0.1 Justification for the Form of the Non-Relativistic Lattice Action

The continuum action for bosons with a non-relativistic dispersion, a rotating external potential, a non-zero chemical potential, an external trap potential, and an interaction term is as follows:

$$S = \int_V \phi^* \left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu - i\omega_z(x\partial_y - y\partial_x) - \frac{m\omega_{\text{trap}}^2}{2}(x^2+y^2) \right)\phi + \lambda \int_V (\phi^*\phi)^2. \tag{1}$$

To convert this to a lattice action, we must first discretize the derivatives. We will use a backwards finite difference discretization for the single derivative and a central difference approximation for the double derivative, such that:

$$\begin{aligned} \partial_i \phi_r &= \frac{1}{a}\left(\phi_r - \phi_{r-\hat{i}}\right) && (2) \\ \nabla^2 \phi_r &= \sum_i \frac{1}{a^2}\left(\phi_{r+\hat{i}} - 2\phi_r + \phi_{r-\hat{i}}\right), && (3) \end{aligned}$$

where $r = (x, y, \tau)$ and the discretization length a (lattice spacing) is 1 for spatial derivatives and $d\tau$ for temporal ones.

In order to treat the finite chemical potential, the external trapping potential, the rotation, and the interaction we must shift our indices on the field that is acted on by μ and ω_{trap} by one step in the time direction. This is to make these potentials gauge invariant in the lattice formulation. Since we have periodic boundary conditions in time, we don't have to worry about boundaries is. Therefore, our lattice action becomes, at each lattice site, r:

$$\begin{aligned} S_{\text{lat},r} &= \phi_r^* \left[\phi_r - \phi_{r-\hat{\tau}} - d\tau\mu\phi_{r-\hat{\tau}} - \frac{d\tau}{2m}\sum_{i=1}^{d}\left(\phi_{r+\hat{i}} - 2\phi_r + \phi_{r-\hat{i}}\right) - \frac{d\tau m\omega_{\text{trap}}^2}{2}(x^2+y^2)\phi_{r-\hat{\tau}} \right] && (4) \\ &\quad -\phi_r^*\left[i d\tau\omega_z \left(x\phi_{r-\hat{\tau}} - x\phi_{r-\hat{y}-\hat{\tau}} - y\phi_{r-\hat{\tau}} + y\phi_{r-\hat{x}-\hat{\tau}} \right) \right] + d\tau\lambda(\phi_r^*\phi_{r-\hat{\tau}})^2. \end{aligned}$$

We can then combine our time derivative and our chemical potential in the following way:

$$\begin{aligned} S_{\text{lat},r} &= \phi_r^* \left[\phi_r - (1 + d\tau\mu)\phi_{r-\hat{\tau}} - \frac{d\tau}{2m}\sum_{i=1}^{d}\left(\phi_{r+\hat{i}} - 2\phi_r + \phi_{r-\hat{i}}\right) - d\tau\frac{m\omega_{\text{trap}}^2}{2}(x^2+y^2)\phi_{r-\hat{\tau}} \right] && (5) \\ &\quad -\phi_r^*\left[i d\tau\omega_z \left(x\phi_{r-\hat{\tau}} - x\phi_{r-\hat{y}-\hat{\tau}} - y\phi_{r-\hat{\tau}} + y\phi_{r-\hat{x}-\hat{\tau}} \right) \right] + d\tau\lambda(\phi_r^*\phi_{r-\hat{\tau}})^2. \end{aligned}$$

Note that to second order in the lattice size, this is equivalent to:

$$\begin{aligned} S_{\text{lat},r} &= \phi_r^* \left[\phi_r - e^{d\tau\mu}\phi_{r-\hat{\tau}} - \frac{d\tau}{2m}\sum_{i=1}^{d}\left(\phi_{r+\hat{i}} - 2\phi_r + \phi_{r-\hat{i}}\right) - d\tau\frac{m\omega_{\text{trap}}^2}{2}(x^2+y^2)\phi_{r-\hat{\tau}} \right] && (6) \\ &\quad - i d\tau\omega_z\phi_r^*\left[(x-y)\phi_{r-\hat{\tau}} - x\phi_{r-\hat{y}-\hat{\tau}} + y\phi_{r-\hat{x}-\hat{\tau}} \right] + d\tau\lambda(\phi_r^*\phi_{r-\hat{\tau}})^2. \end{aligned}$$

This will be our lattice action, which we will complexify and use to evolve our system in Langevin time. To simplify, let's divide the lattice action into smaller components:

$$S_{\text{lat}} = \sum_r \left(S_{\tau,r} + S_{\nabla,r} - S_{\text{trap},r} - S_{\omega,r} + S_{\text{int},r} \right) \tag{7}$$

with

$$
\begin{aligned}
S_{\tau,r} &= \phi_r^*\phi_r - e^{d\tau\mu}\phi_r^*\phi_{r-\hat{\tau}} && (8)\\
S_{\nabla,r} &= d\tau\frac{d}{m}\phi_r^*\phi_r - \frac{d\tau}{2m}\sum_{i=1}^{d}\left(\phi_r^*\phi_{r+\hat{i}} + \phi_r^*\phi_{r-\hat{i}}\right) && (9)\\
S_{\mathrm{trap},r} &= d\tau\frac{m\omega_{\mathrm{trap}}^2}{2}(x^2+y^2)\phi_r^*\phi_{r-\hat{\tau}} && (10)\\
S_{\omega,r} &= id\tau\omega_z\left[(x-y)\phi_r^*\phi_{r-\hat{\tau}} - x\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} + y\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}\right] && (11)\\
S_{\mathrm{int},r} &= d\tau\lambda(\phi_r^*\phi_{r-\hat{\tau}})^2. && (12)
\end{aligned}
$$

Note that we are restricting ourselves to two spatial dimensions at this point in the work. The extension of this method to three-dimensional rotating systems is saved for future work.

This action must first be rewritten with the complex fields expressed in terms of two real fields, $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and $\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$. Each piece of the action is computed below:

First, the time derivative and chemical potential part of the action:

$$
\begin{aligned}
S_{\tau,r} &= \phi_r^*\phi_r - e^{d\tau\mu}\phi_r^*\phi_{r-\hat{\tau}}\\
&= \frac{1}{2}\left[\phi_{1,r}^2 + \phi_{2,r}^2 - e^{d\tau\mu}\left(\phi_{1,r}\phi_{1,r-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right)\right] \qquad (13)\\
&= \frac{1}{2}\sum_{a=1}^{2}\left[\phi_{a,r}^2 - e^{d\tau\mu}\phi_{a,r}\phi_{a,r-\hat{\tau}} - ie^{d\tau\mu}\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right].
\end{aligned}
$$

Next, the spatial derivative part (corresponding to the kinetic energy):

$$
\begin{aligned}
S_{\nabla,r} &= \frac{d\tau}{2m}\left[2d\phi_r^*\phi_r - \sum_{i=1}^{d}\left(\phi_r^*\phi_{r+\hat{i}} + \phi_r^*\phi_{r-\hat{i}}\right)\right]\\
&= \frac{d\tau}{4m}\left[2d\left(\phi_{1,r}^2 + \phi_{2,r}^2\right) - \sum_{i=\pm1}^{d}\left(\phi_{1,r}\phi_{1,r+\hat{i}} + i\phi_{1,r}\phi_{2,r+\hat{i}} - i\phi_{2,r}\phi_{1,r+\hat{i}} + \phi_{2,r}\phi_{2,r+\hat{i}}\right)\right] \qquad (14)\\
&= \frac{d\tau}{4m}\sum_{a=1}^{2}\left[2d\phi_{a,r}^2 - \left(\sum_{i=\pm1}^{d}\phi_{a,r}\phi_{a,r+\hat{i}} + i\sum_{b=1}^{2}\sum_{i=\pm1}^{d}\epsilon_{ab}\phi_{a,r}\phi_{b,r+\hat{i}}\right)\right].
\end{aligned}
$$

Then, for the part of the action due to the external trapping potential:

$$
\begin{aligned}
S_{\mathrm{trap},r} &= d\tau\frac{m\omega_{\mathrm{trap}}^2}{2}(x^2+y^2)\phi_r^*\phi_{r-\hat{\tau}}\\
&= \frac{d\tau m\omega_{\mathrm{tr}}^2(x^2+y^2)}{4}\left[\phi_{1,r}\phi_{1,r-\tau} + i\phi_{1,r}\phi_{2,r-\tau} - i\phi_{2,r}\phi_{1,r-\tau} + \phi_{2,r}\phi_{2,r-\tau}\right] \qquad (15)\\
&= \frac{d\tau\omega_{\mathrm{tr}}(x^2+y^2)}{4}\sum_{a=1}^{2}\left(\phi_{a,r}\phi_{a,r-\hat{\tau}} + i\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right).
\end{aligned}
$$

Next, the rotational piece:

$$
\begin{aligned}
S_{\omega,r} &= id\tau\omega_z\left[(x-y)\phi_r^*\phi_{r-\hat{\tau}} - x\phi_r^*\phi_{r-\hat{y}-\hat{\tau}} + y\phi_r^*\phi_{r-\hat{x}-\hat{\tau}}\right] \\
&= \frac{id\tau\omega_z}{2}\left[(x-y)\left(\phi_{1,r}\phi_{1,r-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{\tau}}\right) + i(x-y)\left(\phi_{1,r}\phi_{2,r-\hat{\tau}} + \phi_{2,r}\phi_{1,r-\hat{\tau}}\right)\right] \\
&\quad -\frac{id\tau\omega_z}{2}\left[x\left(\phi_{1,r}\phi_{1,r-\hat{y}-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{y}-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{y}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{y}-\hat{\tau}}\right)\right] \\
&\quad +\frac{id\tau\omega_z}{2}\left[y\left(\phi_{1,r}\phi_{1,r-\hat{x}-\hat{\tau}} + i\phi_{1,r}\phi_{2,r-\hat{x}-\hat{\tau}} - i\phi_{2,r}\phi_{1,r-\hat{x}-\hat{\tau}} + \phi_{2,r}\phi_{2,r-\hat{x}-\hat{\tau}}\right)\right] \\
&= \frac{d\tau\omega_z}{2}\sum_{a=1}^{2}\left[\sum_{b=1}^{2}\epsilon_{ab}\left((x-y)\phi_{a,r}\phi_{b,r-\hat{\tau}} - x\phi_{a,r}\phi_{b,r-\hat{y}-\hat{\tau}} + y\phi_{a,r}\phi_{b,r-\hat{x}-\hat{\tau}}\right)\right] \\
&\quad +i\frac{d\tau\omega_z}{2}\sum_{a=1}^{2}\left[\sum_{b=1}^{2}\left((y-x)\phi_{a,r}\phi_{a,r-\hat{\tau}} + x\phi_{a,r}\phi_{a,r-\hat{y}-\hat{\tau}} - y\phi_{a,r}\phi_{a,r-\hat{x}-\hat{\tau}}\right)\right].
\end{aligned}
\tag{16}
$$

And finally, the interaction term in the action:

$$
\begin{aligned}
S_{\text{int},r} &= d\tau\lambda(\phi_r^*\phi_{r-\hat{\tau}})^2 \\
&= \dots \\
&= \frac{d\tau\lambda}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\left[2\phi_{a,r}\phi_{a,r-\hat{\tau}}\phi_{b,r}\phi_{b,r-\hat{\tau}} - \phi_{a,r}^2\phi_{b,r-\hat{\tau}}^2\right] \\
&\quad +i\frac{d\tau\lambda}{2}\sum_{a=1}^{2}\sum_{b=1}^{2}\left[\epsilon_{ab}\left(\phi_{a,r}^2\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}} - \phi_{a,r}\phi_{a,r-\hat{\tau}}^2\phi_{b,r}\right)\right].
\end{aligned}
\tag{17}
$$

We will work with the lattice action in this form in order to derive the Langevin drift function.

0.2 Writing the Complex Action in Terms of Real Fields

Because we want to test our free, nonrotating, and non interacting system in multiple dimensions and leave open the possibility of future work in three dimensions, we leave the dimension number d general in the following derivations.

First, we take our complex field, ϕ, and represent it as the complex sum of two real fields: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. For each of the action contributions, this gives us:

$$
\text{Re}[S_{\tau,r}] \rightarrow \frac{1}{2}\sum_{a=1}^{2}\left(\phi_{a,r}^2 - e^{d\tau\mu}\ \phi_{a,r}\phi_{a,r-\hat{\tau}}\right) \tag{18}
$$

$$
\text{Im}[S_{\tau,r}] \rightarrow \frac{-e^{d\tau\mu}}{2}\sum_{a,b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}} \tag{19}
$$

$$
\text{Re}[S_{\nabla,r}] \rightarrow \sum_{a=1}^{2}\left(\frac{d}{m}\phi_{a,r}^2 - \frac{1}{4m}\sum_{i=\pm 1}^{d}\phi_{a,r}\phi_{a,r+\hat{i}}\right) \tag{20}
$$

$$
\text{Im}[S_{\nabla,r}] \rightarrow \frac{-1}{4m}\sum_{a,b=1}^{2}\sum_{i=\pm 1}^{d}\epsilon_{ab}\phi_{a,r}\phi_{b,r+\hat{i}} \tag{21}
$$

$$\mathrm{Re}[S_{\mathrm{trap},r}] \;\rightarrow\; \frac{m}{4}\omega_{\mathrm{trap}}^2\left(x^2+y^2\right)\sum_{a=1}^{2}\phi_{a,r}\phi_{a,r-\hat{\tau}} \tag{22}$$

$$\mathrm{Im}[S_{\mathrm{trap},r}] \;\rightarrow\; \frac{m}{4}\omega_{\mathrm{trap}}^2\left(x^2+y^2\right)\sum_{a,b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}} \tag{23}$$

$$\mathrm{Re}[S_{\omega,r}] \;\rightarrow\; \frac{\omega_z}{2}\sum_{a,b=1,2}\epsilon_{ab}\left((y-x)\phi_{a,r}\phi_{b,r-\hat{\tau}}+\widetilde{x}\phi_{a,r}\phi_{b,r-\hat{y}-\hat{\tau}}-\widetilde{y}\phi_{a,r}\phi_{b,r-\hat{x}-\hat{\tau}}\right) \tag{24}$$

$$\mathrm{Im}[S_{\omega,r}] \;\rightarrow\; \frac{\omega_z}{2}\sum_{a=1}^{2}\left((x-y)\phi_{a,r}\phi_{a,r-\hat{\tau}}-\widetilde{x}\phi_{a,r}\phi_{a,r-\hat{y}-\hat{\tau}}+\widetilde{y}\phi_{a,r}\phi_{a,r-\hat{x}-\hat{\tau}}\right) \tag{25}$$

where $\widetilde{x}$ and $\widetilde{y}$ are our x and x coordinates shifted by the center of the trap:

$$\widetilde{x} \;=\; x-\frac{N_x-1}{2}$$
$$\widetilde{y} \;=\; y-\frac{N_x-1}{2}$$

$$\mathrm{Re}[S_{\mathrm{int},r}] \;\rightarrow\; \frac{\lambda}{4}\sum_{a,b=1}^{2}\left(2\phi_{a,r}\phi_{b,r}\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}-\phi_{a,r}^2\phi_{b,r-\hat{\tau}}^2\right) \tag{26}$$

$$\mathrm{Im}[S_{\mathrm{int},r}] \;\rightarrow\; \frac{\lambda}{2}\sum_{a,b=1}^{2}\epsilon_{ab}\left(\phi_{a,r}^2\phi_{a,r-\hat{\tau}}\phi_{b,r-\hat{\tau}}-\phi_{a,r}\phi_{b,r}\phi_{a,r-\hat{\tau}}^2\right), \tag{27}$$

where $S_{j,r}=\mathrm{Re}[S_{j,r}]+i\mathrm{Im}[S_{j,r}]$, and $\epsilon_{12}=1$, $\epsilon_{21}=-1$, and $\epsilon_{11}=\epsilon_{22}=0$.

From here, we can compute the drift function.

0.3 Complexifying the Real Fields

Now we take our real fields, ϕ_a, where $a=1,2$, and rewrite them as two complex fields: $\phi_a=\phi_a^R+i\phi_a^I$

The time derivative piece, $S_{\tau,r}$ becomes:

$$\begin{aligned} S_{\tau,r}^R \;\rightarrow\; & \frac{1}{2}\sum_{a=1}^{2}\left((\phi_{a,r}^R)^2-(\phi_{a,r}^I)^2-e^{d\tau\mu}\phi_{a,r}^R\phi_{a,r-\hat{\tau}}^R+e^{d\tau\mu}\phi_{a,r}^I\phi_{a,r-\hat{\tau}}^I\right) \\ & +\frac{1}{2}\sum_{a,b=1}^{2}\epsilon_{ab}\left(e^{d\tau\mu}\phi_{a,r}^R\phi_{b,r-\hat{\tau}}^I+e^{d\tau\mu}\phi_{a,r}^I\phi_{b,r-\hat{\tau}}^R\right) \end{aligned} \tag{28}$$

$$\begin{aligned} S_{\tau,r}^I \;\rightarrow\; & \frac{1}{2}\sum_{a=1}^{2}\left(2\phi_{a,r}^R\phi_{a,r}^I-e^{d\tau\mu}\phi_{a,r}^R\phi_{a,r-d\hat{\tau}\tau}^I-e^{d\tau\mu}\phi_{a,r}^I\phi_{a,r-\hat{\tau}}^R\right) \\ & -\frac{1}{2}\sum_{a,b=1}^{2}\epsilon_{ab}\left(e^{d\tau\mu}\phi_{a,r}^R\phi_{b,r-\hat{\tau}}^R-e^{d\tau\mu}\phi_{a,r}^I\phi_{b,r-\hat{\tau}}^I\right), \end{aligned} \tag{29}$$

while the spatial derivative piece $S_{\nabla,r}$ (from the kinetic energy) becomes:

$$
\begin{aligned}
S^R_{\nabla,r} &\rightarrow \sum_{a=1}^{2}\left[\frac{d}{m}(\phi^R_{a,r})^2 - \frac{d}{m}(\phi^I_{a,r})^2 - \frac{1}{4m}\sum_{i=\pm x,y}\left(\phi^R_{a,r}\phi^R_{a,r+\hat{i}} - \phi^I_{a,r}\phi^I_{a,r+\hat{i}}\right)\right] \\
&\quad + \frac{1}{4m}\sum_{a,b=1}^{2}\sum_{i=\pm x,y}\epsilon_{ab}\left(\phi^R_{a,r}\phi^I_{b,r+\hat{i}} + \phi^I_{a,r}\phi^R_{b,r+\hat{i}}\right) \qquad (30)\\
S^I_{\nabla,r} &\rightarrow \sum_{a=1}^{2}\left[\frac{2d}{m}\phi^R_{a,r}\phi^I_{a,r} - \frac{1}{4m}\sum_{i=\pm x,y}\left(\phi^R_{a,r}\phi^I_{a,r+\hat{i}} + \phi^I_{a,r}\phi^R_{a,r+\hat{i}}\right)\right] \\
&\quad - \frac{1}{4m}\sum_{a,b=1}^{2}\sum_{i=\pm x,y}\epsilon_{ab}\left(\phi^R_{a,r}\phi^R_{b,r+\hat{i}} - \phi^I_{a,r}\phi^I_{b,r+\hat{i}}\right). \qquad (31)
\end{aligned}
$$

The trapping potential term, S_{trap}, becomes:

$$
S^R_{\text{trap},r} \rightarrow \frac{m}{4}\omega^2_{\text{trap}}\left(x^2+y^2\right)\sum_{a=1}^{2}\left[\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{a,r-\hat{\tau}} - \sum_{b=1}^{2}\epsilon_{ab}\left(\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}} + \phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}\right)\right] \qquad (32)
$$

$$
S^I_{\text{trap},r} \rightarrow \frac{m}{4}\omega^2_{\text{trap}}\left(x^2+y^2\right)\sum_{a=1}^{2}\left[\phi^R_{a,r}\phi^I_{a,r-\hat{\tau}} + \phi^I_{a,r}\phi^R_{a,r-\hat{\tau}} + \sum_{b=1}^{2}\epsilon_{ab}\left(\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{b,r-\hat{\tau}}\right)\right]. \qquad (33)
$$

The rotating term, $S_{\omega,r}$ becomes:

$$
\begin{aligned}
S^R_{\omega,r} &\rightarrow \frac{\omega_z}{2}\sum_{a=1}^{2}(y-x)\left[(\phi^I_{a,r}\phi^R_{a,r-\hat{\tau}} + \phi^R_{a,r}\phi^I_{a,r-\hat{\tau}}) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{b,r-\hat{\tau}})\right] \\
&\quad + \frac{\omega_z}{2}\sum_{a=1}^{2}\widetilde{x}\left[(\phi^I_{a,r}\phi^R_{a,r-\hat{y}-\hat{\tau}} + \phi^R_{a,r}\phi^I_{a,r-\hat{y}-\hat{\tau}}) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{a,r}\phi^R_{b,r-\hat{y}-\hat{\tau}} - \phi^I_{a,r}\phi^I_{b,r-\hat{y}-\hat{\tau}})\right] \\
&\quad - \frac{\omega_z}{2}\sum_{a=1}^{2}\widetilde{y}\left[(\phi^I_{a,r}\phi^R_{a,r-\hat{x}-\hat{\tau}} + \phi^R_{a,r}\phi^I_{a,r-\hat{x}-\hat{\tau}}) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{a,r}\phi^R_{b,r-\hat{x}-\hat{\tau}} - \phi^I_{a,r}\phi^I_{b,r-\hat{x}-\hat{\tau}})\right] \qquad (34)\\
S^I_{\omega,r} &\rightarrow \frac{\omega_z}{2}\sum_{a=1}^{2}(x-y)\left[(\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}} - \phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}) - \sum_{b=1}^{2}\epsilon_{ab}(\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}} + \phi^R_{a,r}\phi^I_{b,r-\hat{\tau}})\right] \\
&\quad - \frac{\omega_z}{2}\sum_{a=1}^{2}\widetilde{x}\left[(\phi^R_{a,r}\phi^R_{a,r-\hat{y}-\hat{\tau}} - \phi^I_{a,r}\phi^I_{a,r-\hat{y}-\hat{\tau}}) - \sum_{b=1}^{2}\epsilon_{ab}(\phi^I_{a,r}\phi^R_{b,r-\hat{y}-\hat{\tau}} + \phi^R_{a,r}\phi^I_{b,r-\hat{y}-\hat{\tau}})\right] \\
&\quad + \frac{\omega_z}{2}\sum_{a=1}^{2}\widetilde{y}\left[(\phi^R_{a,r}\phi^R_{a,r-\hat{x}-\hat{\tau}} - \phi^I_{a,r}\phi^I_{a,r-\hat{x}-\hat{\tau}}) - \sum_{b=1}^{2}\epsilon_{ab}(\phi^I_{a,r}\phi^R_{b,r-\hat{x}-\hat{\tau}} + \phi^R_{a,r}\phi^I_{b,r-\hat{x}-\hat{\tau}})\right] \qquad (35)
\end{aligned}
$$

and finally, the interaction term $S_{\text{int},r}$ becomes:

**still to do: copy over these finished calculations from notebook

$$
\begin{aligned}
\mathrm{Re}[S_{\text{int},r}] \;\rightarrow\; & \frac{\lambda}{4}\sum_{a,b=1}^{2}\left[\left(\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}}\right)^2 - \left(\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}}\right)^2\right] \\
& + \frac{\lambda}{4}\sum_{a,b=1}^{2}\left[\phi^R_{a,r}\phi^I_{b,r}\left((\phi^R_{a,r-\hat{\tau}})^2 - (\phi^I_{a,r-\hat{\tau}})^2\right) + \phi^I_{a,r}\phi^R_{b,r}\left((\phi^R_{a,r-\hat{\tau}})^2 - (\phi^I_{a,r-\hat{\tau}})^2\right)\right] \\
& + \frac{\lambda}{4}\sum_{a,b=1}^{2}\left[\phi^I_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}}\left((\phi^I_{a,r})^2 - (\phi^R_{a,r})^2\right) + \phi^R_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}}\left((\phi^I_{a,r})^2 - (\phi^R_{a,r})^2\right)\right] \\
& + \frac{\lambda}{4}\sum_{a,b=1}^{2} 2\left[\phi^I_{a,r}\phi^I_{b,r}\left(\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} - \phi^R_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}}\right) + \phi^R_{a,r}\phi^R_{b,r}\left(\phi^R_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}} - \phi^I_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}}\right)\right] \\
& - \frac{\lambda}{4}\sum_{a,b=1}^{2} 2\left[\phi^I_{a,r}\phi^R_{b,r}\left(\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} + \phi^I_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}}\right) + \phi^R_{a,r}\phi^I_{b,r}\left(\phi^R_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} + \phi^I_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}}\right)\right] \\
& + \frac{\lambda}{4}\sum_{a,b=1}^{2} 2\left[\phi^I_{a,r}\phi^R_{a,r}\left(\phi^I_{a,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}} - \phi^R_{a,r-\hat{\tau}}\phi^R_{b,r-\hat{\tau}}\right) + \phi^I_{a,r-\hat{\tau}}\phi^R_{a,r-\hat{\tau}}\left(\phi^R_{a,r}\phi^R_{b,r} - \phi^I_{a,r}\phi^I_{b,r}\right)\right] \\
& + \frac{\lambda}{4}\sum_{a,b=1}^{2} 4\left[\phi^I_{a,r}\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}}\phi^I_{b,r-\hat{\tau}}\right]
\end{aligned}
\tag{36}
$$

where in all of the above, $S_j = S^R_j + iS^I_j$, and $\epsilon_{12} = 1$, $\epsilon_{21} = -1$, and $\epsilon_{11} = \epsilon_{22} = 0$. Note that we were able to compress the real part of the interaction due to the sum over a and b.

0.4 Generating the NRRB CL Equations

0.4.1 Derivatives on the Lattice

When taking derivatives of this lattice action with respect to the fields, we do the following:

$$
\begin{aligned}
\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2}\phi_{a,q}\phi_{a,q+\hat{i}}\right) &= \sum_{a=1}^{2}\sum_{q=1}^{N_r}\left(\phi_{a,q}\frac{\delta}{\delta\phi_{c,r}}\phi_{a,q+\hat{i}} + \frac{\delta\phi_{a,q}}{\delta\phi_{c,r}}\phi_{a,q+\hat{i}}\right) \\
&= \sum_{a=1}^{2}\sum_{q=1}^{N_r}\left(\phi_{a,q}\delta_{c,a}\delta_{r,q+\hat{i}} + \delta_{c,a}\delta_{q,r}\phi_{a,q+\hat{i}}\right) \\
&= \phi_{c,r-\hat{i}} + \phi_{c,r+\hat{i}}.
\end{aligned}
\tag{37}
$$

Similarly,

$$
\frac{\delta}{\delta\phi_{c,r}}\left(\sum_{q=1}^{N_r}\sum_{a=1}^{2}\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,q}\phi_{b,q+\hat{i}}\right) = \sum_{b=1}^{2}\epsilon_{cb}\left(\phi_{b,r-\hat{i}} + \phi_{b,r+\hat{i}}\right). \tag{38}
$$

0.4.2 Computing the derivative of the action with respect to the real fields

The first step in computing the CL Equations is to find $\frac{\delta S_r}{\delta \phi_{a,r}}$. This is done below, with the sum over $a = 1, 2$ implied:

$$\frac{\delta S_r}{\delta \phi_{a,r}} = \frac{\delta S_{\tau,r}}{\delta \phi_{a,r}} + \frac{\delta S_{\nabla,r}}{\delta \phi_{a,r}} - \frac{\delta S_{\text{trap},r}}{\delta \phi_{a,r}} - \frac{\delta S_{\omega,r}}{\delta \phi_{a,r}} + \frac{\delta S_{\text{int},r}}{\delta \phi_{a,r}} \tag{39}$$

Again, we proceed by modifying each of the 5 parts of the action. First, the time and chemical potential term:

$$\begin{aligned}
2\frac{\delta}{\delta \phi_{a,r}} S_{\tau,r} &= \frac{\delta}{\delta \phi_{a,r}} \sum_{a=1}^{2} \left[\phi_{a,r}^2 - e^{d\tau\mu}\phi_{a,r}\phi_{a,r-\hat{\tau}} - ie^{d\tau\mu} \sum_{b=1}^{2} \epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}} \right] \\
2\frac{\delta}{\delta \phi_{a,r}} S_{\tau,r} &= 2\phi_{a,r} - e^{d\tau\mu}\left(\phi_{a,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}}\right) - ie^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}} + \phi_{b,r+\hat{\tau}}\right) \qquad (40)
\end{aligned}$$

Next, the spatial derivative part:

$$\begin{aligned}
-\frac{4m}{d\tau}\frac{\delta}{\delta \phi_{a,r}} S_{\nabla,r} &= \frac{\delta}{\delta \phi_{a,r}} \sum_{a=1}^{2} \left[\sum_{i=\pm x,y} \phi_{a,r}\phi_{a,r+\hat{i}} - 2\phi_{a,r}^2 + i\sum_{b=1}^{2}\sum_{i=\pm x,y} \epsilon_{ab}\phi_{a,r}\phi_{b,r+\hat{i}} \right] \\
&= \sum_{i=\pm x,y} \left(\phi_{a,r+\hat{i}} + \phi_{a,r-\hat{i}}\right) - 4\phi_{a,r} \\
&= 2\sum_{i=\pm x,y} \phi_{a,r+\hat{i}} - 4\phi_{a,r} \\
-\frac{2m}{d\tau}\frac{\delta}{\delta \phi_{a,r}} S_{\nabla,r} &= \sum_{i=\pm x,y} \phi_{a,r+\hat{i}} - 2\phi_{a,r} \qquad (41)
\end{aligned}$$

Then the part of the action due to the external trapping potential:

$$\begin{aligned}
\frac{4}{d\tau m\omega_{\text{tr}}^2(r_\perp^2)}\frac{\delta}{\delta \phi_{a,r}} S_{\text{trap},r} &= \frac{\delta}{\delta \phi_{a,r}} \sum_{a=1}^{2} \left(\phi_{a,r}\phi_{a,r-\hat{\tau}} + i\sum_{b=1}^{2} \epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}} \right) \\
&= \sum_{a=1}^{2} \left(\phi_{a,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}} + i\sum_{b=1}^{2} \epsilon_{ab}\left(\phi_{b,r+\hat{\tau}} + \phi_{b,r-\hat{\tau}}\right) \right) \\
\frac{4}{d\tau m\omega_{\text{tr}}^2(r_\perp^2)}\frac{\delta}{\delta \phi_{a,r}} S_{\text{trap},r} &= \sum_{a=1}^{2} \left(\left(\phi_{a,r+\hat{\tau}} + \phi_{a,r-\hat{\tau}}\right) + i\sum_{b=1}^{2} \epsilon_{ab}\left(\phi_{b,r+\hat{\tau}} + \phi_{b,r-\hat{\tau}}\right) \right) \qquad (42)
\end{aligned}$$

where $r_\perp^2 = x^2 + y^2$. Next, the rotational piece:

$$\begin{aligned}
\frac{2}{d\tau\omega_z}\frac{\delta}{\delta \phi_{a,r}} S_{\omega,r} &= \frac{\delta}{\delta \phi_{a,r}} \sum_{a=1}^{2} \left[\sum_{b=1}^{2} \epsilon_{ab}\left(x\phi_{a,r}\phi_{b,r-\hat{y}} - y\phi_{a,r}\phi_{b,r-\hat{x}}\right) + i\left((x-y)\phi_{a,r}^2 - x\phi_{a,r}\phi_{a,r-\hat{y}} + y\phi_{a,r}\phi_{a,r-\hat{x}}\right) \right] \\
&= \epsilon_{ab}\left[x\left(\phi_{b,r-\hat{y}} + \phi_{b,r+\hat{y}}\right) - y\left(\phi_{b,r-\hat{x}} + \phi_{b,r+\hat{x}}\right)\right] \\
&\quad + i\left[2(x-y)\phi_{a,r} - x\left(\phi_{a,r-\hat{y}} + \phi_{a,r+\hat{y}}\right) + y\left(\phi_{a,r-\hat{x}} + \phi_{a,r+\hat{x}}\right)\right] \\
\frac{2}{d\tau\omega_z}\frac{\delta}{\delta \phi_{a,r}} S_{\omega,r} &= \epsilon_{ab}\left[x\left(\phi_{b,r-\hat{y}} + \phi_{b,r+\hat{y}}\right) - y\left(\phi_{b,r-\hat{x}} + \phi_{b,r+\hat{x}}\right)\right] \\
&\quad + i\left[2(x-y)\phi_{a,r} - x\left(\phi_{a,r-\hat{y}} + \phi_{a,r+\hat{y}}\right) + y\left(\phi_{a,r-\hat{x}} + \phi_{a,r+\hat{x}}\right)\right] \qquad (43)
\end{aligned}$$

And finally, the interaction term in the action:

$$\begin{aligned}
\frac{4}{d\tau\lambda}\frac{\delta}{\delta\phi_{a,r}}S_{\text{int},r} &= \frac{\delta}{\delta\phi_{a,r}}\sum_{a=1}^{2}\sum_{b=1}^{2}\phi_{a,r}^2\phi_{b,r}^2 \\
&= 4\phi_{a,r}\sum_{b=1}^{2}\phi_{b,r}^2 \\
\frac{1}{d\tau\lambda}\frac{\delta}{\delta\phi_{a,r}}S_{\text{int},r} &= \phi_{a,r}\sum_{b=1}^{2}\phi_{b,r}^2. \qquad (44)
\end{aligned}$$

Here, there is an extra factor of 2 that appears when solving this equation without the summation notation - I'm not really sure why it doesn't appear when using summation, but it is the correct factor.

0.4.3 Second Complexification of Drift Function

The next step is to complexify our real fields, a and b, such that $\phi_a = \phi_a^R + i\phi_a^I$. We do this for each part of the drift function, $K_{a,r} = \frac{\delta S_r}{\delta\phi_{a,r}}$.

First, the time and chemical potential term:

$$\begin{aligned}
2\frac{\delta}{\delta\phi_{a,r}}S_{\tau,r} &= 2\phi_{a,r} - e^{d\tau\mu}\left(\phi_{a,r-\hat{\tau}} + \phi_{a,r+\hat{\tau}}\right) - ie^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}} + \phi_{b,r+\hat{\tau}}\right) \\
&= 2(\phi_{a,r}^R + i\phi_{a,r}^I) - e^{d\tau\mu}\left(\phi_{a,r-\hat{\tau}}^R + i\phi_{a,r-\hat{\tau}}^I + \phi_{a,r+\hat{\tau}}^R + i\phi_{a,r+\hat{\tau}}^I\right) \\
&\quad -ie^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}}^R + i\phi_{b,r-\hat{\tau}}^I + \phi_{b,r+\hat{\tau}}^R + i\phi_{b,r+\hat{\tau}}^I\right) \\
&= 2\phi_{a,r}^R - e^{d\tau\mu}\left(\phi_{a,r-\hat{\tau}}^R + \phi_{a,r+\hat{\tau}}^R\right) + e^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}}^I - \phi_{b,r+\hat{\tau}}^I\right) \\
&\quad +i\left[2\phi_{a,r}^I - e^{d\tau\mu}\left(\phi_{a,r-\hat{\tau}}^I + \phi_{a,r+\hat{\tau}}^I\right) - e^{d\tau\mu}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}}^R - \phi_{b,r+\hat{\tau}}^R\right)\right]
\end{aligned}$$

So

$$\begin{aligned}
\text{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\tau,r}\right] &= \phi_{a,r}^R - \frac{e^{d\tau\mu}}{2}\left(\phi_{a,r-\hat{\tau}}^R + \phi_{a,r+\hat{\tau}}^R\right) + \frac{e^{d\tau\mu}}{2}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}}^I - \phi_{b,r+\hat{\tau}}^I\right) \qquad (45) \\
\text{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\tau,r}\right] &= \phi_{a,r}^I - \frac{e^{d\tau\mu}}{2}\left(\phi_{a,r-\hat{\tau}}^I + \phi_{a,r+\hat{\tau}}^I\right) - \frac{e^{d\tau\mu}}{2}\epsilon_{ab}\left(\phi_{b,r-\hat{\tau}}^R - \phi_{b,r+\hat{\tau}}^R\right) \qquad (46)
\end{aligned}$$

Next, the spatial derivative part:

$$\begin{aligned}
-\frac{2m}{d\tau}\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r} &= \sum_{i=\pm1}^{d}\phi_{a,r+\hat{i}} - 2d\phi_{a,r} \\
&= \sum_{i=\pm1}^{d}\left(\phi_{a,r+\hat{i}}^R + i\phi_{a,r+\hat{i}}^I\right) - 2d\phi_{a,r}^R - 2id\phi_{a,r}^I \\
&= \sum_{i=\pm1}^{d}\phi_{a,r+\hat{i}}^R - 2d\phi_{a,r}^R + i\left(\sum_{i=\pm1}^{d}\phi_{a,r+\hat{i}}^I - 2d\phi_{a,r}^I\right)
\end{aligned}$$

So

$$\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r}\right] = \frac{d\tau}{2m}\left(2d\phi^R_{a,r} - \sum_{i=\pm 1}^{d}\phi^R_{a,r+\hat{i}}\right) \tag{47}$$

$$\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\nabla,r}\right] = \frac{d\tau}{2m}\left(2d\phi^I_{a,r} - \sum_{i=\pm 1}^{d}\phi^I_{a,r+\hat{i}}\right) \tag{48}$$

Then the part of the action due to the external trapping potential:

$$\begin{aligned}
\frac{4}{d\tau m\omega_{\mathrm{tr}}^2(r_\perp^2)}\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r} &= \sum_{a=1}^{2}\left((\phi_{a,r+\hat{\tau}}+\phi_{a,r-\hat{\tau}}) + i\sum_{b=1}^{2}\epsilon_{ab}\left(\phi_{b,r+\hat{\tau}}+\phi_{b,r-\hat{\tau}}\right)\right)\\
&= \sum_{a=1}^{2}\left((\phi^R_{a,r+\hat{\tau}}+\phi^R_{a,r-\hat{\tau}}) - \sum_{b=1}^{2}\epsilon_{ab}\left(\phi^I_{b,r+\hat{\tau}}+\phi^I_{b,r-\hat{\tau}}\right)\right)\\
&\quad +i\sum_{a=1}^{2}\left((\phi^I_{a,r+\hat{\tau}}+\phi^I_{a,r-\hat{\tau}}) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{b,r+\hat{\tau}}+\phi^R_{b,r-\hat{\tau}})\right)
\end{aligned} \tag{49}$$

So

$$\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r}\right] = \frac{d\tau\omega_{\mathrm{tr}}^2(r_\perp^2)}{4}\sum_{a=1}^{2}\left((\phi^R_{a,r+\hat{\tau}}+\phi^R_{a,r-\hat{\tau}}) - \sum_{b=1}^{2}\epsilon_{ab}\left(\phi^I_{b,r+\hat{\tau}}+\phi^I_{b,r-\hat{\tau}}\right)\right) \tag{50}$$

$$\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\mathrm{trap},r}\right] = \frac{d\tau\omega_{\mathrm{tr}}^2(r_\perp^2)}{4}\sum_{a=1}^{2}\left((\phi^I_{a,r+\hat{\tau}}+\phi^I_{a,r-\hat{\tau}}) + \sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{b,r+\hat{\tau}}+\phi^R_{b,r-\hat{\tau}})\right) \tag{51}$$

where $r_\perp^2 = x^2+y^2$. Next, the rotational piece:

$$\begin{aligned}
\frac{2}{d\tau\omega_z}\frac{\delta}{\delta\phi_{a,r}}S_{\omega,r} &= \epsilon_{ab}\left[x\left(\phi_{b,r-\hat{y}}+\phi_{b,r+\hat{y}}\right) - y\left(\phi_{b,r-\hat{x}}+\phi_{b,r+\hat{x}}\right)\right]\\
&\quad +i\left[2(x-y)\phi_{a,r} - x\left(\phi_{a,r-\hat{y}}+\phi_{a,r+\hat{y}}\right) + y\left(\phi_{a,r-\hat{x}}+\phi_{a,r+\hat{x}}\right)\right]\\
&= \epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}}+i\phi^I_{b,r-\hat{y}}+\phi^R_{b,r+\hat{y}}+i\phi^I_{b,r+\hat{y}}\right) - y\left(\phi^R_{b,r-\hat{x}}+i\phi^I_{b,r-\hat{x}}+\phi^R_{b,r+\hat{x}}+i\phi^I_{b,r+\hat{x}}\right)\right]\\
&\quad +i\left[2(x-y)\phi^R_{a,r}+2i(x-y)\phi^I_{a,r} - x\left(\phi^R_{a,r-\hat{y}}+i\phi^I_{a,r-\hat{y}}+\phi^R_{a,r+\hat{y}}+i\phi^I_{a,r+\hat{y}}\right)\right]\\
&\quad +i\left[y\left(\phi^R_{a,r-\hat{x}}+i\phi^I_{a,r-\hat{x}}+\phi^R_{a,r+\hat{x}}+i\phi^I_{a,r+\hat{x}}\right)\right]\\
&= -2(x-y)\phi^I_{a,r} + x\left(\phi^I_{a,r-\hat{y}}+\phi^I_{a,r+\hat{y}}\right) - y\left(\phi^I_{a,r-\hat{x}}+\phi^I_{a,r+\hat{x}}\right)\\
&\quad +\epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}}+\phi^R_{b,r+\hat{y}}\right) - y\left(\phi^R_{b,r-\hat{x}}+\phi^R_{b,r+\hat{x}}\right)\right]\\
&\quad +i\left[2(x-y)\phi^R_{a,r} - x\left(\phi^R_{a,r-\hat{y}}+\phi^R_{a,r+\hat{y}}\right) + y\left(\phi^R_{a,r-\hat{x}}+\phi^R_{a,r+\hat{x}}\right)\right]\\
&\quad +i\epsilon_{ab}\left[x\left(\phi^I_{b,r-\hat{y}}+\phi^I_{b,r+\hat{y}}\right) - y\left(\phi^I_{b,r-\hat{x}}+\phi^I_{b,r+\hat{x}}\right)\right]
\end{aligned} \tag{52}$$

So

$$\begin{aligned}
\mathrm{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\omega_z,r}\right] &= \frac{d\tau\omega_z}{2}\left[x\left(\phi^I_{a,r-\hat{y}}+\phi^I_{a,r+\hat{y}}\right) - y\left(\phi^I_{a,r-\hat{x}}+\phi^I_{a,r+\hat{x}}\right) - 2(x-y)\phi^I_{a,r}\right]\\
&\quad +\frac{d\tau\omega_z}{2}\epsilon_{ab}\left[x\left(\phi^R_{b,r-\hat{y}}+\phi^R_{b,r+\hat{y}}\right) - y\left(\phi^R_{b,r-\hat{x}}+\phi^R_{b,r+\hat{x}}\right)\right]\\
\mathrm{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\omega_z,r}\right] &= \frac{d\tau\omega_z}{2}\left[2(x-y)\phi^R_{a,r} - x\left(\phi^R_{a,r-\hat{y}}+\phi^R_{a,r+\hat{y}}\right) + y\left(\phi^R_{a,r-\hat{x}}+\phi^R_{a,r+\hat{x}}\right)\right]\\
&\quad +\frac{\omega_z}{2}\epsilon_{ab}\left[x\left(\phi^I_{b,r-\hat{y}}+\phi^I_{b,r+\hat{y}}\right) - y\left(\phi^I_{b,r-\hat{x}}+\phi^I_{b,r+\hat{x}}\right)\right]
\end{aligned} \tag{53}$$

And finally, the interaction term in the action:

$$\begin{aligned}
\frac{1}{d\tau\lambda}\frac{\delta}{\delta\phi_{a,r}}S_{\text{int},r} &= \phi_{a,r}\sum_{b=1}^{2}\phi_{b,r}^2 \\
&= (\phi^R_{a,r}+i\phi^I_{a,r})\sum_{b=1}^{2}(\phi^R_{b,r}+i\phi^I_{b,r})^2 \\
&= \phi^R_{a,r}\sum_{b=1}^{2}\left((\phi^R_{b,r})^2+2i\phi^R_{b,r}\phi^I_{b,r}-(\phi^I_{b,r})^2\right)+i\phi^I_{a,r}\sum_{b=1}^{2}\left((\phi^R_{b,r})^2+2i\phi^R_{b,r}\phi^I_{b,r}-(\phi^I_{b,r})^2\right) \\
&= \sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2+2i\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}-\phi^R_{a,r}(\phi^I_{b,r})^2+i\phi^I_{a,r}(\phi^R_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}-i\phi^I_{a,r}(\phi^I_{b,r})^2\right] \\
&= \sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2-\phi^R_{a,r}(\phi^I_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}\right]+i\sum_{b=1}^{2}\left[2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}+\phi^I_{a,r}(\phi^R_{b,r})^2-\phi^I_{a,r}(\phi^I_{b,r})^2\right]
\end{aligned}$$

So

$$\text{Re}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\text{int},r}\right] = d\tau\lambda\sum_{b=1}^{2}\left[\phi^R_{a,r}(\phi^R_{b,r})^2-\phi^R_{a,r}(\phi^I_{b,r})^2-2\phi^I_{a,r}\phi^R_{b,r}\phi^I_{b,r}\right] \tag{54}$$

$$\text{Im}\left[\frac{\delta}{\delta\phi_{a,r}}S_{\text{int},r}\right] = d\tau\lambda\sum_{b=1}^{2}\left[\phi^I_{a,r}(\phi^R_{b,r})^2+2\phi^R_{a,r}\phi^R_{b,r}\phi^I_{b,r}-\phi^I_{a,r}(\phi^I_{b,r})^2\right] \tag{55}$$

0.5 Lattice Observables

The observables:

0.5.1 Average Density

The average density is the sum over the local density at each spatial site, normalized by the spatial lattice volume:

$$\langle\hat{n}\rangle = \frac{1}{N_x^d}\sum_r n_r \tag{56}$$

where n_r can be found by taking the derivative of the lattice action with respect to the chemical potential, and then complexifying the fields as we have done before:

$$\begin{aligned}
n_r &= -\frac{\partial}{\partial\beta\mu}S_{\tau,r} = -\frac{\partial}{\partial\beta\mu}\left(\phi^*_r\phi_r-\phi^*_r e^{\beta\mu/N_\tau}\phi_{r-\hat{\tau}}\right) = \frac{\partial}{\partial\beta\mu}\left(\phi^*_r e^{\beta\mu/N_\tau}\phi_{r-\hat{\tau}}\right) \\
&= \frac{1}{N_\tau}e^{\beta\mu/N_\tau}\left(\phi^*_r\phi_{r-\hat{\tau}}\right) = \frac{1}{2N_\tau}e^{\beta\mu/N_\tau}\left(\phi_{1,r}-i\phi_{2,r}\right)\left(\phi_{1,r-\hat{\tau}}+i\phi_{2,r-\hat{\tau}}\right) \\
&= \frac{1}{N_\tau}e^{\beta\mu/N_\tau}\sum_{a=1}^{2}\left[\phi_{a,r}\phi_{a,r-\hat{\tau}}+i\sum_{b=1}^{2}\epsilon_{ab}\phi_{a,r}\phi_{b,r-\hat{\tau}}\right] \\
&= \frac{1}{2N_\tau}e^{\beta\mu/N_\tau}\sum_{a=1}^{2}\left[\phi^R_{a,r}\phi^R_{a,r-\hat{\tau}}-\phi^I_{a,r}\phi^I_{a,r-\hat{\tau}}-\sum_{b=1}^{2}\epsilon_{ab}(\phi^I_{a,r}\phi^R_{b,r-\hat{\tau}}+\phi^R_{a,r}\phi^I_{b,r-\hat{\tau}})\right] \\
&+ \frac{i}{2N_\tau}e^{\beta\mu/N_\tau}\sum_{a=1}^{2}\left[\phi^I_{a,r-\hat{\tau}}\phi^R_{a,r-\hat{\tau}}+\phi^R_{a,r-\hat{\tau}}\phi^I_{a,r-\hat{\tau}}+\sum_{b=1}^{2}\epsilon_{ab}(\phi^R_{a,r}\phi^R_{b,r-\hat{\tau}}-\phi^I_{a,r}\phi^I_{b,r-\hat{\tau}})\right]
\end{aligned} \tag{57}$$

0.5.2 Average Angular Momentum

The angular momentum operator, L_z can be written

$$L_z = ((x - x_0)p_y - (y - y_0)p_x) = -i\hbar((x - x_0)\partial_y - (y - y_0)\partial_x). \tag{58}$$

We are interested in the expectation value of this operator: $\langle L_z\rangle$, which is found by summing over the lattice:

$$\langle L_z\rangle = -i\sum_r \phi_r^*((x - \frac{N_x - 1}{2})\partial_y - (y - \frac{N_x - 1}{2})\partial_x)\phi_r, \tag{59}$$

where $\hbar \to 1$. First, let us implement our lattice derivative: $\partial_j\phi_r = \frac{1}{a^2}(\phi_{r-\hat{j}} - \phi_r)$ (recall that our lattice spacing is $a = 1$):

$$\begin{aligned}
\langle L_z\rangle &= -i\sum_r \phi_r^*((x - r_c)\phi_{r-\hat{y}} - x\phi_r - (y - r_c)\phi_{r-\hat{x}} + y\phi_r) \\
&= i\sum_r ((y - r_c)\phi_r^*\phi_{r-\hat{x}} - (x - r_c)\phi_r^*\phi_{r-\hat{y}} - (y - x)\phi_r^*\phi_r)
\end{aligned} \tag{60}$$

Here we have also written $\frac{N_x-1}{2}$ as r_c for simplicity. Let us write ϕ as $\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$; then, our angular momentum operator becomes a sum over the lattice sites and the real fields ϕ_1 and ϕ_2:

$$\begin{aligned}
\langle L_z\rangle &= \frac{i}{2}\sum_r (y - r_c)\left(\phi_{1,r}\phi_{1,r-\hat{x}} + i\phi_{1,r}\phi_{2,r-\hat{x}} - i\phi_{2,r}\phi_{1,r-\hat{x}} + \phi_{2,r}\phi_{2,r-\hat{x}}\right) \\
&\quad -\frac{i}{2}\sum_r (x - r_c)\left(\phi_{1,r}\phi_{1,r-\hat{y}} + i\phi_{1,r}\phi_{2,r-\hat{y}} - i\phi_{2,r}\phi_{1,r-\hat{y}} + \phi_{2,r}\phi_{2,r-\hat{y}}\right) \qquad (61) \\
&\quad -\frac{i}{2}\sum_r (y - x)(\phi_{1,r}^2 + \phi_{2,r}^2) \\
&= \frac{1}{2}\sum_r\sum_{a=1}^{2}\left((y - r_c)\phi_{a,r}\phi_{a,r-\hat{x}} - (x - r_c)\phi_{a,r}\phi_{a,r-\hat{y}} - (y - x)\phi_{a,r}^2\right) \qquad (62) \\
&\quad -\frac{i}{2}\sum_r\sum_{a,b=1}^{2}\epsilon_{ab}\left((x - r_c)\phi_{a,r}\phi_{b,r-\hat{y}} - (y - r_c)\phi_{a,r}\phi_{b,r-\hat{x}}\right)
\end{aligned}$$

Next, we must complexify our real fields: $\phi_a = \phi_a^R + i\phi_a^I$, leading us to the following equation for the angular momentum in terms of our four lattice fields:

$$\begin{aligned}
\langle L_z \rangle &= -\frac{i}{2}\sum_r \sum_{a=1}^{2} \left((x-r_c)(\phi^R_{a,r}\phi^R_{a,r-\hat{y}} - \phi^I_{a,r}\phi^I_{a,r-\hat{y}}) - (y-r_c)(\phi^R_{a,r}\phi^R_{a,r-\hat{x}} - \phi^I_{a,r}\phi^I_{a,r-\hat{x}})\right) \\
&\quad -\frac{i}{2}\sum_r \sum_{a=1}^{2} (y-x)\left((\phi^R_{a,r})^2 - (\phi^I_{a,r})^2\right) \\
&\quad -\frac{i}{2}\sum_r \sum_{a,b=1}^{2} \epsilon_{ab}\left((x-r_c)(\phi^R_{a,r}\phi^I_{b,r-\hat{y}} + \phi^I_{a,r}\phi^R_{b,r-\hat{y}}) - (y-r_c)(\phi^R_{a,r}\phi^I_{b,r-\hat{x}} + \phi^I_{a,r}\phi^R_{b,r-\hat{x}})\right) \\
&\quad +\frac{1}{2}\sum_r \sum_{a=1}^{2} \left((x-r_c)(\phi^R_{a,r}\phi^I_{a,r-\hat{y}} + \phi^I_{a,r}\phi^R_{a,r-\hat{y}}) - (y-r_c)(\phi^R_{a,r}\phi^I_{a,r-\hat{x}} + \phi^I_{a,r}\phi^R_{a,r-\hat{x}})\right) \\
&\quad +\sum_r \sum_{a,b=1}^{2} (y-x)\phi^R_{a,r}\phi^I_{a,r} \\
&\quad +\frac{1}{2}\sum_r \sum_{a,b=1}^{2} \epsilon_{ab}\left((x-r_c)(\phi^R_{a,r}\phi^R_{b,r-\hat{y}} - \phi^I_{a,r}\phi^I_{b,r-\hat{y}}) - (y-r_c)(\phi^R_{a,r}\phi^R_{b,r-\hat{x}} - \phi^I_{a,r}\phi^I_{b,r-\hat{x}})\right).
\end{aligned}$$

When this is divided into the real and imaginary parts of the observable, we get:

$$\begin{aligned}
\mathrm{Re}\langle L_z \rangle &= \frac{1}{2}\sum_r \sum_{a=1}^{2} \left((x-r_c)(\phi^R_{a,r}\phi^I_{a,r-\hat{y}} + \phi^I_{a,r}\phi^R_{a,r-\hat{y}}) - (y-r_c)(\phi^R_{a,r}\phi^I_{a,r-\hat{x}} + \phi^I_{a,r}\phi^R_{a,r-\hat{x}})\right) \\
&\quad +\sum_r \sum_{a=1}^{2} (y-x)\phi^R_{a,r}\phi^I_{a,r} \qquad (63) \\
&\quad +\frac{1}{2}\sum_r \sum_{a,b=1}^{2} \epsilon_{ab}\left((x-r_c)(\phi^R_{a,r}\phi^R_{b,r-\hat{y}} - \phi^I_{a,r}\phi^I_{b,r-\hat{y}}) - (y-r_c)(\phi^R_{a,r}\phi^R_{b,r-\hat{x}} - \phi^I_{a,r}\phi^I_{b,r-\hat{x}})\right). \\
\mathrm{Im}\langle L_z \rangle &= -\frac{1}{2}\sum_r \sum_{a=1}^{2} \left((x-r_c)(\phi^R_{a,r}\phi^R_{a,r-\hat{y}} - \phi^I_{a,r}\phi^I_{a,r-\hat{y}}) - (y-r_c)(\phi^R_{a,r}\phi^R_{a,r-\hat{x}} - \phi^I_{a,r}\phi^I_{a,r-\hat{x}})\right) \\
&\quad -\frac{1}{2}\sum_r \sum_{a=1}^{2} (y-x)\left((\phi^R_{a,r})^2 - (\phi^I_{a,r})^2\right) \qquad (64) \\
&\quad -\frac{1}{2}\sum_r \sum_{a,b=1}^{2} \epsilon_{ab}\left((x-r_c)(\phi^R_{a,r}\phi^I_{b,r-\hat{y}} + \phi^I_{a,r}\phi^R_{b,r-\hat{y}}) - (y-r_c)(\phi^R_{a,r}\phi^I_{b,r-\hat{x}} + \phi^I_{a,r}\phi^R_{b,r-\hat{x}})\right)
\end{aligned}$$

0.5.3 Circulation

The circulation is defined as

$$\Gamma[l] = \frac{1}{2\pi}\oint_{l\times l} dx\,(\theta_{t,x+j} - \theta_{t,x}) \qquad (65)$$

$$\theta_{t,x} = \tan^{-1}\left(\frac{\mathrm{Im}[\phi_{t,x}]}{\mathrm{Re}[\phi_{t,x}]}\right). \qquad (66)$$

Given that our field, ϕ, is broken into 4 components ($\phi_{1/2}^{R/I}$, we need to rewrite this quantity in terms of those components:

$$\begin{aligned}
\phi &= \frac{1}{\sqrt{2}}\left(\phi_1^R + i\phi_1^I + i\phi_2^R - \phi_2^I\right) && (67)\\
\mathrm{Re}[\phi_{t,x}] &= \frac{1}{\sqrt{2}}\left(\phi_{1,t,x}^R - \phi_{2,t,x}^I\right) && (68)\\
\mathrm{Im}[\phi_{t,x}] &= \frac{1}{\sqrt{2}}\left(\phi_{1,t,x}^I + \phi_{2,t,x}^R\right) && (69)\\
\theta_{t,x} &= \tan^{-1}\left(\frac{\phi_{1,t,x}^I + \phi_{2,t,x}^R}{\phi_{1,t,x}^R - \phi_{2,t,x}^I}\right). && (70)
\end{aligned}$$

Therefore, our circulation is a sum computed around a loop on the lattice of length l in each direction:

$$\Gamma[l] = \frac{1}{2\pi}\sum_{l\times l}\left(\theta_{t,x+j} - \theta_{t,x}\right) \tag{71}$$

where $\theta_{t,x+j}$ is computed at the next site on the loop from $\theta_{t,x}$ for each point along the loop.

0.5.4 Average Energy

The expectation value of the density is defined as

$$\langle E\rangle = \frac{-\partial \ln \mathcal{Z}}{\partial\beta} = \frac{-\partial \ln(e^{-S})}{\partial\beta} = \frac{\partial S}{\partial\beta}. \tag{72}$$

Since we have written the action in terms of β as

$$S \to \frac{1}{N_\tau}\sum_{x,\tau}\Delta x^d\beta\left[\left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu - i\omega_z(x\partial_y - y\partial_x) - \frac{m\omega_{\text{trap}}^2}{2}(x^2+y^2)\right)\phi + \lambda(\phi^*\phi)^2\right]. \tag{73}$$

We can see that the action is linear in β. Therefore, we can simply calculate the energy in the following way:

$$\langle E\rangle = \frac{S}{\beta} \tag{74}$$

0.6 Some checks on our code

0.6.1 Nonrotating, noninteracting, nonrelativistic, finite chemical potential in 1, 2, and 3 dimensions

The lattice action for a nonrotating, noninteracting, and nonrelativistic system is the following:

$$S_{\text{lat},r} = \phi_r^*\left[\phi_r - e^{d\tau\mu}\phi_{r-\hat{\tau}} - \frac{d\tau}{2m}\sum_{i=1}^{d}\left(\phi_{r+\hat{i}} - 2\phi_r + \phi_{r-\hat{i}}\right)\right]. \tag{75}$$

This can be written as fields multiplying a matrix:

$$S_{\text{lat},r} = \sum_r\sum_{r'}\phi_r^* M\phi_{r'} = \sum_r\sum_{r'}\phi_r^*\left[(1+\frac{d\tau d}{m})\delta_{r,r'} - e^{d\tau\mu}\delta_{r-\hat{\tau},r'} - \frac{d\tau}{2m}\sum_{i=1}^{d}\left(\delta_{r+\hat{i},r'} + \delta_{r-\hat{i},r'}\right)\right]\phi_{r'}, \tag{76}$$

which we can use to determine analytically the density and field modulus squared of this system in order to check against our code's results. Recall that

$$\begin{aligned}
\langle \hat{n} \rangle &= \frac{-1}{V}\frac{\partial ln\mathcal{Z}}{\partial(\beta\mu)} = \frac{-1}{V}\frac{\partial}{\partial(\beta\mu)}(-ln(det(M))) \\
&= \frac{1}{V}\frac{\partial}{\partial(\beta\mu)}Tr(lnM) = \frac{1}{V}\frac{\partial}{\partial(\beta\mu)}\sum_k lnD_{kk} = \frac{1}{V}\sum_k \frac{1}{D_{kk}}\frac{\partial D_{kk}}{\partial(\beta\mu)}
\end{aligned} \tag{77}$$

with $\beta\mu = N_\tau d\tau\mu$, and note that for a nonrelativistic system,

$$\begin{aligned}
\langle \phi^*\phi \rangle &= \frac{-1}{V}\frac{\partial ln\mathcal{Z}}{\partial(d/m)} = \frac{-1}{V}\frac{\partial}{\partial(d/m)}(-ln(det(M))) \\
&= \frac{1}{V}\frac{\partial}{\partial(d/m)}Tr(lnM) = \frac{1}{V}\frac{\partial}{\partial(d/m)}\sum_k lnD_{kk} \\
&= \frac{1}{V}\sum_k \frac{1}{D_{kk}}\frac{\partial D_{kk}}{\partial(d/m)} = \sum_k \frac{1}{D_{kk}}.
\end{aligned} \tag{78}$$

Diagonalizing our matrix, M

We can represent the nonrotating, noninteracting action as

$$S[\lambda = \omega = 0] = \sum_{r,r'} \phi_r^* M_{r,r'}[d,m,\mu]\phi_{r'} \tag{79}$$

where

$$M_{r,r'}[d,m,\mu] = \left[(1+\frac{d\tau d}{m})\delta_{r,r'} - e^{d\tau\mu}\delta_{r-\hat{t},r'} - \frac{d\tau}{2m}\sum_{i=1}^{d}(\delta_{r+\hat{\imath},r'} + \delta_{r-\hat{\imath},r'})\right]. \tag{80}$$

We want to diagonalize M by applying a transformation matrix, such that $D_{kk'} = U^\dagger MU$, where

$$U_{r,k} = \frac{\sqrt{2^d}}{\sqrt{N_x^d N_\tau}}e^{ik_0 t}\prod_{i=1}^{d}\sin(k_i x_i) \tag{81}$$

$$U^\dagger_{r,k} = \frac{\sqrt{2^d}}{\sqrt{N_x^d N_\tau}}e^{-ik_0 t}\prod_{i=1}^{d}\sin(k_i x_i) \tag{82}$$

$$k_0 = \frac{2\pi n_0}{N_\tau},\ \ n_0 \in [1,2,...,N_\tau] \tag{83}$$

$$k_i = \frac{\pi n_i}{(N_x+1)},\ \ n_i \in [1,2,...,N_x]. \tag{84}$$

Applying the transformation matrix, we get

$$
\begin{aligned}
D_{k,k'} &= \frac{2^d}{N_x^d N_t} \sum_{r,r'} e^{-ik_0 t} \prod_{i=1}^{d} \sin(k_i x_i) \left[(1+\frac{d\tau d}{m})\delta_{r,r'}\right] e^{ik_0' t'} \prod_{i=1}^{d} \sin(k_i' x_i') \\
&- \frac{2^d}{N_x^d N_t} \sum_{r,r'} e^{-ik_0 t} \prod_{i=1}^{d} \sin(k_i x_i) \left[e^{d\tau\mu}\delta_{r-\hat{t},r'}\right] e^{ik_0' t'} \prod_{i=1}^{d} \sin(k_i' x_i') \\
&- \frac{2^d}{N_x^d N_t} \sum_{r,r'} e^{-ik_0 t} \prod_{i=1}^{d} \sin(k_i x_i) \left[\frac{d\tau}{2m} \sum_{i=1}^{d} \delta_{r+\hat{\imath},r'}\right] e^{ik_0' t'} \prod_{i=1}^{d} \sin(k_i' x_i') \\
&- \frac{2^d}{N_x^d N_t} \sum_{r,r'} e^{-ik_0 t} \prod_{i=1}^{d} \sin(k_i x_i) \left[\frac{d\tau}{2m} \sum_{i=1}^{d} \delta_{r-\hat{\imath},r'}\right] e^{ik_0' t'} \prod_{i=1}^{d} \sin(k_i' x_i').
\end{aligned} \tag{85}
$$

Resolving the delta functions, performing the sum over r', and pulling everything that does not depend on $r = (t, \vec{x})$ outside the sum, this reduces to

$$
\begin{aligned}
D_{k,k'} &= \frac{2^d}{N_x^d N_t}(1+\frac{d\tau d}{m}) \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i)\sin(k_i' x_i) \\
&- \frac{2^d}{N_x^d N_t} e^{d\tau\mu} e^{-ik_0'} \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i)\sin(k_i' x_i) \\
&- \frac{2^d}{N_x^d N_t}\frac{d\tau}{2m} \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i) \sum_{i=1}^{d} \sin(k_i' x_i + k_i') \\
&- \frac{2^d}{N_x^d N_t}\frac{d\tau}{2m} \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i) \sum_{i=1}^{d} \sin(k_i' x_i - k_i').
\end{aligned} \tag{86}
$$

To further expand the last two lines, we use the following trig identity: $\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$, which gives us:

$$
\begin{aligned}
D_{k,k'} &= \frac{2^d}{N_x^d N_t}(1+\frac{d\tau d}{m} - e^{d\tau\mu} e^{-ik_0'}) \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i)\sin(k_i' x_i) \qquad (87) \\
&- \frac{2^d}{N_x^d N_t}\frac{d\tau}{2m} \sum_{i=1}^{d} \cos(k_i') \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i)\sin(k_i' x_i) \\
&- \frac{2^d}{N_x^d N_t}\frac{d\tau}{2m} \sum_{i=1}^{d} \sin(k_i') \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i)\cos(k_i' x_i) \\
&- \frac{2^d}{N_x^d N_t}\frac{d\tau}{2m} \sum_{i=1}^{d} \cos(k_i') \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i)\sin(k_i' x_i) \\
&+ \frac{2^d}{N_x^d N_t}\frac{d\tau}{2m} \sum_{i=1}^{d} \sin(k_i') \sum_{r} e^{-it(k_0-k_0')} \prod_{i=1}^{d} \sin(k_i x_i)\cos(k_i' x_i).
\end{aligned}
$$

Using the following Fourier identities

$$\begin{aligned}\sum_x \sin(kx)\sin(k'x) &= \frac{N_x}{2}\delta_{k,k'} \\ \sum_x \sin(kx)\cos(k'x) &= 0 \qquad (88)\\ \sum_x e^{-ix(k-k')} &= N_x\delta_{k,k'}\end{aligned} \tag{89}$$

we find that

$$\begin{aligned}D_{k,k'} &= \frac{2^d}{N_x^d N_t}\left(1+\frac{d\tau d}{m} - e^{d\tau\mu}e^{-ik'_0} - \frac{d\tau}{m}\sum_{i=1}^{d}\cos(k'_i)\right) N_t\delta_{k_0,k'_0}\prod_{i=1}^{d}\frac{N_{x_i}}{2}\delta_{k_i,k'_i} \\ &= \frac{2^d}{N_x^d N_t}\left(1+\frac{d\tau d}{m} - e^{d\tau\mu}e^{-ik'_0} - \frac{d\tau}{m}\sum_{i=1}^{d}\cos(k'_i)\right) N_t\left(\frac{N_x}{2}\right)^d\delta_{k,k'} \\ D_{k,k'} &= \left(1+\frac{d\tau d}{m} - e^{d\tau\mu}e^{-ik'_0} - \frac{d\tau}{m}\sum_{i=1}^{d}\cos(k'_i)\right)\delta_{k,k'},\end{aligned}$$

or, slightly rearranged:

$$D_{k,k'} = \left(1 - e^{d\tau\mu}e^{-ik'_0} + \frac{d\tau}{m}\sum_{i=1}^{d}(1-\cos(k'_i))\right)\delta_{k,k'}. \tag{90}$$

Note that this is a complex matrix, with real and imaginary parts:

$$\mathrm{Re}\left[D_{k,k'}\right] = \left(1 - e^{d\tau\mu}\cos(k'_0) + \frac{d\tau}{m}\sum_{i=1}^{d}(1-\cos(k'_i))\right)\delta_{k,k'} \tag{91}$$

$$\mathrm{Im}\left[D_{k,k'}\right] = e^{d\tau\mu}\sin(k'_0)\delta_{k,k'}. \tag{92}$$

Analytical solution for the nonrotating, noninteracting density

We can now use our diagonal matrix $D_{k,k'} = D_{k,k}$ to solve for the density of this system. Recall that

$$\langle\hat{n}\rangle = \frac{1}{VN_\tau}\sum_k \frac{1}{D_{kk}}\frac{\partial D_{kk}}{\partial(d\tau\mu)}. \tag{93}$$

We first need to solve for $\frac{\partial D_{kk}}{\partial(d\tau\mu)}$:

$$\begin{aligned}\frac{\partial D_{kk}}{\partial\mu} &= \frac{\partial}{\partial(d\tau\mu)}\left[\left(1 - e^{d\tau\mu}e^{-ik'_0} + \frac{d\tau}{m}\sum_{i=1}^{d}(1-\cos(k'_i))\right)\delta_{k,k'}\right] \\ &= -e^{d\tau\mu}e^{-ik'_0}\delta_{k,k'}.\end{aligned} \tag{94}$$

Plugging this in to our equation for the density gives us:

$$\langle\hat{n}\rangle = \frac{1}{N_x^d N_t}\sum_k \frac{D_{kk}^*}{|D_{kk}|^2}\left(-e^{d\tau\mu}e^{-ik'_0}\delta_{k,k'}\right). \tag{95}$$

Analytical solution for the nonrotating, noninteracting field modulus squared

$$\langle \phi^* \phi \rangle = \sum_k \frac{1}{D_{kk}} = \sum_k \frac{D^*_{kk}}{|D_{kk}|^2}. \tag{96}$$

Analytical solution for the two-point correlation function

The eigenvalues and eigenvectors can be used to compute the solution for two-point correlation function for this system:

$$G(x, x') = \sum_k \frac{1}{D_{kk}} U^\dagger_k(x) U_k(x') \tag{97}$$