



Deriving the action for the lattice
 Continuum action: $r_1^2 = (x^2 + y^2)$

$$S[\phi] = \int d^4x dt \left[\phi^* \left(\partial_t - \mu - \frac{\nabla^2}{2m} - \frac{m}{2} w_{tr}^2 r_1^2 - i w_2 (x \partial_y - y \partial_x) \right) \phi + \lambda (\phi^* \phi)^2 \right]$$

To discretize:

$$\int d^4x dt \rightarrow \sum_{\vec{r}, t} a^4 dt, \quad a \phi^* \phi \rightarrow a \phi_r^* \phi_{r,\vec{t}} \quad (\text{see appendix})$$

Discrete action:

$$S[\phi] = \sum_{\vec{r}, t} a^4 dt \left[\phi_r^* \left(\partial_t - \frac{\nabla^2}{2m} \right) \phi_r - \phi_r^* \left(\mu + \frac{m}{2} w_{tr}^2 r_1^2 + i w_2 (x \partial_y - y \partial_x) \right) \phi_{r,\vec{t}} + \lambda (\phi_r^* \phi_{r,\vec{t}})^2 \right]$$

Discrete derivatives:

$$\partial_j \phi = \frac{1}{a} (\phi_{r,j} - \phi_{r,j-1}) \quad \text{and} \quad \nabla^2 \phi = \sum_{j=1}^d \frac{1}{a^2} (\phi_{r,j-1} - 2\phi_r + \phi_{r,j+1})$$

$$\phi_r^* \partial_t \phi_r = \frac{1}{a^4} \phi_r^* (\phi_{r,\vec{t}} - \phi_{r,\vec{t}-1})$$

$$\phi_r^* \nabla^2 \phi_r = \frac{1}{a^2} \sum_{j=1}^d (\phi_{r,j-1}^* - 2\phi_r^* \phi_r + \phi_{r,j+1}^*)$$

$$\phi_r^* \partial_x \phi_{r,\vec{t}} = \frac{1}{a} (\phi_{r,\vec{t}}^* \phi_r - \phi_{r,\vec{t}-1}^* \phi_{r,\vec{t}}), \quad \phi_r^* \partial_y \phi_{r,\vec{t}} = \frac{1}{a} (\phi_{r,\vec{t}}^* \phi_r - \phi_{r,\vec{t}-1}^* \phi_{r,\vec{t}})$$

So our discrete (lattice) action is:

$$S = \sum_{\vec{r}, t} \left[a \phi_r^* \phi_r - a \phi_r^* \phi_{r,\vec{t}} - a dt \mu \phi_r^* \phi_{r,\vec{t}} - \frac{a^2 dt}{2m} \sum_{j=1}^d (\phi_{r,j-1}^* - 2\phi_r^* \phi_r + \phi_{r,j+1}^*) - \frac{ma^2 dt}{2} w_{tr}^2 r_1^2 \phi_r^* \phi_{r,\vec{t}} + i w_2 a^2 dt (x \phi_{r,\vec{t}}^* \phi_{r,\vec{t}} - y \phi_{r,\vec{t}}^* \phi_{r,\vec{t}}) + y \phi_{r,\vec{t}}^* \phi_{r,\vec{t}}) + a^4 dt \lambda (\phi_r^* \phi_{r,\vec{t}})^2 \right]$$

Let's update our parameters to incorporate dt and a

$$x = \tilde{x}, \quad y = \tilde{y}, \quad r^2 = \tilde{r}^2, \quad \mu = \frac{\tilde{\mu}}{dt}, \quad m = \frac{\tilde{m} dt}{a^2}, \quad w_{tr} = \frac{\tilde{w}_{tr}}{dt}$$

$$w_2 = \frac{\tilde{w}_2}{dt}, \quad \lambda = \frac{\tilde{\lambda}}{dt}$$

$$S = \sum_{\tilde{x}, \tilde{r}} \left[a^d (\phi_r^* \phi_r - \phi_r^* \phi_{r,\tilde{i}} - \tilde{\mu}^* \phi_{r,\tilde{i}} \phi_{r,\tilde{i}}) - \frac{a}{2\tilde{m}} \sum_{j=1}^d (\phi_r^* \phi_{r,j} - 2\phi_r^* \phi_r + \phi_r^* \phi_{r,j}) \right. \\ \left. - \frac{\tilde{m} a^d}{2} \tilde{\omega}_{tr}^2 \tilde{r}_1^2 \phi_r^* \phi_{r,\tilde{i}} + i \tilde{\omega}_z a^d (\tilde{x} \phi_r^* \phi_{r,\tilde{i}} - \tilde{x} \phi_r^* \phi_{r,\tilde{i}} \tilde{y} \phi_r^* \phi_{r,\tilde{i}} + \tilde{y} \phi_r^* \phi_{r,\tilde{i}}) \right. \\ \left. + a^d \tilde{\lambda} (\phi_r^* \phi_{r,\tilde{i}})^2 \right]$$

We can pull out a constant factor of a^d now:

$$S = \sum_{\tilde{x}, \tilde{r}} a^d \left[(\phi_r^* \phi_r - \phi_r^* \phi_{r,\tilde{i}} - \tilde{\mu}^* \phi_{r,\tilde{i}} \phi_{r,\tilde{i}}) - \frac{1}{2\tilde{m}} \sum_{j=1}^d (\phi_r^* \phi_{r,j} - 2\phi_r^* \phi_r + \phi_r^* \phi_{r,j}) \right. \\ \left. - \frac{\tilde{m}}{2} \tilde{\omega}_{tr}^2 \tilde{r}_1^2 \phi_r^* \phi_{r,\tilde{i}} + i \tilde{\omega}_z (\tilde{x} \phi_r^* \phi_{r,\tilde{i}} - \tilde{x} \phi_r^* \phi_{r,\tilde{i}} \tilde{y} \phi_r^* \phi_{r,\tilde{i}} + \tilde{y} \phi_r^* \phi_{r,\tilde{i}}) + \tilde{\lambda} (\phi_r^* \phi_{r,\tilde{i}})^2 \right]$$

Noting that, to second order in $d\tau$, $(1 + d\tau/\mu) = e^{d\tau/\mu}$, we further simplify:

$$S = \sum_{\tilde{x}, \tilde{r}} a^d \left[\underbrace{(\phi_r^* \phi_r - e^{\tilde{\mu}} \phi_{r,\tilde{i}} \phi_{r,\tilde{i}})}_{S_{tr}} - \frac{1}{2\tilde{m}} \sum_{j=1}^d (\phi_r^* \phi_{r,j} - 2\phi_r^* \phi_r + \phi_r^* \phi_{r,j}) \underbrace{+ S_{\phi}}_{S_{\phi}} \right. \\ \left. - \frac{\tilde{m}}{2} \tilde{\omega}_{tr}^2 \tilde{r}_1^2 \phi_r^* \phi_{r,\tilde{i}} + i \tilde{\omega}_z (\tilde{x} \phi_r^* \phi_{r,\tilde{i}} - \tilde{x} \phi_r^* \phi_{r,\tilde{i}} \tilde{y} \phi_r^* \phi_{r,\tilde{i}} + \tilde{y} \phi_r^* \phi_{r,\tilde{i}}) + \tilde{\lambda} (\phi_r^* \phi_{r,\tilde{i}})^2 \right] \\ \underbrace{- S_w}_{S_{int}}$$

We can make our work easier by writing this out as:

$$S = \sum_{\tilde{x}, \tilde{r}} a^d [S_{tr} + S_{\phi} - S_{\text{trap}} - S_w + S_{\text{int}}]$$

Writing our complex fields in terms of two real components

The next step is to note that $\phi = \phi_1 + i\phi_2$ and apply this coordinate transformation to the lattice.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \text{ and } \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \rightarrow$$

$$S_{\text{tr}} = \phi_r^* \phi_r - e^{\tilde{\mu}} \phi_r^* \phi_{r-\hat{x}} \rightarrow \frac{1}{2} [\phi_{1,r}^2 + \phi_{2,r}^2 - e^{\tilde{\mu}} (\phi_{1,r} \phi_{1,r-\hat{x}} - i\phi_{2,r} \phi_{1,r-\hat{x}} + i\phi_{1,r} \phi_{2,r-\hat{x}} + \phi_{2,r} \phi_{2,r-\hat{x}})]$$

$$S_{\text{tr}} = -\frac{1}{2\tilde{m}} \sum_{j=1}^d (\phi_r^* \phi_{r-\hat{j}} - 2\phi_r^* \phi_r + \phi_r^* \phi_{r+\hat{j}})$$

$$\rightarrow \frac{1}{4\tilde{m}} \sum_{j=1}^d [2\phi_{1,r}^2 + 2\phi_{2,r}^2 - (\phi_{1,r} \phi_{1,r-\hat{j}} - i\phi_{2,r} \phi_{1,r-\hat{j}} + i\phi_{1,r} \phi_{2,r-\hat{j}} + \phi_{2,r} \phi_{2,r-\hat{j}} + \phi_{1,r} \phi_{1,r+\hat{j}} - i\phi_{2,r} \phi_{1,r+\hat{j}} + i\phi_{1,r} \phi_{2,r+\hat{j}} + \phi_{2,r} \phi_{2,r+\hat{j}})]$$

$$S_{\text{trap}} = \frac{\tilde{m}}{2} \tilde{\omega}_{\text{tr}}^2 \tilde{r}_\perp^2 \phi_r^* \phi_{r-\hat{x}} \rightarrow \frac{\tilde{m}}{4} \tilde{\omega}_{\text{tr}}^2 \tilde{r}_\perp^2 (\phi_{1,r} \phi_{1,r-\hat{x}} - i\phi_{2,r} \phi_{1,r-\hat{x}} + i\phi_{1,r} \phi_{2,r-\hat{x}} + \phi_{2,r} \phi_{2,r-\hat{x}})$$

$$S_{\omega} = i\tilde{\omega}_2 (\tilde{x}\phi_r^* \phi_{r-\hat{x}} - \tilde{x}\phi_r^* \phi_{r-\hat{y}} - \tilde{y}\phi_r^* \phi_{r-\hat{x}} - \tilde{y}\phi_r^* \phi_{r-\hat{y}})$$

$$\rightarrow i\frac{\tilde{\omega}_2}{2} \left[\tilde{x}(\phi_{1,r} \phi_{1,r-\hat{x}} - i\phi_{2,r} \phi_{1,r-\hat{x}} + i\phi_{1,r} \phi_{2,r-\hat{x}} + \phi_{2,r} \phi_{2,r-\hat{x}} - \phi_{1,r} \phi_{1,r-\hat{y}} + i\phi_{2,r} \phi_{1,r-\hat{y}} - i\phi_{1,r} \phi_{2,r-\hat{y}} - \phi_{2,r} \phi_{2,r-\hat{y}}) + \tilde{y}(\phi_{1,r} \phi_{1,r-\hat{x}} - i\phi_{2,r} \phi_{1,r-\hat{x}} + i\phi_{1,r} \phi_{2,r-\hat{x}} + \phi_{2,r} \phi_{2,r-\hat{x}} - \phi_{1,r} \phi_{1,r-\hat{y}} + i\phi_{2,r} \phi_{1,r-\hat{y}} - i\phi_{1,r} \phi_{2,r-\hat{y}} - \phi_{2,r} \phi_{2,r-\hat{y}}) \right]$$

$$= i\frac{\tilde{\omega}_2}{2} \left[\tilde{x}(\phi_{1,r} \phi_{1,r-\hat{x}} + \phi_{2,r} \phi_{2,r-\hat{x}} - \phi_{1,r} \phi_{1,r-\hat{y}} - \phi_{2,r} \phi_{2,r-\hat{y}}) + \tilde{y}(\phi_{1,r} \phi_{1,r-\hat{x}} + \phi_{2,r} \phi_{2,r-\hat{x}} - \phi_{1,r} \phi_{1,r-\hat{y}} - \phi_{2,r} \phi_{2,r-\hat{y}}) \right]$$

$$+ \frac{\tilde{\omega}_2}{2} \left[\tilde{x}(\phi_{2,r} \phi_{1,r-\hat{x}} - \phi_{1,r} \phi_{2,r-\hat{x}} - \phi_{2,r} \phi_{1,r-\hat{y}} + \phi_{1,r} \phi_{2,r-\hat{y}}) + \tilde{y}(\phi_{2,r} \phi_{1,r-\hat{x}} - \phi_{1,r} \phi_{2,r-\hat{x}} - \phi_{2,r} \phi_{1,r-\hat{y}} + \phi_{1,r} \phi_{2,r-\hat{y}}) \right]$$

$$= i\frac{\tilde{\omega}_2}{2} [(\tilde{x}-\tilde{y})(\phi_{1,r} \phi_{1,r-\hat{x}} + \phi_{2,r} \phi_{2,r-\hat{x}}) - \tilde{x}(\phi_{1,r} \phi_{1,r-\hat{y}} + \phi_{2,r} \phi_{2,r-\hat{y}}) + \tilde{y}(\phi_{1,r} \phi_{1,r-\hat{x}} + \phi_{2,r} \phi_{2,r-\hat{x}})]$$

$$+ \frac{\tilde{\omega}_2}{2} [(\tilde{x}-\tilde{y})(\phi_{2,r} \phi_{1,r-\hat{x}} - \phi_{1,r} \phi_{2,r-\hat{x}}) - \tilde{x}(\phi_{2,r} \phi_{1,r-\hat{y}} - \phi_{1,r} \phi_{2,r-\hat{y}}) + \tilde{y}(\phi_{2,r} \phi_{1,r-\hat{x}} - \phi_{1,r} \phi_{2,r-\hat{x}})]$$

$$\begin{aligned}
S_{rl} &= \tilde{\gamma} (\phi_r^* \phi_{r\hat{r}})^2 \rightarrow \frac{\tilde{\lambda}}{4} \left(\phi_{1,r} \phi_{1,r\hat{r}} - i \phi_{2,r} \phi_{1,r\hat{r}} + i \phi_{1,r} \phi_{1,r\hat{r}} + \phi_{2,r} \phi_{2,r\hat{r}} \right)^2 \\
&= \frac{\tilde{\lambda}}{4} \left(\phi_{1,r} \phi_{1,r\hat{r}} \phi_{1,r} \phi_{1,r\hat{r}} - i \phi_{1,r} \phi_{1,r\hat{r}} \phi_{2,r} \phi_{1,r\hat{r}} + i \phi_{1,r} \phi_{1,r\hat{r}} \phi_{1,r} \phi_{2,r\hat{r}} + \phi_{1,r} \phi_{1,r\hat{r}} \phi_{2,r} \phi_{2,r\hat{r}} \right. \\
&\quad - i \phi_{2,r} \phi_{1,r\hat{r}} \phi_{1,r} \phi_{1,r\hat{r}} - \phi_{2,r} \phi_{1,r\hat{r}} \phi_{2,r} \phi_{1,r\hat{r}} + \phi_{2,r} \phi_{1,r\hat{r}} \phi_{1,r} \phi_{2,r\hat{r}} - i \phi_{2,r} \phi_{1,r\hat{r}} \phi_{2,r} \phi_{2,r\hat{r}} \\
&\quad + i \phi_{1,r} \phi_{2,r\hat{r}} \phi_{1,r} \phi_{1,r\hat{r}} + \phi_{1,r} \phi_{2,r\hat{r}} \phi_{2,r} \phi_{1,r\hat{r}} - \phi_{1,r} \phi_{2,r\hat{r}} \phi_{1,r} \phi_{2,r\hat{r}} + i \phi_{1,r} \phi_{2,r\hat{r}} \phi_{2,r} \phi_{2,r\hat{r}} \\
&\quad \left. + \phi_{2,r} \phi_{2,r\hat{r}} \phi_{1,r} \phi_{1,r\hat{r}} - i \phi_{2,r} \phi_{2,r\hat{r}} \phi_{2,r} \phi_{1,r\hat{r}} + i \phi_{2,r} \phi_{2,r\hat{r}} \phi_{1,r} \phi_{2,r\hat{r}} + \phi_{2,r} \phi_{2,r\hat{r}} \phi_{2,r} \phi_{2,r\hat{r}} \right)
\end{aligned}$$

