Rotating Bosons Project

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Contents

1 Derivations for the Aarts Paper

1.1 *The Density using the Lattice Fields*

The lattice density is defined as follows:

$$\langle n \rangle = \frac{1}{\Omega} \sum_x n_x \tag{1}$$

$$\begin{aligned} n_x &= \sum_{a,b=1}^{2} \left(\delta_{ab} \sinh \mu - i\epsilon_{ab} \cosh \mu \right) \phi_{a,x} \phi_{b+\hat{4}} \\ &= \sum_{a,b=1}^{2} \left(\delta_{ab} \sinh \mu - i\epsilon_{ab} \cosh \mu \right) (\phi^R_{a,x} \phi^R_{b,x+\hat{4}} - \phi^I_{a,x} \phi^I_{b,x+\hat{4}} \\ &\quad + i \left[\phi^R_{a,x} \phi^I_{b,x+\hat{4}} + \phi^I_{a,x} \phi^R_{b,x+\hat{4}} \right]). \end{aligned} \tag{2}$$

First, we explicitly compute the sum over a and b:

$$\begin{aligned} n_x = & \\ & \sinh \mu \left(\phi^R_{1,x} \phi^R_{1,x+\hat{4}} - \phi^I_{1,x} \phi^I_{1,x+\hat{4}} + i \left[\phi^R_{1,x} \phi^I_{1,x+\hat{4}} + \phi^I_{1,x} \phi^R_{1,x+\hat{4}} \right] \right) \\ & -i \cosh \mu \left(\phi^R_{1,x} \phi^R_{2,x+\hat{4}} - \phi^I_{1,x} \phi^I_{2,x+\hat{4}} + i \left[\phi^R_{1,x} \phi^I_{2,x+\hat{4}} + \phi^I_{1,x} \phi^R_{2,x+\hat{4}} \right] \right) \\ & +i \cosh \mu \left(\phi^R_{2,x} \phi^R_{1,x+\hat{4}} - \phi^I_{2,x} \phi^I_{1,x+\hat{4}} + i \left[\phi^R_{2,x} \phi^I_{1,x+\hat{4}} + \phi^I_{2,x} \phi^R_{1,x+\hat{4}} \right] \right) \\ & + \sinh \mu \left(\phi^R_{2,x} \phi^R_{2,x+\hat{4}} - \phi^I_{2,x} \phi^I_{2,x+\hat{4}} + i \left[\phi^R_{2,x} \phi^I_{2,x+\hat{4}} + \phi^I_{2,x} \phi^R_{2,x+\hat{4}} \right] \right). \end{aligned} \tag{3}$$

Now, separating this into real and imaginary parts:

$$\begin{aligned} \mathrm{Re}[n_x] &= \sinh \mu \left(\phi^R_{1,x} \phi^R_{1,x+\hat{4}} - \phi^I_{1,x} \phi^I_{1,x+\hat{4}} + \phi^R_{2,x} \phi^R_{2,x+\hat{4}} - \phi^I_{2,x} \phi^I_{2,x+\hat{4}} \right) \\ &+ \cosh \mu \left(\phi^R_{1,x} \phi^I_{2,x+\hat{4}} + \phi^I_{1,x} \phi^R_{2,x+\hat{4}} - \phi^R_{2,x} \phi^I_{1,x+\hat{4}} - \phi^I_{2,x} \phi^R_{1,x+\hat{4}} \right) \end{aligned} \tag{4}$$

$$\begin{aligned} \mathrm{Im}[n_x] &= \sinh \mu \left(\phi^R_{1,x} \phi^I_{1,x+\hat{4}} + \phi^I_{1,x} \phi^R_{1,x+\hat{4}} + \phi^R_{2,x} \phi^I_{2,x+\hat{4}} + \phi^I_{2,x} \phi^R_{2,x+\hat{4}} \right) \\ &+ \cosh \mu \left(\phi^R_{2,x} \phi^R_{1,x+\hat{4}} - \phi^I_{2,x} \phi^I_{1,x+\hat{4}} - \phi^R_{1,x} \phi^R_{2,x+\hat{4}} + \phi^I_{1,x} \phi^I_{2,x+\hat{4}} \right). \end{aligned} \tag{5}$$

This form can be plugged directly into the code to compute the density.

1.2 *Diagonalization of the noninteracting action via Fourier Transformations*

The noninteracting action can be expressed:

$$S = \sum_{x,x',a,a'} \phi^*_{x,a} M_{x,a;x'a'} \phi_{x',a'}. \tag{6}$$

The noninteracting lattice action is the following:

$$\begin{aligned} S &= \sum_x \left[(2d+m^2)\phi^*_x \phi_x - \sum_{\nu=1}^{4} (\phi^*_x e^{-\mu\delta_{\nu,4}} \phi_{x+\hat{\nu}} + \phi^*_{x+\hat{\nu}} e^{\mu\delta_{\nu,4}} \phi_x) \right] \\ &= \sum_{x,x'} \phi^*_x \left[(2d+m^2)\delta_{x,x'} - \sum_{j=1}^{d} (\delta_{x,x'-\hat{j}} + \delta_{x,x'+\hat{j}}) - (e^{-\mu}\delta_{x,x'-\hat{4}} + e^{\mu}\delta_{x,x'+\hat{4}}) \right] \phi_{x'}, \end{aligned} \tag{7}$$

which allows us to obtain the matrix, M, proposed in equation **??**. This matrix can be expressed:

$$M[m,d,\mu] = (2d+m^2)\delta_{x,x'} - \sum_{j=1}^{d} (\delta_{x,x'-\hat{j}} + \delta_{x,x'+\hat{j}}) - (e^{-\mu}\delta_{x,x'-\hat{4}} + e^{\mu}\delta_{x,x'+\hat{4}}), \tag{8}$$

which can be diagonalized using a Fourier transformation.

The Fourier transformation is a unitary transformation

$$D_{ij} = [U^\dagger M U]_{ij} \tag{9}$$

where

$$\begin{aligned} U_{xk} &= \frac{1}{\sqrt{N_x^d N_t}} exp(i\vec{k}\cdot\vec{x} - i\omega t) \\ \vec{x} &= L*(x_1,x_2,x_3),\ t = L*x_0 \\ \vec{k} &= \frac{2\pi}{LN_x}(k_1,k_2,k_3),\ \omega = \frac{2\pi}{LN_t}k_0 \\ \delta_{x,x'+\hat{4}} &= \delta_{x,x'}\delta_{y,y'}\delta_{y,y'}\delta_{t,t'+1}. \end{aligned} \tag{10}$$

When these unitary matrices are applied and simplified, we obtain the following diagonal matrix:

(11)

$$
\begin{aligned}
D = &\frac{1}{N_x^d N_t} \sum_x \sum_{x'} e^{-i\vec{k}\cdot\vec{x}+i\omega t} \\
&\left((2d+m^2)\delta_{x,x'} - \sum_{j=1}^{d} (\delta_{x,x'-\hat{j}} + \delta_{x,x'+\hat{j}}) - (e^{-\mu}\delta_{x,x'-\hat{4}} + e^{\mu}\delta_{x,x'+\hat{4}}) \right) e^{i\vec{k}'\cdot\vec{x}'-i\omega' t'} \\
= &\frac{1}{N_x^d N_t} \sum_x e^{-i\vec{k}\cdot\vec{x}+i\omega t} \left((2d+m^2) - \sum_{j=1}^{d} (e^{ik'_j} + e^{-ik'_j}) - (e^{-\mu-i\omega'} + e^{\mu+i\omega'}) \right) e^{i\vec{k}'\cdot\vec{x}-i\omega' t} \\
= &\frac{1}{N_x^d N_t} \sum_x \left((2d+m^2) - \sum_{j=1}^{d} (e^{ik'_j} + e^{-ik'_j}) - (e^{-\mu-i\omega'} + e^{\mu+i\omega'}) \right) e^{-i\vec{x}\cdot(\vec{k}-\vec{k}')+it(\omega-\omega')}
\end{aligned}
$$

Recall that $\sum_{j=1}^{N} e^{i(a-a')j} = N\delta_{a,a'}$, so our sum over x collapses and our factors of $N_x^d N_t$ cancel:

$$
D_{k,k'} = \left((2d+m^2) - \sum_{j=1}^{d} (e^{ik'_j} + e^{-ik'_j}) - (e^{-\mu-i\omega'} + e^{\mu+i\omega'}) \right) \delta_{k,k'} \tag{12}
$$

Now that we have this diagonal matrix, we can determine the density and field modulus squared as a function of this density in the following way. First, the density:

$$
\begin{aligned}
\langle \hat{n} \rangle &= \frac{-1}{V} \frac{\partial ln\mathcal{Z}}{\partial \mu} = \frac{-1}{V} \frac{\partial}{\partial \mu} (-ln(det(M))) \\
&= \frac{1}{V} \frac{\partial}{\partial \mu} Tr(lnM) = \frac{1}{V} \frac{\partial}{\partial \mu} \sum_k lnD_{kk} = \frac{1}{V} \sum_k \frac{1}{D_{kk}} \frac{\partial D_{kk}}{\partial \mu} \\
&= \frac{1}{V} \sum_k \frac{1}{D_{kk}} (\cos k^0 \sinh \mu + i \sin k^0 \cosh \mu).
\end{aligned} \tag{13}
$$

And now, the field modulus squared:

$$
\begin{aligned}
\langle \hat{n} \rangle &= \frac{-1}{V} \frac{\partial ln\mathcal{Z}}{\partial (m^2)} = \frac{-1}{V} \frac{\partial}{\partial (m^2)} (-ln(det(M))) \\
&= \frac{1}{V} \frac{\partial}{\partial (m^2)} Tr(lnM) = \frac{1}{V} \frac{\partial}{\partial (m^2)} \sum_k lnD_{kk} = \frac{1}{V} \sum_k \frac{1}{D_{kk}} \frac{\partial D_{kk}}{\partial (m^2)} = \sum_k \frac{1}{D_{kk}}.
\end{aligned} \tag{14}
$$