$\mathbb{A}$dvanced $\mathbb{C}$alculus

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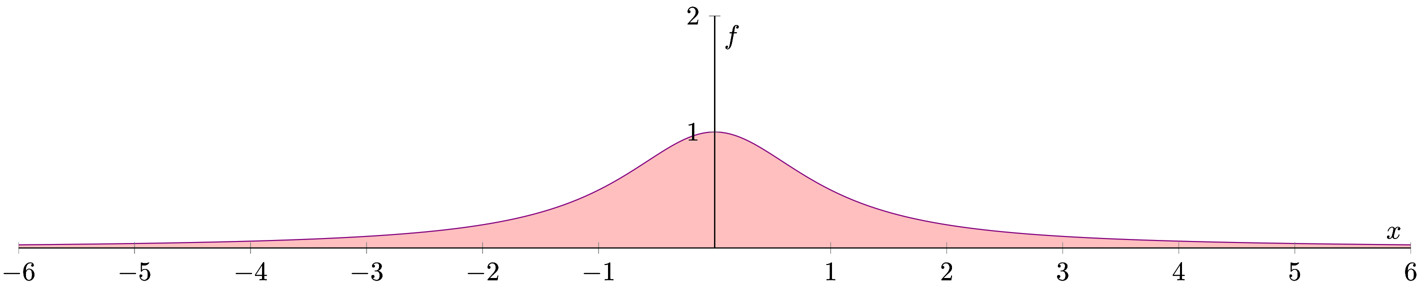
1

Represent $f(x) = \dfrac{1}{1+x^2}$ as a Fourier cosine integral. Use Mathematica to plot the integral on the interval $[-3, 3]$.



Solve the cosine integral given $f(x) = \displaystyle\int_0^\infty A(w)\cos(wx)\,dw$, where $A(w) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(v)\cos(vw)\,dv$.

- Let $\mathcal{F}_c(\gamma) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \gamma(v)\cos(vw)\,dv$ and $\mathcal{F}_c^{-1}(\gamma) = \displaystyle\int_0^\infty \gamma(w)\cos(wx)$.

- $A(w) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{\cos(vw)}{1+v^2}\,dv = \mathcal{F}_c(f)$. † Note $\mathcal{F}_c^{-1}(\mathcal{F}_c(\gamma)) = \gamma \ldots$

  From class we have, for $x > 0$: $e^{-kx} = \dfrac{2k}{\pi}\displaystyle\int_0^\infty \dfrac{\cos(wx)}{k^2+w^2}\,dw$.

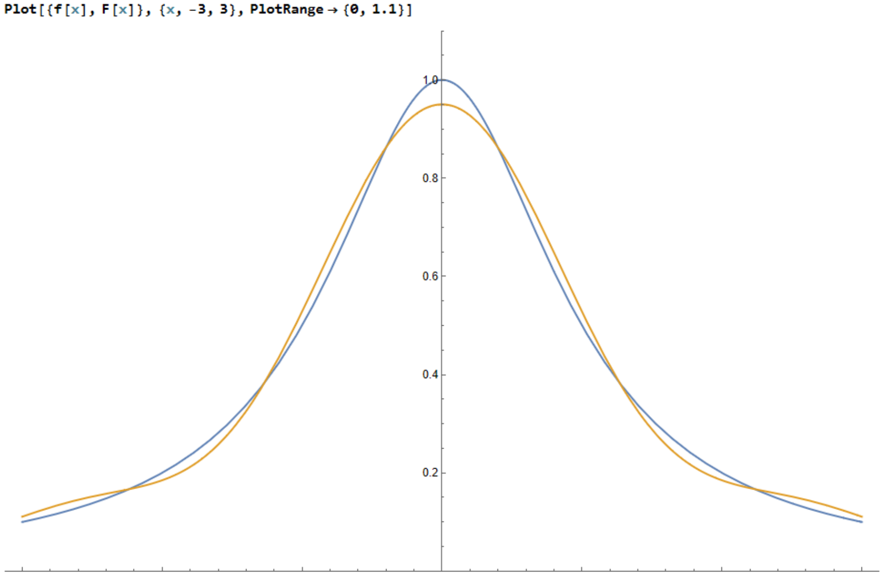
  Thus, for $v > 0$: $\alpha = e^{-v} = \dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{\cos(vw)}{1+v^2}\,dw = \mathcal{F}_c(\mathcal{F}_c^{-1}(\alpha))$.

  $A(w) = \mathcal{F}_c(f) = \mathcal{F}_c(\mathcal{F}_c^{-1}(\alpha)) = \alpha$

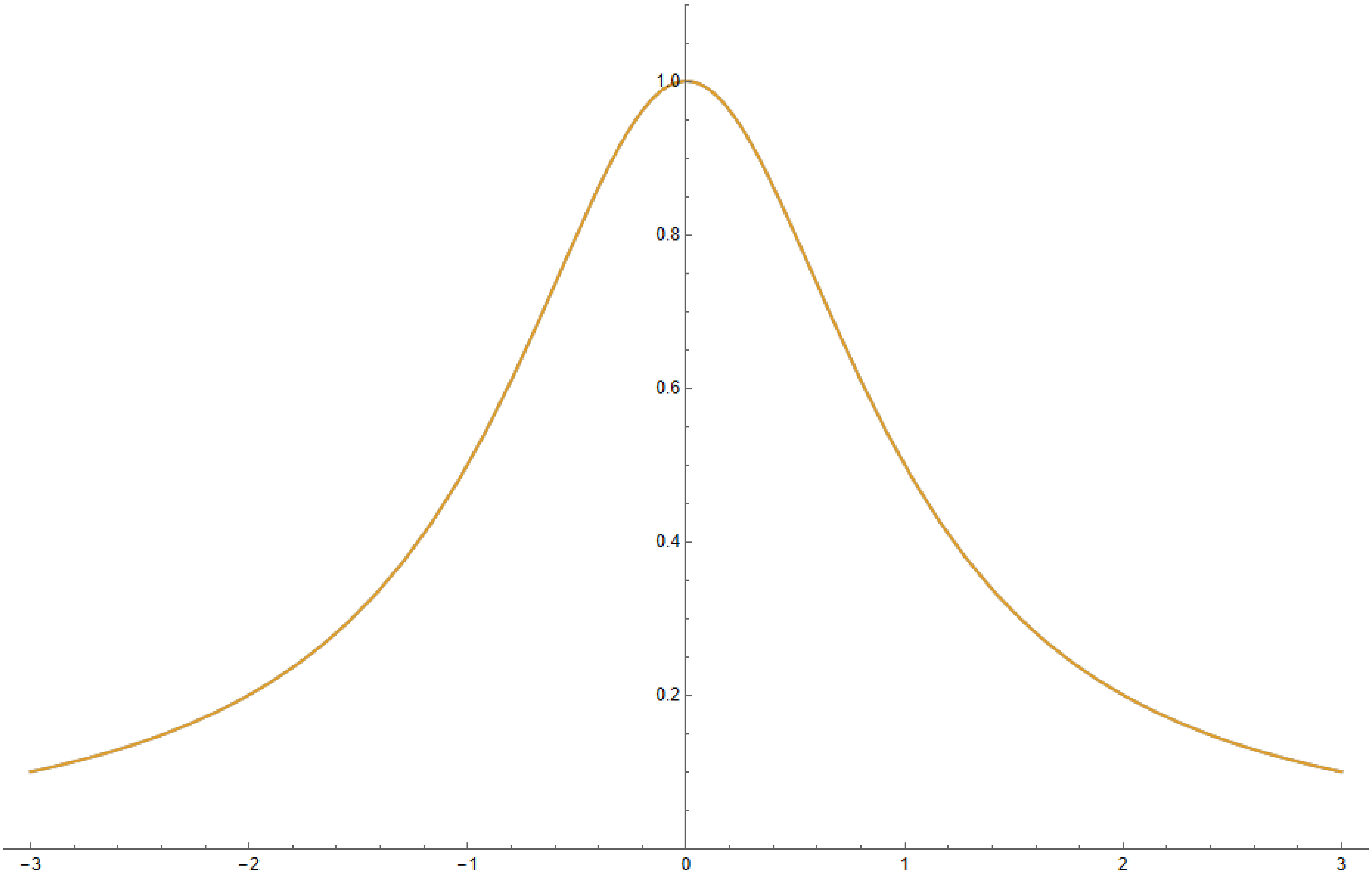
- $f(x) = \mathcal{F}_c^{-1}(A) = \mathcal{F}_c^{-1}(\alpha) = \displaystyle\int_0^\infty e^{-w}\cos(wx)\,dw$.

Plot the integral representations of f. † Use Mathematica . . .

```
f[x_] := 1 / (1 + x^2)
F[x_] := NIntegrate[E^-w * Cos[w * x], {w, 0, 3}];
Plot[{f[x], F[x]}, {x, -3, 3}, PlotRange → {0, 1.1}]
```

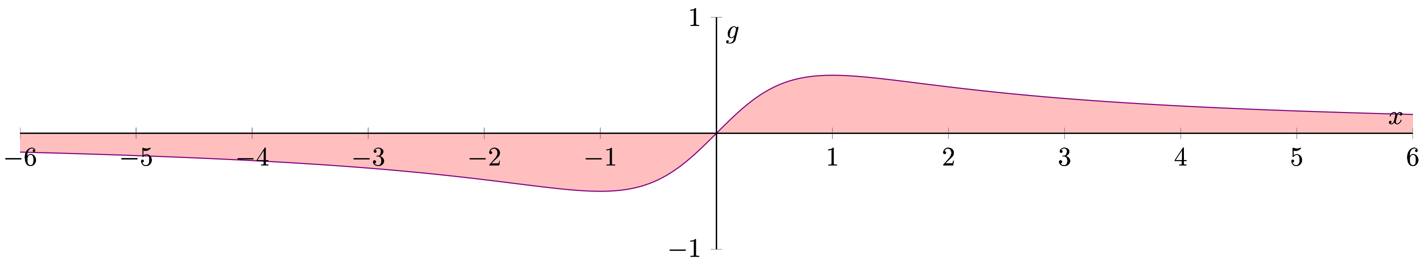


```
f[x_] := 1 / (1 + x^2)
F[x_] := NIntegrate[E^-w * Cos[w * x], {w, 0, Infinity}];
Plot[{f[x], F[x]}, {x, -3, 3}, PlotRange → {0, 1.1}]
```



2

Represent $g(x) = \dfrac{x}{1+x^2}$ as a Fourier sine integral. Use Mathematica to plot the integral on the interval $[-3, 3]$.



Solve the sine integral given $g(x) = \displaystyle\int_0^\infty B(w)\sin(wx)\,dw$, where $B(w) = \dfrac{2}{\pi}\displaystyle\int_0^\infty g(v)\sin(vw)\,dv$.

- Let $\mathcal{F}_s(\gamma) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \gamma(v)\sin(vw)\,dv$ and $\mathcal{F}_s^{-1}(\gamma) = \displaystyle\int_0^\infty \gamma(w)\sin(wx)$.
- $B(w) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{v\sin(vw)}{1+v^2}\,dv$. † Note $\mathcal{F}_s^{-1}(\mathcal{F}_s(\gamma)) = \gamma \ldots$

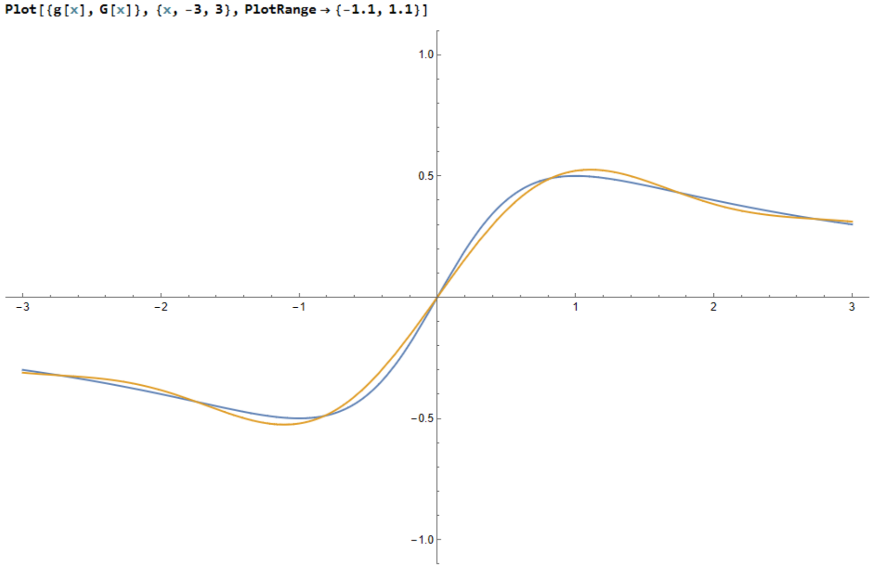
  From class we have, for $x > 0$: $e^{-kx} = \dfrac{2k}{\pi}\displaystyle\int_0^\infty \frac{w\sin(wx)}{k^2+w^2}\,dw$.

  Thus, for $v > 0$: $\beta = e^{-v} = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{v\sin(vw)}{1+v^2}\,dw = \mathcal{F}_s(\mathcal{F}_s^{-1}(\beta))$.

  $B(w) = \mathcal{F}_s(g) = \mathcal{F}_s(\mathcal{F}_s^{-1}(\beta)) = \beta$
- $g(x) = \mathcal{F}_s^{-1}(B) = \mathcal{F}_s^{-1}(\beta) = \displaystyle\int_0^\infty e^{-w}\sin(wx)\,dw$.

Plot the integral representations of g. † Use Mathematica . . .

```
g[x_] := x / (1 + x^2)
G[x_] := NIntegrate[E^-w * Sin[w * x], {w, 0, 3}];
Plot[{g[x], G[x]}, {x, -3, 3}, PlotRange → {-1.1, 1.1}]
```



```
g[x_] := x / (1 + x^2)
G[x_] := NIntegrate[E^-w*Sin[w*x], {w, 0, Infinity}];
Plot[{g[x], G[x]}, {x, -3, 3}, PlotRange → {-1.1, 1.1}]
```