right)$$
$$-2\Phi_{1,r}\left(\Phi_{2,r+\hat{\tau}}^{2}+\Phi_{2,r-\hat{\tau}}^{2}\right)$$

$$\frac{\delta}{\delta\phi_{1,r}}\left(\phi_{1,r'}^2\phi_{1,r'-\hat{\tau}}^2+\phi_{2,r'}^2\phi_{2,r'-\hat{\tau}}^2+4\phi_{1,r'}\phi_{2,r'}\phi_{1,r'-\hat{\tau}}\phi_{2,r'-\hat{\tau}}-\phi_{1,r'}^2\phi_{2,r'-\hat{\tau}}^2-\phi_{1,r'-\hat{\tau}}^2\phi_{2,r'}^2\right)$$

$$\frac{\delta}{\delta\phi_{1,r}}\left(\phi_{1,r'}^2\phi_{1,r'-\hat{\tau}}^2\right)=2\phi_{1,r'}\phi_{1,r'-\hat{\tau}}^2\delta_{rr'}+2\phi_{1,r'}^2\phi_{1,r'-\tau}\delta_{r,r'-\hat{\tau}}$$

$$=2\phi_{1,r}\phi_{1,r-\hat{\tau}}^2+2\phi_{1,r}\phi_{1,r+\hat{\tau}}^2$$

$$\frac{\delta}{\delta\phi_{1,r}}\left(\phi_{2,r'}^2\phi_{2,r'-\hat{\tau}}^2\right)=0$$

$$\frac{\delta}{\delta\phi_{1,r}}\left(4\phi_{1,r'}\phi_{2,r'}\phi_{1,r'-\hat{\tau}}\phi_{2,r'-\hat{\tau}}\right)=4\phi_{2,r'}\phi_{1,r'-\hat{\tau}}\phi_{2,r'-\hat{\tau}}\delta_{rr'}$$

$$+4\phi_{1,r'}\phi_{2,r'}\phi_{2,r'-\tau}\delta_{rr'-\hat{\tau}}=4\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}}+4\phi_{1,r+\hat{\tau}}\phi_{2,r+\hat{\tau}}\phi_{2,r}$$

$$\frac{\delta}{\delta\phi_{1,r}}\left(-\phi_{1,r'}^2\phi_{2,r'-\hat{\tau}}^2\right)=-2\phi_{1,r'}\phi_{2,r'-\hat{\tau}}^2\delta_{rr'}=-2\phi_{1,r}\phi_{2,r-\hat{\tau}}^2$$

$$\frac{\delta}{\delta\phi_{1,r}}\left(-\phi_{1,r'-\hat{\tau}}^2\phi_{2,r'}^2\right)=-2\phi_{1,r'-\tau}\phi_{2,r'}^2\delta_{rr'-\hat{\tau}}=-2\phi_{1,r}\phi_{2,r+\hat{\tau}}^2$$

all together:

$$=2\phi_{1,r}\phi_{1,r-\hat{\tau}}^2+2\phi_{1,r}\phi_{1,r+\hat{\tau}}^2+4\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}}+4\phi_{1,r+\hat{\tau}}\phi_{2,r+\hat{\tau}}\phi_{2,r}$$
$$-2\phi_{1,r}\phi_{2,r-\hat{\tau}}^2-2\phi_{1,r}\phi_{2,r+\hat{\tau}}^2$$

$$=2\phi_{1,r}\left(\phi_{1,r-\hat{\tau}}^2+\phi_{1,r+\hat{\tau}}^2\right)+4\phi_{2,r}\left(\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}}+\phi_{1,r+\hat{\tau}}\phi_{2,r+\hat{\tau}}\right)$$
$$-2\phi_{2,r}\left(\phi_{2,r-\hat{\tau}}^2+\phi_{2,r+\hat{\tau}}^2\right)$$

$$=2\phi_{1,r}\left(\phi_{1,r+\hat{\tau}}^2+\phi_{1,r-\hat{\tau}}^2\right)+4\phi_{2,r}\left(\phi_{2,r+\hat{\tau}}\phi_{1,r+\hat{\tau}}+\phi_{2,r-\hat{\tau}}\phi_{1,r-\hat{\tau}}\right)$$
$$-2\phi_{1,r}\left(\phi_{2,r+\hat{\tau}}^2+\phi_{2,r-\hat{\tau}}^2\right)$$

CHECK

$$(C+D)_{a\to 1} = \frac{\delta}{\delta\Phi_{1,r}}\left[\overbrace{\sum_{a=1}^{2}\sum_{b=1}^{2} 2i\,\varepsilon_{ab}\left(\Phi_{a,r}^{2}\Phi_{a,r-\hat{\tau}}\Phi_{b,r-\hat{\tau}} - \Phi_{a,r}\Phi_{b,r}\Phi_{b,r-\hat{\tau}}^{2}\right)}^{\mathrm{Im}(S_\lambda)(a/4)}\right]$$

$$(C+D)_{a\to 1} = 2i\sum_{b=1}^{2}\varepsilon_{1b}\Big[\Phi_{b,r}\Phi_{1,r+\hat{\tau}}^{2} - \Phi_{b,r}\Phi_{b,r+\hat{\tau}}^{2} + 2\Phi_{1,r}\Phi_{1,r-\hat{\tau}}\Phi_{b,r-\hat{\tau}}$$
$$- \Phi_{b,r}\Phi_{b,r-\hat{\tau}}^{2} + \Phi_{b,r}\Phi_{1,r-\hat{\tau}}^{2} + 2\Phi_{1,r}\Phi_{1,r+\hat{\tau}}\Phi_{b,r+\hat{\tau}}\Big]$$

$$= 2i\Big[\Phi_{2,r}\Phi_{1,r+\hat{\tau}}^{2} - \Phi_{2,r}\Phi_{2,r+\hat{\tau}}^{2} + 2\Phi_{1,r}\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}} - \Phi_{2,r}\Phi_{2,r-\hat{\tau}}^{2}$$
$$+ \Phi_{2,r}\Phi_{1,r-\hat{\tau}}^{2} + 2\Phi_{1,r}\Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}}\Big]$$

$$= 2i\Big[\Phi_{2,r}\left(\Phi_{1,r+\hat{\tau}}^{2} + \Phi_{1,r-\hat{\tau}}^{2}\right) - \Phi_{2,r}\left(\Phi_{2,r+\hat{\tau}}^{2} + \Phi_{2,r-\hat{\tau}}^{2}\right)$$
$$+ 2\Phi_{1,r}\left(\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}} + \Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}}\right)\Big]$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left[\sum_{a=1}^{2}\sum_{b=1}^{2} 2i\,\varepsilon_{ab}\left(\Phi_{a,r'}^{2}\Phi_{a,r'-\hat{\tau}}\Phi_{b,r'-\hat{\tau}} - \Phi_{a,r'}\Phi_{b,r'}\Phi_{b,r'-\hat{\tau}}^{2}\right)\right]$$

$$= 2i\frac{\delta}{\delta\Phi_{1,r}}\left[\Phi_{1,r'}^{2}\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\hat{\tau}} - \Phi_{1,r'}\Phi_{2,r'}\Phi_{2,r'-\hat{\tau}}^{2} - \Phi_{2,r'}^{2}\Phi_{2,r'-\hat{\tau}}\Phi_{1,r'-\hat{\tau}} + \Phi_{2,r'}\Phi_{1,r'}\Phi_{1,r'-\hat{\tau}}^{2}\right]$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(\Phi_{1,r'}^{2}\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\hat{\tau}}\right) = \Phi_{1,r'}^{2}\Phi_{2,r'-\hat{\tau}}\,\delta_{r r'-\hat{\tau}} + 2\Phi_{1,r'}\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\tau}\,\delta_{rr'}$$
$$= \Phi_{1,r+\hat{\tau}}^{2}\Phi_{2,r} + 2\Phi_{1,r}\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}}$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(-\Phi_{1,r'}\Phi_{2,r'}\Phi_{2,r'-\hat{\tau}}^{2}\right) = -\Phi_{2,r'}\Phi_{2,r'-\hat{\tau}}^{2}\,\delta_{rr'} = -\Phi_{2,r}\Phi_{2,r-\hat{\tau}}^{2}$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(-\Phi_{2,r'}^{2}\Phi_{2,r'-\hat{\tau}}\Phi_{1,r'-\hat{\tau}}\right) = -\Phi_{2,r'}^{2}\Phi_{2,r'-\hat{\tau}}\,\delta_{r,r'-\hat{\tau}} = -\Phi_{2,r}\Phi_{2,r+\hat{\tau}}^{2}$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(\Phi_{2,r'}\Phi_{1,r'}\Phi_{1,r'-\hat{\tau}}^{2}\right) = \Phi_{1,r'}\Phi_{2,r'}\left(2\Phi_{1,r'-\hat{\tau}}\right)\delta_{rr'-\hat{\tau}} + \Phi_{2,r'}\Phi_{1,r'-\hat{\tau}}^{2}\,\delta_{rr'}$$
$$= 2\Phi_{1,r}\Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}} + \Phi_{2,r}\Phi_{1,r-\hat{\tau}}^{2}$$

all together:

$$= 2i\Big[\phi_{1,r+\hat{\tau}}^2 \phi_{2,r} + 2\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}^2 - \phi_{2,r}\phi_{2,r+\hat{\tau}}^2$$
$$+ 2\phi_{1,r}\phi_{1,r+\hat{\tau}}\phi_{2,r+\hat{\tau}} + \phi_{2,r}\phi_{1,r-\hat{\tau}}^2\Big]$$

$$= 2i\Big[\phi_{2,r}\left(\phi_{1,r+\hat{\tau}}^2 + \phi_{1,r-\hat{\tau}}^2\right) - \phi_{2,r}\left(\phi_{2,r-\hat{\tau}}^2 + \phi_{2,r+\hat{\tau}}^2\right)$$
$$+ 2\phi_{1,r}\left(\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} + \phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}}\right)\Big]$$

$$= 2i\Big[\phi_{2,r}\left(\phi_{1,r+\hat{\tau}}^2 + \phi_{1,r-\hat{\tau}}^2\right) - \phi_{2,r}\left(\phi_{2,r+\hat{\tau}}^2 + \phi_{2,r-\hat{\tau}}^2\right)$$
$$+ 2\phi_{1,r}\left(\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} + \phi_{1,r+\hat{\tau}}\phi_{2,r+\hat{\tau}}\right)\Big] \quad \checkmark$$

Observables $\rightarrow V = (aN_x)^d$ spatial volume

$$\hat{n} = \frac{1}{V}\hat{N} \quad , \quad \hat{N} = \frac{-\partial}{\partial(\beta\mu)}\ln Z = \frac{\partial}{\partial(\beta\mu)} S_{lat}$$

$$\hat{L}_z = \frac{-\partial}{\partial(\beta\omega_z)}\ln Z = \frac{\partial}{\partial(\beta\omega_z)} S_{lat}$$

$$\Gamma(\ell) = \frac{1}{2\pi}\oint_{x\in\ell} dx\,(\theta_{t,x+\hat{j}} - \theta_{t,x}) \, , \quad \theta_{t,x} = \tan^{-1}\left(\frac{\mathrm{Im}[\phi_{t,x}]}{\mathrm{Re}[\phi_{t,x}]}\right)$$

$$\langle E\rangle = \frac{1}{V}\sum_r \epsilon_r , \quad \epsilon_r = \frac{-\partial}{\partial\beta}\ln Z = \frac{\partial}{\partial\beta}S_{lat} = \frac{1}{\beta}S_{lat}$$

$$\beta = (d\tau N_\tau)^{-1} \rightarrow \beta x = \frac{x}{d\tau N_\tau} \qquad \frac{\partial(\beta x)}{\partial x} = \beta \rightarrow \beta\,\partial x = \partial(\beta x)$$

$$\rightarrow \frac{\partial}{\partial(\beta x)} = \frac{1}{d\tau N_\tau}\frac{\partial}{\partial x}$$

$$\hat{N} = \frac{\partial}{\partial(\beta\mu)}S_{lat} = \frac{\partial}{\partial(\beta\mu)}S_\mu$$ (S_μ is the only part of S that depends on μ)

$$= \frac{\partial}{\partial(\beta\mu)}\sum_r a^d \sum_{a=1}^{2}\frac{1}{2}\left[\phi_{a,r}^2 - e^{d\tau\mu}\phi_{a,r}\phi_{a,r-\hat{\tau}} - i e^{d\tau\mu}\sum_{b=1}^{2}\varepsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right]$$

$$= \sum_r \frac{a^d}{d\tau N_\tau}\sum_{a=1}^{2}\frac{1}{2}\frac{\partial}{\partial\mu}\left[\phi_{a,r}^2 - e^{d\tau\mu}\phi_{a,r}\phi_{a,r-\hat{\tau}} - i e^{d\tau\mu}\sum_{b=1}^{2}\varepsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right]$$

$$= \frac{(N_x a)^d}{2N_\tau}\sum_r\sum_{a=1}^{2}\left[-d\tau e^{d\tau\mu}\phi_{a,r}\phi_{a,r-\hat{\tau}} - i\,d\tau e^{d\tau\mu}\sum_{b=1}^{2}\varepsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right]$$

$$= \frac{-1}{2}\frac{(N_x a)^d}{d\tau N_\tau}(N_\tau d\tau)e^{d\tau\mu}\sum_r\left[\sum_{a=1}^{2}\phi_{a,r}\phi_{a,r-\hat{\tau}} - i\sum_{a,b=1}^{2}\varepsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right]$$

$$\boxed{\hat{n} = -\frac{1}{2}e^{d\tau\mu}\sum_r\sum_{a,b=1}^{2}\left(\delta_{ab}\phi_{a,r}\phi_{a,r-\hat{\tau}} + i\varepsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right)}$$

$$S_\mu = \sum_r a^d\sum_{a=1}^{2}\frac{1}{2}\left[\phi_{a,r}^2 - e^{d\tau\mu}\phi_{a,r}\phi_{a,r-\hat{\tau}} - i e^{d\tau\mu}\sum_{b=1}^{2}\varepsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right]$$

$$* \quad \hat{n} = \frac{1}{(N_x a)^d}\left[S_\mu - \sum_r a^d\sum_{a=1}^{2}\frac{1}{2}\phi_{a,r}^2\right] \qquad \boxed{\hat{N} = S_\mu - \sum_r \frac{a^d}{2}\sum_{a=1}^{2}\phi_{a,r}^2}$$

$$\hat{L}_z = \frac{\partial}{\partial(\beta\omega_z)} S_{lat} = \frac{\partial}{\partial(\beta\omega_z)} S_\omega \quad \text{(same argument)}$$

$$\hat{L}_z = \frac{1}{d\tau N_\tau} \frac{\partial}{\partial \omega_z} \sum_r a^d \sum_{a=1}^{2} \Big[\frac{d\tau\,\omega_z}{2} \sum_{b=1}^{2} \varepsilon_{ab} \Big(\big(\tfrac{y}{a} - \tfrac{x}{a}\big) \phi_{a,r} \phi_{b,r-\hat{\tau}} + \tfrac{x}{a} \phi_{a,r} \phi_{b,r-\hat{y}-\hat{\tau}}$$

$$- \tfrac{y}{a} \phi_{a,r} \phi_{b,r-\hat{x}-\hat{\tau}} \Big) + \frac{i\,d\tau\,\omega_z}{2} \Big(\big(\tfrac{x}{a} - \tfrac{y}{a}\big) \phi_{a,r} \phi_{a,r-\hat{\tau}} - \tfrac{x}{a} \phi_{a,r} \phi_{a,r-\hat{y}-\hat{\tau}}$$

$$+ \tfrac{y}{a} \phi_{a,r} \phi_{a,r-\hat{x}-\hat{\tau}} \Big) \Big]$$

since S_ω is linear in ω_z, we can rewrite this:

$$\boxed{\hat{L}_z = \frac{1}{d\tau N_\tau} \frac{1}{\omega_z} S_\omega}$$

$$= \frac{1}{d\tau N_\tau} \sum_r a^d \sum_{a=1}^{2} \Big[\frac{d\tau}{2} \sum_{b=1}^{2} \varepsilon_{ab} \Big(\big(\tfrac{y}{a} - \tfrac{x}{a}\big) \phi_{a,r} \phi_{b,r-\hat{\tau}} + \tfrac{x}{a} \phi_{a,r} \phi_{b,r-\hat{y}-\hat{\tau}}$$

$$- \tfrac{y}{a} \phi_{a,r} \phi_{b,r-\hat{x}-\hat{\tau}} \Big) + \frac{i\,d\tau}{2} \Big(\big(\tfrac{x}{a} - \tfrac{y}{a}\big) \phi_{a,r} \phi_{a,r-\hat{\tau}} - \tfrac{x}{a} \phi_{a,r} \phi_{a,r-\hat{y}-\hat{\tau}}$$

$$+ \tfrac{y}{a} \phi_{a,r} \phi_{a,r-\hat{x}-\hat{\tau}} \Big) \Big]$$

$$\boxed{\phi^*\phi = \frac{1}{2} \sum_r a^d \sum_{a=1}^{2} \phi_a^2} \longrightarrow S_\mu = \hat{N} + \phi^*\phi$$

$$V_{trap} = \frac{\partial}{\partial(\beta\omega_{tr})} S_{lat} = \frac{\partial}{\partial(\beta\omega_{tr})} S_{trap}$$

$$V_{tr} = \frac{1}{d\tau N_\tau} \frac{\partial}{\partial \omega_{tr}} \sum_r a^d \frac{m a^2}{4 d\tau} (d\tau\,\omega_{tr})^2 \big(\tfrac{x}{a} - \tfrac{y}{a}\big)^2 \sum_{a=1}^{2} \Big(\phi_{a,r} \phi_{a,r-\hat{\tau}} + i \sum_{b=1}^{2} \varepsilon_{ab} \phi_{a,r} \phi_{b,r-\hat{\tau}} \Big)$$

$$= \frac{1}{d\tau N_\tau} \sum_r a^d \frac{m a^2}{4 d\tau} (2\, d\tau^2 \omega_{tr}) \big(\tfrac{x}{a} - \tfrac{y}{a}\big)^2 \sum_{a=1}^{2} \Big(\phi_{a,r} \phi_{a,r-\hat{\tau}} + i \sum_{b=1}^{2} \varepsilon_{ab} \phi_{a,r} \phi_{b,r-\hat{\tau}} \Big)$$

$$\boxed{\hat{V}_{tr} = \frac{2}{d\tau N_\tau \omega_{tr}} S_{trap}}$$

$$V_{int} = \frac{\partial}{\partial(\beta\lambda)} S_{lat} = \frac{\partial}{\partial(\beta\lambda)} S_{int}$$

$$= \frac{1}{d\tau N_\tau} \frac{\partial}{\partial\lambda} \sum_r a^d \frac{d\tau\lambda}{4} \sum_{a=1}^{2}\sum_{b=1}^{2} \Big[2\phi_{a,r}\phi_{a,r-\hat\tau}\phi_{b,r}\phi_{b,r-\hat\tau} - \phi_{a,r}^2 \phi_{b,r-\hat\tau}^2$$

$$+ 2i\epsilon_{ab}\phi_{a,r}^2 \phi_{a,r-\hat\tau}\phi_{b,r-\hat\tau} - \phi_{a,r}\phi_{b,r}\phi_{a,r-\hat\tau}^2 \Big]$$

$$= \frac{1}{d\tau N_\tau} \sum_r a^d \frac{d\tau}{4} \sum_{a=1}^{2}\sum_{b=1}^{2} \Big[2\phi_{a,r}\phi_{a,r-\hat\tau}\phi_{b,r}\phi_{b,r-\hat\tau} - \phi_{a,r}^2 \phi_{b,r-\hat\tau}^2$$

$$+ 2i\epsilon_{ab}\phi_{a,r}^2 \phi_{a,r-\hat\tau}\phi_{b,r-\hat\tau} - \phi_{a,r}\phi_{b,r}\phi_{a,r-\hat\tau}^2 \Big]$$

$$\boxed{\hat{V}_{int} = \frac{1}{d\tau N_\tau \lambda} S_{int}}$$

Kinetic energy, $\hat{T}$, is just S_∇

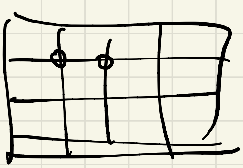
Write S_{lat} in terms of these other observables:

$$S_{lat} = S_\mu + S_\nabla - S_{trap} - S_\omega + S_{int}$$

$$= \hat{N} + \phi^*\phi + \hat{T} - \frac{N_\tau}{2}\tilde{\omega}_{tr}\hat{V}_{tr} - N_\tau \tilde{\omega}_z \hat{L}_z + N_\tau \tilde{\lambda}\hat{V}_{int}$$

$$\hat{n} = -\frac{1}{2} e^{d\tau\mu} \sum_{\vec{x},\tau} \underline{n_r(\vec{x},\tau)}$$

$$n_r = \sum_{a,b=1} \left(\delta_{ab}\phi_{a,r}\phi_{a,r-\hat\tau} + i\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat\tau} \right) \qquad n_r(x,y)$$



$$n_r(x,y) = \frac{1}{N_\tau}\sum_\tau n_r(x,y,\tau)$$

$$n_r(x,y,\tau = N_\tau/2)$$

Circulation

$$\Gamma(\ell) = \frac{1}{2\pi} \oint_{\ell\times\ell} dx \left(\theta_{t,x+\hat{j}} - \theta_{t,x}\right) \qquad \theta_{t,x} = \arctan\left(\frac{\mathrm{Im}[\Phi_{t,x}]}{\mathrm{Re}[\Phi_{t,x}]}\right)$$

$$\phi = \frac{1}{\sqrt{2}}\left(\mathrm{Re}[\phi] + i\,\mathrm{Im}[\phi]\right) = \frac{1}{\sqrt{2}}\left[\phi_1^R + i\phi_1^I + i\phi_2^R - \phi_2^I\right]$$

So $\mathrm{Re}[\phi] = \phi_1^R - \phi_2^I$ & $\mathrm{Im}[\phi] = \phi_1^I + \phi_2^R$

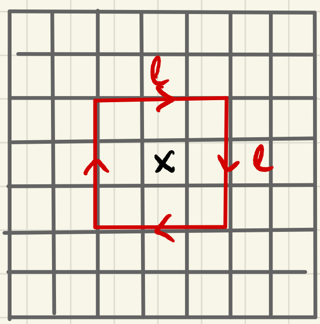
and $\theta_{t,x} = \arctan\left(\dfrac{\phi_1^I + \phi_2^R}{\phi_1^R - \phi_2^I}\right)$

*note a potential issue: singularities can occur if $\phi_1^R = \phi_2^I$. This is extremely unlikely to happen.

We convert our integral to a sum, so we can calculate it on a lattice

$$\int dx \rightarrow \sum_x a$$

We choose an $\ell\times\ell$ sub-section of the lattice centered around the middle and proceed around it in a clockwise fashion.



$$\sum_{x\in\ell\times\ell} a\left(\theta_{t,x+\hat{j}} - \theta_{t,x}\right)$$

where $x+\hat{j}$ is the next site on our $\ell\times\ell$ loop

So $$\Gamma_t(\ell) = \frac{a}{2\pi}\sum_{x\in\ell\times\ell}\left[\arctan\left(\frac{\phi^I_{1,x+\hat{j}} + \phi^R_{2,x+\hat{j}}}{\phi^R_{1,x+\hat{j}} - \phi^I_{2,x+\hat{j}}}\right) - \arctan\left(\frac{\phi^I_{1,x} + \phi^R_{2,x}}{\phi^R_{1,x} - \phi^I_{2,x}}\right)\right]$$

we then average this over our time lattice

$$\Gamma(\ell) = \frac{a}{2\pi N_\tau}\sum_{\tau,x\in\ell\times\ell}\left[\arctan\left(\frac{\phi^I_{1,\tau,x+\hat{j}} + \phi^R_{2,\tau,x+\hat{j}}}{\phi^R_{1,\tau,x+\hat{j}} - \phi^I_{2,\tau,x+\hat{j}}}\right) - \arctan\left(\frac{\phi^I_{1,\tau,x} + \phi^R_{2,\tau,x}}{\phi^R_{1,\tau,x} - \phi^I_{2,\tau,x}}\right)\right]$$

Energy

Traditionally, in stat mech calculations, you can calculate the energy by taking a derivative of the partition function wrt β.

We have to put our action in a different format to do this, as $\beta = N_\tau d\tau$ and we've already absorbed $d\tau$ into our parameters.

So back to an early form of the action:

$$S = \sum_{\vec{x},\tau}\Big[a^d \phi^*_r \phi_r - a^d e^{d\tau\mu}\phi^*_r\phi_{r-\tau} - \frac{a^{d-2}d\tau}{2m}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}} - 2\phi^*_r\phi_r + \phi^*_r\phi_{r+\hat{j}}\right)$$
$$- \frac{ma^d d\tau}{2}\omega_{tr}^2 r_\perp^2 \phi^*_r\phi_{r-\tau} + i\omega_z a^{d-1} d\tau\left(x\phi^*_r\phi_{r-\hat{y}-\hat{\tau}} - x\phi^*_r\phi_{r-\hat{\tau}} - y\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}\right.$$
$$\left. + y\phi^*_r\phi_{r-\hat{\tau}}\right) + a^d d\tau\,\lambda\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2\Big]$$

multiply by $1 = \frac{N_\tau}{N_\tau}$ and take $N_\tau d\tau \to \beta$

$$S = \frac{a^d}{N_\tau}\sum_{\vec{x},\tau}\Big[N_\tau \phi^*_r \phi_r - N_\tau e^{d\tau\mu}\phi^*_r\phi_{r-\tau} - \frac{N_\tau d\tau}{a^2 2m}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}} - 2\phi^*_r\phi_r + \phi^*_r\phi_{r+\hat{j}}\right)$$
$$- \frac{m N_\tau d\tau}{2}\omega_{tr}^2 r_\perp^2 \phi^*_r\phi_{r-\tau} + i\frac{\omega_z}{a} N_\tau d\tau\left(x\phi^*_r\phi_{r-\hat{y}-\hat{\tau}} - x\phi^*_r\phi_{r-\hat{\tau}} - y\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}\right.$$
$$\left. + y\phi^*_r\phi_{r-\hat{\tau}}\right) + N_\tau d\tau\,\lambda\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2\Big]$$

$d\tau = \beta/N_\tau$

$$S = \frac{a^d}{N_\tau}\sum_{\vec{x},\tau}\Big[N_\tau \phi^*_r \phi_r - N_\tau e^{\frac{\beta\mu}{N_\tau}}\phi^*_r\phi_{r-\tau} - \frac{\beta}{a^2 2m}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}} - 2\phi^*_r\phi_r + \phi^*_r\phi_{r+\hat{j}}\right)$$
$$- \frac{m\beta}{2}\omega_{tr}^2 r_\perp^2 \phi^*_r\phi_{r-\tau} + i\frac{\omega_z}{a}\beta\left(x\phi^*_r\phi_{r-\hat{y}-\hat{\tau}} - x\phi^*_r\phi_{r-\hat{\tau}} - y\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}\right.$$
$$\left. + y\phi^*_r\phi_{r-\hat{\tau}}\right) + \beta\lambda\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2\Big]$$

we take a derivative wrt β:

$$\hat{E} = -\frac{d}{d\beta} S[\beta]$$

aside from the exponential, every term is linear in β or a constant

$$\text{so } \hat{E} = \frac{a^d}{N_\tau}\sum_{\vec{x},\tau}\Big[\cancel{N_\tau}\Big(\frac{\mu}{\cancel{N_\tau}}\Big)e^{\beta\mu/N_\tau}\phi_r^*\phi_{r-\hat{\tau}} + \frac{1}{2a^2 m}\sum_{j=1}^{d}\big(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\big)$$
$$+ \frac{m}{2}\omega_{tr}^2 r_\perp^2 \phi_r^*\phi_{r-\tau} - i\frac{\omega_z}{a}\big(x\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - x\phi_r^*\phi_{r-\hat{\tau}} - y\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + y\phi_r^*\phi_{r-\hat{\tau}}\big)$$
$$-\lambda(\phi_r^*\phi_{r-\hat{\tau}})^2\Big]$$

we should be able to write this in terms of other observables:

$$\hat{E} = \frac{a^d}{N_\tau}\sum_{\vec{x},\tau}\Big[\frac{\tilde{\mu}}{d\tau}e^{\tilde{\mu}}\phi_r^*\phi_{r-\hat{\tau}} + \frac{1}{2\tilde{m}d\tau}\sum_{j=1}^{d}\big(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\big)$$
$$+ \frac{1}{d\tau}\frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}^2\phi_r^*\phi_{r-\tau} - i\frac{\tilde{\omega}_z}{d\tau}\big(\tilde{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \tilde{x}\phi_r^*\phi_{r-\hat{\tau}} - \tilde{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \tilde{y}\phi_r^*\phi_{r-\hat{\tau}}\big)$$
$$-\frac{\tilde{\lambda}}{d\tau}(\phi_r^*\phi_{r-\hat{\tau}})^2\Big]$$

$$= \frac{1}{N_\tau d\tau}\Big[\tilde{\mu}\big(\phi_r^*\phi_r - S_\mu\big) - S_\nabla + S_{tr} + S_w - S_{int}\Big]$$

$$S = \sum_{\vec{x},\tau} a^d \Big[\underbrace{\phi_r^*\phi_r - e^{\tilde{\mu}}\phi_r^*\phi_{r-\hat{\tau}}}_{S_\mu} \underbrace{- \frac{1}{2\tilde{m}}\sum_{j=1}^{d}\big(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\big)}_{S_\nabla}$$
$$\underbrace{-\frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi_r^*\phi_{r-\hat{\tau}}}_{S_{tr}} + \underbrace{i\tilde{\omega}_z\big(\tilde{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \tilde{x}\phi_r^*\phi_{r-\hat{\tau}} - \tilde{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \tilde{y}\phi_r^*\phi_{r-\hat{\tau}}\big)}_{-S_w} + \underbrace{\tilde{\lambda}(\phi_r^*\phi_{r-\hat{\tau}})^2}_{S_{int}}\Big]$$

Next, we complexify our real fields, ϕ_a $(a=1,2)$, $\phi_a \to \phi_a^R + i\phi_a^I$

Analysis

$N_x \times N_x \times N_\tau$ lattice and N_s Steps

we need:

- density profiles → one value at each site on $N_x \times N_x$ lattice at each N_s $(\underbrace{N_x \times N_x \times N_s}_{N_x \times N_x \times N_\tau \times N})$

    dp_N_s.h5

- observables (x 8) → 8 values at each N_s

    obs_N_s.h5

HDF5 in AMReX

amrex/Src/Base/AMReX_PlotFileUtil.cpp

useful: numpy.reshape function ⇒ lookup "order" flag normally:

dict [observable name] [Langevin step] [value]

Re density TL = 1000 #

$\hookrightarrow \sum_{x,y,z} n^R(x,y,z)$

now:

dict [ka Re] [Langevin step] [value]

$\hookrightarrow$ array of size

$N_x N_x N_z$

Proposed solution: when testing for errors in Ka, eta, only look at a few Langevin steps (to save memory)

→ histogram the drift function to look for tails

density profiles → $N_x \times N_x$

ERCAP Proposal

Strategy: propose 2x as much time as you think you'll need

N_t range: 320–480

N_x range: 81x81 or 101x101 ← justify using ω_{tr}^2

$nL = 10^6$, save every 100 steps

Storage space?

include 2-3 runs w/ very large N_x to get density profiles (to see vortices)

↳ argue for the importance of supercomputing for resolving these lattices

try to come up with an estimate for how many sites you need to see vortices

maybe plot the density profiles for a trapped, not rotating system to show how the trap confines most of the fluid into a smaller area in the center.

Check old free gas data → does it match new stuff w/ right parameters?

Outstanding questions

- [x] Program → nuclear, HEP, condensed matter?

Personnel:

ask Rich

- [x] Should Rich be included as a senior investigator

Funding

- [x] DOE funding - who do I pick?
  Exascale grant number, university funding

Security
- none

Project Details

- [ ] how important is the website?

Resources

- [x] how much justification is needed for the space request?
  - I'm doing a statistical physics calculation, which requires saving lots of data for analysis.
  - 700 TB? 300 TB? what is too much?
  "we don't normally restrict people on how much space they

Codes

- [x] Should I link to my own GitHub? leave off

Scaling notes

$a=1$, $\lambda_T = \sqrt{2\pi\beta} = \sqrt{\frac{2\pi}{N_\tau d\tau}}$, $\lambda_{HO} = \frac{1}{\sqrt{\omega_{tr}}}$

$a \ll \lambda_T$, $\omega_{tr} \ll 1$

research notes 1/28/2019

⚠ this is wrong! $\beta = \frac{1}{T} = N_\tau d\tau$

$a \ll \lambda_T \ll N_x$

C. Shill defense 2/14/2019

ω_{tr}^2 should be 0.001 − 0.005 (justification??)

μ_{crit} w/ V_{trap} present is $\frac{d}{2}\pi\omega_{tr}$

plot $\langle \hat{n} \rangle$ v. ω_{tr} for each μ

research notes 2/15/2019

renormalization for λ → use virial coefficients

research notes 2/20/2019

$a=1 \ll a_{HO} = \frac{1}{\sqrt{\omega_{tr}}}$ for $\lambda=0$

$\lambda>0$ will change this

research notes 3/7/19

$\lambda_n \sim n^{-1/2}$.

$\lambda_T \gg \lambda_n$ deep quantum regime (vortices)

$\lambda_T \ll \lambda_n$ classical regime

we want $\beta\omega_{tr} \ll 1$, $\beta\omega_{tr} = (a_{tr}/\lambda_T)^2$

research notes 6/4/2019

Stopped July 2019

Reading List

- [ ] Timo Lähde & Tom Liu paper on HMC and discretization

From Rothe's Lattice Gauge Theories:

$\boxed{\text{Requirement for numerical calculations:}}$

the scales relevant to the particular problem under investigation should be large compared to the lattice spacing, but small compared to the lattice size