HW 4 - Cason Konzer

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Hi all, HW 4 will be due Tuesday, right before the exam on Thursday. You can start most of it now, but some of the problems technically won't have that material covered until this upcoming Tuesday's class.
I'd also like to stress again the value of reading the book. I don't like to formalize reading as an official 'assignment', but as we transition to writing more proofs it's helpful to get not only the technical aspects of writing them but also the cultural aspects -- what do people expect to see when they read a proof? What are the words that signal the flow of your argument.

So to this end I highly recommend reading the "Additional Writing Guidelines" in Section 3.1 of Sundstrom, and the entirety of Section 3.2.

Okay, and then a few problems.

2.4: 2cde, 3bdfg, 4bcdf, 7, 10

3.1: 5, 15, 19

3.2: 5, 9

Section 2.4: Quantifiers & Negations

$x=-1$

ⓒ $-1 \in \mathbb{R}$; $\sqrt{-1} = i$; $i \notin \mathbb{R}$; $i \in \mathbb{C}$

2. For each of the following, use a counterexample to show that the statement is false. Then write the negation of the statement in English, without using symbols for quantifiers.

- * **(a)** $(\forall m \in \mathbb{Z})$ (m^2 is even).
- * **(b)** $(\forall x \in \mathbb{R})$ $(x^2 > 0)$.
- **(c)** For each real number x, $\sqrt{x} \in \mathbb{R}$.
- **(d)** $(\forall m \in \mathbb{Z}) \left(\frac{m}{3} \in \mathbb{Z}\right)$.
- **(e)** $(\forall a \in \mathbb{Z}) \left(\sqrt{a^2} = a\right)$.
- * **(f)** $(\forall x \in \mathbb{R})$ $(\tan^2 x + 1 = \sec^2 x)$.

There exists a real number x with an unreal root.

ⓓ $m = 1$; $1 \in \mathbb{Z}$; $1/3 \notin \mathbb{Z}$; $1/3 \in \mathbb{Q}$

- There exists an integer m such that m divided by 3 is not an integer.

ⓔ $a=-5$; $-5 \in \mathbb{Z}$; $-5^2 = 25$; $\sqrt{25} = \pm 5$; $5 \neq -5$

There exists an integer that is no longer itself if squared & then rooted.

3. For each of the following statements

- Write the statement as an English sentence that does not use the symbols for quantifiers.
- Write the negation of the statement in symbolic form in which the negation symbol is not used.
- Write a useful negation of the statement in an English sentence that does not use the symbols for quantifiers.

- * **(a)** $(\exists x \in \mathbb{Q}) \left(x > \sqrt{2}\right)$.
- **(b)** $(\forall x \in \mathbb{Q})$ $(x^2 - 2 \neq 0)$.
- * **(c)** $(\forall x \in \mathbb{Z})$ (x is even or x is odd).
- **(d)** $(\exists x \in \mathbb{Q}) \left(\sqrt{2} < x < \sqrt{3}\right)$. **Note**: The sentence "$\sqrt{2} < x < \sqrt{3}$" is actually a conjuction. It means $\sqrt{2} < x$ and $x < \sqrt{3}$.
- * **(e)** $(\forall x \in \mathbb{Z})$ (If x^2 is odd, then x is odd).
- **(f)** $(\forall n \in \mathbb{N})$ [If n is a perfect square, then $(2^n - 1)$ is not a prime number].
- **(g)** $(\forall n \in \mathbb{N})$ ($n^2 - n + 41$ is a prime number).
- * **(h)** $(\exists x \in \mathbb{R})$ $(\cos(2x) = 2(\cos x))$.

ⓑ For each rational number x, x squared minus 2 is non-zero.

$(\exists x \in \mathbb{Q})(x^2 - 2 = 0)$: There exists a rational number x that is zero when it is squared & 2 is subtracted.

ⓓ There exists a rational number x, such that x is greater than the square root of 2 and less than the square root of 3.

$(\forall x \in \mathbb{Q})(x < \sqrt{2} \vee x > \sqrt{3})$: There exists a rational number x less than root 2 or greater than root 3.

ⓕ For all natural numbers n, If n is a perfect square, then 2 to the n^{th} power, minus 1, is composite.

$(\exists n \in \mathbb{N})$ (n is a perfect square $\wedge$ $(2^n - 1)$ is prime): There exists a natural number n, such that n is a perfect square and 2 to the n^{th} power, minus 1, is prime.

ⓖ For all natural numbers, n, n squared, minus n, plus 41, is prime.

$(\exists n \in \mathbb{N})$ ($n^2 - n + 41$ is composite): There exists a natural number, n, such that, n squared, minus n, plus 41, is composite.

4. Write each of the following statements as an English sentence that does not use the symbols for quantifiers.

- * **(a)** $(\exists m \in \mathbb{Z})$ $(\exists n \in \mathbb{Z})$ $(m > n)$
- **(b)** $(\exists m \in \mathbb{Z})$ $(\forall n \in \mathbb{Z})$ $(m > n)$
- **(c)** $(\forall m \in \mathbb{Z})$ $(\exists n \in \mathbb{Z})$ $(m > n)$
- **(d)** $(\forall m \in \mathbb{Z})$ $(\forall n \in \mathbb{Z})$ $(m > n)$
- * **(e)** $(\exists n \in \mathbb{Z})$ $(\forall m \in \mathbb{Z})$ $(m^2 > n)$
- **(f)** $(\forall n \in \mathbb{Z})$ $(\exists m \in \mathbb{Z})$ $(m^2 > n)$

ⓑ There exists an integer m that is greater than any integer n.

ⓒ There exists an integer n that is less than any integer m.

ⓓ any integer m is greater than any integer n.

(d) any integer m is greater than any integer n.

(f) There exists an integer m, that when squared, is greater than any integer n.

7. Assume that the universal set is $\mathbb{R}$. Consider the following sentence:

$$(\exists t \in \mathbb{R})\,(t \cdot x = 20).$$

(a) Explain why this sentence is an open sentence and not a statement.

(b) If 5 is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(c) If π is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(d) If 0 is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(e) What is the truth set of the open sentence $(\exists t \in \mathbb{R})\,(t \cdot x = 20)$?

(a) we do not know what x is.

(b) yes, true, (c) yes, false

(d) yes, false

(e) $\{x, t \in \mathbb{R} \mid xt = 20\}$

x is any $\mathbb{R}$.

10. In calculus, we define a function f with domain $\mathbb{R}$ to be **strictly increasing** provided that for all real numbers x and y, $f(x) < f(y)$ whenever $x < y$. Complete each of the following sentences using the appropriate symbols for quantifiers:

* **(a)** A function f with domain $\mathbb{R}$ is strictly increasing provided that . . . $(\forall x, y \in \mathbb{R})((x < y) \rightarrow (f(x) < f(y)))$

(b) A function f with domain $\mathbb{R}$ is not strictly increasing provided that . . . . $(\exists x, y \in \mathbb{R})((x < y) \wedge (f(x) > f(y)))$

Complete the following sentence in English without using symbols for quantifiers:

(c) A function f with domain $\mathbb{R}$ is not strictly increasing provided that . . . . There exists real numbers x and y, such that x is less than y, but the function evaluated at x is greater than the function evaluated at y.

Section 3.1: Direct Proofs

* **5.** Prove the following proposition:

Let a be an integer. If there exists an integer n such that $a \mid (4n + 3)$ and $a \mid (2n + 1)$, then $a = 1$ or $a = -1$.

Hint: Use the fact that the only divisors of 1 are 1 and -1.

Let a and n be integers.

Assume a divides $(2n+1) \wedge (4n+3)$

Because $a \mid (2n+1) \Rightarrow a \mid 2(2n+1) \Rightarrow a \mid (4n+2)$ let $4n+2 = j$; $j \in \mathbb{Z}$

Since we know $a \mid j$ & $a \mid j+1$, it follows

$aK = j$ & $al = j+1$; $K, l \in \mathbb{Z}$

now $al - aK = j - j + 1 = 1 = a(l - K)$

This shows $a \mid 1$ as 1 is a multiple of a.

$\therefore$ a is 1 or -1 as no other integers divide 1. ∎

15. Let r be a positive real number. The equation for a circle of radius r whose center is the origin is $x^2 + y^2 = r^2$.

(a) Use implicit differentiation to determine $\dfrac{dy}{dx}$.

(b) Let (a, b) be a point on the circle with $a \neq 0$ and $b \neq 0$. Determine the slope of the line tangent to the circle at the point (a, b).

(c) Prove that the radius of the circle to the point (a, b) is perpendicular to the line tangent to the circle at the point (a, b). **Hint**: Two lines (neither

(a) $\dfrac{d}{dx}\left(x^2 + y^2(x)\right) = \dfrac{d}{dx} r^2$

$\dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(y^2(x)) = 0$

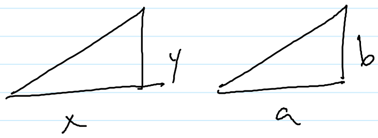
the slope of the line tangent to the circle at the point (a, b).

(c) Prove that the radius of the circle to the point (a, b) is perpendicular to the line tangent to the circle at the point (a, b). **Hint**: Two lines (neither of which is horizontal) are perpendicular if and only if the products of their slopes is equal to -1.

$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2(x)) = 0$

$2x + 2y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = \frac{-2x}{2y}$; $\boxed{\frac{dy}{dx} = \frac{-x}{y}}$

(b) (x, y) ; (a, b) $x = a$; $y = b$; $\frac{dy}{dx} = \frac{-x}{y}$; $\boxed{\frac{dy}{dx} = \frac{-a}{b}}$

(c) 

$m = \frac{b}{a}$; $m \cdot \frac{dy}{dx} = \frac{b}{a} \cdot \frac{-a}{b} = \frac{-a}{a} \cdot \frac{b}{b} = -1 \cdot 1 = -1$ ■

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19. Evaluation of proofs

This type of exercise will appear frequently in the book. In each case, there is a proposed proof of a proposition. However, the proposition may be true or may be false.

- If a proposition is false, the proposed proof is, of course, incorrect. In this situation, you are to find the error in the proof and then provide a counterexample showing that the proposition is false.
- If a proposition is true, the proposed proof may still be incorrect. In this case, you are to determine why the proof is incorrect and then write a correct proof using the writing guidelines that have been presented in this book.
- If a proposition is true and the proof is correct, you are to decide if the proof is well written or not. If it is well written, then you simply must indicate that this is an excellent proof and needs no revision. On the other hand, if the proof is not well written, then you must then revise the proof by writing it according to the guidelines presented in this text.

(a) Proposition. If m is an even integer, then $(5m + 4)$ is an even integer.

Proof. We see that $5m + 4 = 10n + 4 = 2(5n + 2)$. Therefore, $(5m + 4)$ is an even integer. ■

(b) Proposition. For all real numbers x and y, if $x \neq y$, $x > 0$, and $y > 0$, then $\frac{x}{y} + \frac{y}{x} > 2$.

Proof. Since x and y are positive real numbers, xy is positive and we can multiply both sides of the inequality by xy to obtain

$$\left(\frac{x}{y} + \frac{y}{x}\right) \cdot xy > 2 \cdot xy$$
$$x^2 + y^2 > 2xy.$$

By combining all terms on the left side of the inequality, we see that $x^2 - 2xy + y^2 > 0$ and then by factoring the left side, we obtain $(x - y)^2 > 0$. Since $x \neq y$, $(x - y) \neq 0$ and so $(x - y)^2 > 0$. This proves that if $x \neq y$, $x > 0$, and $y > 0$, then $\frac{x}{y} + \frac{y}{x} > 2$. ■

(c) Proposition. For all integers a, b, and c, if $a \mid (bc)$, then $a \mid b$ or $a \mid c$.

Proof. We assume that a, b, and c are integers and that a divides bc. So, there exists an integer k such that $bc = ka$. We now factor k as $k = mn$, where m and n are integers. We then see that

$$bc = mna.$$

This means that $b = ma$ or $c = na$ and hence, $a \mid b$ or $a \mid c$. ■

(d) Proposition. For all positive integers a, b, and c, $\left(a^b\right)^c = a^{(b^c)}$.

This proposition is false as is shown by the following counterexample: If we let $a = 2$, $b = 3$, and $c = 2$, then

$$\left(a^b\right)^c = a^{(b^c)}$$
$$\left(2^3\right)^2 = 2^{(3^2)}$$
$$8^2 = 2^9$$
$$64 \neq 512$$

(a) True. define $m = 2n \mid n \in \mathbb{Z}$

so $5m + 4 = 5(2n) + 4 = 10n + 4 = 2(5n + 2)$

Thus $5m+4$ is even. ■

(b) True, this proof assumes the conclusion!

To fix this the proof needs to be written in the other direction.

Consider $(x-y)^2$; This is inherently positive as the square of any real number is positive.

Because $x \neq y$ it is non-zero.

So $(x-y)^2 > 0$ $\therefore$ $x^2 - 2xy + y^2 > 0$

Following, $x^2 + y^2 > 2xy$. as $x, y > 0$, $xy > 0$, $\therefore x^2 + y^2 > 0$.

dividing by xy,

$\frac{x^2 + y^2}{xy} > \frac{2xy}{xy} \Rightarrow \frac{x}{y} + \frac{y}{x} > 2$ ■

(c) False! error: Assumption $bc = mna$ implies $b = ma \lor c = na$

b or c could be 1, thus $b = mna$ or $c = mna$.

C.E. $a = 2$; $b = 2$; $c = 1$ $\therefore$ $a \mid bc$ as $2 \mid 2$,

$a \mid b$ as $2 \mid 2$,

but & $a \nmid c$ as $2 \nmid 1$ ■

(d) Correct!

The proposition is False & the C.E. is Correct ✓

The proposition is False & the C.E. is Correct ✓

Section 3.2 : More Methods of Proof

5. Is the following proposition true or false?

For all integers a and b, if ab is even, then a is even or b is even.

Justify your conclusion by writing a proof if the proposition is true or by providing a counterexample if it is false.

TRUE ✓

It is given ab is even

We will prove by contradiction $(\exists a,b \in \mathbb{Z})(ab$ is even & a is odd & b odd$)$

let $a = 2k+1 \wedge b = 2l+1 \mid k,l \in \mathbb{Z}$

now $ab = (2k+1)(2l+1) = 4kl + 2k + 2l + 1 = 2(2kl+k+l)+1$

as $2kl+k+l$ is some integer m, $ab = 2m+1$.

Thus ab must be odd ∎

*** 9.** A real number x is defined to be a **rational number** provided

there exist integers m and n with $n \neq 0$ such that $x = \frac{m}{n}$.

A real number that is not a rational number is called an **irrational number.**

It is known that if x is a positive rational number, then there exist positive integers m and n with $n \neq 0$ such that $x = \frac{m}{n}$.

Is the following proposition true or false? Explain.

For each positive real number x, if x is irrational, then $\sqrt{x}$ is irrational.

This statement is True. I provide a proof by Contradiction.

$\neg((\forall x \in \mathbb{R}^+)(x \notin \mathbb{Q} \Rightarrow \sqrt{x} \notin \mathbb{Q}))$: $(\exists x \in \mathbb{R}^+)(x \notin \mathbb{Q} \wedge \sqrt{x} \in \mathbb{Q})$

let $\sqrt{x}$ be some rational number a/b with $a,b \in \mathbb{Z} \mid b \neq 0$

$\sqrt{x} = a/b$ thus $x = (a/b)^2 = \frac{a^2}{b^2}$.

Let a^2 be some integer m & b^2 some integer n.

as b is non-zero, b^2 and n are non-zero.

Now $x = \frac{m}{n}$, $m,n \in \mathbb{Z} \mid n \neq 0$ Thus $x \in \mathbb{Q}$ $(\Rightarrow\Leftarrow)$

This is a contradiction thus the original statement is true. ∎