$\mathbb{A}$dvanced $\mathbb{C}$alculus

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1

Find the Fourier series representation of $u(x,t)$, the solution to the wave equation with $c = 1$, $L = 1$, $g(x) = 0$, and $f(x) = 0.01x(1-x)$ on $0 \leq x < 1$, extended as an odd function. List the first 4 non-zero terms. You may use Mathematica to find B_n but only if you submit the notebook file showing your work.

- *Solution via seperation of variables:* $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$$u_n(x,t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{L}$$
$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$
$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$
$$\lambda_n = \frac{cn\pi}{L}$$

- Solving for given values.. Integration in corresponding notebook

$$\lambda_n = \frac{1n\pi}{1} = n\pi$$
$$B_n^* = \frac{2}{1n\pi} \int_0^1 0 \sin \frac{1\pi x}{1}\, dx = 0$$
$$B_n = \frac{2}{1} \int_0^1 (0.01x - 0.01x^2) \sin \frac{n\pi x}{1}\, dx = 0.02\Big[\int_0^1 x \sin n\pi x\, dx - \int_0^1 x^2 \sin n\pi x\, dx\Big]$$
$$B_n = 0.02\Big[\frac{(\pi n)\sin - \pi n((\pi n)\cos)}{\pi^2 n^2} - \frac{\left(2 - \pi^2 n^2\right)((\pi n)\cos) + 2\pi n((\pi n)\sin) - 2}{\pi^3 n^3}\Big]$$
$$u_n(x,t) = \Big(0.02\Big[\frac{(\pi n)\sin - \pi n((\pi n)\cos)}{\pi^2 n^2} - \frac{\left(2 - \pi^2 n^2\right)((\pi n)\cos) + 2\pi n((\pi n)\sin) - 2}{\pi^3 n^3}\Big] \cos n\pi t + 0 \sin n\pi t\Big) \sin \frac{n\pi x}{1}$$
$$u_n(x,t) = 0.02\Big(\Big[\frac{(\pi n)\sin - \pi n((\pi n)\cos)}{\pi^2 n^2} - \frac{\left(2 - \pi^2 n^2\right)((\pi n)\cos) + 2\pi n((\pi n)\sin) - 2}{\pi^3 n^3}\Big] \cos n\pi t\Big) \sin n\pi x$$
$$u(x,t) = 0.02 \sum_{n=1}^{\infty} \Big(\Big[\frac{(\pi n)\sin - \pi n((\pi n)\cos)}{\pi^2 n^2} - \frac{\left(2 - \pi^2 n^2\right)((\pi n)\cos) + 2\pi n((\pi n)\sin) - 2}{\pi^3 n^3}\Big] \cos n\pi t\Big) \sin n\pi x$$

- Solving for first 4 non-zero terms.. Table in corresponding notebook

$n = 1$: $0.00258012 \cos \pi t \sin \pi x$

$n = 3$: $0.0000955601 \cos 3\pi t \sin 3\pi x$

$n = 5$: $0.000020641 \cos 5\pi t \sin 5\pi x$

$n = 7$: $7.52222 * 10^{-6} \cos 7\pi t \sin 7\pi x$

- Approximate solution up to n = 8

$$u_n(x,t) = 0.00258 \cos \pi t \sin \pi x + 0.0000956 \cos 3\pi t \sin 3\pi x + 0.000021 \cos 5\pi t \sin 5\pi x + 7.52 * 10^{-6} \cos 7\pi t \sin 7\pi x$$

2

Using your solution to #1, plot $f(x)$, $u(x,0)$, and $u(x,0.5)$ on the same graph.

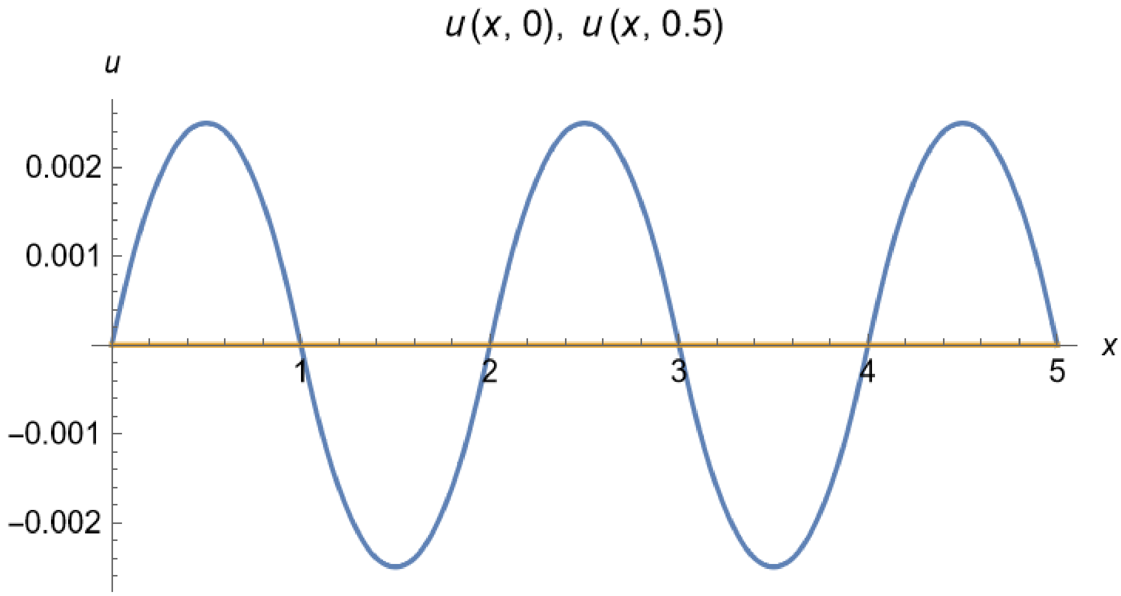

Figure 1: static time

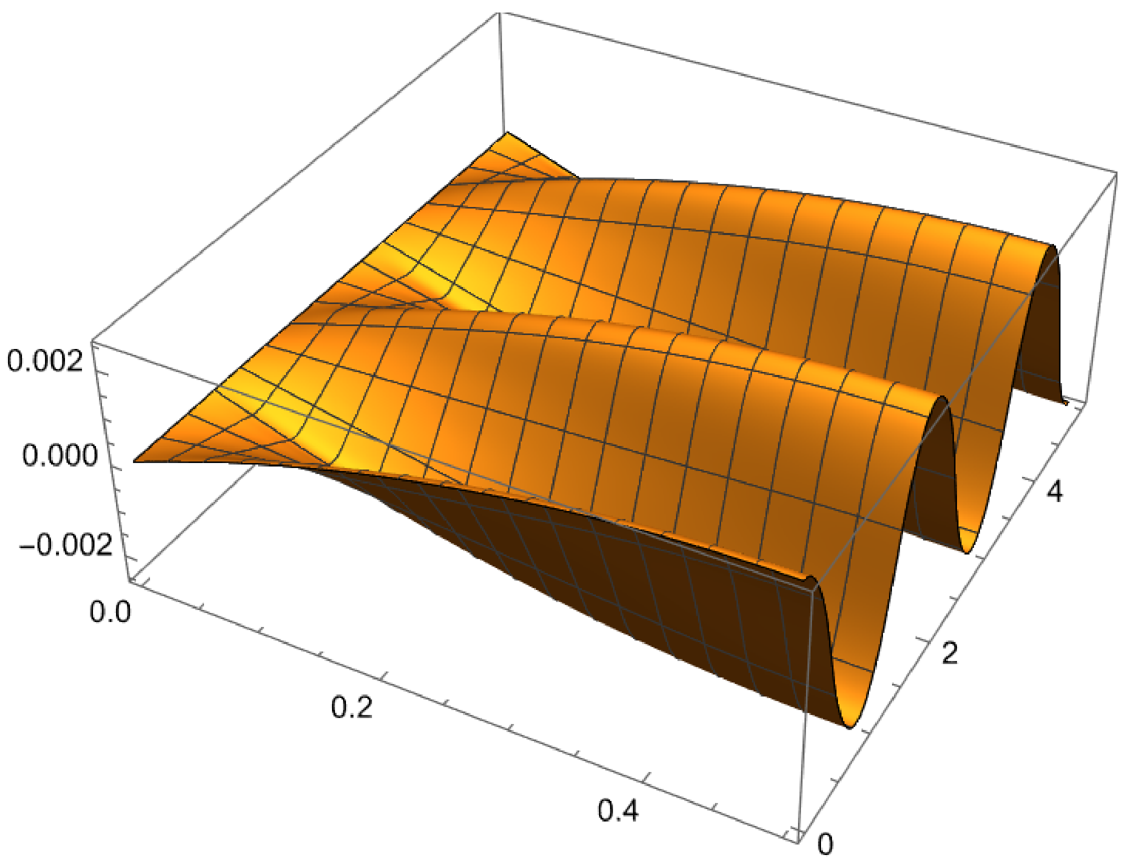

Figure 2: dynamic time

3

Using D'Alembert's solution to #1 (i.e. $u(x,t) = \frac{1}{2}(f(x+t) + f(x-t)))$, plot $u(x,t)$, $\frac{1}{2}f(x+t)$, and $\frac{1}{2}f(x-t)$ on the same graph, creating one graph for each of $t = 0, 0.2., 0.4, 0.6, 0.8$, and 1. You probably want to use Mathematica for this. Note, you MUST extend $f(x)$ to be odd with period 2.

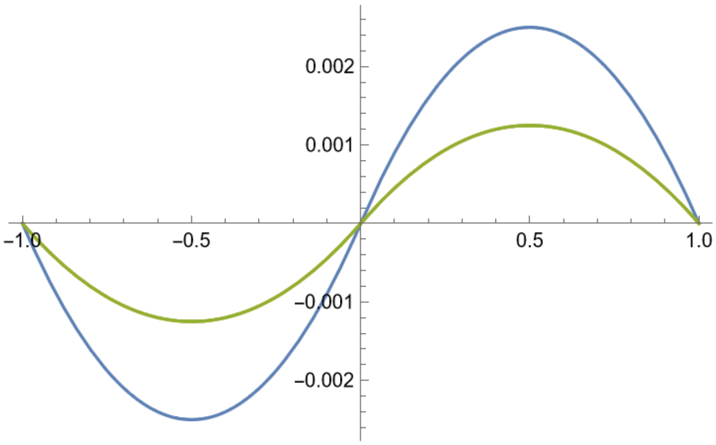

Figure 3: t=0.0

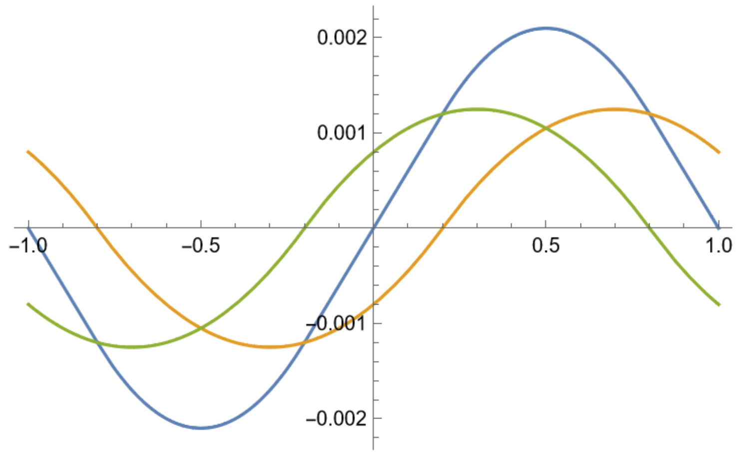

Figure 4: t=0.2

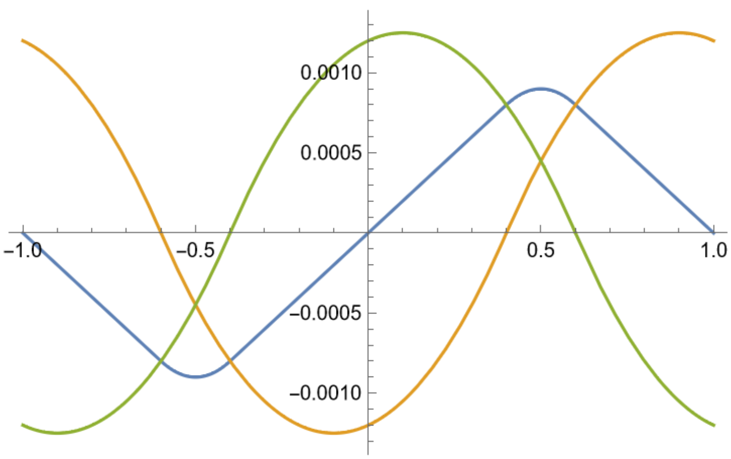

Figure 5: t=0.4

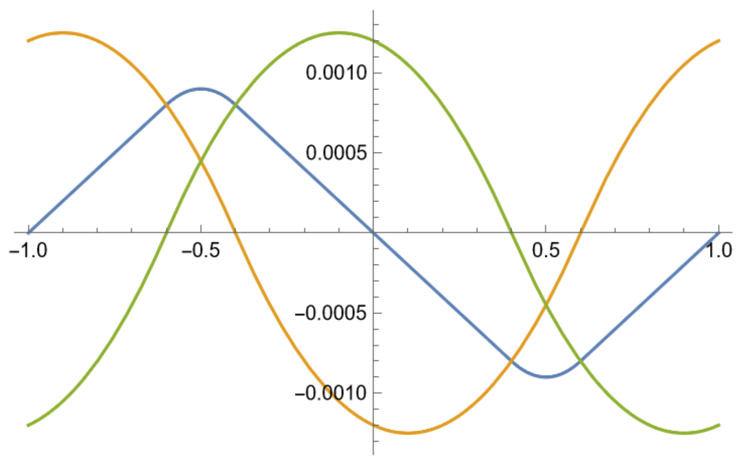

Figure 6: t=0.6

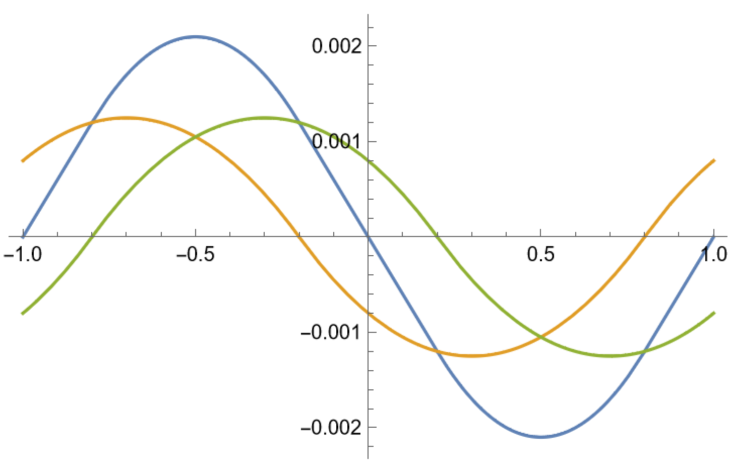

Figure 7: t=0.8

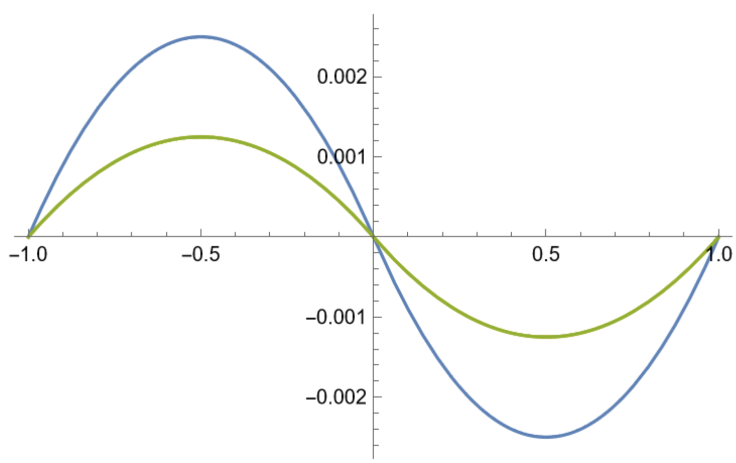

Figure 8: t=1.0