Bernstein Polynomials: Properties.

A Bernstein polynomial of order N is defined as

$$x_N(t) = \sum_{i=0}^{N} \bar{x}_{i,N} b_{i,N}(t), \quad t \in [0, t_f]$$

where $\bar{x}_{i,N} \in \mathbb{R}$ is the Bernstein coefficient and $b_{i,N}(t)$ is the Bernstein polynomials basis defined as

$$b_{i,N}(t) = \binom{N}{i} \frac{t^i (t_f - t)^{N-i}}{t_f^n},$$

where

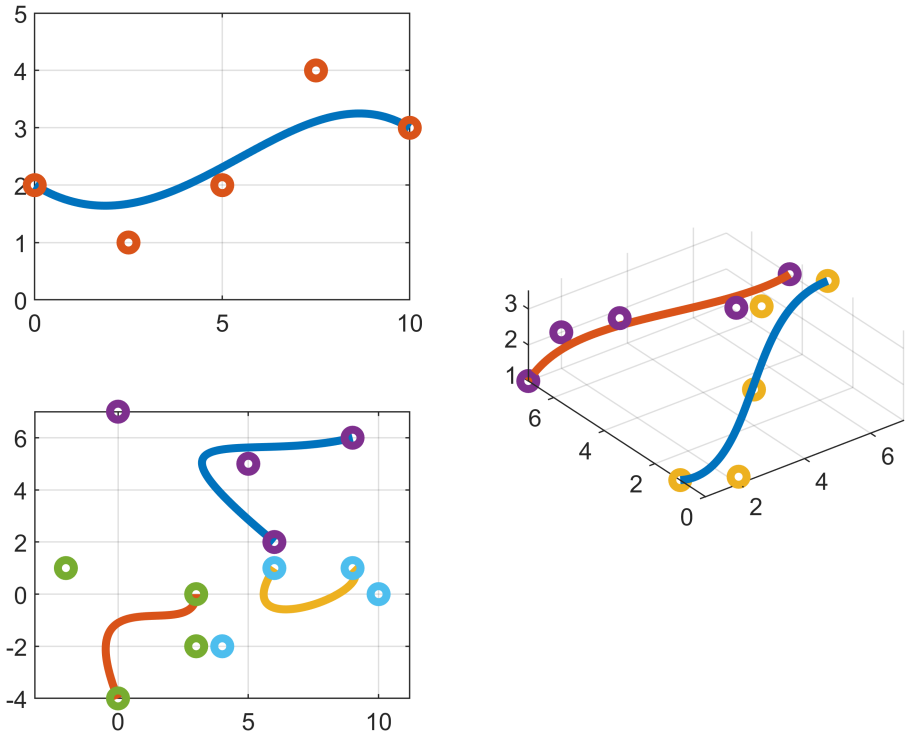
$$\binom{N}{i} = \frac{N!}{i!(N-i)!}$$

is the binomial coefficient. Notice that, for a given range of argument $[0, t_f]$, Bernstein polynomials are uniquely defined by the Bernstein coefficients: unlike Lagrange polynomials, Bernstein polynomials do NOT satisfy the Kronecker property and they are NOT interpolating polynomials. Bernstein polynomials were popularized by Pierre Bezier in the early 1960s as useful tools for geometric design (Bezier used 2D and 3D Bernstein polynomials to design the shape of the cars at the Renault company in France), and are now widely used in computer graphics, animations, and type fonts such as postscript fonts and true type fonts. For this reason, Bernstein polynomials are often referred to as a Bezier curves, especially when used to describe spatial curves.

Bernstein polynomials possess favorable geometric and numerical properties that can be exploited in many application domains, including motion planning and trajectory generation. In this chapter, we state the main geometric properties of Bernstein polynomials and present the BeBOT MATLAB toolkit, where these properties are implemented. The following code implements examples of 1D, 2D and 3D Bernstein polynomials. The Bernstein polynomials illustrated are obtained as follows

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Examples of 1,2,3D Bernstein polynomials
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all
close all

tf = 10;
t = linspace(0,tf,1000);

% 1D Bernstein Polynomial
x_N = [2 1 2 4 3];
xN = BernsteinPoly(x_N,t);


% 2D Bernstein Polynomials
x1_N = [6 0 5 9; 2 7 5 6];
x2_N = [0 -2 3 3; -4 1 -2 0];
x3_N = [6 4 10 9; 1 -2 -0 1];
```

```
x1N = BernsteinPoly(x1_N,t);
x2N = BernsteinPoly(x2_N,t);
x3N = BernsteinPoly(x3_N,t);


% 3D Bernstein Polynomials
p1_N = [1,2,4,5,7; 1,0,2,3,3; 1,1.2,2,3.5,3.5];
p2_N = [1,2,3,5,7; 7,7,6,4,4.5; 1,2,2.5,3,3];
p1N = BernsteinPoly(p1_N,t);
p2N = BernsteinPoly(p2_N,t);


% Plot the polynomials above
figure
subplot(2,2,1);
plot(t,xN,'LineWidth',3); hold on
plot(linspace(0,tf,length(x_N)),x_N,'o','LineWidth',3);
%set(gca,'fontsize', 24);
axis([0 10 0 5])
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.52, 0.45, 0.45]);
subplot(2,2,3)
plot(x1N(1,:),x1N(2,:),'LineWidth',3); hold on
plot(x2N(1,:),x2N(2,:),'LineWidth',3);
plot(x3N(1,:),x3N(2,:),'LineWidth',3);
plot(x1_N(1,:),x1_N(2,:),'o','LineWidth',3);
plot(x2_N(1,:),x2_N(2,:),'o','LineWidth',3);
plot(x3_N(1,:),x3_N(2,:),'o','LineWidth',3);
%set(gca,'fontsize', 40);
axis equal
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.02, 0.45, 0.45]);
subplot(2,2,[2,4])
plot3(p1N(1,:),p1N(2,:),p1N(3,:),'LineWidth',3); hold on
plot3(p2N(1,:),p2N(2,:),p2N(3,:),'LineWidth',3);
plot3(p1_N(1,:),p1_N(2,:),p1_N(3,:),'o','LineWidth',3);
plot3(p2_N(1,:),p2_N(2,:),p2_N(3,:),'o','LineWidth',3);
%set(gca,'fontsize', 40);
axis equal
grid on
```



```
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.35, 0.15, 0.8, 0.8]);
```

End point value property

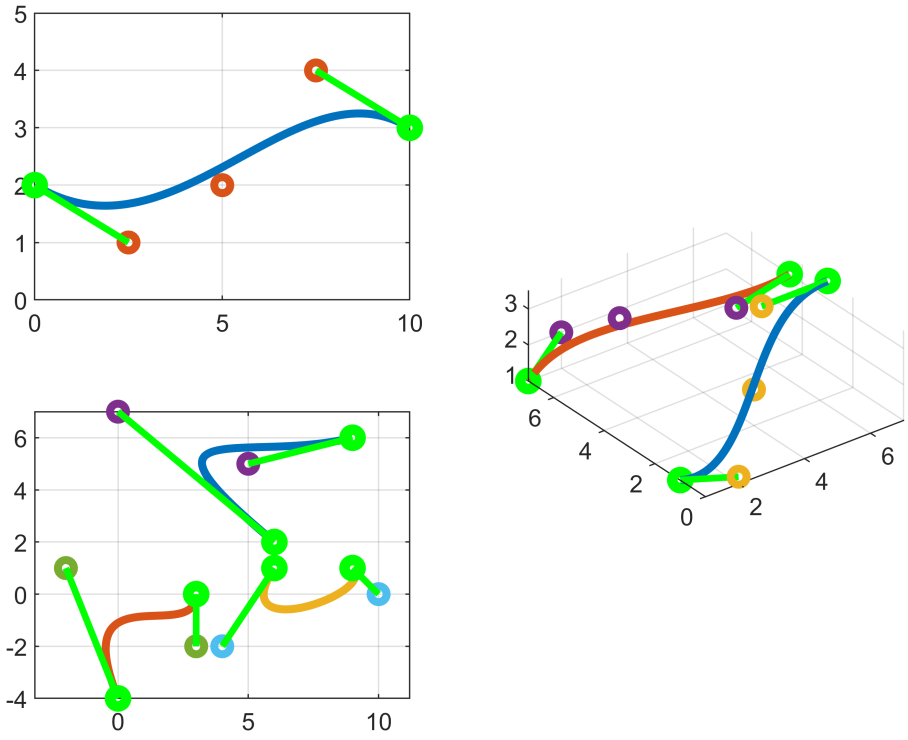
The Bernstein polynomial

$$x_N(t) = \sum_{j=0}^{N} \bar{x}_{j,N} b_{j,N}(t)$$

satisfies $x_N(0) = \bar{x}_{0,N}$ and $x_N(t_f) = \bar{x}_{N,N}$. Moreover, the tangent (the derivative) of a Bernstein polynomial at the initial and final points lies on the vectors $\bar{x}_{1,N} - \bar{x}_{0,N}$ and $\bar{x}_{N,N} - \bar{x}_{N-1,N}$, respectively. A graphical illustration of the end point value property is provided here

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Depiction of End point value property
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


% Plot
figure
subplot(2,2,1);
plot(t,xN,'LineWidth',3); hold on
tnodes = linspace(0,tf,length(x_N));
plot(tnodes,x_N,'o','LineWidth',3);
plot(tnodes(1),x_N(1),'o','LineWidth',4,'Color','g');
```

```
plot(tnodes(end),x_N(end),'o','LineWidth',4,'Color','g');
plot(tnodes(1:2),x_N(1:2),'LineWidth',2.5,'Color','g');
plot(tnodes(end-1:end),x_N(end-1:end),'LineWidth',2.5,'Color','g');
%set(gca,'fontsize', 40);
axis([0 10 0 5])
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.52, 0.45, 0.45]);
subplot(2,2,3)
plot(x1N(1,:),x1N(2,:),'LineWidth',3); hold on
plot(x2N(1,:),x2N(2,:),'LineWidth',3);
plot(x3N(1,:),x3N(2,:),'LineWidth',3);
plot(x1_N(1,:),x1_N(2,:),'o','LineWidth',3);
plot(x2_N(1,:),x2_N(2,:),'o','LineWidth',3);
plot(x3_N(1,:),x3_N(2,:),'o','LineWidth',3);
plot(x1_N(1,1),x1_N(2,1),'o','LineWidth',4,'Color','g');
plot(x1_N(1,end),x1_N(2,end),'o','LineWidth',4,'Color','g');
plot(x1_N(1,1:2),x1_N(2,1:2),'LineWidth',2.5,'Color','g');
plot(x1_N(1,end-1:end),x1_N(2,end-1:end),'LineWidth',2.5,'Color','g');
plot(x2_N(1,1),x2_N(2,1),'o','LineWidth',4,'Color','g');
plot(x2_N(1,end),x2_N(2,end),'o','LineWidth',4,'Color','g');
plot(x2_N(1,1:2),x2_N(2,1:2),'LineWidth',2.5,'Color','g');
plot(x2_N(1,end-1:end),x2_N(2,end-1:end),'LineWidth',2.5,'Color','g');
plot(x3_N(1,1),x3_N(2,1),'o','LineWidth',4,'Color','g');
plot(x3_N(1,end),x3_N(2,end),'o','LineWidth',4,'Color','g');
plot(x3_N(1,1:2),x3_N(2,1:2),'LineWidth',2.5,'Color','g');
plot(x3_N(1,end-1:end),x3_N(2,end-1:end),'LineWidth',2.5,'Color','g');
%set(gca,'fontsize', 40);
axis equal
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.02, 0.45, 0.45]);
subplot(2,2,[2,4])
plot3(p1N(1,:),p1N(2,:),p1N(3,:),'LineWidth',3); hold on
plot3(p2N(1,:),p2N(2,:),p2N(3,:),'LineWidth',3);
plot3(p1_N(1,:),p1_N(2,:),p1_N(3,:),'o','LineWidth',3);
plot3(p2_N(1,:),p2_N(2,:),p2_N(3,:),'o','LineWidth',3);
plot3(p1_N(1,1),p1_N(2,1),p1_N(3,1),'o','LineWidth',4,'Color','g');
plot3(p1_N(1,end),p1_N(2,end),p1_N(3,end),'o','LineWidth',4,'Color','g');
plot3(p1_N(1,1:2),p1_N(2,1:2),p1_N(3,1:2),'LineWidth',2.5,'Color','g');
plot3(p1_N(1,end-1:end),p1_N(2,end-1:end),p1_N(3,end-1:end),'LineWidth',2.5,'Color','g');
plot3(p2_N(1,1),p2_N(2,1),p2_N(3,1),'o','LineWidth',4,'Color','g');
plot3(p2_N(1,end),p2_N(2,end),p2_N(3,end),'o','LineWidth',4,'Color','g');
plot3(p2_N(1,1:2),p2_N(2,1:2),p2_N(3,1:2),'LineWidth',2.5,'Color','g');
plot3(p2_N(1,end-1:end),p2_N(2,end-1:end),p2_N(3,end-1:end),'LineWidth',2.5,'Color','g');
%set(gca,'fontsize', 40);
axis equal
grid on
```



```
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.35, 0.15, 0.8, 0.8]);
```

Convex Hull Property

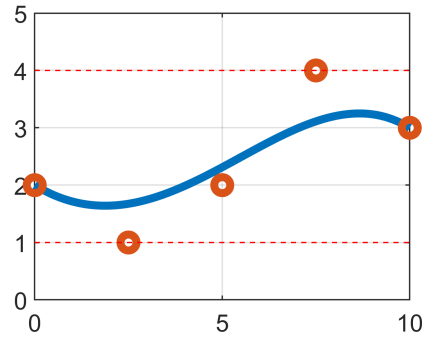
A Bernstein polynomial is completely contained in the convex hull of its Bernstein coefficients. Given a 1D Bernstein polynomial $x_N(t) : [0, t_f] \to \mathbb{R}$ with Bernstein coefficients $\bar{x}_{j,N} \in \mathbb{R}, \quad j = 0, \ldots, N$, the convex hull property implies that the following inequality holds:

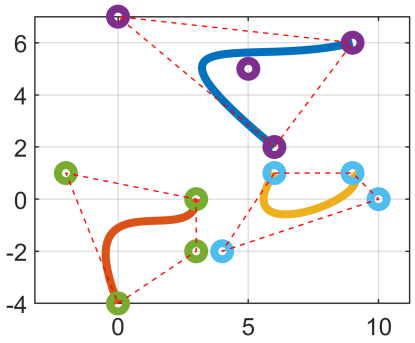
$$\min_j \bar{x}_{j,N} \leq x_N(t) \leq \max_j \bar{x}_{j,N}$$

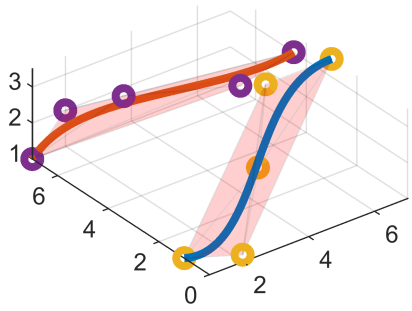
for all $t \in [0, t_f]$. The following code provides a depiction of this property.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Depiction of convex hull property
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


% Plot
figure
subplot(2,2,1);
plot(t,xN,'LineWidth',3); hold on
plot(linspace(0,tf,length(x_N)),x_N,'o','LineWidth',3);
plot(linspace(0,tf,length(x_N)),ones(length(x_N))*max(x_N),'--','LineWidth',0.5,'Color','r');
plot(linspace(0,tf,length(x_N)),ones(length(x_N))*min(x_N),'--','LineWidth',0.5,'Color','r');
%set(gca,'fontsize', 40);
```

```
axis([0 10 0 5])
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.52, 0.45, 0.45]);
subplot(2,2,3)
plot(x1N(1,:),x1N(2,:),'LineWidth',3); hold on
plot(x2N(1,:),x2N(2,:),'LineWidth',3);
plot(x3N(1,:),x3N(2,:),'LineWidth',3);
plot(x1_N(1,:),x1_N(2,:),'o','LineWidth',3);
plot(x2_N(1,:),x2_N(2,:),'o','LineWidth',3);
plot(x3_N(1,:),x3_N(2,:),'o','LineWidth',3);
[k,av] = convhull(x1_N');
plot(x1_N(1,k),x1_N(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x2_N');
plot(x2_N(1,k),x2_N(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x3_N');
plot(x3_N(1,k),x3_N(2,k),'--','LineWidth',0.5,'Color','r');
%set(gca,'fontsize', 40);
axis equal
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.02, 0.45, 0.45]);
subplot(2,2,[2,4])
plot3(p1N(1,:),p1N(2,:),p1N(3,:),'LineWidth',3); hold on
plot3(p2N(1,:),p2N(2,:),p2N(3,:),'LineWidth',3);
plot3(p1_N(1,:),p1_N(2,:),p1_N(3,:),'o','LineWidth',3);
plot3(p2_N(1,:),p2_N(2,:),p2_N(3,:),'o','LineWidth',3);
[k,av] = convhull(p1_N');
trisurf(k,p1_N(1,:),p1_N(2,:),p1_N(3,:),'FaceColor','r','FaceAlpha',.1,'EdgeAlpha',.1)
[k,av] = convhull(p2_N');
trisurf(k,p2_N(1,:),p2_N(2,:),p2_N(3,:),'FaceColor','r','FaceAlpha',.1,'EdgeAlpha',.1)
%set(gca,'fontsize', 40);
axis equal
grid on
```








```
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.35, 0.15, 0.8, 0.8]);
```

Degree Elevation

A Bernstein polynomial $x_N(t)$ of degree N with Bernstein coefficients $\bar{x}_{j,N},\ j = 0, \ldots, N$, can be expressed as a Bernstein polynomial $x_M(t)$ of degree $M,\ M > N$, with Bernstein coefficients given by

$$\bar{x}_{k,M} = \sum_{j=\max(0,k+M-N)}^{\min(N,k)} \frac{\binom{M-N}{k-j}\binom{N}{j}}{\binom{M}{k}} \bar{x}_{j,N},$$

with $k = 0, \ldots, M$. The above equation can be rewritten in matrix form as

$$\bar{x}_M = \bar{x}_N \mathbf{E}_N^M,$$

where $\bar{x}_N = [\bar{x}_{0,N}, \ldots, \bar{x}_{N,N}]$ and $\bar{x}_M = [\bar{x}_{0,M}, \ldots, \bar{x}_{M,M}]$ are the vectors of Bernstein coefficients, and $\mathbf{E}_N^M = \{e_{j,k}\} \in \mathbb{R}^{N+1 \times M+1}$ is the matrix of degree elevation from order N to order M with

$$e_{i,i+j} = \frac{\binom{M-N}{j}\binom{N}{i}}{\binom{M}{i+j}},$$

$i = 0, \ldots, N$ and $j = 0, \ldots, M - N$.

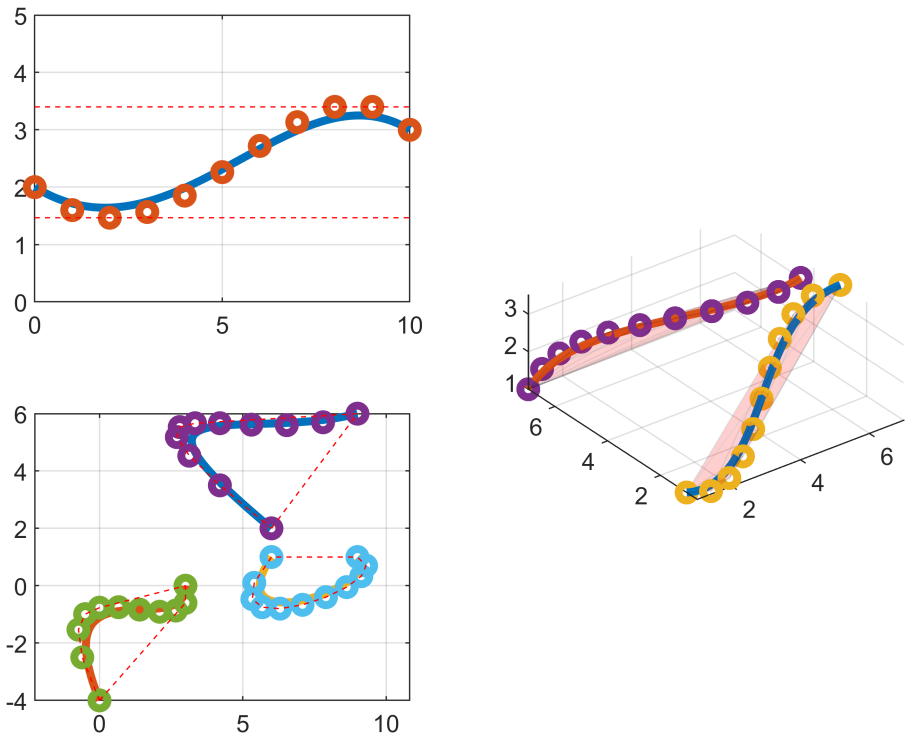
In BeBOT, the function

```
% DegElevMatrix(N,M)
```

generates the degree elevation matrix given above. The following script shows an example where the 1D, 2D and 3D Bernstein polynomials defined earlier are degree elevated.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Degree evelation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


M = 10; % Elevate to Mth order


% Degree elevate the coefficients of the 1D Bernstein polynomial
x_N_elev = x_N*DegElevMatrix(length(x_N)-1,M);


% Degree elevate the coefficients of the 2D Bernstein polynomials
x1_N_elev = x1_N*DegElevMatrix(length(x1_N)-1,M);
x2_N_elev = x2_N*DegElevMatrix(length(x2_N)-1,M);
x3_N_elev = x3_N*DegElevMatrix(length(x3_N)-1,M);


% Degree elevate the coefficients of the 3D Bernstein polynomials
p1_N_elev = p1_N*DegElevMatrix(length(p1_N)-1,M);
p2_N_elev = p2_N*DegElevMatrix(length(p2_N)-1,M);




% Plot
figure
subplot(2,2,1);
plot(t,xN,'LineWidth',3); hold on
plot(linspace(0,tf,length(x_N_elev)),x_N_elev,'o','LineWidth',3);
plot(linspace(0,tf,length(x_N_elev)),ones(length(x_N_elev))*max(x_N_elev),'--','LineWidth',0.5,
plot(linspace(0,tf,length(x_N_elev)),ones(length(x_N_elev))*min(x_N_elev),'--','LineWidth',0.5,
%set(gca,'fontsize', 40);
axis([0 10 0 5])
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.52, 0.45, 0.45]);
subplot(2,2,3)
plot(x1N(1,:),x1N(2,:),'LineWidth',3); hold on
plot(x2N(1,:),x2N(2,:),'LineWidth',3);
plot(x3N(1,:),x3N(2,:),'LineWidth',3);
plot(x1_N_elev(1,:),x1_N_elev(2,:),'o','LineWidth',3);
```

```
plot(x2_N_elev(1,:),x2_N_elev(2,:),'o','LineWidth',3);
plot(x3_N_elev(1,:),x3_N_elev(2,:),'o','LineWidth',3);
[k,av] = convhull(x1_N_elev');
plot(x1_N_elev(1,k),x1_N_elev(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x2_N_elev');
plot(x2_N_elev(1,k),x2_N_elev(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x3_N_elev');
plot(x3_N_elev(1,k),x3_N_elev(2,k),'--','LineWidth',0.5,'Color','r');
%set(gca,'fontsize', 40);
axis equal
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.02, 0.45, 0.45]);
subplot(2,2,[2,4])
plot3(p1N(1,:),p1N(2,:),p1N(3,:),'LineWidth',3); hold on
plot3(p2N(1,:),p2N(2,:),p2N(3,:),'LineWidth',3);
plot3(p1_N_elev(1,:),p1_N_elev(2,:),p1_N_elev(3,:),'o','LineWidth',3);
plot3(p2_N_elev(1,:),p2_N_elev(2,:),p2_N_elev(3,:),'o','LineWidth',3);
[k,av] = convhull(p1_N_elev');
trisurf(k,p1_N_elev(1,:),p1_N_elev(2,:),p1_N_elev(3,:),'FaceColor','r','FaceAlpha',.1,'EdgeAlph
[k,av] = convhull(p2_N_elev');
trisurf(k,p2_N_elev(1,:),p2_N_elev(2,:),p2_N_elev(3,:),'FaceColor','r','FaceAlpha',.1,'EdgeAlph
%set(gca,'fontsize', 40);
axis equal
grid on
```




```
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.35, 0.15, 0.8, 0.8]);
```

It can be noted that the Bernstein coefficients of a degree elevated Bernstein polynomial converge to the polynomial itself. Mathematically, we have

$$\max_j \left\| \bar{x}_{j,M} - x_M\left(\frac{j}{M} t_f\right) \right\| \leq \frac{C_d}{M}$$

where C_d is a positive constant independent of M. The equation above implies that by applying degree elevation and using the convex hull property, one can easily compute a convex shape that is less conservative than the original convex hull and that still contains the Bernstein polynomial.

de Casteljau Algorithm

The de Casteljau algorithm is an efficient and numerically stable recursive method to evaluate a Bernstein polynomial at any given point. The algorithm is also used to split a Bernstein polynomial into two independent ones.

Given an Nth order Bernstein polynomial $x_N : [0, t_f] \to \mathbb{R}^n$, and a scalar $\lambda \in [0, 1]$, the Bernstein polynomial at $t_{\text{div}} = \lambda t_f$ can be computed using the following recursive relation:

$$\bar{x}_{i,N}^{[0]} = \bar{x}_{i,N}\,, \quad i = 0, \ldots, N\,,$$

$$\bar{x}_{i,N}^{[j]} = \bar{x}_{i,N}^{[j-1]}(1-\lambda) + \bar{x}_{i+1,N}^{[j-1]}\lambda\,,$$

with $i = 0, \ldots, N-j$, and $j = 1, \ldots, N$. Then, the Bernstein polynomial evaluated at t_{div} is given by

$$x_N(t_{\text{div}}) = \bar{x}_{0,N}^{[N]}\,.$$

Moreover, the Bernstein polynomial can be subdivided at t_{div} into two Nth order Bernstein polynomials with Bernstein coefficients

$$\bar{x}_{0,N}^{[0]}, \bar{x}_{0,N}^{[1]}, \ldots, \bar{x}_{0,N}^{[N]}\,, \quad \text{and} \quad \bar{x}_{0,N}^{[N]}, \bar{x}_{1,N}^{[N-1]}, \ldots, \bar{x}_{N,N}^{[0]}\,.$$

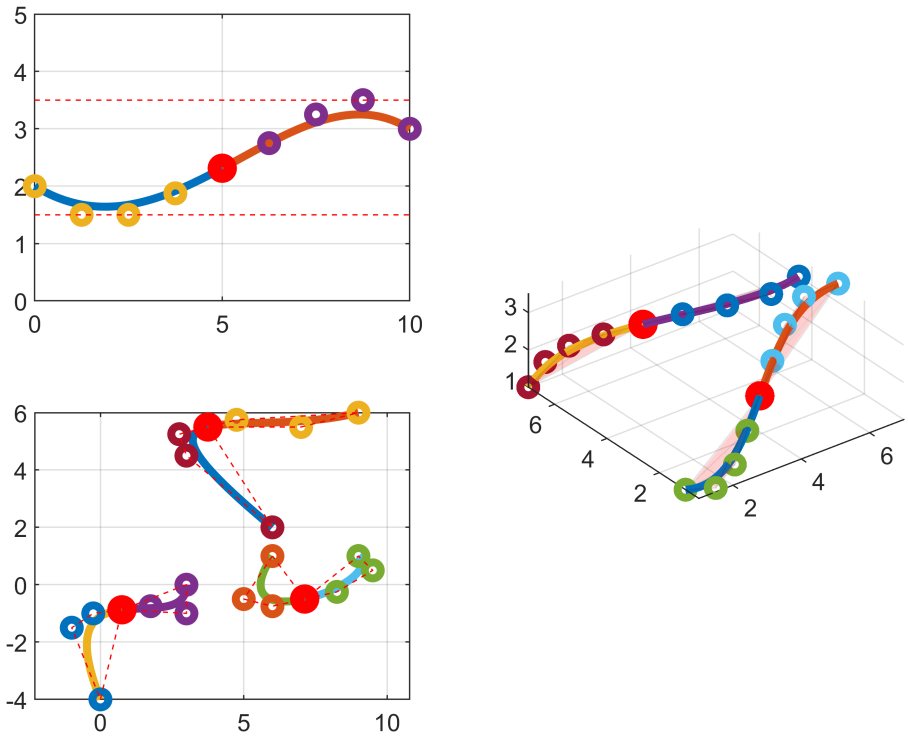
The following code is used to generate figures that depict Bernstein polynomials (with Bernstein coefficients defined earlier) that are subdivided into two 5th order Bernstein polynomials at $\lambda = 0.5$ (i.e. they are split at $t_f/2$).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% de Casteljau algorithm
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


lambda = 0.5; % Value at which we subdivide (lambda \in [0,1])


% Subdivide the 1D Bernstein polynomials at lambda
[Cpout Pos] = deCasteljau(x_N,lambda);
% Pos is the value of xN at lambda*tf (or xN(lambda*tf))
x_NA = Cpout(:, 1:length(x_N(1,:)));
```

```
x_NB = Cpout(:, length(x_N(1,:)):end);
% x_NA and x_NB are the coefficients of the two Bernstein polynomials
xNA = BernsteinPoly(x_NA,t);
xNB = BernsteinPoly(x_NB,t);


% Plot
figure
subplot(2,2,1);
tA = linspace(0,lambda,length(xNA))*tf;
tB = linspace(lambda,1,length(xNA))*tf;
plot(tA,xNA,'LineWidth',3); hold on
plot(tB,xNB,'LineWidth',3);
plot(linspace(0,lambda*tf,length(x_NA)),x_NA,'o','LineWidth',3);
plot(linspace(lambda*tf,tf,length(x_NB)),x_NB,'o','LineWidth',3);
plot(lambda*tf,Pos,'o','LineWidth',5,'Color','r');
plot(linspace(0,tf,length(x_N)),ones(length(x_N))*max(max(x_NA),max(x_NB)),'--','LineWidth',0.5
plot(linspace(0,tf,length(x_N)),ones(length(x_N))*min(min(x_NA),min(x_NB)),'--','LineWidth',0.5
%set(gca,'fontsize', 40);
axis([0 10 0 5])
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.52, 0.45, 0.45]);




% Subdivide the 2D Bernstein polynomials at lambda
[Cpout Pos1] = deCasteljau(x1_N,lambda);
x1_NA = Cpout(:, 1:length(x1_N(1,:)));
x1_NB = Cpout(:, length(x1_N(1,:)):end);
x1NA = BernsteinPoly(x1_NA,t);
x1NB = BernsteinPoly(x1_NB,t);
[Cpout Pos2] = deCasteljau(x2_N,lambda);
x2_NA = Cpout(:, 1:length(x2_N(1,:)));
x2_NB = Cpout(:, length(x2_N(1,:)):end);
x2NA = BernsteinPoly(x2_NA,t);
x2NB = BernsteinPoly(x2_NB,t);
[Cpout Pos3] = deCasteljau(x3_N,lambda);
x3_NA = Cpout(:, 1:length(x3_N(1,:)));
x3_NB = Cpout(:, length(x3_N(1,:)):end);
x3NA = BernsteinPoly(x3_NA,t);
x3NB = BernsteinPoly(x3_NB,t);


% Plot
subplot(2,2,3)
plot(x1NA(1,:),x1NA(2,:),'LineWidth',3); hold on
plot(x1NB(1,:),x1NB(2,:),'LineWidth',3);
```

```
plot(x2NA(1,:),x2NA(2,:),'LineWidth',3);
plot(x2NB(1,:),x2NB(2,:),'LineWidth',3);
plot(x3NA(1,:),x3NA(2,:),'LineWidth',3);
plot(x3NB(1,:),x3NB(2,:),'LineWidth',3);
plot(x1_NA(1,:),x1_NA(2,:),'o','LineWidth',3);
plot(x2_NA(1,:),x2_NA(2,:),'o','LineWidth',3);
plot(x3_NA(1,:),x3_NA(2,:),'o','LineWidth',3);
plot(x1_NB(1,:),x1_NB(2,:),'o','LineWidth',3);
plot(x2_NB(1,:),x2_NB(2,:),'o','LineWidth',3);
plot(x3_NB(1,:),x3_NB(2,:),'o','LineWidth',3);
plot(Pos1(1),Pos1(2),'o','LineWidth',5,'Color','r');
plot(Pos2(1),Pos2(2),'o','LineWidth',5,'Color','r');
plot(Pos3(1),Pos3(2),'o','LineWidth',5,'Color','r');
[k,av] = convhull(x1_NA');
plot(x1_NA(1,k),x1_NA(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x1_NB');
plot(x1_NB(1,k),x1_NB(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x2_NA');
plot(x2_NA(1,k),x2_NA(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x2_NB');
plot(x2_NB(1,k),x2_NB(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x3_NA');
plot(x3_NA(1,k),x3_NA(2,k),'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(x3_NB');
plot(x3_NB(1,k),x3_NB(2,k),'--','LineWidth',0.5,'Color','r');
%set(gca,'fontsize', 40);
axis equal
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.02, 0.45, 0.45]);




% Subdivide the 3D Bernstein polynomials at lambda
[Cpout Pos1] = deCasteljau(p1_N,lambda);
p1_NA = Cpout(:, 1:length(p1_N(1,:)));
p1_NB = Cpout(:, length(p1_N(1,:)):end);
p1NA = BernsteinPoly(p1_NA,t);
p1NB = BernsteinPoly(p1_NB,t);
[Cpout Pos2] = deCasteljau(p2_N,lambda);
p2_NA = Cpout(:, 1:length(p2_N(1,:)));
p2_NB = Cpout(:, length(p2_N(1,:)):end);
p2NA = BernsteinPoly(p2_NA,t);
p2NB = BernsteinPoly(p2_NB,t);
subplot(2,2,[2,4])
plot3(p1NA(1,:),p1NA(2,:),p1NA(3,:),'LineWidth',3); hold on
plot3(p1NB(1,:),p1NB(2,:),p1NB(3,:),'LineWidth',3);
plot3(p2NA(1,:),p2NA(2,:),p2NA(3,:),'LineWidth',3);
```

```
plot3(p2NB(1,:),p2NB(2,:),p2NB(3,:),'LineWidth',3);
plot3(p1_NA(1,:),p1_NA(2,:),p1_NA(3,:),'o','LineWidth',3);
plot3(p1_NB(1,:),p1_NB(2,:),p1_NB(3,:),'o','LineWidth',3);
plot3(p2_NA(1,:),p2_NA(2,:),p2_NA(3,:),'o','LineWidth',3);
plot3(p2_NB(1,:),p2_NB(2,:),p2_NB(3,:),'o','LineWidth',3);
plot3(Pos1(1),Pos1(2),Pos1(3),'o','LineWidth',5,'Color','r');
plot3(Pos2(1),Pos2(2),Pos2(3),'o','LineWidth',5,'Color','r');
[k,av] = convhull(p1_NA');
trisurf(k,p1_NA(1,:),p1_NA(2,:),p1_NA(3,:),'FaceColor','r','FaceAlpha',.1,'EdgeAlpha',.1)
[k,av] = convhull(p1_NB');
trisurf(k,p1_NB(1,:),p1_NB(2,:),p1_NB(3,:),'FaceColor','r','FaceAlpha',.1,'EdgeAlpha',.1)
[k,av] = convhull(p2_NA');
trisurf(k,p2_NA(1,:),p2_NA(2,:),p2_NA(3,:),'FaceColor','r','FaceAlpha',.1,'EdgeAlpha',.1)
[k,av] = convhull(p2_NB');
trisurf(k,p2_NB(1,:),p2_NB(2,:),p2_NB(3,:),'FaceColor','r','FaceAlpha',.1,'EdgeAlpha',.1)
%set(gca,'fontsize', 40);
axis equal
grid on
```




```
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.35, 0.15, 0.8, 0.8]);
```

Notice that by employing the de Casteljau algorithm, smaller convex hulls containing the Bernstein polynomials are obtained.

Minimum Distance Algorithm

By exploiting the convex-hull property, the end point values property, and the de Casteljau algorithm, in combination with the Gilbert-Johnson-Keerthi (GJK) distance algorithm, one can implement computationally efficient algorithms to evaluate the minimum distance between two Bernstein polynomials, $f_N(t)$ and $g_N(t)$, namely

$$\min_{t_a,t_b\in[0,t_f]} ||f_N(t_a) - g_N(t_b)||$$

$$\text{argmin}_{t_a,t_b\in[0,t_f]} ||f_N(t_a) - g_N(t_b)||$$

In few words, the GJK algorithm, widely used in computer graphics and video games to compute the minimum distance between convex shapes, can be used to compute the distance between two convex hulls containing Bernstein polynomials. The de Casteljau algorithm can be used to split these convex hulls in smaller objects. These two algorithms can be repeated iteratively until the points of minimum distance are the end points of the polynomials' subdivisions. The same procedure can be also used to compute the extrema of a 1D Bernstein polynomials, or the distance between a Bernstein polynomial and a convex shape.

BeBOT offers functions to compute the minimum distance between two 2D or 3D Bernstein polynomials

```
% MinDistBernstein2Bernstein.m
```

the minimum distance between a 2D or 3D Bernstein polynomials and a convex shape

```
% MinDistBernstein2Polygon.m
```

and the extrema of a 1D Bernstein polynomial

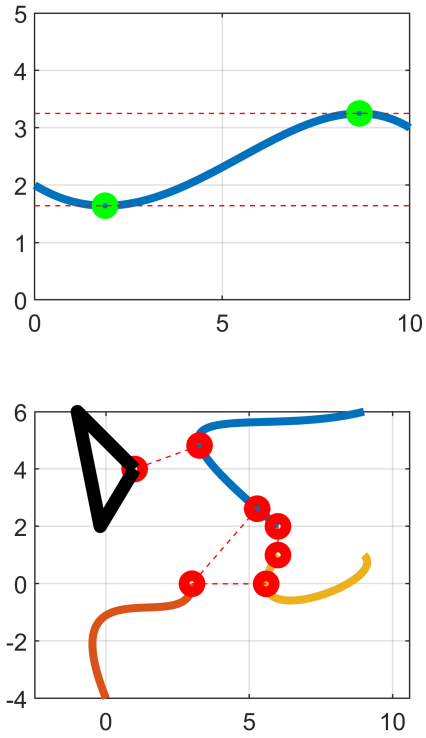
```
% MaximumBernstein.m
% MinimumBernstein.m
```

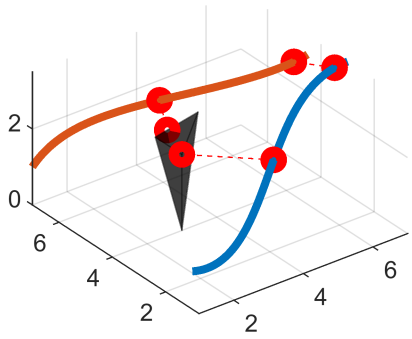
An example showing how these functions are used is given below.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Minimum distance algorithm
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


% Compute max and min of a 1D Bernstein polynomial
[max, tmax] = MaximumBernstein(x_N);
[~, pmax] = deCasteljau(x_N,tmax);
[min, tmin] = MinimumBernstein(x_N);
[~, pmin] = deCasteljau(x_N,tmin);


% Plot
figure
subplot(2,2,1);
plot(t,xN,'LineWidth',3); hold on
tnodes = linspace(0,tf,length(x_N));
plot(tmax*tf,max,'o','LineWidth',4,'Color','g');
```

```
plot(linspace(0,10,tf),max*ones(10,1),'--','LineWidth',0.5,'Color','r');
plot(tmin*tf,min,'o','LineWidth',4,'Color','g');
plot(linspace(0,10,tf),min*ones(10,1),'--','LineWidth',0.5,'Color','r');
%set(gca,'fontsize', 40);
axis([0 10 0 5])
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.52, 0.45, 0.45]);


% Compute distance between 2D Bernstein polynomials
% and between 2D Bernstein polynomials and convex shapes
[dist, t1, t2] = MinDistBernstein2Bernstein(x1_N, x2_N);
[~,x12col1] = deCasteljau(x1_N,t1);
[~,x12col2] = deCasteljau(x2_N,t2);
[dist, t1, t3] = MinDistBernstein2Bernstein(x1_N, x3_N);
[~,x13col1] = deCasteljau(x1_N,t1);
[~,x13col3] = deCasteljau(x3_N,t3);
[dist, t2, t3] = MinDistBernstein2Bernstein(x2_N, x3_N);
[~,x23col2] = deCasteljau(x2_N,t2);
[~,x23col3] = deCasteljau(x3_N,t3);
shape = [1 -1  -0.2; 4 6 2];
[dist, t, pt] = MinDistBernstein2Polygon(x1_N, shape);
[~,x1shapecol1] = deCasteljau(x1_N,t);


% Plot
subplot(2,2,3)
plot(x1N(1,:),x1N(2,:),'LineWidth',3); hold on
plot(x2N(1,:),x2N(2,:),'LineWidth',3);
plot(x3N(1,:),x3N(2,:),'LineWidth',3);
plot(x12col1(1),x12col1(2),'o','LineWidth',4,'Color','r');
plot(x12col2(1),x12col2(2),'o','LineWidth',4,'Color','r');
plot([x12col1(1) x12col2(1)],[x12col1(2) x12col2(2)],'--','LineWidth',0.5,'Color','r');
plot(x13col1(1),x13col1(2),'o','LineWidth',4,'Color','r');
plot(x13col3(1),x13col3(2),'o','LineWidth',4,'Color','r');
plot([x13col1(1) x13col3(1)],[x13col1(2) x13col3(2)],'--','LineWidth',0.5,'Color','r');
plot(x23col2(1),x23col2(2),'o','LineWidth',4,'Color','r');
plot(x23col3(1),x23col3(2),'o','LineWidth',4,'Color','r');
plot([x23col2(1) x23col3(1)],[x23col2(2) x23col3(2)],'--','LineWidth',0.5,'Color','r');
plot(x1shapecol1(1),x1shapecol1(2),'o','LineWidth',4,'Color','r');
plot(pt(1),pt(2),'o','LineWidth',4,'Color','r');
plot([x1shapecol1(1) pt(1)],[x1shapecol1(2) pt(2)],'--','LineWidth',0.5,'Color','r');
[k,av] = convhull(shape');
plot(shape(1,k),shape(2,k),'-','LineWidth',5,'Color','k');
%set(gca,'fontsize', 40);
axis equal
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.02, 0.02, 0.45, 0.45]);
```

```
% Compute distance between 3D Bernstein polynomials
[dist, t1, t2] = MinDistBernstein2Bernstein(p1_N, p2_N);
[~,pcol1] = deCasteljau(p1_N,t1);
[~,pcol2] = deCasteljau(p2_N,t2);
shape = [3 3 3 5; 4 4 5 6; 0 2 2 1.5];
[dist, t, pt1] = MinDistBernstein2Polygon(p1_N, shape);
[~,pcol1shape] = deCasteljau(p1_N,t);
[dist, t, pt2] = MinDistBernstein2Polygon(p2_N, shape);
[~,pcol2shape] = deCasteljau(p2_N,t);


% Plot
subplot(2,2,[2,4])
plot3(p1N(1,:),p1N(2,:),p1N(3,:),'LineWidth',3); hold on
plot3(p2N(1,:),p2N(2,:),p2N(3,:),'LineWidth',3);
plot3(pcol1(1),pcol1(2),pcol1(3),'o','LineWidth',4,'Color','r');
plot3(pcol2(1),pcol2(2),pcol2(3),'o','LineWidth',4,'Color','r');
plot3([pcol1(1) pcol2(1)],[pcol1(2) pcol2(2)],[pcol1(3) pcol2(3)],'--','LineWidth',0.5,'Color',
%set(gca,'fontsize', 40);
axis equal
grid on
%set(gca, 'Units', 'Normalized', 'OuterPosition', [0.35, 0.15, 0.8, 0.8]);
plot3(pcol1shape(1),pcol1shape(2),pcol1shape(3),'o','LineWidth',4,'Color','r');
plot3(pcol2shape(1),pcol2shape(2),pcol2shape(3),'o','LineWidth',4,'Color','r');
plot3(pt1(1),pt1(2),pt1(3),'o','LineWidth',4,'Color','r');
plot3(pt2(1),pt2(2),pt2(3),'o','LineWidth',4,'Color','r');
plot3([pcol1shape(1) pt1(1)],[pcol1shape(2) pt1(2)],[pcol1shape(3) pt1(3)],'--','LineWidth',0.5
plot3([pcol2shape(1) pt2(1)],[pcol2shape(2) pt2(2)],[pcol2shape(3) pt2(3)],'--','LineWidth',0.5
[k,av] = convhull(shape');
trisurf(k,shape(1,:),shape(2,:),shape(3,:),'FaceColor','k','FaceAlpha',.5,'EdgeAlpha',.5)
```






```
%set(gca,'fontsize', 40);
```

Check Collision

BeBOT can be used to check whether the minimum distance between two Bernstein polynomials, or between a Bernstein polynomials and a convex shape, is greater than a minimun specified distance. The check collision algorithms returns a 0 (collision occurs) or 1 (collision does not occur) and is more computationally efficient than the minimum distance algorithm. Here is an example

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Check collision algorithm
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
collision = CollCheckBernstein2Bernstein(p1_N, p2_N, 1.6)
```

```
collision = logical
   1
```

```
collision = CollCheckBernstein2Bernstein(p1_N, p2_N, 1.5)
```

```
collision = logical
   0
```

```
collision = CollCheckBernstein2Polygon(p1_N, shape, 2.4)
```

```
collision = logical
   1
```

```
collision = CollCheckBernstein2Polygon(p1_N, shape, 2.3)
```

```
collision = logical
   0
```

Arithmetic Operations

The sum (difference) of two polynomials of the same order can be performed by simply adding (subtracting) their Bernstein coefficients. Let $f_M(t)$ and $g_N(t)$ be two 1-dimensional Bernstein polynomials of degree M and N, respectively, with coefficients $\bar{f}_M = [\bar{f}_{0,M}, \ldots, \bar{f}_{M,M}]$ and $\bar{g}_N = [\bar{g}_{0,N}, \ldots, \bar{g}_{N,N}]$. The product $y_{M+N}(t) = f_M(t) g_N(t)$ is a Bernstein polynomial of degree $(M+N)$ with coefficients $\bar{y}_{M+N} = [\bar{y}_{0,M+N}, \ldots, \bar{y}_{M+N,M+N}]$, given by

$$\bar{y}_{k,M+N} = \sum_{j=\max(0,k-N)}^{\min(M,k)} \frac{\binom{M}{j}\binom{N}{k-j}}{\binom{M+N}{k}} \bar{f}_{j,M} \bar{g}_{k-j,N}.$$

With BeBOT, the Bernstein coefficients of the product between two Bernstein polynomials are easily computed as follows:

```
% Compute Bernstein coefficients of product
f_M = [2 4 6 9 4];
g_N = [3 2 1 1 3 8 2];
y_MpN = BernsteinProduct(f_M,g_N);
```

The ratio between two 1D Bernstein polynomials, $f_N(t)$ and $g_N(t)$, with Bernstein coefficients $\bar{f}_{0,N}, \ldots, \bar{f}_{N,N}$ and $\bar{g}_{0,N}, \ldots, \bar{g}_{N,N}$, i.e., $r_N(t) = f_N(t)/g_N(t)$, can be expressed as a Nth order *rational* Bernstein polynomial defined as

$$r_N(t) = \frac{\sum_{i=0}^{n} \bar{r}_{i,N} w_{i,n} b_{i,N}(t)}{\sum_{i=0}^{n} w_{i,N} b_{i,N}(t)}, \quad t \in [0, t_f],$$

with coefficients and weights $\bar{r}_{i,N} = \dfrac{\bar{f}_{i,N}}{\bar{g}_{i,N}}, \quad w_{i,N} = \bar{f}_{i,N}.$

The end point values, convex hull, and degree elevation properties apply also to rational Bernstein polynomials. The de Casteljau and minimum distance algorithms can be easily extended to *rational* Bernstein polynomials.

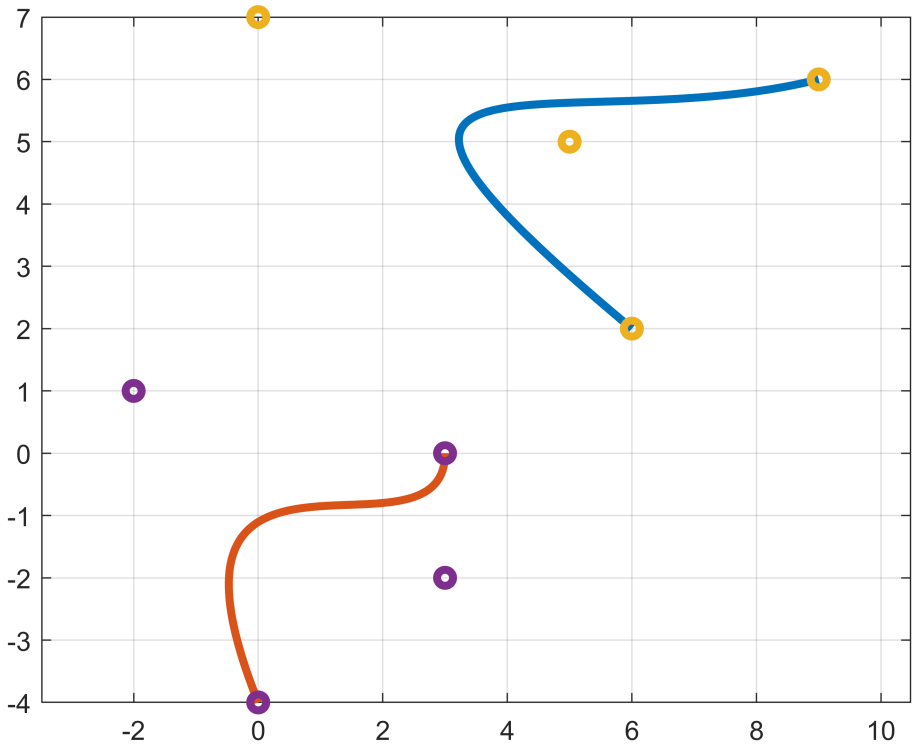
Bernstein Polynomials as Vehicles' Trajectories

In what follows we present the use of Bernstein polynomials as a tool to represent vehicle trajectories.

2D Trajectories

```
clear all

tf = 10;
t = linspace(0,tf,1000);
```

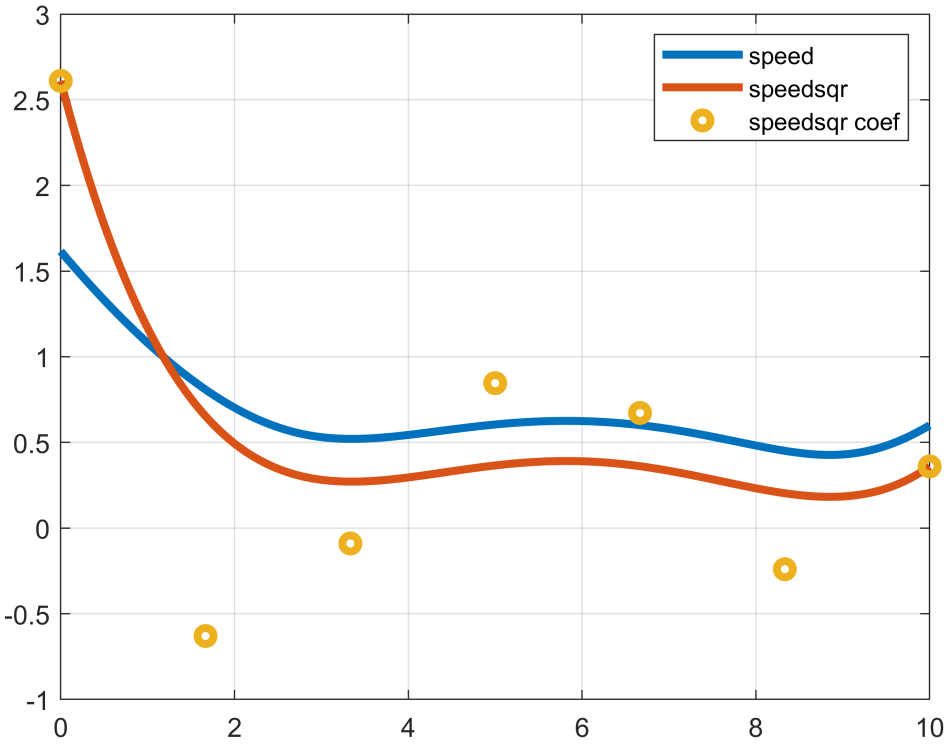
```
%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 2D Trajectories
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 2D Bernstein Polynomials
x1_N = [6 0 5 9; 2 7 5 6];
x2_N = [0 -2 3 3; -4 1 -2 0];
x1N = BernsteinPoly(x1_N,t);
x2N = BernsteinPoly(x2_N,t);

% Plot
figure
plot(x1N(1,:),x1N(2,:),'LineWidth',3); hold on
plot(x2N(1,:),x2N(2,:),'LineWidth',3);
plot(x1_N(1,:),x1_N(2,:),'o','LineWidth',3);
plot(x2_N(1,:),x2_N(2,:),'o','LineWidth',3);
%set(gca,'fontsize', 40);
axis equal
grid on
```



Given the 2D trajectories above, the speed (square) of the vehicles can be computed as follows

```
%% Speed square
[~,N] = size(x2_N);
N = N - 1;
```

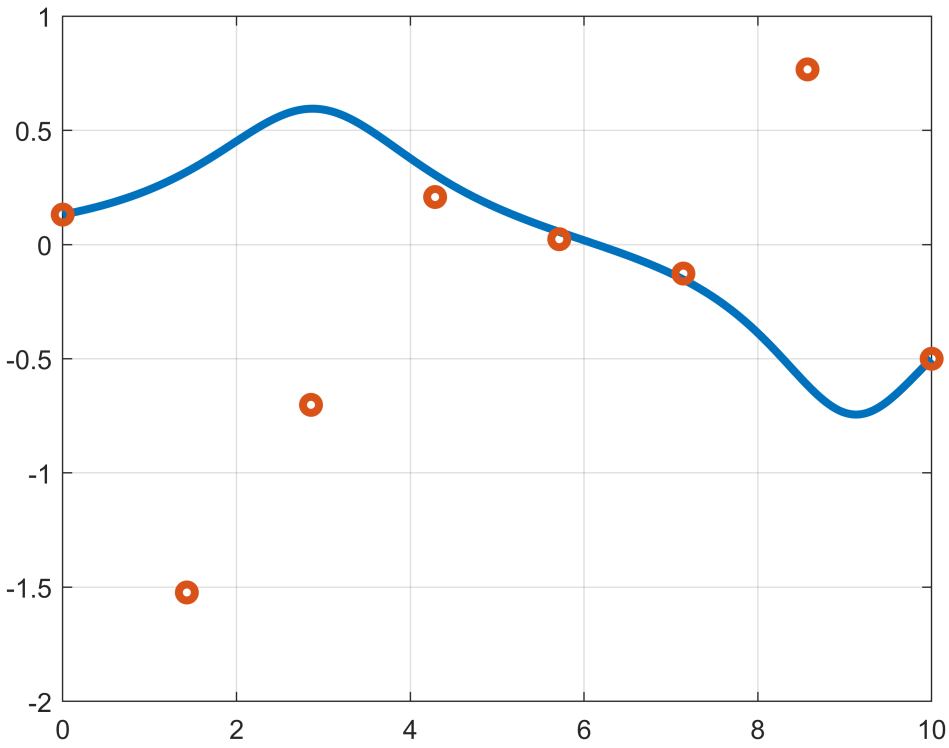
```
[tnodes,w,Dm] = BeBOT(N,tf);
x2_Ndot = x2_N*Dm;
speedsqr_2N = BernsteinProduct(x2_Ndot(1,:),x2_Ndot(1,:))+BernsteinProduct(x2_Ndot(2,:),x2_Ndot
speedsqr2N = BernsteinPoly(speedsqr_2N,t);
speed2N = sqrt(speedsqr2N);

% Plot
figure
plot(t,speed2N,'LineWidth',3); hold on
plot(t,speedsqr2N,'LineWidth',3);
plot(linspace(0,tf,2*N+1),speedsqr_2N,'o','LineWidth',3);
%set(gca,'fontsize', 40);
legend('speed','speedsqr','speedsqr coef')
grid on
```



The angular rates can be computed as follows

```
%% Psidot
x2_Nddot = x2_Ndot*Dm;
psidot_num = (BernsteinProduct(x2_Nddot(1,:),x2_Ndot(2,:))-BernsteinProduct(x2_Nddot(2,:),x2_N
psidot_num = psidot_num*DegElevMatrix(length(psidot_num)-1,length(psidot_num));
psidot_den = speedsqr_2N*DegElevMatrix(length(speedsqr_2N)-1,length(speedsqr_2N));
psidot_coeff = psidot_num./(psidot_den);
psidot_weights = (psidot_den);
psidot = BernsteinPoly(psidot_num,t)./BernsteinPoly(psidot_den,t);

% Plot
```

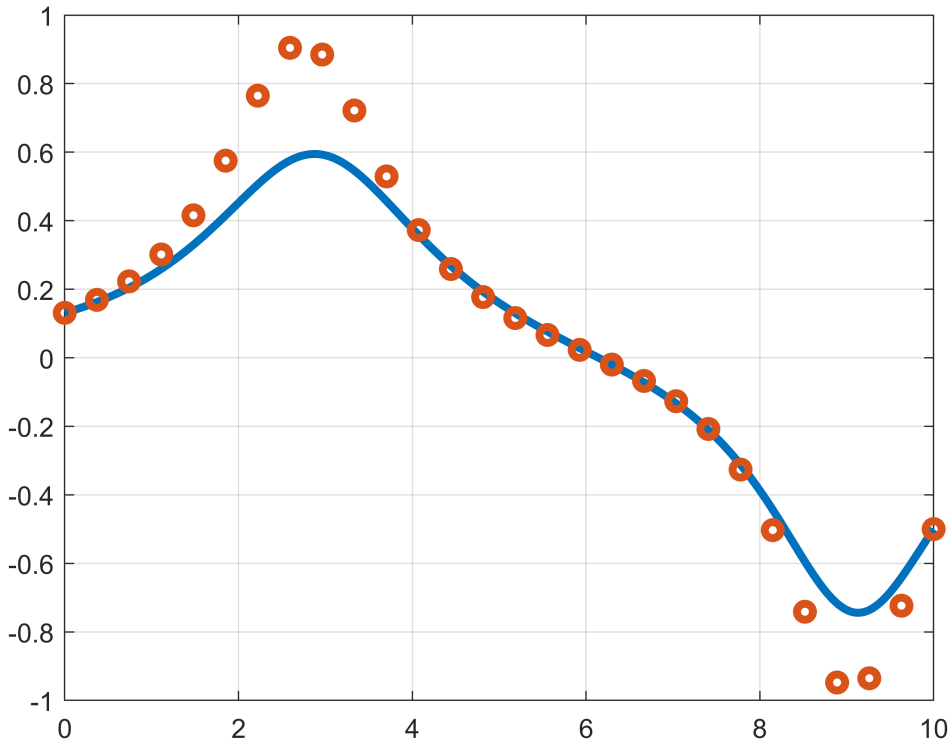
```
figure
plot(t,psidot,'LineWidth',3); hold on
plot(linspace(0,tf,length(psidot_coeff)),psidot_coeff,'o','LineWidth',3);
%set(gca,'fontsize', 40);
grid on
```




```
```

These can be degree elevated

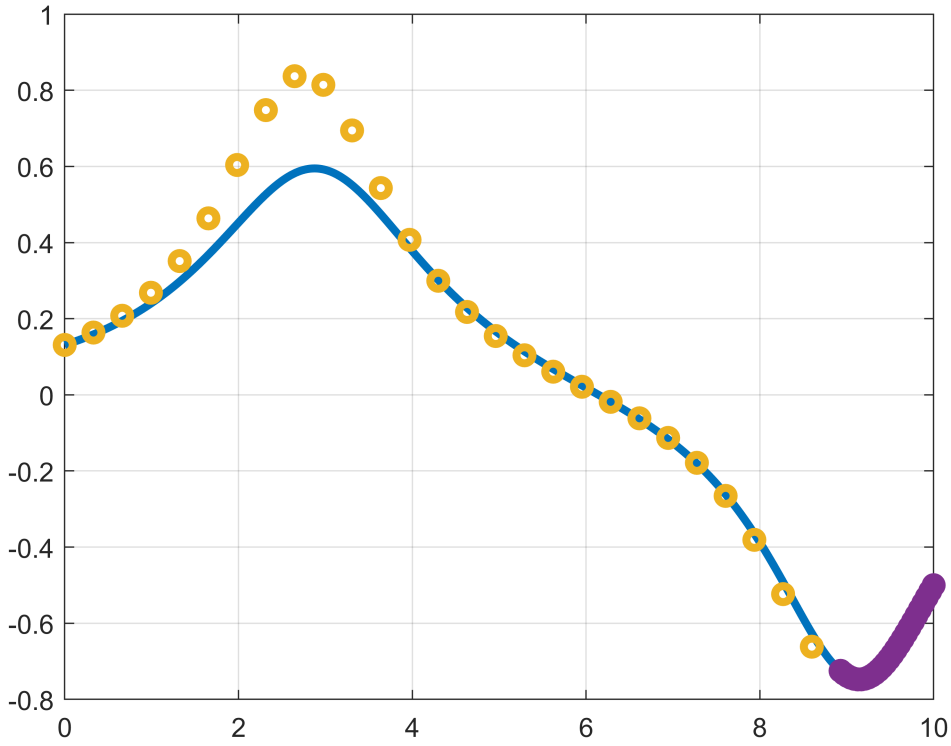
```
% Degree elevate
[~,N] = size(psidot_num);
N = N - 1;
psidot_num = psidot_num*DegElevMatrix(N,N+20);
psidot_den = psidot_den*DegElevMatrix(N,N+20);
psidot_coeff = psidot_num./psidot_den;
psidot_weights = psidot_den;
psidot = BernsteinPoly(psidot_num,t)./BernsteinPoly(psidot_den,t);

% Plot
figure
plot(t,psidot,'LineWidth',3); hold on
[~,M] = size(psidot_coeff);
plot(linspace(0,tf,M),psidot_coeff,'o','LineWidth',3);
%set(gca,'fontsize', 40);
grid on
```



or subdivided

```
% Subdivide at min
[minv,iminv] = min(psidot_coeff);
lambdamin = iminv/length(psidot_coeff);
[Cpnum,Pout] = deCasteljau(psidot_num,lambdamin);
[Cpden,Pout] = deCasteljau(psidot_den,lambdamin);
psidot_numA = Cpnum(:, 1:length(psidot_num(1,:)));
psidot_numB = Cpnum(:, length(psidot_num(1,:)):end);
psidot_denA = Cpden(:, 1:length(psidot_num(1,:)));
psidot_denB = Cpden(:, length(psidot_num(1,:)):end);
psidot_coeffA = psidot_numA./psidot_denA;
psidot_weightsA = psidot_denA;
psidot_coeffB = psidot_numB./psidot_denB;
psidot_weightsB = psidot_denB;
tA = linspace(0,lambdamin*tf,1000);
tB = linspace(lambdamin*tf,tf,1000);
psidotA = BernsteinPoly(psidot_numA,tA)./BernsteinPoly(psidot_denA,tA);
psidotB = BernsteinPoly(psidot_numB,tB)./BernsteinPoly(psidot_denB,tB);

% Plot
figure
plot(tA,psidotA,'LineWidth',3); hold on
```

```
plot(tB,psidotB,'LineWidth',3); hold on
[~,M] = size(psidot_coeffA);
plot(linspace(tA(1),tA(end),M),psidot_coeffA,'o','LineWidth',3);
plot(linspace(tB(1),tB(end),M),psidot_coeffB,'o','LineWidth',3);
%set(gca,'fontsize', 40);
grid on
```

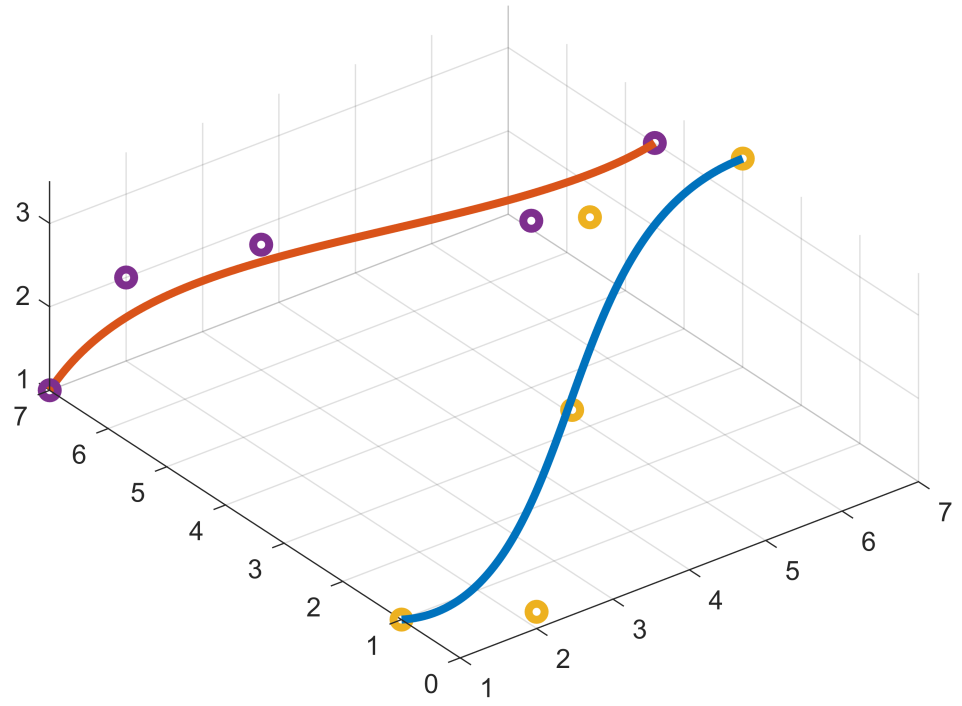



```
```

3D Trajectories

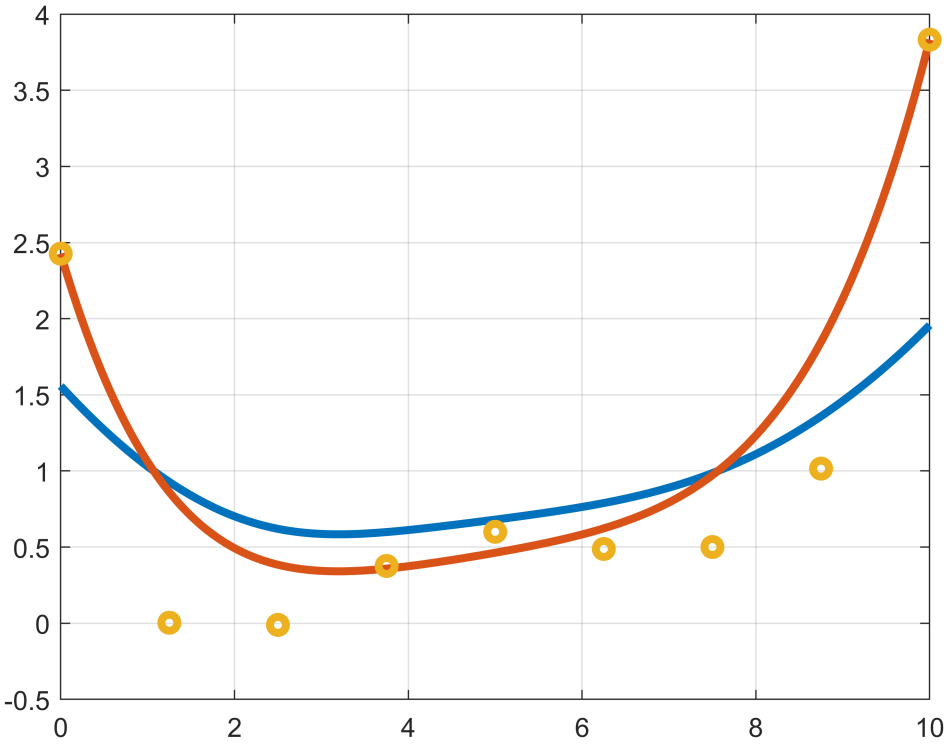
```
clear all

tf = 10;
t = linspace(0,tf,1000);

%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 3D trajectories
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


%% 3D Bernstein Polynomials
p1_N = [1,2,4,5,7; 1,0,2,3,3; 1,1.2,2,3.5,3.5];
p2_N = [1,2,3,5,7; 7,7,6,4,4.5; 1,2,2.5,3,3];
p1N = BernsteinPoly(p1_N,t);
```

```
p2N = BernsteinPoly(p2_N,t);


% Plot
figure
plot3(p1N(1,:),p1N(2,:),p1N(3,:),'LineWidth',3); hold on
plot3(p2N(1,:),p2N(2,:),p2N(3,:),'LineWidth',3);
plot3(p1_N(1,:),p1_N(2,:),p1_N(3,:),'o','LineWidth',3);
plot3(p2_N(1,:),p2_N(2,:),p2_N(3,:),'o','LineWidth',3);
%set(gca,'fontsize', 40);
axis equal
grid on
```

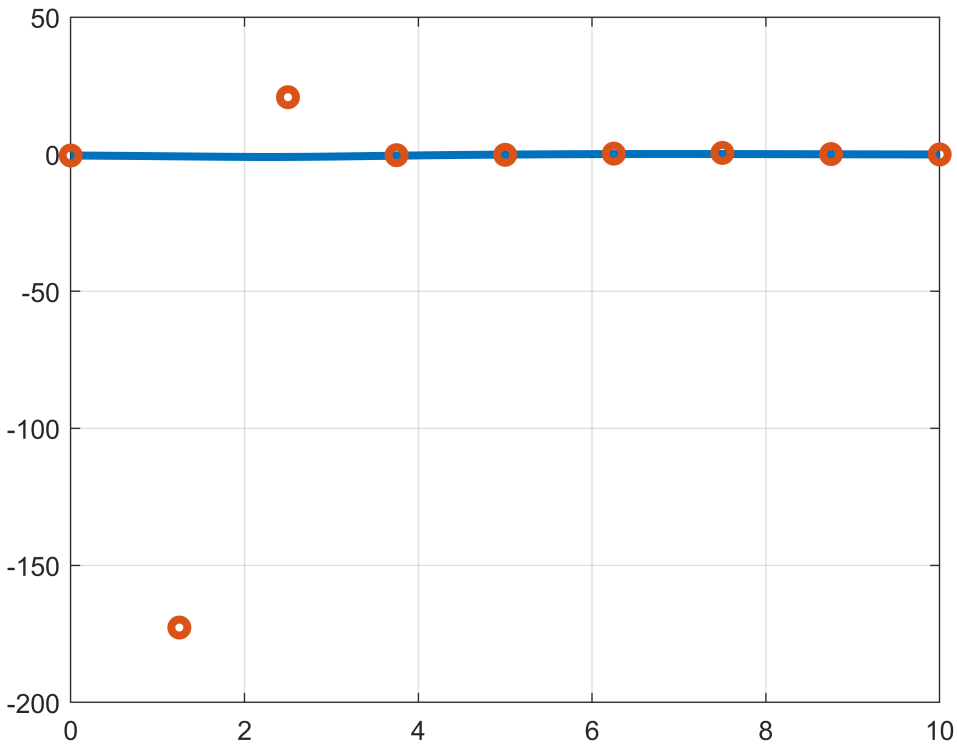



```


%% Speed square
[~,N] = size(p1_N);
N = N - 1;
[tnodes,w,Dm] = LGL_PS(N,tf);
p1_Ndot = p1_N*Dm;
speedsqr_2N = BernsteinProduct(p1_Ndot(1,:),p1_Ndot(1,:))+BernsteinProduct(p1_Ndot(2,:),p1_Ndot
speedsqr2N = BernsteinPoly(speedsqr_2N,t);
speed2N = sqrt(speedsqr2N);


% Plot
figure
```

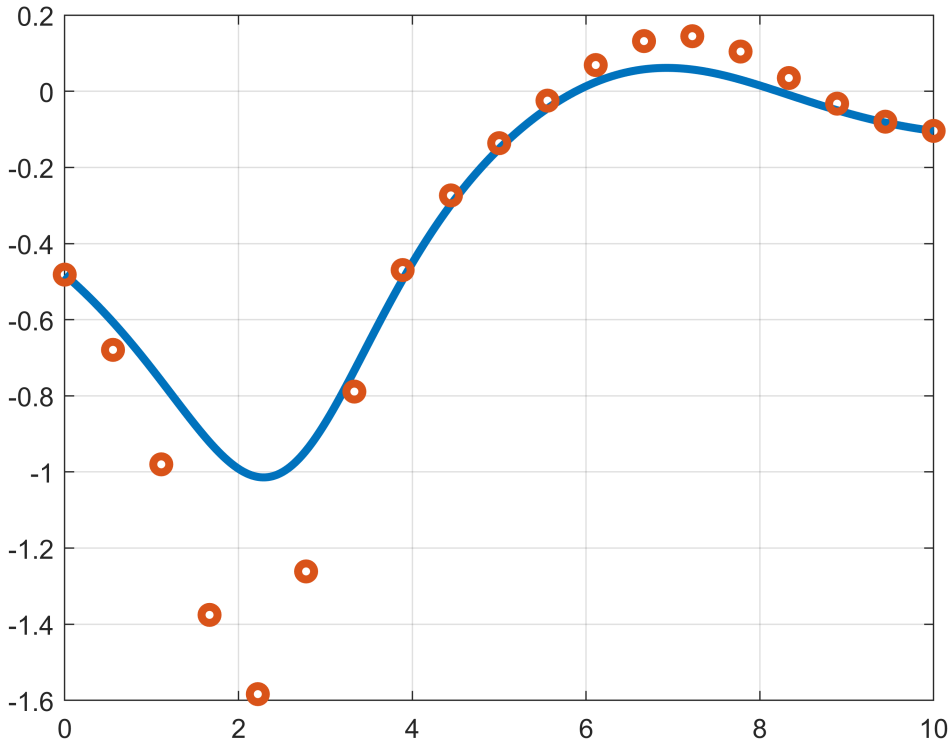
```
plot(t,speed2N,'LineWidth',3); hold on
plot(t,speedsqr2N,'LineWidth',3);
plot(linspace(0,tf,2*N+1),speedsqr_2N,'o','LineWidth',3);
%set(gca,'fontsize', 40);
grid on
```




```
%% Psidot
p1_Nddot = p1_Ndot*Dm;
psidot_num = (BernsteinProduct(p1_Nddot(1,:),p1_Ndot(2,:))-BernsteinProduct(p1_Nddot(2,:),p1_N
psidot_den = speedsqr_2N;
psidot_coeff = psidot_num./psidot_den;
psidot_weights = psidot_den;
psidot = BernsteinPoly(psidot_num,t)./BernsteinPoly(psidot_den,t);

% Plot
figure
plot(t,psidot,'LineWidth',3); hold on
plot(linspace(0,tf,2*N+1),psidot_coeff,'o','LineWidth',3);
%set(gca,'fontsize', 40);
grid on
```

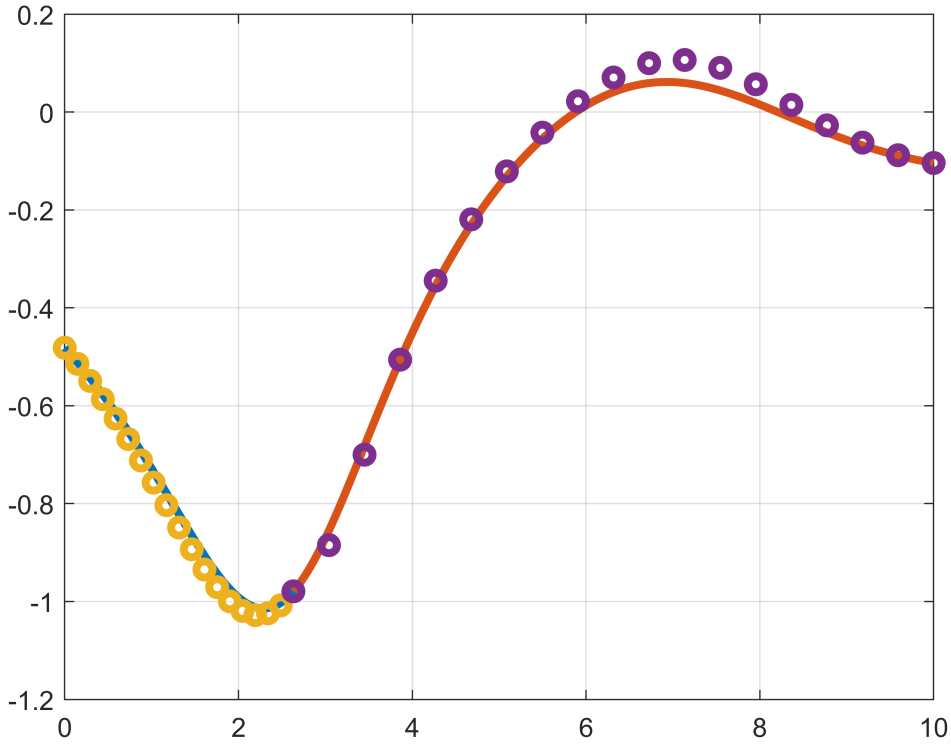



```
% Degree elevate
[~,N] = size(psidot_num);
N = N - 1;
psidot_num = psidot_num*DegElevMatrix(N,N+10);
psidot_den = psidot_den*DegElevMatrix(N,N+10);
psidot_coeff = psidot_num./psidot_den;
psidot_weights = psidot_den;
psidot = BernsteinPoly(psidot_num,t)./BernsteinPoly(psidot_den,t);

% Plot
figure
plot(t,psidot,'LineWidth',3); hold on
[~,M] = size(psidot_coeff);
plot(linspace(0,tf,M),psidot_coeff,'o','LineWidth',3);
%set(gca,'fontsize', 40);
grid on
```




```
% Subdivide at min
[minv,iminv] = min(psidot_coeff);
lambdamin = iminv/length(psidot_coeff);
[Cpnum,Pout] = deCasteljau(psidot_num,lambdamin);
[Cpden,Pout] = deCasteljau(psidot_den,lambdamin);
psidot_numA = Cpnum(:, 1:length(psidot_num(1,:)));
psidot_numB = Cpnum(:, length(psidot_num(1,:)):end);
psidot_denA = Cpden(:, 1:length(psidot_num(1,:)));
psidot_denB = Cpden(:, length(psidot_num(1,:)):end);
psidot_coeffA = psidot_numA./psidot_denA;
psidot_weightsA = psidot_denA;
psidot_coeffB = psidot_numB./psidot_denB;
psidot_weightsB = psidot_denB;
tA = linspace(0,lambdamin*tf,1000);
tB = linspace(lambdamin*tf,tf,1000);
psidotA = BernsteinPoly(psidot_numA,tA)./BernsteinPoly(psidot_denA,tA);
psidotB = BernsteinPoly(psidot_numB,tB)./BernsteinPoly(psidot_denB,tB);

% Plot
figure
plot(tA,psidotA,'LineWidth',3); hold on
plot(tB,psidotB,'LineWidth',3); hold on
[~,M] = size(psidot_coeffA);
plot(linspace(tA(1),tA(end),M),psidot_coeffA,'o','LineWidth',3);
```

```
plot(linspace(tB(1),tB(end),M),psidot_coeffB,'o','LineWidth',3);
%set(gca,'fontsize', 40);
grid on
```



Bernstein Approximation

In what follows we introduce a few properties of Bernstein polynomials as function approximators.

Bernstein Quadrature

Recall that numerical quadrature is an approximation method for definite integrals. Bernstein quadrature establishes weights w_i for given equadistant nodes $t_i = \frac{i}{N} t_f$ to approximate an integral as a finite sum:

$$\int_a^b f(t)dt \approx \sum_{i=1}^{N} w_i f(t_i)$$

Note that the above finite sum involves evaluation of the function $f(t)$ at the equidistant nodes $\{t_i\}_{i=0}^{N}$. The weights are given by

$$w_i = \frac{t_f}{N+1} \qquad \forall i \in \{0, \dots, N\}$$

Example

Define the following functions

$$u_1(t) = \frac{1}{\sqrt{1+(T-t)^2}}$$

$$u_2(t) = \frac{T-t}{\sqrt{1+(T-t)^2}}$$

with $T = 12$.

```
clear all

T = 12;
u1 = @(t) 1./(sqrt(1+(T-t).^2));
u2 = @(t) (T-t)./(sqrt(1+(T-t).^2));
```

We plot these functions for $t \in [0, T]$

```
t = 0:0.01:T;
u1fig = figure;
plot(t,u1(t),'LineWidth',3,'Color','k'); hold on
grid on
u2fig = figure;
plot(t,u2(t),'Linewidth',3,'Color','k'); hold on
grid on
```

The integrals of $u_1(t)$and $u_2(t)$ are computed analytically as

$$\int_0^T u_1(t)dt = \text{asinh}(T), \quad \int_0^T u_2(t)dt = \sqrt{T^2+1} - 1$$

which are defined on Matlab as follows

```
intu1 = asinh(T);
intu2 = sqrt(T^2+1)-1;
```

The following code computes the integral numerically using Bernstein quadrature. First, we define the order of approximation:

```
N = 80;
```

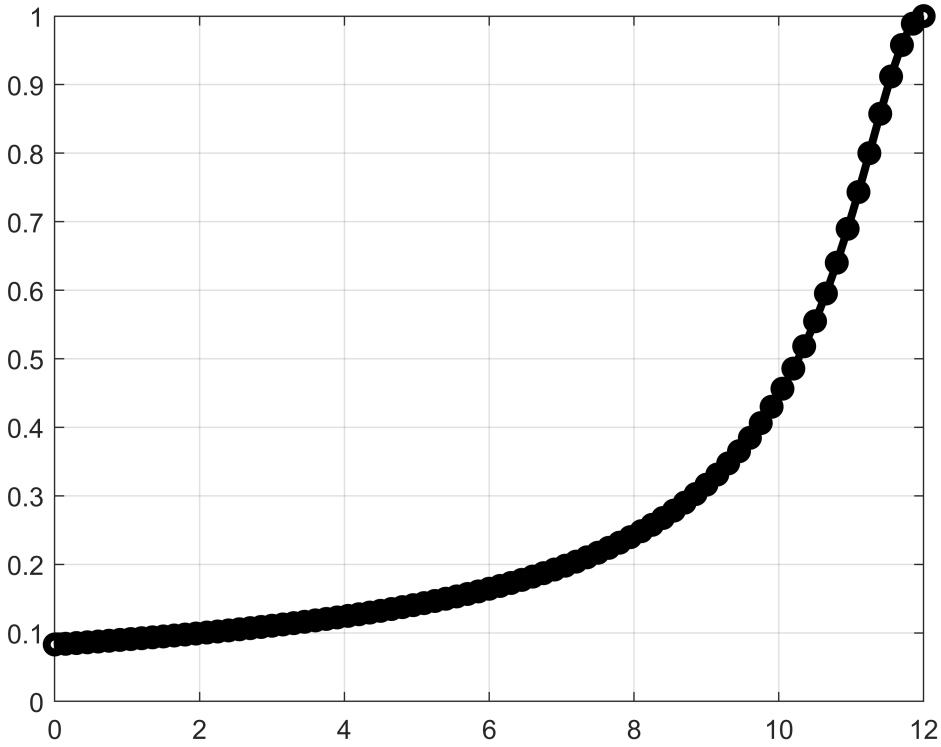
Then, we compute the integration weights and the equidistant nodes (though this should be quite straightforward, we use the BeBOT package)

```
[tnodes, w, D] = BeBOT(N,T);
```

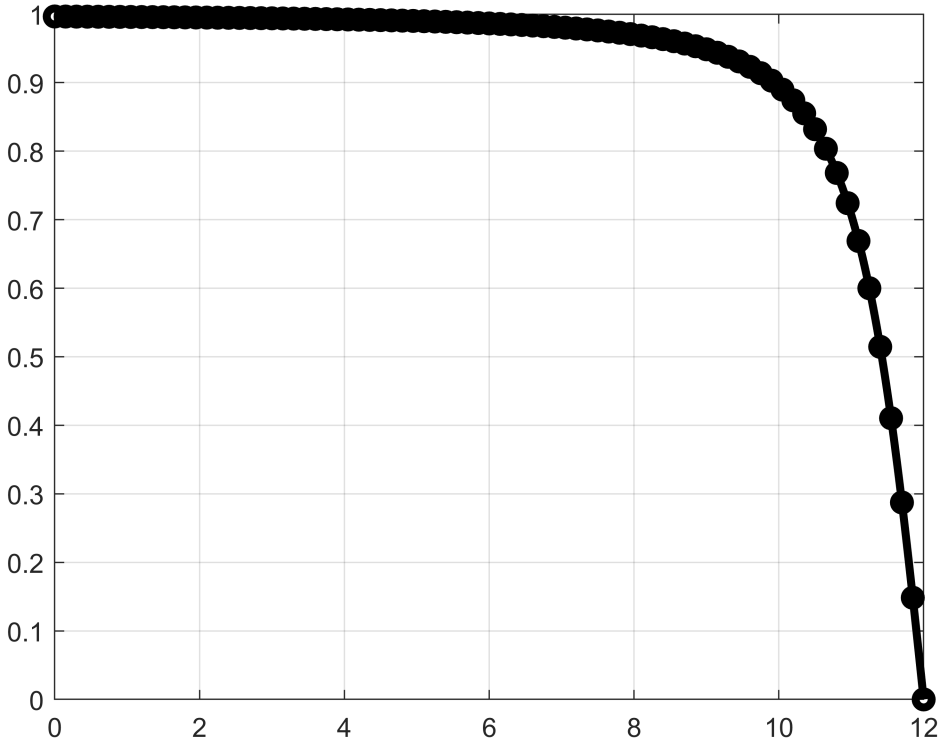
Then, we evaluate the functions $u_1(t)$ and $u_2(t)$at the time nodes

```
u1_N = u1(tnodes);
figure(u1fig);
plot(tnodes,u1_N,'o','LineWidth',3,'Color','k'); hold on
```

```
grid on
```




```
u2_N = u2(tnodes);
figure(u2fig);
plot(tnodes,u2_N,'o','LineWidth',3,'Color','k'); hold on
grid on
```



Finally, we apply the quadrature rule

```
int_u1 = u1_N*w;
int_u2 = u2_N*w;
```

and compare the analytical solution with the numerical one

```
err1 = int_u1 - intu1
```

```
err1 = 0.0410
```

```
err2 = int_u2 - intu2
```

```
err2 = -0.0644
```

You are welcome to play with the order of approximation to verify how the integration error decreases when increasing N.

Convergence properties

The following results hold for Bernstein quadrature:

if $f(t) \in \mathscr{C}^0$ on $[0, t_f]$ then we have

$$\left\| \int_0^{t_f} f(t)dt - \sum_{j=0}^{N} w_j f(t_j) \right\| \leq C_I W_f(N^{-1/2})$$

where W_f is the modulus of continuity of $f(t)$ and $C_I > 0$ is independent of N. Moreover, if $f(t) \in \mathscr{C}^2$ then

$$\left\| \int_0^{t_f} f(t)dt - \sum_{j=0}^{N} w_j f(t_j) \right\| \leq \frac{C_I}{N}$$

Bernstein Approximation

Given any function $f(t)$, the Bernstein polynomial $f_N(t)$ of degree N with Bernstein coefficients given by the function computed at equidistant nodes is called the Bernstein approximation of $f(t)$. For points

$$t_i = \frac{i}{N} t_f$$

this can be written as

$$f_N(t) = \sum_{i=0}^{N} f(t_i) b_{i,N}(t)$$

Example

For the functions $u_1(t)$ and $u_2(t)$ defined earlier and with order of approximation and equidistant time nodes

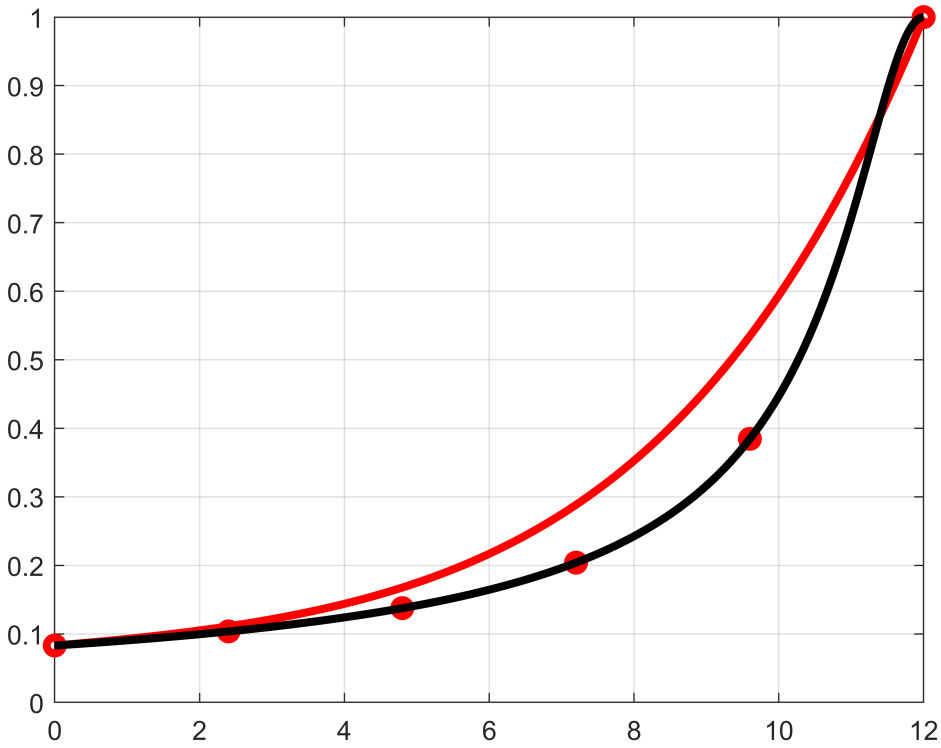
```
N = 5;
[t_nodes, ~, ~] = BeBOT(N,T);
```

the Bernstein approximations are given by

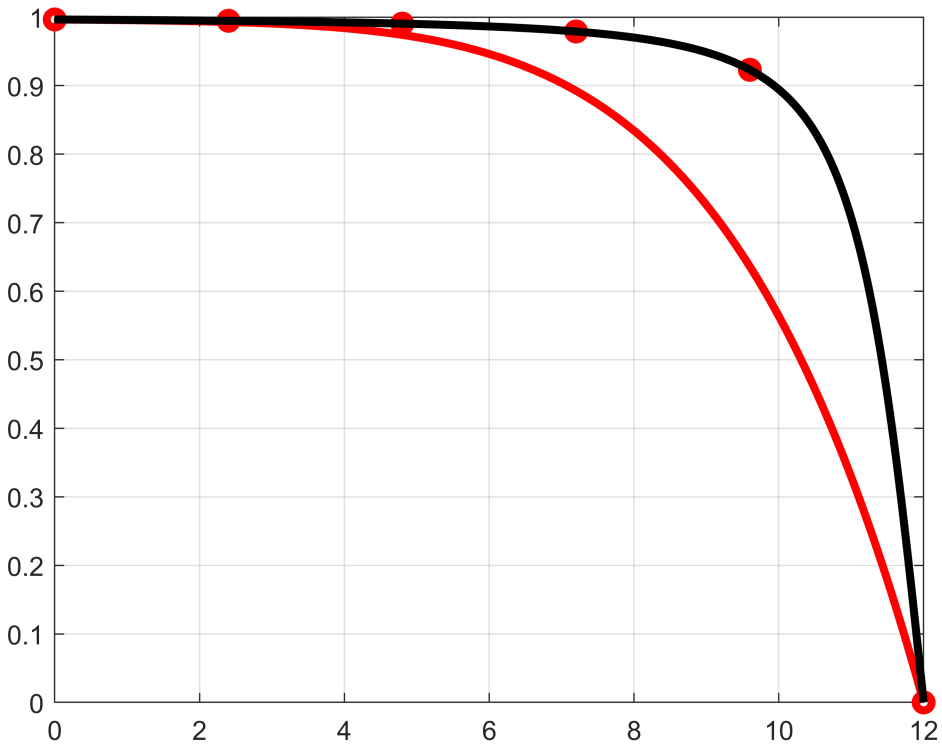
```
u1_N = BernsteinPoly(u1(t_nodes),t);
u2_N = BernsteinPoly(u2(t_nodes),t);
```

The Bernstein approximation and the nodes are plotted here

```
figure
plot(t_nodes,u1(t_nodes),'o','Color','r','LineWidth',3); hold on;
plot(t,u1_N,'Color','r','LineWidth',3);
plot(t,u1(t),'Color','k','LineWidth',3);
grid on
```




```
figure
plot(t_nodes,u2(t_nodes),'o','Color','r','LineWidth',3); hold on;
plot(t,u2_N,'Color','r','LineWidth',3);
plot(t,u2(t),'Color','k','LineWidth',3);
grid on
```



Convergence properties

The following results hold for Bernstein approximation. Let $f(t) \in \mathscr{C}^0$ on $[0, t_f]$. Then, the Bernstein approximant satisfies

$$||f_N(t) - f(t)|| \leq C_0 W_f(N^{-1/2})$$

where $C_0 > 0$ is independent on N. Moreover, if $f(t) \in \mathscr{C}^2$ then

$$||f_N(t) - f(t)|| \leq C_0/N$$

Example: Non-smooth functions

The convergence of the Bernstein approximation is much slower than that of Lagrange interpolating polynomials. However, Bernstein approximation exhibits uniform convergence properties even when the function that is being approximated is non smooth (no Gibbs phenomenon).

Define the following function

$$u(t) = 1, \quad t \leq \bar{T}/2$$

$$u(t) = -1, \quad t > \bar{T}/2$$

for $\bar{T} = 2$. In Matlab we write

```
syms u(t)
Tbar = 2;
```

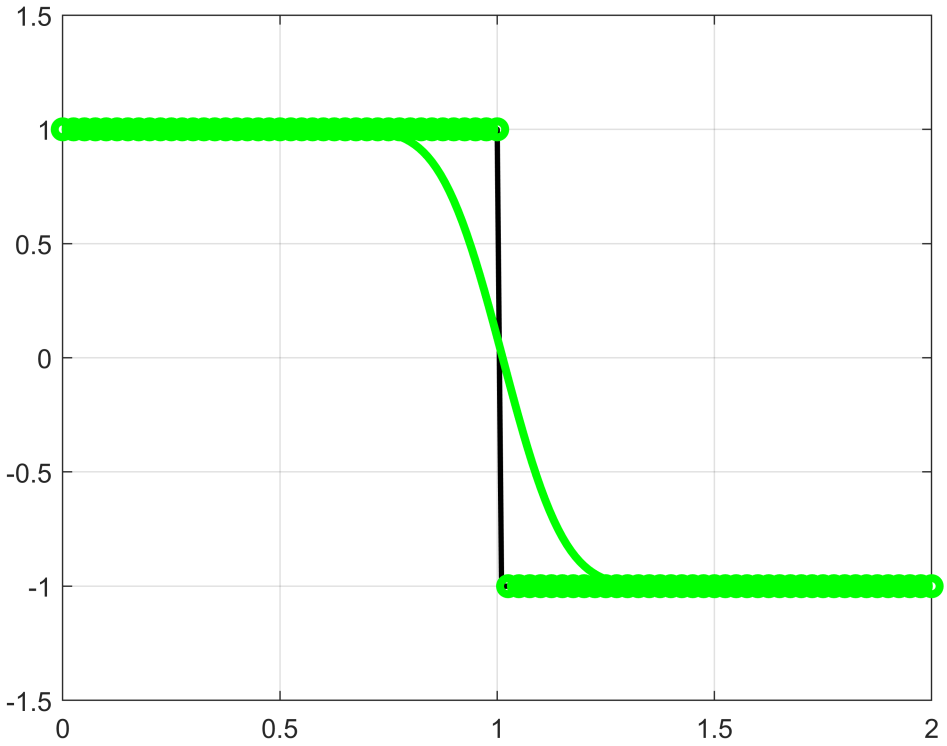
```
u(t) =@(t) piecewise(t<=Tbar/2, 1, Tbar/2<t, -1);
t = 0:0.01:Tbar;
uval = u(t);
```

which is plotted here

```
figure
plot(t,uval,'Linewidth',2,'Color','k'); hold on
grid on
```

Then, the Bernstein approximation of this function is computed and plotted as follows

```
N = 80;
[t_nodes,w,Dm] = BeBOT(N,Tbar);
u_N = BernsteinPoly(u(t_nodes),t);
plot(t_nodes,u(t_nodes),'o','Color','g','LineWidth',3);
plot(t,u_N,'Color','g','LineWidth',3);
```



You can modify the order of approximation to verify its dependence on the performance of the Bernstein approximant.

Bernstein Differentiation

Similarly to Lagrange interpolating polynomials, the Bernstein approximation of a function $f(t)$ by the Nth order Bernstein polynomial $f_N(t)$ also provides an estimate for the derivative $\dot{f}(t)$ through the derivative of $\dot{f}_N(t)$:

$$\dot{f}(t) \approx \dot{f}_N(t) = \sum_{i=0}^{N} f(t_i)\dot{b}_{i,N}(t)$$

The above equation can be written in matrix form as

$$\dot{f}_N(t) = \sum_{i=0}^{N}\sum_{j=0}^{N} f(t_j)D_{ji}b_{i,N}(t)$$

where D_{ji} is the (ij)entry of the differentiation matrix.

Consider the functions $u_1(t)$and $u_2(t)$defined earlier. Their derivatives are

$$\dot{u}_1(t) = \frac{T-t}{((T-t)^2+1)^{3/2}}$$

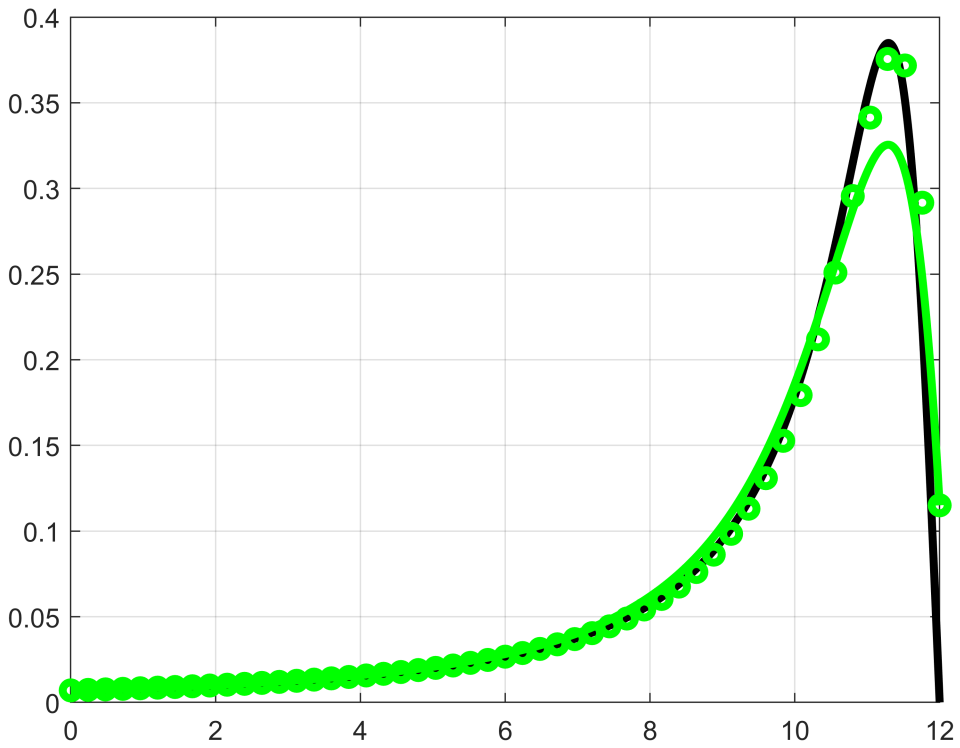
$$\dot{u}_2(t) = -\frac{1}{(T^2-2Tt+t^2+1)^{3/2}}$$

These are defined and plotted in Matlab as follows:

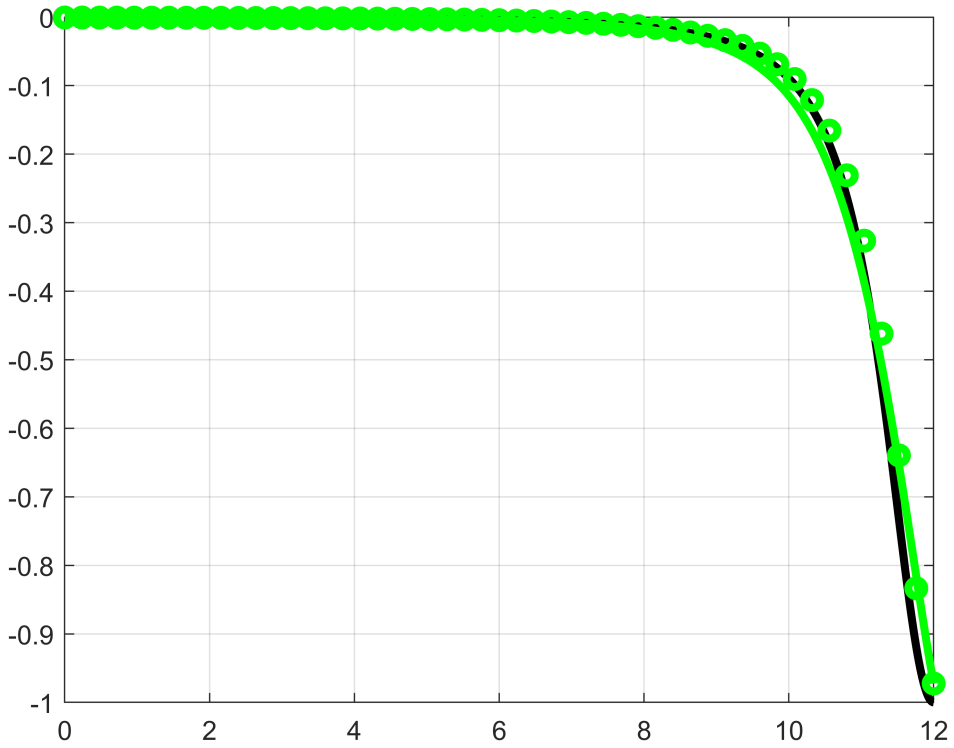
```
u1dot = @(t) (T-t)./((T-t).^2+1).^(3/2);
u2dot = @(t) -1./(T^2-2*T*t+t.^2+1).^(3/2);
t = 0:0.01:T;
fu1dot = figure;
plot(t,u1dot(t),'Linewidth',3,'Color','k'); hold on
grid on
fu2dot = figure;
plot(t,u2dot(t),'Linewidth',3,'Color','k'); hold on
grid on
```

These derivatives are then approximated by Bernstein polynomials as follows:

```
N = 50;
[t_nodes,w,Dm] = BeBOT(N,T);
u1dot_N = BernsteinPoly(u1(t_nodes)*Dm,t);
u2dot_N = BernsteinPoly(u2(t_nodes)*Dm,t);
figure(fu1dot);
plot(t_nodes,u1(t_nodes)*Dm,'o','Color','g','LineWidth',3); hold on
plot(t,u1dot_N,'Color','g','LineWidth',3);
```




```
figure(fu2dot);
plot(t_nodes,u2(t_nodes)*Dm,'o','Color','g','LineWidth',3); hold on
plot(t,u2dot_N,'Color','g','LineWidth',3);
```



Convergence properties

The following is true about the derivatives of Bernstein approximants. Let $f(t) \in \mathscr{C}^{r+2}$ on $[0, t_f]$, $r > 0$, and let $f_N(t)$ be the Bernstein approximant. Let $f^{(r)}(t)$ denote the rth derivative of $f(t)$. Then, the following inequalities hold:

$$||f_N(t) - f(t)|| \leq C_0/N$$

$$||\dot{f}_N(t) - \dot{f}(t)|| \leq C_1/N$$

$$\vdots$$

$$||f^r_N(t) - f^r(t)|| \leq C_r/N$$

Bernstein Integration

The integral of a Bernstein polynomial is also a Bernstein polynomial with control points computed as follows:

$$f(\bar{t}) = \dot{f} * I + f(0) * \mathbf{1}$$

where I is an integration matrix. In Matlab this can be computed as follows. We first define Bernstein coefficients

```
xn = [0 2 4 6 4 6 8 2 3 4 6];
```

describing a Bernstein polynomial of order

```
N = length(xn) - 1;
```

The derivative of this polynomial has control points

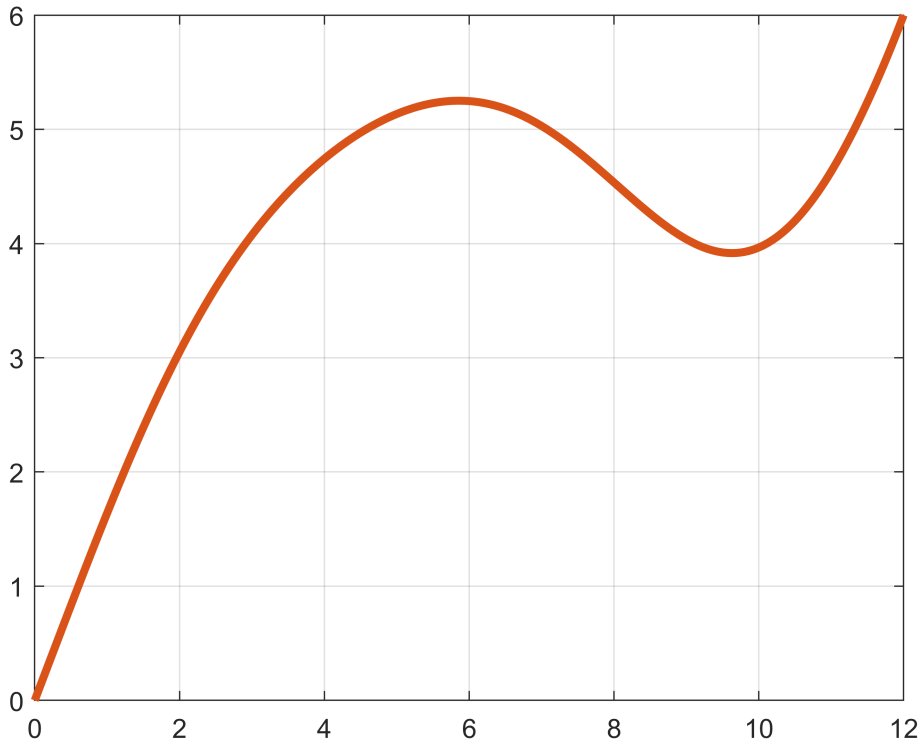
```
[t_nodes,w,Dm] = BeBOT(N,T);
xndot = xn*Dm;
```

If we integrate xndot we should recover xn. This is done as follows

```
I = BernsteinIntegrationMatrix(N,T);
xndot_int = xndot*I + ones(1,N+2)*xn(1);
```

We compare the results

```
figure
plot(t,BernsteinPoly(xn,t),'LineWidth',3); hold on
plot(t,BernsteinPoly(xndot_int,t),'LineWidth',3)
grid on
```

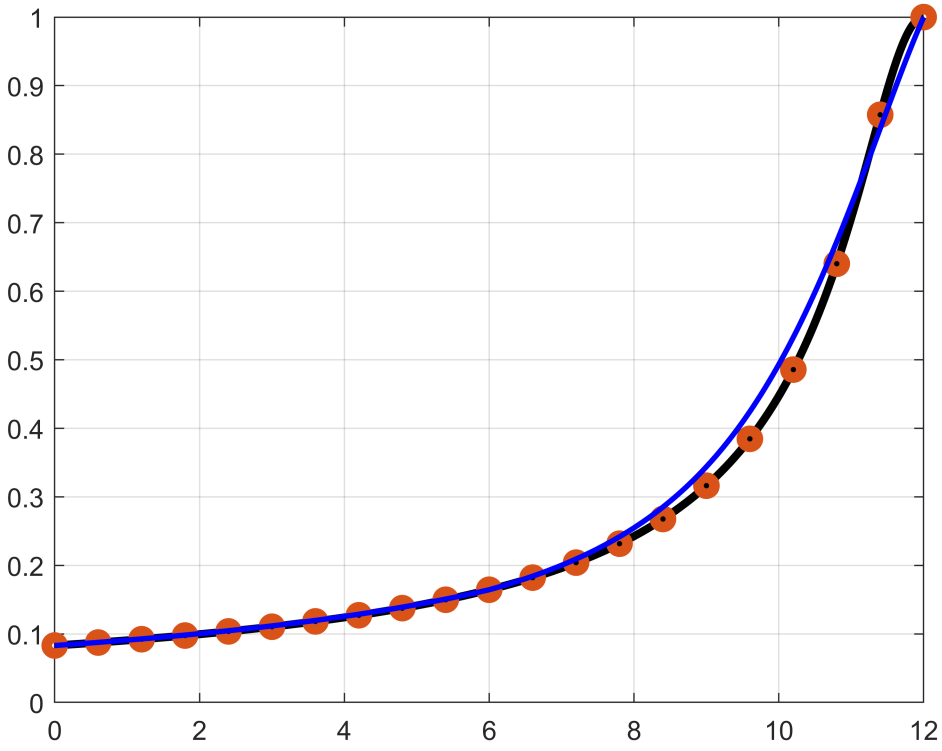


Approximation using Piecewise Bernstein Polynomials

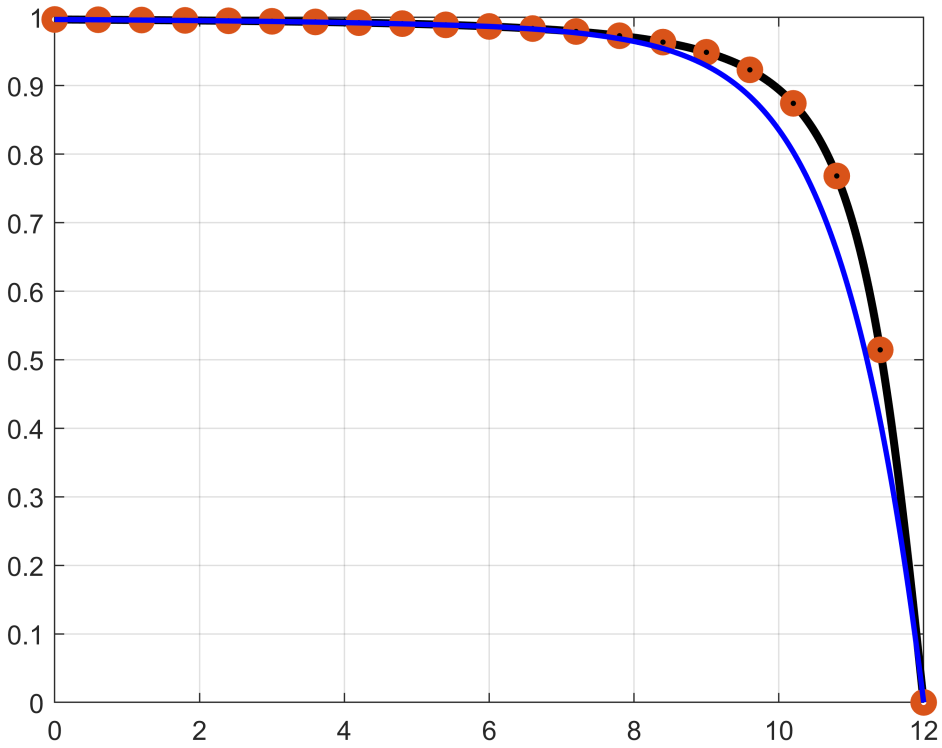
```
clear all

%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Bernstein Approximation of smooth functions
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


N = 10; % Order of approximation (N+1 nodes)


%% Define the following functions
T = 12;
u1 = @(t) 1./(sqrt(1+(T-t).^2));
u2 = @(t) (T-t)./(sqrt(1+(T-t).^2));
% Plot the functions in [0,T]
t = 0:0.01:T;
u1figcomp = figure;
plot(t,u1(t),'LineWidth',3,'Color','k'); hold on
%set(gca,'fontsize', 26);
grid on
u2figcomp = figure;
plot(t,u2(t),'Linewidth',3,'Color','k'); hold on
grid on
%set(gca,'fontsize', 26);


% Their derivatives are
u1dot = @(t) (T-t)./((T-t).^2+1).^(3/2);
u2dot = @(t) -1./(T^2-2*T*t+t.^2+1).^(3/2);
% Plot the derivatives in [0,T]
u1dotfigcomp = figure;
plot(t,u1dot(t),'Linewidth',3,'Color','k'); hold on
%set(gca,'fontsize', 26);
grid on
u2dotfigcomp = figure;
plot(t,u2dot(t),'Linewidth',3,'Color','k'); hold on
%set(gca,'fontsize', 26);
grid on

% Their integrals are
intu1 = asinh(T);
intu2 = sqrt(T^2+1)-1;

%% Compute Bern approx nodes/weights/Diff
tknots = linspace(0,T,3);
[tnodes,w,D] = PiecewiseBeBOT(N,tknots);

%% Plot the function at the equidistant nodes
u1nodes = u1(tnodes);
u2nodes = u2(tnodes);
figure(u1figcomp)
plot(tnodes,u1nodes,'o','Linewidth',4);
figure(u2figcomp)
plot(tnodes,u2nodes,'o','Linewidth',4);
```

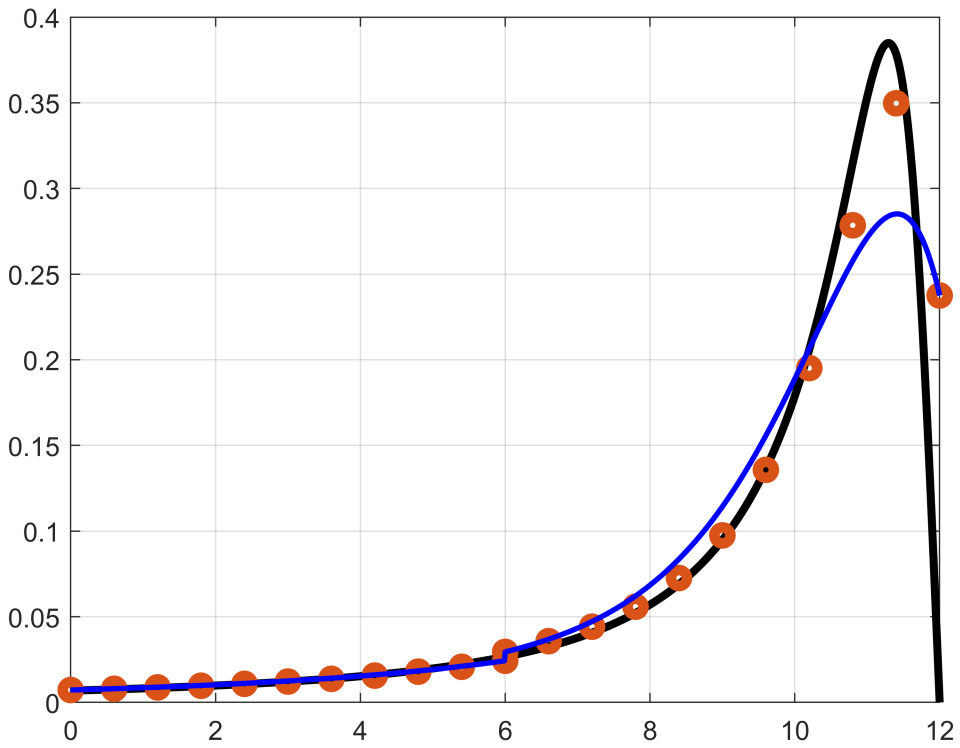
```
%% Compute the Piecewise Bernstein Polynomial at these nodes
u1N = PiecewiseBernsteinPoly(u1nodes,tknots,t);
u2N = PiecewiseBernsteinPoly(u2nodes,tknots,t);
figure(u1figcomp)
plot(t,u1N,'Linewidth',2,'Color','b');
```



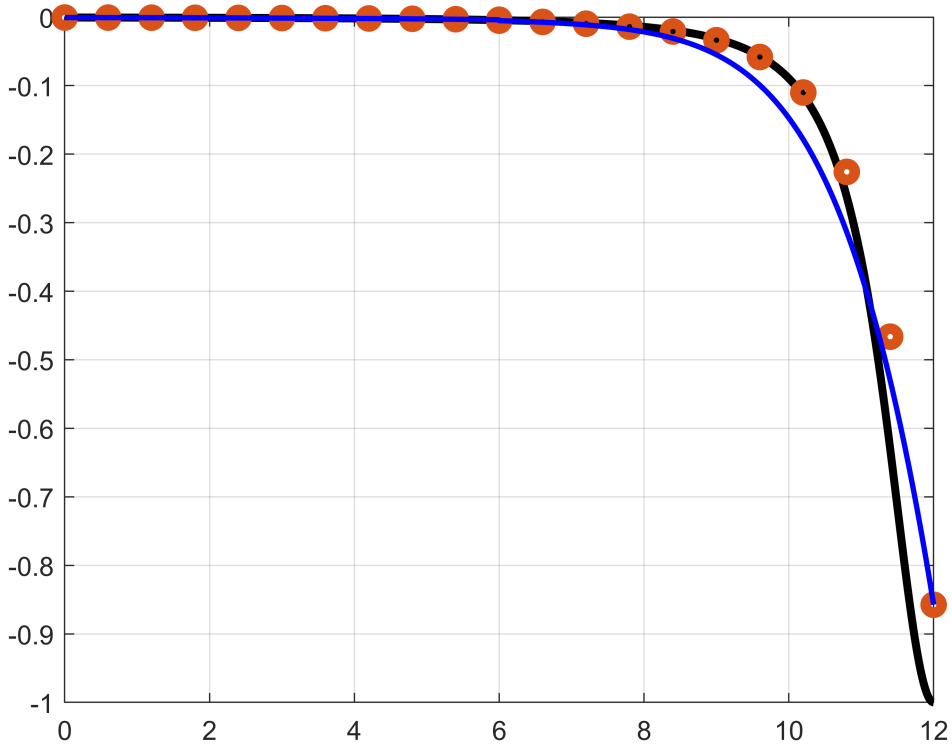

```
figure(u2figcomp)
plot(t,u2N,'Linewidth',2,'Color','b');
```




```
%% Compute the derivative of Bernstein poly
u1nodes_dot = u1nodes*D;
u2nodes_dot = u2nodes*D;
u1Ndot = PiecewiseBernsteinPoly(u1nodes_dot,tknots,t);
u2Ndot = PiecewiseBernsteinPoly(u2nodes_dot,tknots,t);
figure(u1dotfigcomp)
plot(tnodes,u1nodes_dot,'o','Linewidth',4);
plot(t,u1Ndot,'Linewidth',2,'Color','b');
```

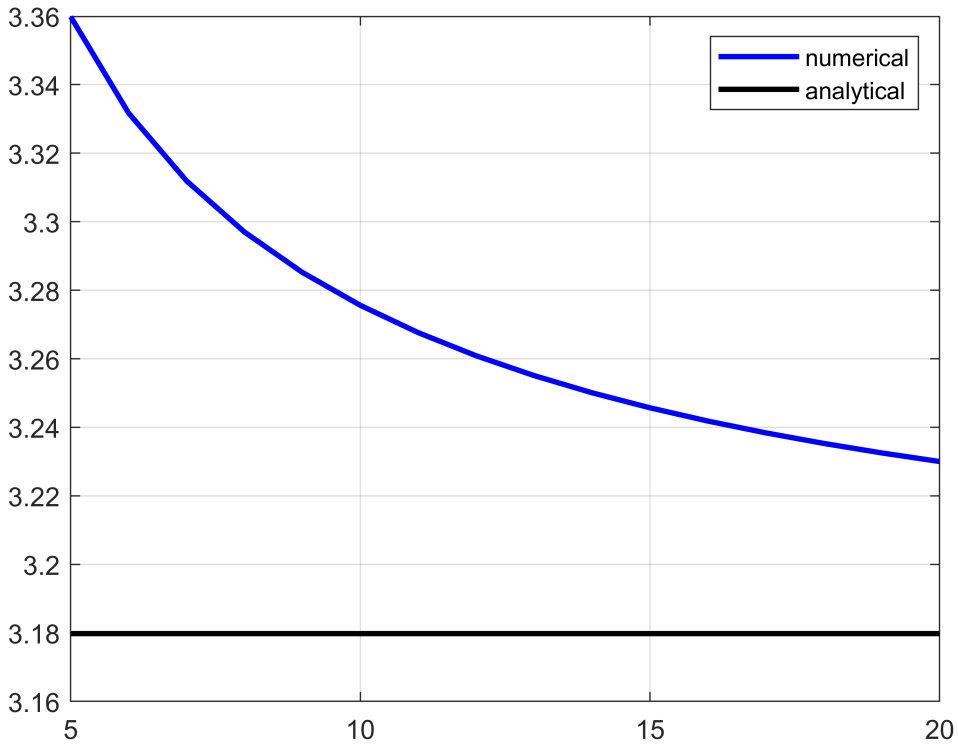



```
figure(u2dotfigcomp)
plot(tnodes,u2nodes_dot,'o','Linewidth',4);
plot(t,u2Ndot,'Linewidth',2,'Color','b');
```

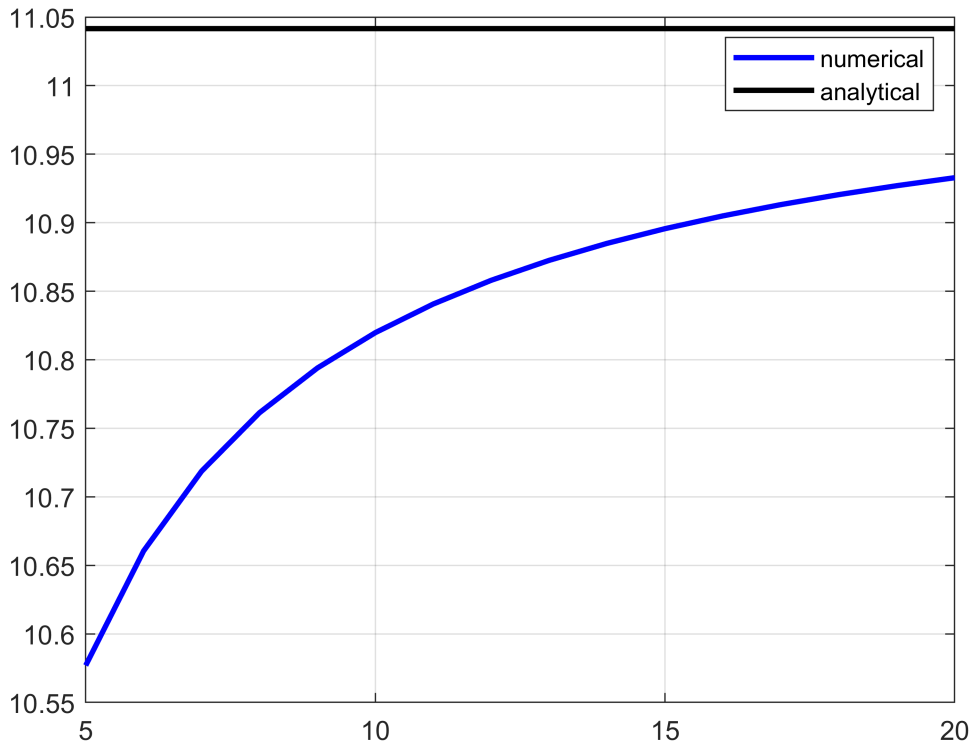



```
%% Compute the integral of Bernstein poly
intu1N = u1nodes*w;
intu2N = u2nodes*w;
% Compare to analytical answer
intu1;
intu2;



% Compare the integral of Bernstein approximation for different N
i = 0;
for N = 5:20
    i = i + 1;
    [tnodes,w,~] = PiecewiseBeBOT(N,tknots);
    u1nodes = u1(tnodes);
    u2nodes = u2(tnodes);
    intu1N(i) = u1nodes*w;
    intu2N(i) = u2nodes*w;
    order(i) = N;
end
figure
```

```
plot(order,intu1N,'Linewidth',2,'Color','b'); hold on
plot(order,intu1*ones(length(order),1),'Linewidth',2,'Color','k');
legend('numerical','analytical')
%set(gca,'fontsize', 26);
grid on
```

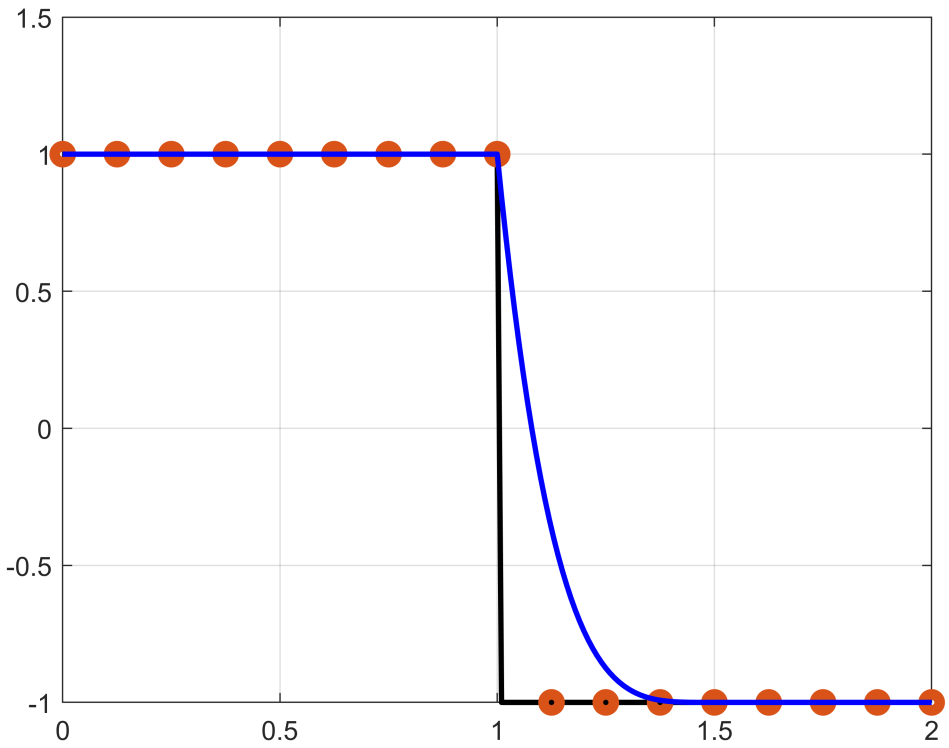



```
figure
plot(order,intu2N,'Linewidth',2,'Color','b'); hold on
plot(order,intu2*ones(length(order),1),'Linewidth',2,'Color','k');
legend('numerical','analytical')
%set(gca,'fontsize', 26);
grid on
```




```
%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Bernstein Approximation of non-smooth functions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


N = 4; % Order of approximation (N+1 nodes)
```

```
%% Define the following function
syms u(t)
T = 2;
u(t) =@(t) piecewise(t<=T/2, 1, T/2<t, -1);
t = 0:0.01:T;
uval = u(t);
figure
plot(t,uval,'Linewidth',2,'Color','k'); hold on
%set(gca,'fontsize', 26);
grid on


%% Compute Bern approx nodes/weights/Diff
tknots = linspace(0,T,5);
[tnodes,~,~] = PiecewiseBeBOT(N,tknots);


%% Plot the function at the equidistant nodes
unodes = u(tnodes);
plot(tnodes,unodes,'o','Linewidth',4); hold on




%% Compute the Bernstein Polynomial at these nodes
uN = PiecewiseBernsteinPoly(unodes,tknots,t);
plot(t,uN,'Linewidth',2,'Color','b');
```




```
%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Lagrange approximation of non-smooth functions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


%% Plot the function on different figures
figure
```

```
plot(t,uval,'Linewidth',2,'Color','k'); hold on
%set(gca,'fontsize', 26);
grid on


%% Compute LGL nodes
[tnodes,~,~] = LGL_PS(N,T); % Compute tnodes,w,Diff in [0,T]


%% Plot the function at LGL nodes
unodes = u(tnodes);
plot(tnodes,unodes,'o','Linewidth',4);


%% Compute the Lagrange polynomial at these nodes
uN = LagrangePoly(unodes,tnodes,t);
plot(t,uN,'Linewidth',2,'Color','b');
```