CSCI567 Machine Learning (Spring 2023)
Week 7: Support Vector Machines

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Outline

1. Support vector machines (primal formulation)
2. A detour of Lagrangian duality
3. Support vector machines (dual formulation)

Support vector machines (SVM)

- One of the most commonly used classification algorithms
- Works well with the kernel trick
- Strong theoretical guarantees

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We focus on **binary classification** here.

Primal formulation

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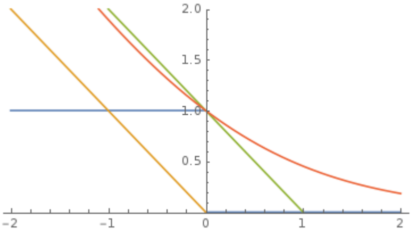

- perceptron loss $\ell_{\text{perceptron}}(z) = \max\{0, -z\} \rightarrow$ Perceptron
- logistic loss $\ell_{\text{logistic}}(z) = \log(1 + \exp(-z)) \rightarrow$ logistic regression
- hinge loss $\ell_{\text{hinge}}(z) = \max\{0, 1 - z\} \rightarrow$ **SVM**

Primal formulation

For a linear model $(\boldsymbol{w}, b)$, this means

$$\min_{\boldsymbol{w},b} \sum_n \max\left\{0, 1 - y_n(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b)\right\} + \frac{\lambda}{2}\|\boldsymbol{w}\|_2^2$$

- recall $y_n \in \{-1, +1\}$
- a nonlinear mapping $\boldsymbol{\phi}$ is applied
- the bias/intercept term b is used explicitly (think about why after this lecture)

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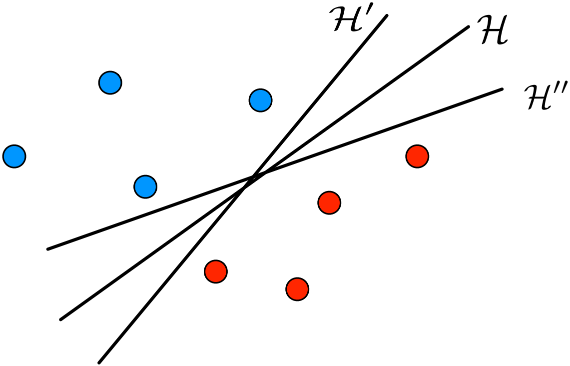
So why L2 regularized hinge loss?

Geometric motivation: separable case

When data is **linearly separable**, there are *infinitely many hyperplanes with zero training error*:



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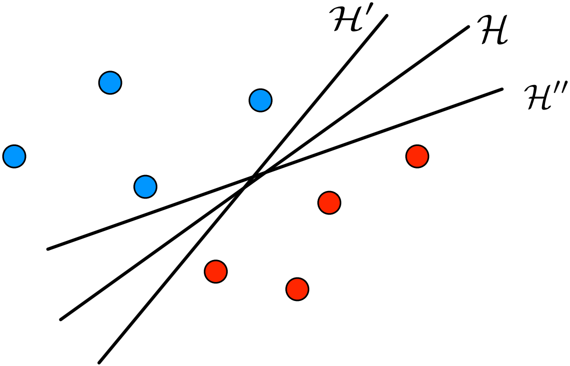
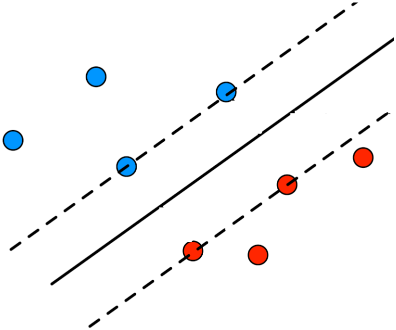


So which one should we choose?

Intuition

The further away from data points the better.



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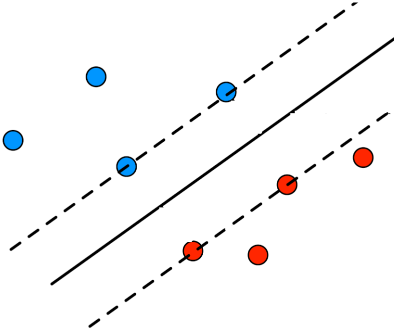

How to formalize this intuition?

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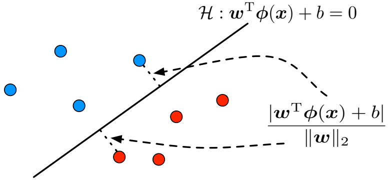
For a hyperplane that correctly classifies $(\boldsymbol{x}, y)$, the distance becomes

$$\frac{y(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{x} + b)}{\|\boldsymbol{w}\|_2}$$

Maximizing margin

Margin: the *smallest* distance from all training points to the hyperplane

$$\text{MARGIN OF } (\boldsymbol{w},\ b) = \min_n \frac{y_n(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b)}{\|\boldsymbol{w}\|_2}$$



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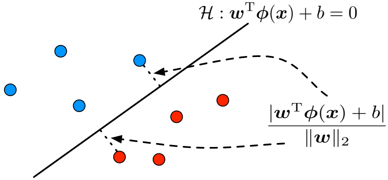

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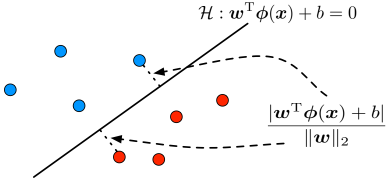

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&\textsc{margin of } (\boldsymbol{w},\ b) \\
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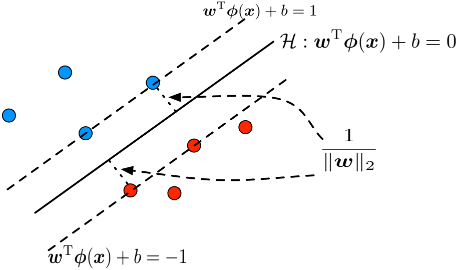

Summary for separable data

For a separable training set, we aim to solve

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SVM is thus also called *max-margin* classifier. The constraints above are called *hard-margin* constraints.

General non-separable case

If data is not linearly separable, the previous constraint

$$y_n(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b) \geq 1, \quad \forall\ n$$

is obviously *not feasible*.

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To deal with this issue, we relax them to **soft-margin** constraints:

$$y_n(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b) \geq 1 - \xi_n, \quad \forall\ n$$

where we introduce **slack variables** $\xi_n \geq 0$.

SVM Primal formulation

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We want ξ_n to be as small as possible too. The objective becomes

$$\begin{aligned} \min_{\boldsymbol{w}, b, \{\xi_n\}} \quad & \frac{1}{2}\|\boldsymbol{w}\|_2^2 + C \sum_n \xi_n \\ \text{s.t.} \quad & y_n(\boldsymbol{w}^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}_n) + b) \geq 1 - \xi_n, \quad \forall\ n \\ & \xi_n \geq 0, \quad \forall\ n \end{aligned}$$

where C is a hyperparameter to balance the two goals.

Equivalent form

Formulation

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\min_{\boldsymbol{w},b,\{\xi_n\}} \quad & C\sum_n \xi_n + \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\
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with $\lambda = 1/C$. *This is exactly minimizing L2 regularized hinge loss!*

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- It is a convex (**quadratic** in fact) problem
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- there are **more specialized and efficient** algorithms
- but usually we apply kernel trick, which requires solving the *dual problem*

Outline

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Applying it to SVM reveals an important aspect of the algorithm

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Suppose we want to solve

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where functions $h_1, \ldots, h_{\mathsf{J}}$ define J **constraints**.

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SVM primal formulation is clearly of this form with $\mathsf{J} = 2\mathsf{N}$ constraints:

$$F(\boldsymbol{w}, b, \{\xi_n\}) = C \sum_n \xi_n + \frac{1}{2}\|\boldsymbol{w}\|_2^2$$

$$h_n(\boldsymbol{w}, b, \{\xi_n\}) = 1 - y_n(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b) - \xi_n \quad \forall\, n \in [\mathsf{N}]$$

$$h_{\mathsf{N}+n}(\boldsymbol{w}, b, \{\xi_n\}) = -\xi_n \quad \forall\, n \in [\mathsf{N}]$$

Lagrangian

The **Lagrangian** of the previous problem is defined as:

$$L(\boldsymbol{w}, \{\lambda_j\}) = F(\boldsymbol{w}) + \sum_{j=1}^{\mathsf{J}} \lambda_j h_j(\boldsymbol{w})$$

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and thus,

$$\min_{\boldsymbol{w}} \max_{\{\lambda_j\} \geq 0} L\left(\boldsymbol{w}, \{\lambda_j\}\right) \iff \min_{\boldsymbol{w}} F(\boldsymbol{w}) \ \text{ s.t. } \ h_j(\boldsymbol{w}) \leq 0 \quad \forall\, j \in [\mathsf{J}]$$

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$$\begin{aligned}\max_{\{\lambda_j\}\geq 0} \min_{\boldsymbol{w}} L\left(\boldsymbol{w}, \{\lambda_j\}\right) &= \min_{\boldsymbol{w}} L\left(\boldsymbol{w}, \{\lambda_j^*\}\right) \leq L\left(\boldsymbol{w}^*, \{\lambda_j^*\}\right) \\ &\leq \max_{\{\lambda_j\}\geq 0} L\left(\boldsymbol{w}^*, \{\lambda_j\}\right) = \min_{\boldsymbol{w}} \max_{\{\lambda_j\}\geq 0} L\left(\boldsymbol{w}, \{\lambda_j\}\right)\end{aligned}$$

This is called "**weak duality**".

Strong duality

When $F, h_1, \ldots, h_m$ are convex, under some mild conditions:

$$\min_{\boldsymbol{w}} \max_{\{\lambda_j\} \geq 0} L\left(\boldsymbol{w}, \{\lambda_j\}\right) = \max_{\{\lambda_j\} \geq 0} \min_{\boldsymbol{w}} L\left(\boldsymbol{w}, \{\lambda_j\}\right)$$

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Deriving the Karush-Kuhn-Tucker (KKT) conditions

Observe that if strong duality holds:

$$F(\boldsymbol{w}^*) = \min_{\boldsymbol{w}} \max_{\{\lambda_j\} \geq 0} L\left(\boldsymbol{w}, \{\lambda_j\}\right) = \max_{\{\lambda_j\} \geq 0} \min_{\boldsymbol{w}} L\left(\boldsymbol{w}, \{\lambda_j\}\right)$$

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- equality $\min_{\boldsymbol{w}} L(\boldsymbol{w}, \{\lambda_j^*\}) = L(\boldsymbol{w}^*, \{\lambda_j^*\})$ implies $\boldsymbol{w}^*$ is a **minimizer** of $L(\boldsymbol{w}, \{\lambda_j^*\})$ and thus has **zero gradient**:

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}^*, \{\lambda_j^*\}) = \nabla F(\boldsymbol{w}^*) + \sum_{j=1}^{\mathsf{J}} \lambda_j^* \nabla h_j(\boldsymbol{w}^*) = \mathbf{0}$$

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These are *necessary conditions*.

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These are *necessary conditions*. They are also *sufficient* when F is convex and $h_1, \ldots, h_{\mathsf{J}}$ are continuously differentiable convex functions.

Outline

Writing down the Lagrangian

Recall the primal formulation

$$\begin{aligned} \min_{\boldsymbol{w},b,\{\xi_n\}} \quad & C\sum_n \xi_n + \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\ \text{s.t.} \quad & 1 - y_n[\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b] \leq \xi_n, \quad \forall\ n \\ & \xi_n \geq 0, \quad \forall\ n \end{aligned}$$

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Lagrangian is

$$
\begin{aligned}
L\left(\boldsymbol{w}, b, \{\xi_n\}, \{\alpha_n\}, \{\lambda_n\}\right) = {} & C\sum_n \xi_n + \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_n \lambda_n \xi_n \\
& + \sum_n \alpha_n \left(1 - y_n(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b) - \xi_n\right)
\end{aligned}
$$

where $\alpha_1, \ldots, \alpha_{\mathsf{N}} \ge 0$ and $\lambda_1, \ldots, \lambda_{\mathsf{N}} \ge 0$ are Lagrangian multipliers.

Applying the stationarity condition

$$L = C \sum_n \xi_n + \frac{1}{2}\|\boldsymbol{w}\|_2^2 - \sum_n \lambda_n \xi_n + \sum_n \alpha_n \left(1 - y_n(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b) - \xi_n\right)$$

$\exists$ primal and dual variables $\boldsymbol{w}, b, \{\xi_n\}, \{\alpha_n\}, \{\lambda_n\}$ s.t. $\nabla_{\boldsymbol{w}, b, \{\xi_n\}} L = \mathbf{0}$,

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Replacing w by $\sum_n y_n \alpha_n \phi(\boldsymbol{x}_n)$ in the Lagrangian gives

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To find the dual solutions, it amounts to solving

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Note the last three constraints can be written as $0 \leq \alpha_n \leq C$ for all n.

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Note the last three constraints can be written as $0 \leq \alpha_n \leq C$ for all n. So the final **dual formulation of SVM** is:

$$\begin{aligned} \max_{\{\alpha_n\}} \quad & \sum_n \alpha_n - \frac{1}{2}\sum_{m,n} y_m y_n \alpha_m \alpha_n \boldsymbol{\phi}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}_n) \\ \text{s.t.} \quad & \sum_n \alpha_n y_n = 0 \quad \text{and} \quad 0 \leq \alpha_n \leq C, \quad \forall\ n \end{aligned}$$

Kernelizing SVM

Now it is clear that with a **kernel function** k for the mapping ϕ, we can kernelize SVM as:

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Again, no need to compute $\phi(\boldsymbol{x})$. It is a **quadratic program** and many efficient optimization algorithms exist.

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A point with $\alpha_n^* > 0$ is called a "**support vector**". Hence the name SVM.

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To identify b, we need to apply complementary slackness.

Applying complementary slackness

For all n we should have

$$\lambda_n^* \xi_n^* = 0, \quad \alpha_n^* \left(1 - \xi_n^* - y_n(\boldsymbol{w}^{*\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}_n) + b^*)\right) = 0$$

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For some support vector $\boldsymbol{\phi}(\boldsymbol{x}_n)$ if we have $0 < \alpha_n^* < C$, then

$$\lambda_n^* = C - \alpha_n^* > 0$$

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The prediction on a new point $\boldsymbol{x}$ is therefore

$$\mathrm{SGN}\left(\boldsymbol{w^*}^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}) + b^*\right) = \mathrm{SGN}\left(\sum_m y_m \alpha_m^* k(\boldsymbol{x}_m, \boldsymbol{x}) + b^*\right)$$

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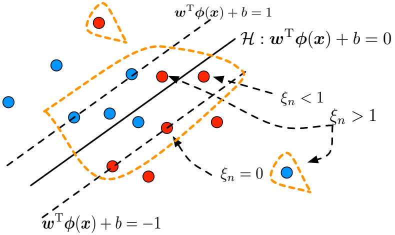

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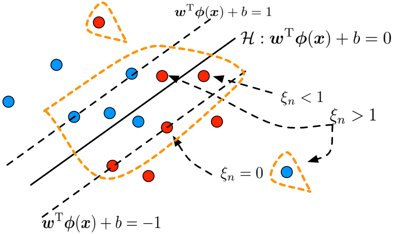

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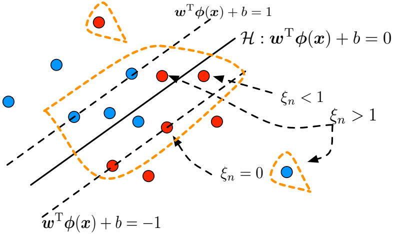

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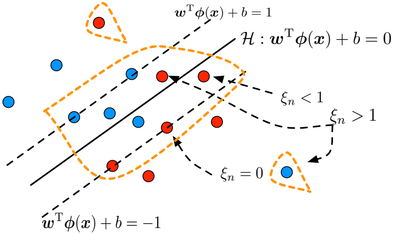

Support vectors (circled with the orange line) are *the only points that matter!*

An example

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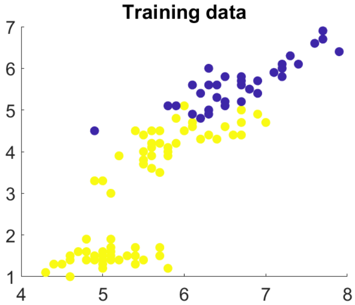

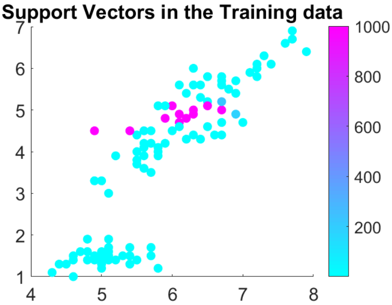

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SVM: **max-margin linear classifier**

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\min_{\boldsymbol{w}, b, \{\xi_n\}} \quad & C \sum_n \xi_n + \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\
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Dual (kernelizable, reveals what training points are support vectors):

$$\begin{aligned} \max_{\{\alpha_n\}} \quad & \sum_n \alpha_n - \frac{1}{2}\sum_{m,n} y_m y_n \alpha_m \alpha_n \boldsymbol{\phi}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) \\ \text{s.t.} \quad & \sum_n \alpha_n y_n = 0 \quad \text{and} \quad 0 \le \alpha_n \le C, \quad \forall\ n \end{aligned}$$

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