Math 200 Exam 1

Mathematicians are like Frenchmen: whatever you say to them, they translate it into their own language, and forthwith it means something entirely different.
– Goethe

You have the full class period (75 minutes) for the exam. If you can't decide what something on the test means, send me a message on zoom and I'll see if I can answer it. The exam technically ends at 10:45am, and by 11:00am should be submitted to BlackBoard. This can be done either by uploading scanned versions of your written papers using a decent scanning app, or submitting text saying that your test write-up is done in OneNote. No notes, calculators, textbooks, phones, internet, etc., allowed.

Problem 1. Find the cardinality $|S|$ of each of the sets described below:

- $S = \{1, 2, 1, \{1, 2\}, \emptyset, \{\emptyset\}\}$.
- $S = \{x \in \mathbb{R} \mid x^2 + 1 = 6\} \cap \{x \in \mathbb{Z} \mid -10 \leq x \leq 10\}$
- $S = \{x \in \mathbb{R} \mid x^2 + 1 = 5\} \cup \{x \in \mathbb{Z} \mid -10 \leq x \leq 10\}$
- $S = \{x \in \mathbb{Z} \mid 2x - 4 \geq -10\} - \{x \in \mathbb{R} \mid x > 0\}$

  In that last bullet, the "minus" operation $-$ is the *difference* of two sets A and B, defined by

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

Include very brief justifications, just enough that I can tell you're not making up numbers at random.

Problem 2. For a given real number a, consider the statement

$$P(a) = \Big((a + \sqrt{2}) \notin \mathbb{Q}\Big) \vee \Big((-a + \sqrt{2}) \notin \mathbb{Q}\Big)$$

- Write down the negation $\neg P(a)$. (As usual, simplify: don't just put a $\neg$ sign at the start of it).

- Use a proof by contradiction to prove the statement

  $$\forall a \in \mathbb{R}, P(a).$$

  As always, write your proof using precise language and complete sentences.

Problem 3. Consider the following statement, for a positive integer n:

If n is even, then $n^2 + n$ is even.

of the form $P \to Q$. Decide which of the following are true:

- $Q \to P$
- $\neg Q \to \neg P$
- $\neg P \to \neg Q$
- $P \longleftrightarrow Q$
- $\neg(P \to Q)$

You do not need to give formal proofs of the true statements, only brief justifications of how you decided whether each is true or false.

Problem 4. We have Villagers A, B, and C, one of whom is a knave (always lies), one of whom is a knight (always tells the truth), and one of whom is normal (sometimes lies and sometimes tells the truth), but we don't know which is which.

- A says "C is a knave."
- B says "A is a knight."
- C says "C is normal."

Figure out all possible configurations of who is who.

Problem 5. We define the "not and" connector, denoted by the symbol $\uparrow$, by the truth table

A	B	$A \uparrow B$
T	T	F
T	F	T
F	T	T
F	F	T

Write down truth tables for each of the three following expressions involving $\uparrow$'s and use the truth tables to interpret each as a simpler, logically equivalent, statement:

$$p \uparrow p \qquad\qquad (p \uparrow q) \uparrow (p \uparrow q) \qquad\qquad (p \uparrow p) \uparrow (q \uparrow q)$$

Problem 6. Consider the following chain of reasoning:

> If I study, then I will pass.
> If I do not go to a movie, then I will study.
> I failed.
> Therefore, I went to a movie.

Convert the argument from English statements into the language of propositional logic (introducing sentence names P, Q, R, etc.), and then analyze the argument using the tools of propositional logic. Conclude with an assessment of the validity of the argument.
(Note: I can imagine several ways of doing this – I am not looking for one specific way).