Exam 1

Thursday, March 4, 2021 9:22 AM

200Exam1New

Math 200 Exam 1

Mathematicians are like Frenchmen: whatever you say to them, they translate it into their own language, and forthwith it means something entirely different.
– Goethe

You have the full class period (75 minutes) for the exam. If you can't decide what something on the test means, send me a message on zoom and I'll see if I can answer it. The exam technically ends at 10:45am, and by 11:00am should be submitted to BlackBoard. This can be done either by uploading scanned versions of your written papers using a decent scanning app, or submitting text saying that your test write-up is done in OneNote. No notes, calculators, textbooks, phones, internet, etc., allowed.

Problem 1. Find the cardinality $|S|$ of each of the sets described below:

1 • $S = \{1, 2, 1, \{1, 2\}, \emptyset, \{\emptyset\}\}$. $|S| = 6$ 1

2 • $S = \{x \in \mathbb{R} \mid x^2 + 1 = 6\} \cap \{x \in \mathbb{Z} \mid -10 \leq x \leq 10\}$ $\{\pm\sqrt{5}\} \cap \{-10, -9 \ldots -1, 0, 1 \cdots 9, 10\} \therefore |S| = 0$ 2

3 • $S = \{x \in \mathbb{R} \mid x^2 + 1 = 5\} \cup \{x \in \mathbb{Z} \mid -10 \leq x \leq 10\}$ $\{-2, 2\} \cup \{-10, -9, -8, -7, -6, -5, -3, -2, -1, 0, \ldots, 9, 10\} \therefore |S| = 2$ 3

4 • $S = \{x \in \mathbb{Z} \mid 2x - 4 \geq -10\} - \{x \in \mathbb{R} \mid x > 0\}$ $\{-3\} - \{0.1, 1, 2, \pi, 5 \ldots\}$

In that last bullet, the "minus" operation $-$ is the *difference* of two sets A and B, defined by

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

Include very brief justifications, just enough that I can tell you're not making up numbers at random.

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$-3 \in A \wedge -3 \notin B$ $|S| = 1$

Problem 2. For a given real number a, consider the statement

$$P(a) = \left(\overset{X}{(a+\sqrt{2}) \notin \mathbb{Q}}\right) \vee \left(\overset{Y}{(-a+\sqrt{2}) \notin \mathbb{Q}}\right) \quad \neg(X \vee Y) \equiv (\neg X \wedge \neg Y)$$

- Write down the negation $\neg P(a)$. (As usual, simplify: don't just put a $\neg$ sign at the start of it).

$\neg X = ((a+\sqrt{2}) \in \mathbb{Q}) \quad \neg Y = ((-a+\sqrt{2}) \in \mathbb{Q})$

$\neg P(a) = ((a+\sqrt{2}) \in \mathbb{Q}) \wedge ((-a+\sqrt{2}) \in \mathbb{Q})$

$\neg P(a) = (\pm a + \sqrt{2}) \in \mathbb{Q}$

- Use a proof by contradiction to prove the statement

$$\forall a \in \mathbb{R}, P(a).$$

As always, write your proof using precise language and complete sentences.

Hypothesis: For all real numbers a, $((a+\sqrt{2}) \notin \mathbb{Q})$ or $((a-\sqrt{2}) \notin \mathbb{Q})$

Negation: There exists a real number a, such that positive or negative a plus root 2 is rational.

We will prove this by contradiction.

let $a = \sqrt{2}$, Thus $\sqrt{2}+\sqrt{2} = 2\sqrt{2}$ and $2\sqrt{2}$ is rational.

in case 2, $\sqrt{2}-\sqrt{2} = 0$ and 0 is rational.

We have thus proved that a real number, $\sqrt{2}$, exists and is rational. Thus proved by contradiction ∎

$n \in \mathbb{Z}^+$

Problem 3. Consider the following statement, for a positive integer n:

P If n is even, then $n^2 + n$ is even. Q

of the form $P \to Q$. Decide which of the following are true:

- $Q \to P$ False
- $\neg Q \to \neg P$ True
- $\neg P \to \neg Q$ False
- $P \longleftrightarrow Q$ False
- $\neg(P \to Q)$ False $P \wedge \neg Q$

You do not need to give formal proofs of the true statements, only brief justifications of how you decided whether each is true or false.

$(n^2+n$ even$) \to (n$ even$)$ - $1(2)=2$ & $n=1$

$n(n+1)$ even $\to$ n even

$(n^2+n$ odd$) \to (n$ odd$)$ - True as n^2+n is never odd

$n(n+1)$ odd $\to$ n odd

odd · even or vice versa = even

$1 \to 1(2) = 2$ - n^2+n

n odd $\to$ $n(n+1)$ odd

odd $\to$ even always even

n even $\to$ $n(n+1)$ even & $n(n+1)$ even $\to$ n even is false

false as $n(n+1) \longrightarrow n$ even

$(n$ even$) \wedge (n(n+1)$ odd$)$ - $n(n+1)$ always even

Problem 4. We have Villagers A, B, and C, one of whom is a knave (always lies), one of whom is a knight (always tells the truth), and one of whom is normal (sometimes lies and sometimes tells the truth), but we don't know which is which.

- A says "C is a knave."
- B says "A is a knight."
- C says "C is normal."

Figure out all possible configurations of who is who.

✓ 1. A Knight
   ∴ C is Knave
   ∴ B is normal

Statements

A ~~true~~ ok
B ~~false~~ ok
C ~~true~~ ok ✓

2. A Knave
   ∴ C ≠ Knave
   ∴ ✗ a. B Knight, B statement thus ~~false~~ contradicts
   ✗ b. B Normal, B statement ok
      ∴ C Knight, C statement contradicts

3. A normal
   ✗ a. B Knight, B statement ~~false~~ contradicts
   ✗ b. B Knave, B statement ok
      ∴ C Knight, C statement contradicts

Possibilities Thus: $\boxed{\begin{matrix} A = \text{Knight} \\ B = \text{normal} \\ C = \text{Knave} \end{matrix}}$ ~~A Knave C normal~~

Problem 5. We define the "not and" connector, denoted by the symbol $\uparrow$, by the truth table

A	B	$A \uparrow B$
T	T	F
T	F	T
F	T	T
F	F	T

$A \wedge B$
T
F
F
F

Write down truth tables for each of the three following expressions involving $\uparrow$'s and use the truth tables to interpret each as a simpler, logically equivalent, statement:

$$p \uparrow p \qquad (p \uparrow q) \uparrow (p \uparrow q) \qquad (p \uparrow p) \uparrow (q \uparrow q)$$

p		q	P and p	P and q	q and q	Not (P and p)	Not (P and q)	Not (q and q)	Not(P and q) and Not(P and q)	Not (Not(P and q) and Not(P and q))	Not(P and p) and Not(q and q)	Not (Not(P and p) and Not(q and q))
t		t	t	t	t	f	f	f	f	t	f	t
t		f	t	f	f	f	t	t	t	f	f	t
f		t	f	f	t	t	t	f	t	f	f	t
f		f	f	f	f	t	t	t	t	f	t	f

$p \uparrow p$ $\qquad$ $(p \uparrow q) \uparrow (p \uparrow q)$ $\qquad$ $(p \uparrow p) \uparrow (q \uparrow q)$

p		q	Not p	Not q	P or q
t		t	f	f	t
t		f	f	t	t
f		t	t	f	t
f		f	t	t	f

$$p \uparrow p \equiv \neg p$$
$$(p \uparrow q) \uparrow (p \uparrow q) \equiv p \wedge q$$
$$(p \uparrow p) \uparrow (q \uparrow q) \equiv p \vee q$$

I study := P

I pass := Q

go to movie := R

Problem 6. Consider the following chain of reasoning:

1 If I study, then I will pass.
2 If I do not go to a movie, then I will study.
3 I failed.
4 Therefore, I went to a movie.

Convert the argument from English statements into the language of propositional logic (introducing sentence names P, Q, R, etc.), and then analyze the argument using the tools of propositional logic. Conclude with an assessment of the validity of the argument.
(Note: I can imagine several ways of doing this – I am not looking for one specific way).

(1) $P \to Q$ (2) $\neg R \to P$ (3) $\neg Q$ (4) R

(2) $\equiv R \vee P$

$\neg Q$ is True.

By (1) Thus $P \to Q$ is false only if they studied & failed
as they failed they did not study.

By (2) $\neg R \to P$ is false only if the did not go to a movie and did not study

There fore as they did not study they had to go to the movie
as $\neg R$ must be false.

In Conclusion, the student failed the test, thus they did not study, thus they did go to a movie

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∴ The Argument is Valid ✓