NonRelativistic Rotating Bosons via Complex Langevin

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Contents

Chapter 1

The Method: Complex Langevin

1.1 Motivation

In 1946, Fritz London first proposed that superconductivity and superfluidity were "quantum mechanisms on a macroscopic scale" [5]. Additionally, he linked superfluidity with the (then only theoretical and not observed) mechanism of Bose-Einstein condensation. Since London's early insights into superfluid behavior, the system has been studied extensively using super-cooled atoms [5].

In 1949, Lars Onsager predicted that vortices would form in rotating superfluids [4]. Richard Feynman expanded on Onsager's prediction a few years later, reiterating the expectation that quantized vortices would appear when superfluids were forced to rotate [2]. Another 30 years after these predictions, the first direct observation of quantum vortices was made in rotating superfluid helium [6].

Superfluid velocity has no curl, and therefore is irrotational. However, nonzero hydrodynamic circulation can exist, and must be quantized in units of $2\pi\hbar/m$ [5]. This value can be measured in experiment, and will provide a useful point of comparison for this project. This disconnect between the finite circulation and the irrotational superfluid velocity forces the superfluid to form singular regions in the density, leading to the formation of vortices. The lowest energy configuration of these vortices is a triangular lattice - a Wigner crystal.

Experimentally, great progress has been made in studying rotating superfluids since the first direct observation of vortex formation. In 2000, vortex formation was observed in stirred, magnetically-trapped rubidium atoms [3]. The next year, the Ketterle Group at MIT observed triangular vortex lattices of up to 130 vortices in rotating ultracold sodium atoms [1]. Ultracold atoms provide a highly controlled, tuneable setting for studying vortex formation and other properties of rotating superfluids. Theoretically, treatment of these systems has stalled due to the presence of the sign problem. This project is an attempt to apply the complex Langevin method to circumvent the sign problem in a rotating superfluid.

1.2 Action and formalism

We examine in this project a $2+1$ dimensional system of nonrelativistic bosons of mass m with a contact interaction λ in an external harmonic trap of frequency ω_{tr} and experiencing a rotation of frequency ω_z. Both the harmonic trap and the rotation are centered around the midpoint of the lattice, and we use hard-wall boundary conditions in the spatial extent of the lattice and periodic boundary conditions in Euclidean time.

For rotation, we need at least $2+1$ dimensions. I will occasionally express equations for the general $d+1$ dimensional case, but initial computational tests will all be in $2+1$ dimensions. We can easily test $1+1$ dimensional nonrotating systems, but going to $3+1$ dimensions will require a significant expansion of the code.

Our path integral for the system is

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]} \tag{1.1}$$

where our action is defined in the next section.

1.2.1 The Action

The action for a trapped, rotating, and interacting non-relativistic system in $d+1$ Euclidean dimensions follows this general formula:

$$S = \int \mathrm{d}^d x \, \mathrm{d}\tau \left[\phi^* \left(\mathcal{H} - \mu - V_{\mathrm{tr}} - \omega_z L_z\right)\phi + \lambda(\phi^*\phi)^2\right]. \tag{1.2}$$

Specifically, in two spatial dimensions, this becomes

$$S = \int \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau \left[\phi^* \left(\mathcal{H} - \mu - \frac{m}{2}\omega_{\mathrm{tr}}^2 r_\perp^2 - \omega_z L_z\right)\phi + \lambda(\phi^*\phi)^2\right] \tag{1.3}$$

where $\mathcal{H} = \partial_\tau - \frac{\nabla^2}{2m}$. The trapping potential is harmonic, $\omega_{\mathrm{tr}} r_\perp^2$ where $r_\perp^2 = x^2 + y^2$, and the angular momentum is defined in terms of the quantum mechanical operator, $\omega_z L_z = i\omega_z(x\partial_y - y\partial_x)$. For both the trap and the angular momentum, x and y are measured relative to the origin.

To take this to a lattice representation, we must discretize the space. The origin becomes the center of the lattice, and therefore x and y are measured relative to the point $(N_x/2, N_x/2)$ on the lattice.

Using a backward-difference derivative and denoting the position on the $2+1$ dimensional lattice as r, we define: $\partial_j \phi = \frac{1}{a}(\phi_{r-\hat{j}} - \phi_r)$ and $\partial_j^2 \phi = \frac{1}{a^2}(\phi_{r-\hat{j}} - 2\phi_r + \phi_{r-\hat{j}})$. We combine $\partial_\tau - \mu$ in the lattice representation and similarly represent the angular momentum, interaction, and external trap as external gauge fields in order to avoid divergences in the continuum limit (see Appendix A.1 for details and justification of these steps). Finally, using spatial lattice site separation $a = 1$ and temporal lattice spacing $d\tau$, and after scaling our parameters to be dimensionless lattice parameters (again, see Appendix A.1), our lattice action becomes:

$$\begin{aligned} S_{\mathrm{lat}} \;=\; \sum_{\vec{x},\tau} a^d \Bigg[& \phi_r^*\phi_r - e^{\bar{\mu}}\phi_r^*\phi_{r-\hat{\tau}} - \frac{1}{2\bar{m}}\sum_{j=1}^{d}\left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right) - \frac{\bar{m}}{2}\bar{\omega}_{\mathrm{tr}}^2 \bar{r}_\perp^2 \phi_r^*\phi_{r-\hat{\tau}} \\ & + i\bar{\omega}_z\left(\bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{\tau}}\right) + \bar{\lambda}\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2\Bigg] \end{aligned} \tag{1.4}$$

Note that for $d < 2$, we omit the rotational term, as it requires at least two spatial dimensions. We only consider cases of dimensionality less than two for testing our algorithm against exactly-soluble cases such as the free gas. In addition, going to $d > 2$ would require significant restructuring of the code, so we reserve that for future work.

To simplify our future work, let's write S as a sum of the different contributions to the action:

$$S_{\mathrm{lat}} = (S_\mu + S_\nabla - S_{\mathrm{tr}} - S_\omega + S_{\mathrm{int}}) \tag{1.5}$$

with

$$S_{\mu,r} = \sum_{\vec{x},\tau} a^d \left(\phi_r^*\phi_r - e^{\bar{\mu}}\phi_r^*\phi_{r-\hat{\tau}}\right) \tag{1.6}$$

$$S_{\nabla,r} = \sum_{\vec{x},\tau} a^d \left(\frac{1}{2\bar{m}}\sum_{j=1}^{d}\left(2\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{j}} - \phi_r^*\phi_{r+\hat{j}}\right)\right) \tag{1.7}$$

$$S_{\mathrm{tr},r} = \sum_{\vec{x},\tau} a^d \left(\frac{\bar{m}}{2}\bar{\omega}_{\mathrm{tr}}^2\bar{r}_\perp^2\phi_r^*\phi_{r-\hat{\tau}}\right) \tag{1.8}$$

$$S_{\omega,r} = \sum_{\vec{x},\tau} a^d \left(i\bar{\omega}_z\left(\bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}\right)\right) \tag{1.9}$$

$$S_{\mathrm{int},r} = \sum_{\vec{x},\tau} a^d \left(\bar{\lambda}\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2\right). \tag{1.10}$$

From here on, we will be working with these individual contributions to the action.

1.3 The Langevin Equations

In order to treat this action composed of complex-valued fields, we use a method called complex Langevin (CL). This method uses a stochastic evolution of the complex fields in a fictitious time – Langevin time – in order to produce sets of solutions distributed according to the weight e^{-S}. Just as standard Monte Carlo methods operate by sampling from the distribution e^{-S}, this method allows us to stochastically evaluate observables whose physical behavior is governed by our action, S.

First, we must write our complex fields as a complex sum of two real fields (i.e. $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. This is worked out in Appendix A.2 for all the contributions to our action. We then complexify the real fields and evolve the four resulting components (ϕ_1^R, ϕ_1^I, ϕ_2^R, and ϕ_2^I) according to the complex Langevin equations, a set of coupled stochastic differential equations shown here:

$$\phi_{a,r}^R(n+1) = \phi_{a,r}^R(n) + \epsilon K_{a,r}^R(n) + \sqrt{\epsilon}\eta_{a,r}(n) \tag{1.11}$$

$$\phi_{a,r}^I(n+1) = \phi_{a,r}^I(n) + \epsilon K_{a,r}^I(n), \tag{1.12}$$

where $a = 1, 2$ labels our two real fields, η is Gaussian-distributed real noise with mean of 0 and standard deviation of $\sqrt{2}$. The drift functions, K, are derived from the action:

$$K_{a,r}^R = -\mathrm{Re}\left[\frac{\delta S}{\delta\phi_{a,r}}|_{\phi_a\to\phi_a^R+i\phi_a^I}\right] \tag{1.13}$$

$$K_{a,r}^I = -\mathrm{Im}\left[\frac{\delta S}{\delta\phi_{a,r}}|_{\phi_a\to\phi_a^R+i\phi_a^I}\right] \tag{1.14}$$

The derivation of the Langevin equations is worked out in Appendix A.3, and the results are shown below with the sums over a and b implied:

$$
\begin{aligned}
-K^R_{a,r} \;=\;& \phi^R_{a,r} - \frac{e^{d\tau\mu}}{2}\left(\phi^R_{a,r-\hat{\tau}} + \phi^R_{a,r+\hat{\tau}}\right) + \frac{e^{d\tau\mu}}{2}\epsilon_{ab}\left(\phi^I_{b,r-\hat{\tau}} - \phi^I_{b,r+\hat{\tau}}\right) \\
&+\frac{d\tau}{2m}\left(2d\phi^R_{a,r} - \sum_{i=\pm 1}^{d}\phi^R_{a,r+\hat{i}}\right) \\
&-\frac{d\tau\omega^2_{\mathrm{tr}}(r^2_\perp)}{4}\left(\left(\phi^R_{a,r+\hat{\tau}} + \phi^R_{a,r-\hat{\tau}}\right) - \epsilon_{ab}\left(\phi^I_{b,r+\hat{\tau}} + \phi^I_{b,r-\hat{\tau}}\right)\right) \\
&-\frac{d\tau\omega_z}{2}\left[\epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}} - \phi^R_{b,r+\hat{y}}\right) - y\left(\phi^R_{b,r-\hat{x}} - \phi^R_{b,r+\hat{x}}\right)\right] - 2(x-y)\phi^I_{a,r}\right] \\
&-\frac{\omega_z}{2}\left[x\left(\phi^I_{a,r-\hat{y}} + \phi^I_{a,r+\hat{y}}\right) - y\left(\phi^I_{a,r-\hat{x}} + \phi^I_{a,r+\hat{x}}\right)\right] \\
&+d\tau\lambda\left[\phi^R_{a,r}(\phi^R_{b,r})^2 - \phi^R_{a,r}(\phi^I_{b,r})^2 - 2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}\right] \qquad (1.15) \\
-K^I_{a,r} \;=\;& \phi^I_{a,r} - \frac{e^{d\tau\mu}}{2}\left(\phi^I_{a,r-\hat{\tau}} + \phi^I_{a,r+\hat{\tau}}\right) - \frac{e^{d\tau\mu}}{2}\epsilon_{ab}\left(\phi^R_{b,r-\hat{\tau}} - \phi^R_{b,r+\hat{\tau}}\right) \\
&+\frac{d\tau}{2m}\left(2d\phi^I_{a,r} - \sum_{i=\pm 1}^{d}\phi^I_{a,r+\hat{i}}\right) \\
&-\frac{d\tau\omega^2_{\mathrm{tr}}(r^2_\perp)}{4}\left(\left(\phi^I_{a,r+\hat{\tau}} + \phi^I_{a,r-\hat{\tau}}\right) + \epsilon_{ab}(\phi^R_{b,r+\hat{\tau}} + \phi^R_{b,r-\hat{\tau}})\right) \\
&-\frac{d\tau\omega_z}{2}\left[\epsilon_{ab}\left(x(\phi^I_{b,r-\hat{y}} - \phi^I_{b,r+\hat{y}}) - y(\phi^I_{b,r-\hat{x}} - \phi^I_{b,r+\hat{x}})\right) + 2(x-y)\phi^R_{a,r}\right] \\
&-\frac{\omega_z}{2}\left[y\left(\phi^R_{a,r-\hat{x}} + \phi^R_{a,r+\hat{x}}\right) - x\left(\phi^R_{a,r-\hat{y}} + \phi^R_{a,r+\hat{y}}\right)\right] \\
&+d\tau\lambda\left[\phi^I_{a,r}(\phi^R_{b,r})^2 + 2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r} - \phi^I_{a,r}(\phi^I_{b,r})^2\right] \qquad (1.16)
\end{aligned}
$$

This evolution is repeated until we have evolved for a long period in Langevin time (determined by the observation of thermalization followed by enough steps to produce good statistical error). Observables of interest can be calculated as functions of the fields at each point in Langevin time and averaged to find the expectation value.

1.4 Observables

1.4.1 Observables averaged over the volume

The observables of interest in this simulation are the density $\langle\hat{n}\rangle$, the field modulus $\langle\phi^*\phi\rangle$, the angular momentum $\langle\hat{L}_z\rangle$, the moment of inertia $\langle\hat{I}_z\rangle$, and the circulation of the fields around the center of the lattice. The individual contributions to the energy (kinetic, trapping potential, and interaction potential) have also been derived, and can be found in Appendix A.4.

The sum over all lattice sites and subsequent normalization by the lattice volume $V = N_x^d N_\tau$ is implied:

$$\mathrm{Re}\langle n\rangle = \frac{e^{\bar{\mu}}}{2}\sum_{a,b=1}^{2}\left[\delta_{ab}(\phi^R_{a,r}\phi^R_{r,b-\hat{\tau}} - \phi^I_{a,r}\phi^I_{r,b-\hat{\tau}}) - \epsilon_{ab}(\phi^R_{a,r}\phi^I_{r,b-\hat{\tau}} + \phi^I_{a,r}\phi^R_{r,b-\hat{\tau}})\right] \tag{1.17}$$

$$\mathrm{Im}\langle n\rangle = \frac{e^{\bar{\mu}}}{2}\sum_{a,b=1}^{2}\left[\delta_{ab}(\phi^R_{a,r}\phi^I_{r,b-\hat{\tau}} + \phi^I_{a,r}\phi^R_{r,b-\hat{\tau}}) + \epsilon_{ab}(\phi^R_{a,r}\phi^R_{r,b-\hat{\tau}} - \phi^I_{a,r}\phi^I_{r,b-\hat{\tau}})\right] \tag{1.18}$$

$$\mathrm{Re}\langle\phi^*\phi\rangle = \frac{1}{2}\sum_{a=1}^{2}\left(\phi^R_{a,r}\phi^R_{a,r} - \phi^I_{a,r}\phi^I_{a,r}\right) \tag{1.19}$$

$$\mathrm{Im}\langle\phi^*\phi\rangle = \sum_{a=1}^{2}\phi^R_{a,r}\phi^I_{a,r} \tag{1.20}$$

$$\begin{aligned}\mathrm{Re}\langle L_z\rangle = {} & \frac{1}{2}\sum_{a=1}^{2}\Big[(y-x)\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}} + (y-x)\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}} \\ & +x\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}-\hat{y}} + x\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}-\hat{y}} - y\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}-\hat{x}} - y\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}-\hat{x}} \\ & +\epsilon_{ab}\sum_{b=1}^{2}\Big((x-y)\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{y}} + (y-x)\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{y}} \\ & +x\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{y}} - x\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{y}} + y\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{x}} - y\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{x}}\Big)\Big]\end{aligned} \tag{1.21}$$

$$\begin{aligned}\mathrm{Im}\langle L_z\rangle = {} & \frac{1}{2}\sum_{a=1}^{2}\Big[(x-y)\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}} + (y-x)\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}} - x\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}-\hat{y}} \\ & +x\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}-\hat{y}} + y\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}-\hat{x}} - y\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}-\hat{x}} \\ & +\epsilon_{ab}\sum_{b=1}^{2}\Big((y-x)\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}} + (y-x)\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}} + x\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{y}} \\ & +x\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{y}} - y\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{x}} - y\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{x}}\Big)\Big]\end{aligned} \tag{1.22}$$

The moment of inertia can be calculated from the angular momentum by differentiating with respect to the angular frequency. We can do this by taking a numerical gradient.

The circulation is computed along an $l \times l$ loop on the lattice:

$$\Gamma[l] = \frac{1}{2\pi}\sum_{l\times l}\left(\theta_{t,x+j} - \theta_{t,x}\right) \tag{1.23}$$

where

$$\theta_{t,x} = \tan^{-1}\left(\frac{\phi^I_{1,t,x} + \phi^R_{2,t,x}}{\phi^R_{1,t,x} - \phi^I_{2,t,x}}\right) \tag{1.24}$$

and $\theta_{t,x+j}$ is computed at the next site on the loop from $\theta_{t,x}$ for each point along the loop.

See Appendix A.4 for the full derivation of the observables.

1.4.2 Observables per site

The next phase of this project will move to larger lattices in order to calculate the density profiles of the rotating system. We expect to see the formation of vortices in the density of the fluid (localized sites of 0 density). We expect these vortices to be concentrated closer to the center of the trap. In order to observe these vortices and their structure, we need a finer lattice mesh in the center of the system. This will involve a great deal more computational intensity.

Chapter 2

Results and Analysis

2.1 Testing

2.1.1 Running the code

We have two allocations we can use, one through XSEDE to run on GPUs on Bridges-2, and the other through NERSC to run on cori. In the GitHub, there are notes on how to compile and run the code.

2.1.2 The Free Gas

We can compute the solution exactly for the case of no trapping potential, no interaction, and no rotation (i.e. the free gas). In order to confirm that our CL simulation is producing reasonable answers, we can compare the simulation with $\omega_{\mathrm{tr}} = \lambda = \omega_z = 0$ to the analytical solution for the free gas.

2.1.3 Systematics

[NOTE: To be filled in with more detail later...] We need to run some tests to determine autocorrelation of the observables and the ideal Langevin step size for our simulations.

2.1.4 Parameters

One of our first objectives once we've tested the free gas and the systematics is to determine optimal parameter ranges. As we are performing a lattice simulation, our lattice parameters will be constrained by the size of the spatial lattice (N_x), the size of the temporal lattice (N_τ), and the lattice spacing ($a = 1$). The parameters we need to be concerned about are μ, ω_{tr}, ω_z, and λ. These will all be constrained by their interactions with each other and with the lattice size and spacing.

The thermal wavelength is defined as $\lambda_T = \sqrt{2\pi\beta} = \sqrt{2\pi N_\tau d\tau}$, the harmonic oscillator wavelength is defined as $\lambda = \frac{1}{\sqrt{\omega_{\mathrm{tr}}}}$.

To ensure that our system can fit on our lattice, we require that

$$a \ll \lambda_T \ll N_x \tag{2.1}$$

So we can resolve the effects of the trap, we require that

$$a \ll \lambda_{HO} \tag{2.2}$$

In order to be in a quantum (as opposed to classical) regime, we also require that

$$\hat{n}^{-d} \ll \lambda_T \tag{2.3}$$

where $\hat{n}$ is the density of the system, which will depend on μ.

We will also need to make N_x sufficiently large that the the effects of the trap, the rotation, and the interaction keep the system confined to the lattice without causing problems at the boundaries (as we have imposed that the fields go to zero at the boundaries). This will require careful testing, setting a condition first for $\omega_{\rm tr}$ and then adjusting the lattice size as we add rotation and then interaction.

Appendices

Appendix A

NRRB Derivations

A.1 Justification for the Form of the Non-Relativistic Lattice Action

The continuum action for bosons with a non-relativistic dispersion, a rotating external potential, a non-zero chemical potential, an external trap potential, and an interaction term is as follows:

$$S = \int_V \phi^* \left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu - i\omega_z(x\partial_y - y\partial_x) - \frac{m\omega_{\text{trap}}^2}{2}(x^2+y^2) \right) \phi + \lambda \int_V (\phi^*\phi)^2. \tag{A.1}$$

To convert this to a lattice action, we must first discretize the derivatives. We will use a backwards finite difference discretization for the single derivative and a central difference approximation for the double derivative, such that:

$$\begin{aligned} \partial_i \phi_r &= \frac{1}{a}\left(\phi_r - \phi_{r-\hat{i}}\right) && \text{(A.2)} \\ \nabla^2 \phi_r &= \sum_i \frac{1}{a^2}\left(\phi_{r+\hat{i}} - 2\phi_r + \phi_{r-\hat{i}}\right), && \text{(A.3)} \end{aligned}$$

where $r = (x, y, \tau)$ and the discretization length is a for spatial derivatives and $d\tau$ for temporal ones.

In order to treat the finite chemical potential, the external trapping potential, the rotation, and the interaction we must shift our indices on the field that is acted on by these external parameters by one step in the time direction. This is to make these potentials gauge invariant in the lattice formulation. Since we have periodic boundary conditions in time, we don't have to worry about boundaries in time.

When we go from the continuous action to the discrete action, we must account for the role of finite spacing.

$$\int \mathrm{d}^d x \, \mathrm{d}\tau \to \sum_{\vec{x},\tau} a^d d\tau. \tag{A.4}$$

We then scale our parameters to their lattice versions, incorporating the lattice spacing, denoted by a bar:

$$
\begin{aligned}
\bar{x} &= x/a \\
\bar{y} &= y/a \\
\bar{r}^2 &= r^2/a^2 \\
\bar{\mu} &= d\tau\mu \\
\bar{m} &= ma^2/d\tau \\
\bar{\omega}_{\rm tr} &= d\tau\omega_{\rm tr} \\
\bar{\omega}_z &= d\tau\omega_z \\
\bar{\lambda} &= d\tau\lambda
\end{aligned}
$$

This allows us to cancel factors of $d\tau$ and leaves us with an overall a^d that we can divide out. Therefore, our lattice action becomes:

$$
\begin{aligned}
S_{\rm lat} = & \sum_{\vec{x},\tau} a^d \Bigg[\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{\tau}} - \bar{\mu}\phi_r^*\phi_{r-\hat{\tau}} - \frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}} \right) \\
& - \frac{\bar{m}}{2} \bar{\omega}_{\rm tr}^2 \bar{r}_\perp^2 \phi_r^*\phi_{r-\hat{\tau}} + i\bar{\omega}_z \left(\bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{\tau}} \right) \\
& + \bar{\lambda} \left(\phi_r^*\phi_{r-\hat{\tau}} \right)^2 \Bigg]
\end{aligned} \tag{A.5}
$$

We can then combine our time derivative and our chemical potential in the following way:

$$
\begin{aligned}
S_{\rm lat} = & \sum_{\vec{x},\tau} a^d \Bigg[\phi_r^*\phi_r - (1+\bar{\mu})\phi_r^*\phi_{r-\hat{\tau}} - \frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}} \right) \\
& - \frac{\bar{m}}{2} \bar{\omega}_{\rm tr}^2 \bar{r}_\perp^2 \phi_r^*\phi_{r-\hat{\tau}} + i\bar{\omega}_z \left(\bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{\tau}} \right) \\
& + \bar{\lambda} \left(\phi_r^*\phi_{r-\hat{\tau}} \right)^2 \Bigg]
\end{aligned} \tag{A.6}
$$

Note that to second order in the lattice size, this is equivalent to:

$$
\begin{aligned}
S_{\rm lat} = & \sum_{\vec{x},\tau} a^d \Bigg[\phi_r^*\phi_r - e^{\bar{\mu}}\phi_r^*\phi_{r-\hat{\tau}} - \frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}} \right) - \frac{\bar{m}}{2} \bar{\omega}_{\rm tr}^2 \bar{r}_\perp^2 \phi_r^*\phi_{r-\hat{\tau}} \\
& + i\bar{\omega}_z \left(\bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{\tau}} \right) + \bar{\lambda} \left(\phi_r^*\phi_{r-\hat{\tau}} \right)^2 \Bigg]
\end{aligned} \tag{A.7}
$$

This will be our lattice action, which we will complexify and use to evolve our system in Langevin time. To simplify, let's divide the lattice action into smaller components:

$$
S_{\rm lat} = \left(S_\mu + S_\nabla - S_{\rm trap} - S_\omega + S_{\rm int} \right) \tag{A.8}
$$

with

$$
\begin{aligned}
S_{\mu,r} &= \sum_{\vec{x},\tau} a^d \left(\phi_r^* \phi_r - e^{\bar{\mu}} \phi_r^* \phi_{r-\hat{\tau}}\right) && \text{(A.9)} \\
S_{\nabla,r} &= \sum_{\vec{x},\tau} a^d \left(\frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(2\phi_r^* \phi_r - \phi_r^* \phi_{r-\hat{j}} - \phi_r^* \phi_{r+\hat{j}}\right)\right) && \text{(A.10)} \\
S_{\text{trap},r} &= \sum_{\vec{x},\tau} a^d \left(\frac{\bar{m}}{2} \bar{\omega}_{\text{tr}}^2 \bar{r}_\perp^2 \phi_r^* \phi_{r-\hat{\tau}}\right) && \text{(A.11)} \\
S_{\omega,r} &= \sum_{\vec{x},\tau} a^d \left(i\bar{\omega}_z \left(\bar{x}\phi_r^* \phi_{r-\hat{\tau}} - \bar{x}\phi_r^* \phi_{r-\hat{y}-\hat{\tau}} - \bar{y}\phi_r^* \phi_{r-\hat{\tau}} + \bar{y}\phi_r^* \phi_{r-\hat{x}-\hat{\tau}}\right)\right) && \text{(A.12)} \\
S_{\text{int},r} &= \sum_{\vec{x},\tau} a^d \left(\bar{\lambda} \left(\phi_r^* \phi_{r-\hat{\tau}}\right)^2\right). && \text{(A.13)}
\end{aligned}
$$

Note that we are restricting ourselves to two spatial dimensions at this point in the work. The extension of this method to three-dimensional rotating systems is saved for future work.

A.2 Writing the Complex Action in Terms of Real Fields

This action must first be rewritten with the complex fields expressed in terms of two real fields, $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and $\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$. Each piece of the action is computed below. Note from this point on, we drop the external sum over the entire spacetime lattice and its weighting (a^d), as it is the same for all components of the action.

First, the time derivative and chemical potential part of the action at any given site $\vec{x}, \tau$ on the lattice:

$$
\begin{aligned}
S_{\mu,r} &= \phi_r^* \phi_r - e^{\bar{\mu}} \phi_r^* \phi_{r-\hat{\tau}} \\
&= \frac{1}{2}\left[\phi_{1,r}^2 + \phi_{2,r}^2 - e^{\bar{\mu}}\left(\phi_{1,r}\phi_{1,r-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right)\right] \\
&= \frac{1}{2}\sum_{a=1}^{2}\left[\phi_{a,r}^2 - e^{\bar{\mu}}\phi_{a,r}\phi_{a,r-\hat{\tau}} - ie^{\bar{\mu}}\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right], && \text{(A.14)}
\end{aligned}
$$

where $\epsilon_{12} = 1,\ \epsilon_{21} = -1, \epsilon_{11} = \epsilon_{22} = 0$. Next, the spatial derivative part (corresponding to the kinetic energy):

$$
\begin{aligned}
S_{\nabla,r} &= \frac{1}{2\bar{m}}\sum_{j=1}^{d}\left(2\phi_r^*\phi_r - \phi_r^*\phi_{r-\hat{j}} - \phi_r^*\phi_{r+\hat{j}}\right) \\
&= \frac{1}{4\bar{m}}\sum_{j=1}^{d}\left[2\left(\phi_{1,r}^2 + \phi_{2,r}^2\right) - \left(\phi_{1,r}\phi_{1,r+\hat{i}} + i\phi_{1,r}\phi_{2,r+\hat{i}} - i\phi_{2,r}\phi_{1,r+\hat{i}} + \phi_{2,r}\phi_{2,r+\hat{i}}\right)\right. \\
&\quad \left. - \left(\phi_{1,r}\phi_{1,r-\hat{i}} + i\phi_{1,r}\phi_{2,r-\hat{i}} - i\phi_{2,r}\phi_{1,r-\hat{i}} + \phi_{2,r}\phi_{2,r-\hat{i}}\right)\right] \\
&= \frac{1}{4\bar{m}}\sum_{j=1}^{d}\sum_{a=1}^{2}\left[2\phi_{a,r}^2 - \left(\phi_a\phi_{a,r-\hat{j}} + \phi_a\phi_{a,r+\hat{j}}\right) - i\sum_{b=1}^{2}\epsilon_{ab}\left(\phi_{a,r}\phi_{b,r-\hat{j}} + \phi_{a,r}\phi_{b,r+\hat{j}}\right)\right] && \text{(A.15)}
\end{aligned}
$$

Then, for the part of the action due to the external trapping potential:

$$\begin{aligned}
S_{\text{trap},r} &= \frac{\bar{m}}{2}\bar{\omega}_{\text{tr}}^2\bar{r}_\perp^2\phi_r^*\phi_{r-\hat{\tau}} \\
&= \frac{\bar{m}}{4}\bar{\omega}_{\text{tr}}^2\bar{r}_\perp^2\left[\phi_{1,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{2,r-\tau}\right] \\
&= \frac{\bar{m}}{4}\bar{\omega}_{\text{tr}}^2\bar{r}_\perp^2\sum_{a=1}^{2}\left[\phi_{a,r}\phi_{a,r-\hat{\tau}} + i\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right].
\end{aligned} \tag{A.16}$$

Next, the rotational piece:

$$\begin{aligned}
S_{\omega,r} &= i\bar{\omega}_z\left(\bar{x}\phi_r^*\phi_{r-\hat{\tau}} - \bar{x}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \bar{y}\phi_r^*\phi_{r-\hat{\tau}} + \bar{y}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}\right) \\
&= \frac{\bar{\omega}_z}{2}\left[(\bar{y}-\bar{x})\left(\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\right) + \bar{x}(\phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}})\right. \\
&\quad \left.-\bar{y}(\phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}})\right] \\
&\quad +i\frac{\bar{\omega}_z}{2}\left[(\bar{x}-\bar{y})(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}) - \bar{x}(\phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{y}-\hat{\tau}})\right. \\
&\quad \left.+\bar{y}(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}})\right] \\
&= \frac{\bar{\omega}_z}{2}\sum_{a=1}^{2}\left[\sum_{b=1}^{2}\epsilon_{ab}\left((\bar{y}-\bar{x})\phi_{a,r}\phi_{b,r-\hat{\tau}} + \bar{x}\phi_{a,r}\phi_{b,r-\hat{y}-\hat{\tau}} - \bar{y}\phi_{a,r}\phi_{b,r-\hat{x}-\hat{\tau}}\right)\right] \\
&\quad +i\frac{\bar{\omega}_z}{2}\sum_{a=1}^{2}\left[(\bar{x}-\bar{y})\phi_{a,r}\phi_{a,r-\hat{\tau}} - \bar{x}\phi_{a,r}\phi_{a,r-\hat{y}-\hat{\tau}} + \bar{y}\phi_{a,r}\phi_{a,r-\hat{x}-\hat{\tau}}\right]
\end{aligned} \tag{A.17}$$

And finally, the interaction term in the action:

$$\begin{aligned}
S_{\text{int},r} &= \bar{\lambda}(\phi_r^*\phi_{r-\hat{\tau}})^2 \\
&= \frac{\bar{\lambda}}{4}\left[\phi_{1,r}^2\phi_{1,r-\hat{\tau}}^2 + \phi_{2,r}^2\phi_{2,r-\hat{\tau}}^2 + 4\phi_{1,r}\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} - \phi_{1,r}^2\phi_{2,r-\hat{\tau}}^2 - \phi_{2,r}^2\phi_{1,r-\hat{\tau}}^2\right] \\
&\quad +i\frac{\bar{\lambda}}{4}\left[2\phi_{1,r}^2\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} - 2\phi_{2,r}^2\phi_{1,r-\hat{\tau}}\phi_{2,r-\hat{\tau}} - 2\phi_{1,r}\phi_{2,r}\phi_{1,r-\hat{\tau}}^2 + 2\phi_{1,r}\phi_{2,r}\phi_{2,r-\hat{\tau}}^2\right] \\
&= \frac{\bar{\lambda}}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left[2\phi_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r}\phi_{b,r-\hat{\tau}} - \phi_{a,r}^2\phi_{b,r-\hat{\tau}}^2\right] \\
&\quad +i\frac{\bar{\lambda}}{2}\sum_{a=1}^{2}\sum_{b=1}^{2}\left[\epsilon_{ab}\left(\phi_{a,r}^2\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}} - \phi_{a,r}\phi_{b,r}\phi_{a,r-\hat{\tau}}^2\right)\right].
\end{aligned}$$

We will work with the lattice action in this form in order to derive the Langevin drift function.

A.3 Generating the NRRB CL Equations

A.3.1 Derivatives on the Lattice

When taking derivatives of this lattice action with respect to the fields, we do the following:

$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2}\phi_{a,q}\phi_{a,q+\hat{i}}\right) = \sum_{a=1}^{2}\sum_{q=1}^{N_r}\left(\phi_{a,q}\frac{\delta}{\delta\phi_{c,r}}\phi_{a,q+\hat{i}}+\frac{\delta\phi_{a,q}}{\delta\phi_{c,r}}\phi_{a,q+\hat{i}}\right) \tag{A.18}$$

$$= \sum_{a=1}^{2}\sum_{q=1}^{N_r}\left(\phi_{a,q}\delta_{c,a}\delta_{r,q+\hat{i}}+\delta_{c,a}\delta_{q,r}\phi_{a,q+\hat{i}}\right)$$

$$= \phi_{c,r-\hat{i}}+\phi_{c,r+\hat{i}}.$$

Similarly,

$$\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2}\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,q}\phi_{b,q+\hat{i}}\right)=\sum_{b=1}^{2}\epsilon_{cb}\left(\phi_{b,r-\hat{i}}+\phi_{b,r+\hat{i}}\right). \tag{A.19}$$

A.3.2 Computing the derivative of the action with respect to the real fields

The first step in computing the CL Equations is to find $\frac{\delta S_r}{\delta\phi_{a,r}}$. This is done below, with the sum over $a=1,2$ implied:

$$\frac{\delta S_r}{\delta\phi_{a,r}} = \frac{\delta S_{\mu,r}}{\delta\phi_{a,r}}+\frac{\delta S_{\nabla,r}}{\delta\phi_{a,r}}-\frac{\delta S_{\text{trap},r}}{\delta\phi_{a,r}}-\frac{\delta S_{\omega,r}}{\delta\phi_{a,r}}+\frac{\delta S_{\text{int},r}}{\delta\phi_{a,r}} \tag{A.20}$$

Again, we proceed by modifying each of the 5 parts of the action. First, the time and chemical potential term:

$$\frac{\delta}{\delta\phi_{a,r}}S_{\mu,r} = \frac{1}{2}\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\left[\phi_{c,r}^2-e^{\bar{\mu}}\phi_{c,r}\phi_{c,r-\hat{\tau}}-ie^{\bar{\mu}}\sum_{b=1}^{2}\epsilon_{cb}\phi_{c,r}\phi_{b,r-\hat{\tau}}\right]$$

$$= \phi_{a,r}-\frac{1}{2}e^{\bar{\mu}}\left(\phi_{a,r-\hat{\tau}}+\phi_{a,r+\hat{\tau}}\right)-\frac{i}{2}e^{\bar{\mu}}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}}+\phi_{b,r+\hat{\tau}}\right) \tag{A.21}$$

Next, the spatial derivative part:

$$\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r} = \frac{1}{4\bar{m}}\frac{\delta}{\delta\phi_{a,r}}\sum_{j=1}^{d}\sum_{c=1}^{2}\Big[2\phi_{c,r}^2-(\phi_{c,r}\phi_{c,r+\hat{j}}+\phi_{c,r}\phi_{c,r-\hat{j}})$$

$$-i\sum_{b=1}^{2}\epsilon_{cb}(\phi_{c,r}\phi_{b,r+\hat{j}}+\phi_{c,r}\phi_{b,r-\hat{j}})\Big]$$

$$= \frac{1}{4\bar{m}}\sum_{j=1}^{d}\left[4\phi_{a,r}-2(\phi_{a,r+\hat{j}}+\phi_{c,r-\hat{j}})-2i\sum_{b=1}^{2}\epsilon_{ab}(\phi_{b,r+\hat{j}}+\phi_{b,r-\hat{j}})\right]$$

$$= \frac{1}{2\bar{m}}\sum_{j=1}^{d}\left[2\phi_{a,r}-(\phi_{a,r+\hat{j}}+\phi_{c,r-\hat{j}})-i\sum_{b=1}^{2}\epsilon_{ab}(\phi_{b,r+\hat{j}}+\phi_{b,r-\hat{j}})\right] \tag{A.22}$$

Then the part of the action due to the external trapping potential:

$$\frac{\delta}{\delta\phi_{a,r}}S_{\text{trap},r} = \frac{\bar{m}\bar{\omega}_{\text{tr}}^2\bar{r}_\perp^2}{4}\frac{\delta}{\delta\phi_{a,r}}\sum_{c=1}^{2}\left(\phi_{c,r}\phi_{c,r-\hat{\tau}}+i\sum_{b=1}^{2}\epsilon_{cb}\phi_{c,r}\phi_{b,r-\hat{\tau}}\right)$$

$$= \frac{\bar{m}\bar{\omega}_{\text{tr}}^2\bar{r}_\perp^2}{4}\left(\phi_{a,r-\hat{\tau}}+\phi_{a,r+\hat{\tau}}+i\sum_{b=1}^{2}\epsilon_{ab}(\phi_{b,r-\hat{\tau}}+\phi_{b,r+\hat{\tau}})\right)$$

where $\bar{r}_\perp^2 = \bar{x}^2 + \bar{y}^2$, or in $1d$, $\bar{r}_\perp^2 = \bar{x}^2$. Next, the rotational piece:

$$\begin{aligned}
\frac{\delta}{\delta\phi_{a,r}} S_{\omega,r} &= \frac{\bar{\omega}_z}{2}\frac{\delta}{\delta\phi_{a,r}} \sum_{c=1}^{2} \left[\sum_{b=1}^{2} \epsilon_{cb}\left((\bar{y}-\bar{x})\phi_{c,r}\phi_{b,r-\hat{\tau}} + \bar{x}\phi_{c,r}\phi_{b,r-\hat{y}-\hat{\tau}} - \bar{y}\phi_{c,r}\phi_{b,r-\hat{x}-\hat{\tau}}\right)\right. \\
&\left. +i\Big((\bar{x}-\bar{y})\phi_{c,r}\phi_{c,r-\hat{\tau}} - \bar{x}\phi_{c,r}\phi_{c,r-\hat{y}-\hat{\tau}} + \bar{y}\phi_{c,r}\phi_{c,r-\hat{x}-\hat{\tau}}\Big)\right] \\
&= \frac{\bar{\omega}_z}{2}\left[\sum_{b=1}^{2}\epsilon_{ab}\Big((\bar{y}-\bar{x})(\phi_{b,r-\hat{\tau}}+\phi_{b,r+\hat{\tau}}) + \bar{x}(\phi_{b,r-\hat{y}-\hat{\tau}}+\phi_{b,r+\hat{y}+\hat{\tau}})\right. \\
&-\bar{y}(\phi_{b,r-\hat{x}-\hat{\tau}}+\phi_{b,r+\hat{x}+\hat{\tau}})\Big) + i\Big((\bar{x}-\bar{y})(\phi_{a,r-\hat{\tau}}+\phi_{a,r+\hat{\tau}}) \\
&\left.-\bar{x}(\phi_{a,r-\hat{y}-\hat{\tau}}+\phi_{a,r+\hat{y}+\hat{\tau}}) + \bar{y}(\phi_{a,r-\hat{x}-\hat{\tau}}+\phi_{a,r+\hat{x}+\hat{\tau}})\Big)\right] \qquad \text{(A.23)}
\end{aligned}$$

Note, this term is set to zero unless $d > 1$. And finally, the interaction term in the action:

$$\begin{aligned}
\frac{\delta}{\delta\phi_{a,r}} S_{\text{int},r} &= \frac{\bar{\lambda}}{4}\frac{\delta}{\delta\phi_{a,r}} \sum_{c=1}^{2}\sum_{b=1}^{2}\left[2\phi_{c,r}\phi_{c,r-\hat{\tau}}\phi_{b,r}\phi_{b,r-\hat{\tau}} - \phi_{c,r}^2\phi_{b,r-\hat{\tau}}^2\right. \\
&\left.+2i\epsilon_{cb}\left(\phi_{c,r}^2\phi_{c,r-\hat{\tau}}\phi_{b,r-\hat{\tau}} - \phi_{c,r}\phi_{b,r}\phi_{c,r-\hat{\tau}}^2\right)\right] \\
&= \frac{\bar{\lambda}}{4}\sum_{b=1}^{2}\left[4\phi_{b,r}(\phi_{a,r+\hat{\tau}}\phi_{b,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}) - 2\phi_{a,r}(\phi_{b,r+\hat{\tau}}^2 + \phi_{b,r-\hat{\tau}}^2)\right. \qquad \text{(A.24)} \\
&\left.+2i\epsilon_{ab}(\phi_{b,r}(\phi_{a,r+\hat{\tau}}^2 + \phi_{a,r-\hat{\tau}}^2) - \phi_{b,r}(\phi_{b,r+\hat{\tau}}^2 + \phi_{b,r-\hat{\tau}}^2) + 2\phi_{a,r}(\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}}\phi_{b,r+\hat{\tau}}))\right]
\end{aligned}$$

So our final drift function is given by:

$$\begin{aligned}
K_a &= \sum_r \left[\phi_{a,r} - \frac{1}{2}e^{\bar{\mu}}\left(\phi_{a,r-\hat{\tau}}+\phi_{a,r+\hat{\tau}}\right) - \frac{i}{2}e^{\bar{\mu}}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}}+\phi_{b,r+\hat{\tau}}\right)\right. \\
&+\frac{1}{2\bar{m}}\sum_{j=1}^{d}\left[2\phi_{a,r} - (\phi_{a,r+\hat{j}}+\phi_{c,r-\hat{j}}) - i\sum_{b=1}^{2}\epsilon_{ab}(\phi_{b,r+\hat{j}}+\phi_{b,r-\hat{j}})\right] \\
&+\frac{\bar{m}\bar{\omega}_{\text{tr}}^2\bar{r}_\perp^2}{4}\left(\phi_{a,r-\hat{\tau}}+\phi_{a,r+\hat{\tau}} + i\sum_{b=1}^{2}\epsilon_{ab}(\phi_{b,r-\hat{\tau}}+\phi_{b,r+\hat{\tau}})\right) \\
&+\frac{\bar{\omega}_z}{2}\left(\sum_{b=1}^{2}\epsilon_{ab}\Big((\bar{y}-\bar{x})(\phi_{b,r-\hat{\tau}}+\phi_{b,r+\hat{\tau}}) + \bar{x}(\phi_{b,r-\hat{y}-\hat{\tau}}+\phi_{b,r+\hat{y}+\hat{\tau}})\right. \\
&-\bar{y}(\phi_{b,r-\hat{x}-\hat{\tau}}+\phi_{b,r+\hat{x}+\hat{\tau}})\Big) + i\Big((\bar{x}-\bar{y})(\phi_{a,r-\hat{\tau}}+\phi_{a,r+\hat{\tau}}) \\
&\left.-\bar{x}(\phi_{a,r-\hat{y}-\hat{\tau}}+\phi_{a,r+\hat{y}+\hat{\tau}}) + \bar{y}(\phi_{a,r-\hat{x}-\hat{\tau}}+\phi_{a,r+\hat{x}+\hat{\tau}})\Big)\right) \\
&+\frac{\bar{\lambda}}{4}\sum_{b=1}^{2}\left[4\phi_{b,r}(\phi_{a,r+\hat{\tau}}\phi_{b,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}) - 2\phi_{a,r}(\phi_{b,r+\hat{\tau}}^2 + \phi_{b,r-\hat{\tau}}^2)\right. \\
&\left.\left.+2i\epsilon_{ab}(\phi_{b,r}(\phi_{a,r+\hat{\tau}}^2 + \phi_{a,r-\hat{\tau}}^2) - \phi_{b,r}(\phi_{b,r+\hat{\tau}}^2 + \phi_{b,r-\hat{\tau}}^2) + 2\phi_{a,r}(\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}}\phi_{b,r+\hat{\tau}}))\right]\right]
\end{aligned}$$

Which we can divide into real and imaginary contributions: $K_a = K_a^{\mathrm{R}} + iK_a^{\mathrm{I}}$:

$$\begin{aligned}
K_a^{\mathrm{R}} &= \sum_r \Bigg[\phi_{a,r} - \frac{1}{2} e^{\bar{\mu}} \left(\phi_{a,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}} \right) + \frac{1}{2\bar{m}} \sum_{j=1}^{d} \Big(2\phi_{a,r} - (\phi_{a,r+\hat{j}} + \phi_{c,r-\hat{j}}) \Big) \\
&\quad + \frac{\bar{m}\bar{\omega}_{\mathrm{tr}}^2 \bar{r}_\perp^2}{4} \left(\phi_{a,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}} \right) + \frac{\bar{\omega}_z}{2} \sum_{b=1}^{2} \epsilon_{ab} \Bigg((\bar{y} - \bar{x})(\phi_{b,r-\hat{\tau}} + \phi_{b,r+\hat{\tau}}) \\
&\quad + \bar{x}(\phi_{b,r-\hat{y}-\hat{\tau}} + \phi_{b,r+\hat{y}+\hat{\tau}}) - \bar{y}(\phi_{b,r-\hat{x}-\hat{\tau}} + \phi_{b,r+\hat{x}+\hat{\tau}}) \Bigg) \\
&\quad + \frac{\bar{\lambda}}{4} \sum_{b=1}^{2} \Big(4\phi_{b,r}(\phi_{a,r+\hat{\tau}}\phi_{b,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}) - 2\phi_{a,r}(\phi_{b,r+\hat{\tau}}^2 + \phi_{b,r-\hat{\tau}}^2) \Big) \Bigg] \\
K_a^{\mathrm{I}} &= -\sum_r \Bigg[\frac{1}{2} e^{\bar{\mu}} \epsilon_{ab} \left(\phi_{b,r-\hat{\tau}} + \phi_{b,r+\hat{\tau}} \right) + \frac{1}{2\bar{m}} \sum_{j=1}^{d} \Big(\sum_{b=1}^{2} \epsilon_{ab}(\phi_{b,r+\hat{j}} + \phi_{b,r-\hat{j}}) \Big) \\
&\quad - \frac{\bar{m}\bar{\omega}_{\mathrm{tr}}^2 \bar{r}_\perp^2}{4} \left(\sum_{b=1}^{2} \epsilon_{ab}(\phi_{b,r-\hat{\tau}} + \phi_{b,r+\hat{\tau}}) \right) - \frac{\bar{\omega}_z}{2} \Bigg(\Big((\bar{x} - \bar{y})(\phi_{a,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}}) \\
&\quad - \bar{x}(\phi_{a,r-\hat{y}-\hat{\tau}} + \phi_{a,r+\hat{y}+\hat{\tau}}) + \bar{y}(\phi_{a,r-\hat{x}-\hat{\tau}} + \phi_{a,r+\hat{x}+\hat{\tau}}) \Big) \Bigg) \\
&\quad - \frac{\bar{\lambda}}{2} \sum_{b=1}^{2} \Bigg(\epsilon_{ab}(\phi_{b,r}(\phi_{a,r+\hat{\tau}}^2 + \phi_{a,r-\hat{\tau}}^2) - \phi_{b,r}(\phi_{b,r+\hat{\tau}}^2 + \phi_{b,r-\hat{\tau}}^2) \\
&\quad + 2\phi_{a,r}(\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}}\phi_{b,r+\hat{\tau}})) \Bigg) \Bigg] \qquad \text{(A.25)}
\end{aligned}$$

A.3.3 NOT UPDATED YET:Second Complexification of Drift Function

The next step is to complexify our real fields, a and b, such that $\phi_a = \phi_a^R + i\phi_a^I$. We do this for each part of the drift function, $K_{a,r} = \frac{\delta S_r}{\delta \phi_{a,r}}$.

First, the time and chemical potential term:

$$\begin{aligned}
2\frac{\delta}{\delta\phi_{a,r}} S_{\mu,r} &= 2\phi_{a,r} - e^{d\tau\mu} \left(\phi_{a,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}} \right) - ie^{d\tau\mu} \epsilon_{ab} \left(\phi_{b,r-\hat{\tau}} + \phi_{b,r+\hat{\tau}} \right) \\
&= 2(\phi_{a,r}^R + i\phi_{a,r}^I) - e^{d\tau\mu} \left(\phi_{a,r-\hat{\tau}}^R + i\phi_{a,r-\hat{\tau}}^I + \phi_{a,r+\hat{\tau}}^R + i\phi_{a,r+\hat{\tau}}^I \right) \\
&\quad - ie^{d\tau\mu} \epsilon_{ab} \left(\phi_{b,r-\hat{\tau}}^R + i\phi_{b,r-\hat{\tau}}^I + \phi_{b,r+\hat{\tau}}^R + i\phi_{b,r+\hat{\tau}}^I \right) \\
&= 2\phi_{a,r}^R - e^{d\tau\mu} \left(\phi_{a,r-\hat{\tau}}^R + \phi_{a,r+\hat{\tau}}^R \right) + e^{d\tau\mu} \epsilon_{ab} \left(\phi_{b,r-\hat{\tau}}^I - \phi_{b,r+\hat{\tau}}^I \right) \\
&\quad + i \left[2\phi_{a,r}^I - e^{d\tau\mu} \left(\phi_{a,r-\hat{\tau}}^I + \phi_{a,r+\hat{\tau}}^I \right) - e^{d\tau\mu} \epsilon_{ab} \left(\phi_{b,r-\hat{\tau}}^R - \phi_{b,r+\hat{\tau}}^R \right) \right]
\end{aligned}$$

So

$$\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}} S_{\mu,r} \right] = \phi_{a,r}^R - \frac{e^{d\tau\mu}}{2} \left(\phi_{a,r-\hat{\tau}}^R + \phi_{a,r+\hat{\tau}}^R \right) + \frac{e^{d\tau\mu}}{2} \epsilon_{ab} \left(\phi_{b,r-\hat{\tau}}^I - \phi_{b,r+\hat{\tau}}^I \right) \qquad \text{(A.26)}$$

$$\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}} S_{\mu,r} \right] = \phi_{a,r}^I - \frac{e^{d\tau\mu}}{2} \left(\phi_{a,r-\hat{\tau}}^I + \phi_{a,r+\hat{\tau}}^I \right) - \frac{e^{d\tau\mu}}{2} \epsilon_{ab} \left(\phi_{b,r-\hat{\tau}}^R - \phi_{b,r+\hat{\tau}}^R \right) \qquad \text{(A.27)}$$

Next, the spatial derivative part:

$$
\begin{aligned}
-\frac{2m}{d\tau}\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r} &= \sum_{i=\pm1}^{d}\phi_{a,r+\hat{i}} - 2d\phi_{a,r} \\
&= \sum_{i=\pm1}^{d}\left(\phi^R_{a,r+\hat{i}} + i\phi^I_{a,r+\hat{i}}\right) - 2d\phi^R_{a,r} - 2id\phi^I_{a,r} \\
&= \sum_{i=\pm1}^{d}\phi^R_{a,r+\hat{i}} - 2d\phi^R_{a,r} + i\left(\sum_{i=\pm1}^{d}\phi^I_{a,r+\hat{i}} - 2d\phi^I_{a,r}\right)
\end{aligned}
$$

So

$$
\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r}\right] = \frac{d\tau}{2m}\left(2d\phi^R_{a,r} - \sum_{i=\pm1}^{d}\phi^R_{a,r+\hat{i}}\right) \tag{A.28}
$$

$$
\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r}\right] = \frac{d\tau}{2m}\left(2d\phi^I_{a,r} - \sum_{i=\pm1}^{d}\phi^I_{a,r+\hat{i}}\right) \tag{A.29}
$$

Then the part of the action due to the external trapping potential:

$$
\begin{aligned}
\frac{4}{d\tau m\omega^2_{\mathrm{tr}}(r^2_\perp)}\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r} &= \sum_{a=1}^{2}\left((\phi_{a,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}) + i\sum_{b=1}^{2}\epsilon_{ab}\left(\phi_{b,r+\hat{\tau}} + \phi_{b,r-\hat{\tau}}\right)\right) \\
&= \sum_{a=1}^{2}\left((\phi^R_{a,r+\hat{\tau}} + \phi^R_{a,r-\hat{\tau}}) - \sum_{b=1}^{2}\epsilon_{ab}\left(\phi^I_{b,r+\hat{\tau}} + \phi^I_{b,r-\hat{\tau}}\right)\right) \\
&\quad + i\sum_{a=1}^{2}\left((\phi^I_{a,r+\hat{\tau}} + \phi^I_{a,r-\hat{\tau}}) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{b,r+\hat{\tau}} + \phi^R_{b,r-\hat{\tau}})\right)
\end{aligned} \tag{A.30}
$$

So

$$
\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r}\right] = \frac{d\tau\omega^2_{\mathrm{tr}}(r^2_\perp)}{4}\sum_{a=1}^{2}\left((\phi^R_{a,r+\hat{\tau}} + \phi^R_{a,r-\hat{\tau}}) - \sum_{b=1}^{2}\epsilon_{ab}\left(\phi^I_{b,r+\hat{\tau}} + \phi^I_{b,r-\hat{\tau}}\right)\right) \tag{A.31}
$$

$$
\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r}\right] = \frac{d\tau\omega^2_{\mathrm{tr}}(r^2_\perp)}{4}\sum_{a=1}^{2}\left((\phi^I_{a,r+\hat{\tau}} + \phi^I_{a,r-\hat{\tau}}) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{b,r+\hat{\tau}} + \phi^R_{b,r-\hat{\tau}})\right) \tag{A.32}
$$

where $r^2_\perp = x^2 + y^2$. Next, the rotational piece:

$$
\begin{aligned}
\frac{2}{d\tau\omega_z}\frac{\delta}{\delta\phi_{a,r}}S_{\omega,r} &= \epsilon_{ab}\left[x\left(\phi_{b,r-\hat{y}} + \phi_{b,r+\hat{y}}\right) - y\left(\phi_{b,r-\hat{x}} + \phi_{b,r+\hat{x}}\right)\right] \\
&\quad +i\left[2(x-y)\phi_{a,r} - x\left(\phi_{a,r-\hat{y}} + \phi_{a,r+\hat{y}}\right) + y\left(\phi_{a,r-\hat{x}} + \phi_{a,r+\hat{x}}\right)\right] \\
&= \epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}} + i\phi^I_{b,r-\hat{y}} + \phi^R_{b,r+\hat{y}} + i\phi^I_{b,r+\hat{y}}\right) - y\left(\phi^R_{b,r-\hat{x}} + i\phi^I_{b,r-\hat{x}} + \phi^R_{b,r+\hat{x}} + i\phi^I_{b,r+\hat{x}}\right)\right] \\
&\quad +i\left[2(x-y)\phi^R_{a,r} + 2i(x-y)\phi^I_{a,r} - x\left(\phi^R_{a,r-\hat{y}} + i\phi^I_{a,r-\hat{y}} + \phi^R_{a,r+\hat{y}} + i\phi^I_{a,r+\hat{y}}\right)\right] \\
&\quad +i\left[y\left(\phi^R_{a,r-\hat{x}} + i\phi^I_{a,r-\hat{x}} + \phi^R_{a,r+\hat{x}} + i\phi^I_{a,r+\hat{x}}\right)\right] \\
&= -2(x-y)\phi^I_{a,r} + x\left(\phi^I_{a,r-\hat{y}} + \phi^I_{a,r+\hat{y}}\right) - y\left(\phi^I_{a,r-\hat{x}} + \phi^I_{a,r+\hat{x}}\right) \\
&\quad +\epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}} + \phi^R_{b,r+\hat{y}}\right) - y\left(\phi^R_{b,r-\hat{x}} + \phi^R_{b,r+\hat{x}}\right)\right] \\
&\quad +i\left[2(x-y)\phi^R_{a,r} - x\left(\phi^R_{a,r-\hat{y}} + \phi^R_{a,r+\hat{y}}\right) + y\left(\phi^R_{a,r-\hat{x}} + \phi^R_{a,r+\hat{x}}\right)\right] \\
&\quad +i\epsilon_{ab}\left[x\left(\phi^I_{b,r-\hat{y}} + \phi^I_{b,r+\hat{y}}\right) - y\left(\phi^I_{b,r-\hat{x}} + \phi^I_{b,r+\hat{x}}\right)\right]
\end{aligned} \tag{A.33}
$$

So

$$\begin{aligned}
\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\omega_z,r}\right] &= \frac{d\tau\omega_z}{2}\left[x\left(\phi^I_{a,r-\hat{y}}+\phi^I_{a,r+\hat{y}}\right)-y\left(\phi^I_{a,r-\hat{x}}+\phi^I_{a,r+\hat{x}}\right)-2(x-y)\phi^I_{a,r}\right] \\
&\quad +\frac{d\tau\omega_z}{2}\epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}}+\phi^R_{b,r+\hat{y}}\right)-y\left(\phi^R_{b,r-\hat{x}}+\phi^R_{b,r+\hat{x}}\right)\right] \\
\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\omega_z,r}\right] &= \frac{d\tau\omega_z}{2}\left[2(x-y)\phi^R_{a,r}-x\left(\phi^R_{a,r-\hat{y}}+\phi^R_{a,r+\hat{y}}\right)+y\left(\phi^R_{a,r-\hat{x}}+\phi^R_{a,r+\hat{x}}\right)\right] \quad \text{(A.34)} \\
&\quad +\frac{\omega_z}{2}\epsilon_{ab}\left[x\left(\phi^I_{b,r-\hat{y}}+\phi^I_{b,r+\hat{y}}\right)-y\left(\phi^I_{b,r-\hat{x}}+\phi^I_{b,r+\hat{x}}\right)\right]
\end{aligned}$$

And finally, the interaction term in the action:

$$\begin{aligned}
\frac{1}{d\tau\lambda}\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{int},r} &= \phi_{a,r}\sum_{b=1}^{2}\phi^2_{b,r} \\
&= (\phi^R_{a,r}+i\phi^I_{a,r})\sum_{b=1}^{2}(\phi^R_{b,r}+i\phi^I_{b,r})^2 \\
&= \phi^R_{a,r}\sum_{b=1}^{2}\left((\phi^R_{b,r})^2+2i\phi^R_{b,r}\phi^I_{b,r}-(\phi^I_{b,r})^2\right)+i\phi^I_{a,r}\sum_{b=1}^{2}\left((\phi^R_{b,r})^2+2i\phi^R_{b,r}\phi^I_{b,r}-(\phi^I_{b,r})^2\right) \\
&= \sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2+2i\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}-\phi^R_{a,r}(\phi^I_{b,r})^2+i\phi^I_{a,r}(\phi^R_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}-i\phi^I_{a,r}(\phi^I_{b,r})^2\right] \\
&= \sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2-\phi^R_{a,r}(\phi^I_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}\right]+i\sum_{b=1}^{2}\left[2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}+\phi^I_{a,r}(\phi^R_{b,r})^2-\phi^I_{a,r}(\phi^I_{b,r})^2\right]
\end{aligned}$$

So

$$\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{int},r}\right] = d\tau\lambda\sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2-\phi^R_{a,r}(\phi^I_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}\right] \quad \text{(A.35)}$$

$$\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{int},r}\right] = d\tau\lambda\sum_{b=1}^{2}\left[\phi^I_{a,r}(\phi^R_{b,r})^2+2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}-\phi^I_{a,r}(\phi^I_{b,r})^2\right] \quad \text{(A.36)}$$

A.4 Lattice Observables

The observables we are interested in calculating are:

- particle density (local and average density)
- angular momentum
- field modulus squared
- harmonic trapping potential energy
- interaction potential energy
- kinetic energy

- total energy
- circulation

Most of our observables can be determined by taking appropriate derivatives of $\ln \mathcal{Z}$, with the exception of the field modulus squared and the circulation. The general form for this argument is shown below.

We have a partition function, $\mathcal{Z}$,

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]} \tag{A.37}$$

Observables are calculated via a path integral as well, where the integrand includes the observable as a function of the fields, weighted by e^{-S}:

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \mathcal{O}[\phi] e^{-S[\phi]} \tag{A.38}$$

If we take a derivative of the log of $\mathcal{Z}$ with respect to some parameter α, we get something that looks very similar to this expression:

$$\frac{\partial \ln \mathcal{Z}}{\partial \alpha} = \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \alpha} = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \left(-\frac{\partial S[\phi, \alpha]}{\partial \alpha}\right) e^{-S[\phi,\alpha]} \tag{A.39}$$

which suggests that we can compute observables by taking derivatives of the action

$$\hat{\mathcal{O}}_\alpha = -\frac{\partial}{\partial \alpha} S[\phi, \alpha] \tag{A.40}$$

In our statistical simulation, we seek to compute average values of these observables, weighted by e^{-S}. The complex Langevin simulation ensures that our samples are distributed according to the proper weight, so we are able to perform a simple average of the observables after the simulation has thermalized.

[NOTE: where does the beta come in? Or do we just need to take derivatives wrt our lattice parameters (e.g. $\bar{\mu}$ instead of μ)?]

A.4.1 Density

To calculate the density, first we need to determine the total number of particles in the system. We can do this by taking a derivative of the lattice action with respect to the chemical potential. Since the only part of the lattice action that depends on $\bar{\mu}$ is S_μ, we can ignore the rest of the action:

$$\begin{aligned} \hat{N}[\phi_i] &= \frac{\partial}{\partial \bar{\mu}} S_{lat} = \frac{\partial}{\partial \bar{\mu}} S_\mu = \frac{\partial}{\partial \bar{\mu}} \sum_{\vec{x},\tau} a^d \left(\phi_r^* \phi_r - e^{\bar{\mu}} \phi_r^* \phi_{r-\hat{\tau}}\right) \\ &= \sum_{\vec{x},\tau} a^d \left(e^{\bar{\mu}} \phi_r^* \phi_{r-\hat{\tau}}\right). \end{aligned} \tag{A.41}$$

This can be written in terms of S_μ:

$$\hat{N}[\phi_i] = \sum_{\vec{x},\tau} a^d \phi_r^* \phi_r - S_\mu \tag{A.42}$$

To determine the density, we simply normalize the particle number by the spatial lattice volume:

$$\hat{n}[\phi_i] = \frac{1}{(aN_x)^d} \hat{N}[\phi_i] \tag{A.43}$$

Complexification

When we apply the complexification to the real fields, $\phi_{a/b} \rightarrow \phi^R_{a/b} + i\phi^I_{a/b}$, we get:

$$
\begin{aligned}
\hat{n} &= \frac{1}{2N_x^d} e^{\bar{\mu}} \sum_{\vec{x},\tau} \sum_{a,b=1}^{2} \left(\delta_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}} + i\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}} \right) \\
&= \frac{1}{2N_x^d} e^{\bar{\mu}} \sum_{\vec{x},\tau} \sum_{a,b=1}^{2} \left[\delta_{ab}(\phi^R_{a,r}\phi^R_{r,b-\hat{\tau}} - \phi^I_{a,r}\phi^I_{r,b-\hat{\tau}}) - \epsilon_{ab}(\phi^R_{a,r}\phi^I_{r,b-\hat{\tau}} + \phi^I_{a,r}\phi^R_{r,b-\hat{\tau}}) \right] \\
& \quad \frac{i}{2N_x^d} e^{\bar{\mu}} \sum_{\vec{x},\tau} \sum_{a,b=1}^{2} \left[\delta_{ab}(\phi^R_{a,r}\phi^I_{r,b-\hat{\tau}} + \phi^I_{a,r}\phi^R_{r,b-\hat{\tau}}) + \epsilon_{ab}(\phi^R_{a,r}\phi^R_{r,b-\hat{\tau}} - \phi^I_{a,r}\phi^I_{r,b-\hat{\tau}}) \right] \qquad \text{(A.44)}
\end{aligned}
$$

This was done in Mathematica. **[NOTE: outstanding questions – where did this extra overall minus sign come from, and is the $1/Nx^d$ the same one we use when taking the average?]**

A.4.2 Local Density

The local density is simply the density function without the sum over the lattice. Since we are concerned with spatial density, we can either average our results over the extent of the time lattice

$$\hat{n}_r[\phi_i, \vec{x}] = \frac{1}{N_\tau (aN_x)^d} \sum_\tau a^d \left(e^{\bar{\mu}} \phi^*_r \phi_{r-\hat{\tau}} \right) \qquad \text{(A.45)}$$

or take the density profile of a single time slice:

$$\hat{n}_r[\phi_i, \vec{x}, \tau] = \frac{1}{(aN_x)^d} a^d \left(e^{\bar{\mu}} \phi^*_r \phi_{r-\hat{\tau}} \right) \qquad \text{(A.46)}$$

Averaging over the extent of the time lattice should produce better results (i.e. we will have a mean and a standard error, which gives us a statistical value to evaluate), so we will do this.

Complexification

The difference between this observable and the density is simply that we evaluate the local density at each site and then do not average over the entire spatial lattice. The complexified observable value is therefore the same, after removing the sum over $\vec{x}$.

A.4.3 Field Modulus Squared

The field modulus squared is simply a measure of the magnitude of the complex field, ϕ. This can be written straightforwardly as

$$\phi^*\phi = \sum_{\vec{x},\tau} a^d d\tau \phi^*_{r,\tau} \phi_{r,\tau} \qquad \text{(A.47)}$$

While this is not an incredibly useful quantity to calculate, it's very simple and can be used in some early testing to ensure stability of the method and check against other results.

Complexification

When we apply the complexification to the real fields, $\phi_{a/b} \to \phi^R_{a/b} + i\phi^I_{a/b}$, we get:

$$\begin{aligned}\phi^*\phi &= \frac{1}{2}\sum_{\vec{x},\tau} a^d \sum_{a=1}^{2} d\tau\,(\phi_{a,r}\phi_{a,r}) \\ &= \frac{1}{2}\sum_{\vec{x},\tau} a^d \sum_{a=1}^{2} d\tau\big[\big((\phi^R_{a,r})^2 - (\phi^I_{a,r})^2\big) + i\phi^R_{a,r}\phi^I_{a,r}\big]\end{aligned} \tag{A.48}$$

This was done in Mathematica.

[NOTE: outstanding questions – this extra factor of $d\tau$ should be divided out – where does that come from? Is it just part of the observable? You'll want to prove this before submitting the paper – look at the analytical solutions for the field modulus]

A.4.4 Angular Momentum

We can compute the angular momentum by taking a derivative of the lattice action with respect to the rotation frequency. Since the only part of the lattice action that depends on $\bar{\omega}_z$ is S_ω, we can ignore the rest of the action:

$$\begin{aligned}\hat{L}_z[\phi_i] &= -\frac{\partial}{\partial\bar{\omega}_z}S_{lat} = \frac{\partial}{\partial\bar{\omega}_z}S_\omega \\ &= \frac{\partial}{\partial\bar{\omega}_z}\sum_{\vec{x},\tau} a^d\left(i\bar{\omega}_z\left(\bar{x}\phi^*_r\phi_{r-\hat{\tau}} - \bar{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}} - \bar{y}\phi^*_r\phi_{r-\hat{\tau}} + \bar{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}\right)\right) \\ &= \sum_{\vec{x},\tau} a^d\left(i\left(\bar{x}\phi^*_r\phi_{r-\hat{\tau}} - \bar{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}} - \bar{y}\phi^*_r\phi_{r-\hat{\tau}} + \bar{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}\right)\right)\end{aligned} \tag{A.49}$$

This can be written in terms of S_ω:

$$\hat{L}_z[\phi_i] = \frac{1}{\bar{\omega}_z}S_\omega \tag{A.50}$$

[NOTE: Check the sign here – you may have lost a minus sign]

Complexification

When we apply the complexification to the real fields, $\phi_{a/b} \to \phi^R_{a/b} + i\phi^I_{a/b}$, we get:

$$\begin{aligned}
\hat{L}_z[\phi_i] &= \frac{1}{2}\sum_{a=1}^{2}\left[\sum_{b=1}^{2}\epsilon_{ab}\left((\bar{y}-\bar{x})\phi_{a,r}\phi_{b,r-\hat{\tau}} + \bar{x}\phi_{a,r}\phi_{b,r-\hat{y}-\hat{\tau}} - \bar{y}\phi_{a,r}\phi_{b,r-\hat{x}-\hat{\tau}}\right)\right] \\
&\quad + i\frac{1}{2}\sum_{a=1}^{2}\left[(\bar{x}-\bar{y})\phi_{a,r}\phi_{a,r-\hat{\tau}} - \bar{x}\phi_{a,r}\phi_{a,r-\hat{y}-\hat{\tau}} + \bar{y}\phi_{a,r}\phi_{a,r-\hat{x}-\hat{\tau}}\right] \\
&= \frac{1}{2}\sum_{a=1}^{2}\Big[(y-x)\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}} + (y-x)\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}} \\
&\quad + x\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}-\hat{y}} + x\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}-\hat{y}} - y\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}-\hat{x}} - y\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}-\hat{x}} \\
&\quad + \epsilon_{ab}\sum_{b=1}^{2}\Big((x-y)\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}} + (y-x)\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}} \\
&\quad + x\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{y}} - x\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{y}} + y\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{x}} - y\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{x}}\Big)\Big] \\
&+ \frac{i}{2}\sum_{a=1}^{2}\Big[(x-y)\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}} + (y-x)\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}} - x\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}-\hat{y}} \\
&\quad + x\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}-\hat{y}} + y\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}-\hat{x}} - y\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}-\hat{x}} \\
&\quad + \epsilon_{ab}\sum_{b=1}^{2}\Big((y-x)\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}} + (y-x)\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}} + x\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{y}} \\
&\quad + x\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{y}} - y\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}-\hat{x}} - y\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}-\hat{x}}\Big)\Big]
\end{aligned} \tag{A.51}$$

[NOTE: You lost your $\bar{x}$ and $\bar{y}$ along the way, so check for that. It should be correct except for not having the bar over it, but better to confirm...]

A.4.5 Harmonic Trapping Potential Energy

As is likely obvious by now, the way to calculate the potential energy due to the harmonic trap is by taking a derivative with respect to $\bar{\omega}_{\text{tr}}$. The only part of the action that depends on $\bar{\omega}_{\text{tr}}$ is S_{tr}:

$$\begin{aligned}
\hat{V}_{\text{tr}}[\phi_i] &= -\frac{\partial}{\partial\bar{\omega}_{\text{tr}}}S_{lat} = \frac{\partial}{\partial\bar{\omega}_{\text{tr}}}S_{\text{tr}} \\
&= \frac{\partial}{\partial\bar{\omega}_{\text{tr}}}\sum_{\vec{x},\tau}a^d\left(\frac{\bar{m}}{2}\bar{\omega}^2_{\text{tr}}\bar{r}^2_\perp\phi^*_r\phi_{r-\hat{\tau}}\right) = \sum_{\vec{x},\tau}a^d\left(\frac{\bar{m}}{2}(2\bar{\omega}_{\text{tr}})\bar{r}^2_\perp\phi^*_r\phi_{r-\hat{\tau}}\right) \\
&= \sum_{\vec{x},\tau}a^d\left(\bar{m}\bar{\omega}_{\text{tr}}\bar{r}^2_\perp\phi^*_r\phi_{r-\hat{\tau}}\right)
\end{aligned} \tag{A.52}$$

In terms of S_{tr}, this is

$$V_{\text{tr}} = \frac{2}{\bar{\omega}_{\text{tr}}}S_{\text{tr}} \tag{A.53}$$

[NOTE: Check the sign here as well]

Complexification

When we apply the complexification to the real fields, $\phi_{a/b} \to \phi^R_{a/b} + i\phi^I_{a/b}$, we get:

$$
\begin{aligned}
V_{\rm tr} &= \sum_{\vec{x},\tau} a^d \frac{\bar{m}}{2}\bar{\omega}_{\rm tr}\bar{r}_\perp^2 \sum_{a=1}^{2}\left[\phi_{a,r}\phi_{a,r-\hat{\tau}} + i\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right] \\
&= \frac{\bar{m}}{2}\bar{\omega}_{\rm tr}\bar{r}_\perp^2 \sum_{\vec{x},\tau} a^d \sum_{a=1}^{2}\left[\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{a,r-\hat{\tau}} - \epsilon_{ab}(\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}} + \phi^I_{a,r}\phi^R_{b,r-\hat{\tau}})\right] \\
&\quad + i\frac{\bar{m}}{2}\bar{\omega}_{\rm tr}\bar{r}_\perp^2 \sum_{\vec{x},\tau} a^d \sum_{a=1}^{2}\left[\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}} + \phi^I_{a,r}\phi^R_{a,r-\hat{\tau}} + \epsilon_{ab}(\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{b,r-\hat{\tau}})\right] \quad \text{(A.54)}
\end{aligned}
$$

A.4.6 Interaction Potential Energy

The interaction also generates a potential energy, and we can calculate that by taking a derivative as well. The interaction parameter, $\bar{\lambda}$, is only present in the interaction term, $S_{\rm int}$:

$$
\begin{aligned}
\hat{V}_{\rm int}[\phi_i] &= -\frac{\partial}{\partial\bar{\lambda}}S_{lat} = -\frac{\partial}{\partial\bar{\lambda}}S_{\rm int} \\
&= -\frac{\partial}{\partial\bar{\lambda}}\sum_{\vec{x},\tau} a^d\left(\bar{\lambda}\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2\right) = -\sum_{\vec{x},\tau} a^d\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2. \quad \text{(A.55)}
\end{aligned}
$$

In terms of $S_{\bar{\lambda}}$, this is:

$$
\hat{V}_{\rm int} = \frac{1}{\bar{\lambda}}S_{\bar{\lambda}} \quad \text{(A.56)}
$$

Complexification

When we apply the complexification to the real fields, $\phi_{a/b} \to \phi^R_{a/b} + i\phi^I_{a/b}$, we get:

$$
\begin{aligned}
\hat{V}_{\text{int}} &= \frac{1}{4}\sum_{\vec{x},\tau} a^d \sum_{a,b=1}^{2} \left[2\phi_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r}\phi_{b,r-\hat{\tau}} - \phi^2_{a,r}\phi^2_{b,r-\hat{\tau}}\right] \\
&\quad + i\frac{\bar{\lambda}}{2}\sum_{\vec{x},\tau} a^d \sum_{a,b=1}^{2} \left[\epsilon_{ab}\left(\phi^2_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}} - \phi_{a,r}\phi_{b,r}\phi^2_{a,r-\hat{\tau}}\right)\right] \\
&= \frac{1}{4}\sum_{\vec{x},\tau} a^d \sum_{a,b=1}^{2} \left[4\phi^R_{a,r}\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}}\right. \\
&\quad +\phi^R_{a,r}\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} + \phi^I_{a,r}\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} \\
&\quad -\phi^R_{a,r}\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad +2\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r}\phi^R_{b,r-\hat{\tau}} + 2\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r}\phi^I_{b,r-\hat{\tau}} \\
&\quad -2\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r}\phi^I_{b,r-\hat{\tau}} - 2\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r}\phi^R_{b,r-\hat{\tau}} \\
&\quad -2\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r}\phi^I_{b,r-\hat{\tau}} - 2\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r}\phi^R_{b,r-\hat{\tau}} \\
&\quad -2\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r}\phi^R_{b,r-\hat{\tau}} - 2\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r}\phi^I_{b,r-\hat{\tau}} \\
&\quad +\epsilon_{ab}(\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r} + \phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r} \\
&\quad -\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r} - \phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r} \\
&\quad +\phi^I_{a,r}\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} + \phi^I_{a,r}\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad -\phi^R_{a,r}\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} - \phi^R_{a,r}\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad +2\phi^R_{a,r}\phi^R_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}} + 2\phi^R_{a,r}\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad \left.-2\phi^R_{a,r}\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} - 2\phi^I_{a,r}\phi^I_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}})\right] \\
&\quad + i\frac{1}{4}\sum_{\vec{x},\tau} a^d \sum_{a,b=1}^{2} \left[\right. \\
&\quad 2\phi^I_{a,r}\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} - 2\phi^R_{a,r}\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad +2\phi^R_{a,r}\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} - 2\phi^R_{a,r}\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} \\
&\quad +2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} - 2\phi^I_{a,r}\phi^I_{b,r}\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} \\
&\quad +2\phi^R_{a,r}\phi^R_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} - 2\phi^I_{a,r}\phi^I_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad +2\phi^R_{a,r}\phi^I_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} - 2\phi^R_{a,r}\phi^I_{b,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad +2\phi^I_{a,r}\phi^R_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} - 2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad +\epsilon_{ab}(\phi^R_{a,r}\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} - \phi^R_{a,r}\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} \\
&\quad +\phi^I_{a,r}\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} \\
&\quad +\phi^I_{a,r}\phi^I_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{a,r-\hat{\tau}} - \phi^R_{a,r}\phi^R_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^R_{a,r-\hat{\tau}} \\
&\quad +\phi^R_{a,r}\phi^R_{b,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{b,r}\phi^I_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}} \\
&\quad +2\phi^R_{a,r}\phi^I_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}} + 2\phi^I_{a,r}\phi^R_{b,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}} \\
&\quad \left.-2\phi^R_{a,r}\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} - 2\phi^R_{a,r}\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}})\right]
\end{aligned} \tag{A.57}
$$

A.4.7 Kinetic Energy

Of course, in addition to the potential energies, our system has a kinetic energy, which we can calculate. The kinetic energy is actually one of the terms in the action, and we don't need to perform any derivatives to calculate it:

$$\hat{T} = S_{\nabla} = \sum_{\vec{x},\tau} a^d \left(\frac{1}{2\bar{m}} \sum_{j=1}^{d} \left(2\phi_r^* \phi_r - \phi_r^* \phi_{r-\hat{j}} - \phi_r^* \phi_{r+\hat{j}} \right) \right) \tag{A.58}$$

Complexification

$$\begin{aligned}
\hat{T} &= \frac{1}{4\bar{m}} \sum_{\vec{x},\tau} a^d \sum_{j=1}^{d} \sum_{a=1}^{2} \left[2\phi_{a,r}^2 - \left(\phi_a \phi_{a,r-\hat{j}} + \phi_a \phi_{a,r+\hat{j}} \right) \right. \\
&\quad \left. -i \sum_{b=1}^{2} \epsilon_{ab} \left(\phi_{a,r} \phi_{b,r-\hat{j}} + \phi_{a,r} \phi_{b,r+\hat{j}} \right) \right] \\
&= \frac{1}{4\bar{m}} \sum_{\vec{x},\tau} a^d \sum_{j=1}^{d} \sum_{a=1}^{2} \left[2\phi_{a,r}^R \phi_{a,r}^R - 2\phi_{a,r}^I \phi_{a,r}^I \right. \\
&\quad + \phi_{a,r}^I \phi_{a,r-\hat{j}}^I + \phi_{a,r}^I \phi_{a,r+\hat{j}}^I - \phi_{a,r}^R \phi_{a,r-\hat{j}}^R - \phi_{a,r}^R \phi_{a,r+\hat{j}}^R \\
&\quad \left. + \epsilon_{ab} \left(\phi_{a,r}^R \phi_{b,r-\hat{j}}^I + \phi_{a,r}^R \phi_{b,r+\hat{j}}^I + \phi_{a,r}^I \phi_{b,r-\hat{j}}^R + \phi_{a,r}^I \phi_{b,r+\hat{j}}^R \right) \right] \\
&\quad + \frac{i}{4\bar{m}} \sum_{\vec{x},\tau} a^d \sum_{j=1}^{d} \sum_{a=1}^{2} \left[4\phi_{a,r}^R \phi_{a,r}^I - \phi_{a,r}^R \phi_{a,r-\hat{j}}^I - \phi_{a,r}^R \phi_{a,r+\hat{j}}^I \right. \\
&\quad - \phi_{a,r}^I \phi_{a,r-\hat{j}}^R - \phi_{a,r}^I \phi_{a,r+\hat{j}}^R + \epsilon_{ab} \left(\phi_{a,r}^I \phi_{b,r-\hat{j}}^I + \phi_{a,r}^I \phi_{b,r+\hat{j}}^I \right. \\
&\quad \left. \left. - \phi_{a,r}^R \phi_{b,r-\hat{j}}^R - \phi_{a,r}^R \phi_{b,r+\hat{j}}^R \right) \right]
\end{aligned} \tag{A.59}$$

A.4.8 Action

We see from the above work that the full action can actually be calculated by summing many of the other observables.

$$\begin{aligned}
S_{\text{lat}} &= S_\mu + S_\nabla - S_{\text{tr}} - S_\omega + S_{\text{int}} \\
&= \hat{N} + \phi^* \phi + \hat{T} - \frac{\bar{\omega_{\text{tr}}}}{2} \hat{V}_{\text{tr}} - \bar{\omega}_z \hat{L}_z + \bar{\lambda} \hat{V}_{\text{int}}
\end{aligned} \tag{A.60}$$

Complexification

As we have already developed a complexified expression for each of the observables that make up the action, we won't re-derive the expressions here. We can construct a complexified action from the observables already derived above.

A.4.9 Total Energy

Traditionally, in stat mech calculations, you can calculate the energy by taking a derivative of the partition function with respect to the inverse temperature, β. In lattice units, $\beta = N_\tau d\tau$, so we can rewrite our lattice action in terms of β. We cancelled out factors of $d\tau$ and a, so we need to put those back in:

$$
\begin{aligned}
S_{\text{lat}} &= \sum_{\vec{x},\tau} a^d \Bigg[\phi_r^*\phi_r - e^{d\tau\mu}\phi_r^*\phi_{r-\hat{\tau}} - \frac{d\tau}{2ma^2}\sum_{j=1}^{d}\left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right) - \frac{m}{2}d\tau\omega_{\text{tr}}^2 r^2\phi_r^*\phi_{r-\hat{\tau}} \\
&\quad + id\tau\omega_z\left(\frac{x}{a}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \frac{x}{a}\phi_r^*\phi_{r-\hat{\tau}} - \frac{y}{a}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \frac{y}{a}\phi_r^*\phi_{r-\hat{\tau}}\right) + d\tau\lambda\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2\Bigg] \qquad \text{(A.61)}
\end{aligned}
$$

Note that only one term in the action has no dependence on $d\tau$, and it came from a derivative with respect to $d\tau$ **[NOTE: Possibly, you have to walk this back one more step... to before you combined terms to get the exponential. It may not make a big difference, and right now we're not very interested in this observable, but look into it.]**

If we multiply the whole thing by $1 = \frac{N_\tau}{N_\tau}$ (and use the same trick on the $d\tau\mu$ in the exponential), we can see where β appears:

$$
\begin{aligned}
S_{\text{lat}} &= \frac{1}{N_\tau}\sum_{\vec{x},\tau} a^d \Bigg[N_\tau\phi_r^*\phi_r - N_\tau e^{\beta\mu/N_\tau}\phi_r^*\phi_{r-\hat{\tau}} - \frac{\beta}{2ma^2}\sum_{j=1}^{d}\left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right) - \frac{m}{2}\beta\omega_{\text{tr}}^2 r^2\phi_r^*\phi_{r-\hat{\tau}} \\
&\quad + i\beta\omega_z\left(\frac{x}{a}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \frac{x}{a}\phi_r^*\phi_{r-\hat{\tau}} - \frac{y}{a}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \frac{y}{a}\phi_r^*\phi_{r-\hat{\tau}}\right) + \beta\lambda\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2\Bigg] \qquad \text{(A.62)}
\end{aligned}
$$

We then take a derivative with respect to β, giving us:

$$
\begin{aligned}
\hat{E} &= -\frac{1}{N_\tau}\sum_{\vec{x},\tau} a^d \Bigg[\mu e^{\beta\mu/N_\tau}\phi_r^*\phi_{r-\hat{\tau}} + \frac{1}{2ma^2}\sum_{j=1}^{d}\left(\phi_r^*\phi_{r-\hat{j}} - 2\phi_r^*\phi_r + \phi_r^*\phi_{r+\hat{j}}\right) + \frac{m}{2}\omega_{\text{tr}}^2 r^2\phi_r^*\phi_{r-\hat{\tau}} \\
&\quad - i\omega_z\left(\frac{x}{a}\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} - \frac{x}{a}\phi_r^*\phi_{r-\hat{\tau}} - \frac{y}{a}\phi_r^*\phi_{r-\hat{x}-\hat{\tau}} + \frac{y}{a}\phi_r^*\phi_{r-\hat{\tau}}\right) - \lambda\left(\phi_r^*\phi_{r-\hat{\tau}}\right)^2\Bigg] \qquad \text{(A.63)}
\end{aligned}
$$

This can be written in terms of our other observables, and we bring back our parameters in lattice units:

$$
\begin{aligned}
\hat{E} &= -\frac{1}{N_\tau d\tau}\sum_{\vec{x},\tau} a^d\left[\bar{\mu}\left(\phi_r^*\phi_r - S_\mu\right) - S_\nabla + S_{\text{tr}} + S_\omega - S_{\text{int}}\right] \\
&= \sum_{\vec{x},\tau} a^d\left[\frac{1}{N_\tau d\tau}\bar{\mu}\hat{N} + \frac{1}{N_\tau d\tau}\hat{T} - \frac{\bar{\omega}_{\text{tr}}}{2}\hat{V}_{\text{tr}} - \bar{\omega}_z\hat{L}_z + \bar{\lambda}\hat{V}_{\text{int}}\right] \qquad \text{(A.64)}
\end{aligned}
$$

Complexification

Just as with the action, we can express the complexified total energy using our other observables.

A.4.10 Circulation

The circulation is defined as

$$
\begin{aligned}
\Gamma[l] &= \frac{1}{2\pi}\oint_{l\times l} dx\left(\theta_{\tau,x+j} - \theta_{\tau,x}\right) \qquad \text{(A.65)} \\
\theta_{t,x} &= \arctan\left(\frac{\mathrm{Im}[\phi_{\tau,x}]}{\mathrm{Re}[\phi_{\tau,x}]}\right). \qquad \text{(A.66)}
\end{aligned}
$$

Given that our field, ϕ, is broken into 4 components ($\phi^{R/I}_{1/2}$, we need to rewrite this quantity in terms of those components:

$$\begin{aligned}
\phi &= \frac{1}{\sqrt{2}}\left(\phi_1^R + i\phi_1^I + i\phi_2^R - \phi_2^I\right) \\
\mathrm{Re}[\phi_{\tau,x}] &= \frac{1}{\sqrt{2}}\left(\phi^R_{1,\tau,x} - \phi^I_{2,\tau,x}\right) \\
\mathrm{Im}[\phi_{\tau,x}] &= \frac{1}{\sqrt{2}}\left(\phi^I_{1,\tau,x} + \phi^R_{2,\tau,x}\right) \\
\theta_{\tau,x} &= \arctan\left(\frac{\phi^I_{1,\tau,x} + \phi^R_{2,\tau,x}}{\phi^R_{1,\tau,x} - \phi^I_{2,\tau,x}}\right).
\end{aligned} \tag{A.67}$$

Note, we have the potential to encounter singularities here – in the extremely unlikely event that $\phi^R_{1,\tau,x} = \phi^I_{2,\tau,x}$, we will have a zero in the denominator. This probably won't ever occur, but we should take it into account when troubleshooting.

We need to convert our integral to a sum, in order to calculate it on a lattice. So we choose an $l \times l$ sub-section of the lattice centered around the middle and proceed around it in a clockwise fashion.

$$\begin{aligned}
\Gamma[l] &= \frac{a}{2\pi}\sum_{l\times l}\left(\theta_{\tau,x+j} - \theta_{\tau,x}\right) \\
&= \frac{a}{2\pi}\sum_{l\times l}\left[\arctan\left(\frac{\phi^I_{1,\tau,x+\hat{j}} + \phi^R_{2,\tau,x+\hat{j}}}{\phi^R_{1,\tau,x+\hat{j}} - \phi^I_{2,\tau,x+\hat{j}}}\right) - \arctan\left(\frac{\phi^I_{1,\tau,x} + \phi^R_{2,t,x}}{\phi^R_{1,\tau,x} - \phi^I_{2,\tau,x}}\right)\right]
\end{aligned} \tag{A.68}$$

where $\phi_{\tau,x+j}$ is the value of the field at the next site on the loop from $\phi_{\tau,x}$ for each point along the loop.

We can then average this value over the extent of our temporal lattice to produce a value with a mean and standard error.

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