$\mathbb{A}$dvanced $\mathbb{C}$alculus

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1

Find the Fourier integral representation of the solution, $u(x,t)$, to the heat equation on a metal bar of infinite length in both directions, with arbitrary c and $u(x,0) = f(x) = e^{-|x|}$. Simplify the inner integral.

- Solve for $A(p) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(v)\cos pv\,dv$.

  $$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|x|}\cos pv\,dv$$

  $$A(p) = \frac{2}{\pi(p^2+1)}$$; Integration in notebook.

- Solve for $B(p) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(v)\sin pv\,dv$.

  $$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|x|}\sin pv\,dv$$

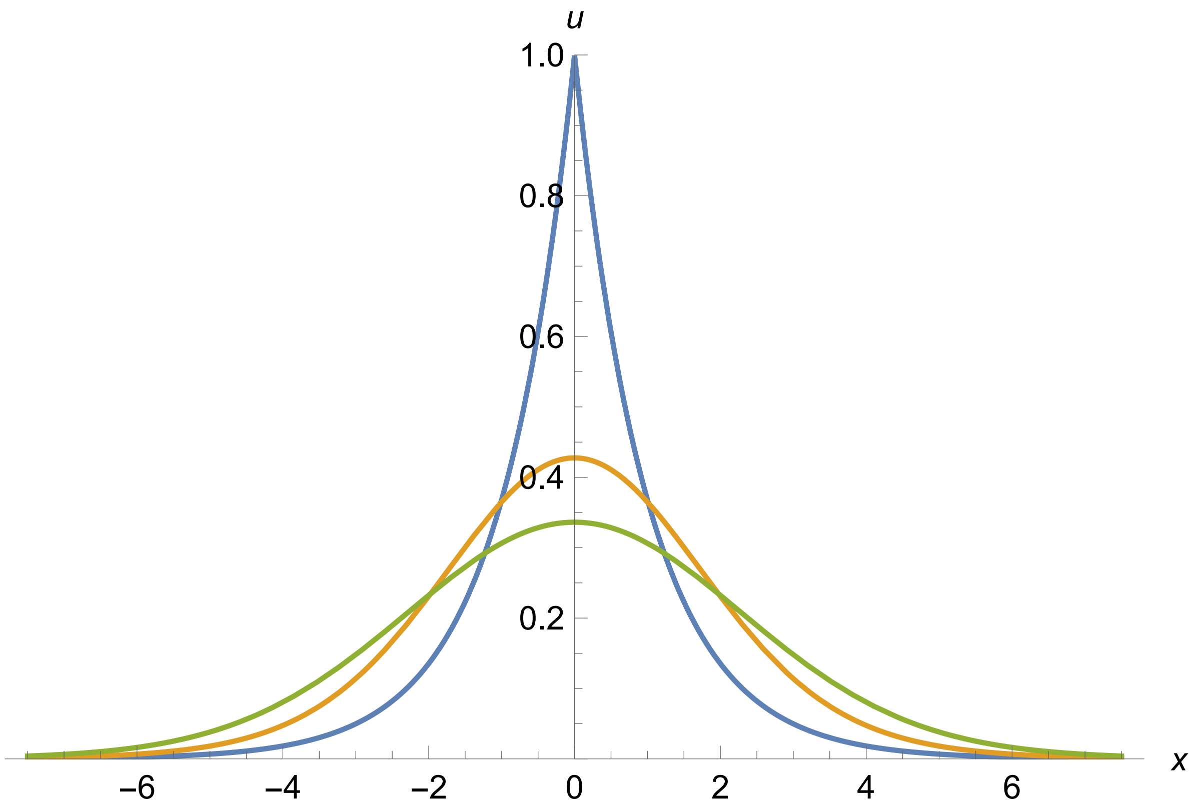
  $B(p) = 0$; Integration in notebook.

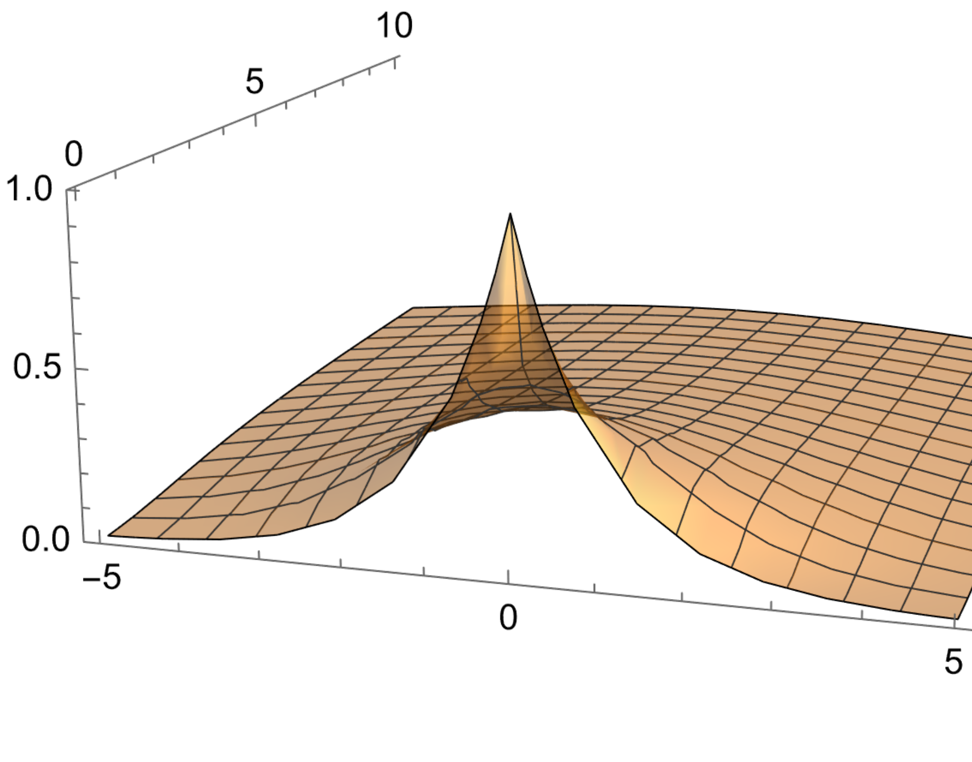
- Substitute $A(p), B(p)$ into $u(x,t) = \displaystyle\int_{0}^{\infty}\Big[A(p)\cos px + B(p)\sin px\Big]e^{-c^2p^2t}\,dp$.

  $$u(x,t) = \frac{2}{\pi}\int_{0}^{\infty}\left[\frac{\cos px}{(p^2+1)}\right]e^{-c^2p^2t}\,dp$$

2

Use Mathematica to plot your solution from #1 (with $c = 1$), plotting $u(x, 0)$, $u(x, 1)$, and $u(x, 2)$ on the same graph.





3

Find the Fourier integral representation of the solution, $u(x,t)$, to the heat equation on a half-infinite metal bar of, with arbitrary c, $u(0,t) = 0$, and $u(x,0) = f(x) = Boole[0 < x < 2](1-(x-1)^2)$. Simplify the inner integral.

- Solve for $\hat{f}_s(w) = \int_0^\infty f(p) \sin wp \, dp$.

$$\hat{f}_s(w) = \int_0^2 (1-(p-1)^2) \sin wp \, dp$$

$$\hat{f}_s(w) = -\frac{4 \sin(w)(w \cos(w) - \sin(w))}{w^3} \text{ ; Integration in notebook.}$$

- Substitute $\hat{f}_s(w)$ into $u(x,t) = \int_0^\infty \hat{f}_s(w) \sin(wx) e^{-c^2 w^2 t} \, dw$.

$$u(x,t) = \frac{2}{\pi} \int_0^\infty -\frac{4 \sin(w)(w \cos(w) - \sin(w))}{w^3} \sin(wx) e^{-c^2 w^2 t} \, dw$$

- Simplify the solution.

$$u(x,t) = -\frac{8}{\pi} \int_0^\infty \frac{\sin(w)(w \cos(w) - \sin(w))}{w^3} \sin(wx) e^{-c^2 w^2 t} \, dw$$

4

Use Mathematica to plot your solution from #3 (with $c = 1$), plotting $u(x, 0)$ and $u(x, 0.5)$ on the same graph.