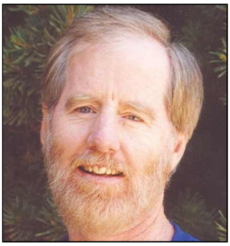
Triangular Numbers, Gaussian Integers, and KenKen

John J. Watkins



John Watkins (jwatkins@coloradocollege.edu) is Professor Emeritus at Colorado College. He has written several books, including *Graphs: An Introductory Approach* (with Robin Wilson), *Across the Board: The Mathematics of Chessboard Problems*, and *Topics in Commutative Ring Theory*. When not skiing or solving KenKen puzzles, he is at work on two books, one on number theory and the other (again with Robin Wilson) on the history of combinatorics.

One of the first of Martin Gardner's "Mathematical Games" columns I ever read was "Euler's Spoilers," in November 1959 [**1**], and after all these years I think it is still my favorite (maybe this is just because I love the way he rhymed 'Euler' and 'spoiler'). It dealt with what we now call Latin squares, $n \times n$ arrays using n symbols such that each symbol appears exactly once in each row and column.

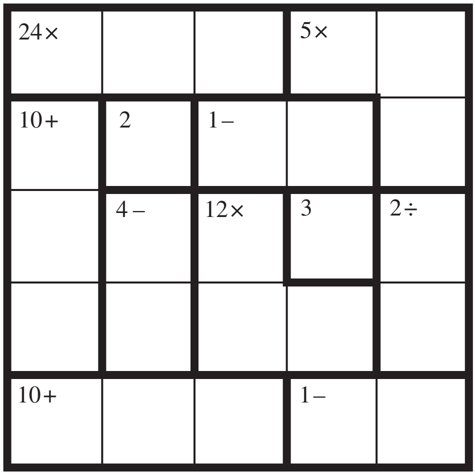
Latin squares are extremely useful in the design of statistical experiments, but they are perhaps better known now for their appearance in recreational puzzles such as sudoku where each row and each column of a 9×9 array contains each of the integers $1, 2, \dots, 9$ exactly once. In 2004, a Japanese mathematics teacher, Tetsuya Miyamoto, invented a sudoku-like puzzle called KenKen (loosely translated, this means "wisdom squared"). KenKen quickly became so popular that it is now a standard feature in newspapers around the world, and there are two very good web sites ([**2**], [**3**]) that offer new puzzles daily online.

As in sudoku, the goal in KenKen is to fill an $n \times n$ grid with the numbers 1 through n so as not to repeat a number in any row or column. Despite the fact that it uses numbers, sudoku is not an arithmetical puzzle; any nine symbols could be used in place of the integers 1–9. Solving a KenKen puzzle, on the other hand, depends heavily upon several important ideas about numbers.

Triangular numbers

In Figure 1, we see a typical KenKen where each heavily outlined 'cage' indicates what must happen inside that particular cage. For example, inside the cage labeled '$24\times$' the product of the three numbers must be 24; hence, the three numbers in this cage are forced to be 4, 3, and 2. The cage labeled '$5\times$' must contain the three numbers 5, 1, and 1 (because 5 is prime); moreover, we can immediately place these three numbers in this cage in their correct positions since the two 1s must appear in different rows and different columns. Since the only ways to partition the integer 10 into a sum of three distinct integers 1, 2, 3, 4, and 5 are as $10 = 5 + 4 + 1$ or as $10 = 5 + 3 + 2$, we see that we have two choices for the numbers to put in each of the cages labeled '$10+$'.

http://dx.doi.org/10.4169/college.math.j.43.1.037
MSC: 00A08, 05B15, 11N80



(a)

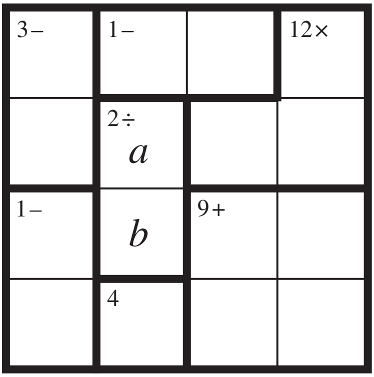
Figure 1. A typical 5×5 KenKen.

It almost goes without saying that prime factorization and partitions of integers are important in solving KenKen puzzles. What is quite surprising is that another important notion in number theory, the ancient Greek concept of triangular numbers, can also often be used to solve a KenKen because in any solution to a KenKen the sum of the numbers in any row or column of the grid is $1 + 2 + \cdots + n$; that is, the sum is the nth triangular number. In Figure 1, the sum of the numbers in any row or column must be 15 (that is, the triangular number $15 = 1 + 2 + 3 + 4 + 5$). For example, since in the bottom row the sum in the first cage is 10, the sum in the second cage must be 5. Hence, that cage contains 2 and 3. But, there is already a 3 in the fourth column, so we can immediately place the 2 and 3 in their correct positions in the bottom row.

At this point we can use a similar argument for the left-hand column to conclude that the sum of the two numbers in the upper and lower left-hand corners must also be 5. Thus, these numbers are either 2 and 3, or 1 and 4. But, they can't be 2 and 3 because we have already placed both 2 and 3 in the bottom row. Therefore, we know these two numbers must be 1 and 4, and we even know that the 4 goes in the upper corner since the cage labeled '24×' contains 4, 3, and 2. It is now a straightforward matter to finish solving this puzzle. (Solutions for the ordinary KenKen puzzles in this article are on page 42.)

In Figure 2, we see a 4×4 KenKen puzzle that normally would require a lot of trial and error to solve. For example, it is clear that the '3−' cage contains a 1 and 4 and the cage below contains a 2 and 3, but a good deal of experimenting would need to take place to determine the exact placement of these four numbers. Fortunately, we can use triangular numbers to avoid a tedious trial and error process.

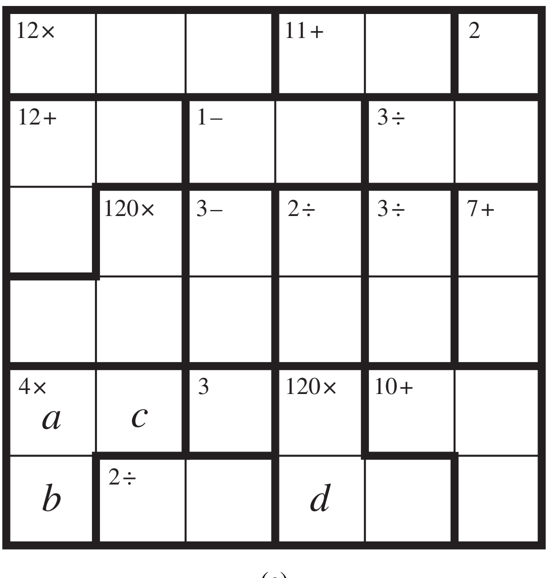
It is clear that the two numbers a and b in the '2÷' cage must be 1 and 2 in some order. Since the sum of the numbers in the two bottom rows is an *even* number ($20 = 10 + 10$), we immediately conclude that b is also even (because 9 is odd and the sum of the two numbers in the '1−' cage must also be odd). Therefore, $b = 2$, and it is now easy to solve this puzzle without any trial and error (the first column can now be completed and the '12×' cage cannot contain a 1). Note that we could have added all the known quantities in the bottom two rows and then subtracted the resulting sum, 18, from 20 to find b, but that takes more work, and often, as in this case, a simple parity argument provides the needed information.



(b)

Figure 2. Using triangular numbers to find b.

In Figure 3, we have a rather difficult 6×6 KenKen that can also be solved with the aid of triangular numbers. We focus first on the '$4\times$' cage in the bottom two rows. The question is whether the 4 factors as $4 \cdot 1 \cdot 1$ or as $2 \cdot 2 \cdot 1$. Note that, in either case, since the 120 in the '$120\times$' cage in the two bottom rows must factor as $6 \cdot 5 \cdot 4$, we can conclude that the '$2\div$' cage in the bottom row does not contain both 2 and 4 (because in the first case there are already two 4s in the bottom two rows, and in the second case there are already two 2s in the bottom two rows). Hence, this '$2\div$' cage contains either 1 and 2, or 3 and 6, and either way the sum of the two numbers in this cage is odd. Now, 3 is also odd, as is the sum of the numbers in the '$120\times$' cage ($6 + 5 + 4 = 15$); however, the sum in the '$10+$' cage is even, as is the entire sum of the bottom two rows (being twice the triangular number 21). Therefore, we immediately see that $a + b + c$ must be odd. Hence, the numbers in the '$4\times$' cage are 2, 2, and 1. With this information we can quickly fill in the bottom two rows (using the



(c)

Figure 3. Using triangular numbers to find a, b and c.

fact that the '11+' cage in the top row must contain a 5 and 6), and easily complete the rest of the puzzle without ever having to resort to trial and error.

More tricks

The puzzle in Figure 3 can also be used to illustrate a very sneaky technique I learned from Barry Cipra in Atlanta at the eighth *Gathering for Gardner* conference. He used his idea to solve sudoku puzzles, but the same idea can also be helpful for KenKen. So, let's start from scratch on the puzzle in Figure 3, and try to determine the value of d based on an argument involving the assumption that there is a *unique* solution to this problem. It is usually implicit, though rarely stated, in puzzles such as KenKen and sudoku, that there is a unique solution to the puzzle, and this fact can sometimes be invoked as an aid in finding that solution. Again, we consider the situation in the '120×' cage at the bottom of the puzzle. This time we look at the '11+' cage in the top row directly above this cage. Since the '11+' cage must contain 5 and 6, if we *assume* that there is a unique solution to this puzzle then there *cannot* also be a 5 and 6 in the two squares in the bottom row in the '120×' cage (otherwise, simply switching the 5s and 6s in the top and bottom rows yields two distinct solutions). And, of course, 5 and 6 can't both appear in the 4th column in the '120×' cage since either a 5 or a 6 appears in that same column in the '11+' cage in the top row. The only remaining option in the '120×' cage then is if $d = 4$.

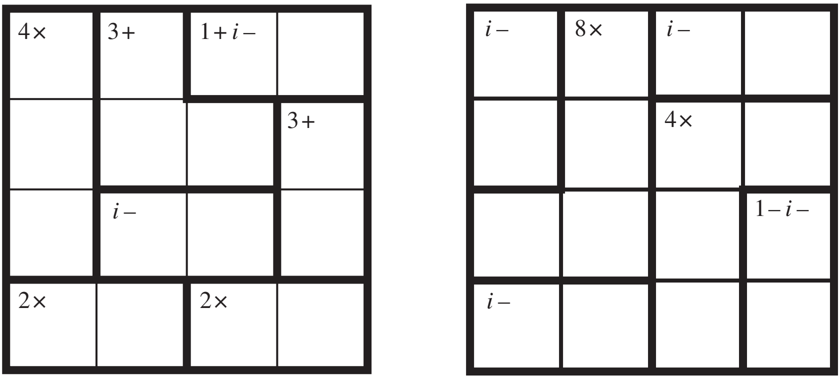
Factorials, the multiplicative analog of triangular numbers, can also be used in solving KenKen puzzles. For example, in a 6×6 puzzle, if there is a straight '120×' cage with four squares in a row, then the product of the two extra squares in this row is 6 (because $\frac{6!}{120} = 6$). So, if this '120×' cage is next to an angled '12×' cage with two squares in the same row and one square in the row above, then the single square in the row above must contain a 2.

Gaussian puzzles

One of the most enjoyable features of KenKen puzzles, especially larger ones, is the way in which the various possibilities that arise from the prime factorization of an integer come into play when dealing with irregularly shaped cages. Another number system with unique factorization into primes is the *Gaussian integers*, all complex numbers of the form $a + bi$ where a and b are integers. What is fun about the Gaussian integers is that some familiar numbers such as 2 and 5 are no longer prime since they can be factored: $2 = (1 + i)(1 - i)$ and $5 = (1 + 2i)(1 - 2i)$.

We conclude this tribute to Martin Gardner with several KenKen puzzles for the reader to solve featuring the Gaussian integers (solutions are on page 57). Here are a few hints to get you started. Consider the '4×' cage in puzzle (d). Since $4 = 2 \cdot 2$ and we are not allowed to repeat a 2 in a column, there is only one useful way to factor 4 as a product of three distinct numbers: $4 = 2(1 + i)(1 - i)$. Thus, we know which three numbers belong in this cage (though not their order). This tells us immediately that a 1 goes in the lower left-hand corner (and then a 2 next to it in the '2×' cage). There are two '3+' cages in this puzzle, one with three squares and one with two squares; it is easy to see that the only possibility is for the cage with three squares to contain 1, $1 + i$, and $1 - i$, and for the cage with two squares to contain 1 and 2.

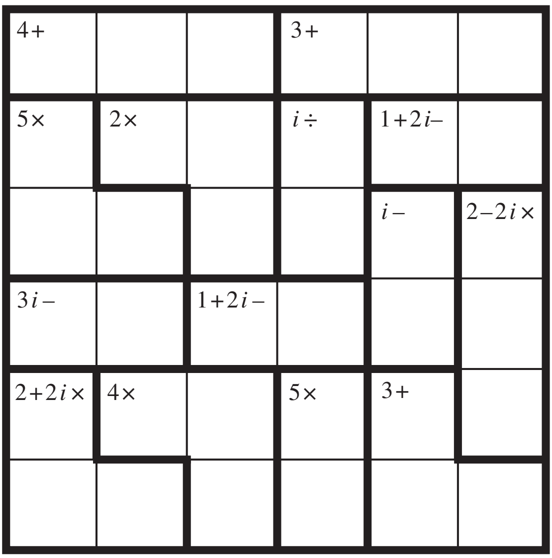
What about the '$i-$' cage in this puzzle? Clearly 2 cannot appear here since 2 minus any of the other three numbers leaves a 1 behind. Also, this cage cannot contain both



(d) (e)

Figure 4. Solve using the four numbers 1, $1+i$, $1-i$, and 2.

$1+i$ and $1-i$ since $(1+i)-(1-i)=2i$. So, the only options for this cage are for it to contain 1 and $1+i$ (since $(1+i)-1=i$), or for it to contain 1 and $1-i$ (since $1-(1-i)=i$). Similarly, the '$1+i-$' cage must contain a 2 (since, otherwise, the 1s completely cancel during subtraction). Thus, the only possibility is for this cage to contain 2 and $1-i$ (since $2-(1-i)=1+i$). The Gaussian integers, as lattice points in the complex plane, are also vectors. For example, $1+i$ can be thought of as the *vector* from the origin (0, 0) to (1, 1). Geometrically, the difference $\alpha-\beta$ between any two complex numbers α and β is just the vector from point β to point α. Thus, in puzzle (d), it is easy to spot which pairs of numbers can go in the '$i-$' cage because the i vector can only be drawn from (1, 0) to (1, 1), or from $(1,-1)$ to (1, 0); similarly, we can determine the pair of numbers that must go in the '$1+i-$' cage merely by observing that we can only draw a $1+i$ vector from $(1,-1)$ to (2, 0).



(f)

Figure 5. Solve using the six numbers 1, $1+i$, $1-i$, $1+2i$, $1-2i$, and 2.

Summary. Latin squares form the basis for the recreational puzzles sudoku and KenKen. In this article we show how useful several ideas from number theory are in solving a KenKen puzzle. For example, the simple notion of triangular number is surprisingly effective. We also introduce a variation of KenKen that uses the Gaussian integers in order to illustrate the concept of unique factorization.

References

1. M. Gardner, Euler's spoilers: the discovery of an order-10 Graeco-Latin square, *New Mathematical Diversions*, The Mathematical Association of America, Washington DC, 1995, 162–172.
2. The official KENKEN web site with puzzles posted daily; available at `http://www.kenken.com`.
3. *The New York Times* puzzle web site with puzzles posted daily; available at `http://www.nytimes.com/kenken`.

Ordinary KenKen Solutions

(a)

4	3	2	1	5
3	2	4	5	1
2	5	1	3	4
5	1	3	4	2
1	4	5	2	3

(b)

1	3	4	2
4	1	2	3
3	2	1	4
2	4	3	1

(c)

3	4	1	5	6	2
5	1	4	3	2	6
6	5	2	1	3	4
4	6	5	2	1	3
1	2	3	6	4	5
2	3	6	4	5	1