Deriving the action for the lattice

Continuum action:

$r_\perp^2 = (x^2+y^2)$

$$S[\phi] = \int d^dx\, d\tau \left[\phi^*\left(\partial_\tau - \mu - \frac{\nabla^2}{2m} - \frac{m}{2}\omega_{tr}^2 r_\perp^2 - i\omega_z(x\partial_y - y\partial_x)\right)\phi + \lambda(\phi^*\phi)^2 \right]$$

To discretize:

$$\int d^dx\, d\tau \rightarrow \sum_{\vec{x},\tau} a^d d\tau \quad , \quad \alpha\phi^*\phi \rightarrow \alpha\phi_r^*\phi_{r-\hat{\tau}} \text{ (see appendix)}$$

Discrete action:

$$S[\phi] = \sum_{\vec{x},\tau} a^d d\tau \left[\phi_r^*\left(\partial_\tau - \frac{\nabla^2}{2m}\right)\phi_r - \phi_r^*\left(\mu + \frac{m}{2}\omega_{tr}^2 r_\perp^2 + i\omega_z(x\partial_y - y\partial_x)\right)\phi_{r-\hat{\tau}} + \lambda(\phi_r^*\phi_{r-\hat{\tau}})^2 \right]$$

Discrete derivatives:

$$\partial_j\phi = \frac{1}{a}\left(\phi_r - \phi_{r-\hat{j}}\right) \text{ and } \nabla^2\phi = \sum_{j=1}^{d} \frac{1}{a^2}\left(\phi_{r-\hat{j}} - 2\phi_r + \phi_{r+\hat{j}}\right)$$

$$\phi_r^*\partial_\tau\phi_r = \frac{1}{d\tau}\left(\phi_r^*\phi_{r-\hat{\tau}} - \phi_r^*\phi_r\right)$$

$$\phi_r^*\nabla^2\phi_r = \frac{1}{a^2}\sum_{j=1}^{d}\left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right)$$

$$\phi_r^*\partial_x\phi_{r-\hat{\tau}} = \frac{1}{a}\left(\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{x}}\right), \quad \phi_r^*\partial_y\phi_{r-\hat{\tau}} = \frac{1}{a}\left(\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{y}}\right)$$

So our discrete (lattice) action is:

$$S = \sum_{\vec{x},\tau}\Big[a^d\phi_r^*\phi_r - a^d\phi_r^*\phi_{r-\hat{\tau}} - a^d d\tau\,\mu\,\phi_r^*\phi_{r-\hat{\tau}} - \frac{a^{d-2}d\tau}{2m}\sum_{j=1}^{d}\left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right)$$
$$- \frac{m a^d d\tau}{2}\omega_{tr}^2 r_\perp^2\phi_r^*\phi_{r-\hat{\tau}} + i\omega_z a^{d-1}d\tau\left(x\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - x\phi_r^*\phi_{r-\hat{\tau}} - y\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}\right.$$
$$\left. + y\phi_r^*\phi_{r-\hat{\tau}}\right) + a^d d\tau\,\lambda(\phi_r^*\phi_{r-\hat{\tau}})^2\Big]$$

Let's update our parameters to incorporate $d\tau$ and a

$$x = a\tilde{x} \quad y = a\tilde{y} \quad r^2 = a^2\tilde{r}^2 \quad \mu = \frac{\tilde{\mu}}{d\tau} \quad m = \frac{\tilde{m}\,d\tau}{a^2} \quad \omega_{tr} = \frac{\tilde{\omega}_{tr}}{d\tau}$$

$$\omega_z = \frac{\tilde{\omega}_z}{d\tau} \quad \lambda = \frac{\tilde{\lambda}}{d\tau}$$

$$S = \sum_{\vec{x},\tau} \Big[a^d \phi^*_r \phi_r - a^d \phi^*_r \phi_{r-\hat{\tau}} - a^d \tilde{\mu}\, \phi^*_r \phi_{r-\tau} - \frac{a^d}{2\tilde{m}} \sum_{j=1}^{d} \left(\phi^*_r \phi_{r-\hat{j}} - 2\phi^*_r \phi_r + \phi^*_r \phi_{r+\hat{j}} \right)$$

$$- \frac{\tilde{m} a^d}{2} \tilde{\omega}_{tr}^2 \tilde{r}_\perp^2 \phi^*_r \phi_{r-\tau} + i \tilde{\omega}_z a^d \left(\tilde{x} \phi^*_r \phi_{r-\hat{y}-\hat{\tau}} - \tilde{x} \phi^*_r \phi_{r-\tau} - \tilde{y} \phi^*_r \phi_{r-\hat{x}-\hat{\tau}} + \tilde{y} \phi^*_r \phi_{r-\hat{\tau}} \right)$$

$$+ a^d \tilde{\lambda} \left(\phi^*_r \phi_{r-\hat{\tau}} \right)^2 \Big]$$

We can pull out a constant factor of a^d now:

$$S = \sum_{\vec{x},\tau} a^d \Big[\phi^*_r \phi_r - \phi^*_r \phi_{r-\hat{\tau}} - \tilde{\mu}\, \phi^*_r \phi_{r-\tau} - \frac{1}{2\tilde{m}} \sum_{j=1}^{d} \left(\phi^*_r \phi_{r-\hat{j}} - 2\phi^*_r \phi_r + \phi^*_r \phi_{r+\hat{j}} \right)$$

$$- \frac{\tilde{m}}{2} \tilde{\omega}_{tr}^2 \tilde{r}_\perp^2 \phi^*_r \phi_{r-\tau} + i \tilde{\omega}_z \left(\tilde{x} \phi^*_r \phi_{r-\hat{y}-\hat{\tau}} - \tilde{x} \phi^*_r \phi_{r-\tau} - \tilde{y} \phi^*_r \phi_{r-\hat{x}-\hat{\tau}} + \tilde{y} \phi^*_r \phi_{r-\hat{\tau}} \right) + \tilde{\lambda} \left(\phi^*_r \phi_{r-\hat{\tau}} \right)^2 \Big]$$

Noting that, to second order in $d\tau$, $(1 + d\tau\mu) = e^{d\tau\mu}$, we further simplify:

$$S = \sum_{\vec{x},\tau} a^d \Big[\overbrace{\phi^*_r \phi_r - e^{\tilde{\mu}} \phi^*_r \phi_{r-\tau}}^{S_\mu} \; \overbrace{- \frac{1}{2\tilde{m}} \sum_{j=1}^{d} \left(\phi^*_r \phi_{r-\hat{j}} - 2\phi^*_r \phi_r + \phi^*_r \phi_{r+\hat{j}} \right)}^{S_\nabla}$$

$$- \underbrace{\frac{\tilde{m}}{2} \tilde{\omega}_{tr}^2 \tilde{r}_\perp^2 \phi^*_r \phi_{r-\tau}}_{S_{tr}} + \underbrace{i \tilde{\omega}_z \left(\tilde{x} \phi^*_r \phi_{r-\hat{y}-\hat{\tau}} - \tilde{x} \phi^*_r \phi_{r-\tau} - \tilde{y} \phi^*_r \phi_{r-\hat{x}-\hat{\tau}} + \tilde{y} \phi^*_r \phi_{r-\hat{\tau}} \right)}_{-S_w} + \underbrace{\tilde{\lambda} \left(\phi^*_r \phi_{r-\hat{\tau}} \right)^2}_{S_{int}} \Big]$$

We can make our work easier by writing this out as:

$$S = \sum_{\vec{x},\tau} a^d \left[S_\mu + S_\nabla - S_{trap} - S_w + S_{int} \right]$$

Writing our complex fields in terms of two real components

The next step is to note that $\phi = \phi_1 + i\phi_2$ and apply this coordinate transformation to the lattice.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \text{ and } \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \longrightarrow$$

$$S_\mu = \phi^*_r\phi_r - e^{\tilde{\mu}}\phi^*_r\phi_{r-\tau} \longrightarrow \frac{1}{2}\left[\phi_{1,r}^2 + \phi_{2,r}^2 - e^{\mu}\left(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right)\right]$$

$$S_\nabla = -\frac{1}{2\tilde{m}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}} - 2\phi^*_r\phi_r + \phi^*_r\phi_{r+\hat{j}}\right)$$

$$\begin{aligned}\rightarrow \frac{1}{4\tilde{m}}\sum_{j=1}^{d}\Big[2\phi_{1,r}^2 + 2\phi_{2,r}^2 - \big(&\phi_{1,r}\phi_{1,r-j} - i\phi_{2,r}\phi_{1,r-j} + i\phi_{1,r}\phi_{2,r-j} + \phi_{2,r}\phi_{2,r-j} \\ &+ \phi_{1,r}\phi_{1,r+j} - i\phi_{2,r}\phi_{1,r+j} + i\phi_{1,r}\phi_{2,r+j} + \phi_{2,r}\phi_{2,r+j}\big)\Big]\end{aligned}$$

$$S_{trap} = \frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau} \rightarrow \frac{\tilde{m}}{4}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\left(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right)$$

$$S_\omega = i\tilde{\omega}_z\left(\tilde{x}\phi^*_r\phi_{r-\tau} - \tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}} + \tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}} - \tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)$$

$$\begin{aligned}\rightarrow i\frac{\tilde{\omega}_z}{2}\Big[&\tilde{x}\big(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{1,r}\phi_{1,r-y-\hat{\tau}} + i\phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}} \\ &- i\phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big) + \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}} \\ &+ i\phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\tau} + i\phi_{2,r}\phi_{1,r-\tau} - i\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)\Big]\end{aligned}$$

$$\begin{aligned}= i\frac{\tilde{\omega}_z}{2}\Big[&\tilde{x}\big(\phi_{1,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{1,r}\phi_{1,r-y-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big) \\ &+ \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\big) \\ + \frac{\tilde{\omega}_z}{2}\Big[&\tilde{x}\big(\phi_{2,r}\phi_{1,r-\tau} - \phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}} + \phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}}\big) \\ &+ \tilde{y}\big(\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\tau} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\big)\Big]\end{aligned}$$

$$\begin{aligned}= i\frac{\tilde{\omega}_z}{2}\Big[&(\tilde{x}-\tilde{y})\big(\phi_{1,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\big) - \tilde{x}\big(\phi_{1,r}\phi_{1,r-y-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big) + \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}}\big)\Big] \\ + \frac{\tilde{\omega}_z}{2}\Big[&(\tilde{x}-\tilde{y})\big(\phi_{2,r}\phi_{1,r-\tau} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\big) - \tilde{x}\big(\phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}}\big) + \tilde{y}\big(\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}}\big)\Big]\end{aligned}$$

$$
\begin{aligned}
S_{int} &= \tilde{\lambda}(\phi^*_r \phi_{r-\hat{\tau}})^2 \to \frac{\tilde{\hat{\lambda}}}{4}\left(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right)^2 \\
&= \frac{\tilde{\hat{\lambda}}}{4}\Big(\phi_{1,r}\phi_{1,r-\tau}\phi_{1,r}\phi_{1,r-\tau} - i\phi_{1,r}\phi_{1,r-\tau}\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{1,r-\tau}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{1,r}\phi_{1,r-\tau}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&\quad - i\phi_{2,r}\phi_{1,r-\tau}\phi_{1,r}\phi_{1,r-\tau} - \phi_{2,r}\phi_{1,r-\tau}\phi_{2,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{1,r-\tau}\phi_{1,r}\phi_{2,r-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\tau}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&\quad + i\phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\tau} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\tau} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&\quad + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\tau} + i\phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}}\Big) \\
&= \frac{\tilde{\hat{\lambda}}}{4}\Big(\phi_{1,r}\phi_{1,r-\tau}\phi_{1,r}\phi_{1,r-\tau} + \phi_{1,r}\phi_{1,r-\tau}\phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\tau}\phi_{2,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{1,r-\tau}\phi_{1,r}\phi_{2,r-\hat{\tau}} \\
&\qquad + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\tau} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}}\Big) \\
&\quad + i\frac{\tilde{\hat{\lambda}}}{4}\Big(-\phi_{1,r}\phi_{1,r-\tau}\phi_{2,r}\phi_{1,r-\tau} + \phi_{1,r}\phi_{1,r-\tau}\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\tau}\phi_{1,r}\phi_{1,r-\tau} - \phi_{2,r}\phi_{1,r-\tau}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&\qquad + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\tau} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}}\Big)
\end{aligned}
$$

Now, we can derive the drift functions by taking a complex derivative of the action, S

$$K_a = \frac{\delta}{\delta\phi_a} S[\phi]$$

not sure we even need this second sum...

When taking derivatives on the lattice, recall that $\delta_i \phi_r = \frac{1}{a}(\phi_r - \phi_{r-\hat{i}})$

Therefore:
$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{a=1}^{2}\sum_{q=1}^{N}\phi_{a,q}\,\phi_{a,q+\hat{i}}\right) = \sum_{a=1}^{2}\sum_{q=1}^{N}\left(\phi_{a,q}\frac{\delta}{\delta\phi_{c,r}}\phi_{a,q+\hat{i}} + \frac{\delta}{\delta\phi_{c,r}}\phi_{a,q}\,\phi_{a,q+\hat{i}}\right)$$
$$= \sum_{a=1}^{2}\sum_{q=1}^{N}\left(\phi_{a,q}\,\delta_{c,a}\,\delta_{r,q+\hat{i}} + \delta_{c,a}\,\delta_{r,q}\,\phi_{a,q+\hat{i}}\right)$$

the sums collapse, due to the Kronecker deltas

$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{a=1}^{2}\sum_{q=1}^{N}\phi_{a,q}\,\phi_{a,q+\hat{i}}\right) = \phi_{c,r-\hat{i}}\ \text{if}(r>0) + \phi_{c,r+\hat{i}}\ \text{if}\left(r+\hat{i}<N\right)$$

similarly
$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2}\phi_{a,q}\,\phi_{a,q-\hat{i}}\right) = \phi_{c,r+\hat{i}}\ \text{if}(r+\hat{i}<N) + \phi_{c,r-\hat{i}}\ \text{if}(r>0)$$

and
$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N}\sum_{a=1}^{2}\sum_{b=1}^{2}\varepsilon_{ab}\,\phi_{a,q}\,\phi_{b,q+\hat{i}}\right) = \sum_{b=1}^{2}\varepsilon_{cb}\left(\phi_{b,r-\hat{i}} + \phi_{b,r+\hat{i}}\right)$$

and
$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N}\sum_{a=1}^{2}\sum_{b=1}^{2}\varepsilon_{ab}\,\phi_{a,q}\,\phi_{b,q-\hat{i}}\right) = \sum_{b=1}^{2}\varepsilon_{cb}\left(\phi_{b,r+\hat{i}} + \phi_{b,r-\hat{i}}\right)$$

Question: don't our boundary conditions already satisfy the constraints?
If $r=N$, $\phi_{r+\hat{i}}=0$ automatically

$$K_a = \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau} S_A + \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau} S_D - \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau} S_{trap} - \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau} S_w + \frac{\delta}{\delta\phi_a}\sum_{\vec{x},\tau} S_{int}$$

Let's take this derivative for each part of the action individuall

$$K_{a,\mu,r} = \frac{\delta}{\delta\phi_{ar}} S_{A_r} = \frac{1}{2}\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\left[\phi_{c,r}^{2} - e^{\tilde{\mu}}\phi_{c,r}\,\phi_{c,r-\hat{\tau}} - i e^{\tilde{\mu}}\sum_{b=1}^{2}\varepsilon_{cb}\,\phi_{c,r}\,\phi_{b,r-\hat{\tau}}\right]$$
$$= \frac{1}{2}\left[2\phi_{a,r} - e^{\tilde{\mu}}\left(\phi_{a,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\right) - i e^{\tilde{\mu}}\sum_{b=1}^{2}\varepsilon_{ab}\left(\phi_{b,r+\hat{\tau}} + \phi_{b,r-\hat{\tau}}\right)\right]$$
$$K_{a,\mu,r} = \phi_{a,r} - \frac{1}{2}e^{\tilde{\mu}}\left(\phi_{a,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\right) - \frac{i}{2}e^{\tilde{\mu}}\sum_{b=1}^{2}\varepsilon_{ab}\left(\phi_{b,r+\hat{\tau}} + \phi_{b,r-\hat{\tau}}\right)$$

$$K_{a,\nabla,r} = \frac{\delta}{\delta\Phi_{a,r}} S_{\nabla,r} = \frac{\delta}{\delta\Phi_{a,r}} \frac{1}{4\tilde{m}} \sum_{j=1}^{d}\sum_{c=1}^{2}\Big[2\Phi_{c,r}^2 - \left(\Phi_{c,r}\Phi_{c,r+\hat{j}} + \Phi_{c,r}\Phi_{c,r-\hat{j}}\right)$$
$$- i\sum_{b=1}^{2}\varepsilon_{cb}\left(\Phi_{c,r}\Phi_{b,r-\hat{j}} + \Phi_{c,r}\Phi_{b,r+\hat{j}}\right)\Big]$$
$$= \frac{1}{4\tilde{m}}\sum_{j=1}^{d}\Big[4\Phi_{a,r} - \left(\Phi_{a,r+\hat{j}} + \Phi_{a,r-\hat{j}} + \Phi_{a,r+\hat{j}} + \Phi_{a,r-\hat{j}}\right)$$
$$- i\sum_{b=1}^{2}\varepsilon_{ab}\left(\Phi_{b,r-\hat{j}} + \Phi_{b,r+\hat{j}} + \Phi_{b,r+\hat{j}} + \Phi_{b,r-\hat{j}}\right)\Big]$$
$$= \frac{1}{4\tilde{m}}\sum_{j=1}^{d}\Big[4\Phi_{a,r} - 2\Phi_{a,r+\hat{j}} - 2\Phi_{a,r-\hat{j}} - 2i\sum_{b=1}^{2}\varepsilon_{ab}\left(\Phi_{b,r+\hat{j}} + \Phi_{b,r-\hat{j}}\right)\Big]$$
$$= \frac{1}{2\tilde{m}}\sum_{j=1}^{d}\Big[2\Phi_{a,r} - \left(\Phi_{a,r+\hat{j}} + \Phi_{a,r-\hat{j}}\right) - i\sum_{b=1}^{2}\varepsilon_{ab}\left(\Phi_{b,r+\hat{j}} + \Phi_{b,r-\hat{j}}\right)\Big]$$

$$K_{a,tr,r} = \frac{\delta}{\delta\Phi_{a,r}} S_{tr,r} = \frac{\delta}{\delta\Phi_{a,r}} \frac{\tilde{m}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2}{4}\sum_{c=1}^{2}\Big[\Phi_{c,r}\Phi_{c,r-\hat{\tau}} + i\sum_{b=1}^{2}\varepsilon_{cb}\Phi_{c,r}\Phi_{b,r-\hat{\tau}}\Big]$$
$$= \frac{\tilde{m}\tilde{\omega}_{tr}^2\hat{r}_\perp^2}{4}\Big[\Phi_{a,r-\hat{\tau}} + \Phi_{a,r+\hat{\tau}} + i\sum_{b=1}^{2}\varepsilon_{ab}\left(\Phi_{b,r-\hat{\tau}} + \Phi_{b,r+\hat{\tau}}\right)\Big]$$

$$K_{a,\omega,r} = \frac{d}{\delta\Phi_{a,r}} S_{\omega,r} = \frac{\delta}{\delta\Phi_{a,r}}\frac{\tilde{\omega}_z}{2}\sum_{c=1}^{2}\Big[\sum_{b=1}^{2}\varepsilon_{cb}\Big((\hat{y}-\tilde{x})\Phi_{c,r}\Phi_{b,r-\hat{\tau}} + \tilde{x}\Phi_{c,r}\Phi_{b,r-\hat{y}-\hat{\tau}}$$
$$- \tilde{y}\Phi_{c,r}\Phi_{b,r-\hat{x}-\hat{\tau}}\Big) + i\Big((\tilde{x}-\hat{y})\Phi_{c,r}\Phi_{c,r-\hat{\tau}} - \hat{x}\Phi_{c,r}\Phi_{c,r-\hat{y}-\hat{\tau}} + \tilde{y}\Phi_{c,r}\Phi_{c,r-\hat{x}-\hat{\tau}}\Big)\Big]$$
$$= \frac{\tilde{\omega}_z}{2}\Big[\sum_{b=1}^{2}\varepsilon_{ab}\Big((\tilde{y}-\tilde{x})\left(\Phi_{b,r+\hat{\tau}} + \Phi_{b,r-\hat{\tau}}\right) + \tilde{x}\left(\Phi_{b,r-\hat{y}-\hat{\tau}} + \Phi_{b,r+\hat{y}+\hat{\tau}}\right)$$
$$- \tilde{y}\left(\Phi_{b,r-\hat{x}-\hat{\tau}} + \Phi_{b,r+\hat{x}+\hat{\tau}}\right)\Big) + i\Big((\tilde{x}-\hat{y})\left(\Phi_{a,r-\hat{\tau}} + \Phi_{a,r+\hat{\tau}}\right)$$
$$- \hat{x}\left(\Phi_{a,r-\hat{y}-\hat{\tau}} + \Phi_{a,r+\hat{y}+\hat{\tau}}\right) + \tilde{y}\left(\Phi_{a,r-\hat{x}-\hat{\tau}} + \Phi_{a,r+\hat{x}+\hat{\tau}}\right)\Big)\Big]$$

$$K_{a,\lambda,r} = \frac{\delta}{\delta\Phi_{a,r}} S_{\lambda,r} = \frac{\delta}{\delta\Phi_{a,r}}\frac{\bar{\lambda}}{4}\sum_{c=1}^{2}\sum_{b=1}^{2}\Big[2\Phi_{c,r}\Phi_{c,r-\hat{\tau}}\Phi_{b,r}\Phi_{b,r-\hat{\tau}} - \Phi_{c,r}^2\Phi_{b,r-\hat{\tau}}^2$$
$$+ 2i\varepsilon_{cb}\left(\Phi_{c,r}^2\Phi_{c,r-\hat{\tau}}\Phi_{b,r-\hat{\tau}} - \Phi_{c,r}\Phi_{b,r}\Phi_{b,r-\hat{\tau}}^2\right)\Big]$$

$$= \frac{\bar{\lambda}}{4}\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\sum_{b=1}^{2}\Big[\underbrace{2\phi_{c,r}\phi_{c,r-\hat{\tau}}\phi_{b,r}\phi_{b,r-\hat{\tau}}}_{A} - \underbrace{\phi_{c,r}^{2}\phi_{b,r-\hat{\tau}}^{2}}_{B}$$
$$+2i\varepsilon_{cb}\Big(\underbrace{\phi_{c,r}^{2}\phi_{c,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}}_{C} - \underbrace{\phi_{c,r}\phi_{b,r}\phi_{b,r-\hat{\tau}}^{2}}_{D}\Big)\Big]$$

A:

$$\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\sum_{b=1}^{2}\left(2\phi_{c,r'}\phi_{c,r'-\hat{\tau}}\phi_{b,r'}\phi_{b,r'-\tau}\right)$$
$$=2\sum_{b,c=1}^{2}\left[\phi_{c,r'}\phi_{b,r'}\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\tau}\right)+\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'}\phi_{b,r'}\right)\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}\right]$$
$$=2\sum_{b,c=1}^{2}\Big[\phi_{c,r'}\phi_{b,r'}\phi_{c,r'-\hat{\tau}}\frac{\delta\phi_{b,r'-\tau}}{\delta\phi_{a,r}}+\phi_{c,r'}\phi_{b,r'}\frac{\delta\phi_{c,r'-\hat{\tau}}}{\delta\phi_{a,r}}\phi_{b,r'-\hat{\tau}}$$
$$+\phi_{c,r'}\frac{\delta\phi_{b,r'}}{\delta\phi_{a,r}}\phi_{c,r'-\tau}\phi_{b,r'-\tau}+\frac{\delta\phi_{c,r'}}{\delta\phi_{a,r}}\phi_{b,r'}\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\tau}\Big]$$
$$=2\sum_{b,c=1}^{2}\Big[\phi_{c,r'}\phi_{b,r'}\phi_{c,r'-\hat{\tau}}\delta_{r,r'-\tau}\delta_{ab}+\phi_{c,r'}\phi_{b,r'}\delta_{r,r'-\hat{\tau}}\delta_{ac}\phi_{b,r'-\hat{\tau}}$$
$$+\phi_{c,r'}\delta_{rr'}\delta_{ab}\phi_{c,r'-\tau}\phi_{b,r'-\tau}+\delta_{rr'}\delta_{ac}\phi_{b,r'}\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\tau}\Big]$$
$$=2\sum_{b,c=1}^{2}\Big(\phi_{c,r+\hat{\tau}}\phi_{b,r+\hat{\tau}}\phi_{c,r}\delta_{ab}+\phi_{c,r+\hat{\tau}}\phi_{b,r+\hat{\tau}}\phi_{b,r}\delta_{ac}$$
$$+\phi_{c,r}\phi_{c,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\delta_{ab}+\phi_{b,r}\phi_{c,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\delta_{ac}\Big)$$
$$=2\Big(\sum_{c=1}^{2}\phi_{c,r+\hat{\tau}}\phi_{a,r+\hat{\tau}}\phi_{c,r}+\sum_{b=1}^{2}\phi_{a,r+\hat{\tau}}\phi_{b,r+\hat{\tau}}\phi_{b,r}$$
$$+\sum_{c=1}^{2}\phi_{c,r}\phi_{c,r-\hat{\tau}}\phi_{a,r-\hat{\tau}}+\sum_{b=1}^{2}\phi_{b,r}\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\Big)$$

since independent sums have independent indices, we can now change our sums over c to be sums over b

$$=2\Big(2\sum_{b=1}^{2}\phi_{b,r}\phi_{b,r+\hat{\tau}}\phi_{a,r+\hat{\tau}}+2\sum_{b=1}^{2}\phi_{b,r}\phi_{b,r-\hat{\tau}}\phi_{a,r-\hat{\tau}}\Big)$$
$$=4\sum_{b=1}^{2}\phi_{b,r}\left(\phi_{b,r+\hat{\tau}}\phi_{a,r+\hat{\tau}}+\phi_{b,r-\hat{\tau}}\phi_{a,r-\hat{\tau}}\right) \quad A$$

B: $$-\frac{\delta}{\delta\phi_{a,r}}\sum_{b=1}^{2}\left(\phi_{c,r'}^{2}\phi_{b,r-\hat{\tau}}^{2}\right)$$

$$= -\sum_{b=1}^{2}\sum_{c=1}^{2}\left[\phi_{c,r'}^{2}\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{b,r'-\hat{\tau}}^{2}\right)+\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'}^{2}\right)\phi_{b,r'\hat{\tau}}^{2}\right]$$

$$= -\sum_{b=1}^{2}\sum_{c=1}^{2}\left[2\phi_{c,r'}^{2}\phi_{b,r'-\hat{\tau}}\delta_{a,b}\delta_{r,r'-\hat{\tau}}+2\phi_{c,r'}\delta_{ac}\delta_{rr'}\phi_{b,r'\hat{\tau}}^{2}\right]$$

$$= -2\left[\sum_{c=1}^{2}\phi_{c,r+\hat{\tau}}^{2}\phi_{a,r}+\sum_{b=1}^{2}\phi_{a,r}\phi_{b,r-\hat{\tau}}^{2}\right]$$

sum over $c \to$ sum over b

$$= -2\sum_{b=1}^{2}\phi_{a,r}\left(\phi_{b,r+\hat{\tau}}^{2}+\phi_{b,r-\hat{\tau}}^{2}\right)$$ B

C: $$\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\sum_{b=1}^{2}2i\varepsilon_{cb}\,\phi_{c,r'}^{2}\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}$$

$$= \sum_{c=1}^{2}\sum_{b=1}^{2}2i\varepsilon_{cb}\left[\phi_{c,r'}^{2}\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}\right)+\frac{\delta}{\delta\phi_{a,r}}\left(\phi_{c,r'}^{2}\right)\phi_{c,r'-\hat{\tau}}\phi_{b,r'\hat{\tau}}\right]$$

$$= \sum_{c=1}^{2}\sum_{b=1}^{2}2i\varepsilon_{cb}\Big[\phi_{c,r'}^{2}\phi_{c,r'-\hat{\tau}}\delta_{ab}\delta_{r,r'-\hat{\tau}}+\phi_{c,r'}^{2}\delta_{ac}\delta_{r,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}$$

$$+2\phi_{c,r'}\delta_{ac}\delta_{rr'}\phi_{c,r'-\hat{\tau}}\phi_{b,r'-\hat{\tau}}\Big]$$

$$= 2i\left[\sum_{c=1}^{2}\varepsilon_{ca}\phi_{c,r+\hat{\tau}}^{2}\phi_{c,r}+\sum_{b=1}^{2}\varepsilon_{ab}\phi_{a,r+\hat{\tau}}^{2}\phi_{b,r}+2\sum_{b=1}^{2}\varepsilon_{ab}\phi_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\right]$$

sum over $c \to$ sum over b C

$$= 2i\sum_{b=1}^{2}\varepsilon_{ab}\left[\phi_{b,r}\phi_{a,r+\hat{\tau}}^{2}-\phi_{b,r}\phi_{b,r+\hat{\tau}}^{2}+2\phi_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}\right]$$

NOTE: minus sign here is because $\varepsilon_{ab}=-\varepsilon_{ba}$!

$$D: \; -\frac{\delta}{\delta\Phi_{a,r}} \sum_{c=1}^{2}\sum_{b=1}^{2}\left[2i\,\varepsilon_{cb}\,\Phi_{c,r'}\,\Phi_{b,r'}\,\Phi^2_{b,r'-\hat{\tau}}\right]$$

$$= -2i \sum_{c=1}^{2}\sum_{b=1}^{2} \varepsilon_{cb}\left[\Phi_{c,r'}\Phi_{b,r'}\frac{\delta}{\delta\Phi_{a,r}}\left(\Phi^2_{b,r'-\hat{\tau}}\right) + \frac{\delta}{\delta\Phi_{a,r}}\left(\Phi_{c,r'}\Phi_{b,r'}\right)\Phi^2_{b,r'-\hat{\tau}}\right]$$

$$= -2i \sum_{c=1}^{2}\sum_{b=1}^{2} \varepsilon_{cb}\left[2\Phi_{c,r'}\Phi_{b,r'}\,\delta_{ab}\,\delta_{r,r'-\hat{\tau}}\,\Phi_{b,r'-\hat{\tau}} + \Phi_{c,r'}\,\delta_{ab}\,\delta_{rr'}\,\Phi^2_{b,r'-\hat{\tau}} + \delta_{ac}\,\delta_{rr'}\,\Phi_{b,r'}\,\Phi^2_{b,r'-\hat{\tau}}\right]$$

$$= -2i\left[\sum_{c=1}^{2}\varepsilon_{ca}\,2\Phi_{c,r+\hat{\tau}}\,\Phi_{a,r+\hat{\tau}}\,\Phi_{a,r} + \sum_{c=1}^{2}\varepsilon_{ca}\,\Phi_{c,r}\,\Phi^2_{a,r-\hat{\tau}} + \sum_{b=1}^{2}\varepsilon_{ab}\,\Phi_{b,r}\,\Phi^2_{b,r-\hat{\tau}}\right]$$

sum over $c \to$ sum over b

$$D \quad = -2i\sum_{b=1}^{2}\varepsilon_{ab}\left[\Phi_{b,r}\,\Phi^2_{b,r-\hat{\tau}} - \Phi_{b,r}\,\Phi^2_{a,r-\hat{\tau}} - 2\Phi_{a,r}\,\Phi_{a,r+\hat{\tau}}\,\Phi_{b,r+\hat{\tau}}\right]$$

NOTE: minus signs again come from $\varepsilon_{ab} = -\varepsilon_{ba}$

CHECK:

$$(A+B)_{a\to 1} = \frac{\delta}{\delta\Phi_{1,r}}\Big(\underbrace{\Phi^2_{1,r}\Phi^2_{1,r-\hat{\tau}} + \Phi^2_{2,r}\Phi^2_{2,r-\hat{\tau}} + 4\Phi_{1,r}\Phi_{2,r}\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}} - \Phi^2_{1,r}\Phi^2_{2,r-\hat{\tau}} - \Phi^2_{1,r-\hat{\tau}}\Phi^2_{2,r}}_{\mathrm{Re}[S_\lambda]/(\lambda/4)}\Big)$$

$$(A+B)_{a\to 1} = 4\Phi_{1,r}\left(\Phi^2_{1,r+\hat{\tau}} + \Phi^2_{1,r-\hat{\tau}}\right) + 4\Phi_{2,r}\left(\Phi_{2,r+\hat{\tau}}\Phi_{1,r+\hat{\tau}} + \Phi_{2,r-\hat{\tau}}\Phi_{1,r-\hat{\tau}}\right)$$
$$-2\Phi_{1,r}\left(\Phi^2_{1,r+\hat{\tau}} + \Phi^2_{1,r-\hat{\tau}}\right) - 2\Phi_{1,r}\left(\Phi^2_{2,r+\hat{\tau}} + \Phi^2_{2,r-\hat{\tau}}\right)$$

$$= 2\Phi_{1,r}\left(\Phi^2_{1,r+\hat{\tau}} + \Phi^2_{1,r-\hat{\tau}}\right) + 4\Phi_{2,r}\left(\Phi_{2,r+\hat{\tau}}\Phi_{1,r+\hat{\tau}} + \Phi_{2,r-\hat{\tau}}\Phi_{1,r-\hat{\tau}}\right)$$
$$-2\Phi_{1,r}\left(\Phi^2_{2,r+\hat{\tau}} + \Phi^2_{2,r-\hat{\tau}}\right)$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(\Phi_{1,r'}^2\Phi_{1,r'-\hat{\tau}}^2+\Phi_{2,r'}^2\Phi_{2,r'-\hat{\tau}}^2+4\Phi_{1,r'}\Phi_{2,r'}\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\hat{\tau}}-\Phi_{1,r'}^2\Phi_{2,r'-\hat{\tau}}^2-\Phi_{1,r'-\hat{\tau}}^2\Phi_{2,r'}^2\right)$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(\Phi_{1,r'}^2\Phi_{1,r'-\hat{\tau}}^2\right)=2\Phi_{1,r'}\Phi_{1,r'-\hat{\tau}}^2\delta_{rr'}+2\Phi_{1,r'}^2\Phi_{1,r'-\tau}\delta_{r,r'-\hat{\tau}}$$

$$=2\Phi_{1,r}\Phi_{1,r-\hat{\tau}}^2+2\Phi_{1,r}\Phi_{1,r+\hat{\tau}}^2$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(\Phi_{2,r'}^2\Phi_{2,r'-\hat{\tau}}^2\right)=0$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(4\Phi_{1,r'}\Phi_{2,r'}\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\hat{\tau}}\right)=4\Phi_{2,r'}\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\hat{\tau}}\delta_{rr'}$$

$$+4\Phi_{1,r'}\Phi_{2,r'}\Phi_{2,r'-\tau}\delta_{rr'-\hat{\tau}}=4\Phi_{2,r}\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}}+4\Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}}\Phi_{2,r}$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(-\Phi_{1,r'}^2\Phi_{2,r'-\hat{\tau}}^2\right)=-2\Phi_{1,r'}\Phi_{2,r'-\hat{\tau}}^2\delta_{rr'}=-2\Phi_{1,r}\Phi_{2,r-\hat{\tau}}^2$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(-\Phi_{1,r'-\hat{\tau}}^2\Phi_{2,r'}^2\right)=-2\Phi_{1,r'-\tau}\Phi_{2,r'}^2\delta_{rr'-\hat{\tau}}=-2\Phi_{1,r}\Phi_{2,r+\hat{\tau}}^2$$

all together:

$$=2\Phi_{1,r}\Phi_{1,r-\hat{\tau}}^2+2\Phi_{1,r}\Phi_{1,r+\hat{\tau}}^2+4\Phi_{2,r}\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}}+4\Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}}\Phi_{2,r}$$
$$-2\Phi_{1,r}\Phi_{2,r-\hat{\tau}}^2-2\Phi_{1,r}\Phi_{2,r+\hat{\tau}}^2$$

$$=2\Phi_{1,r}\left(\Phi_{1,r-\hat{\tau}}^2+\Phi_{1,r+\hat{\tau}}^2\right)+4\Phi_{2,r}\left(\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}}+\Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}}\right)$$
$$-2\Phi_{2,r}\left(\Phi_{2,r-\hat{\tau}}^2+\Phi_{2,r+\hat{\tau}}^2\right)$$

$$=2\Phi_{1,r}\left(\Phi_{1,r+\hat{\tau}}^2+\Phi_{1,r-\hat{\tau}}^2\right)+4\Phi_{2,r}\left(\Phi_{2,r+\hat{\tau}}\Phi_{1,r+\hat{\tau}}+\Phi_{2,r-\hat{\tau}}\Phi_{1,r-\hat{\tau}}\right)$$
$$-2\Phi_{1,r}\left(\Phi_{2,r+\hat{\tau}}^2+\Phi_{2,r-\hat{\tau}}^2\right)$$

CHECK

$$(C+D)_{a\to1} = \frac{\delta}{\delta\Phi_{1,r}}\overbrace{\left[\sum_{a=1}^{2}\sum_{b=1}^{2} 2i\,\varepsilon_{ab}\left(\Phi_{a,r}^2\Phi_{a,r-\hat{\tau}}\Phi_{b,r-\hat{\tau}} - \Phi_{a,r}\Phi_{b,r}\Phi_{b,r-\hat{\tau}}^2\right)\right]}^{\mathrm{Im}(S_\lambda)(A.14)}$$

$$(C+D)_{a\to1} = 2i\sum_{b=1}^{2}\varepsilon_{1b}\Big[\Phi_{b,r}\Phi_{1,r+\hat{\tau}}^2 - \Phi_{b,r}\Phi_{b,r+\hat{\tau}}^2 + 2\Phi_{1,r}\Phi_{1,r-\hat{\tau}}\Phi_{b,r-\hat{\tau}}$$
$$- \Phi_{b,r}\Phi_{b,r-\hat{\tau}}^2 + \Phi_{b,r}\Phi_{1,r-\hat{\tau}}^2 + 2\Phi_{1,r}\Phi_{1,r+\hat{\tau}}\Phi_{b,r+\hat{\tau}}\Big]$$

$$= 2i\Big[\Phi_{2,r}\Phi_{1,r+\hat{\tau}}^2 - \Phi_{2,r}\Phi_{2,r+\hat{\tau}}^2 + 2\Phi_{1,r}\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}} - \Phi_{2,r}\Phi_{2,r-\hat{\tau}}^2$$
$$+ \Phi_{2,r}\Phi_{1,r-\hat{\tau}}^2 + 2\Phi_{1,r}\Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}}\Big]$$

$$= 2i\Big[\Phi_{2,r}\left(\Phi_{1,r+\hat{\tau}}^2 + \Phi_{1,r-\hat{\tau}}^2\right) - \Phi_{2,r}\left(\Phi_{2,r+\hat{\tau}}^2 + \Phi_{2,r-\hat{\tau}}^2\right)$$
$$+ 2\Phi_{1,r}\left(\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}} + \Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}}\right)\Big]$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left[\sum_{a=1}^{2}\sum_{b=1}^{2} 2i\,\varepsilon_{ab}\left(\Phi_{a,r'}^2\Phi_{a,r'-\hat{\tau}}\Phi_{b,r'-\hat{\tau}} - \Phi_{a,r'}\Phi_{b,r'}\Phi_{b,r'-\hat{\tau}}^2\right)\right]$$

$$= 2i\frac{\delta}{\delta\Phi_{1,r}}\left[\Phi_{1,r'}^2\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\hat{\tau}} - \Phi_{1,r'}\Phi_{2,r'}\Phi_{2,r'-\hat{\tau}}^2 - \Phi_{2,r'}^2\Phi_{2,r'-\hat{\tau}}\Phi_{1,r'-\hat{\tau}} + \Phi_{2,r'}\Phi_{1,r'}\Phi_{1,r'-\hat{\tau}}^2\right]$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(\Phi_{1,r'}^2\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\hat{\tau}}\right) = \Phi_{1,r'}^2\Phi_{2,r'-\hat{\tau}}\,\delta_{r,r'-\hat{\tau}} + 2\Phi_{1,r'}\Phi_{1,r'-\hat{\tau}}\Phi_{2,r'-\tau}\,\delta_{rr'}$$
$$= \Phi_{1,r+\hat{\tau}}^2\Phi_{2,r} + 2\Phi_{1,r}\Phi_{1,r-\hat{\tau}}\Phi_{2,r-\hat{\tau}}$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(-\Phi_{1,r'}\Phi_{2,r'}\Phi_{2,r'-\hat{\tau}}^2\right) = -\Phi_{2,r'}\Phi_{2,r'-\hat{\tau}}^2\,\delta_{rr'} = -\Phi_{2,r}\Phi_{2,r-\hat{\tau}}^2$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(-\Phi_{2,r'}^2\Phi_{2,r'-\hat{\tau}}\Phi_{1,r'-\hat{\tau}}\right) = -\Phi_{2,r'}^2\Phi_{2,r'-\hat{\tau}}\,\delta_{r,r'-\hat{\tau}} = -\Phi_{2,r}\Phi_{2,r+\hat{\tau}}^2$$

$$\frac{\delta}{\delta\Phi_{1,r}}\left(\Phi_{2,r'}\Phi_{1,r'}\Phi_{1,r'-\hat{\tau}}^2\right) = \Phi_{1,r'}\Phi_{2,r'}\left(2\Phi_{1,r'-\hat{\tau}}\right)\delta_{r,r'-\hat{\tau}} + \Phi_{2,r'}\Phi_{1,r'-\hat{\tau}}^2\,\delta_{rr'}$$
$$= 2\Phi_{1,r}\Phi_{1,r+\hat{\tau}}\Phi_{2,r+\hat{\tau}} + \Phi_{2,r}\Phi_{1,r-\hat{\tau}}^2$$

all together:

$$= 2i\Big[\phi_{1,r+\hat{\tau}}^2 \phi_{2,r} + 2\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}^2 - \phi_{2,r}\phi_{2,r+\hat{\tau}}^2$$
$$+ 2\phi_{1,r}\phi_{1,r+\hat{\tau}}\phi_{2,r+\hat{\tau}} + \phi_{2,r}\phi_{1,r-\hat{\tau}}^2\Big]$$

$$= 2i\Big[\phi_{2,r}\left(\phi_{1,r+\hat{\tau}}^2 + \phi_{1,r-\hat{\tau}}^2\right) - \phi_{2,r}\left(\phi_{2,r-\hat{\tau}}^2 + \phi_{2,r+\hat{\tau}}^2\right)$$
$$+ 2\phi_{1,r}\left(\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} + \phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}}\right)\Big]$$

$$= 2i\Big[\phi_{2,r}\left(\phi_{1,r+\hat{\tau}}^2 + \phi_{1,r-\hat{\tau}}^2\right) - \phi_{2,r}\left(\phi_{2,r+\hat{\tau}}^2 + \phi_{2,r-\hat{\tau}}^2\right)$$
$$+ 2\phi_{1,r}\left(\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} + \phi_{1,r+\hat{\tau}}\phi_{2,r+\hat{\tau}}\right)\Big]$$

Next, we complexify our real fields, ϕ_a $(a=1,2)$, $\phi_a \to \phi_a^R + i\phi_a^I$

ERCAP Proposal

Strategy: propose 2x as much time as you think you'll need

N_t range: 320-480

N_x range: 81x81 or 101x101 ← justify using ω_{tr}^2

$nL = 10^6$, save every 100 steps

Storage space?

include 2-3 runs w/ very large N_x to get density profiles (to see vortices)

↳ argue for the importance of supercomputing for resolving these lattices

try to come up with an estimate for how many sites you need to see vortices
maybe plot the density profiles for a trapped, not rotating system to show how the trap confines most of the fluid unto a smaller area in the center.

Check old free gas data → does it match new stuff w/ right parameters?