R23B01T0

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Hollow cylinder with heating through convection at outer surface and is insulated at inner surface [1]

Contents

1. Problem description

This problem is for a homogeneous annulus of inner radius R_1 and the outer radius R_2. It is subjected to heating through convection with an environment temperature T_∞. The inner surface is insulated. At time t=0 temperature at every point inside the cylinder is 0.

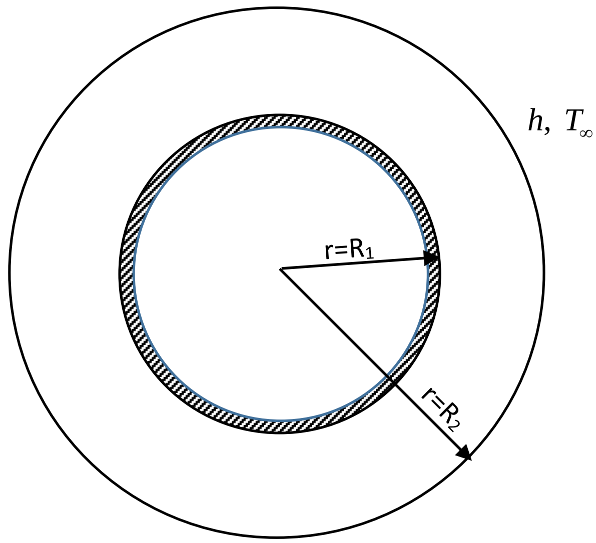

Figure 1. Schematic of R23B10T0 problem

2. Dimensional R23B10T0 problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}; \qquad R_1<r<R_2, \quad t>0 \qquad \text{(R23B01T0-1)}$$

$$\frac{\partial T}{\partial r}(R_1,t)=0 \qquad \text{(R23B01T0-2)}$$

$$-k\frac{\partial T}{\partial r}(R_2,t)=h\left(T(R_2,t)-T_\infty\right) \qquad \text{(R23B01T0-3)}$$

$$T(r,0)=0 \qquad \text{(R23B01T0-4)}$$

3. Dimensional Solution

The dimensional temperature solution is given as

$$T_{\text{R23B01T0}}(r,t) = T_\infty - \frac{\pi^2 T_\infty}{2}\frac{hR_2}{k}\sum_{m=1}^{\infty} e^{\frac{-\beta_m^2 \alpha t}{R_1^2}} \frac{J_1^2(\beta_m)\begin{bmatrix} S_0 J_0\left(\beta_m \frac{r}{R_1}\right) \\ -V_0 Y_0\left(\beta_m \frac{r}{R_1}\right)\end{bmatrix}\begin{bmatrix} S_0 J_0\left(\beta_m \frac{R_2}{R_1}\right) \\ -V_0 Y_0\left(\beta_m \frac{R_2}{R_1}\right)\end{bmatrix}}{\left(Bi^2+\beta_m^2\right)J_1^2(\beta_m) - V_0^2} \quad \text{(R23B01T0-5)}$$

where,

$$V_o = -\beta_m J_1\left(\beta_m \frac{R_2}{R_1}\right) + BiJ_0\left(\beta_m \frac{R_2}{R_1}\right)$$
$$S_o = -\beta_m Y_1\left(\beta_m \frac{R_2}{R_1}\right) + BiY_0\left(\beta_m \frac{R_2}{R_1}\right) \quad \text{(R23B01T0-6)}$$

The eigenvalues are found from the eigencondition

$$S_0 J_1(\beta_m) - V_0 Y_1(\beta_m) = 0 \quad \text{(R23B01T0-7)}$$

Appendix A contains a Matlab program for finding the eigenvalues. Appendix B contains a Matlab program for the solution of the temperature and heat flux. Appendix C is for plots of the temperature and heat flux.

4. Dimensionless groups

$$\tilde{T}(\tilde{r},\tilde{t}) = \frac{T(r,t)}{T_\infty},\ \tilde{q} = \frac{q(r,t)}{(kT_\infty / R_1)},\ \tilde{r} = \frac{r}{R_1},\ \tilde{t} = \frac{\alpha t}{R_1^2},\ \tilde{R} = \frac{R_2}{R_1}, Bi = \frac{hR_1}{k} \quad \text{(R23B01T0-8)}$$

5. Dimensionless R23B01T0 problem

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}}{\partial \tilde{r}}\right) = \frac{\partial \tilde{T}}{\partial \tilde{t}};\quad 1 < r < \tilde{R},\quad \tilde{t} > 0 \quad \text{(R23B01T0-9)}$$

$$\frac{\partial \tilde{T}}{\partial \tilde{r}}(1,\tilde{t}) = 0 \quad \text{(R23B01T0-10)}$$

$$\left[\frac{1}{Bi}\frac{\partial \tilde{T}}{\partial \tilde{r}} + \tilde{T}\right]_{(\tilde{R},\tilde{t})} = 1 \quad \text{(R23B01T0-11)}$$

$$\tilde{T}(\tilde{r},0) = 0 \quad \text{(R23B01T0-12)}$$

6. Dimensionless Solution

The transient problem(R23B01T0-9) is solved by using "Green's functions".

$$\tilde{T}_{\text{R23B01T0}}(\tilde{r},\tilde{t})=\frac{T_{\text{R23B01T0}}(r,t)}{T_\infty}$$

$$=1-\frac{\pi^2}{2}Bi.\tilde{R}\sum_{m=1}^{\infty}e^{-\beta_m^2\tilde{t}}\frac{J_1^2(\beta_m)\begin{bmatrix}S_0J_0(\beta_m\tilde{r})\\-V_0Y_0(\beta_m\tilde{r})\end{bmatrix}\begin{bmatrix}S_0J_0(\beta_m\tilde{R})\\-V_0Y_0(\beta_m\tilde{R})\end{bmatrix}}{\left(Bi^2+\beta_m^2\right)J_1^2(\beta_m)-V_0^2}$$

(R23B01T0-13)

This equation can also be written as

$$\tilde{T}_{\text{R23B01T0}}(\tilde{r},\tilde{t})=1-\frac{\pi^2}{2}Bi.\tilde{R}\sum_{m=1}^{\infty}e^{-\beta_m^2\tilde{t}}\frac{J_1^2(\beta_m)N_m(\tilde{r})N_m(\tilde{R})}{\left(Bi^2+\beta_m^2\right)J_1^2(\beta_m)-V_0^2}\quad\text{(R23B01T0-14)}$$

where,

$$N_m(\tilde{r})=S_0J_0(\beta_m\tilde{r})-V_0Y_0(\beta_m\tilde{r})\quad\text{(R23B01T0-15)}$$

$$\begin{aligned}V_o&=-\beta_mJ_1(\beta_m\tilde{R})+BiJ_0(\beta_m\tilde{R})\\S_o&=-\beta_mY_1(\beta_m\tilde{R})+BiY_0(\beta_m\tilde{R})\end{aligned}\quad\text{(R23B01T0-16)}$$

The non-dimensional heat flux is given by the negative of the first derivative of eq. (R23B01T0-13) with respect to $\tilde{r}$ which produces

$$\tilde{q}_{\text{R23B01T0}}(\tilde{r},\tilde{t})=-\frac{\pi^2}{2}Bi.\tilde{R}\sum_{m=1}^{\infty}e^{-\beta_m^2\tilde{t}}\frac{\beta_mJ_1^2(\beta_m)\tilde{N}_m(\tilde{R})\begin{pmatrix}S_0J_1(\beta_m\tilde{r})\\-V_0Y_1(\beta_m\tilde{r})\end{pmatrix}}{\left(Bi^2+\beta_m^2\right)J_1^2(\beta_m)-V_0^2}$$

(R23B01T0-17)

7. Dimensionless eigenvalues

The dimensionless eigenvalues are found from the eigencondition

$$S_0J_1(\beta_m)-V_0Y_1(\beta_m)=0\quad\text{(R23B01T0-18)}$$

The eigencondition for R23B10T0 and R23B01T0 is the same. So the same Matlab program feigR23 which is used for R23B10T0 case will be used for calculating R23B01T0 solution as well.

Maximum number of terms required in the summation is to get a solution accurate up to 10^{-A} is obtained by rounding off the above expression to the nearest integer.

$$m_{\max} = floor\left(\frac{1}{2}+\left(\frac{\tilde{R}-1}{\pi}\sqrt{\frac{A\ln(10)}{\tilde{t}}}\right)\right) \qquad \text{(R23B01T0-19)}$$

The number of terms increases linearly with the aspect ratio $\tilde{R}$ and inversely with the square root of time. As $\tilde{R}$ increases, the number of terms indicated by this equation becomes large and can even go to infinity. Notice that the number of terms is independent of location $\tilde{r}$. Although, a simpler expression is obtained for m_{max} by calculating in this way, the effect of Bi is being neglected.

8. Discussion

The variation of temperature and heat flux with respect to time and position for the radius ratio of 1.1 and 2 is shown in Tables 1 to 4 in Appendix C.

Intrinsic Verification:

One way to verify the correctness of the solution is to check if the boundary conditions are satisfied. The inner surface is insulated which means the flux values at the inner boundary must be equal to zero. This can be readily verified through Tables 2 and 4, that is the dimensionless heat flux values at the inner boundary are zeroes at all times. Notice that these are not manually placed there, but computed zeroes. Since the outer surface has the boundary condition of third kind, it cannot be verified as simply as first or second kind. Hence, we look at solutions for extreme cases.

The boundary condition of third kind turns into first kind as $Bi \to 0$ and R23B01T0 resolves into R22B00T0 case which results a zero temperature and flux solutions. Although, the temperature and heat flux solutions cannot be compared, eigenvalues can be compared. A comparison of eigenvalues of R23 case with Bi=0 with those of R22 case with same radius ratio is done in R23B10T0 case. It is observed that giving zero as an input for Biot number leads to division by zero in the computation of the solution which results an error in the solution. After rigorous testing of the program for various cases, it is justified to say that the program gives accurate solutions for $Bi > 5\times10^{-6}$.

Similarly, as $Bi \to \infty$, R23B01T0 resolves into R21B01T0 case. Practically, substituting $Bi = \infty$ is not possible and hence a Biot of 10^{-6} is chosen for comparison. A comparison of eigenvalues of R23 case with Bi=10^{-6} with those of R21 case with same radius ratio is shown in Table 5. The comparison of the temperature and heat flux solutions for both the cases is shown in Tables 6 and 7. It can be observed that at higher times, the solution varies slightly and this is because the Biot number is not exactly ∞. Also, It is observed that the number of eigenvalues required, calculated by eq. (R23B01T0-19), is not adequate for accurate heat flux solutions in case of larger Biot numbers. This can be attributed to the fact that the heat flux solution contains β_m term in the summation and as Biot numbers become larger, the eigenvalues also become large. In order to overcome this, a safety factor of 1.2 is introduced in calculating the required number of eigenvalues.

An even more powerful way to verify the accuracy of the solution is to use knowledge of the deviation time. Deviation time is the shortest time, t_{dev}, for heating at one boundary to cause a deviation in the temperature rise (to a specified level such as 10^{-A} of the surface temperature rise) at a specified interior location r caused by a homogeneous boundary condition at the other boundary.

Prof. de Monte gave an expression to calculate the deviation time in Cartesian coordinate system. [3, eq. (19)] The expression is modified to calculate deviation time in radial coordinate system.

The deviation time is calculated using the equation

$$\frac{\alpha t_{dev}}{\left(R_2 + r - 2R_1\right)^2} = \frac{1}{10A} \qquad \text{(R23B01T0-20)}$$

Let's calculate deviation time for an interior location of $\tilde{r} = 1.25$ for the R23B01T0 problem with a radius ration of 2. The dimensionless deviation time is calculated by

$$\frac{\tilde{t}_{dev}}{\left(\tilde{R} + \tilde{r} - 2\right)^2} = \frac{1}{10A} \qquad \text{(R23B01T0-21)}$$

$$\tilde{t}_{dev} = \frac{\left(\tilde{R} + \tilde{r} - 2\right)^2}{10A} = \frac{\left(2 + 1.25 - 2\right)^2}{10A}$$

$$\text{for A=10;}\ \ \tilde{t}_{dev} = \frac{1.5625}{100} = 0.015625 \qquad \text{(R23B01T0-22)}$$

In other words, for a specified interior location $\tilde{r}$, and $\tilde{t} < \tilde{t}_{dev}$, the effect of boundary condition at the inner surface is not felt. This implies that the solution should the same in that specified region and time for any kind of boundary condition at the inner surface. This is verified for one particular region, $\tilde{r} > 1.25$ and $\tilde{t} < 0.015625$ using two different boundary conditions (1st and 2nd) through Tables 8 and 9.

9. References

1. Cole, K.D., Beck, J.V., Haji-Sheikh, A. and Litkouhi, B., Heat Conduction Using Green's Functions, CRC Press, 2011, 2nd Edition.
2. The formulation used for calculating the eigenvalues in feigR21(num,R) is provided by Prof. Haji-Sheikh, University of Texas, Arlington.
3. de Monte, F., Beck, J. V., and Amos, D. E., Diffusion of thermal disturbances in two-dimensional Cartesian transient heat conduction, Int. J. Heat Mass Transfer, Vol. 51, No. 25-26, pp. 5931-5941, December 2008.
4. Satish Nallapaneni & James V. Beck, Hollow cylinder with jump in temperature at inner radius and convection at outer radius, Exact Analytical Conduction Toolbox, exact.unl.edu, March 27, 2014.
5. Satish Nallapaneni & James V. Beck, Hollow cylinder with jump in heat flux at inner boundary and convection at outer boundary, Exact Analytical Conduction Toolbox, exact.unl.edu, April 21, 2014
6. Satish Nallapaneni & James V. Beck, Hollow cylinder insulated at inner radius and heating through a step change in temperature at outer radius, Exact Analytical Conduction Toolbox, exact.unl.edu, June 26, 2014

Appendix A. MATLAB function feigR23(num,R,Bi) for finding the eigenvalues of R23B10T0 case.

```
% feigR23.m
% Author: Satish Nallapaneni
% Revision April 21, 2014
% R must  be greater than 1
% Bi must be greater than or equal to 0
% Eigen condition:
% f = S0*besselj(1,beta) - V0*bessely(1,beta) where,
% V0 = -beta*besselj(1,beta*R)+Bi*besselj(0,beta*R) and
% S0 = -beta*bessely(1,beta*R)+Bi*bessely(0,beta*R)
function BB=feigR23(num,R,Bi) %
    x=R-1;
if Bi>0
for j= 1:num
    eig21 = (j-1/2)*pi/x + ((0.267 - 0.007*x)/(1 + 0.5*x) +...
        (0.05 + 0.04*x*j^(2/3))/(x + 1))/j^1.667 + (j - 1)*0.06/j^2;
    n=j-1;
    if n>0
        eig22=n*pi/x + 0.058*x^0.25/n;
    else
        eig22=Bi/(100+Bi);
    end
    eigi=(eig22+1.25*Bi^0.6*eig21/j^1.5)/(1+1.25*Bi^0.6/j^1.5);
    eigi=roundn(((eigi*10^10)/10^10),-30);
    x0 = eigi;
    dx=pi/x/10;
    for pp=1:2
        UL = (x0 + dx);
        LL = (x0 - dx);
        f0=eigenfunction(x0,R,Bi);
        f1=eigenfunction(LL,R,Bi);
        f2=eigenfunction(UL,R,Bi);
        fp = ((f2 - f1)/2)/dx;
        % fp: first derivative using first order central difference
        fpp = ((f1 + f2 - 2*f0)/dx^2);
        % fpp:  second derivative using first order central difference
        h = (-f0/fp);
        eps=-fpp*h^2/2/(fp + h*fpp);
        x0 = x0 + h + eps; % Newton Raphson method for finding eigenvalues
        dx = h/5;
    end
    for n=1:3
        V0x=-x0*besselj(1,x0*R)+Bi*besselj(0,x0*R);
        S0x=-x0*bessely(1,x0*R)+Bi*bessely(0,x0*R);
        fx=(S0x)*besselj(1,x0)-(V0x)*bessely(1,x0);
        DV0=-R*(x0*besselj(0,x0*R)+Bi*besselj(1,x0*R));
        DS0=-R*(x0*bessely(0,x0*R)+Bi*bessely(1,x0*R));
        fpx=DS0*besselj(1,x0)+S0x*(besselj(0,x0)-besselj(1,x0)/x0)-...
            DV0*bessely(1,x0)-V0x*(bessely(0,x0)-bessely(1,x0)/x0);
```

```
        % fpx: actual first derivative of the eigencondition w.r.t. x0
        x0=x0-(fx/fpx); % Newton Raphson method for finding eigenvalues
    end
    eig(j)=x0;
end
for ii=1:num
    index(ii)=ii; x0=eig(ii); Bv(ii)=Bi; Ra(ii)=R;
    fz(ii)=eigenfunction(x0,R,Bi);
    n(ii)=x0/((ii-0.5)*pi/x);
end
else
for ii= 1:num
    x0=1/(R-1)*(ii+.01)*pi;
    for it=1:5
        fxn=besselj(1,x0)*bessely(1,x0*R)-besselj(1,x0*R)*bessely(1,x0);% =0
for eigencondition
        f1=(besselj(0,x0)-besselj(2,x0))*bessely(1,x0*R);
        f2=R*besselj(1,x0)*(bessely(0,x0*R)-bessely(2,x0*R));
        f2=f2-R*bessely(1,x0)*(besselj(0,x0*R)-besselj(2,x0*R));
        f3=-(bessely(0,x0)-bessely(2,x0))*besselj(1,x0*R);% First derivative
        fpxn=(f1+f2+f3)/2;
        delt=fxn/fpxn; x0=x0-delt;%Newton Raphson method for finding
eigenvalues
    end%it
eig(ii)=x0;
fz(ii)=besselj(1,x0)*bessely(1,x0*R)-besselj(1,x0*R)*bessely(1,x0);
index(ii)=ii;Bv(ii)=0; Ra(ii)=R;
n(ii)=eig(ii)/pi*(R-1)/ii;
end%ii
end %if Bi>0
sprintf(' index      R        Biot#         beta          m         Root Value')
BBB=[index' Ra' Bv'  eig' n' fz'];
fprintf('%5.0f  %10.5f   %5.3f    %12.10f  %2.5f   %12.5e\n',BBB')

BB=eig;

% function that calculates eigencondition for a given eigenvalue
function f=eigenfunction(beta,R,Bi)
S0 = -beta*bessely(1, beta*R) + Bi*bessely(0, beta*R);
V0 = -beta*besselj(1, beta*R) + Bi*besselj(0, beta*R);
f = (S0*besselj(1, beta) - V0*bessely(1, beta));
```

Appendix B. MATLAB function fdR23B01T0.

- **fdR23B01T0**

  Heat conduction function for the R23B01T0 case.

- **Syntax**

```
[Td,qd] = fdR23B01T0(rv,tv,R,Bi,A)
```

- **Description**

fdR23B01T0(*rv,tv,R,Bi,A*) returns the dimensionless temperature *Td* and heat flux *qd* solutions at a given dimensionless location $\tilde{r}$ from the center, between 1 and $\tilde{R}$, and at a given dimensionless time $\tilde{t}$, with an accuracy of 10^{-A} (*A = 2,3,…,15*), for the R23B01T0 problem, where *Bi* is the Biot number.

If *rv* and *tv* are not single values but arrays (length(*rv*) = *n* and length(*tv*) = *m*) defining the dimensionless locations and times of interest, respectively, the above function returns the dimensionless temperature *Td* and heat flus *qd* as double scripted arrays, where size (*Td*) = size(*qd*)=[*m, n*].

- **Examples**

Example 1

```
>> [Td, qd] = fdR23B01T0(1.5,0.1,2,10,15)

Td =

   0.224299308813856


qd =

   0.797065231874290
```

Example 2

```
A =

    15
```

```
>> rv=[1.15 1.5 1.75]'

rv =

   1.150000000000000
   1.500000000000000
   1.750000000000000

>> tv=[0.01 0.1 0.3]'

tv =

   0.010000000000000
   0.100000000000000
   0.300000000000000

>> R=2

R =

     2

>> [Td, qd] = fdR23B01T0(rv,tv,R,10,A)

Td =

   0.000000000450283   0.000124631932252   0.030929537508239
   0.057510101009556   0.224299308813856   0.486910478216232
   0.415631508407497   0.565323210875686   0.735053179208392


qd =

   0.000000019843602   0.003445789550221   0.514245894590471
   0.198887334941900   0.797065231874290   1.289060361387466
   0.221728529493741   0.600033032841435   0.738190871841669
```

```
% fdR23B01T0 function
% Author: Satish Nallapaneni
% Revision June 27, 2014
% INPUTS:
% R: radius ratio R2/R1
% rv: dimensionless location starting at rd=r/R1=1 and ending at rd=R2/R1=R
% tv: dimensionless time starting at td=0
% A = desired accuracy (1E-A =10^-A); A=2,3, ..., 15
% Bi: Biot number
% OUTPUTS:
% Td: dimensionless temperature calculated at (xd,td) to desired accuracy A
% qd: dimensionless heat flux calculated at (xd,td) to desired accuracy A
function [Td,qd]=fdR23B01T0(rv,tv,R,Bi,A)
   srv=length(rv);
   stv=length(tv);
   Temp=zeros(stv,srv);      % Preallocating arrays for speed
   flux=zeros(stv,srv);      % Preallocating arrays for speed
 % calculate number of eigenvalues required to obtain solution with accuracy
A
   mmax1=floor(2*(0.5+(R-1)/pi*sqrt(A*log(10)/min(tv))));
   bet=feigR23(mmax1,R,Bi);% Call the function to get eigenvalues
for ir = 1:srv    % begin space loop
    r=rv(ir);
    term1=1;  % steady-state temperature solution
    term2=0;              % steady-state heat flux solution
    for it=1:stv          % begin time loop
        t=tv(it);
      % calculate m_max for every timestep as it reduces computational cost
        mmax=floor(1.2*(0.5+(R-1)/pi*sqrt(A*log(10)/t)));
        term3 = 0;
        term4 = 0;
        for ii=1:mmax
            bt=bet(ii);
            V0=-bt*besselj(1,bt*R)+Bi*besselj(0,bt*R);
            S0=-bt*bessely(1,bt*R)+Bi*bessely(0,bt*R);
            Nr = (S0*besselj(0,bt*r)-V0*bessely(0,bt*r));
            L1 = (S0*besselj(0,bt*R)-V0*bessely(0,bt*R));
            Denominator = (Bi^2+bt^2)*besselj(1,bt)*besselj(1,bt)-V0^2;
            Norm = Denominator/(besselj(1,bt)*besselj(1,bt));
            term3 = term3 - (pi*pi/2)*Bi*R*(exp(-bt*bt*t))*Nr*L1/Norm;
            Nm = (S0*besselj(1,bt*r)-V0*bessely(1,bt*r));
            term4 = term4 - (pi*pi/2)*Bi*R*exp(-bt*bt*t)*bt*Nm*L1/Norm;
        end
        T(ir)=term1;
        Q(ir)=term2;
        Temp(it,ir) = term1 + term3;  % total temperature solution
        flux(it,ir) = term2 + term4;  % total heat flux solution
    end
end
Td=abs(Temp); qd=abs(flux);
```

Appendix C. Tables and plots of dimensionless temperatures and heat fluxes

Table 1. Temperatures as a function of time and position for R23B01T0 case for R_2/R_1 = 1.1 and Bi=10.

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1,\tilde{t})$	$\tilde{T}(1.025,\tilde{t})$	$\tilde{T}(1.05,\tilde{t})$	$\tilde{T}(1.075,\tilde{t})$	$\tilde{T}(\tilde{R},\tilde{t})$
1.1000	0.00001	0.0000000000	0.0000000000	0.0000000000	0.0000000002	0.0347492083
1.1000	0.00005	0.0000000000	0.0000000000	0.0000000108	0.0003941871	0.0752478680
1.1000	0.00010	0.0000000000	0.0000000029	0.0000142341	0.0042209772	0.1039374584
1.1000	0.00020	0.0000000433	0.0000083523	0.0007760682	0.0189419246	0.1422657239
1.1000	0.00030	0.0000048248	0.0001466117	0.0035306781	0.0358843533	0.1700137674
1.1000	0.00040	0.0000563940	0.0006748081	0.0081584500	0.0524632044	0.1923587722
1.1000	0.00050	0.0002610635	0.0017795784	0.0140933612	0.0681174472	0.2112935478
1.1000	0.00060	0.0007518843	0.0035188922	0.0208571039	0.0827728896	0.2278341286
1.1000	0.00070	0.0016407181	0.0058733646	0.0281175786	0.0964856309	0.2425806664
1.1000	0.00080	0.0029986714	0.0087913972	0.0356559859	0.1093434839	0.2559220656
1.1000	0.00090	0.0048576006	0.0122124904	0.0433309795	0.1214360912	0.2681271138
1.1000	0.00100	0.0072193680	0.0160771169	0.0510524310	0.1328460691	0.2793907290
1.1000	0.00200	0.0516259016	0.0694887520	0.1249351574	0.2219849231	0.3616720532
1.1000	0.00300	0.1129965593	0.1327245350	0.1913603052	0.2875467261	0.4178991851
1.1000	0.00400	0.1762559253	0.1955035989	0.2520318946	0.3430102242	0.4639727113
1.1000	0.00500	0.2367978275	0.2549161657	0.3079284083	0.3927285620	0.5047719611
1.1000	0.00600	0.2934424311	0.3103040569	0.3595787674	0.4382419331	0.5419606142
1.1000	0.00700	0.3460532952	0.3616865705	0.4073530804	0.4802073536	0.5762003087
1.1000	0.00800	0.3947993277	0.4092756734	0.4515569323	0.5189955126	0.6078321804
1.1000	0.00900	0.4399280194	0.4533274839	0.4924617116	0.5548761426	0.6370881628
1.1000	0.01000	0.4816965680	0.4940975402	0.5303150432	0.5880762137	0.6641570183
1.1000	0.02000	0.7612373178	0.7669501590	0.7836345992	0.8102433035	0.8452908184
1.1000	0.03000	0.8900113789	0.8926430612	0.9003289297	0.9125865181	0.9287315363
1.1000	0.04000	0.9493325478	0.9505448609	0.9540854396	0.9597320308	0.9671694086
1.1000	0.05000	0.9766594882	0.9772179534	0.9788489594	0.9814501229	0.9848762317
1.1000	0.06000	0.9892479399	0.9895052029	0.9902565436	0.9914547978	0.9930330720
1.1000	0.07000	0.9950469468	0.9951654578	0.9955115710	0.9960635598	0.9967906090
1.1000	0.08000	0.9977183223	0.9977729157	0.9979323565	0.9981866361	0.9985215592
1.1000	0.09000	0.9989489204	0.9989740694	0.9990475176	0.9991646542	0.9993189402
1.1000	0.10000	0.9995158088	0.9995273939	0.9995612286	0.9996151889	0.9996862624
1.1000	steady state	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000

Table 2. Heat Fluxes as a function of time and position for R21B01T0 case for R_2/R_1 = 1.1 and Bi=10.

$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1,\tilde{t})$	$\tilde{q}(1.025,\tilde{t})$	$\tilde{q}(1.05,\tilde{t})$	$\tilde{q}(1.075,\tilde{t})$	$\tilde{q}(\tilde{R},\tilde{t})$
1.1000	0.00001	0.0000000000	0.0000000000	0.0000000000	0.0000002277	9.6525079168
1.1000	0.00005	0.0000000000	0.0000000000	0.0000057602	0.1216841460	9.2475213197
1.1000	0.00010	0.0000000000	0.0000011492	0.0040227170	0.7376695529	8.9606254155
1.1000	0.00020	0.0000000000	0.0017481663	0.1193428398	1.9478582298	8.5773427610
1.1000	0.00030	0.0000000000	0.0213141446	0.3866073573	2.7507655077	8.2998623255
1.1000	0.00040	0.0000000000	0.0761225425	0.7074283017	3.2861168367	8.0764122784
1.1000	0.00050	0.0000000000	0.1648933327	1.0240806770	3.6598821600	7.8870645218
1.1000	0.00060	0.0000000000	0.2767858138	1.3151626513	3.9309017529	7.7216587137
1.1000	0.00070	0.0000000000	0.4004089964	1.5750534282	4.1330267671	7.5741933356
1.1000	0.00080	0.0000000000	0.5269054828	1.8041163203	4.2868595005	7.4407793440
1.1000	0.00090	0.0000000000	0.6502675860	2.0047374349	4.4055709529	7.3187288622
1.1000	0.00100	0.0000000000	0.7668055707	2.1798013216	4.4979252926	7.2060927100
1.1000	0.00200	0.0000000000	1.4413097936	3.0252458705	4.7489773641	6.3832794684
1.1000	0.00300	0.0000000000	1.5729140073	3.1100642756	4.5631727273	5.8210081487
1.1000	0.00400	0.0000000000	1.5296695511	2.9739918756	4.2746553011	5.3602728869
1.1000	0.00500	0.0000000000	1.4384666798	2.7818043126	3.9718460175	4.9522803892
1.1000	0.00600	0.0000000000	1.3382519543	2.5834817631	3.6805621664	4.5803938580
1.1000	0.00700	0.0000000000	1.2406264244	2.3936276529	3.4075867812	4.2379969127
1.1000	0.00800	0.0000000000	1.1487729883	2.2159800567	3.1539151486	3.9216781960
1.1000	0.00900	0.0000000000	1.0633040053	2.0509783857	2.9188368035	3.6291183717
1.1000	0.01000	0.0000000000	0.9840654499	1.8980964382	2.7011902916	3.3584298173
1.1000	0.02000	0.0000000000	0.4533352850	0.8743979528	1.2443430142	1.5470918156
1.1000	0.03000	0.0000000000	0.2088338196	0.4028009078	0.5732201158	0.7126846372
1.1000	0.04000	0.0000000000	0.0962015657	0.1855546103	0.2640600682	0.3283059137
1.1000	0.05000	0.0000000000	0.0443162954	0.0854777453	0.1216421365	0.1512376826
1.1000	0.06000	0.0000000000	0.0204147825	0.0393762511	0.0560357705	0.0696692801
1.1000	0.07000	0.0000000000	0.0094042911	0.0181390975	0.0258134859	0.0320939101
1.1000	0.08000	0.0000000000	0.0043321888	0.0083559721	0.0118912625	0.0147844080
1.1000	0.09000	0.0000000000	0.0019956698	0.0038492693	0.0054778391	0.0068105980
1.1000	0.10000	0.0000000000	0.0009193269	0.0017732077	0.0025234261	0.0031373759
1.1000	steady state	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 3. Temperatures as a function of time and position for R23B01T0 case for R_2/R_1 = 2 and Bi=10.

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1,\tilde{t})$	$\tilde{T}(1.25,\tilde{t})$	$\tilde{T}(1.5,\tilde{t})$	$\tilde{T}(1.75,\tilde{t})$	$\tilde{T}(\tilde{R},\tilde{t})$
2.0000	0.00001	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0347296483
2.0000	0.00010	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.1037594713
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000000	0.0000000017	0.2780427693
2.0000	0.00200	0.0000000000	0.0000000000	0.0000000000	0.0000104807	0.3589705953
2.0000	0.00300	0.0000000000	0.0000000000	0.0000000000	0.0002322598	0.4116606305
2.0000	0.00400	0.0000000000	0.0000000000	0.0000000035	0.0011877947	0.4508536217
2.0000	0.00500	0.0000000000	0.0000000000	0.0000001049	0.0033071125	0.4819842305
2.0000	0.00600	0.0000000000	0.0000000000	0.0000010583	0.0067269773	0.5077209972
2.0000	0.00700	0.0000000000	0.0000000000	0.0000056540	0.0113757094	0.5295870971
2.0000	0.00800	0.0000000000	0.0000000007	0.0000202252	0.0170860782	0.5485386504
2.0000	0.00900	0.0000000000	0.0000000054	0.0000552305	0.0236653885	0.5652168541
2.0000	0.01000	0.0000000000	0.0000000291	0.0001246319	0.0309295375	0.5800732584
2.0000	0.02000	0.0000004393	0.0000735871	0.0057998856	0.1163506543	0.6744468328
2.0000	0.03000	0.0000441868	0.0011614823	0.0234958592	0.1955718492	0.7251760221
2.0000	0.04000	0.0004732757	0.0049012899	0.0495505149	0.2611693015	0.7584545453
2.0000	0.05000	0.0020308210	0.0120141136	0.0793971025	0.3153093199	0.7825482552
2.0000	0.06000	0.0054697681	0.0223079735	0.1102543628	0.3606098775	0.8010732044
2.0000	0.07000	0.0112411626	0.0352395503	0.1406741344	0.3991109817	0.8159101074
2.0000	0.08000	0.0194611078	0.0502309562	0.1699703344	0.4323091180	0.8281511247
2.0000	0.09000	0.0300055728	0.0667781489	0.1978664099	0.4613025890	0.8384816113
2.0000	0.10000	0.0426164190	0.0844698464	0.2242993088	0.4869104782	0.8473574513
2.0000	0.20000	0.2215248201	0.2768017659	0.4268486447	0.6458777160	0.8982097296
2.0000	0.30000	0.3985652212	0.4443561500	0.5653232109	0.7350531792	0.9243711631
2.0000	0.40000	0.5397911077	0.5752754683	0.6685654131	0.7985129290	0.9425641703
2.0000	0.50000	0.6484971721	0.6756646893	0.7470245711	0.8462877282	0.9561943344
2.0000	0.60000	0.7316192598	0.7523717751	0.8068721975	0.8826635138	0.9665625608
2.0000	0.70000	0.7950985860	0.8109439509	0.8525558011	0.9104206401	0.9744727686
2.0000	0.80000	0.8435653204	0.8556628743	0.8874323180	0.9316100233	0.9805110959
2.0000	0.90000	0.8805681768	0.8898042205	0.9140590092	0.9477869821	0.9851210047
2.0000	1.00000	0.9088184679	0.9158698308	0.9343874216	0.9601374123	0.9886404724
2.0000	steady state	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000

Table 4. Heat Fluxes as a function of time and position for R23B01T0 case for R_2/R_1 = 2 and Bi=10.

$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1,\tilde{t})$	$\tilde{q}(1.25,\tilde{t})$	$\tilde{q}(1.5,\tilde{t})$	$\tilde{q}(1.75,\tilde{t})$	$\tilde{q}(\tilde{R},\tilde{t})$
2.0000	0.00001	0.0000000000	0.0000000000	0.0000000000	0.0000000000	9.6527035173
2.0000	0.00010	0.0000000000	0.0000000000	0.0000000000	0.0000000000	8.9624052874
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000000	0.0000002254	7.2195723065
2.0000	0.00200	0.0000000000	0.0000000000	0.0000000000	0.0007205130	6.4102940474
2.0000	0.00300	0.0000000000	0.0000000000	0.0000000011	0.0110221270	5.8833936948
2.0000	0.00400	0.0000000000	0.0000000000	0.0000002271	0.0435612807	5.4914637827
2.0000	0.00500	0.0000000000	0.0000000000	0.0000055645	0.0996169022	5.1801576954
2.0000	0.00600	0.0000000000	0.0000000001	0.0000472095	0.1728791272	4.9227900282
2.0000	0.00700	0.0000000000	0.0000000025	0.0002180952	0.2559786351	4.7041290288
2.0000	0.00800	0.0000000000	0.0000000319	0.0006883309	0.3430287608	4.5146134958
2.0000	0.00900	0.0000000000	0.0000002329	0.0016840197	0.4299723369	4.3478314587
2.0000	0.01000	0.0000000000	0.0000011444	0.0034457896	0.5142458946	4.1992674161
2.0000	0.02000	0.0000000000	0.0014938325	0.0851148612	1.0931451389	3.2555316723
2.0000	0.03000	0.0000000000	0.0161035256	0.2400198659	1.3276540635	2.7482397792
2.0000	0.04000	0.0000000000	0.0518700030	0.3927425635	1.4123340920	2.4154545470
2.0000	0.05000	0.0000000000	0.1027601297	0.5177080587	1.4314319051	2.1745174481
2.0000	0.06000	0.0000000000	0.1593834048	0.6133805233	1.4200560624	1.9892679557
2.0000	0.07000	0.0000000000	0.2147274631	0.6844043901	1.3941112631	1.8408989262
2.0000	0.08000	0.0000000000	0.2647875926	0.7359594614	1.3613231370	1.7184887527
2.0000	0.09000	0.0000000000	0.3077781703	0.7723879584	1.3256232049	1.6151838874
2.0000	0.10000	0.0000000000	0.3432813174	0.7970652319	1.2890603614	1.5264254865
2.0000	0.20000	0.0000000000	0.4259858041	0.7579435371	0.9704040048	1.0179027040
2.0000	0.30000	0.0000000000	0.3501599800	0.6000330328	0.7381908718	0.7562883690
2.0000	0.40000	0.0000000000	0.2709780842	0.4612189044	0.5631861217	0.5743582967
2.0000	0.50000	0.0000000000	0.2074131266	0.3525775235	0.4299130791	0.4380566556
2.0000	0.60000	0.0000000000	0.1584294446	0.2692455377	0.3282134183	0.3343743917
2.0000	0.70000	0.0000000000	0.1209660113	0.2055681199	0.2505768731	0.2552723139
2.0000	0.80000	0.0000000000	0.0923544685	0.1569446274	0.1913054555	0.1948890409
2.0000	0.90000	0.0000000000	0.0705092631	0.1198212861	0.1460542009	0.1487899529
2.0000	1.00000	0.0000000000	0.0538310976	0.0914788919	0.1115066615	0.1135952756
2.0000	steady state	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 5. Comparison of eigenvalues of R23 case for Bi=10^{-6} with those of R22 case for R_2/R_1=2.

m	$\tilde{R}$	β_m **(R21)**	$\frac{\beta_m(\tilde{R}-1)}{(m-0.5)\pi}$	Root Value
1	2.00000	3.1965783808	1.01750	1.38778e-17
2	2.00000	6.3123495104	1.00464	-2.08167e-17
3	2.00000	9.4444649255	1.00209	-3.12250e-17
4	2.00000	12.5812028101	1.00118	-2.08167e-17
5	2.00000	15.7198542694	1.00076	2.42861e-17
6	2.00000	18.8594766201	1.00053	2.94903e-17
7	2.00000	21.9996580212	1.00039	2.25514e-17
8	2.00000	25.1401904069	1.00030	-1.38778e-17
9	2.00000	28.2809574583	1.00023	-2.77556e-17
10	2.00000	31.4218890982	1.00019	-1.73472e-17
11	2.00000	34.5629406055	1.00016	0.00000e+00
12	2.00000	37.7040821059	1.00013	-6.93889e-18
13	2.00000	40.8452928854	1.00011	1.73472e-18
14	2.00000	43.9865581315	1.00010	3.46945e-17
15	2.00000	47.1278669717	1.00008	-2.42861e-17

m	$\tilde{R}$	*Bi*	β_m **(R23)**	$\frac{\beta_m(\tilde{R}-1)}{(m-0.5)\pi}$	Root Value
1	2.00000	1.0e-06	0.0011547004	0.00074	4.91789e-14
2	2.00000	1.0e-06	3.1965786862	0.67834	-1.94289e-16
3	2.00000	1.0e-06	6.3123496677	0.80371	-2.77556e-17
4	2.00000	1.0e-06	9.4444650310	0.85893	-1.38778e-16
5	2.00000	1.0e-06	12.5812028894	0.88994	2.77556e-17
6	2.00000	1.0e-06	15.7198543330	0.90978	-1.94289e-16
7	2.00000	1.0e-06	18.8594766731	0.92356	4.44089e-16
8	2.00000	1.0e-06	21.9996580666	0.93369	-1.66533e-16
9	2.00000	1.0e-06	25.1401904466	0.94146	4.44089e-16
10	2.00000	1.0e-06	28.2809574937	0.94759	-1.38778e-16
11	2.00000	1.0e-06	31.4218891300	0.95256	-1.66533e-16
12	2.00000	1.0e-06	34.5629406344	0.95667	1.05471e-15
13	2.00000	1.0e-06	37.7040821324	0.96013	6.66134e-16
14	2.00000	1.0e-06	40.8452929099	0.96307	1.11022e-16
15	2.00000	1.0e-06	43.9865581543	0.96561	-1.38778e-15

Note: In case of third kind boundary condition, the first eigenvalue is closer to zero and hence the jump in index.

Table 6. Comparison of temperature solution R23B01T0 as $Bi \to \infty$ with R21B01T0 for $\tilde{R} = 2$

R23B01T0 temperature solution for *Bi=5x10*^{5}

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1,\tilde{t})$	$\tilde{T}(1.25,\tilde{t})$	$\tilde{T}(1.5,\tilde{t})$	$\tilde{T}(1.75,\tilde{t})$	$\tilde{T}(\mathrm{R},\tilde{t})$
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000000	0.0000000242	0.9999648197
2.0000	0.01000	0.0000000000	0.0000001440	0.0004702351	0.0824637237	0.9999892236
2.0000	0.02000	0.0000015956	0.0002240834	0.0143602176	0.2260874162	0.9999925318
2.0000	0.04000	0.0011168566	0.0101787852	0.0892506963	0.4033715426	0.9999948736
2.0000	0.06000	0.0105490804	0.0389838887	0.1725725490	0.5039753654	0.9999959129
2.0000	0.08000	0.0332737661	0.0794267336	0.2452600742	0.5701072411	0.9999965336
2.0000	0.10000	0.0671914360	0.1250448975	0.3067731373	0.6177893645	0.9999969582
2.0000	0.20000	0.2885940539	0.3511594312	0.5170028098	0.7487255211	0.9999980459
2.0000	0.40000	0.6233236924	0.6579393830	0.7478807010	0.8700818564	0.9999989936
2.0000	0.60000	0.8020843105	0.8202869866	0.8675655273	0.9317683877	0.9999994715
2.0000	0.80000	0.8960249215	0.9055878497	0.9304258565	0.9641548040	0.9999997223
2.0000	1.00000	0.9453768047	0.9504006802	0.9634493023	0.9811687680	0.9999998541
2.0000	2.00000	0.9978141709	0.9980152088	0.9985373690	0.9992464400	0.9999999942
2.0000	steady state	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000

R21B01T0 temperature solution

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1,\tilde{t})$	$\tilde{T}(1.25,\tilde{t})$	$\tilde{T}(1.5,\tilde{t})$	$\tilde{T}(1.75,\tilde{t})$	$\tilde{T}(\mathrm{R},\tilde{t})$
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000000	0.0000000243	1.0000000000
2.0000	0.01000	0.0000000000	0.0000001440	0.0004702601	0.0824662121	1.0000000000
2.0000	0.02000	0.0000015957	0.0002240922	0.0143606158	0.2260912112	1.0000000000
2.0000	0.04000	0.0011168859	0.0101789934	0.0892520205	0.4033754256	1.0000000000
2.0000	0.06000	0.0105492680	0.0389844392	0.1725743462	0.5039789121	1.0000000000
2.0000	0.08000	0.0332742164	0.0794276039	0.2452620726	0.5701104666	1.0000000000
2.0000	0.10000	0.0671921726	0.1250460292	0.3067752126	0.6177923208	1.0000000000
2.0000	0.20000	0.2885957043	0.3511612077	0.5170048342	0.7487276709	1.0000000000
2.0000	0.40000	0.6233254969	0.6579411467	0.7478823352	0.8700832466	1.0000000000
2.0000	0.60000	0.8020857239	0.8202883354	0.8675666961	0.9317692773	1.0000000000
2.0000	0.80000	0.8960259080	0.9055887798	0.9304266338	0.9641553555	1.0000000000
2.0000	1.00000	0.9453774511	0.9504012852	0.9634497964	0.9811691019	1.0000000000
2.0000	2.00000	0.9978142224	0.9980152563	0.9985374059	0.9992464622	1.0000000000
2.0000	steady state	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000

Table 7. Comparison of heat flux solution R23B01T0 as $Bi \to \infty$ with R21B01T0 for $\tilde{R} = 2$

R23B01T0 heat flux solution for *Bi=5x10^5*

$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1,\tilde{t})$	$\tilde{q}(1.25,\tilde{t})$	$\tilde{q}(1.5,\tilde{t})$	$\tilde{q}(1.75,\tilde{t})$	$\tilde{q}(\tilde{R},\tilde{t})$
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000000	0.0000031155	17.5901279688
2.0000	0.01000	0.0000000000	0.0000055219	0.0124280205	1.2412291289	5.3882091738
2.0000	0.02000	0.0000000000	0.0043779897	0.1978648921	1.8891811725	3.7340986911
2.0000	0.04000	0.0000000000	0.1021420956	0.6544044264	1.9264409562	2.5631878525
2.0000	0.06000	0.0000000000	0.2611659701	0.8830656170	1.7542076769	2.0435582077
2.0000	0.08000	0.0000000000	0.3882730458	0.9737235549	1.5905000563	1.7332157114
2.0000	0.10000	0.0000000000	0.4663895684	0.9967565717	1.4524059702	1.5208883298
2.0000	0.20000	0.0000000000	0.4791111636	0.8230857602	0.9993637938	0.9770351616
2.0000	0.40000	0.0000000000	0.2635519390	0.4404917563	0.5205434870	0.5032105088
2.0000	0.60000	0.0000000000	0.1385744787	0.2314921252	0.2734225197	0.2642610633
2.0000	0.80000	0.0000000000	0.0728011187	0.1216149290	0.1436417768	0.1388282636
2.0000	1.00000	0.0000000000	0.0382459987	0.0638902762	0.0754620441	0.0729332648
2.0000	2.00000	0.0000000000	0.0015304710	0.0025566653	0.0030197269	0.0029185340
2.0000	steady state	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

R21B01T0 heat flux solution

$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1,\tilde{t})$	$\tilde{q}(1.25,\tilde{t})$	$\tilde{q}(1.5,\tilde{t})$	$\tilde{q}(1.75,\tilde{t})$	$\tilde{q}(\tilde{R},\tilde{t})$
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000000	0.0000031163	17.5901101621
2.0000	0.01000	0.0000000000	0.0000055223	0.0124286352	1.2412594026	5.3882035292
2.0000	0.02000	0.0000000000	0.0043781515	0.1978697242	1.8892035216	3.7340946964
2.0000	0.04000	0.0000000000	0.1021439483	0.6544121917	1.9264515566	2.5631850231
2.0000	0.06000	0.0000000000	0.2611690375	0.8830723740	1.7542135796	2.0435558934
2.0000	0.08000	0.0000000000	0.3882763188	0.9737289200	1.5905036796	1.7332137027
2.0000	0.10000	0.0000000000	0.4663925350	0.9967607387	1.4524083061	1.5208865245
2.0000	0.20000	0.0000000000	0.4791120620	0.8230866723	0.9993637459	0.9770336044
2.0000	0.40000	0.0000000000	0.2635516107	0.4404910298	0.5205422432	0.5032086795
2.0000	0.60000	0.0000000000	0.1385739772	0.2314911972	0.2734212255	0.2642594857
2.0000	0.80000	0.0000000000	0.0728006843	0.1216141561	0.1436407599	0.1388271092
2.0000	1.00000	0.0000000000	0.0382456808	0.0638897203	0.0754613328	0.0729324871
2.0000	2.00000	0.0000000000	0.0015304404	0.0025566130	0.0030196630	0.0029184687
2.0000	steady state	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 8. Comparison of R13B01T0 and R23B01T0 temperature solutions for $\tilde{R}=2$ with Bi=10 in the region $\tilde{r}>1.25$ and $\tilde{t}<0.015625$

R13B01T0:

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1.25,\tilde{t})$	$\tilde{T}(1.5,\tilde{t})$	$\tilde{T}(1.75,\tilde{t})$	$\tilde{T}(2,\tilde{t})$
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000017	0.2780427693
2.0000	0.00200	0.0000000000	0.0000000000	0.0000104807	0.3589705953
2.0000	0.00400	0.0000000000	0.0000000035	0.0011877947	0.4508536217
2.0000	0.00600	0.0000000000	0.0000010583	0.0067269773	0.5077209972
2.0000	0.00800	0.0000000007	0.0000202252	0.0170860782	0.5485386504
2.0000	0.01000	0.0000000291	0.0001246319	0.0309295375	0.5800732584
2.0000	0.01100	0.0000001175	0.0002445118	0.0387164833	0.5934380897
2.0000	0.01200	0.0000003787	0.0004315120	0.0468895384	0.6055602222
2.0000	0.01300	0.0000010259	0.0007014940	0.0553361525	0.6166318846
2.0000	0.01400	0.0000024231	0.0010685850	0.0639649978	0.6268046235
2.0000	0.01500	0.0000051267	0.0015446173	0.0727026732	0.6362000021

R23B01T0:

$\tilde{R}$	$\tilde{t}$	$\tilde{T}(1.25,\tilde{t})$	$\tilde{T}(1.5,\tilde{t})$	$\tilde{T}(1.75,\tilde{t})$	$\tilde{T}(2,\tilde{t})$
2.0000	0.00100	0.0000000000	0.0000000000	0.0000000017	0.2780427693
2.0000	0.00200	0.0000000000	0.0000000000	0.0000104807	0.3589705953
2.0000	0.00400	0.0000000000	0.0000000035	0.0011877947	0.4508536217
2.0000	0.00600	0.0000000000	0.0000010583	0.0067269773	0.5077209972
2.0000	0.00800	0.0000000007	0.0000202252	0.0170860782	0.5485386504
2.0000	0.01000	0.0000000291	0.0001246319	0.0309295375	0.5800732584
2.0000	0.01100	0.0000001175	0.0002445118	0.0387164833	0.5934380897
2.0000	0.01200	0.0000003787	0.0004315120	0.0468895384	0.6055602222
2.0000	0.01300	0.0000010259	0.0007014940	0.0553361525	0.6166318846
2.0000	0.01400	0.0000024231	0.0010685850	0.0639649978	0.6268046235
2.0000	0.01500	0.0000051267	0.0015446173	0.0727026732	0.6362000021

Table 9. Comparison of R13B01T0 and R23B01T0 heat flux solutions for $\tilde{R}=2$ with Bi=10 in the region $\tilde{r}<1.75$ and $\tilde{t}<0.030625$

R13B01T0:

$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1.25,\tilde{t})$	$\tilde{q}(1.5,\tilde{t})$	$\tilde{q}(1.75,\tilde{t})$	$\tilde{q}(2,\tilde{t})$
2.0000	0.00100	0.0000000000	0.0000000000	0.0000002254	7.2195723065
2.0000	0.00200	0.0000000000	0.0000000000	0.0007205130	6.4102940474
2.0000	0.00400	0.0000000000	0.0000002271	0.0435612807	5.4914637827
2.0000	0.00600	0.0000000001	0.0000472095	0.1728791272	4.9227900282
2.0000	0.00800	0.0000000319	0.0006883309	0.3430287608	4.5146134958
2.0000	0.01000	0.0000011444	0.0034457896	0.5142458946	4.1992674161
2.0000	0.01100	0.0000042145	0.0061896041	0.5943525919	4.0656191033
2.0000	0.01200	0.0000124968	0.0100815129	0.6695109828	3.9443977782
2.0000	0.01300	0.0000313596	0.0152274654	0.7394016199	3.8336811542
2.0000	0.01400	0.0000690131	0.0216743555	0.8039942544	3.7319537655
2.0000	0.01500	0.0001367210	0.0294176133	0.8634333163	3.6379999791

R23B01T0:

$\tilde{R}$	$\tilde{t}$	$\tilde{q}(1.25,\tilde{t})$	$\tilde{q}(1.5,\tilde{t})$	$\tilde{q}(1.75,\tilde{t})$	$\tilde{q}(2,\tilde{t})$
2.0000	0.00100	0.0000000000	0.0000000000	0.0000002254	7.2195723065
2.0000	0.00200	0.0000000000	0.0000000000	0.0007205130	6.4102940474
2.0000	0.00400	0.0000000000	0.0000002271	0.0435612807	5.4914637827
2.0000	0.00600	0.0000000001	0.0000472095	0.1728791272	4.9227900282
2.0000	0.00800	0.0000000319	0.0006883309	0.3430287608	4.5146134958
2.0000	0.01000	0.0000011444	0.0034457896	0.5142458946	4.1992674161
2.0000	0.01100	0.0000042145	0.0061896041	0.5943525919	4.0656191033
2.0000	0.01200	0.0000124968	0.0100815129	0.6695109828	3.9443977782
2.0000	0.01300	0.0000313596	0.0152274654	0.7394016199	3.8336811542
2.0000	0.01400	0.0000690131	0.0216743555	0.8039942544	3.7319537655
2.0000	0.01500	0.0001367210	0.0294176133	0.8634333163	3.6379999791

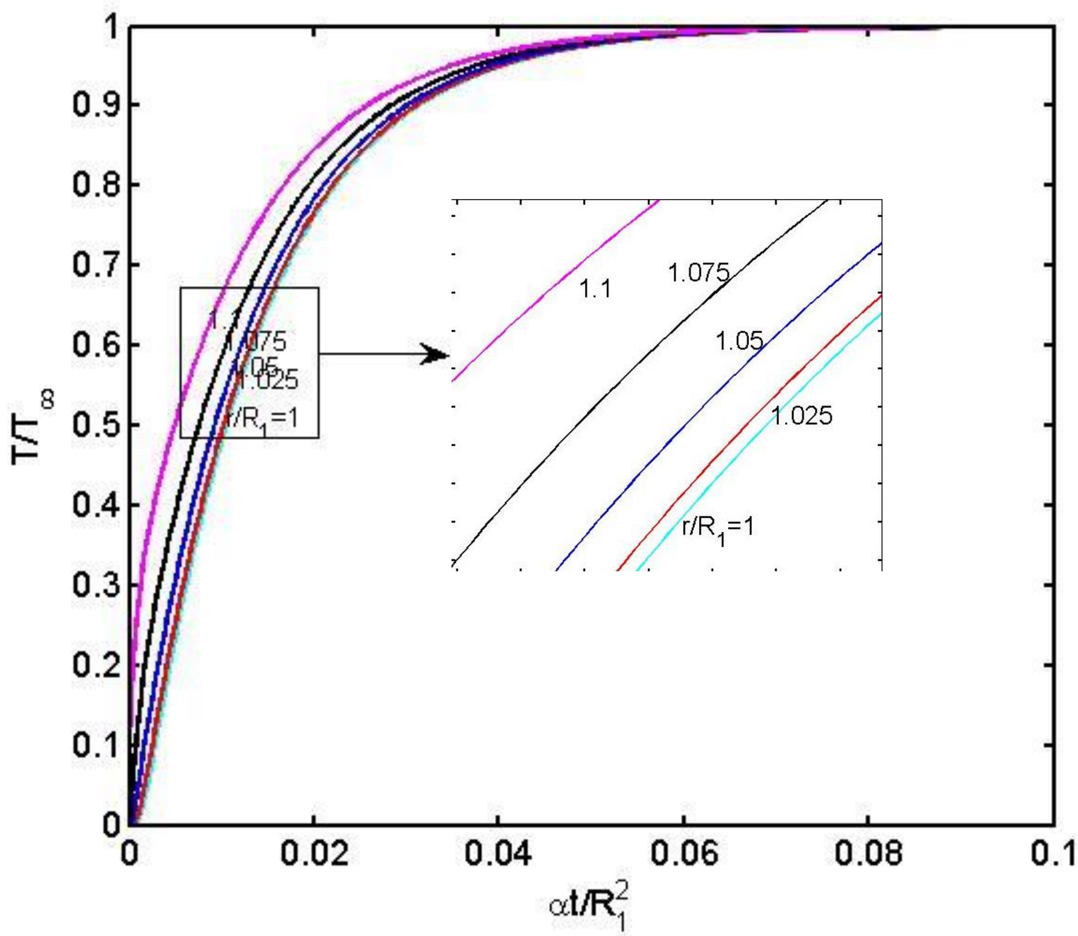

Fig. 2. Dimensionless transient temperatures versus dimensionless time for the R23B01T0 case with $\tilde{R} = 1.1$ and $Bi = 10$

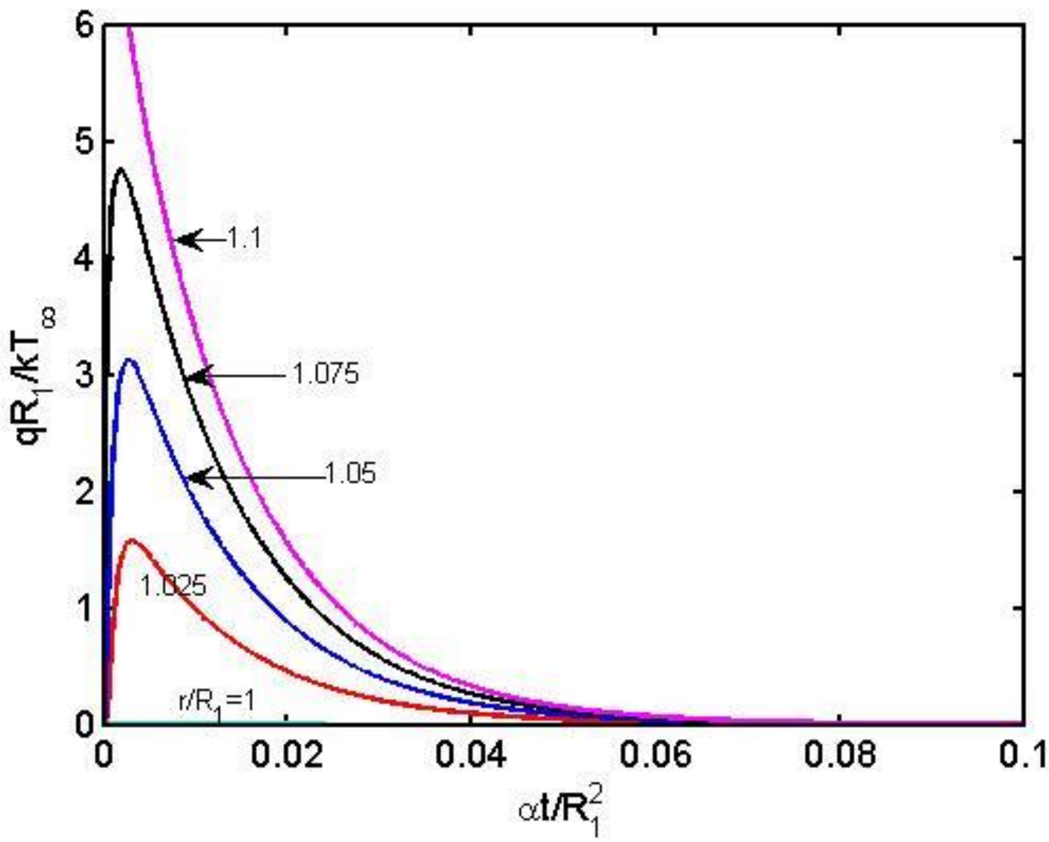

Fig. 3. Dimensionless transient heat fluxes versus dimensionless time for the R23B01T0 case with $\tilde{R}=1.1$ and $Bi=10$

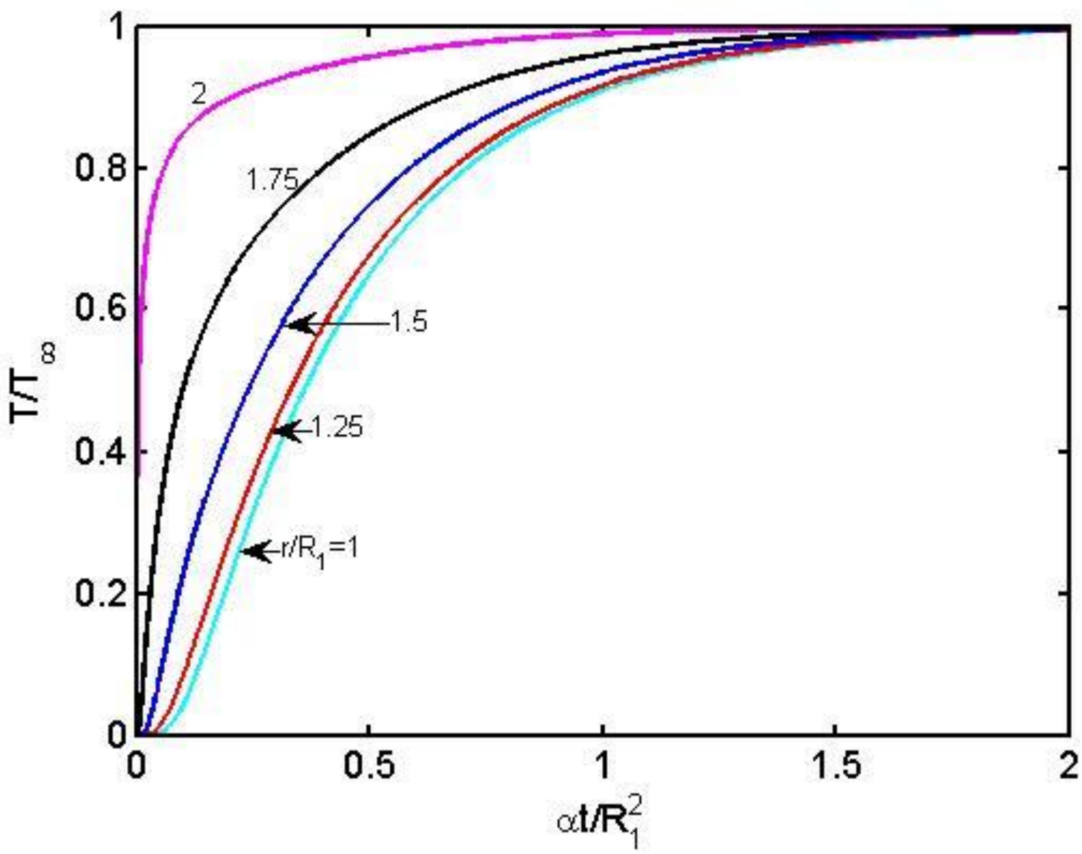

Fig. 4. Dimensionless transient temperatures versus dimensionless time for the R23B01T0 case with $\tilde{R} = 2$ and $Bi = 10$.

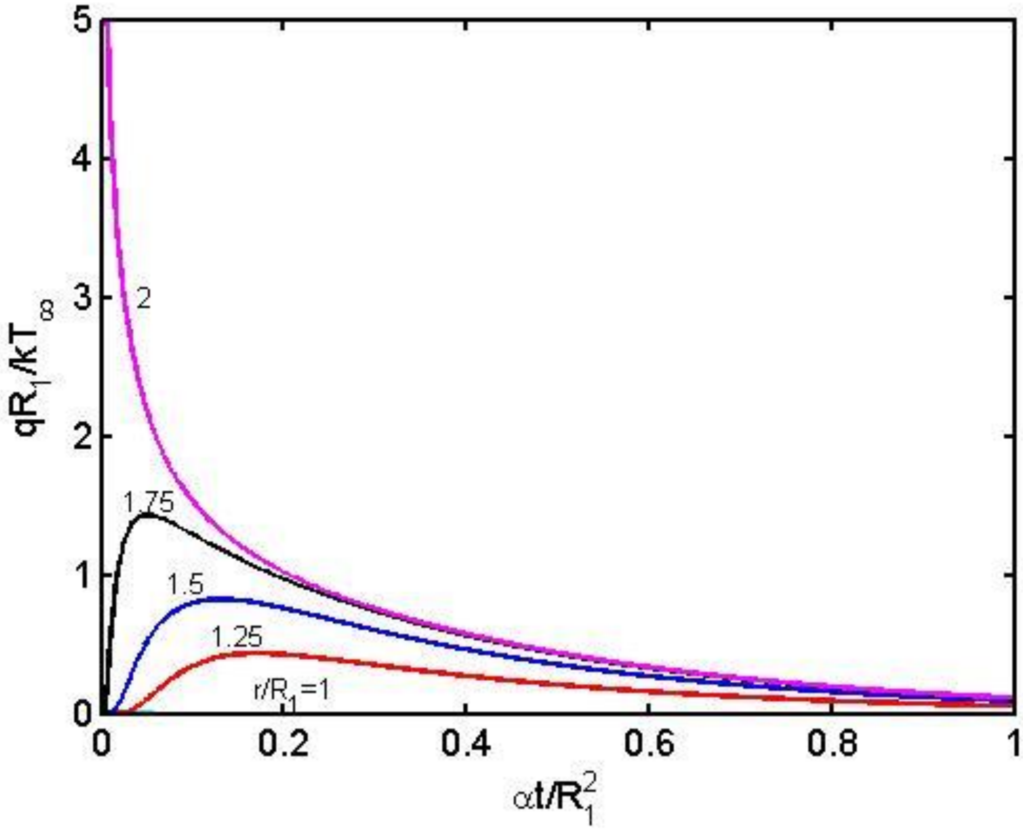

Fig. 5. Dimensionless transient heat fluxes versus dimensionless time for the R23B01T0 case with $\tilde{R}=2$ and $Bi=10$.

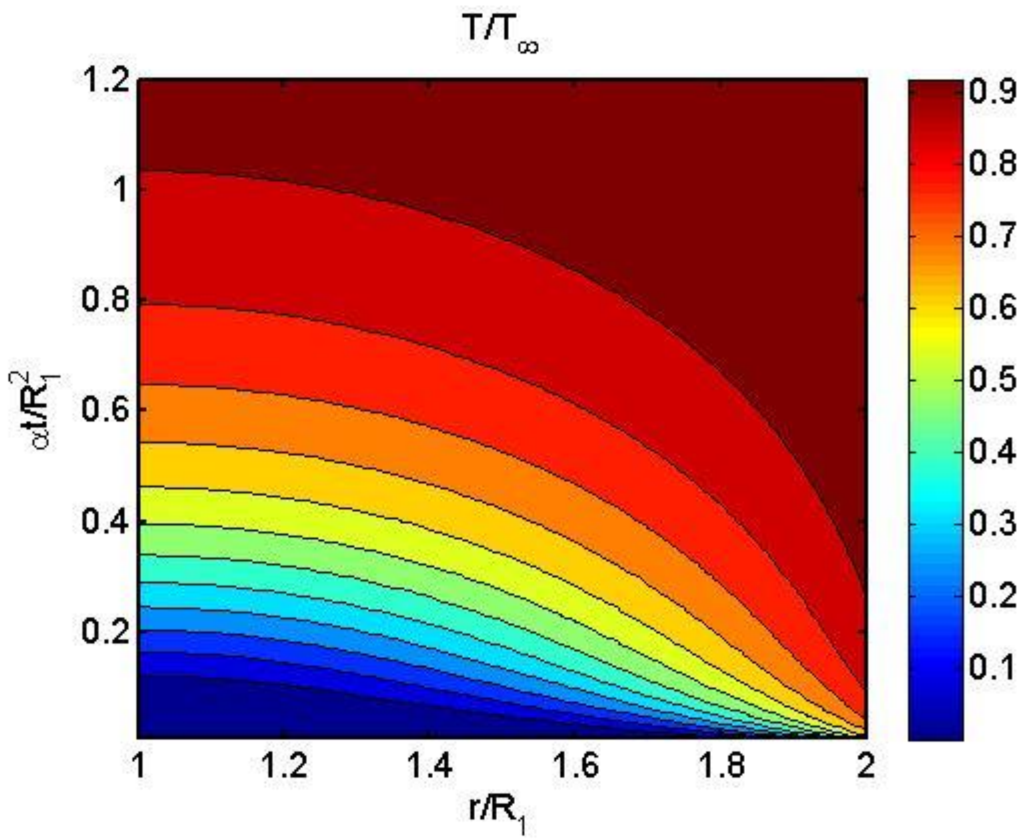

Fig. 6. Variation of dimensionless transient temperatures with respect to dimensionless time and dimensional radius for the R23B01T0 case with $\tilde{R} = 2$.

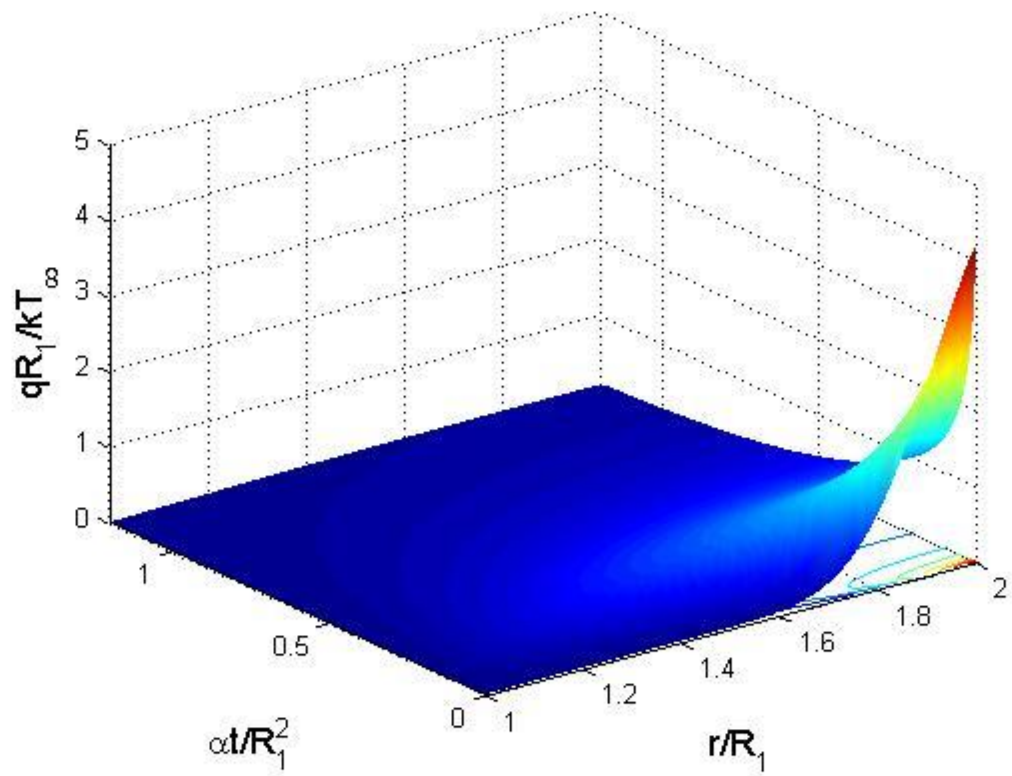

Fig. 7. Variation of dimensionless heat fluxes with respect to dimensionless time and dimensional radius for the R23B01T0 case with $\tilde{R} = 2$.