$\mathbb{A}$dvanced $\mathbb{C}$alculus

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November 28, 2021

1

Find the general solution of $u_{xx} - 6u_{xy} + 9u_{yy} = 0$ by the following: let $x = v$ and $y = w - 3v$, or equivalently, $v = x$ and $w = y + 3x$; define $U(v, w)$ to be $U(v, w) = u(v, w - 3v) = u(x, y)$; derive and solve a PDE for $U(v, w)$; convert back to $u(x, y)$. Hint: the solution will involve two arbitrary functions. Use your solution to provide a non-trivial example of a solution.

- Solve the characteristic $B^2 - 4AC$.

  $A = 1$

  $B = -6$

  $C = 9$

  $(-6)^2 - 4(1)(9) = 36 - 36 = 0$

  The characteristic is parabolic $\Rightarrow$ Tartget $U_{vv} = 0$

- Solve for the partials.

  $U_v = U_x x_v + U_y y_v$

  $= U_x - 3U_y$

  $U_{vv} = (U_x - 3U_y)_x x_v + (U_x - 3U_y)_y y_v$

  $= (U_{xx} - 3U_{yx}) - 3(U_{xy} - 3U_{yy})$

  $= U_{xx} - 6U_{xy} + 9U_{yy} = 0$

- Solve for U.

  $U_v = j(w) + C$

  $U = j(w)v + k(w) + C$

  $u = j(y + 3x)x + k(y + 3x) + C$

- Example non-trivial solution.

  $u = \sin(y + 3x)x + e^{y+3x}$

  $u_x = 3\cos(y + 3x)x + \sin(y + 3x) + 3e^{y+3x}$

  $u_{xx} = -9\sin(y + 3x)x + 3\cos(y + 3x) + 3\cos(y + 3x) + 9e^{y+3x} = -9\sin(y + 3x)x + 6\cos(y + 3x) + 9e^{y+3x}$

  $u_{xy} = -3\sin(y + 3x)x + \cos(y + 3x) + 3e^{y+3x}$

  $u_y = \cos(y + 3x)x + e^{y+3x}$

  $u_{yy} = -\sin(y + 3x)x + e^{y+3x}$

- Solving the non-trivial solution.

  $(-9\sin(y + 3x)x + 6\cos(y + 3x) + 9e^{y+3x}) - 6(-3\sin(y + 3x)x + \cos(y + 3x) + 3e^{y+3x}) + 9(-\sin(y + 3x)x + e^{y+3x})$

  $-9\sin(y + 3x)x + 6\cos(y + 3x) + 9e^{y+3x} + 18\sin(y + 3x)x - 6\cos(y + 3x) - 18e^{y+3x} - 9\sin(y + 3x)x + 9e^{y+3x}$

  $(18\sin(y + 3x)x - 18\sin(y + 3x)x) + (6\cos(y + 3x) - 6\cos(y + 3x)) + (18e^{y+3x} - 18e^{y+3x}) = 0$

2

Find the Fourier series representation of $u(x,t)$, the solution to the heat equation on a metal bar of length $L = 10$ with $\rho = 10.6$, $K = 1.04$, $\sigma = 0.056$, and $u(x,0) = f(x) = 4 - 0.8|x-5|$ on $0 \le x \le 10$, extended as an odd function. Enforce the Dirichlet boundary condition of $u(0,t) = u(10,t) = 0$.

- Solve for $c = \dfrac{K}{\rho\sigma}$ and $\lambda_n = \dfrac{cn\pi}{L}$.

$$c = \frac{1.04}{10.6 * 0.056} = 1.7520$$
$$\lambda_n = \frac{1.7520n\pi}{10} = 0.1752\pi n$$

- Solve for $B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$$B_n = \frac{2}{10}\int_0^{10}(4-0.8|x-5|)\sin\frac{n\pi x}{10}\,dx$$
$$B_n = \frac{1}{5}\Big[\int_0^5 (4-0.8(5-x))\sin\frac{n\pi x}{10}\,dx + \int_5^{10}(4-0.8(x-5))\sin\frac{n\pi x}{10}\,dx\Big]$$
$$B_n = \frac{1}{5}\Big[\int_0^5 (4-4+0.8x)\sin\frac{n\pi x}{10}\,dx + \int_5^{10}(4+4-0.8x)\sin\frac{n\pi x}{10}\,dx\Big]$$
$$B_n = \frac{1}{5}\Big[\int_0^5 0.8x\sin\frac{n\pi x}{10}\,dx + \int_5^{10} 8\sin\frac{n\pi x}{10}\,dx - \int_5^{10} 0.8x\sin\frac{n\pi x}{10}\,dx\Big]$$
$$B_n = \frac{1}{5}\Big[\frac{0.8(10^2\sin\frac{n\pi x}{10} - 10\pi x\cos\frac{n\pi x}{10})}{n^2\pi^2}\Big|_0^5 + \frac{80\cos\frac{n\pi x}{10}}{n\pi}\Big|_5^{10} - \frac{0.8(10^2\sin\frac{n\pi x}{10} - 10\pi x\cos\frac{n\pi x}{10})}{n^2\pi^2}\Big|_5^{10}\Big]$$
$$B_n = \frac{4}{25}\Big[\frac{100\sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{100\cos n\pi}{n\pi} + \frac{100\pi\cos n\pi}{n^2\pi^2} + \frac{100\sin\frac{n\pi}{2}}{n^2\pi^2}\Big]$$
$$B_n = \frac{400}{25}\Big[\frac{\sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{\cos n\pi}{n\pi} + \frac{\pi\cos n\pi}{n^2\pi^2} + \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\Big]$$
$$B_n = \frac{16}{n^2\pi^2}\Big[2\sin\frac{n\pi}{2} + (n+1)\pi\cos n\pi\Big]$$

- Under the given boundary condition we know the form of u_n and u.
- Substitute B_n, L, λ_n into $u_n = B_n \sin\dfrac{n\pi x}{L}e^{-\lambda_n^2 t}$

$$u_n = \frac{16}{n^2\pi^2}\Big[2\sin\frac{n\pi}{2} + (n+1)\pi\cos n\pi\Big]\sin(\frac{n\pi x}{10})e^{-0.0307\pi^2 n^2 t}$$

- Substitute u_n into $u = \displaystyle\sum_{n=1}^{\infty} u_n$

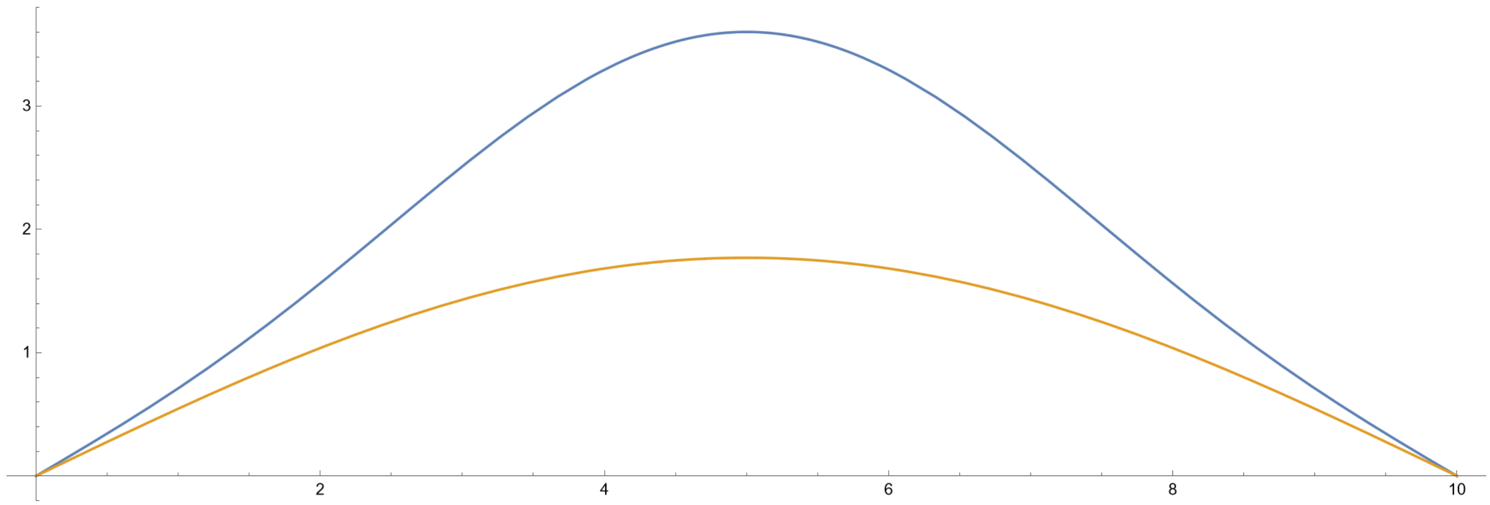
$$u = 16\sum_{n=1}^{\infty}\frac{2\sin(\frac{n\pi}{2}) + (n+1)\pi\cos(n\pi)}{n^2\pi^2}\sin(\frac{n\pi x}{10})e^{-0.0307\pi^2 n^2 t}$$

- With $x = 0$ and $x = 10$

$$u(x = \{0,10\}, t = t) = 16\sum_{n=1}^{\infty} 0 = 0$$

3

Use Mathematica and your solution from the last problem to plot $u(x,0)$ and $u(x,2)$ on the same graph. When defining u, use the first four non-zero terms of the series.



4

Find the Fourier series representation of the steady-state solution $u(x, y)$ of the heat equation on the square metal plate with corners $(0, 0)$ and $(2, 2)$ in the plane, satisfying the following boundary conditions: $u_y(x, 0) = u_x(0, y) = u_x(2, y) = 0$ and $u(x, 2) = \pi$.

- Solve for $p_n = \dfrac{n\pi}{a}$

  $$p_n = \frac{n\pi}{2}$$

- Solve for $a_0 = \dfrac{1}{a}\displaystyle\int_0^a f(x)\,dx$

  $$a_0 = \frac{1}{2}\int_0^2 \pi\,dx = \frac{1}{2}\Big[\pi x\Big|_0^2\Big] = \frac{1}{2}\Big[2\pi - 0\Big]$$
  $$a_0 = \pi = A_0$$

- Solve for $a_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$

  $$a_n = \frac{2}{2}\int_0^2 \pi\cos\frac{n\pi x}{2}\,dx = \frac{2\sin\frac{n\pi x}{2}}{n}\Big|_0^2 = \Big[\frac{2\sin n\pi}{n} - 0\Big]$$
  $$a_n = 0 = A_n(e^{p_n y} - e^{-p_n y}) = A_n$$

- Under the given boundary condition we know the form of u.

- Substitute A_0, A_n, L, p_n into $u = A_0 + \displaystyle\sum_{n=1}^{\infty} A_n \cos(p_n x)\Big[e^{p_n y} - e^{-p_n y}\Big]$

  $$u = \pi + \sum_{n=1}^{\infty} 0$$
  $$u(x, y) = \pi$$