Category Theory I

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1 Motivating Examples

In many fields of math, we construct some kind of object from another. Usually, these constructions have special properties to justify them. We give a few of these constructions which have similar properties to justify the language of Category Theory.

Definition 1.1. *Given a set S, the* ***free group*** *$F(S)$ is the group consisting of all words that can be built from elements of S.*

Example 1.2. *For $S = \{ * \}$, $F(S) = \mathbb{Z}$. For $S = \{ x, y \}$, $F(S) = \mathbb{Z} * \mathbb{Z}$, where some words are x, y, xy, xyy, yxy, etc.*

We note that for sets S, T, any set-function $f : S \to T$ gives us a unique group homomorphism $Ff : F(S) \to F(T)$. Constructing this homomorphism is left as an exercise.

$$\begin{array}{c} S \xrightarrow{\qquad f \qquad} T \\ \big\downarrow \\ F(S) \xrightarrow{\qquad Ff \qquad} F(T) \end{array}$$

Definition 1.3. *Given a set S, we may define the* ***free $\mathbb{R}$-vector space*** *$G(S)$ to be an $\mathbb{R}$ vector space with basis S.*

Example 1.4. *For $S = \{ * \}$, $G(S) \cong \mathbb{R}$. For $S = \{ x, y \}$, $G(S) \cong \mathbb{R}^2$, where a vector would look like $3x + 2y$ or $4x - y\sqrt{2}$. Note that $G(\mathbb{Z})$ is infinite-dimensional.*

Similarly, a set-function $f : S \to T$ induces a unique linear map $Gf : G(S) \to G(T)$, whose construction is also left as an exercise.

Definition 1.5. *Given a set S, we may define the* ***free commutative ring*** *$R(S)$ to be $\mathbb{Z}[S]$.*

Example 1.6. *For $S = \emptyset$, $R(S) = \mathbb{Z}$. For $S = \{ x, y \}$, $R(S) \cong \mathbb{Z}[x, y]$.*

Again, a set-function $f : S \to T$ induces a unique linear map $Rf : R(S) \to R(T)$, whose construction is again left as an exercise.

These "map-preserving" constructions are not limited to "free" objects:

Definition 1.7. *For any topological space X and point $x \in X$, the **fundamental group** $\pi_1(X, x)$ is defined to be the collection of loops in X based at x, up to homotopy. Composition is defined by concatenation.*

For topological spaces X and Y, with some point $x \in X$, any continuous function $f : X \to Y$ induces a unique linear map $\bar{f} : \pi_1(X, x) \to \pi_1(Y, f(x))$.

Definition 1.8. *For any ring R, the **general linear group** $GL_n(R)$ is the set of $n \times n$ matrices with entries in R.*

By now, you should know what's coming. For rings R, S and a ring homomorphism $\varphi : R \to S$, we get a group homomorphism $\bar{\varphi} : GL_n(R) \to GL_n(S)$. What is it?

If a construction preserves functions in this way, it is called **functorial**. Here are a few more examples:

Definition 1.9. *Any ring R defines a **multiplicative group of units** $R^\times$.*

Definition 1.10. *Fix a group G. Any group H defines a set $\mathrm{Hom}(G, H)$, the set of group homomorphisms $G \to H$.*

For any group homomorphism $\varphi : H \to I$, we get a set-function $\bar{\varphi} : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, I)$. How does this work?

This idea of consider both objects and the maps between them will just the definition of a category.

2 What is a category?

2.1 Definition

Definition 2.1. *A **category** $\mathcal{C}$ consists of:*

- *A collection of objects, ob($\mathcal{C}$)*
- *For each pair of objects $A, B \in ob(\mathcal{C})$, there is a collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(A, B)$. We will often write $f : A \to B$ instead of $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$*
- *We can compose morphisms. For any morphisms $f : A \to B$ and $g : B \to C$, there is a morphism $g \circ f : A \to C$.*
- *For any $A \in \mathrm{ob}(\mathcal{C})$, There is an identity morphism $1_A : A \to A$ such that for $f : B \to A$, we have $1_A \circ f = f$ and for $g : A \to B$, we have $g \circ 1_A = g$*
- *Associativity: $f \circ (g \circ h) = (f \circ g) \circ h$*

2.2 Examples

- Set is the category of sets with morphisms being functions
- Group is the category of groups with morphisms being group homomorphisms
- Ab is the category of abelian groups with group homomorphisms
- CRing is the category of commutative rings with ring homomorphisms
- Top is the category of topological spaces with continuous maps as morphisms
- Vect_k is the category of vector spaces over a field k with linear maps
- Categories don't need an underlying set structure. Any poset is a category where the relation is a morphism. That is, if $A \leq B$ then $\mathrm{Hom}_S(A, B)$ contains one element called $\leq$

3 What is a functor?

In the nature of category theory, we like to have maps between our objects. For categories, those maps are called functors.

Definition 3.1. *A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a map between the objects of a category together with a map between the morphisms of the categories defined so that for $f : A \to B$, there is a morphism $F(f) : F(A) \to F(B)$. Functors need to have the following properties:*

- $F(f \circ g) = F(f) \circ F(g)$
- $F(1_A) = 1_{F(A)}$

Remark 3.2. *Categories, together with functors, form the category Cat.*

Example 3.3. *The **forgetful functor** $U : \mathsf{Grp} \to \mathsf{Set}$ sends a group to its underlying set.*

All of our examples from section 1 are functors. For the next few examples, make sure to figure out for yourself how the functor acts on the morphisms and verify that it satisfies the necessary properties of a functor.

Example 3.4. *The **powerset functor** $P : \mathsf{Set} \to \mathsf{Set}$ sends a set S to its set $P(S)$ of subsets, and a function $f : S \to T$ to the direct-image functor $\bar{f} : P(S) \to P(T)$ sending $A \in P(S)$ to the image $f(A) \in P(T)$.*

Example 3.5. *The **abelianization functor** $Ab : \mathsf{Grp} \to \mathsf{Ab}$ sends a group G to its abelianization $G/[G, G]$.*

Example 3.6. *The **path-components functor** $\pi_0 : \mathsf{Top} \to \mathsf{Set}$ sends a topological space to its set of path-connected components.*

The fact that functors compose associatively, and the existence of a unique *identity functor* $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ for any category $\mathcal{C}$ indicates that we can define a **category of categories**. There are some set-theoretic issues which we will not address, but this construction is still possible.

3.1 Contravariant functors

Some constructions seem functorial, but with the arrows in the wrong direction.

Example 3.7. *Fix a group G. The functor* $\mathrm{Hom}(-, G) : \mathsf{Grp} \to \mathsf{Set}$ *sends a group H to the set of group homomorphisms* $\mathrm{Hom}(H, G)$.

This is similar to Definition 1.10. But here, a group homomorphism $\varphi : H \to I$ induces a natural set-function $\mathrm{Hom}(I, G) \to \mathrm{Hom}(H, G)$, sending $f : I \to G$ to $f \circ \varphi : H \to G$.

$$\begin{tikzcd} H \arrow[rr, "\varphi"] & & I \\ & {} \arrow[d, mapsto] & \\ & {} & \\ \mathrm{Hom}(H,G) & & \mathrm{Hom}(I,G) \arrow[ll, "- \circ \varphi"'] \end{tikzcd}$$

There are many more examples.

Definition 3.8. *Let V be a vector space over a field k. The dual vector space V^* of V is the space of linear maps $V \to k$.*

These functors that reverse the arrows are called **contravariant**. A functor that is not contravariant, as in the original definition, is called **covariant**. Contravariant functors motivate a new categorical definition:

Definition 3.9. *Given a category $\mathcal{C}$, the* ***opposite or dual category*** *$\mathcal{C}^{op}$ is defined to be $\mathcal{C}$ with the arrows reversed.*

Remark 3.10. *A contravariant functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is typically written as a (covariant) functor $\mathcal{C}^{op} \to \mathcal{D}$.*

4 Closing Thoughts

4.1 Philosophical Aside

Functors are a core facet of what makes category theory relevant: its ability to describe different fields of mathematics with the same universal language. This allows us to see deep underlying mathematical patterns and define them rigorously.

4.2 Isomorphisms

Definition 4.1. *An* ***isomorphism*** *is a morphism $f : A \to B$ such that there is an inverse morphism $g : B \to A$ which satisfies $f \circ g = 1_B$ and $g \circ f = 1_A$. Objects A and B which have an isomorphism between them are called* ***isomorphic****.*

This notion of isomorphism is a nice generalization of how isomorphisms are defined in various fields. This definition coincides with the definition of isomorphism in Grp and Top

Lemma 4.2. *Functors preserve isomorphisms. That is, suppose $f : x \to y$ is an isomorphism in a category $\mathcal{C}$, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. $Ff : Fx \to Fy$ is an isomorphism.*

Proof. Let $g : y \to x$ be the inverse of f, so $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$. Then $Ff \circ Fg = F\mathrm{id}_y = \mathrm{id}_{Fy}$, and $Fg \circ Ff = F\mathrm{id}_x = \mathrm{id}_{Fx}$. Thus Ff is an isomorphism. □

4.3 An exercise

Given a group G, we may construct the **center** $Z(G)$, the **commutator subgroup** $[G, G]$, and the **automorphism group** $\mathrm{Aut}(G)$. Which of these constructions are functorial?