Math 200 Final Exam

"Joke" of the Day:

> Q: *How do mathematicians induce good behavior in their children?*
> A: "If I've told you n times, I've told you $n+1$ times..."

You have the full exam period (150 minutes) to complete and submit the exam.

Problem 1. For each of $p = 3$, $p = 5$, $p = 7$, and $p = 11$, find the smallest natural number $n > 0$ such that

$$2^n \equiv 1 \pmod{p}.$$

Use your answer (and more examples, if you like) to conjecture a general pattern.

Problem 2. Short answer problems on sets:

(a) (Carefully) find $|S|$ for the set $S = \{\{1, 2\}, 3, \{\emptyset\}, \{4\}, 3, 4\}$.

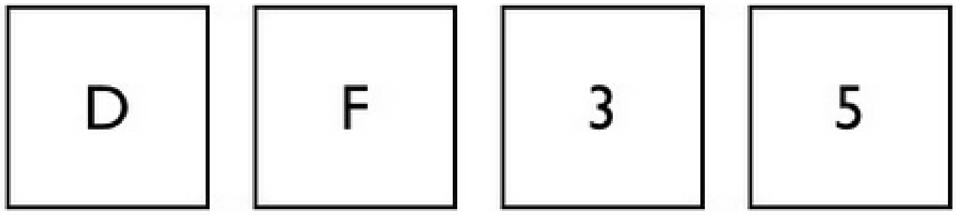
(b) T/F for each of the below, using the set S of the previous problem ($\mathcal{P}$ denotes the power set). No justification needed.

$$\emptyset \in S \qquad \{\emptyset\} \in S \qquad \emptyset \subset S \qquad \{\emptyset\} \subset S \qquad \emptyset \in \mathcal{P}(S) \qquad \emptyset \subset \mathcal{P}(S)$$

(c) T/F with very brief justification: If A and B are uncountable sets, then $|A| = |B|$.

(d) T/F with very brief justification: If A and B are countable infinite sets, then $|A| = |B|$.

Problem 3. The following problem involves four pieces of paper (let's call them *cards*). One side of each of the cards is shown below.



Each card has a number on one side and a letter on the other. Consider the claim

Claim: Every card with a D on one side has a 3 on the other side.

- Write down a simple form for the negation of the claim. (That is, don't just write "It is not true that...")

- Find the smallest number of cards you would have to turn over to decide whether the claim is true or not. Briefly justify your answer. (And be careful!)

Problem 4. Let $f\colon A \to B$ be a function. For a subset S of B, we define its inverse image under f as follows:

$$f^{-1}(S) = \{x \in A : f(x) \in S\}.$$

(Note that f does not need to be invertible for that definition to make sense).

- Consider the case $A = B = \mathbb{R}$ and f is given by $f(x) = x^2$. Determine $f^{-1}(S)$ for the half-open interval $S = (1, 4]$. Include a picture illustrating your answer.

- Returning to the general setting, prove that for subsets R and S of B, we have the following fact:

  If $R \subseteq S$, then $f^{-1}(R) \subseteq f^{-1}(S)$.

- Use an arrow diagram (or any other format) to give an example showing the following claim is false:

  If $R \neq S$, then $f^{-1}(R) \neq f^{-1}(S)$.

Problem 5. The notation $\mathbb{R}[x]$ is commonly used for the set of polynomials with real coefficients, i.e.,

$$\mathbb{R}[x] = \{x^3 + 7x + 2\,,\; 2x^6 - 2x^4 - 3x + \pi\,,\, 5x + 3\,,\; x^{21} - 5\,, \ldots\}.$$

Consider the derivative function $\phi\colon \mathbb{R}[x] \to \mathbb{R}[x]$, which takes in a polynomial and outputs its derivative. So, for example:

$$\begin{aligned}
\phi(x^3 + 7x + 2) &= 3x^2 + 7 \\
\phi(5x + 3) &= 5 \\
\phi(2x^6 - 2x^4 - 3x + \pi) &= 12x^5 - 8x^3 - 3 \\
\phi(7) &= 0 \\
&\text{etc.}
\end{aligned}$$

- Decide with proof whether ϕ is injective.
- Decide with proof whether ϕ is surjective.

Problem 6. Consider a relation $\sim$ on the set $\mathbb{R} \times \mathbb{R}$ defined by

$$(a, b) \sim (c, d) \text{ if and only if } a^2 + b^2 = c^2 + d^2.$$

a) Prove that this $\sim$ is an equivalence relation.

b) Consider the set

$$S = \{(a, b) \in \mathbb{R} \times \mathbb{R} : (a, b) \sim (3, 4)\}.$$

Provide four explicit elements of S and give a geometric description of the set S inside the plane.

Problem 7. If p is a prime number, say as much as you can about the possible values of $p \bmod 6$.

Use the previous part to prove that 3, 5, and 7 form the only instance of three odd integers in a row which are all prime.

Problem 8. Recall that $n!$ (or "n factorial") is the product of the natural numbers from 1 through n. For example, $1! = 1$, $2! = 1 \times 2 = 2$, and $3! = 1 \times 2 \times 3 = 6$.

Consider a new recursive sequence defiend by $a_1 = 1$ and for $n \geq 1$ by

$$a_{n+1} = a_n + n \cdot n!.$$

a) Compute a_2, a_3, a_4, and a_5. (Note: Be *very* careful with indices. What should n be for you to be able to compute a_2 using the above formula?) Make a prediction about a closed formula for a_n.

b) Give a proof by induction that your formula above is correct.