How to Write a Proof

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1 July 2025

1 Why write proofs?

"We must know. We will know."

David Hilbert

As we learn in number theory, mathematical knowledge comes from lived experience. We *know* things are true, based on our many experiences with them, and the intuition we've built over time. But every mathematical culture, from ancient Greece to medieval Islam to the present day, has wanted a way to clarify, confirm, and *prove* their intuition. How do we know that we're right?

We will take a classic theorem - that $\sqrt{2}$ is irrational - and try to prove it in many different ways. By interrogating each proof we will tease out its essential elements and its flaws. In doing so, we will see mathematical proof emerge as a natural way to convince ourself of truths and communicate them to others.

Exercise. What is a proof? Come up with properties that a proof should and shouldn't have.

Taking inspiration from homotopy type theory, our overarching idea will be that **a proof is a path**. Given a point of departure (hypotheses) and a destination (result), a proof is a method to clearly and logically move from one to the other.

2 Finding a proof

We will prove the following statement:

Theorem. $\sqrt{2}$ is irrational.

For each of the following proofs, critique it and describe precisely what makes it or prevents it from being a proof.

Proof. Obviously, $\sqrt{2}$ is irrational. □

There is no path! The proof describes no clear point of departure and skips straight to the result.

Proof. Since $\sqrt{2}$ is irrational, $\sqrt{2}$ is irrational. $\square$

There is a path here, which is a start. But seems to go from the result straight to the result! This begs the question: what is our point of departure?

> "Most of my work consists of coming up with the right definitions."
>
> Peter Scholze

The definitions you choose can have a big impact on the path you take in your proof.

Exercise. What does irrational mean? Give as many distinct, rigorous definitions as you can. Which one will set you up for the cleanest proof?

We choose the following:

Definition. A **rational number** is a real number that can be written as $\frac{p}{q}$, for p, q integers with $q \neq 0$. A real number is **irrational** if it is not rational.

We want to show that $\sqrt{2}$ is not rational - that it cannot be written as $\frac{p}{q}$ for integers p, q. Now, we have our point of departure and our destination, let's try another proof!

Proof. For all integers p, q, $\frac{p}{q} \neq \sqrt{2}$. $\square$

There is a point of departure (integers p, q) and a destination ($\frac{p}{q} \neq \sqrt{2}$). But there is nothing in the middle! A path has to connect the two points, filling in the gap with logical steps in a way the reader can understand. So, let's fill in those gaps:

Proof. $\sqrt{2} = \frac{p}{q} \implies 2 = \frac{p^2}{q^2} \implies 2q^2 = p^2 \implies 2 \mid p^2 \implies 2 \mid p$
$2 = \frac{(2k)^2}{q^2} = \frac{4k^2}{q^2} \implies 2q^2 = 4k^2 \implies q^2 = 2k^2 \implies 2 \mid q^2 \implies 2 \mid q$
$\implies\Leftarrow$ $\square$

Got that?
The biggest problem here is clarity. What is our point of departure - what are p and q? What is k? Even if this proof has implications which are *technically* correct, it is certainly not a *good* proof. Why is that?
A good proof should provide, among other things, explanation and intuition. The reader should understand why you are taking each step that you do. A good proof is also aware of its audience, choosing which steps to include or omit based on the reader's experience. Let's give it another go.

Proof. Suppose $\sqrt{2} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$, $\frac{p}{q}$ in simplest form. Then $2 = \frac{p^2}{q^2}$ so $2q^2 = p^2$. Thus p^2 is even. If p was odd, p^2 would be odd, so p is even. Thus $p = 2k$ for $k \in \mathbb{Z}$. Fix k. Then $2 = \frac{4k^2}{q^2}$ so $2q^2 = 4k^2 \implies q^2 = 2k^2$. Thus q is also even. Then $\frac{p}{q}$ is not in simplest form, so $\sqrt{2}$ is irrational. $\square$

From a technical standpoint, this proof is correct. But mathematical writing, ultimately, is just like any other writing. The proof needs direction, signposting, readability, and a better sense of flow. The reader must know the purpose of each step of the proof. We don't judge novels by their grammar, but rather by the story they tell. The same goes for proofs. Let's put on our writing hats and fix this up!

Proof. We will show that $\sqrt{2}$ is irrational. Suppose for the sake of contradiction that $\sqrt{2}$ is rational, so it may be written as $\sqrt{2} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $\frac{p}{q}$ in simplest form. Fix such p and q. Squaring both sides, we see that $2 = \frac{p^2}{q^2}$, and thus $2q^2 = p^2$. This means that p^2 is an even number, so p is also even. Therefore, we can write $p = 2k$ for some $k \in \mathbb{Z}$. Fix such a k. Then, we can rewrite the previous expression as

$$\begin{aligned} 2q^2 &= (2k)^2 \\ &= 4k^2. \end{aligned}$$

Dividing by 2, we get $q^2 = 2k^2$, and conclude that q^2 is even, so q is even. Since both p and q are even, $\frac{p}{q}$ is not in simplest form! This contradicts our assumption that $\sqrt{2}$ was rational, so $\sqrt{2}$ must be irrational. $\square$

In examining these proofs, we see a style of writing that comes from the desire for math that is correct, clear, and convincing. These are the burdens of the proof-writer.