$\mathbb{A}$dvanced $\mathbb{C}$alculus

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1

Derive a formula for $\mathcal{F}_c\{f^{(4)}(x)\}$ in terms of $\mathcal{F}_c\{f(x)\}$ and $\mathcal{F}_s\{f(x)\}$.

Solve iteratively.

- $\mathcal{F}_c\{f^{(4)}(x)\} = \mathcal{F}_c\{(f^{(3)}(x))'\} = \mathcal{F}_c\{(f''(x))''\} = \mathcal{F}_c\{(f'(x))^{(3)}\}$.

- $\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\dfrac{2}{\pi}}f(0) \;\;;\;\; \mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}$.

- $\mathcal{F}_c\{f''(x)\} = \mathcal{F}_c\{(f'(x))'\} = w\mathcal{F}_s\{f'(x)\} - \sqrt{\dfrac{2}{\pi}}f'(0)$.

  $= -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\dfrac{2}{\pi}}f'(0)$.

- $\mathcal{F}_c\{f^{(3)}(x)\} = \mathcal{F}_c\{(f''(x))'\} = -w^2\mathcal{F}_c\{f'(x)\} - \sqrt{\dfrac{2}{\pi}}f''(0)$.

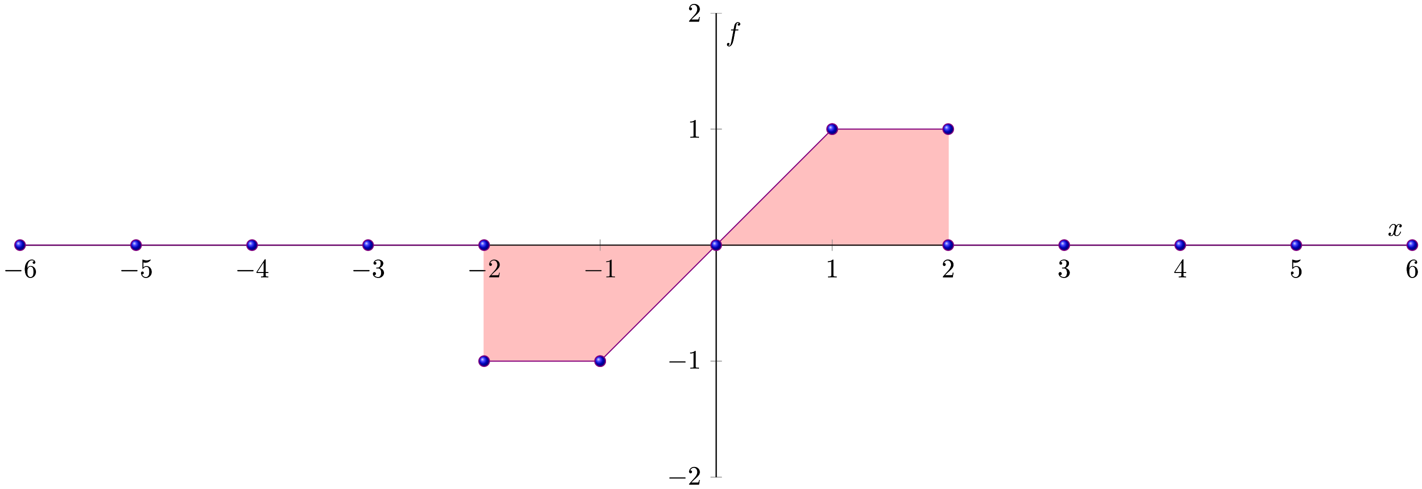
  $= -w^3\mathcal{F}_s\{f(x)\} + w^2\sqrt{\dfrac{2}{\pi}}f(0) - \sqrt{\dfrac{2}{\pi}}f''(0)$.

- $\mathcal{F}_c\{f^{(4)}(x)\} = \mathcal{F}_c\{(f^{(3)}(x))'\} = -w^3\mathcal{F}_s\{f'(x)\} + w^2\sqrt{\dfrac{2}{\pi}}f'(0) - \sqrt{\dfrac{2}{\pi}}f'''(0)$.

  $= w^4\mathcal{F}_c\{f(x)\} + w^2\sqrt{\dfrac{2}{\pi}}f'(0) - \sqrt{\dfrac{2}{\pi}}f'''(0)$.

2

Find $\mathcal{F}_s\{f(x)\}$ for an odd function with $f(x) = \begin{cases} x & \text{if} \quad -1 < x < 1 \\ -1 & \text{if} \quad -2 < x < -1 \\ 1 & \text{if} \quad 1 < x < 2 \\ 0 & \text{if} \quad x < -2,\ x > 2 \end{cases}$.



Solve

- $\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(vx)\,dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 x\sin(vx)\,dx + \sqrt{\frac{2}{\pi}} \int_1^2 1\sin(vx)\,dx + \sqrt{\frac{2}{\pi}} \int_2^\infty 0\sin(vx)\,dx$$

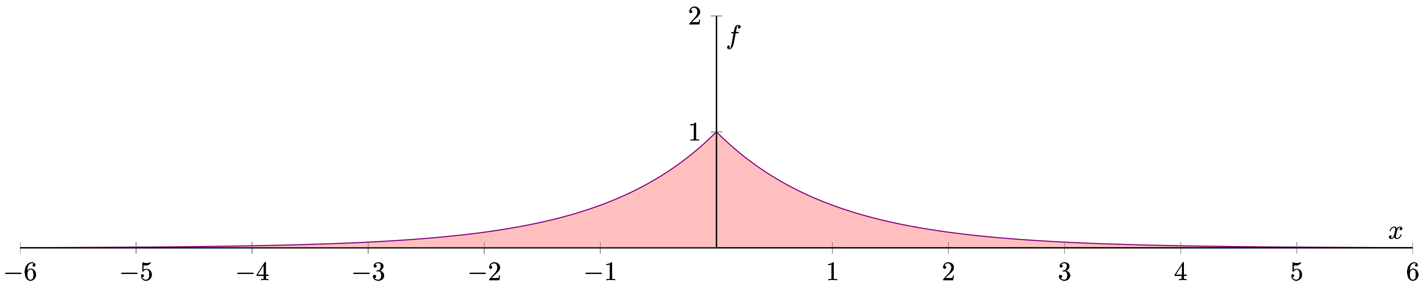
$$= \sqrt{\frac{2}{\pi}} x \int_0^1 \sin(vx)\,dx + \sqrt{\frac{2}{\pi}} \int_1^2 \sin(vx)\,dx = \sqrt{\frac{2}{\pi}} \Big[\frac{\sin wx}{w^2} - \frac{x\cos wx}{w}\Big|_0^1\Big]\,dx - \sqrt{\frac{2}{\pi}} \Big[\frac{\cos wx}{w}\Big|_1^2\Big]$$

$$= \sqrt{\frac{2}{\pi}} x \Big[\frac{\sin w}{w^2} - \frac{\cos w}{w} - \frac{\cos 2w}{w} + \frac{\cos w}{w}\Big]$$

$$= \sqrt{\frac{2}{\pi}} x \Big[\frac{\sin w}{w^2} - \frac{\cos 2w}{w}\Big]$$

3

Find $\mathcal{F}\{f(x)\}$, where $f(x) = e^{-|x|}$ for $-\infty < x < \infty$. (Hint: you cannot use a table for this).



Solve

- $\mathcal{F}\{f(x)\} = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}e^{-iwx}\,dx$

$$= \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{0} e^{x-iwx}\,dx + \sqrt{\frac{1}{2\pi}}\int_{0}^{\infty} e^{-x-iwx}\,dx = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{0} e^{x(1-iw)}\,dx + \sqrt{\frac{1}{2\pi}}\int_{0}^{\infty} e^{-x(1+iw)}\,dx$$

$$= \sqrt{\frac{1}{2\pi}}\Big[\frac{e^{x(1-iw)}}{1-iw}\Big|_{-\infty}^{0}\Big] + \sqrt{\frac{1}{2\pi}}\Big[\frac{e^{-x(1+iw)}}{-1-iw}\Big|_{0}^{\infty}\Big] = \sqrt{\frac{1}{2\pi}}\Big[\frac{e^{0(1-iw)}}{1-iw} - \frac{e^{-\infty(1-iw)}}{1-iw} + \frac{e^{-\infty(1+iw)}}{-1-iw} - \frac{e^{0(1+iw)}}{-1-iw}\Big]$$

$$= \sqrt{\frac{1}{2\pi}}\Big[\frac{1}{1-iw} - \frac{0}{1-iw} + \frac{0}{-1-iw} - \frac{1}{-1-iw}\Big] = \sqrt{\frac{1}{2\pi}}\Big[\frac{(-1-iw)-(1-iw)}{(1-iw)(-1-iw)}\Big] = \sqrt{\frac{1}{2\pi}}\Big[\frac{-2}{-1-w^2}\Big]$$

$$= \frac{2}{\sqrt{2\pi}(1+w^2)}.$$