HW 8 - CASON KONZER

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6.3 #4cd, 10

6.4 #7efgh (note examples of functions can either be like f(x)=x^2 or arrow diagrams)

6.5 #1

7.2 #15

(1->1) (onto) (1->1 & onto)

6.3: Injections, Surjections, & Bijections

4. For each of the following functions, determine if the function is a bijection. Justify all conclusions.

(c) $f: (\mathbb{R} - \{4\}) \to \mathbb{R}$ defined by $f(x) = \dfrac{3x}{x-4}$, for all $x \in (\mathbb{R} - \{4\})$.

Injection? let $x_1, x_2 \in (\mathbb{R} - \{4\})$ so $f(x_1) = \frac{3x_1}{x_1 - 4}$ & $f(x_2) = \frac{3x_2}{x_2-4}$

let $f(x_1) = f(x_2)$ so $\frac{3x_1}{x_1-4} = \frac{3x_2}{x_2-4}$ Now $\frac{x_1}{x_1-4} = \frac{x_2}{x_2-4}$, Xmultiplying..

$x_1(x_2-4) = x_2(x_1-4)$ & $x_1x_2 - 4x_1 = x_1x_2 - 4x_2$, Thus

$-4x_1 = -4x_2$, so yes $x_1 = x_2$ & f is injective.

Surjective? $\forall y \in \mathbb{R}, \exists x \in (\mathbb{R} - \{4\}) \mid f(x) = b$?

let $y \in \mathbb{R}$, we need $f(x) = \frac{3x}{x-4} = y = \frac{3}{1-4/x}$ So

$1 - \frac{4}{x} = \frac{3}{y}$ & $\frac{-4}{x} = \frac{3}{y} - 1 = \frac{3-y}{y}$ Now $\frac{x}{-4} = \frac{y}{3-y}$ &

$x = \frac{-4y}{3-y}$, But this numerer $x \notin (\mathbb{R} - \{0\})$

As $3 \in \mathbb{R}$ & @ $y = 3$ $x = \frac{-12}{0}$, Undefined.

$\therefore$ f is not onto or Bijective ∎

(d) $g: (\mathbb{R} - \{4\}) \to (\mathbb{R} - \{3\})$ defined by $g(x) = \dfrac{3x}{x-4}$, for all $x \in (\mathbb{R} - \{4\})$.

$x - 4$

$x \in (\mathbb{R} - \{4\})$.

One-to-One? | Yes, From before all elements map to a unique image, Given any $f(x_1) = f(x_2)$, $x_1 = x_2$.

Onto | yes, $\forall y \in (\mathbb{R} - \{3\})$, $\exists x \in (\mathbb{R} - \{4\}) \mid f(x) = y$.

Let $x = \frac{-4y}{3-y}$, Thus $f(x) = \frac{3\left(\frac{-4y}{3-y}\right)}{\left(\frac{-4y}{3-y}\right) - 4} = \frac{3}{1 - 4\left(\frac{3-y}{-4y}\right)} = \dots$

Continuing, $\dots \frac{3}{1 + \frac{3-y}{y}} = \frac{3}{\frac{y+3-y}{y}} = 3 \cdot \frac{y}{3} = \boxed{y = f(x)}$ ☒

As $f : (\mathbb{R} - \{4\}) \to (\mathbb{R} - \{3\})$ is one-to-one & onto,

f is A Bijection :)

10. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x, y) = -x^2 y + 3y$, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Is the function f an injection? Is the function f a surjection? Justify your conclusions.

One-to-One? | Consider $f(1,1)$ & $f(-1,1)$, here $(1,1) \neq (-1,1)$; $f(1,1) = -(1^2)(1) + 3(1) = -1 + 3 = 2$, similarly $f(-1,1) = -(-1^2)(1) + 3(1) = -1 + 3 = 2$; so $f(1,1) = f(-1,1)$, but $(1,1) \neq (-1,1)$ Thus f is <u>not</u> injective ☒

Onto? | $\forall (z) \in \mathbb{R}$, $\exists (x, y) \in \mathbb{R} \times \mathbb{R} \mid f(x,y) = z$?

well $f(x,y) = -x^2 y + 3y = y(3 - x^2)$

For $f(x,y) = z$, $z = y(3 - x^2)$ thus $y = \frac{z}{3-x^2}$ &

$\frac{z}{y} - 3 = -x^2$ so $x = \sqrt{3 - \frac{z}{y}}$ thus $f(x,y)$ is

$f\left(\sqrt{3 - \frac{z}{y}}, y\right)$ & $f\left(x, \frac{z}{3-x^2}\right)$ following we have

$f(\sqrt{3-\frac{z}{y}}, y)$ & $f(x, \frac{z}{3-x^2})$ following we have

$\Downarrow$ $\Downarrow$

$y(3-(3-\frac{z}{y}))$ & $(\frac{z}{3-x^2})(3-x^2) \Rightarrow z$ In both Cases we can find an x, y where $f(x,y)=z$.

$\Downarrow$

$y(\frac{z}{y}) \Rightarrow z$

Further More, $\forall z \in \mathbb{R}, \exists (0,y) \in \mathbb{R}\times\mathbb{R} \mid f(x,y)=z$,
as $f(0,y)=3y$ & $f(0,\frac{z}{3}) = \frac{3z}{3} = z$ ■

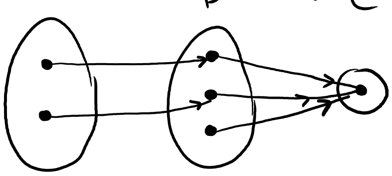
$\boxed{\text{6.4: Composition of Functions}}$

7. For each of the following, give an example of functions $f: A \to B$ and $g: B \to C$ that satisfy the stated conditions, or explain why no such example exists.

(e) The function f is not a surjection, but the function $g \circ f$ is a surjection.

$\exists b \in B, \forall a \in A, f(a) \neq b$ here f is not onto as there exists a $b \in B$ such that no $a \in A$ maps to b. But $g \circ f$ is a surjection as all $c \in C$ have an element $a \in A$ that map to C.

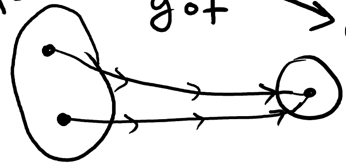
ex. $A \xrightarrow{f} B \xrightarrow{g} C$



$f(a)=b$; $g(b)=c$; $g(f(a))=g(b)=c$ $\left[\text{let } f:\{1,2\}\to\{1,2,3\} \; g:\{1,2,3\}\to\{1\}\right]$

explicitly, $f(x)=x$, $g(\dot{x})=y=g(f(x))$
$x \in A$ $\quad \dot{x} \in B, y \in C$

$A \xrightarrow{g \circ f} C$

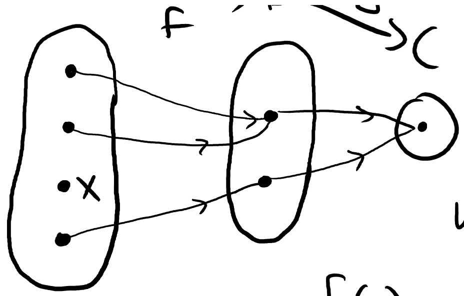


$[g \circ f: \{1,2\} \to \{1\}]$

(f) The function f is not an injection, but the function $g \circ f$ is an injection.

$A \xrightarrow{f} B \xrightarrow{g} C$

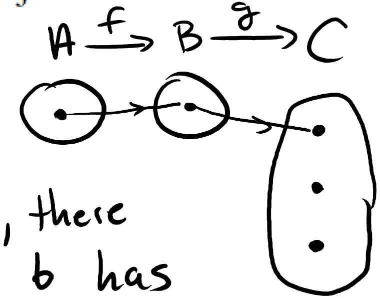
Impossible ; Given an element



Impossible: Given an element $x \in A$, such that $f(x) \notin B$, when composing $g \circ f := g(f(x))$, $f(x)$ could not be used as an input for g because it is not in the Domain of B. It follows that as the images in C are mapped from B, & the Given $x \in A$ is not in B, it does not map to an image in C. Thus $g \circ f$ CANNOT BE INJECTIVE as there is no mapping $g \circ f : A \to C$ for the $x \in A$.

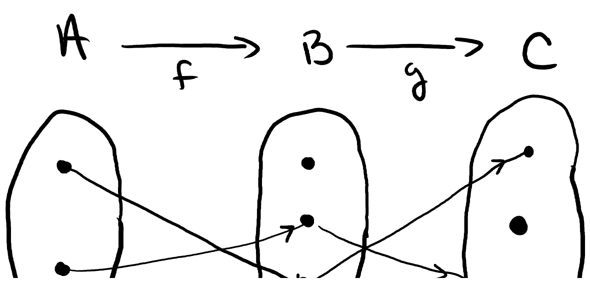
(g) The function g is not a surjection, but the function $g \circ f$ is a surjection.

$g : B \to C$ & $f : A \to B$, $g \circ f : A \to C = g(f)$



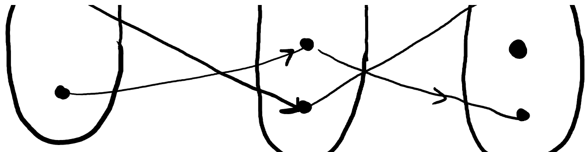
This is not possible. Given any element $b \in B$, there is always some element $c \in C$ such that no b has the image c, as g is not surjective. It is straight forward to see that as f has a CoDomain of B, any image for any element $a \in A$ will be some element $b \in B$, again no element $b \in B$ will map to some element $c \in C$, Thus $g \circ f$ is not surjective either.

(h) The function g is not an injection, but the function $g \circ f$ is an injection.



This is possible!

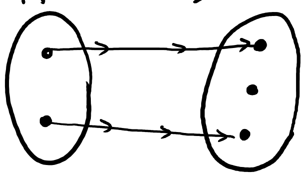
ex. $f : \{1,2\} \to \{1,2,3\}$
$g : \{1,2,3\} \to \{1,2,3\}$



ex. $f: \{1,2\} \to \{1,2,3\}$
$g: \{1,2,3\} \to \{1,2,3\}$
$g \circ f: \{1,2\} \to \{1,2,3\}$

$g \circ f$

$A \longrightarrow C$



here $f \begin{cases} 3a \; ; \; a=1 \\ a \; ; \; a=2 \end{cases}$ $a \in A$, $f(a) \in B$ & $g \begin{cases} \frac{3}{2} b \; ; \; b=2 \\ \frac{b}{3} \; ; \; b=3 \end{cases}$ $b \in B$, $g(b) \in C$

& $g \circ f \begin{cases} a \; ; \; a=1 \\ \frac{3}{2} a \; ; \; a=2 \end{cases}$ $a \in A$, $g \circ f(a) \in C$

$\boxed{\text{6.5: Inverse Functions}}$

1. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

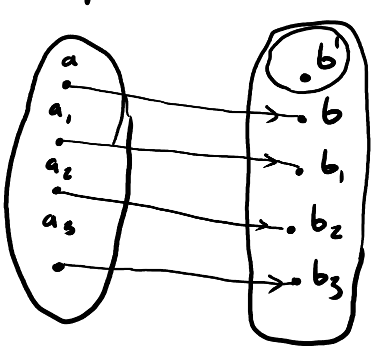
(a) Construct an example of a function $f: A \to B$ that is not a bijection. Write the inverse of this function as a set of ordered pairs. Is the inverse of f a function? Explain. If so, draw an arrow diagram for f and f^{-1}.

$\forall a \in A \; ; \; \exists b \in B \mid (a,b) \in f$. $\forall a \in A$ & $\forall b, c \in B$, If $((a,b) \in f$ & $(a,c) \in f)$ then $b = c$

let $b' \in B \mid (a, b') \notin f$; thus ther is no $f(a) = b'$ & the function is neither on to or Bijective

ex. $A = \{a, a_1, a_2, a_3\}$ & $B = \{b', b, b_1, b_2, b_3\}$ so $(a, f(a))$

$A \xrightarrow{f} B$



here $f = \{(a,b), (a_1,b_1), (a_2,b_2), (a_3,b_3)\}$

$(b, f^{-1}(b))$

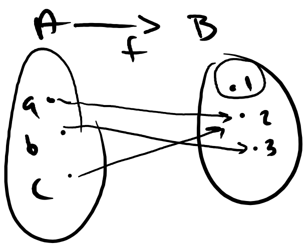
thus $f^{-1} = \{(b,a), (b_1,a_1), (b_2,a_2), (b_3,a_3)\}$

$f^{-1}: B \to A$ is not a function as,

$\exists b' \in B \mid \nexists a' \in A$ where $(b', a') \in f$. [no $(b', f^{-1}(b'))$]

b' must have some image in A for $f^{-1}: B \to A$ to be considered a function, but it does not ∎

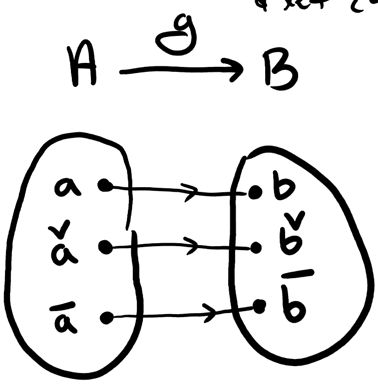
✲ w/ $A = \{a, b, c\}$ & $B = \{1, 2, 3\}$



Consider "1" instead of b; the argument is the same & this is not a function when inverted ✲

(b) Construct an example of a function $g: A \to B$ that is a bijection. Write the inverse of this function as a set of ordered pairs. Is the inverse of g a function? Explain. If so, draw an arrow diagram for g and g^{-1}.

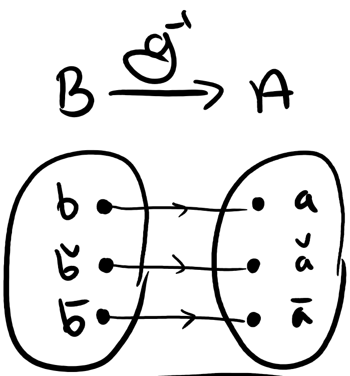
✲ let $\{a, \check{a}, \bar{a}\} = \{a, b, c\}$ & $\{b, \check{b}, \bar{b}\} = \{1, 2, 3\}$



$g = \{(a, b), (\check{a}, \check{b}), (\bar{a}, \bar{b})\} \leftarrow (a, g(a))$

$A = \{a, \check{a}, \bar{a}\}$ $B = \{b, \check{b}, \bar{b}\}$

$g^{-1} = \{(b, a), (\check{b}, \check{a}), (\bar{b}, \bar{a})\} \leftarrow (b, g^{-1}(b))$



yes, g^{-1} is indeed a function such that $g^{-1}: B \to A$, where

$\forall b \in B, \exists a \in A \mid (b, a) \in g^{-1}$ &

$\forall b \in B, \exists a, \bar{a} \in A \mid [((b, a) \in g^{-1} \land (b, \bar{a}) \in g^{-1}) \to a = \bar{a}]$

7.2: Equivalence Relations

15. Define the relation $\approx$ on $\mathbb{R} \times \mathbb{R}$ as follows: For $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$, $(a, b) \approx (c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$.

(a) Prove that $\approx$ is an equivalence relation on $\mathbb{R} \times \mathbb{R}$.

Reflexivity: let $a, b, c, d = x \in \mathbb{R}$ thus $x^2 + x^2 = x^2 + x^2$ & $2x^2 = 2x^2$, Similarly let $a, c = x \in \mathbb{R}$ & $b, d = y \in \mathbb{R}$ thus $x^2+y^2 = x^2+y^2$ ✓

$(x,x) \approx (x,x)$ & $(x,y) \approx (x,y)$ Thus $\approx$ is Reflexive

<u>Symmetric</u>) let $a, c = x \in \mathbb{R}$ & $b, d = y \in \mathbb{R}$ as before, $x^2+y^2 = x^2+y^2$,

now let $a, c = y \in \mathbb{R}$ & $b, d = x \in \mathbb{R}$; now $y^2+x^2 = y^2+x^2$

We Can see $(x,y) \approx (x,y)$; $(y,x) \approx (y,x)$; & last that $(x,y) \approx (y,x)$ as $x^2+y^2 = y^2+x^2$ ✓ Thus $\approx$ is Symmetric

<u>TRANSITIVE?</u>) Consider $i, j, k, l, m, n \in \mathbb{R} \mid (i,j), (k,l), (m,n) \in \mathbb{R} \times \mathbb{R}$

let $m^2+n^2 = k^2+l^2$ & $k^2+l^2 = i^2+j^2$; Thus $(m,n) \approx (k,l)$ & $(k,l) \approx (i,j)$

It is Straight forward that

$m^2+n^2 = k^2+l^2 = i^2+j^2$ & following $m^2+n^2 = i^2+j^2$

We can See $(m,n) \approx (i,j)$; As this is an iff relation, we Know that this relation is Transitive ✓

Such relation $(\approx)$ satisfys Reflexivity, Symmetry, & Transitivity, Thus $(\approx)$ is an Equivalence Relation ☒

(**b**) List four different elements of the set

$$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x, y) \approx (4, 3)\}.$$

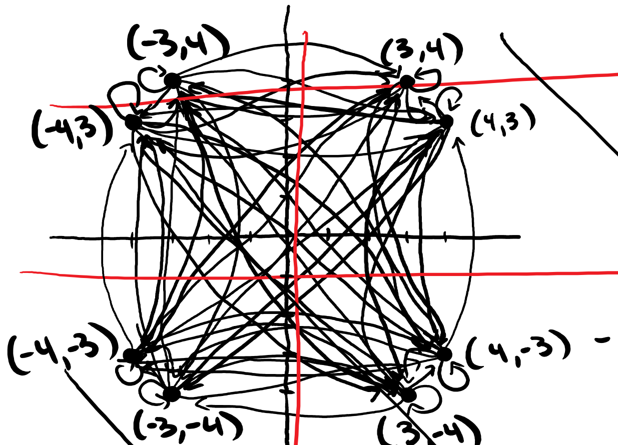
here $(x,y) \approx (4,3)$ states $x^2+y^2 = 4^2+3^2$.
$= 16+9 = 25$

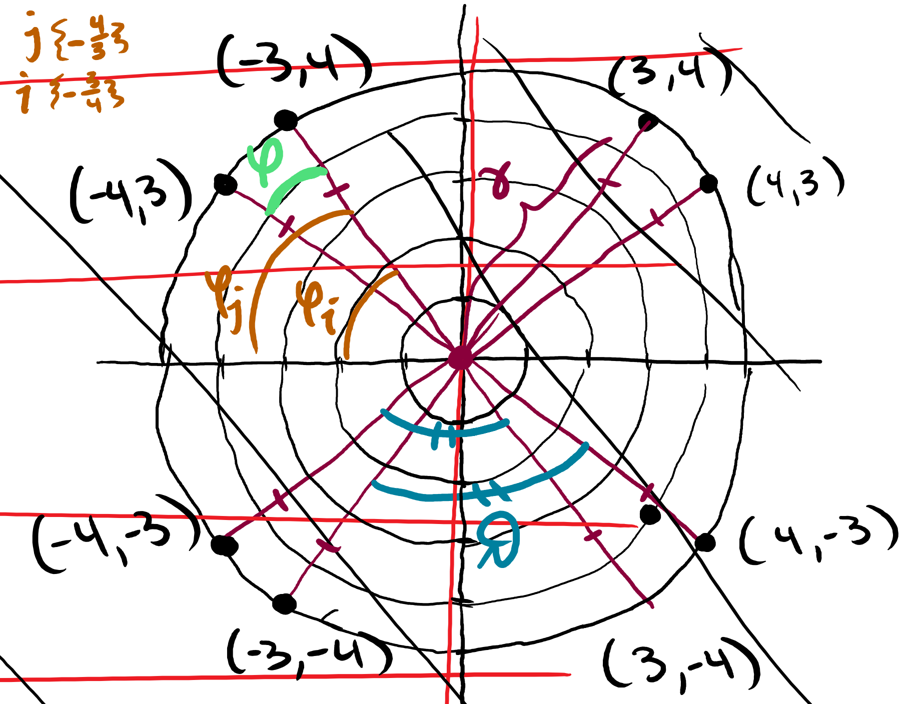
$\therefore$ $(4,3), (3,4), (-4,3)$ & $(-4,-3) \in C$

As each element satisfys the given relation.

As each element satisfys the given relation. $\nparallel$

(c) Give a geometric description of the set C.





here, C contains 8 nodes; Each Node is Related to Every node w/ C. The Relation ($\aleph$), represented by Edges in the above graph is a equivalence relation as prior Stated.

Each node is an equivalent distance γ from the origin sceen in the Cartesian Plane.

The nodes form 2 crosses WRT The origin, an angle $\theta = \frac{\pi}{2}$ separates the nodes on each respective cross

An angle φ seperates the two crosses, denoting a

As the Reals are a Continuum; The Geometric Picuture is in fact a Circle!

Here $\gamma = \sqrt{4^2+3^2}$

Here $r = \sqrt{4^2 + 3^2}$

$= \sqrt{16+9} = \sqrt{25}$

So we have a Circle of radius $(r) = 5$

two crosses, denoting a phase-like shift within the relation.

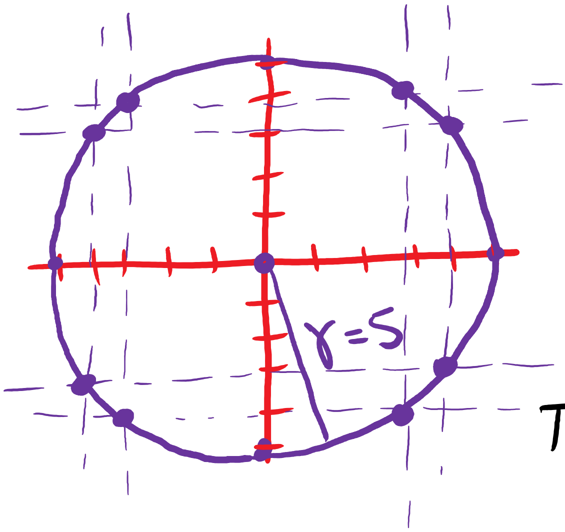
$$= |\varphi_j - \varphi_i|$$

$$\varphi_j = \tan^{-1}\left(-\frac{4}{3}\right)$$

$$\varphi_i = \tan^{-1}\left(-\frac{3}{4}\right)$$

$$\varphi = |-0.93 + 0.64|$$

$$\varphi = 0.29 \text{ RAD}$$



Each Point is Related To Itself & Every Other Point!