Basics of Fourier Series / Transforms

$f(x) \quad x \in [\frac{L}{2}, \frac{L}{2}]$

$$f(x) = \sum_{j=-\infty}^{\infty} a_j \; e^{i 2\pi j \frac{x}{L}}$$

$i = \sqrt{-1}$

Use orthogonality to find a_j:

$$\int_{-\frac{L}{2}}^{L/2} dx \; e^{-i 2\pi n \frac{x}{L}} \cdot e^{i 2\pi j \frac{x}{L}} = L \, \delta_{n,j}$$

$$a_j = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \; f(x) \, e^{-i 2\pi j \frac{x}{L}}$$

$$\tilde{f}(k) = \frac{1}{L} \int_{-\frac{L}{2}}^{L/2} dx \; f(x) \, e^{-ikx}$$

$$k = \frac{2\pi j}{L}$$

Continuum limit: Let $L \to \infty$

$$f(x) = \int_{-\infty}^{\infty} dj \; a_j \; e^{i 2\pi j \frac{x}{L}}$$

$$= \frac{1}{2\pi} \int dk \; \tilde{f}(k) \, e^{ikx}$$

Useful Properties of FT:

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$k=0$ mode is the integral of the function

$$\tilde{f}(k=0) = \int_{-\infty}^{\infty} dx\, f(x)\cdot e^{-0} = \int_{-\infty}^{\infty} dx\, f(x)$$

Derivatives of $f(x)$:

$$\frac{df}{dx} = \frac{d}{dx}\left[\frac{1}{2\pi}\int dk\, \tilde{f}(k)\, \underline{e^{ikx}}\right]$$

$$= \frac{1}{2\pi}\int dk\, ik\, \tilde{f}(k)\, e^{ikx}$$

$$\mathcal{F}[f'] = ik\, \tilde{f}(k)$$

$$\mathcal{F}[f''] = -k^2\, \tilde{f}(k)$$

Convolutions: $$f(x) = \int_{-\infty}^{\infty} dx'\, g(x-x')\, h(x')$$

$$= (g * h)(x)$$

$$\tilde{f}(k) = \tilde{g}(k)\cdot \tilde{h}(k)$$

Integral of a product of functions:

$$\int dx\, h(x)\cdot g(x) = \frac{1}{V}\sum_k \tilde{g}(k)\tilde{h}(-k)$$

Gaussian Fourier Transform:

$$f(x) = \left(\frac{1}{2\pi a^2}\right)^{1/2} e^{-\frac{x^2}{2a^2}}$$

$$\tilde{f}(k) = e^{-k^2 a^2/2}$$

In higher dimension:

$$\tilde{f} \ldots \int \ldots e^{-ik\cdot r}$$

In higher dimension:

$$\tilde{f}(\underline{k}) = \int d\underline{r}\; f(\underline{r})\, e^{-i\underline{k}\cdot\underline{r}}$$

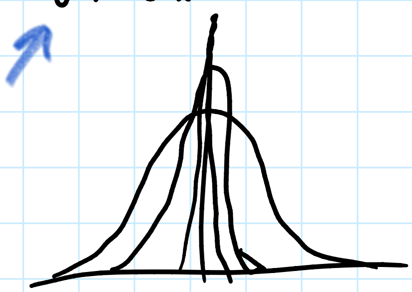
$$f(\underline{r}) = \frac{1}{V}\sum_{\underline{k}} \tilde{f}(\underline{k})\, e^{i\underline{k}\cdot\underline{r}}$$

$$f(\underline{r}) = \frac{1}{(2\pi)^3}\int d\underline{k}\; \tilde{f}(k)\, e^{i\underline{k}\cdot\underline{r}}$$

Dirac delta function:

$$f(a) = \int dx \cdot \delta(x-a)\cdot f(x)$$

$$\int dx\; \delta(x) = 1$$



$\delta(x)$ always returns 0 is $x \neq 0$

$\delta(x)$ carries unit $\quad \delta(x)\,[=]\,\frac{1}{x}$

$h(x)$ has N_h zeros:

$$\int_{-\infty}^{\infty} dx\; \delta[h(x)] = N_h$$

Classical density of states:

$$\Omega(E) = \frac{1}{h^{3N} N!}\int d\underline{p}^N \int d\underline{r}^N\, \delta\left[E - \mathcal{K}(\underline{p}^N) - \mathcal{U}(\underline{r}^N)\right]$$

Fourier transform of delta function:

Fourier transform of delta function:

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, e^{ik(x-a)}$$

Functional Calculus

Functional: mapping b/t a function $f(x)$ defined on $x \in [a,b]$ + a number F

$F[f]$

$$F_1[f] = \int_a^b dx \, f(x)$$

$$F_2[f] = \int_a^b dx \, [f(x)]^2$$

$$F_3[f] = \int_a^b dx \left(-[f(x)]^2 + [f(x)]^4 + \left[\frac{df}{dx}\right]^2\right)$$

$$F_4[f] = \int_a^b dx \int_a^b dx' \, f(x) \, K(x,x') \, f(x')$$

Functional Derivatives

$f(x)$ is perturbed by $\delta f(x)$

$$F[f+\delta f] = F[f] + \int_a^b dx \, \Gamma_1(x) \cdot \delta f(x) + \frac{1}{2!}\int dx \int dx' \, \Gamma_2(x,x') \cdot \delta f(x) \cdot \delta f(x')$$

$$\frac{\delta F}{\delta f(x)} \equiv \Gamma_1(x) \qquad \frac{\delta^2 F}{\delta f(x) \, \delta f(x')} \equiv \Gamma_2(x,x')$$

$$\frac{\delta F}{\delta f(x)} \equiv \Gamma_1(x) \qquad \frac{\delta^2 F}{\delta f(x)\,\delta f(x')} \equiv \Gamma_2(x,x')$$

$$F_2[f+\delta f] = \int_a^b dx\,[f(x)+\delta f(x)]^2$$

$$= \int dx \int dx'\,[f(x)+\delta f(x)]\cdot\delta(x-x')\cdot[f(x')+\delta f(x')]$$

$$= \int dx \int dx' \Big\{ \delta(x-x')\big[f(x)\,f(x') + f(x)\,\delta f(x') + f(x')\,\underline{\delta f(x)} + \delta f(x)\cdot\delta f(x')\big]\Big\}$$

$$= \underline{\int dx \int dx'\, f(x)\,f(x')\cdot\delta(x-x')} \rightarrow \int dx\,[f(x)]^2 = F[f]$$

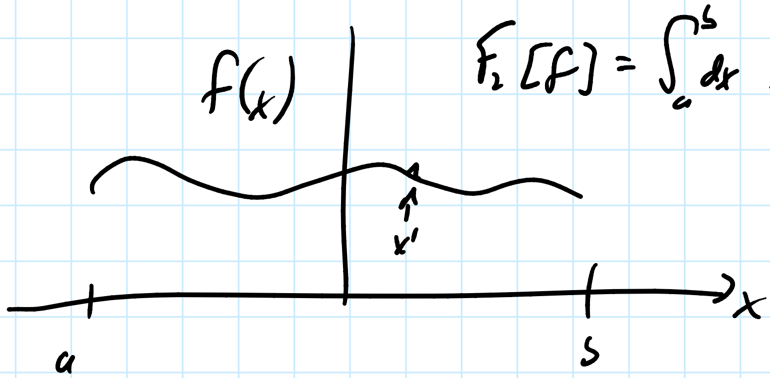
$$+\, 2\int dx \int dx'\,\delta(x-x')\,f(x)\cdot\delta f(x') \rightarrow 2\int dx\; f(x)\,\delta f(x)$$

$$+\, \frac{2}{2}\int dx \int dx'\,\delta(x-x')\,\delta f(x)\cdot\delta f(x')$$

$$\frac{\delta F_2}{\delta f(x)} = 2\,f(x)$$

$$\frac{\delta^2 F_2}{\delta f(x)\;\delta f(x')} = 2\cdot\delta(x-x')$$

$$F_2[f] = \int_a^b dx\; f(x)$$



$$\frac{\delta F_2}{\delta f(x')}$$

$$F_4[f] = \int dx \int dx' \; f(x) \; K(x,x') \; f(x')$$

$$\frac{\delta F_4}{\delta f(x)} = 2 \int dx' \, K(x,x') \, f(x')$$

Can also show: for a system w/ PBCs:

$f(a) = f(b)$

$$F_7[f] = \int_a^b dx \left[\frac{\partial f}{\partial x}\right]^2$$

$$\frac{\delta F_7}{\delta f(x)} = -2 \frac{\partial^2 f}{\partial x^2}$$

Functional chain rule:

$$\frac{\delta F[g]}{\delta f(x)} = \int dx' \, \frac{\delta F[g]}{\delta g(x')} \cdot \frac{\delta g(x')}{\delta f(x)}$$

$$\frac{\delta f(x)}{\delta f(x')} = \delta(x - x')$$

Externa Problems:

The Functional $F[f]$ has an extremum when

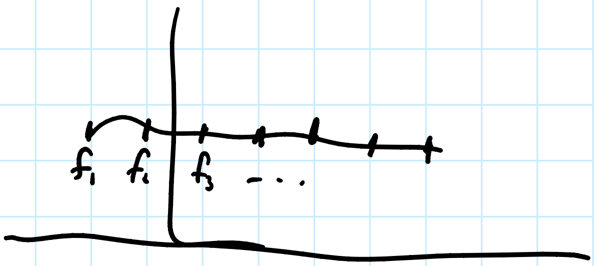
$$\left.\frac{\delta F[f]}{\delta f(x)}\right|_{f=f^*} = 0$$

f^* is the function evaluated at the extremum

Functional Integration

Integrating $F[f]$ means evaluating F for all shapes of $f(x)$

$$I = \int \mathcal{D}f \; F[f]$$



Discretize the function,

$$I_N = \int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 \cdots \int_{-\infty}^{\infty} df_N \; F(\underline{f})$$

# of discritization pts

Dirac delta functional:

$$\delta[f-g] = 0 \text{ unless } f(x) = g(x) \;\forall\, x \in [a,b]$$

$$\int \mathcal{D}f \;\delta[f-g] \cdot F[f] = F[g]$$

"Discretized" delta functional:

$$\delta[f-g] \approx \prod_{j=1}^{M} \delta\big(f(x_j) - g(x_j)\big)$$

Write each term in Fourier space:

Write each term in Fourier space:

$$\delta[f-g] = \frac{1}{(2\pi)^n} \prod_j \left[\int_{-\infty}^{\infty} d\omega(x_i)\, e^{i\omega(x_j)[f(x_j)-g(x_j)]} \right]$$

Upon restoring continuum limit:

$$\delta[f-g] = \int \mathcal{D}\omega\, e^{i\int dx\, \omega(x)[f(x)-g(x)]}$$

Definition of functional inverse:

$$\int dx'\, A(x,x')\cdot A^{-1}(x',x'') = \delta(x-x'')$$

In Fourier Space:

$$\tilde{A}(k)\cdot \tilde{A}^{-1}(k) = 1$$

Gaussian Functional Integrals:

$$\exp\left[-\frac{1}{2}\int dx \int dx'\, J(x)\, A(x,x')\, J(x')\right]$$

$$\left(\frac{1}{2\pi a^2}\right)^{1/2} e^{-\frac{x^2}{2a^2}} = \int dk\, e^{-k^2 a^2/2}\cdot e^{ikx}$$

$$= \frac{\int \mathcal{D}f\, \exp\left[-\frac{1}{2}\int dx \int dx'\, f(x)\cdot A^{-1}(x,x')\cdot f(x') + i\int dx\, J(x)\cdot f(x)\right]}{\int \mathcal{D}f\, \exp\left[-\frac{1}{2}\int dx \int dx'\, f(x)\cdot A^{-1}(x,x')\cdot f(x')\right]}$$

$A(x,x')$ has a functional inverse

$$= \exp\left[+\frac{1}{2}\int dx \int dx'\, J(x)\, A(x,x')\, J(x')\right]$$

$$\exp\left[+\frac{1}{2}\int dx\int dx'\, J(x)\, A(x,x')\, J(x')\right]$$

$$= \frac{\int \mathcal{D}f\, \exp\left[-\frac{1}{2}\int dx\int dx'\, f(x)\cdot A^{-1}(x,x')\cdot f(x') + \int dx\, J(x)\cdot f(x)\right]}{\int \mathcal{D}f\, \exp\left[-\frac{1}{2}\int dx\int dx'\, f(x)\cdot A^{-1}(x,x')\cdot f(x')\right]}$$