Math 200 Exam 2 (with Solutions)

Problem 1. Short computation:

- Decide if the integer 251 is prime. Justify your conclusion.

  Solution. We need only check divisibility by primes up to $\sqrt{251}$. Since $\sqrt{251} < \sqrt{256} = 16$, this means we have to check divisibility by primes less than 16, which is only 2, 3, 5, 7, 11, and 13. Since we can immediately eliminate 2, 3, and 5 by their divisibility tests, the only computation is to check that 251 is not divisibly by 7, 11, or 13, after which we conlclude that indeed 251 is prime. $\square$

- Compute the remainder when the number $3^{60003} + 56^{56000}$ is divided by 28.

  Solution. Since $56 \equiv 0 \bmod 28$, the second summand will disappear completely mod 28. For the first term, we have $3^3 = 27 \equiv -1 \bmod 28$. Therefore $3^6 = (3^3)^2 \equiv (-1)^2 \equiv 1 \bmod 28$, and so

$$3^{60003} + 56^{56000} \equiv 3^{60000} \cdot 3^3 + 0^{56000}$$
$$1 \cdot -1 + 0 \equiv -1 \equiv \boxed{27} \pmod{28}.$$

$\square$

Problem 2. The *perfect squares* are the well-known numbers

$$0, 1, 4, 9, 16, 25, 36, 49, \ldots$$

- Choose a careful definition of what it means for an integer n to be a perfect square.

  Solution. An integer n is a *perfect square* if there exists an integer k such that $n = k^2$. $\square$

- Let m and n be integers. Use argument by contrapositive to prove the following claim, using *your* definition of a perfect square:

  If mn is *not* a perfect square, then at least one of m or n is also *not* a perfect square.

  Solution. The contrapositive is the claim that if *both* m and n are perfect squares, then so is mn. But this is easy: Write $m = k^2$ and $n = \ell^2$. Then $mn = k^2\ell^2 = (k\ell)^2$ is a perfect square. $\square$

Problem 3. Euclidean Algorithm and gcd's:

- Compute $\gcd(1547, 819)$.

  Solution. By the Euclidean Algorithm,

  $$\gcd(1547, 819) = \gcd(819, 728) = \gcd(728, 91) = 91.$$

  $\square$

- The number 1009 is prime. Say as much as possible about $\gcd(a, a + 1009)$ for an integer a.

  Solution. By the Euclidean Algorithm,

  $$\gcd(a, a + 1009) = \gcd(a, (a + 1009) - a) = \gcd(a, 1009),$$

  so this gcd must be a divisor of 1009, so since 1009 is prime, must be 1 or 1009. We conclude that

  $$\boxed{\gcd(a, a + 1009) = \begin{cases} 1009 & \text{if } a \text{ is a multiple of 1009.} \\ 1 & \text{if } a \text{ otherwise.} \end{cases}}$$

  $\square$

- Define a relation $\sim$ on the natural numbers $\mathbb{N}$ by

  $$a \sim b \quad \text{if and only if} \quad \gcd(a, b) = 1.$$

  Decide with brief justification which of the three equivalence relation conditions $\sim$ satisfies.

  Solution. The relation $\sim$ is not reflexive since for any integer $a > 1$, we have $\gcd(a, a) = a \neq 1$. It is symmetric since $\gcd(a, b) = \gcd(b, a)$, so they are either both 1 or both not 1. Finally, the relation is not transitive since we could take, for example, $a = c = 2$ and $b = 3$. $\square$

Problem 4. (a) Let A, B, and C be sets. Give a careful proof that $A - B = A \cap B^c$.

Proof. We can proceed by mutual inclusion, or do both inclusions at once through a series of if-and-only-ifs:

$$\begin{aligned} x \in A - B &\longleftrightarrow (x \in A) \wedge (x \notin B) \\ &\longleftrightarrow (x \in A) \wedge (x \in B^c) \\ &\longleftrightarrow x \in A \cap B^c \end{aligned}$$

Since the two sets have the same elements, they are equal. $\square$

(b) Below is a 'proof' that attempts to use the result above to prove the identity $A - (B - C) = A - (B \cup C)$. Identify the error(s) in the proof and use Venn diagrams to illustrates why the result is false.

Proof.

$$\begin{aligned} A - (B - C) &= (A - B) - (A - C) \\ &= \big(A \cap B^c\big) \cap \big(A \cap C^c\big) \\ &= A \cap \big(B^c \cap C^c\big) \\ &= A \cap (B \cup C)^c \\ &= A - (B \cup C) \quad \blacksquare \end{aligned}$$

Proof. Both of the first two equalities are incorrect:

- It is not true that that $A - (B - C) = (A - B) - (A - C)$. The latter is the elements of A which are in C but not in B, which is a which is a small subset of the former, which includes every element of A not in B.
- The second step is also incorrect, but could be fixed by replacing the middle $\cap$ with a $-$.

$\square$

Problem 5. For this problem, the universal set U is the set of all people that have ever lived. Let $P(a, b)$ denote the statement "a is a parent of b." Decode the following logical expressions as a short and naturally phrased English sentence about Bob.

1. $\exists z,\ \ P(z, \text{Bob}) \wedge (\forall y,\ P(z, y) \longrightarrow (y = \text{Bob}))$

   Solution. Preliminary version: Theres exists a person z who is a parent of x. That person z has the property that if y is any of their children, then $y = x$.

   Final version: $\boxed{\text{Bob is an only child (of at least one of his parents).}}$ □

2. $\forall y, z,\ \ P(\text{Bob}, y) \longrightarrow \neg P(y, z)$.

   Solution. Preliminary version: For any two people in the universe y and z, if x is y's parent, then y is not z's parent.

   Final version: $\boxed{\text{Bob has no grandchildren.}}$ □

Problem 6 (Optional Extra Credit)**.** Two is the only even prime number!

1. How cool is that?

   Solution. Actually....NOT VERY! "Two is the only even prime number" literally means "Two is the only prime number divisible by two." But that's silly, and true for every prime:

   - Three is the only prime number divisible by 3.
   - Five is the only prime number divisibile by 5.
   - $\vdots$

   So the *only* thing special about 2 in this case is that we've given a special name, "even", to being divisible by 2. Whoop de doo. □

2. Complete the joke: This fact makes it the ____________________ prime.

   Solution. *Oddest*! Get it? *Odd*est? 'cuz it's even? Har dee har har. :) □