Category Theory Minicourse II
Functors and Natural Transformations

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1 Functors

1.1 Last Time

Last time, we generalized familiar constructions like products and greatest common divisor, introducing the definition of a category along the way.

Definition 1.1. *A **category** $\mathcal{C}$ consists of:*

- *A collection of objects, $ob(\mathcal{C})$*
- *For each pair of objects $A, B \in ob(\mathcal{C})$, there is a collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$. We will often write $f : A \to B$ instead of $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$*
- *We can compose morphisms. For any morphisms $f : A \to B$ and $g : B \to C$, there is a morphism $g \circ f : A \to C$.*
- *For any $A \in \mathrm{ob}(\mathcal{C})$, There is an identity morphism $1_A : A \to A$ such that for $f : B \to A$, we have $1_A \circ f = f$ and for $g : A \to B$, we have $g \circ 1_A = g$*
- *Associativity: $f \circ (g \circ h) = (f \circ g) \circ h$*

Example 1.2. *Some examples of categories include:*

1. Set *is the category of sets with morphisms being functions*
2. Group *is the category of groups with morphisms being group homomorphisms*
3. Ab *is the category of abelian groups with group homomorphisms*
4. CRing *is the category of commutative rings with ring homomorphisms*
5. Top *is the category of topological spaces with continuous maps as morphisms*
6. Vect_k *is the category of vector spaces over a field k with linear maps*

7. *Categories don't need an underlying set structure. Any poset is a category where the relation is a morphism. That is, if* $A \leq B$ *then* $\mathrm{Hom}_S(A, B)$ *contains one element called* $\leq$

It is easy to think that categories are defined by their objects, but this is a misconception.[1] The central idea of category theory is that **mathematical objects are defined by their morphisms**. We defined a category in order to precisely describe this idea, but it raises the question: what is a morphism between categories?

objects	morphisms
categories	?

1.2 Motivating Examples

Definition 1.3. *Given a set* S*, the* ***free group*** $F(S)$ *is the group consisting of all words that can be built from elements of* S*.*

Example 1.4. *For* $S = \{ * \}$*,* $F(S) = \mathbb{Z}$*. For* $S = \{ x, y \}$*,* $F(S) = \mathbb{Z} * \mathbb{Z}$*, where some words are* x, y, xy, xyy, yxy*, etc.*

We note that for sets S, T, any set-function $f : S \to T$ gives us a unique group homomorphism $Ff : F(S) \to F(T)$. Constructing this homomorphism is left as an exercise.

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ & \Big\downarrow & \\ F(S) & \xrightarrow{\quad Ff \quad} & F(T) \end{array}$$

Definition 1.5. *Given a set* S*, we may define the* ***free*** $\mathbb{R}$***-vector space*** $G(S)$ *to be an* $\mathbb{R}$ *vector space with basis* S*.*

Example 1.6. *For* $S = \{ * \}$*,* $G(S) \cong \mathbb{R}$*. For* $S = \{ x, y \}$*,* $G(S) \cong \mathbb{R}^2$*, where a vector would look like* $3x + 2y$ *or* $4x - y\sqrt{2}$*. Note that* $G(\mathbb{Z})$ *is infinite-dimensional.*

Similarly, a set-function $f : S \to T$ induces a unique linear map $Gf : G(S) \to G(T)$, whose construction is also left as an exercise.

Definition 1.7. *Given a set* S*, we may define the* ***free commutative ring*** $R(S)$ *to be* $\mathbb{Z}[S]$*.*

Example 1.8. *For* $S = \emptyset$*,* $R(S) = \mathbb{Z}$*. For* $S = \{ x, y \}$*,* $R(S) \cong \mathbb{Z}[x, y]$*.*

Again, a set-function $f : S \to T$ induces a unique linear map $Rf : R(S) \to R(T)$, whose construction is again left as an exercise.

[1] The fact that categories are named for the objects is not helpful.

Definition 1.9. *For any ring R, the **general linear group** $GL_n(R)$ is the set of $n \times n$ matrices with entries in R.*

By now, you should know what's coming. For rings R, S and a ring homomorphism $\varphi : R \to S$, we get a group homomorphism $\bar{\varphi} : GL_n(R) \to GL_n(S)$. What is it?

If a construction preserves functions in this way, it is called **functorial**. Here are a few more examples:

Definition 1.10. *Any ring R defines a **multiplicative group of units** $R^\times$.*

Definition 1.11. *Fix a group G. Any group H defines a set $\mathrm{Hom}(G, H)$, the set of group homomorphisms $G \to H$.*

Exercise 1.12. *For any group homomorphism $\varphi : H \to I$, we get a set-function $\bar{\varphi} : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, I)$. How does this work?*

1.3 The Idea of Functors

If objects aren't the core building blocks of a category, then what is? Looking at the definition, there is very little structure. Effectively, all we require is that morphisms between objects compose - for morphisms $f : A \to B$ and $g : B \to C$, we need a morphism $g \circ f : A \to C$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

In some sense, the core building blocks of a category are pairs of objects A, B and morphisms $A \to B$ between them, and the behavior of these blocks is defined by their composition. We can think of morphisms and composition as defining a category in the same way that multiplication defines a group, or addition and scalar multiplication define a vector space.

In the same way that group homomorphisms take elements to elements respecting multiplication, a morphism between categories $\mathcal{C}$ and $\mathcal{D}$ should take morphisms $c \to c'$ to morphisms $d \to d'$ in a way that respects composition. This motivates the definition of a **functor**.

Definition 1.13. *A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a map between the objects of a category together with a map between the morphisms of the categories defined so that for $f : A \to B$, there is a morphism $F(f) : F(A) \to F(B)$. Functors need to have the following properties:*

- $F(f \circ g) = F(f) \circ F(g)$

- $F(1_A) = 1_{F(A)}$

Example 1.14. *The **forgetful functor** U : $\mathsf{Grp} \to \mathsf{Set}$ sends a group to its underlying set.*

Example 1.15. *The **powerset functor** P : $\mathsf{Set} \to \mathsf{Set}$ sends a set S to its set $P(S)$ of subsets, and a function $f : S \to T$ to the direct-image functor $\bar{f} : P(S) \to P(T)$ sending $A \in P(S)$ to the image $f(A) \in P(T)$.*

Remark 1.16. *Categories, together with functors, form the category Cat.*

1.4 Question

objects	morphisms
categories	functors
functors	?

We defined categories to understand morphisms between mathematical objects, which motivated us to define functors to understand morphisms between categories. So, what are morphisms between functors?

2 Natural Transformations

2.1 The Determinant

Recall the functors $GL_n(-)$: $\mathsf{Ring} \to \mathsf{Group}$, taking a ring to the group of $n \times n$ matrices over it, and $(-)^\times$: $\mathsf{Ring} \to \mathsf{Group}$, taking a ring to its group of units. These are functors, so for rings R, S, they preserve ring homomorphisms $\varphi : R \to S$. What are the maps $GL_n(\varphi)$ and $\varphi^\times$?

$$\begin{tikzcd} R \arrow[d, "\varphi"] & GL_n(R) \arrow[d, "GL_n(\varphi)"] & R^\times \arrow[d, "\varphi^\times"] \\ S & GL_n(S) & S^\times \end{tikzcd}$$

Both of these functors go from Ring to Group, so we might wonder if they are related. This is the point of category theory, examining relationships between objects to tease out more information about them.

Example 2.1. *Given a ring R, the **determinant** is a group homomorphism $det : GL_n(R) \to R^\times$. We omit the definition, although for $n = 2$, $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.*

We can draw determinants into the diagram as follows:

$$
\begin{tikzcd}
R \arrow[d, "\varphi"] & \longmapsto & GL_n(R) \arrow[r, "\det"] \arrow[d, "GL_n(\varphi)"] & R^\times \arrow[d, "\varphi^\times"] \\
S & & GL_n(S) \arrow[r, "\det"'] & S^\times
\end{tikzcd}
$$

By composition, this diagram gives us two group homomorphisms $GL_n(R) \to S^\times$: $\det \circ GL_n(\varphi)$ and $\varphi^\times \circ \det$. It would be very "natural" if these were the same. Are they?

Exercise 2.2. *When $n = 2$, prove $\det \circ GL_n(\varphi) = \varphi^\times \circ \det$.*

When all function compositions between the same objects in a diagram are equal, we say the diagram **commutes**. This is a very well-behaved relationship between functors![2]

2.2 Abelianization

A similar example is abelianization.

Definition 2.3. *Given a group G, the **commutator subgroup** $[G, G]$ is defined to be $\{ aba^{-1}b^{-1} : a, b \in G \}$. The **abelianization** of G is $G/[G, G]$. Note that any group has a quotient group homomorphism $\pi : G \to G/[G, G]$.*

Exercise 2.4. *Show that abelianization is a functor.*

The abelianization of a group is basically what happens if you take a group and then assume all elements commute. It is the most "natural" way to turn a group into an abelian group, and category theory lets us see why.

Remark 2.5. *For any category $\mathcal{C}$, there is an **identity functor** $\mathcal{C} \to \mathcal{C}$.*

Exercise 2.6. *Show that the following diagram commutes for all groups G, H and homomorphisms $\varphi : G \to H$:*

$$
\begin{tikzcd}
G \arrow[r, "\pi_G"] \arrow[d, "\varphi"] & G/[G, G] \arrow[d, "\varphi^{ab}"] \\
H \arrow[r, "\pi_H"] & H/[H, H]
\end{tikzcd}
$$

This example motivates a new definition of a "morphism between functors," which we call a **natural transformation**.

[2] By the way, notice that $(-)^\times$ is the same as $GL_1(-)$.

2.3 Defining Natural Transformations

Definition 2.7. *Given categories $\mathcal{C}, \mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$, a* ***natural transformation*** $\alpha : F \Rightarrow G$ *is a collection of morphisms $\alpha_c : F(c) \to G(c)$ for all objects $c \in ob(\mathcal{C})$, such that for all objects $c, d \in ob(\mathcal{C})$ and morphisms $\varphi : c \to d$, the following diagram commutes:*

$$\begin{tikzcd}
F(c) \arrow[r, "\alpha_c"] \arrow[d, "F\varphi"] & G(c) \arrow[d, "G\varphi"] \\
F(d) \arrow[r, "\alpha_d"'] & G(d)
\end{tikzcd}$$

This is a lot of words, but fundamentally we just want a natural transformation between functors to preserve all the structure of the functors.

Exercise 2.8. *Verify that for categories $\mathcal{C}, \mathcal{D}$, the functors $\mathcal{C} \to \mathcal{D}$ form a category, with morphisms being natural transformations.*

Here are a few more examples of natural transformations:

Exercise 2.9. *Consider the identity functor $id : \mathsf{Set} \to \mathsf{Set}$ and power set functor $\mathscr{P} : \mathsf{Set} \to \mathsf{Set}$. For any set S, what is the most natural map $S \to \mathscr{P}(S)$? Is it a natural transformation?*

A natural transformation $\alpha : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a **natural isomorphism** if α_c is an isomorphism for all $c \in \mathrm{ob}(\mathcal{C})$.

Exercise 2.10. *Consider the open-set and closed-set functors $\mathcal{O}, \mathcal{C} : \mathsf{Top} \to \mathsf{Set}$. For a topological space X, what is a natural map $\mathcal{O}(X) \to \mathcal{C}(X)$? Is it a natural transformation? A natural isomorphism?*

2.4 Closing Remarks

Category theory, as previously mentioned, derives information about mathematical objects by analyzing their relationship with other objects. Categories and morphisms allow us to understand the relationships between objects, functors allow us to understand the relationships between categories, and now, natural transformations allow us to understand the relationships between functors. One might wonder if there is a similar way to understand relationships between natural transformations.

objects	morphisms
categories	functors
functors	natural transformations
natural transformations	?

Exercise 2.11. *Find some construction that behaves as a morphism of natural transformations in a way that makes sense in the table above. (Hint: You may want to work in the category of categories, or the category of functors between two categories.)*