Final Exam

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200Final

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Math 200 Final Exam

"Joke" of the Day:

Q: *How do mathematicians induce good behavior in their children?*
A: "If I've told you n times, I've told you $n+1$ times..."

You have the full exam period (150 minutes) to complete and submit the exam.

Problem 1. For each of $p = 3$, $p = 5$, $p = 7$, and $p = 11$, find the smallest natural number $n > 0$ such that

$$2^n \equiv 1 \pmod{p}.$$

Use your answer (and more examples, if you like) to conjecture a general pattern.

$n =$ 1 2 3 4 5 6 7 8 ...
$2^n =$ 2, 4, 8, 16, 32, 64, 128, 256, ...
3 5 7

$\boxed{2^n \equiv 1 \bmod 3 \quad [n=2]}$ as $3+1=4=2^2$

5, 10, 15

$\boxed{2^n \equiv 1 \bmod 5 \quad [n=4]}$ as $15+1=16=2^4$

$\boxed{2^n \equiv 1 \bmod 7 \quad [n=6]}$ as $63+1=64=2^6$

7, 14, 21, 28, 35, 42, 49, 56, 63

$2^n \equiv 1 \bmod 9 \quad [n=8]$ as $255+1=256=2^8$

$\boxed{2^n = 1 \bmod 11 \quad [n=10]}$

in general $2^n \equiv 1 \bmod [m]$ | $n = m-1$

Problem 2. Short answer problems on sets: $\{\{\}\}$

(a) (Carefully) find $|S|$ for the set $S = \{\{1,2\},3,\{\emptyset\},\{4\},3,4\}$. $= \{\underbrace{\{1,2\}}_{1}, \underbrace{3}_{2}, \underbrace{\{\emptyset\}}_{3}, \underbrace{\{4\}}_{4}, \underbrace{4}_{5}\}$

$|S| = 5$

(b) T/F for each of the below, using the set S of the previous problem ($\mathcal{P}$ denotes the power set). No justification needed.

$\emptyset \in S$	$\{\emptyset\} \in S$	$\emptyset \subset S$	$\{\emptyset\} \subset S$	$\emptyset \in \mathcal{P}(S)$	$\emptyset \subset \mathcal{P}(S)$
F	T	T	T	T	T

(c) T/F with very brief justification: If A and B are uncountable sets, then $|A| = |B|$.

T: for each set A & B one can find an element from one set to pair with the other, forever. each have ∞ elements to select from @ all times.

(d) T/F with very brief justification: If A and B are countable infinite sets, then $|A| = |B|$.

T: Because each set is Countable, one can index both sets with the natural numbers, pairing their elements to ∞, & beyond.

Problem 3. The following problem involves four pieces of paper (let's call them *cards*). One side of each of the cards is shown below.

3			
D	F	3	5

Each card has a number on one side and a letter on the other. Consider the claim

Claim: Every card with a D on one side has a 3 on the other side.

- Write down a simple form for the negation of the claim. (That is, don't just write "It is not true that...")

Claim: $(D \rightarrow 3)$ $(\neg D \vee 3)$ Negation: $\neg(D \rightarrow 3) = D \wedge \neg 3$ $\neg(\neg D \vee 3)$

Formally; There exists a Card with D on one side & 3 on the other.

- Find the smallest number of cards you would have to turn over to decide whether the claim is true or not. Briefly justify your answer. (And be careful!)

2: you must check the card D & 5; This is because if D does not have an opposing 3, or if 5 has an opposing D; The Claim is FALSE.

3 & F can have any letter/#.

$S \subseteq B$

Problem 4. Let $f\colon A \to B$ be a function. For a subset S of B, we define its inverse image under f as follows:

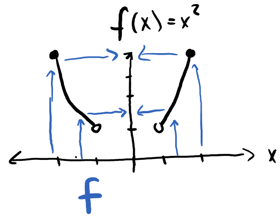
$$f^{-1}(S) = \{x \in A : f(x) \in S\}.$$

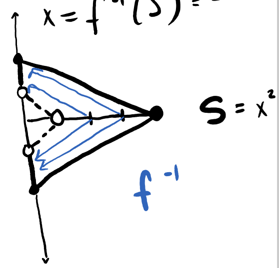
(Note that f does not need to be invertible for that definition to make sense).

- Consider the case $A = B = \mathbb{R}$ and f is given by $f(x) = x^2$. Determine $f^{-1}(S)$ for the half-open interval $S = (1, 4]$. Include a picture illustrating your answer.

$S \subseteq \mathbb{R}$; $f(x) \in S$. & $x \in A$

So $f^{-1}(S) = f^{-1}(f(x)) = x$





- Returning to the general setting, prove that for subsets R and S of B, we have the following fact:

  If $R \subseteq S$, then $f^{-1}(R) \subseteq f^{-1}(S)$.

let $e \in R$ & $R \subseteq S$; ∴ $e \in S$. Now $e = f(x) \mid x_e \in A$. & $f(x_e) = x_e^2$

So $f^{-1}(R) = f^{-1}(e) = f^{-1}(f(x_e)) = x_e$, $x_e \in f^{-1}(R)$

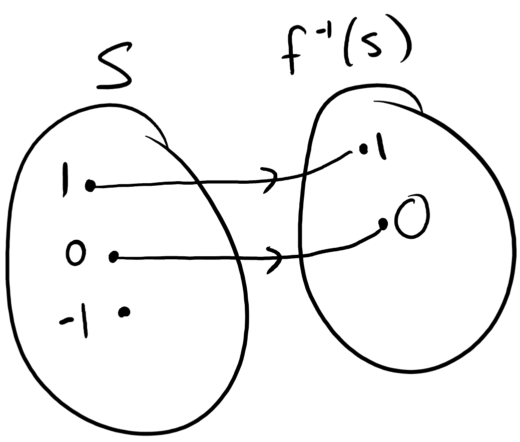
thus $f^{-1}(S) = f^{-1}(e) = f^{-1}(f(x_e)) = x_e$, $x_e \in f^{-1}(S)$

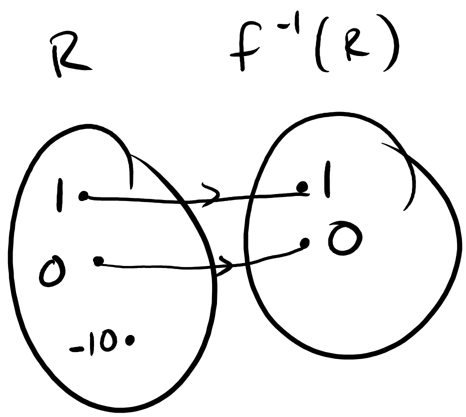
we can see that for all elements, e, in R, e is in S, & all elements x_e in $f^{-1}(R)$, x_e is in $f^{-1}(S)$ ∴ $f^{-1}(R) \subseteq f^{-1}(S)$

- Use an arrow diagram (or any other format) to give an example showing the following claim is false:

  If $R \neq S$, then $f^{-1}(R) \neq f^{-1}(S)$.

$S, R \subseteq \mathbb{R}$





here $R \neq S$; yet $f^{-1}(R) = f^{-1}(S)$

as both -1 & -10 have no inverse image.

Problem 5. The notation $\mathbb{R}[x]$ is commonly used for the set of polynomials with real coefficients, i.e.,

$$\mathbb{R}[x] = \{x^3 + 7x + 2,\ 2x^6 - 2x^4 - 3x + \pi,\ 5x + 3,\ x^{21} - 5, \ldots\}.$$

Consider the derivative function $\phi\colon \mathbb{R}[x] \to \mathbb{R}[x]$, which takes in a polynomial and outputs its derivative. So,

$$\mathbb{R}[x] = \{x^3 + 7x + 2\,,\ 2x^6 - 2x^4 - 3x + \pi\,,\ 5x + 3\,,\ x^{21} - 5\,, \ldots\}.$$

Consider the derivative function $\phi\colon \mathbb{R}[x] \to \mathbb{R}[x]$, which takes in a polynomial and outputs its derivative. So, for example:

$$\phi(x^3 + 7x + 2) = 3x^2 + 7$$
$$\phi(5x + 3) = 5$$
$$\phi(2x^6 - 2x^4 - 3x + \pi) = 12x^5 - 8x^3 - 3$$
$$\phi(7) = 0$$
etc.

* • Decide with proof whether ϕ is injective. one - to - one

• Decide with proof whether ϕ is surjective. on to

Injective: Yes, All Polynomials have a derivative, Thus all elements in the domain have an image in the co-domain

$\forall\, i \in \mathbb{R}[x]\ \exists\, \theta \in \mathbb{R}[x] \mid \phi[i] = \theta$. $\theta = \frac{d}{dx} i = i_x$;

We know i_x exists as all polynomials are differentiable $\therefore \phi[i] = i_x = \theta$

Surjective: No, All constants map to the same image 0, thus we do not have unique inputs for all images.

C.E. $\phi[422] = 0 = \phi[969]$

Problem 6. Consider a relation $\sim$ on the set $\mathbb{R} \times \mathbb{R}$ defined by

$$(a, b) \sim (c, d) \text{ if and only if } a^2 + b^2 = c^2 + d^2.$$

a) Prove that this $\sim$ is an equivalence relation.

Reflexivity: $(a,b) \sim (a,b)$ as $a^2 + b^2 = a^2 + b^2$
or $(a,a) \sim (a,a)$ as $a^2 + a^2 = a^2 + a^2 \Rightarrow 2a^2 = 2a^2$

Symmetry: $(a,b) \sim (b,a)$ as $a^2 + b^2 = b^2 + a^2$

Transitivity: $((a,b) \sim (c,d)\ \&\ (c,d) \sim (e,f)) \to ((a,b) \sim (e,f))$
here $a^2 + b^2 = c^2 + d^2\ \&\ c^2 + d^2 = e^2 + f^2$
simply $a^2 + b^2 = c^2 + d^2 = e^2 + f^2$
$\Downarrow$
$a^2 + b^2 = e^2 + f^2 \therefore (a,b) \sim (e,f)$

Reflexivity, Symmetry & Transitivity are Satisfied for $\sim$ $\therefore$ $\sim$ is an equivalence relation ■

b) Consider the set

$$S = \{(a, b) \in \mathbb{R} \times \mathbb{R} : (a, b) \sim (3, 4)\}.$$

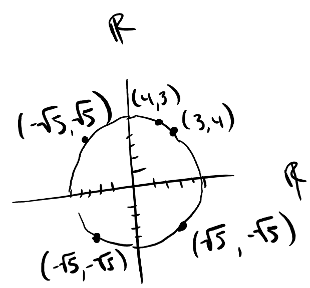
Provide four explicit elements of S and give a geometric description of the set S inside the plane.

$a^2 + b^2 \sim 3^2 + 4^2$
$= 9 + 16$
$= 25$

$a^2 + b^2 \sim 25$

$e_1 : (3,4)$
$e_2 : (4,3)$
$e_3 : (\sqrt{5}, \sqrt{5})$
$e_4 : (-\sqrt{5}, -\sqrt{5})$

S IS A CIRCLE OF RADIUS 5; in $\mathbb{R} \times \mathbb{R}$



Problem 7. If p is a prime number, say as much as you can about the possible values of p mod 6.

First Few Primes: 2, 3, 5, 7, 11, 13, 17, 19, ...

(1) $p \bmod 6 \not\equiv 0$; as if $6 \mid p$ then p is not prime!

$2 \bmod 6 \equiv 2$; $3 \bmod 6 \equiv 3$; $5 \bmod 6 \equiv 5$; $7 \bmod 6 \equiv 1$; $11 \bmod 6 \equiv 5$; $13 \bmod 6 \equiv 1$; $17 \bmod 6 \equiv 5$
$19 \bmod 6 \equiv 1$

(2) $(p \bmod 6 \equiv 2) \leftrightarrow (p=2)$ (3) $(p \bmod 6 \equiv 3) \leftrightarrow (p=3)$ (5) for $p>3$,

(4) $\forall p : [p \bmod 6 \equiv 1 \vee 2 \vee 3 \vee 5] \Rightarrow [p \bmod 6 \not\equiv 0 \vee 4]$ $p \bmod 6 \equiv 1 \vee 5$

Use the previous part to prove that 3, 5, and 7 form the only instance of three odd integers in a row which are all prime.

an odd integer is of the form $(2k+1) \mid k \in \mathbb{Z}$

We know that as k increases, a pattern arises.

w/ $k=1$: $\{1, 7, 13, 19, 25 \ldots\} \equiv (2k+1) \bmod 6$, $\{3, 9, 15, 21, 27 \ldots\} \equiv (2(k+1)+1) \bmod 6$

& $\{5, 11, 17, 23, 29 \ldots\} \equiv (2(k+2)+1) \bmod 6$

"Circled" are the primes. It is clear that @ $(2(k+3)+1)$ we cycle back to $2k+1$ as $2k+7 = 2k+6+1$ & $2k+1+6 \bmod 6$
$= 2k+1 \bmod 6$.

$\therefore$ Since 3 is the only prime congruent to $(2(k+1)+1) \bmod 6$;
3, 5, 7 are the only 3 consecutive primes that are odd.,

or vice-versa, 3 consecutive odds that are prime

Problem 8. Recall that $n!$ (or "n factorial") is the product of the natural numbers from 1 through n. For example, $1! = 1$, $2! = 1 \times 2 = 2$, and $3! = 1 \times 2 \times 3 = 6$.

Consider a new recursive sequence defiend by $a_1 = 1$ and for $n \geq 1$ by

$$a_{n+1} = a_n + n \cdot n!.$$

a) Compute a_2, a_3, a_4, and a_5. (Note: Be *very* careful with indices. What should n be for you to be able to compute a_2 using the above formula?) Make a prediction about a closed formula for a_n.

$a_2 = a_{1+1} = a_1 + 1 \cdot 1! = 1 + 1 \times 1 = \boxed{2 = a_2}$

$a_3 = a_{2+1} = a_2 + 2 \cdot 2! = 2 + 4 = \boxed{6 = a_3}$

$a_4 = a_3 + 3 \cdot 3! = 6 + 3 \times 6 = 6 + 18 = \boxed{24 = a_4}$

$a_5 = a_4 + 4 \cdot 4! = 24 + 4 \cdot 24 = 24 + 96 = \boxed{120 = a_5}$

$48 + 48$

prediction: $a_n = n!$

b) Give a proof by induction that your formula above is ~~correct~~.

$\boxed{P(1) : a_1 = 1! = 1}$

Assume $P(n) : a_n = n!$

We will proove $P(n+1) = (n+1)!$

$P(n+1) : a_{n+1} = a_n + n \cdot n!$

$a_{n+1} = n! + n \cdot n!$

$a_{n+1} = n!(1+n)$

$= (n+1)!$ ☺

To Clarify $\prod_{i=1}^{n} i = 1 \times 2 \times \cdots \times n$ $(n!)$ & $\prod_{i=1}^{n+1} = 1 \times 2 \times \cdots \times n \times (n+1) = (n+1) \prod_{i=1}^{n} = (n+1)\, n!$ $(n+1)!$