HW 5 - Cason Konzer

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"Okay, we're now into the real proofs part of the homeworks. Each (well, most) of the following problems have you prove something, and it's not always clear how to get started. Looking at the section name sometimes helps (e.g., 3.3 on proofs by contradiction), but in general, don't expect to sit down and write a proof from start to finish. Take your time, do scratch work off to the side, and email me for hints if you get stuck. Usually once you figure out the rough flow of how the proof should go, it's just an exercise in writing to convert your idea into an argument up to the standards of a mathematical proof. As always, I'm around to help."

3.2, #11, 16, 19
3.3, #6, 18
3.4, #5

Section 3.2 (More Methods of Proofs)

11. Prove that for each integer a, if $a^2 - 1$ is even, then 4 divides $a^2 - 1$.

We are assuming we have any integer a.

Consider a to be an even integer, $2i$ with i in the integers.

Thus $a^2 - 1$ is $(2i)^2 - 1$ and after some algebra we can see this statement is odd a $2(2i^2) - 1$ is an even minus 1.

So we now see a must be odd if we want $a^2 - 1$ even.

Lets have $a = 2j - 1$ with j in the integers.

Now $a^2 - 1 = (2j-1)^2 - 1 = 4j^2 - 4j + 1 - 1 = 4j^2 - 4j$.

$2j^2 - 2j$ is an integer so $a^2 - 1$ is even as $a^2 - 1 = 2(2j^2 - 2j)$

LAST, As $4j^2 - 4j = 4(j^2 - j)$ we can say $4 \mid a^2 - 1$, as $4(j^2 - j) = a^2 - 1$ ∎

16. Let y_1, y_2, y_3, y_4 be real numbers. The **mean**, $\overline{y}$, of these four numbers is defined to be the sum of the four numbers divided by 4. That is,

$$\overline{y} = \frac{y_1 + y_2 + y_3 + y_4}{4}.$$

Prove that there exists a y_i with $1 \le i \le 4$ such that $y_i \ge \overline{y}$.

Hint: One way is to let y_{max} be the largest of y_1, y_2, y_3, y_4.

It is clear that $4\overline{y} = y_1 + y_2 + y_3 + y_4$.

Let us impose $y_1 \le y_2 \le y_3 \le y_4$. This orders our elements.

We can now see $y_{max} = y_4$. Also $1 \le 4 \le 4$.

it follows that there exists a $y_i = y_4 = y_{max}$.

this $y_4 \equiv 4\overline{y} - y_3 - y_2 - y_1 \Rightarrow \overline{y}_4 = 4\overline{y} - y_3 - y_2 - y_1$

this $y_4 \equiv 4\bar{y} - y_3 - y_2 - y_1 \Rightarrow \bar{y}_4 = \frac{4\bar{y} - y_3 - y_2 - y_1}{4}$

Now $\bar{y}_4 = \frac{4y_4 + (4y_3 - y_3) + (4y_2 - y_2) + (4y_1 - y_1)}{4}$

Going Back to Original form, $\bar{y}_4 = \frac{3y_1 + 3y_2 + 3y_3 + 4y_4}{4}$

it is clear $\bar{y}_4 \geq \bar{y}$ as $\frac{3y_1 + 3y_2 + 3y_3 + 4y_4}{4} \geq \frac{y_1 + y_2 + y_3 + y_4}{4}$

We can also see that $\bar{y}_4 = \frac{y_4}{1}$ $\therefore y_4 \geq \bar{y}$

As $1 \leq i \leq 4$; we have proved there exists an $i = 4$ such that $y_i \geq \bar{y}$ ■

19. Evaluation of Proofs

See the instructions for Exercise (19) on page 100 from Section 3.1.

(a) Proposition. If m is an odd integer, then $(m + 6)$ is an odd integer.

Proof. For $m + 6$ to be an odd integer, there must exist an integer n such that

$$m + 6 = 2n + 1.$$

By subtracting 6 from both sides of this equation, we obtain

$$\begin{aligned} m &= 2n - 6 + 1 \\ &= 2(n - 3) + 1. \end{aligned}$$

By the closure properties of the integers, $(n - 3)$ is an integer, and hence, the last equation implies that m is an odd integer. This proves that if m is an odd integer, then $m + 6$ is an odd integer. ■

• The proposition is true, although this proof needs some adjustment to be well written. This proof as of current assumes the conclusion and proves the proposition, it must be worked the other way.

Proof/ let m be an odd integer defined as $2n+1 \mid n \in \mathbb{Z}$.

Proof/ Let m be an odd integer defined as $2n+1 \mid n \in \mathbb{Z}$.
Now $m+6 = 2n+1+6$. Rearranging, $m+6 = 2(n+3)+1$.
Last, As n is an integer, $n+3$ is another integer, say k.
So $m+6 = 2k+1$, which is odd by definition. ∎

(b) Proposition. For all integers m and n, if mn is an even integer, then m is even or n is even.

Proof. For either m or n to be even, there exists an integer k such that $m = 2k$ or $n = 2k$. So if we multiply m and n, the product will contain a factor of 2 and, hence, mn will be even. ■

The proposition is true! Again this proof works from the conclusion back to the hypothesis.

I would prove this by contrapositive

Proof/ Assume some m & n are odd integers.
Lets assign $m = 2k+1$ & $n = 2l+1$ with k & l integers.
Now $mn = (2k+1)(2l+1) = 4kl + 2k + 2l + 1$.
We can re-write this as $2(2kl + k + l) + 1$.
Because $2kl + k + l$ is some other integer, say i, $mn = 2i+1$ and thus mn is odd.
There fore NO TWO ODDS CAN MULTIPLY TO AN EVEN, thus m or n is even. it is easy to see that some 2 integers will multiply to an even. For example, $(2)(1) = (2)$ & $(2)(2) = (4)$.

The Contrapositive states that there exists some integers m & n such that if m & n are odd, then integers m & n such that if m & n are odd, then mn is odd. This is logically equivalent to the origional proposition & proved above. ■

$\boxed{\text{SECTION 3.3 : ProoF By Contradiction}}$

6. Are the following statements true or false? Justify each conclusion.

* **(a)** For each positive real number x, if x is irrational, then x^2 is irrational.

If x is irrational, $x \neq y/z$ for some $y, z \in \mathbb{Z}$ $z \neq 0$

But, $\sqrt{2}$ is irrational, yet $\sqrt{2}^2 = 2$, which is rational.

Thus this statement is $\boxed{\text{false}}$ by the C.E. above

* **(b)** For each positive real number x, if x is irrational, then $\sqrt{x}$ is irrational.

We will prove the Contrapositive: For each positive real number x, if $\sqrt{x}$ is rational, then x is rational.

let $\sqrt{x}$ be some y/z ; $y, z \in \mathbb{Z}$; $z \neq 0$. Now $x = \sqrt{x}^2$

so $x = \left(\frac{y}{z}\right)^2 = \frac{y^2}{z^2}$ and yes we can say that because y & z are integers, y^2 & z^2 are integers. Lets call them λ & η, as $z \neq 0$, $\eta \neq 0$ thus $x = \lambda/\eta$ & x is rational.

$\boxed{\text{TRUE}}$

(c) For every pair of real numbers x and y, if $x + y$ is irrational, then x is irrational and y is irrational.

$\boxed{\text{FALSE}}$ C.E. $\pi + 1$ is irrational. yet 1 is rational.

letting $x = \pi$ & $y = 1$, $x \notin \mathbb{Q}$ & $y \in \mathbb{Q}$.

(d) For every pair of real numbers x and y, if $x + y$ is irrational, then x is irrational or y is irrational.

$\boxed{\text{True}}$ I will prove this via the contrapositive.

- For all pairs of Reals, if x and y are rational, then x+y is rational.

Let $x = a/b$ and $y = c/d$ such that $a,b,c,d \in \mathbb{Z}$ & $b,d \neq 0$.

So $x+y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$.

As both b & d are non-zero ints., bd is a non-zero int.
as a,d,b,c are ints, ad & bc are ints., thus ad+bc is an int.
Now as $x+y = \frac{ad+bc}{bd}$ & the numerator is an int., and the denominator a non-zero int., x+y is rational ∎

18. A **magic square** is a square array of natural numbers whose rows, columns, and diagonals all sum to the same number. For example, the following is a 3 by 3 magic square since the sum of the 3 numbers in each row is equal to 15, the sum of the 3 numbers in each column is equal to 15, and the sum of the 3 numbers in each diagonal is equal to 15.

8	3	4
1	5	9
6	7	2

Prove that the following 4 by 4 square cannot be completed to form a magic square.

$\{11, 12, 13, 14, 15, 16\}$

Looking at the rows, we have.

	D_1			D_2
R_1	A	1	B	2
R_2	3	4	5	C
R_3	6	7	D	8
R_4	9	E	10	F
	C_1	C_2	C_3	C_4

$R_1 = A + B + 3$

$R_2 = C + 12$

$R_3 = D + 21$

Looking at the column, we have.

$C_1 = A + 18$

$C_2 = E + 12$

$C_3 = B + D + 15$

$C_4 = C + F + 10$

$R_4 = E+F+19$

LAST, LOOKING AT THE DIAGONALS, WE HAVE... $D_1 = A+D+F+4$
$D_2 = 23$

As $D_2 = 23$, All other columns, Rows, & Diagonals must be 23.

Looking @ R_3; $D+21=23$ so $D=2$, but 2 is already in use.

Looking @ R_2 & C_2, $C+12=E+12$, so $C=E$, but these also must differ.

Above C & E would equal 11; so $C_4 = 21+F$, this is the same issue as R_3, as $F=2$ & 2 is in use.

Every one of these arguments prove the magic square impossible ∎

SECTION 3.4: Using Cases in Proofs

5. **(a)** Prove the following proposition:

For all integers a, b, and d with $d \neq 0$, if d divides a or d divides b, then d divides the product ab. $((d|a) \vee (d|b)) \rightarrow (d|ab)$

Hint: Notice that the hypothesis is a disjunction. So use two cases.

Case 1) Assume $d|a$; thus $a = da'$ for some a' integer. therefore $ab = da'b$, so as $(a'b)$ is an integer, d divides ab as $(d)(a'b) = ab$.

Case 2) Assume $d|b$; thus $b = db'$ for some b' integer. therefore, $ab = db'a$, so as $(b'a)$ is an integer, d divides ab as $(d)(b'a) = ab$. ∎

(b) Write the contrapositive of the proposition in Exercise (5a).

$\neg(d|ab) \rightarrow \neg((d|a) \vee (d|b)) \equiv (\neg(d|a) \wedge \neg(d|b))$

If d does not divide ab, then d does not divide a & d does not divide b, for all ints. a, b, d.

* **(c)** Write the converse of the proposition in Exercise (5a). Is the converse true or false? Justify your conclusion.

$(d \mid ab) \to ((d \mid a) \vee (d \mid b))$ if d divides ab, then d divides a or d divides b.

$\boxed{\text{FALSE}}$ C.E. let $a=2$, $b=3$, $d=6$

thus $ab = 2 \cdot 3 = 6$ & $d \mid 6$ as $6 \mid 6$, $6 \times 1 = 6$

but $6 \nmid 2$ as $6 \cdot \frac{2}{6} = 2$ & $6 \nmid 3$ as $6 \cdot \frac{3}{6} = 3$ $\blacksquare$