NonRelativistic Rotating Bosons via Complex Langevin

Casey E. Berger

September 9, 2020

Contents

Chapter 1

Notes To Self and To-Do List

1.1 Updates to make in the notes

1.1.1 TO DO

- Determine a more natural way to write the interaction term in Appendix A.2
- Update notes to be more explicit about multidimensional cases... We're using $d = 2$, but we want to generalize this. Is the best way to write it generally? Or to derive it in all three dimensions seperately?
- Double check the derivation of the drift functions (in Appendix A.4)
- Double check the derivation of the observables (in Appendix A.5)

1.1.2 DONE

- Double check the discretization of $S_{\tau,r} = \phi_r^* \phi_r - \phi_r^* e^{d\tau\mu} \phi_{r-\hat{\tau}}$
- Double check the discretization of $S_{\nabla,r} = \frac{d}{m} \phi_r^* \phi_r - \frac{1}{2m} \sum_{i=\pm x,y} \phi_r^* \phi_{r+\hat{i}}$
- Double check the discretization of $S_{\mathrm{tr},r} = \frac{m}{2} \omega_{\mathrm{trap}}^2 (x^2 + y^2) \phi_r^* \phi_{r-\hat{\tau}}$
- Double check the discretization of $S_{\omega,r} = i\omega_z \left[(x - y) \phi_r^* \phi_{r-\hat{\tau}} - \widetilde{x} \phi_r^* \phi_{r-\hat{y}-\hat{\tau}} + \widetilde{y} \phi_r^* \phi_{r-\hat{x}-\hat{\tau}} \right]$
- Double check the discretization of $S_{\mathrm{int},r} = \lambda \left(\phi_r^* \phi_{r-\hat{\tau}} \right)^2$
- Copy derivation from GoodNotes – discretization updated (the version in this doc is correct, the new one is just more clear where the factors of a go, which don't matter for now because we've set $a = 1$)
- check signs in AppendixA.1
- Double check the math when you write $\phi = \phi_1 + i\phi_2$ (in Appendix A.2)

1.2 General Notes and Reminders

*note that all interaction in this system is entirely governed by λ

1.2.1 Stuff to look into

- From Ref. [**?**] (see master lit review for citation):

  "A balance between the kinetic energy [**see ref in lit review**] and the interaction energy [**see ref in lit review**], viz. Eq. (10), of a BEC leads to a typical length scale called coherence length [**see ref in lit review**] for a weakly interacting BEC [11]. This quantity is relevant for super-fluid effects. For instance, it provides the typical size of the core of quantized vortices [7, 11, 13]. "

- From ref. [**?**] (see master lit review for citation):

  "The vortex nucleation process in a rotating scalar Bose- Einstein condensate provides a unique model system to study the emergence of a quantum phase transition from the microscopic few-body regime to the thermodynamic limit. Even in the presence of interactions, the nucleation process of the first vortex is associated with exact linearity of the ground state energy as a function of angular momentum which leads to a discontinuous transition between the non-rotating ground state and the unit vortex."

1.2.2 Ideas

Might it be feasible to program a GPU for the CL evolution? It could then simultaneously update the lattice very quickly.

Other methods:

- Path Optimization Method (Ohnishi et al)
- Tensor Network Systems (Banuls et al)
- World Line Methods??

1.2.3 Goals:

1. Reproduce results of Gert Aarts paper - COMPLETED
2. Reproduce the free gas - COMPLETED (Not for AMReX yet)
3. Check against virial theorems for trapped bosons - IN PROGRESS

   I'm not sure that I can check the virial theorem here... how would I calculate kinetic energy only? Also we have rotation, interaction, and trap, not just trap.
4. Compare with Hayata/Yamamoto paper - IN PROGRESS
5. Some new directions to take it:
   - Fixed values of angular momentum
   - Polarized bosons
   - Mass-imbalanced bosons
   - Efimov effect and 3-body forces
   - Some interesting problems in $2 + 1$ dimensions

- Virial coefficients for bosons at unitarity (maybe talk to Chris)
- Auxiliary field to represent interaction - will allow us to study phase transitions as a function of N_f ***
- Can we ever find a way to vary the dimensionality continuously?

Chapter 2

The Method: Complex Langevin

2.1 The rotating bosonic gas

Hard wall boundary conditions in space and periodic (bosonic) in time. Nonrelativistic dispersion, harmonic trapping potential centered at the midpoint of the lattice, rotation about the midpoint of the lattice in 2 dimensions, and contact interaction (delta function).

For rotation, we need at least $2+1$ dimensions. I will derive all equations for the general $d+1$ dimensional case, but initial computational tests will all be in $2+1$ dimensions.

Our path integral for the system is

$$\mathcal{Z} = \int \mathrm{d}\phi e^{-S[\phi]} \tag{2.1}$$

where our action is defined in the next section.

2.1.1 The Action

The action for a trapped, rotating, and interacting non-relativistic system in $d+1$ Euclidean dimensions follows this general formula:

$$S = \int \mathrm{d}^d x \, \mathrm{d}\tau \left[\phi^* \left(\mathcal{H} - \mu - V_{\text{trap}} - \omega_z L_z\right)\phi + \lambda(\phi^*\phi)^2\right]. \tag{2.2}$$

Specifically, in two spatial dimensions, this becomes

$$S = \int \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau \left[\phi^* \left(\mathcal{H} - \mu - \frac{m}{2}\omega_{\text{trap}}^2 r_\perp^2 - \omega_z L_z\right)\phi + \lambda(\phi^*\phi)^2\right] \tag{2.3}$$

where $\mathcal{H} = \partial_\tau - \frac{\nabla^2}{2m}$ and λ represents a contact interaction coupling. The trapping potential is harmonic, $\omega_{\text{trap}} r_\perp^2$ where $r_\perp^2 = x^2 + y^2$, and the angular momentum is defined in terms of the quantum mechanical operator, $\omega_z L_z = i\omega_z(x\partial_y - y\partial_x)$. For both the trap and the angular momentum, x and y are measured relative to the origin.

To take this to a lattice representation, we must discretize the space. The origin becomes the center of the lattice, and therefore x and y are measured relative to the point $(N_x/2, N_x/2)$ on the lattice.

Using a backward-difference derivative and denoting the position on the $2+1$ dimensional lattice as r, we define: $\partial_j \phi = \frac{1}{a}(\phi_{r-\hat{j}} - \phi_r)$ and $\partial_j^2 \phi = \frac{1}{a^2}(\phi_{r-\hat{j}} - 2\phi_r + \phi_{r-\hat{j}})$. We combine $\partial_\tau - \mu$ in the lattice representation and similarly represent the angular momentum, interaction, and external trap as external gauge fields in order to avoid divergences in the continuum limit (see Appendix A.1 for details and justification of these steps). Finally, using spatial lattice site separation $a = 1$ and temporal lattice spacing $d\tau$, our lattice action becomes (sum over $r = (x, y, \tau)$ implied):

$$\textit{see appendix} \tag{2.4}$$

Note that for $d < 2$, we omit the rotational term, as it requires at least two spatial dimensions. We only consider cases of dimensionality less than two for testing our algorithm against exactly-soluble cases such as the free gas.

To simplify our future work, let's write S as a sum of the different contributions to the action:

$$S = \sum_r (S_{\tau,r} + d\tau S_{\nabla,r} - d\tau S_{tr,r} - d\tau S_{\omega,r} + d\tau S_{\text{int},r}) \tag{2.5}$$

with

$$\begin{aligned}
S_{\tau,r} &= \phi_r^* \phi_r - \phi_r^* e^{d\tau\mu} \phi_{r-\hat{\tau}} & (2.6)\\
S_{\nabla,r} &= \frac{d}{m}\phi_r^*\phi_r - \frac{1}{2m}\sum_{i=1}^{d} \phi_r^* \left(\phi_{r+\hat{i}} + \phi_{r-\hat{i}}\right) & (2.7)\\
S_{tr,r} &= \frac{m}{2}\omega_{\text{trap}}^2 (x^2 + y^2)\phi_r^*\phi_{r-\hat{\tau}} & (2.8)\\
S_{\omega,r} &= i\omega_z \left[(x-y)\phi_r^*\phi_{r-\hat{\tau}} - \widetilde{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} + \widetilde{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}\right] & (2.9)\\
S_{\text{int},r} &= \lambda \left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2 . & (2.10)
\end{aligned}$$

where $\widetilde{x}$ and $\widetilde{y}$ are our x and x coordinates shifted by the center of the trap:

$$\begin{aligned}
\widetilde{x} &= x - \frac{N_x - 1}{2}\\
\widetilde{y} &= y - \frac{N_x - 1}{2},
\end{aligned}$$

And we number lattice sites from 0 to $N_x - 1$. From here on, we will be working with these individual contributions to the action.

2.2 The Langevin Equations

In order to treat this action composed of complex-valued fields, we use a method called complex Langevin (CL). This method uses a stochastic evolution of the complex fields in a fictitious time – Langevin time – in order to produce sets of solutions distributed according to the weight e^{-S}. Just as standard Monte Carlo methods operate by sampling from the distribution e^{-S}, this method allows us to stochastically evaluate observables whose physical behavior is governed by our action, S.

First, we must write our complex fields as a complex sum of two real fields (i.e. $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. This is worked out in Appendix A.2 for all the contributions to our action. We then complexify the fields a second time and evolve the real and imaginary part of each field according to the complex Langevin equations, a set of coupled stochastic differential equations shown here:

$$\begin{aligned}
\phi_{a,r}^R(n+1) &= \phi_{a,r}^R(n) + \epsilon K_{a,r}^R(n) + \sqrt{\epsilon}\eta_{a,r}(n) & (2.11)\\
\phi_{a,r}^I(n+1) &= \phi_{a,r}^I(n) + \epsilon K_{a,r}^I(n), & (2.12)
\end{aligned}$$

where $a = 1, 2$ labels our two real fields, η is Gaussian-distributed real noise with mean of 0 and standard deviation of $\sqrt{2}$. The drift functions, K, are derived from the action:

$$
\begin{aligned}
K^R_{a,r} &= -\mathrm{Re}\left[\frac{\delta S}{\delta \phi_{a,r}}|_{\phi_a \to \phi^R_a + i\phi^I_a}\right] && (2.13)\\
K^I_{a,r} &= -\mathrm{Im}\left[\frac{\delta S}{\delta \phi_{a,r}}|_{\phi_a \to \phi^R_a + i\phi^I_a}\right] && (2.14)
\end{aligned}
$$

The derivation of the Langevin equations is worked out in Appendix A.4, and the results are shown below with the sums over a and b implied:

$$
\begin{aligned}
-K^R_{a,r} &= \phi^R_{a,r} - \frac{e^{d\tau\mu}}{2}\left(\phi^R_{a,r-\hat{\tau}} + \phi^R_{a,r+\hat{\tau}}\right) + \frac{e^{d\tau\mu}}{2}\epsilon_{ab}\left(\phi^I_{b,r-\hat{\tau}} - \phi^I_{b,r+\hat{\tau}}\right)\\
&+\frac{d\tau}{2m}\left(2d\phi^R_{a,r} - \sum_{i=\pm 1}^{d}\phi^R_{a,r+\hat{i}}\right)\\
&-\frac{d\tau\omega^2_{\mathrm{tr}}(r^2_\perp)}{4}\left(\left(\phi^R_{a,r+\hat{\tau}} + \phi^R_{a,r-\hat{\tau}}\right) - \epsilon_{ab}\left(\phi^I_{b,r+\hat{\tau}} + \phi^I_{b,r-\hat{\tau}}\right)\right)\\
&-\frac{d\tau\omega_z}{2}\left[\epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}} - \phi^R_{b,r+\hat{y}}\right) - y\left(\phi^R_{b,r-\hat{x}} - \phi^R_{b,r+\hat{x}}\right)\right] - 2(x-y)\phi^I_{a,r}\right]\\
&-\frac{\omega_z}{2}\left[x\left(\phi^I_{a,r-\hat{y}} + \phi^I_{a,r+\hat{y}}\right) - y\left(\phi^I_{a,r-\hat{x}} + \phi^I_{a,r+\hat{x}}\right)\right]\\
&+d\tau\lambda\left[\phi^R_{a,r}(\phi^R_{b,r})^2 - \phi^R_{a,r}(\phi^I_{b,r})^2 - 2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}\right] && (2.15)\\
-K^I_{a,r} &= \phi^I_{a,r} - \frac{e^{d\tau\mu}}{2}\left(\phi^I_{a,r-\hat{\tau}} + \phi^I_{a,r+\hat{\tau}}\right) - \frac{e^{d\tau\mu}}{2}\epsilon_{ab}\left(\phi^R_{b,r-\hat{\tau}} - \phi^R_{b,r+\hat{\tau}}\right)\\
&+\frac{d\tau}{2m}\left(2d\phi^I_{a,r} - \sum_{i=\pm 1}^{d}\phi^I_{a,r+\hat{i}}\right)\\
&-\frac{d\tau\omega^2_{\mathrm{tr}}(r^2_\perp)}{4}\left(\left(\phi^I_{a,r+\hat{\tau}} + \phi^I_{a,r-\hat{\tau}}\right) + \epsilon_{ab}(\phi^R_{b,r+\hat{\tau}} + \phi^R_{b,r-\hat{\tau}})\right)\\
&-\frac{d\tau\omega_z}{2}\left[\epsilon_{ab}\left(x(\phi^I_{b,r-\hat{y}} - \phi^I_{b,r+\hat{y}}) - y(\phi^I_{b,r-\hat{x}} - \phi^I_{b,r+\hat{x}})\right) + 2(x-y)\phi^R_{a,r}\right]\\
&-\frac{\omega_z}{2}\left[y\left(\phi^R_{a,r-\hat{x}} + \phi^R_{a,r+\hat{x}}\right) - x\left(\phi^R_{a,r-\hat{y}} + \phi^R_{a,r+\hat{y}}\right)\right]\\
&+d\tau\lambda\left[\phi^I_{a,r}(\phi^R_{b,r})^2 + 2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r} - \phi^I_{a,r}(\phi^I_{b,r})^2\right] && (2.16)
\end{aligned}
$$

This evolution is repeated until we have evolved for a long period in Langevin time (determined by the observation of thermalization followed by enough steps to produce good statistical error). Observables of interest can be calculated as functions of the fields at each point in Langevin time and averaged to find the expectation value.

2.3 Observables

2.3.1 Observables averaged over the volume

The observables of interest in this simulation are the density $\langle\hat{n}\rangle$, the field modulus $\langle\phi^2\phi\rangle$, the angular momentum $\langle\hat{L}_z\rangle$, the moment of inertia $\langle\hat{I}_z\rangle$,the energy $\langle\hat{E}\rangle$, and the circulation of the fields around the center of the lattice.

The sum over all lattice sites and subsequent normalization by the lattice volume $V = N_x^d N_\tau$ is implied:

$$\mathrm{Re}\langle n\rangle = \frac{e^{d\tau\mu}}{2}\left[\sum_{a,b=1}^{2}\delta_{a,b}\left(\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{b,r-\hat{\tau}}\right) - \sum_{a,b=1}^{2}\epsilon_{ab}\left(\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}} + \phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\right)\right] \tag{2.17}$$

$$\mathrm{Im}\langle n\rangle = \frac{e^{d\tau\mu}}{2}\left[\sum_{a,b=1}^{2}\delta_{a,b}\left(\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}} + \phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\right) + \sum_{a,b=1}^{2}\epsilon_{ab}\left(\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{b,r-\hat{\tau}}\right)\right] \tag{2.18}$$

$$\mathrm{Re}\langle\phi^*\phi\rangle = \frac{1}{2}\sum_{a=1}^{2}\left(\phi^R_{a,r}\phi^R_{a,r} - \phi^I_{a,r}\phi^I_{a,r}\right) \tag{2.19}$$

$$\mathrm{Im}\langle\phi^*\phi\rangle = \sum_{a=1}^{2}\phi^R_{a,r}\phi^I_{a,r} \tag{2.20}$$

$$\begin{aligned}\mathrm{Re}\langle L_z\rangle = {} & \frac{1}{2}\sum_{a=1}^{2}\left(\widetilde{x}(\phi^R_{a,r}\phi^I_{a,r-\hat{y}} + \phi^I_{a,r}\phi^R_{a,r-\hat{y}}) - \widetilde{y}(\phi^R_{a,r}\phi^I_{a,r-\hat{x}} + \phi^I_{a,r}\phi^R_{a,r-\hat{x}})\right) \\ & + \sum_{r}\sum_{a=1}^{2}(y-x)\phi^R_{a,r}\phi^I_{a,r} \\ & + \frac{1}{2}\sum_{r}\sum_{a,b=1}^{2}\epsilon_{ab}\left(\widetilde{x}(\phi^R_{a,r}\phi^R_{b,r-\hat{y}} - \phi^I_{a,r}\phi^I_{b,r-\hat{y}}) - \widetilde{y}(\phi^R_{a,r}\phi^R_{b,r-\hat{x}} - \phi^I_{a,r}\phi^I_{b,r-\hat{x}})\right).\end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathrm{Im}\langle L_z\rangle = {} & -\frac{1}{2}\sum_{a=1}^{2}\left(\widetilde{x}(\phi^R_{a,r}\phi^R_{a,r-\hat{y}} - \phi^I_{a,r}\phi^I_{a,r-\hat{y}}) - \widetilde{y}(\phi^R_{a,r}\phi^R_{a,r-\hat{x}} - \phi^I_{a,r}\phi^I_{a,r-\hat{x}})\right) \\ & -\frac{1}{2}\sum_{r}\sum_{a=1}^{2}(y-x)\left((\phi^R_{a,r})^2 - (\phi^I_{a,r})^2\right) \\ & -\frac{1}{2}\sum_{r}\sum_{a,b=1}^{2}\epsilon_{ab}\left(\widetilde{x}(\phi^R_{a,r}\phi^I_{b,r-\hat{y}} + \phi^I_{a,r}\phi^R_{b,r-\hat{y}}) - \widetilde{y}(\phi^R_{a,r}\phi^I_{b,r-\hat{x}} + \phi^I_{a,r}\phi^R_{b,r-\hat{x}})\right)\end{aligned} \tag{2.22}$$

The moment of inertia can be calculated from this by differentiating with respect to the angular frequency. Since L_z is linear in ω_z, we can simply divide by the frequency to obtain I_z.

The energy can be calculated by taking a derivative as well.

$$\hat{E} = -\frac{\partial}{\partial\beta}\ln\mathcal{Z} = \frac{\partial}{\partial\beta}S. \tag{2.23}$$

When we go to a lattice representation of the action, we write it in the following way:

$$S \to \sum_{x,\tau}\Delta x^d\Delta\tau\left[\left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu - i\omega_z(x\partial_y - y\partial_x) - \frac{m\omega^2_{\mathrm{trap}}}{2}(x^2+y^2)\right)\phi + \lambda(\phi^*\phi)^2\right]. \tag{2.24}$$

Recall that $\beta = \Delta\tau N_\tau$. That means we can also write this in terms of β:

$$S \to \frac{1}{N_\tau}\sum_{x,\tau}\Delta x^d\beta\left[\left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu - i\omega_z(x\partial_y - y\partial_x) - \frac{m\omega^2_{\mathrm{trap}}}{2}(x^2+y^2)\right)\phi + \lambda(\phi^*\phi)^2\right]. \tag{2.25}$$

Therefore, the action is linear in β, and so its derivative with respect to beta is simply S/β.

The circulation is computed along an $l \times l$ loop on the lattice:

$$\Gamma[l] = \frac{1}{2\pi} \sum_{l \times l} (\theta_{t,x+j} - \theta_{t,x}) \tag{2.26}$$

where

$$\theta_{t,x} = \tan^{-1}\left(\frac{\phi^I_{1,t,x} + \phi^R_{2,t,x}}{\phi^R_{1,t,x} - \phi^I_{2,t,x}} \right) \tag{2.27}$$

and $\theta_{t,x+j}$ is computed at the next site on the loop from $\theta_{t,x}$ for each point along the loop.

See Appendix A.5 for the full derivation of the observables.

2.3.2 Observables per site

The next phase of this project will move to larger lattices in order to calculate the density profiles of the rotating system. We expect to see the formation of vortices in the density of the fluid (localized sites of 0 density). We expect these vortices to be concentrated closer to the center of the trap. In order to observe these vortices and their structure, we need a finer lattice mesh in the center of the system. This will involve a great deal more computational intensity.

Appendices

Appendix A

NRRB Derivations

A.1 Justification for the Form of the Non-Relativistic Lattice Action

The continuum action for bosons with a non-relativistic dispersion, a rotating external potential, a non-zero chemical potential, an external trap potential, and an interaction term is as follows:

$$S = \int_V \phi^* \left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu - i\omega_z(x\partial_y - y\partial_x) - \frac{m\omega_{\text{trap}}^2}{2}(x^2+y^2) \right) \phi + \lambda \int_V (\phi^*\phi)^2. \tag{A.1}$$

To convert this to a lattice action, we must first discretize the derivatives. We will use a backwards finite difference discretization for the single derivative and a central difference approximation for the double derivative, such that:

$$\partial_i \phi_r = \frac{1}{a}\left(\phi_r - \phi_{r-\hat{i}}\right) \tag{A.2}$$

$$\nabla^2 \phi_r = \sum_i \frac{1}{a^2}\left(\phi_{r+\hat{i}} - 2\phi_r + \phi_{r-\hat{i}}\right), \tag{A.3}$$

where $r = (x, y, \tau)$ and the discretization length is a for spatial derivatives and $d\tau$ for temporal ones.

In order to treat the finite chemical potential, the external trapping potential, the rotation, and the interaction we must shift our indices on the field that is acted on by these external parameters by one step in the time direction. This is to make these potentials gauge invariant in the lattice formulation. Since we have periodic boundary conditions in time, we don't have to worry about boundaries in time.

When we go from the continuous action to the discrete action, we must account for the role of finite spacing.

$$\int \mathrm{d}^d x \, \mathrm{d}\tau \to \sum_{\vec{x},\tau} a^d d\tau. \tag{A.4}$$

We then scale our parameters to their lattice versions, incorporating the lattice spacing, denoted by a bar:

$$\begin{aligned}
\bar{x} &= x/a \\
\bar{y} &= y/a \\
\bar{r}^2 &= r^2/a^2 \\
\bar{\mu} &= d\tau\mu \\
\bar{m} &= ma^2/d\tau \\
\bar{\omega}_{\rm tr} &= d\tau\omega_{\rm tr} \\
\bar{\omega}_z &= d\tau\omega_z \\
\bar{\lambda} &= d\tau\lambda
\end{aligned}$$

This allows us to cancel factors of $d\tau$ and leaves us with an overall a^d that we can divide out. Therefore, our lattice action becomes:

$$\begin{aligned}
S_{\rm lat} = \sum_{\vec{x},\tau} a^d \Bigg[& \phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{\tau}} - \bar{\mu}\phi_r^*\phi_{r-\hat{\tau}} - \frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right) \\
& - \frac{\bar{m}}{2}\bar{\omega}_{\rm tr}^2 \bar{r}_\perp^2 \phi_r^*\phi_{r-\hat{\tau}} + i\bar{\omega}_z \left(\bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{\tau}}\right) \\
& + \bar{\lambda}\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2 \Bigg]
\end{aligned} \tag{A.5}$$

We can then combine our time derivative and our chemical potential in the following way:

$$\begin{aligned}
S_{\rm lat} = \sum_{\vec{x},\tau} a^d \Bigg[& \phi_r^*\phi_r - (1+\bar{\mu})\phi_r^*\phi_{r-\hat{\tau}} - \frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right) \\
& - \frac{\bar{m}}{2}\bar{\omega}_{\rm tr}^2 \bar{r}_\perp^2 \phi_r^*\phi_{r-\hat{\tau}} + i\bar{\omega}_z \left(\bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{\tau}}\right) \\
& + \bar{\lambda}\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2 \Bigg]
\end{aligned} \tag{A.6}$$

Note that to second order in the lattice size, this is equivalent to:

$$\begin{aligned}
S_{\rm lat} = \sum_{\vec{x},\tau} a^d \Bigg[& \phi_r^*\phi_r - e^{\bar{\mu}}\phi_r^*\phi_{r-\hat{\tau}} - \frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right) - \frac{\bar{m}}{2}\bar{\omega}_{\rm tr}^2 \bar{r}_\perp^2 \phi_r^*\phi_{r-\hat{\tau}} \\
& + i\bar{\omega}_z \left(\bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{\tau}}\right) + \bar{\lambda}\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2 \Bigg]
\end{aligned} \tag{A.7}$$

This will be our lattice action, which we will complexify and use to evolve our system in Langevin time. To simplify, let's divide the lattice action into smaller components:

$$S_{\rm lat} = \sum_{\vec{x},\tau} a^d \left(S_{\tau,r} + S_{\nabla,r} - S_{{\rm trap},r} - S_{\omega,r} + S_{{\rm int},r}\right) \tag{A.8}$$

with

$$S_{\mu,r} = \phi_r^*\phi_r - e^{\bar{\mu}}\phi_r^*\phi_{r-\hat{\tau}} \tag{A.9}$$

$$S_{\nabla,r} = \frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(2\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{j}} - \phi_r^*\phi_{r+\hat{j}}\right) \tag{A.10}$$

$$S_{{\rm trap},r} = \frac{\bar{m}}{2}\bar{\omega}_{\rm tr}^2 \bar{r}_\perp^2 \phi_r^*\phi_{r-\hat{\tau}} \tag{A.11}$$

$$S_{\omega,r} = i\bar{\omega}_z \left(\bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}\right) \tag{A.12}$$

$$S_{{\rm int},r} = \bar{\lambda}\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2. \tag{A.13}$$

Note that we are restricting ourselves to two spatial dimensions at this point in the work. The extension of this method to three-dimensional rotating systems is saved for future work.

A.2 Writing the Complex Action in Terms of Real Fields

This action must first be rewritten with the complex fields expressed in terms of two real fields, $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and $\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$. Each piece of the action is computed below:

First, the time derivative and chemical potential part of the action:

$$\begin{aligned} S_{\tau,r} &= \phi_r^*\phi_r - e^{\bar{\mu}}\phi_r^*\phi_{r-\hat{\tau}} \\ &= \frac{1}{2}\left[\phi_{1,r}^2 + \phi_{2,r}^2 - e^{\bar{\mu}}\left(\phi_{1,r}\phi_{1,r-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right)\right] \\ &= \frac{1}{2}\sum_{a=1}^{2}\left[\phi_{a,r}^2 - e^{\bar{\mu}}\phi_{a,r}\phi_{a,r-\hat{\tau}} - ie^{\bar{\mu}}\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right], \end{aligned} \tag{A.14}$$

where $\epsilon_{12} = 1,\ \epsilon_{21} = -1, \epsilon_{11} = \epsilon_{22} = 0$. Next, the spatial derivative part (corresponding to the kinetic energy):

$$\begin{aligned} S_{\nabla,r} &= \frac{1}{2\bar{m}}\sum_{j=1}^{d}\left(2\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{j}} - \phi_r^*\phi_{r+\hat{j}}\right) \\ &= \frac{1}{4\bar{m}}\sum_{j=1}^{d}\Big[2\left(\phi_{1,r}^2 + \phi_{2,r}^2\right) - \left(\phi_{1,r}\phi_{1,r+\hat{i}} + i\phi_{1,r}\phi_{2,r+\hat{i}} - i\phi_{2,r}\phi_{1,r+\hat{i}} + \phi_{2,r}\phi_{2,r+\hat{i}}\right) \\ &\quad - \left(\phi_{1,r}\phi_{1,r-\hat{i}} + i\phi_{1,r}\phi_{2,r-\hat{i}} - i\phi_{2,r}\phi_{1,r-\hat{i}} + \phi_{2,r}\phi_{2,r-\hat{i}}\right)\Big] \\ &= \frac{1}{4\bar{m}}\sum_{j=1}^{d}\sum_{a=1}^{2}\left[2\phi_{a,r}^2 - \left(\phi_a\phi_{a,r-\hat{j}} + \phi_a\phi_{a,r+\hat{j}}\right) - i\sum_{b=1}^{2}\epsilon_{ab}\left(\phi_{a,r}\phi_{b,r-\hat{j}} + \phi_{a,r}\phi_{b,r+\hat{j}}\right)\right] \end{aligned} \tag{A.15}$$

Then, for the part of the action due to the external trapping potential:

$$\begin{aligned} S_{\mathrm{trap},r} &= \frac{\bar{m}}{2}\bar{\omega}_{\mathrm{tr}}^2\bar{r}_\perp^2\phi_r^*\phi_{r-\hat{\tau}} \\ &= \frac{\bar{m}}{4}\bar{\omega}_{\mathrm{tr}}^2\bar{r}_\perp^2\left[\phi_{1,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{2,r-\tau}\right] \\ &= \frac{\bar{m}}{4}\bar{\omega}_{\mathrm{tr}}^2\bar{r}_\perp^2\sum_{a=1}^{2}\left[\phi_{a,r}\phi_{a,r-\hat{\tau}} + i\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right]. \end{aligned} \tag{A.16}$$

Next, the rotational piece:

$$
\begin{aligned}
S_{\omega,r} &= i\bar{\omega}_z\left(\bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}\right) \\
&= \frac{\bar{\omega}_z}{2}\left[(\bar{y}-\bar{x})\left(\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\right) + \bar{x}(\phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}})\right. \\
&\quad \left. -\bar{y}(\phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}})\right] \\
&\quad +i\frac{\bar{\omega}_z}{2}\left[(\bar{x}-\bar{y})(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}) - \bar{x}(\phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{y}-\hat{\tau}})\right. \\
&\quad \left. +\bar{y}(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}})\right] \\
&= \frac{\bar{\omega}_z}{2}\sum_{a=1}^{2}\left[\sum_{b=1}^{2}\epsilon_{ab}\left((\bar{y}-\bar{x})\phi_{a,r}\phi_{b,r-\hat{\tau}} + \bar{x}\phi_{a,r}\phi_{b,r-\hat{y}-\hat{\tau}} - \bar{y}\phi_{a,r}\phi_{b,r-\hat{x}-\hat{\tau}}\right)\right] \\
&\quad +i\frac{\bar{\omega}_z}{2}\sum_{a=1}^{2}\left[(\bar{x}-\bar{y})\phi_{a,r}\phi_{a,r-\hat{\tau}} - \bar{x}\phi_{a,r}\phi_{a,r-\hat{y}-\hat{\tau}} + \bar{y}(\phi_{a,r}\phi_{a,r-\hat{x}-\hat{\tau}}\right]
\end{aligned}
\tag{A.17}
$$

And finally, the interaction term in the action:

$$
\begin{aligned}
S_{\mathrm{int},r} &= \bar{\lambda}(\phi_r^*\phi_{r-\hat{\tau}})^2 \\
&= \frac{\bar{\lambda}}{4}\left[\phi_{1,r}\phi_{1,r-\hat{\tau}}(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}) + \phi_{2,r}\phi_{2,r-\hat{\tau}}(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}})\right. \\
&\quad -\phi_{1,r}\phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r-\hat{\tau}} \\
&\quad \left. +\phi_{1,r}\phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}}\right] \\
&\quad +i\frac{\bar{\lambda}}{4}\left[\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}}\right. \\
&\quad -\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&\quad +\phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&\quad \left. -\phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}}\right] \\
&= \frac{\bar{\lambda}}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left[2\phi_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r}\phi_{b,r-\hat{\tau}} - \phi_{a,r}^2\phi_{b,r-\hat{\tau}}^2\right] \\
&\quad +i\frac{d\tau\lambda}{2}\sum_{a=1}^{2}\sum_{b=1}^{2}\left[\epsilon_{ab}\left(\phi_{a,r}^2\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}} - \phi_{a,r}\phi_{a,r-\hat{\tau}}^2\phi_{b,r}\right)\right].
\end{aligned}
$$

We will work with the lattice action in this form in order to derive the Langevin drift function.

A.3 NOT UPDATED YET: Complexifying the Real Fields

Now we take our real fields, ϕ_a, where $a = 1, 2$, and rewrite them as two complex fields: $\phi_a = \phi_a^R + i\phi_a^I$

The time derivative piece, $S_{\tau,r}$ becomes:

$$S^R_{\tau,r} \;\to\; \frac{1}{2}\sum_{a=1}^{2}\left((\phi^R_{a,r})^2-(\phi^I_{a,r})^2-e^{d\tau\mu}\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}}+e^{d\tau\mu}\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}\right) \tag{A.18}$$
$$+\frac{1}{2}\sum_{a,b=1}^{2}\epsilon_{ab}\left(e^{d\tau\mu}\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}}+e^{d\tau\mu}\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\right)$$

$$S^I_{\tau,r} \;\to\; \frac{1}{2}\sum_{a=1}^{2}\left(2\phi^R_{a,r}\phi^I_{a,r}-e^{d\tau\mu}\phi^R_{a,r}\phi^I_{a,r-d\hat{\tau}\tau}-e^{d\tau\mu}\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}\right) \tag{A.19}$$
$$-\frac{1}{2}\sum_{a,b=1}^{2}\epsilon_{ab}\left(e^{d\tau\mu}\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}}-e^{d\tau\mu}\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}}\right),$$

while the spatial derivative piece $S_{\nabla,r}$ (from the kinetic energy) becomes:

$$S^R_{\nabla,r} \;\to\; \sum_{a=1}^{2}\left[\frac{d}{m}(\phi^R_{a,r})^2-\frac{d}{m}(\phi^I_{a,r})^2-\frac{1}{4m}\sum_{i=\pm x,y}\left(\phi^R_{a,r}\phi^R_{a,r+\hat{i}}-\phi^I_{a,r}\phi^I_{a,r+\hat{i}}\right)\right] \tag{A.20}$$
$$+\frac{1}{4m}\sum_{a,b=1}^{2}\sum_{i=\pm x,y}\epsilon_{ab}\left(\phi^R_{a,r}\phi^I_{b,r+\hat{i}}+\phi^I_{a,r}\phi^R_{b,r+\hat{i}}\right)$$

$$S^I_{\nabla,r} \;\to\; \sum_{a=1}^{2}\left[\frac{2d}{m}\phi^R_{a,r}\phi^I_{a,r}-\frac{1}{4m}\sum_{i=\pm x,y}\left(\phi^R_{a,r}\phi^I_{a,r+\hat{i}}+\phi^I_{a,r}\phi^R_{a,r+\hat{i}}\right)\right] \tag{A.21}$$
$$-\frac{1}{4m}\sum_{a,b=1}^{2}\sum_{i=\pm x,y}\epsilon_{ab}\left(\phi^R_{a,r}\phi^R_{b,r+\hat{i}}-\phi^I_{a,r}\phi^I_{b,r+\hat{i}}\right).$$

The trapping potential term, S_{trap}, becomes:

$$S^R_{\text{trap},r} \;\to\; \frac{m}{4}\omega^2_{\text{trap}}\left(x^2+y^2\right)\sum_{a=1}^{2}\left[\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}}-\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}-\sum_{b=1}^{2}\epsilon_{ab}\left(\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}}+\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\right)\right] \tag{A.22}$$

$$S^I_{\text{trap},r} \;\to\; \frac{m}{4}\omega^2_{\text{trap}}\left(x^2+y^2\right)\sum_{a=1}^{2}\left[\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}}+\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}+\sum_{b=1}^{2}\epsilon_{ab}\left(\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}}-\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}}\right)\right] \tag{A.23}$$

The rotating term, $S_{\omega,r}$ becomes:

$$\begin{aligned}
S_{\omega,r}^R \quad \to \quad & \frac{\omega_z}{2}\sum_{a=1}^{2}(y-x)\left[(\phi_{a,r}^I\phi_{a,r-\hat{\tau}}^R+\phi_{a,r}^R\phi_{a,r-\hat{\tau}}^I)+\sum_{b=1}^{2}\epsilon_{ab}(\phi_{a,r}^R\phi_{b,r-\hat{\tau}}^R-\phi_{a,r}^I\phi_{b,r-\hat{\tau}}^I)\right] \\
&+\frac{\omega_z}{2}\sum_{a=1}^{2}\widetilde{x}\left[(\phi_{a,r}^I\phi_{a,r-\hat{y}-\hat{\tau}}^R+\phi_{a,r}^R\phi_{a,r-\hat{y}-\hat{\tau}}^I)+\sum_{b=1}^{2}\epsilon_{ab}(\phi_{a,r}^R\phi_{b,r-\hat{y}-\hat{\tau}}^R-\phi_{a,r}^I\phi_{b,r-\hat{y}-\hat{\tau}}^I)\right] \\
&-\frac{\omega_z}{2}\sum_{a=1}^{2}\widetilde{y}\left[(\phi_{a,r}^I\phi_{a,r-\hat{x}-\hat{\tau}}^R+\phi_{a,r}^R\phi_{a,r-\hat{x}-\hat{\tau}}^I)+\sum_{b=1}^{2}\epsilon_{ab}(\phi_{a,r}^R\phi_{b,r-\hat{x}-\hat{\tau}}^R-\phi_{a,r}^I\phi_{b,r-\hat{x}-\hat{\tau}}^I)\right] \qquad (A.24) \\
S_{\omega,r}^I \quad \to \quad & \frac{\omega_z}{2}\sum_{a=1}^{2}(x-y)\left[(\phi_{a,r}^R\phi_{a,r-\hat{\tau}}^R-\phi_{a,r}^I\phi_{a,r-\hat{\tau}}^I)-\sum_{b=1}^{2}\epsilon_{ab}(\phi_{a,r}^I\phi_{b,r-\hat{\tau}}^R+\phi_{a,r}^R\phi_{b,r-\hat{\tau}}^I)\right] \\
&-\frac{\omega_z}{2}\sum_{a=1}^{2}\widetilde{x}\left[(\phi_{a,r}^R\phi_{a,r-\hat{y}-\hat{\tau}}^R-\phi_{a,r}^I\phi_{a,r-\hat{y}-\hat{\tau}}^I)-\sum_{b=1}^{2}\epsilon_{ab}(\phi_{a,r}^I\phi_{b,r-\hat{y}-\hat{\tau}}^R+\phi_{a,r}^R\phi_{b,r-\hat{y}-\hat{\tau}}^I)\right] \\
&+\frac{\omega_z}{2}\sum_{a=1}^{2}\widetilde{y}\left[(\phi_{a,r}^R\phi_{a,r-\hat{x}-\hat{\tau}}^R-\phi_{a,r}^I\phi_{a,r-\hat{x}-\hat{\tau}}^I)-\sum_{b=1}^{2}\epsilon_{ab}(\phi_{a,r}^I\phi_{b,r-\hat{x}-\hat{\tau}}^R+\phi_{a,r}^R\phi_{b,r-\hat{x}-\hat{\tau}}^I)\right] \qquad (A.25)
\end{aligned}$$

and finally, the interaction term $S_{\text{int},r}$ becomes:

**still to do: copy over these finished calculations from notebook

$$\begin{aligned}
\text{Re}[S_{\text{int},r}] \quad \to \quad & \frac{\lambda}{4}\sum_{a,b=1}^{2}\left[\left(\phi_{a,r}^R\phi_{b,r-\hat{\tau}}^I\right)^2-\left(\phi_{a,r}^I\phi_{b,r-\hat{\tau}}^I\right)^2\right] \\
&+\frac{\lambda}{4}\sum_{a,b=1}^{2}\left[\phi_{a,r}^R\phi_{b,r}^I\left((\phi_{a,r-\hat{\tau}}^R)^2-(\phi_{a,r-\hat{\tau}}^I)^2\right)+\phi_{a,r}^I\phi_{b,r}^R\left((\phi_{a,r-\hat{\tau}}^R)^2-(\phi_{a,r-\hat{\tau}}^I)^2\right)\right] \\
&+\frac{\lambda}{4}\sum_{a,b=1}^{2}\left[\phi_{a,r-\hat{\tau}}^I\phi_{b,r-\hat{\tau}}^R\left((\phi_{a,r}^I)^2-(\phi_{a,r}^R)^2\right)+\phi_{a,r-\hat{\tau}}^R\phi_{b,r-\hat{\tau}}^I\left((\phi_{a,r}^I)^2-(\phi_{a,r}^R)^2\right)\right] \\
&+\frac{\lambda}{4}\sum_{a,b=1}^{2}2\left[\phi_{a,r}^I\phi_{b,r}^I\left(\phi_{a,r-\hat{\tau}}^I\phi_{b,r-\hat{\tau}}^I-\phi_{a,r-\hat{\tau}}^R\phi_{b,r-\hat{\tau}}^R\right)+\phi_{a,r}^R\phi_{b,r}^R\left(\phi_{a,r-\hat{\tau}}^R\phi_{b,r-\hat{\tau}}^R-\phi_{a,r-\hat{\tau}}^I\phi_{b,r-\hat{\tau}}^I\right)\right] \\
&-\frac{\lambda}{4}\sum_{a,b=1}^{2}2\left[\phi_{a,r}^I\phi_{b,r}^R\left(\phi_{a,r-\hat{\tau}}^R\phi_{b,r-\hat{\tau}}^I+\phi_{a,r-\hat{\tau}}^I\phi_{b,r-\hat{\tau}}^R\right)+\phi_{a,r}^R\phi_{b,r}^I\left(\phi_{a,r-\hat{\tau}}^R\phi_{b,r-\hat{\tau}}^I+\phi_{a,r-\hat{\tau}}^I\phi_{b,r-\hat{\tau}}^R\right)\right] \\
&+\frac{\lambda}{4}\sum_{a,b=1}^{2}2\left[\phi_{a,r}^I\phi_{a,r}^R\left(\phi_{a,r-\hat{\tau}}^I\phi_{b,r-\hat{\tau}}^I-\phi_{a,r-\hat{\tau}}^R\phi_{b,r-\hat{\tau}}^R\right)+\phi_{a,r-\hat{\tau}}^I\phi_{a,r-\hat{\tau}}^R\left(\phi_{a,r}^R\phi_{b,r}^R-\phi_{a,r}^I\phi_{b,r}^I\right)\right] \\
&+\frac{\lambda}{4}\sum_{a,b=1}^{2}4\left[\phi_{a,r}^I\phi_{a,r}^R\phi_{b,r-\hat{\tau}}^R\phi_{b,r-\hat{\tau}}^I\right]
\end{aligned} \qquad (A.26)$$

where in all of the above, $S_j = S_j^R + iS_j^I$, and $\epsilon_{12} = 1$, $\epsilon_{21} = -1$, and $\epsilon_{11} = \epsilon_{22} = 0$. Note that we were able to compress the real part of the interaction due to the sum over a and b.

A.4 NOT UPDATED YET:Generating the NRRB CL Equations

A.4.1 Derivatives on the Lattice

When taking derivatives of this lattice action with respect to the fields, we do the following:

$$\begin{aligned}
\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2}\phi_{a,q}\phi_{a,q+\hat{i}}\right) &= \sum_{a=1}^{2}\sum_{q=1}^{N_r}\left(\phi_{a,q}\frac{\delta}{\delta\phi_{c,r}}\phi_{a,q+\hat{i}}+\frac{\delta\phi_{a,q}}{\delta\phi_{c,r}}\phi_{a,q+\hat{i}}\right) \\
&= \sum_{a=1}^{2}\sum_{q=1}^{N_r}\left(\phi_{a,q}\delta_{c,a}\delta_{r,q+\hat{i}}+\delta_{c,a}\delta_{q,r}\phi_{a,q+\hat{i}}\right) \\
&= \phi_{c,r-\hat{i}}+\phi_{c,r+\hat{i}}.
\end{aligned} \tag{A.27}$$

Similarly,

$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2}\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,q}\phi_{b,q+\hat{i}}\right)=\sum_{b=1}^{2}\epsilon_{cb}\left(\phi_{b,r-\hat{i}}+\phi_{b,r+\hat{i}}\right). \tag{A.28}$$

A.4.2 Computing the derivative of the action with respect to the real fields

The first step in computing the CL Equations is to find $\frac{\delta S_r}{\delta\phi_{a,r}}$. This is done below, with the sum over $a=1,2$ implied:

$$\frac{\delta S_r}{\delta\phi_{a,r}} = \frac{\delta S_{\tau,r}}{\delta\phi_{a,r}}+\frac{\delta S_{\nabla,r}}{\delta\phi_{a,r}}-\frac{\delta S_{\text{trap},r}}{\delta\phi_{a,r}}-\frac{\delta S_{\omega,r}}{\delta\phi_{a,r}}+\frac{\delta S_{\text{int},r}}{\delta\phi_{a,r}} \tag{A.29}$$

Again, we proceed by modifying each of the 5 parts of the action. First, the time and chemical potential term:

$$\begin{aligned}
2\frac{\delta}{\delta\phi_{a,r}}S_{\tau,r} &= \frac{\delta}{\delta\phi_{a,r}}\sum_{a=1}^{2}\left[\phi_{a,r}^2-e^{d\tau\mu}\phi_{a,r}\phi_{a,r-\hat{\tau}}-ie^{d\tau\mu}\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right] \\
2\frac{\delta}{\delta\phi_{a,r}}S_{\tau,r} &= 2\phi_{a,r}-e^{d\tau\mu}\left(\phi_{a,r-\hat{\tau}}+\phi_{a,r+\hat{\tau}}\right)-ie^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}}+\phi_{b,r+\hat{\tau}}\right)
\end{aligned} \tag{A.30}$$

Next, the spatial derivative part:

$$\begin{aligned}
-\frac{4m}{d\tau}\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r} &= \frac{\delta}{\delta\phi_{a,r}}\sum_{a=1}^{2}\left[\sum_{i=\pm x,y}\phi_{a,r}\phi_{a,r+\hat{i}}-2\phi_{a,r}^2+i\sum_{b=1}^{2}\sum_{i=\pm x,y}\epsilon_{ab}\phi_{a,r}\phi_{b,r+\hat{i}}\right] \\
&= \sum_{i=\pm x,y}\left(\phi_{a,r+\hat{i}}+\phi_{a,r-\hat{i}}\right)-4\phi_{a,r} \\
&= 2\sum_{i=\pm x,y}\phi_{a,r+\hat{i}}-4\phi_{a,r} \\
-\frac{2m}{d\tau}\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r} &= \sum_{i=\pm x,y}\phi_{a,r+\hat{i}}-2\phi_{a,r}
\end{aligned} \tag{A.31}$$

Then the part of the action due to the external trapping potential:

$$\begin{aligned}
\frac{4}{d\tau m\omega_{\rm tr}^2(r_\perp^2)}\frac{\delta}{\delta\phi_{a,r}}S_{{\rm trap},r} &= \frac{\delta}{\delta\phi_{a,r}}\sum_{a=1}^{2}\left(\phi_{a,r}\phi_{a,r-\hat\tau}+i\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat\tau}\right)\\
&= \sum_{a=1}^{2}\left(\phi_{a,r+\hat\tau}+\phi_{a,r-\hat\tau}+i\sum_{b=1}^{2}\epsilon_{ab}\left(\phi_{b,r+\hat\tau}+\phi_{b,r-\hat\tau}\right)\right)\\
\frac{4}{d\tau m\omega_{\rm tr}^2(r_\perp^2)}\frac{\delta}{\delta\phi_{a,r}}S_{{\rm trap},r} &= \sum_{a=1}^{2}\left(\left(\phi_{a,r+\hat\tau}+\phi_{a,r-\hat\tau}\right)+i\sum_{b=1}^{2}\epsilon_{ab}\left(\phi_{b,r+\hat\tau}+\phi_{b,r-\hat\tau}\right)\right) \qquad \text{(A.32)}
\end{aligned}$$

where $r_\perp^2 = x^2+y^2$. Next, the rotational piece:

$$\begin{aligned}
\frac{2}{d\tau\omega_z}\frac{\delta}{\delta\phi_{a,r}}S_{\omega,r} &= \frac{\delta}{\delta\phi_{a,r}}\sum_{a=1}^{2}\left[\sum_{b=1}^{2}\epsilon_{ab}\left(x\phi_{a,r}\phi_{b,r-\hat y}-y\phi_{a,r}\phi_{b,r-\hat x}\right)+i\left((x-y)\phi_{a,r}^2-x\phi_{a,r}\phi_{a,r-\hat y}+y\phi_{a,r}\phi_{a,r-\hat x}\right)\right]\\
&= \epsilon_{ab}\left[x\left(\phi_{b,r-\hat y}+\phi_{b,r+\hat y}\right)-y\left(\phi_{b,r-\hat x}+\phi_{b,r+\hat x}\right)\right]\\
&\quad +i\left[2(x-y)\phi_{a,r}-x\left(\phi_{a,r-\hat y}+\phi_{a,r+\hat y}\right)+y\left(\phi_{a,r-\hat x}+\phi_{a,r+\hat x}\right)\right]\\
\frac{2}{d\tau\omega_z}\frac{\delta}{\delta\phi_{a,r}}S_{\omega,r} &= \epsilon_{ab}\left[x\left(\phi_{b,r-\hat y}+\phi_{b,r+\hat y}\right)-y\left(\phi_{b,r-\hat x}+\phi_{b,r+\hat x}\right)\right]\\
&\quad +i\left[2(x-y)\phi_{a,r}-x\left(\phi_{a,r-\hat y}+\phi_{a,r+\hat y}\right)+y\left(\phi_{a,r-\hat x}+\phi_{a,r+\hat x}\right)\right] \qquad \text{(A.33)}
\end{aligned}$$

And finally, the interaction term in the action:

$$\begin{aligned}
\frac{4}{d\tau\lambda}\frac{\delta}{\delta\phi_{a,r}}S_{{\rm int},r} &= \frac{\delta}{\delta\phi_{a,r}}\sum_{a=1}^{2}\sum_{b=1}^{2}\phi_{a,r}^2\phi_{b,r}^2\\
&= 4\phi_{a,r}\sum_{b=1}^{2}\phi_{b,r}^2\\
\frac{1}{d\tau\lambda}\frac{\delta}{\delta\phi_{a,r}}S_{{\rm int},r} &= \phi_{a,r}\sum_{b=1}^{2}\phi_{b,r}^2. \qquad \text{(A.34)}
\end{aligned}$$

Here, there is an extra factor of 2 that appears when solving this equation without the summation notation - I'm not really sure why it doesn't appear when using summation, but it is the correct factor.

A.4.3 NOT UPDATED YET:Second Complexification of Drift Function

The next step is to complexify our real fields, a and b, such that $\phi_a = \phi_a^R + i\phi_a^I$. We do this for each part of the drift function, $K_{a,r} = \frac{\delta S_r}{\delta\phi_{a,r}}$.

First, the time and chemical potential term:

$$\begin{aligned}
2\frac{\delta}{\delta\phi_{a,r}}S_{\tau,r} &= 2\phi_{a,r}-e^{d\tau\mu}\left(\phi_{a,r-\hat\tau}+\phi_{a,r+\hat\tau}\right)-ie^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat\tau}+\phi_{b,r+\hat\tau}\right)\\
&= 2(\phi_{a,r}^R+i\phi_{a,r}^I)-e^{d\tau\mu}\left(\phi_{a,r-\hat\tau}^R+i\phi_{a,r-\hat\tau}^I+\phi_{a,r+\hat\tau}^R+i\phi_{a,r+\hat\tau}^I\right)\\
&\quad -ie^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat\tau}^R+i\phi_{b,r-\hat\tau}^I+\phi_{b,r+\hat\tau}^R+i\phi_{b,r+\hat\tau}^I\right)\\
&= 2\phi_{a,r}^R-e^{d\tau\mu}\left(\phi_{a,r-\hat\tau}^R+\phi_{a,r+\hat\tau}^R\right)+e^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat\tau}^I-\phi_{b,r+\hat\tau}^I\right)\\
&\quad +i\left[2\phi_{a,r}^I-e^{d\tau\mu}\left(\phi_{a,r-\hat\tau}^I+\phi_{a,r+\hat\tau}^I\right)-e^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat\tau}^R-\phi_{b,r+\hat\tau}^R\right)\right]
\end{aligned}$$

So

$$\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\tau,r}\right] = \phi^R_{a,r} - \frac{e^{d\tau\mu}}{2}\left(\phi^R_{a,r-\hat{\tau}} + \phi^R_{a,r+\hat{\tau}}\right) + \frac{e^{d\tau\mu}}{2}\epsilon_{ab}\left(\phi^I_{b,r-\hat{\tau}} - \phi^I_{b,r+\hat{\tau}}\right) \tag{A.35}$$

$$\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\tau,r}\right] = \phi^I_{a,r} - \frac{e^{d\tau\mu}}{2}\left(\phi^I_{a,r-\hat{\tau}} + \phi^I_{a,r+\hat{\tau}}\right) - \frac{e^{d\tau\mu}}{2}\epsilon_{ab}\left(\phi^R_{b,r-\hat{\tau}} - \phi^R_{b,r+\hat{\tau}}\right) \tag{A.36}$$

Next, the spatial derivative part:

$$\begin{aligned}
-\frac{2m}{d\tau}\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r} &= \sum_{i=\pm1}^{d}\phi_{a,r+\hat{i}} - 2d\phi_{a,r} \\
&= \sum_{i=\pm1}^{d}\left(\phi^R_{a,r+\hat{i}} + i\phi^I_{a,r+\hat{i}}\right) - 2d\phi^R_{a,r} - 2id\phi^I_{a,r} \\
&= \sum_{i=\pm1}^{d}\phi^R_{a,r+\hat{i}} - 2d\phi^R_{a,r} + i\left(\sum_{i=\pm1}^{d}\phi^I_{a,r+\hat{i}} - 2d\phi^I_{a,r}\right)
\end{aligned}$$

So

$$\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r}\right] = \frac{d\tau}{2m}\left(2d\phi^R_{a,r} - \sum_{i=\pm1}^{d}\phi^R_{a,r+\hat{i}}\right) \tag{A.37}$$

$$\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r}\right] = \frac{d\tau}{2m}\left(2d\phi^I_{a,r} - \sum_{i=\pm1}^{d}\phi^I_{a,r+\hat{i}}\right) \tag{A.38}$$

Then the part of the action due to the external trapping potential:

$$\begin{aligned}
\frac{4}{d\tau m\omega^2_{\mathrm{tr}}(r^2_\perp)}\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r} &= \sum_{a=1}^{2}\left(\left(\phi_{a,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\right) + i\sum_{b=1}^{2}\epsilon_{ab}\left(\phi_{b,r+\hat{\tau}} + \phi_{b,r-\hat{\tau}}\right)\right) \\
&= \sum_{a=1}^{2}\left(\left(\phi^R_{a,r+\hat{\tau}} + \phi^R_{a,r-\hat{\tau}}\right) - \sum_{b=1}^{2}\epsilon_{ab}\left(\phi^I_{b,r+\hat{\tau}} + \phi^I_{b,r-\hat{\tau}}\right)\right) \\
&\quad + i\sum_{a=1}^{2}\left(\left(\phi^I_{a,r+\hat{\tau}} + \phi^I_{a,r-\hat{\tau}}\right) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{b,r+\hat{\tau}} + \phi^R_{b,r-\hat{\tau}})\right)
\end{aligned} \tag{A.39}$$

So

$$\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r}\right] = \frac{d\tau\omega^2_{\mathrm{tr}}(r^2_\perp)}{4}\sum_{a=1}^{2}\left(\left(\phi^R_{a,r+\hat{\tau}} + \phi^R_{a,r-\hat{\tau}}\right) - \sum_{b=1}^{2}\epsilon_{ab}\left(\phi^I_{b,r+\hat{\tau}} + \phi^I_{b,r-\hat{\tau}}\right)\right) \tag{A.40}$$

$$\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r}\right] = \frac{d\tau\omega^2_{\mathrm{tr}}(r^2_\perp)}{4}\sum_{a=1}^{2}\left(\left(\phi^I_{a,r+\hat{\tau}} + \phi^I_{a,r-\hat{\tau}}\right) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{b,r+\hat{\tau}} + \phi^R_{b,r-\hat{\tau}})\right) \tag{A.41}$$

where $r^2_\perp = x^2 + y^2$. Next, the rotational piece:

$$
\begin{aligned}
\frac{2}{d\tau\omega_z}\frac{\delta}{\delta\phi_{a,r}}S_{\omega,r} &= \epsilon_{ab}\left[x\left(\phi_{b,r-\hat{y}}+\phi_{b,r+\hat{y}}\right)-y\left(\phi_{b,r-\hat{x}}+\phi_{b,r+\hat{x}}\right)\right] \\
&\quad +i\left[2(x-y)\phi_{a,r}-x\left(\phi_{a,r-\hat{y}}+\phi_{a,r+\hat{y}}\right)+y\left(\phi_{a,r-\hat{x}}+\phi_{a,r+\hat{x}}\right)\right] \\
&= \epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}}+i\phi^I_{b,r-\hat{y}}+\phi^R_{b,r+\hat{y}}+i\phi^I_{b,r+\hat{y}}\right)-y\left(\phi^R_{b,r-\hat{x}}+i\phi^I_{b,r-\hat{x}}+\phi^R_{b,r+\hat{x}}+i\phi^I_{b,r+\hat{x}}\right)\right] \\
&\quad +i\left[2(x-y)\phi^R_{a,r}+2i(x-y)\phi^I_{a,r}-x\left(\phi^R_{a,r-\hat{y}}+i\phi^I_{a,r-\hat{y}}+\phi^R_{a,r+\hat{y}}+i\phi^I_{a,r+\hat{y}}\right)\right] \\
&\quad +i\left[y\left(\phi^R_{a,r-\hat{x}}+i\phi^I_{a,r-\hat{x}}+\phi^R_{a,r+\hat{x}}+i\phi^I_{a,r+\hat{x}}\right)\right] \\
&= -2(x-y)\phi^I_{a,r}+x\left(\phi^I_{a,r-\hat{y}}+\phi^I_{a,r+\hat{y}}\right)-y\left(\phi^I_{a,r-\hat{x}}+\phi^I_{a,r+\hat{x}}\right) \\
&\quad +\epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}}+\phi^R_{b,r+\hat{y}}\right)-y\left(\phi^R_{b,r-\hat{x}}+\phi^R_{b,r+\hat{x}}\right)\right] \\
&\quad +i\left[2(x-y)\phi^R_{a,r}-x\left(\phi^R_{a,r-\hat{y}}+\phi^R_{a,r+\hat{y}}\right)+y\left(\phi^R_{a,r-\hat{x}}+\phi^R_{a,r+\hat{x}}\right)\right] \\
&\quad +i\epsilon_{ab}\left[x\left(\phi^I_{b,r-\hat{y}}+\phi^I_{b,r+\hat{y}}\right)-y\left(\phi^I_{b,r-\hat{x}}+\phi^I_{b,r+\hat{x}}\right)\right] \qquad \text{(A.42)}
\end{aligned}
$$

So

$$
\begin{aligned}
\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\omega_z,r}\right] &= \frac{d\tau\omega_z}{2}\left[x\left(\phi^I_{a,r-\hat{y}}+\phi^I_{a,r+\hat{y}}\right)-y\left(\phi^I_{a,r-\hat{x}}+\phi^I_{a,r+\hat{x}}\right)-2(x-y)\phi^I_{a,r}\right] \\
&\quad +\frac{d\tau\omega_z}{2}\epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}}+\phi^R_{b,r+\hat{y}}\right)-y\left(\phi^R_{b,r-\hat{x}}+\phi^R_{b,r+\hat{x}}\right)\right] \\
\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\omega_z,r}\right] &= \frac{d\tau\omega_z}{2}\left[2(x-y)\phi^R_{a,r}-x\left(\phi^R_{a,r-\hat{y}}+\phi^R_{a,r+\hat{y}}\right)+y\left(\phi^R_{a,r-\hat{x}}+\phi^R_{a,r+\hat{x}}\right)\right] \qquad \text{(A.43)} \\
&\quad +\frac{\omega_z}{2}\epsilon_{ab}\left[x\left(\phi^I_{b,r-\hat{y}}+\phi^I_{b,r+\hat{y}}\right)-y\left(\phi^I_{b,r-\hat{x}}+\phi^I_{b,r+\hat{x}}\right)\right]
\end{aligned}
$$

And finally, the interaction term in the action:

$$
\begin{aligned}
\frac{1}{d\tau\lambda}\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{int},r} &= \phi_{a,r}\sum_{b=1}^{2}\phi^2_{b,r} \\
&= (\phi^R_{a,r}+i\phi^I_{a,r})\sum_{b=1}^{2}(\phi^R_{b,r}+i\phi^I_{b,r})^2 \\
&= \phi^R_{a,r}\sum_{b=1}^{2}\left((\phi^R_{b,r})^2+2i\phi^R_{b,r}\phi^I_{b,r}-(\phi^I_{b,r})^2\right)+i\phi^I_{a,r}\sum_{b=1}^{2}\left((\phi^R_{b,r})^2+2i\phi^R_{b,r}\phi^I_{b,r}-(\phi^I_{b,r})^2\right) \\
&= \sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2+2i\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}-\phi^R_{a,r}(\phi^I_{b,r})^2+i\phi^I_{a,r}(\phi^R_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}-i\phi^I_{a,r}(\phi^I_{b,r})^2\right] \\
&= \sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2-\phi^R_{a,r}(\phi^I_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}\right]+i\sum_{b=1}^{2}\left[2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}+\phi^I_{a,r}(\phi^R_{b,r})^2-\phi^I_{a,r}(\phi^I_{b,r})^2\right]
\end{aligned}
$$

So

$$
\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{int},r}\right] = d\tau\lambda\sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2-\phi^R_{a,r}(\phi^I_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}\right] \qquad \text{(A.44)}
$$

$$
\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{int},r}\right] = d\tau\lambda\sum_{b=1}^{2}\left[\phi^I_{a,r}(\phi^R_{b,r})^2+2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}-\phi^I_{a,r}(\phi^I_{b,r})^2\right] \qquad \text{(A.45)}
$$

A.5 NOT UPDATED YET:Lattice Observables

The observables:

A.5.1 Average Density

The average density is the sum over the local density at each spatial site, normalized by the spatial lattice volume:

$$\langle \hat{n} \rangle = \frac{1}{N_x^d} \sum_r n_r \tag{A.46}$$

where n_r can be found by taking the derivative of the lattice action with respect to the chemical potential, and then complexifying the fields as we have done before:

$$\begin{aligned}
n_r &= -\frac{\partial}{\partial \beta\mu} S_{\tau,r} = -\frac{\partial}{\partial \beta\mu}\left(\phi_r^* \phi_r - \phi_r^* e^{\beta\mu/N_\tau}\phi_{r-\hat{\tau}}\right) = \frac{\partial}{\partial \beta\mu}\left(\phi_r^* e^{\beta\mu/N_\tau}\phi_{r-\hat{\tau}}\right) \\
&= \frac{1}{N_\tau} e^{\beta\mu/N_\tau}\left(\phi_r^* \phi_{r-\hat{\tau}}\right) = \frac{1}{2N_\tau} e^{\beta\mu/N_\tau}\left(\phi_{1,r} - i\phi_{2,r}\right)\left(\phi_{1,r-\hat{\tau}} + i\phi_{2,r-\hat{\tau}}\right) \\
&= \frac{1}{N_\tau} e^{\beta\mu/N_\tau} \sum_{a=1}^{2}\left[\phi_{a,r}\phi_{a,r-\hat{\tau}} + i\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right] \\
&= \frac{1}{2N_\tau} e^{\beta\mu/N_\tau} \sum_{a=1}^{2}\left[\phi_{a,r}^R\phi_{a,r-\hat{\tau}}^R - \phi_{a,r}^I\phi_{a,r-\hat{\tau}}^I - \sum_{b=1}^{2}\epsilon_{ab}(\phi_{a,r}^I\phi_{b,r-\hat{\tau}}^R + \phi_{a,r}^R\phi_{b,r-\hat{\tau}}^I)\right] \\
&+ \frac{i}{2N_\tau} e^{\beta\mu/N_\tau} \sum_{a=1}^{2}\left[\phi_{a,r-\hat{\tau}}^I\phi_{a,r-\hat{\tau}}^R + \phi_{a,r-\hat{\tau}}^R\phi_{a,r-\hat{\tau}}^I + \sum_{b=1}^{2}\epsilon_{ab}(\phi_{a,r}^R\phi_{b,r-\hat{\tau}}^R - \phi_{a,r}^I\phi_{b,r-\hat{\tau}}^I)\right]
\end{aligned} \tag{A.47}$$

A.5.2 Average Angular Momentum

The angular momentum operator, L_z can be written

$$L_z = ((x - x_0)p_y - (y - y_0)p_x) = -i\hbar((x - x_0)\partial_y - (y - y_0)\partial_x). \tag{A.48}$$

We are interested in the expectation value of this operator: $\langle L_z \rangle$, which is found by summing over the lattice:

$$\langle L_z \rangle = -i \sum_r \phi_r^*((x - \frac{N_x - 1}{2})\partial_y - (y - \frac{N_x - 1}{2})\partial_x)\phi_r, \tag{A.49}$$

where $\hbar \to 1$. First, let us implement our lattice derivative: $\partial_j \phi_r = \frac{1}{a^2}(\phi_{r-\hat{j}} - \phi_r)$ (recall that our lattice spacing is $a = 1$):

$$\begin{aligned}
\langle L_z \rangle &= -i \sum_r \phi_r^*((x - r_c)\phi_{r-\hat{y}} - x\phi_r - (y - r_c)\phi_{r-\hat{x}} + y\phi_r) \\
&= i \sum_r ((y - r_c)\phi_r^*\phi_{r-\hat{x}} - (x - r_c)\phi_r^*\phi_{r-\hat{y}} - (y - x)\phi_r^*\phi_r)
\end{aligned} \tag{A.50}$$

Here we have also written $\frac{N_x-1}{2}$ as r_c for simplicity. Let us write ϕ as $\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)$; then, our angular momentum operator becomes a sum over the lattice sites and the real fields ϕ_1 and ϕ_2:

$$\begin{aligned}
\langle L_z\rangle &= \frac{i}{2}\sum_r (y-r_c)\left(\phi_{1,r}\phi_{1,r-\hat{x}}+i\phi_{1,r}\phi_{2,r-\hat{x}}-i\phi_{2,r}\phi_{1,r-\hat{x}}+\phi_{2,r}\phi_{2,r-\hat{x}}\right)\\
&-\frac{i}{2}\sum_r (x-r_c)\left(\phi_{1,r}\phi_{1,r-\hat{y}}+i\phi_{1,r}\phi_{2,r-\hat{y}}-i\phi_{2,r}\phi_{1,r-\hat{y}}+\phi_{2,r}\phi_{2,r-\hat{y}}\right) \qquad (A.51)\\
&-\frac{i}{2}\sum_r (y-x)(\phi_{1,r}^2+\phi_{2,r}^2)\\
&= \frac{1}{2}\sum_r\sum_{a=1}^{2}\left((y-r_c)\phi_{a,r}\phi_{a,r-\hat{x}}-(x-r_c)\phi_{a,r}\phi_{a,r-\hat{y}}-(y-x)\phi_{a,r}^2\right) \qquad (A.52)\\
&-\frac{i}{2}\sum_r\sum_{a,b=1}^{2}\epsilon_{ab}\left((x-r_c)\phi_{a,r}\phi_{b,r-\hat{y}}-(y-r_c)\phi_{a,r}\phi_{b,r-\hat{x}}\right)
\end{aligned}$$

Next, we must complexify our real fields: $\phi_a=\phi_a^R+i\phi_a^I$, leading us to the following equation for the angular momentum in terms of our four lattice fields:

$$\begin{aligned}
\langle L_z\rangle &= -\frac{i}{2}\sum_r\sum_{a=1}^{2}\left((x-r_c)(\phi_{a,r}^R\phi_{a,r-\hat{y}}^R-\phi_{a,r}^I\phi_{a,r-\hat{y}}^I)-(y-r_c)(\phi_{a,r}^R\phi_{a,r-\hat{x}}^R-\phi_{a,r}^I\phi_{a,r-\hat{x}}^I)\right)\\
&-\frac{i}{2}\sum_r\sum_{a=1}^{2}(y-x)\left((\phi_{a,r}^R)^2-(\phi_{a,r}^I)^2\right)\\
&-\frac{i}{2}\sum_r\sum_{a,b=1}^{2}\epsilon_{ab}\left((x-r_c)(\phi_{a,r}^R\phi_{b,r-\hat{y}}^I+\phi_{a,r}^I\phi_{b,r-\hat{y}}^R)-(y-r_c)(\phi_{a,r}^R\phi_{b,r-\hat{x}}^I+\phi_{a,r}^I\phi_{b,r-\hat{x}}^R)\right)\\
&+\frac{1}{2}\sum_r\sum_{a=1}^{2}\left((x-r_c)(\phi_{a,r}^R\phi_{a,r-\hat{y}}^I+\phi_{a,r}^I\phi_{a,r-\hat{y}}^R)-(y-r_c)(\phi_{a,r}^R\phi_{a,r-\hat{x}}^I+\phi_{a,r}^I\phi_{a,r-\hat{x}}^R)\right)\\
&+\sum_r\sum_{a,b=1}^{2}(y-x)\phi_{a,r}^R\phi_{a,r}^I\\
&+\frac{1}{2}\sum_r\sum_{a,b=1}^{2}\epsilon_{ab}\left((x-r_c)(\phi_{a,r}^R\phi_{b,r-\hat{y}}^R-\phi_{a,r}^I\phi_{b,r-\hat{y}}^I)-(y-r_c)(\phi_{a,r}^R\phi_{b,r-\hat{x}}^R-\phi_{a,r}^I\phi_{b,r-\hat{x}}^I)\right).
\end{aligned}$$

When this is divided into the real and imaginary parts of the observable, we get:

$$\begin{aligned}
\mathrm{Re}\langle L_z\rangle &= \frac{1}{2}\sum_r\sum_{a=1}^{2}\left((x-r_c)(\phi^R_{a,r}\phi^I_{a,r-\hat{y}}+\phi^I_{a,r}\phi^R_{a,r-\hat{y}})-(y-r_c)(\phi^R_{a,r}\phi^I_{a,r-\hat{x}}+\phi^I_{a,r}\phi^R_{a,r-\hat{x}})\right)\\
&\quad+\sum_r\sum_{a=1}^{2}(y-x)\phi^R_{a,r}\phi^I_{a,r} \qquad\qquad (A.53)\\
&\quad+\frac{1}{2}\sum_r\sum_{a,b=1}^{2}\epsilon_{ab}\left((x-r_c)(\phi^R_{a,r}\phi^R_{b,r-\hat{y}}-\phi^I_{a,r}\phi^I_{b,r-\hat{y}})-(y-r_c)(\phi^R_{a,r}\phi^R_{b,r-\hat{x}}-\phi^I_{a,r}\phi^I_{b,r-\hat{x}})\right).\\
\mathrm{Im}\langle L_z\rangle &= -\frac{1}{2}\sum_r\sum_{a=1}^{2}\left((x-r_c)(\phi^R_{a,r}\phi^R_{a,r-\hat{y}}-\phi^I_{a,r}\phi^I_{a,r-\hat{y}})-(y-r_c)(\phi^R_{a,r}\phi^R_{a,r-\hat{x}}-\phi^I_{a,r}\phi^I_{a,r-\hat{x}})\right)\\
&\quad-\frac{1}{2}\sum_r\sum_{a=1}^{2}(y-x)\left((\phi^R_{a,r})^2-(\phi^I_{a,r})^2\right) \qquad\qquad (A.54)\\
&\quad-\frac{1}{2}\sum_r\sum_{a,b=1}^{2}\epsilon_{ab}\left((x-r_c)(\phi^R_{a,r}\phi^I_{b,r-\hat{y}}+\phi^I_{a,r}\phi^R_{b,r-\hat{y}})-(y-r_c)(\phi^R_{a,r}\phi^I_{b,r-\hat{x}}+\phi^I_{a,r}\phi^R_{b,r-\hat{x}})\right)
\end{aligned}$$

A.5.3 Circulation

The circulation is defined as

$$\Gamma[l] = \frac{1}{2\pi}\oint_{l\times l} dx\,(\theta_{t,x+j}-\theta_{t,x}) \qquad (A.55)$$

$$\theta_{t,x} = \tan^{-1}\left(\frac{\mathrm{Im}[\phi_{t,x}]}{\mathrm{Re}[\phi_{t,x}]}\right). \qquad (A.56)$$

Given that our field, ϕ, is broken into 4 components ($\phi^{R/I}_{1/2}$, we need to rewrite this quantity in terms of those components:

$$\phi = \frac{1}{\sqrt{2}}\left(\phi^R_1+i\phi^I_1+i\phi^R_2-\phi^I_2\right) \qquad (A.57)$$

$$\mathrm{Re}[\phi_{t,x}] = \frac{1}{\sqrt{2}}\left(\phi^R_{1,t,x}-\phi^I_{2,t,x}\right) \qquad (A.58)$$

$$\mathrm{Im}[\phi_{t,x}] = \frac{1}{\sqrt{2}}\left(\phi^I_{1,t,x}+\phi^R_{2,t,x}\right) \qquad (A.59)$$

$$\theta_{t,x} = \tan^{-1}\left(\frac{\phi^I_{1,t,x}+\phi^R_{2,t,x}}{\phi^R_{1,t,x}-\phi^I_{2,t,x}}\right). \qquad (A.60)$$

Therefore, our circulation is a sum computed around a loop on the lattice of length l in each direction:

$$\Gamma[l] = \frac{1}{2\pi}\sum_{l\times l}(\theta_{t,x+j}-\theta_{t,x}) \qquad (A.61)$$

where $\theta_{t,x+j}$ is computed at the next site on the loop from $\theta_{t,x}$ for each point along the loop.

A.5.4 Average Energy

The expectation value of the density is defined as

$$\langle E \rangle = \frac{-\partial \ln \mathcal{Z}}{\partial \beta} = \frac{-\partial \ln(e^{-S})}{\partial \beta} = \frac{\partial S}{\partial \beta}. \tag{A.62}$$

Since we have written the action in terms of β as

$$S \to \frac{1}{N_\tau} \sum_{x,\tau} \Delta x^d \beta \left[\left(\partial_\tau - \frac{1}{2m} \nabla^2 - \mu - i\omega_z (x\partial_y - y\partial_x) - \frac{m\omega_{\text{trap}}^2}{2}(x^2+y^2) \right) \phi + \lambda(\phi^* \phi)^2 \right]. \tag{A.63}$$

We can see that the action is linear in β. Therefore, we can simply calculate the energy in the following way:

$$\langle E \rangle = \frac{S}{\beta} \tag{A.64}$$

A.6 NOT UPDATED YET:Some checks on our code

A.6.1 Nonrotating, noninteracting, nonrelativistic, finite chemical potential in 1, 2, and 3 dimensions

The lattice action for a nonrotating, noninteracting, and nonrelativistic system is the following:

$$S_{\text{lat},r} = \phi_r^* \left[\phi_r - e^{d\tau\mu} \phi_{r-\hat{\tau}} - \frac{d\tau}{2m} \sum_{i=1}^{d} \left(\phi_{r+\hat{i}} - 2\phi_r + \phi_{r-\hat{i}} \right) \right]. \tag{A.65}$$

This can be written as fields multiplying a matrix:

$$S_{\text{lat},r} = \sum_r \sum_{r'} \phi_r^* M \phi_{r'} = \sum_r \sum_{r'} \phi_r^* \left[(1 + \frac{d\tau d}{m}) \delta_{r,r'} - e^{d\tau\mu} \delta_{r-\hat{\tau},r'} - \frac{d\tau}{2m} \sum_{i=1}^{d} \left(\delta_{r+\hat{i},r'} + \delta_{r-\hat{i},r'} \right) \right] \phi_{r'}, \tag{A.66}$$

which we can use to determine analytically the density and field modulus squared of this system in order to check against our code's results. Recall that

$$\begin{aligned} \langle \hat{n} \rangle &= \frac{-1}{V} \frac{\partial ln\mathcal{Z}}{\partial(\beta\mu)} = \frac{-1}{V} \frac{\partial}{\partial(\beta\mu)} (-ln(det(M))) \\ &= \frac{1}{V} \frac{\partial}{\partial(\beta\mu)} Tr(lnM) = \frac{1}{V} \frac{\partial}{\partial(\beta\mu)} \sum_k lnD_{kk} = \frac{1}{V} \sum_k \frac{1}{D_{kk}} \frac{\partial D_{kk}}{\partial(\beta\mu)} \end{aligned} \tag{A.67}$$

with $\beta\mu = N_\tau d\tau\mu$, and note that for a nonrelativistic system,

$$\begin{aligned} \langle \phi^* \phi \rangle &= \frac{-1}{V} \frac{\partial ln\mathcal{Z}}{\partial(d/m)} = \frac{-1}{V} \frac{\partial}{\partial(d/m)} (-ln(det(M))) \\ &= \frac{1}{V} \frac{\partial}{\partial(d/m)} Tr(lnM) = \frac{1}{V} \frac{\partial}{\partial(d/m)} \sum_k lnD_{kk} \\ &= \frac{1}{V} \sum_k \frac{1}{D_{kk}} \frac{\partial D_{kk}}{\partial(d/m)} = \sum_k \frac{1}{D_{kk}}. \end{aligned} \tag{A.68}$$

Diagonalizing our matrix, M

We can represent the nonrotating, noninteracting action as

$$S[\lambda = \omega = 0] = \sum_{r,r'} \phi^*_r M_{r,r'}[d,m,\mu]\phi_{r'} \tag{A.69}$$

where

$$M_{r,r'}[d,m,\mu] = \left[(1+\frac{d\tau d}{m})\delta_{r,r'} - e^{d\tau\mu}\delta_{r-\hat{t},r'} - \frac{d\tau}{2m}\sum_{i=1}^{d}(\delta_{r+\hat{\imath},r'} + \delta_{r-\hat{\imath},r'})\right]. \tag{A.70}$$

We want to diagonalize M by applying a transformation matrix, such that $D_{kk'} = U^\dagger M U$, where

$$\begin{aligned}
U_{r,k} &= \frac{\sqrt{2^d}}{\sqrt{N_x^d N_\tau}} e^{ik_0 t}\prod_{i=1}^{d}\sin(k_i x_i) && \text{(A.71)}\\
U^\dagger_{r,k} &= \frac{\sqrt{2^d}}{\sqrt{N_x^d N_\tau}} e^{-ik_0 t}\prod_{i=1}^{d}\sin(k_i x_i) && \text{(A.72)}\\
k_0 &= \frac{2\pi n_0}{N_\tau},\ \ n_0 \in [1,2,...,N_\tau] && \text{(A.73)}\\
k_i &= \frac{\pi n_i}{(N_x+1)},\ \ n_i \in [1,2,...,N_x]. && \text{(A.74)}
\end{aligned}$$

Applying the transformation matrix, we get

$$\begin{aligned}
D_{k,k'} &= \frac{2^d}{N_x^d N_t}\sum_{r,r'} e^{-ik_0 t}\prod_{i=1}^{d}\sin(k_i x_i)\left[(1+\frac{d\tau d}{m})\delta_{r,r'}\right] e^{ik'_0 t'}\prod_{i=1}^{d}\sin(k'_i x'_i)\\
&-\frac{2^d}{N_x^d N_t}\sum_{r,r'} e^{-ik_0 t}\prod_{i=1}^{d}\sin(k_i x_i)\left[e^{d\tau\mu}\delta_{r-\hat{t},r'}\right] e^{ik'_0 t'}\prod_{i=1}^{d}\sin(k'_i x'_i)\\
&-\frac{2^d}{N_x^d N_t}\sum_{r,r'} e^{-ik_0 t}\prod_{i=1}^{d}\sin(k_i x_i)\left[\frac{d\tau}{2m}\sum_{i=1}^{d}\delta_{r+\hat{\imath},r'}\right] e^{ik'_0 t'}\prod_{i=1}^{d}\sin(k'_i x'_i)\\
&-\frac{2^d}{N_x^d N_t}\sum_{r,r'} e^{-ik_0 t}\prod_{i=1}^{d}\sin(k_i x_i)\left[\frac{d\tau}{2m}\sum_{i=1}^{d}\delta_{r-\hat{\imath},r'}\right] e^{ik'_0 t'}\prod_{i=1}^{d}\sin(k'_i x'_i).
\end{aligned} \tag{A.75}$$

Resolving the delta functions, performing the sum over r', and pulling everything that does not depend on $r = (t, \vec{x})$ outside the sum, this reduces to

$$\begin{aligned}
D_{k,k'} &= \frac{2^d}{N_x^d N_t}(1+\frac{d\tau d}{m})\sum_{r} e^{-it(k_0-k'_0)}\prod_{i=1}^{d}\sin(k_i x_i)\sin(k'_i x_i)\\
&-\frac{2^d}{N_x^d N_t}e^{d\tau\mu}e^{-ik'_0}\sum_{r} e^{-it(k_0-k'_0)}\prod_{i=1}^{d}\sin(k_i x_i)\sin(k'_i x_i)\\
&-\frac{2^d}{N_x^d N_t}\frac{d\tau}{2m}\sum_{r} e^{-it(k_0-k'_0)}\prod_{i=1}^{d}\sin(k_i x_i)\sum_{i=1}^{d}\sin(k'_i x_i + k'_i)\\
&-\frac{2^d}{N_x^d N_t}\frac{d\tau}{2m}\sum_{r} e^{-it(k_0-k'_0)}\prod_{i=1}^{d}\sin(k_i x_i)\sum_{i=1}^{d}\sin(k'_i x_i - k'_i).
\end{aligned} \tag{A.76}$$

To further expand the last two lines, we use the following trig identity: $\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$, which gives us:

$$\begin{aligned}
D_{k,k'} &= \frac{2^d}{N_x^d N_t}\left(1+\frac{d\tau d}{m} - e^{d\tau\mu}e^{-ik'_0}\right)\sum_r e^{-it(k_0-k'_0)}\prod_{i=1}^d \sin(k_i x_i)\sin(k'_i x_i) \\
&-\frac{2^d}{N_x^d N_t}\frac{d\tau}{2m}\sum_{i=1}^d \cos(k'_i)\sum_r e^{-it(k_0-k'_0)}\prod_{i=1}^d \sin(k_i x_i)\sin(k'_i x_i) \\
&-\frac{2^d}{N_x^d N_t}\frac{d\tau}{2m}\sum_{i=1}^d \sin(k'_i)\sum_r e^{-it(k_0-k'_0)}\prod_{i=1}^d \sin(k_i x_i)\cos(k'_i x_i) \\
&-\frac{2^d}{N_x^d N_t}\frac{d\tau}{2m}\sum_{i=1}^d \cos(k'_i)\sum_r e^{-it(k_0-k'_0)}\prod_{i=1}^d \sin(k_i x_i)\sin(k'_i x_i) \\
&+\frac{2^d}{N_x^d N_t}\frac{d\tau}{2m}\sum_{i=1}^d \sin(k'_i)\sum_r e^{-it(k_0-k'_0)}\prod_{i=1}^d \sin(k_i x_i)\cos(k'_i x_i).
\end{aligned} \tag{A.77}$$

Using the following Fourier identities

$$\begin{aligned}
\sum_x \sin(kx)\sin(k'x) &= \frac{N_x}{2}\delta_{k,k'} \\
\sum_x \sin(kx)\cos(k'x) &= 0 \\
\sum_x e^{-ix(k-k')} &= N_x\delta_{k,k'}
\end{aligned} \tag{A.78}$$

$$\tag{A.79}$$

we find that

$$\begin{aligned}
D_{k,k'} &= \frac{2^d}{N_x^d N_t}\left(1+\frac{d\tau d}{m} - e^{d\tau\mu}e^{-ik'_0} - \frac{d\tau}{m}\sum_{i=1}^d \cos(k'_i)\right) N_t\delta_{k_0,k'_0}\prod_{i=1}^d \frac{N_{x_i}}{2}\delta_{k_i,k'_i} \\
&= \frac{2^d}{N_x^d N_t}\left(1+\frac{d\tau d}{m} - e^{d\tau\mu}e^{-ik'_0} - \frac{d\tau}{m}\sum_{i=1}^d \cos(k'_i)\right) N_t\left(\frac{N_x}{2}\right)^d \delta_{k,k'} \\
D_{k,k'} &= \left(1+\frac{d\tau d}{m} - e^{d\tau\mu}e^{-ik'_0} - \frac{d\tau}{m}\sum_{i=1}^d \cos(k'_i)\right)\delta_{k,k'},
\end{aligned}$$

or, slightly rearranged:

$$D_{k,k'} = \left(1 - e^{d\tau\mu}e^{-ik'_0} + \frac{d\tau}{m}\sum_{i=1}^d (1-\cos(k'_i))\right)\delta_{k,k'}. \tag{A.80}$$

Note that this is a complex matrix, with real and imaginary parts:

$$\mathrm{Re}\left[D_{k,k'}\right] = \left(1 - e^{d\tau\mu}\cos(k'_0) + \frac{d\tau}{m}\sum_{i=1}^d (1-\cos(k'_i))\right)\delta_{k,k'} \tag{A.81}$$

$$\mathrm{Im}\left[D_{k,k'}\right] = e^{d\tau\mu}\sin(k'_0)\delta_{k,k'}. \tag{A.82}$$

Analytical solution for the nonrotating, noninteracting density

We can now use our diagonal matrix $D_{k,k'} = D_{k,k}$ to solve for the density of this system. Recall that

$$\langle \hat{n} \rangle = \frac{1}{VN_\tau} \sum_k \frac{1}{D_{kk}} \frac{\partial D_{kk}}{\partial (d\tau\mu)}. \tag{A.83}$$

We first need to solve for $\frac{\partial D_{kk}}{\partial (d\tau\mu)}$:

$$\begin{aligned} \frac{\partial D_{kk}}{\partial \mu} &= \frac{\partial}{\partial (d\tau\mu)} \left[\left(1 - e^{d\tau\mu} e^{-ik'_0} + \frac{d\tau}{m} \sum_{i=1}^{d} (1 - \cos(k'_i)) \right) \delta_{k,k'} \right] \\ &= -e^{d\tau\mu} e^{-ik'_0} \delta_{k,k'}. \end{aligned} \tag{A.84}$$

Plugging this in to our equation for the density gives us:

$$\langle \hat{n} \rangle = \frac{1}{N_x^d N_t} \sum_k \frac{D^*_{kk}}{|D_{kk}|^2} \left(-e^{d\tau\mu} e^{-ik'_0} \delta_{k,k'} \right). \tag{A.85}$$

Analytical solution for the nonrotating, noninteracting field modulus squared

$$\langle \phi^* \phi \rangle = \sum_k \frac{1}{D_{kk}} = \sum_k \frac{D^*_{kk}}{|D_{kk}|^2}. \tag{A.86}$$

Analytical solution for the two-point correlation function

The eigenvalues and eigenvectors can be used to compute the solution for two-point correlation function for this system:

$$G(x, x') = \sum_k \frac{1}{D_{kk}} U_k^\dagger(x) U_k(x') \tag{A.87}$$