$\mathbb{A}$dvanced $\mathbb{C}$alculus

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1

Verify that $u(x,t) = \cos 4t \sin 3x$ satisfies the wave equation for an appropriate choice of c.

- *Wave Equation* $u_{tt} = c^2 u_{xx}$
- u_{tt}

  $$u_t = -4 \sin 4t \sin 3x$$
  $$utt = -16 \cos 4t \sin 3x$$

- u_{xx}

  $$u_x = 3 \cos 4t \cos 3x$$
  $$u_{xx} = -9 \cos 4t \sin 3x$$

- $c^2 = \dfrac{u_{tt}}{u_{xx}}$

  $$c^2 = \frac{-16 \cos 4t \sin 3x}{-9 \cos 4t \sin 3x}$$
  $$c^2 = \frac{16}{9}$$

- *Verify* $u_{tt} = c^2 u_{xx}$

  $$-16 cos 4t sin 3x = -\frac{16}{9} 9 \cos 4t \sin 3x$$
  $$-16 \cos 4t \sin 3x = -16 \cos 4t \sin 3x$$

2

Verify that $u(x,t) = e^{-25t} \sin \omega x$ satisfies the heat equation for an appropriate choice of c.

- *Heat Equation* $u_t = c^2 u_{xx}$
- u_t
  $$u_t = -25e^{-25t} \sin \omega x$$
- u_{xx}
  $$u_x = \omega e^{-25t} \cos \omega x$$
  $$u_{xx} = -\omega^2 e^{-25t} \sin \omega x$$
- $c^2 = \dfrac{u_t}{u_{xx}}$
  $$c^2 = \frac{-25e^{-25t} \sin \omega x}{-\omega^2 e^{-25t} \sin \omega x}$$
  $$c^2 = \frac{25}{\omega^2}$$
- *Verify* $u_t = c^2 u_{xx}$
  $$-25e^{-25t} \sin \omega x = -\frac{25}{\omega^2} \omega^2 e^{-25t} \sin \omega x$$
  $$-25e^{-25t} \sin \omega x = -25e^{-25t} \sin \omega x$$

3

Verify that $u(x,t) = v(x+ct) + w(x-ct)$ satisfies the wave equation for any twice differentiable functions $v(z)$ and $w(z)$.

- *Wave Equation* $u_{tt} = c^2 u_{xx}$
- u_{tt}

  $$u_t = cv_t(x+ct) - cw_t(x-ct)$$
  $$u_{tt} = c^2 v_{tt}(x+ct) + c^2 w_{tt}(x-ct)$$

- u_{xx}

  $$u_x = v_x(x+ct) + w_x(x-ct)$$
  $$u_{xx} = v_{xx}(x+ct) + w_{xx}(x-ct)$$

- $c^2 = \dfrac{u_{tt}}{u_{xx}}$

  $$c^2 = \frac{c^2 v_{tt} + c^2 w_{tt}}{v_{xx} + w_{xx}}$$

- *Verify* $u_{tt} = c^2 u_{xx}$

  $$c^2 v_{tt} + c^2 w_{tt} = \frac{c^2 v_{tt} + c^2 w_{tt}}{v_{xx} + w_{xx}} v_{xx} + w_{xx}$$
  $$c^2 v_{tt} + c^2 w_{tt} = c^2 v_{tt} + c^2 w_{tt}$$

- We have here $v(z(x,t))$ where $z = x + ct$ and $w(\overline{z}(x,t))$ where $\overline{z} = x - ct$

  $u_{tt} = c^2 v(z_{tt}) + c^2 w(\overline{z}_{tt})$ and $u_{xx} = v(z_{xx}) + w(\overline{z}_{xx})$

  Granted, v_{zz} and $w_{\overline{z}\overline{z}}$ exist, $z_{tt}, z_{xx}, \overline{z}_{tt}$, and $\overline{z}_{xx}$ exist, and u_{tt} and u_{xx} exist.

  It follows $u(x,t) = v(x+ct) + w(x-ct)$ solves the wave equation for any twice differentiable functions $v(z)$ and $w(z)$.

4

Verify that $u(x, y) = A\ln(x^2 + y^2) + B$ satisfies the Laplace equation. Find A and B such that u satisfies the boundary conditions: $u = 7$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = e^2$.

- *Laplace Equation* $\nabla^2 u = 0$
- u_{xx}

$$u_x = \frac{2Ax}{x^2 + y^2}$$
$$u_{xx} = \frac{2A\left(x^2 + y^2 - 2x^2\right)}{\left(x^2 + y^2\right)^2}$$
$$u_{xx} = -\frac{2A\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$$

- u_{yy}

$$u_y = \frac{2Ay}{x^2 + y^2}$$
$$u_{yy} = \frac{2A\left(x^2 + y^2 - 2y^2\right)}{\left(x^2 + y^2\right)^2}$$
$$u_{yy} = \frac{2A\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$$

- *Verify* $\nabla^2 u$

$$\nabla^2 u = u_{xx} + u_{yy}$$
$$\nabla^2 u = -\frac{2A\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} + \frac{2A\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$$
$$\nabla^2 u = 0$$

- *Let* $u = 7$ on $x^2 + y^2 = 1$

$$7 = A\ln 1 + B$$
$$7 = B$$

Any constants A, B such that $B = 7$ satisfy the boundary condition.

- *Let* $u = 0$ on $x^2 + y^2 = e^2$

$$0 = A\ln e^2 + B$$
$$-B = 2A$$

Any constants A, B such that $B = -2A$ satisfy the boundary condition.