Deriving the action for the lattice

Continuum action:

$r_\perp^2 = (x^2 + y^2)$

$$S[\phi] = \int d^d x\, d\tau \left[\phi^* \left(\partial_\tau - \mu - \frac{\nabla^2}{2m} - \frac{m}{2} \omega_{tr}^2 r_\perp^2 - i\omega_z (x\partial_y - y\partial_x) \right) \phi + \lambda (\phi^* \phi)^2 \right]$$

To discretize:

$$\int d^d x\, d\tau \rightarrow \sum_{\vec{x},\tau} a^d d\tau \quad , \quad \alpha \phi^* \phi \rightarrow \alpha \phi_r^* \phi_{r-\hat{\tau}} \text{ (see appendix)}$$

Discrete action:

$$S[\phi] = \sum_{\vec{x},\tau} a^d d\tau \left[\phi_r^* \left(\partial_\tau - \frac{\nabla^2}{2m} \right) \phi_r - \phi_r^* \left(\mu + \frac{m}{2} \omega_{tr}^2 r_\perp^2 + i\omega_z (x\partial_y - y\partial_x) \right) \phi_{r-\hat{\tau}} + \lambda (\phi_r^* \phi_{r-\hat{\tau}})^2 \right]$$

Discrete derivatives:

$$\partial_j \phi = \frac{1}{a} (\phi_r - \phi_{r-\hat{j}}) \quad \text{and} \quad \nabla^2 \phi = \sum_{j=1}^{d} \frac{1}{a^2} (\phi_{r-\hat{j}} - 2\phi_r + \phi_{r+\hat{j}})$$

$$\phi_r^* \partial_\tau \phi_r = \frac{1}{d\tau} (\phi_r^* \phi_{r-\hat{\tau}} - \phi_r^* \phi_r)$$

$$\phi_r^* \nabla^2 \phi_r = \frac{1}{a^2} \sum_{j=1}^{d} (\phi_r^* \phi_{r-\hat{j}} - 2\phi_r^* \phi_r + \phi_r^* \phi_{r+\hat{j}})$$

$$\phi_r^* \partial_x \phi_{r-\hat{\tau}} = \frac{1}{a} (\phi_r^* \phi_r - \phi_r^* \phi_{r-\hat{x}}), \quad \phi_r^* \partial_y \phi_{r-\hat{\tau}} = \frac{1}{a} (\phi_r^* \phi_r - \phi_r^* \phi_{r-\hat{y}})$$

So our discrete (lattice) action is:

$$S = \sum_{\vec{x},\tau} \Big[a^d \phi_r^* \phi_r - a^d \phi_r^* \phi_{r-\hat{\tau}} - a^d d\tau \mu \phi_r^* \phi_{r-\hat{\tau}} - \frac{a^{d-2} d\tau}{2m} \sum_{j=1}^{d} (\phi_r^* \phi_{r-\hat{j}} - 2\phi_r^* \phi_r + \phi_r^* \phi_{r+\hat{j}})$$
$$- \frac{m a^d d\tau}{2} \omega_{tr}^2 r_\perp^2 \phi_r^* \phi_{r-\hat{\tau}} + i\omega_z a^{d-1} d\tau (x\phi_r^* \phi_{r-\hat{y}-\hat{\tau}} - x\phi_r^* \phi_{r-\hat{\tau}} - y\phi_r^* \phi_{r-\hat{x}-\hat{\tau}}$$
$$+ y\phi_r^* \phi_{r-\hat{\tau}}) + a^d d\tau \lambda (\phi_r^* \phi_{r-\hat{\tau}})^2 \Big]$$

Let's update our parameters to incorporate $d\tau$ and a

$$x = a\tilde{x} \quad y = a\tilde{y} \quad r^2 = a^2 \tilde{r}^2 \quad \mu = \frac{\tilde{\mu}}{d\tau} \quad m = \frac{\tilde{m}\, d\tau}{a^2} \quad \omega_{tr} = \frac{\tilde{\omega}_{tr}}{d\tau}$$

$$\omega_z = \frac{\tilde{\omega}_z}{d\tau} \quad \lambda = \frac{\tilde{\lambda}}{d\tau}$$

$$S=\sum_{\vec{x},\tau}\Big[a^d\phi^*_r\phi_r-a^d\phi^*_r\phi_{r-\hat{\tau}}-a^d\tilde{\mu}\,\phi^*_r\phi_{r-\tau}-\frac{a^d}{2\tilde{M}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}}-2\phi^*_r\phi_r+\phi^*_r\phi_{r+\hat{j}}\right)$$
$$-\frac{\tilde{m}a^d}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau}+i\tilde{\omega}_z a^d\left(\tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}}-\tilde{x}\phi^*_r\phi_{r-\tau}-\tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}+\tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)$$
$$+a^d\tilde{\lambda}\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2\Big]$$

We can pull out a constant factor of a^d now:

$$S=\sum_{\vec{x},\tau}a^d\Big[\phi^*_r\phi_r-\phi^*_r\phi_{r-\hat{\tau}}-\tilde{\mu}\,\phi^*_r\phi_{r-\tau}-\frac{1}{2\tilde{M}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}}-2\phi^*_r\phi_r+\phi^*_r\phi_{r+\hat{j}}\right)$$
$$-\frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau}+i\tilde{\omega}_z\left(\tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}}-\tilde{x}\phi^*_r\phi_{r-\tau}-\tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}+\tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)+\tilde{\lambda}\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2\Big]$$

Noting that, to second order in $d\tau$, $(1+d\tau\mu)=e^{d\tau\mu}$, we further simplify:

$$S=\sum_{\vec{x},\tau}a^d\Big[\overbrace{\phi^*_r\phi_r-e^{\tilde{\mu}}\phi^*_r\phi_{r-\tau}}^{S_\mu}\;\overbrace{-\frac{1}{2\tilde{M}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}}-2\phi^*_r\phi_r+\phi^*_r\phi_{r+\hat{j}}\right)}^{S_\nabla}$$
$$-\underbrace{\frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau}}_{S_{tr}}+\underbrace{i\tilde{\omega}_z\left(\tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}}-\tilde{x}\phi^*_r\phi_{r-\tau}-\tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}}+\tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)}_{-S_w}+\underbrace{\tilde{\lambda}\left(\phi^*_r\phi_{r-\hat{\tau}}\right)^2}_{S_{int}}\Big]$$

We can make our work easier by writing this out as:

$$S=\sum_{\vec{x},\tau}a^d\left[S_\mu+S_\nabla-S_{trap}-S_w+S_{int}\right]$$

Writing our complex fields in terms of two real components

The next step is to note that $\phi = \phi_1 + i\phi_2$ and apply this coordinate transformation to the lattice.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \text{ and } \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \longrightarrow$$

$$S_\mu = \phi^*_r\phi_r - e^{\tilde{\mu}}\phi^*_r\phi_{r-\tau} \longrightarrow \frac{1}{2}\Big[\phi_{1,r}^2 + \phi_{2,r}^2 - e^{\mu}\big(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)\Big]$$

$$S_\nabla = -\frac{1}{2\tilde{M}}\sum_{j=1}^{d}\left(\phi^*_r\phi_{r-\hat{j}} - 2\phi^*_r\phi_r + \phi^*_r\phi_{r+\hat{j}}\right)$$

$$\to \frac{1}{4\tilde{m}}\sum_{j=1}^{d}\Big[2\phi_{1,r}^2 + 2\phi_{2,r}^2 - \big(\phi_{1,r}\phi_{1,r-j} - i\phi_{2,r}\phi_{1,r-j} + i\phi_{1,r}\phi_{2,r-j} + \phi_{2,r}\phi_{2,r-j}$$
$$+ \phi_{1,r}\phi_{1,r+j} - i\phi_{2,r}\phi_{1,r+j} + i\phi_{1,r}\phi_{2,r+j} + \phi_{2,r}\phi_{2,r+j}\big)\Big]$$

$$S_{trap} = \frac{\tilde{m}}{2}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\phi^*_r\phi_{r-\tau} \to \frac{\tilde{m}}{4}\tilde{\omega}_{tr}^2\tilde{r}_\perp^2\big(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)$$

$$S_w = i\tilde{\omega}_z\left(\tilde{x}\phi^*_r\phi_{r-\tau} - \tilde{x}\phi^*_r\phi_{r-\hat{y}-\hat{\tau}} + \tilde{y}\phi^*_r\phi_{r-\hat{x}-\hat{\tau}} - \tilde{y}\phi^*_r\phi_{r-\hat{\tau}}\right)$$

$$\to i\frac{\tilde{\omega}_z}{2}\Big[\tilde{x}\big(\phi_{1,r}\phi_{1,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} + i\phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}}$$
$$- i\phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big) + \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}}$$
$$+ i\phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{\tau}} + i\phi_{2,r}\phi_{1,r-\tau} - i\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)\Big]$$

$$= i\frac{\tilde{\omega}_z}{2}\Big[\tilde{x}\big(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big)$$
$$+ \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{1,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\big)$$
$$+ \frac{\tilde{\omega}_z}{2}\Big[\tilde{x}\big(\phi_{2,r}\phi_{1,r-\tau} - \phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}} + \phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}}\big)$$
$$+ \tilde{y}\big(\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\tau} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\big)\Big]$$

$$= i\frac{\tilde{\omega}_z}{2}\Big[(\tilde{x}-\tilde{y})\big(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\big) - \tilde{x}\big(\phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{y}-\tau}\big) + \tilde{y}\big(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}}\big)\Big]$$
$$+ \frac{\tilde{\omega}_z}{2}\Big[(\tilde{x}-\tilde{y})\big(\phi_{2,r}\phi_{1,r-\tau} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\big) - \tilde{x}\big(\phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}}\big) + \tilde{y}\big(\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}}\big)\Big]$$

$$S_{int} = \tilde{\lambda}(\phi^*_r \phi_{r-\hat{\tau}})^2 \to \frac{\hat{\tilde{\lambda}}}{4}\left(\phi_{1,r}\phi_{1,r-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right)^2$$

$$\begin{aligned}
= \frac{\hat{\tilde{\lambda}}}{4}\Big(&\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - i\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + i\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&- i\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&+ i\phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&+ \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - i\phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + i\phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}}\Big)
\end{aligned}$$

$$\begin{aligned}
= \frac{\hat{\tilde{\lambda}}}{4}\Big(&\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} \\
&+ \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} - \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}}\Big) \\
+ i\frac{\hat{\tilde{\lambda}}}{4}\Big(&-\phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} - \phi_{2,r}\phi_{1,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} \\
&+ \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{1,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{2,r-\hat{\tau}} - \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\phi_{1,r}\phi_{2,r-\hat{\tau}}\Big)
\end{aligned}$$

Now, we can derive the drift functions by taking a complex derivative of the action, S

$$K_a = \frac{\delta}{\delta \phi_a} S[\phi]$$

When taking derivatives on the lattice, recall that $\delta_i \phi_r = \frac{1}{a}(\phi_r - \phi_{r-\hat{i}})$

Therefore: $$\frac{\delta}{\delta \phi_{c,r}}\left(\sum_{a=1}^{2}\sum_{q=1}^{N} \phi_{a,q}\, \phi_{a,q+\hat{i}}\right) = \sum_{a=1}^{2}\sum_{q=1}^{N}\left(\phi_{a,q} \frac{\delta}{\delta \phi_{c,r}} \phi_{a,q+\hat{i}} + \frac{\delta}{\delta \phi_{c,r}} \phi_{a,q}\, \phi_{a,q+\hat{i}}\right)$$

$$= \sum_{a=1}^{2}\sum_{q=1}^{N}\left(\phi_{a,q}\, \delta_{c,a}\, \delta_{r,q+\hat{i}} + \delta_{c,a}\, \delta_{r,q}\, \phi_{a,q+\hat{i}}\right)$$

the sums collapse, due to the Kronecker deltas

$$\frac{\delta}{\delta \phi_{c,r}}\left(\sum_{a=1}^{2}\sum_{q=1}^{N} \phi_{a,q}\, \phi_{a,q+\hat{i}}\right) = \phi_{c,r-\hat{i}} \text{ if}(r>0) + \phi_{c,r+\hat{i}} \text{ if}(r+\hat{i}<N)$$

similarly $$\frac{\delta}{\delta \phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2} \phi_{a,q}\, \phi_{a,q-\hat{i}}\right) = \phi_{c,r+\hat{i}} \text{ if}(r+\hat{i}<N) + \phi_{c,r-\hat{i}} \text{ if}(r>0)$$

and $$\frac{\delta}{\delta \phi_{c,r}}\left(\sum_{q=1}^{N}\sum_{a=1}^{2}\sum_{b=1}^{2} \varepsilon_{ab}\, \phi_{a,q}\, \phi_{b,q+\hat{i}}\right) = \sum_{b=1}^{2} \varepsilon_{cb}\left(\phi_{b,r-\hat{i}} + \phi_{b,r+\hat{i}}\right)$$

and $$\frac{\delta}{\delta \phi_{c,r}}\left(\sum_{q=1}^{N}\sum_{a=1}^{2}\sum_{b=1}^{2} \varepsilon_{ab}\, \phi_{a,q}\, \phi_{b,q-\hat{i}}\right) = \sum_{b=1}^{2} \varepsilon_{cb}\left(\phi_{b,r+\hat{i}} + \phi_{b,r-\hat{i}}\right)$$

Question: don't our boundary conditions already satisfy the constraints?
If $r=N$, $\phi_{r+\hat{i}}=0$ automatically

$$K_a = \frac{\delta}{\delta \phi_a}\sum_{\vec{x},\tau} S_\mu + \frac{\delta}{\delta \phi_a}\sum_{\vec{x},\tau} S_D - \frac{\delta}{\delta \phi_a}\sum_{\vec{x},\tau} S_{trap} - \frac{\delta}{\delta \phi_a}\sum_{\vec{x},\tau} S_w + \frac{\delta}{\delta \phi_a}\sum_{\vec{x},\tau} S_{int}$$

Let's take this derivative for each part of the action individually

$$K_{a,\mu} = \frac{\delta}{\delta \phi_a}\sum_{\vec{x},\tau} S_\mu =$$

Next, we complexify our real fields, ϕ_a $(a=1,2)$, $\phi_a \to \phi_a^R + i\phi_a^I$