𝔸dvanced ℂalculus

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1

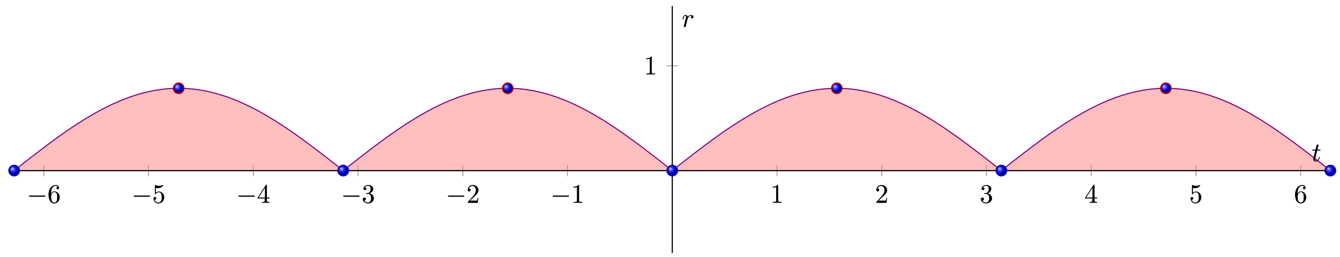
Find the general solution of the ODE $y'' + 9y = r(t)$, where $r(t) = \frac{\pi}{4}|\sin t|$ for $0 < t < 2\pi$ and $r(t+2\pi) = r(t)$.

First solve the homogeneous solution $y_h \mid y'' + 9y = 0$.

- $h^2 + 9 = 0$; $h^2 = -9$; $h = \sqrt{-9}$; $h = \pm 3i$.

- $y_h = c_1 cos(3t) + c_2 sin(3t)$.

Second solve the forcing function $r = \frac{\pi}{4}|\sin t|$ for $0 < t < 2\pi$ with period, $p = 2\pi$; $L = \pi$.



- Notice r is even thus the *Euler* coefficent $b_n = 0$. Solve for a_0 and a_n.

- $a_0 = \frac{1}{\pi}\int_0^\pi \frac{\pi}{4}|\sin t|\, dt$; note $sin(t)$ is positive for all $t \mid 0 < t < \pi$.

  $$a_0 = \frac{1}{4}[-cos(t)\Big|_0^\pi] = \frac{-cos(\pi) + cos(0)}{4} = \frac{1+1}{4} = \frac{1}{2}.$$

- $a_n = \frac{2}{\pi}\int_0^\pi \frac{\pi}{4}|\sin t| \cos(\frac{n\pi t}{\pi})\, dt = \frac{1}{2}\int_0^\pi \sin t \cos(nt)\, dt$. † Invoke Mathematica ...

  $$a_n = \frac{1}{2}[\frac{nsin(t)sin(nt) + cos(t)cos(nt)}{n^2-1}\Big|_0^\pi].$$

  $$a_n = \frac{1}{2}[\frac{(nsin(\pi)sin(n\pi) + cos(\pi)cos(n\pi)) - (nsin(0)sin(0) + cos(0)cos(0))}{n^2-1}].$$

  $$a_n = \frac{1}{2}[\frac{-cos(n\pi) - 1}{n^2-1}] \text{ ; note } cos(n\pi) = 1 \mid n\ even \text{ ; } cos(n\pi) = -1 \mid n\ odd.$$

  $$a_n = \frac{1}{2}[\frac{-1-1}{n^2-1}] = \frac{-1}{n^2-1} \mid n\ even.$$

- $r = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))$

  $$r = \frac{1}{2} - \sum_{n=2\ |n\ even}^{\infty} \frac{1}{n^2-1}\cos(nt).$$

  $$r = \frac{1}{2} - \frac{1}{2^2-1}\cos(2t) - \frac{1}{4^2-1}\cos(4t) - \frac{1}{6^2-1}\cos(6t) - \ldots.$$

  $$r = \frac{1}{2} - \frac{1}{3}\cos(2t) - \frac{1}{15}\cos(4t) - \frac{1}{35}\cos(6t) - \frac{1}{63}\cos(8t) - \ldots.$$

Third solve the particular solution $y_p \mid y'' + 9y = r(t)$.

- $y = A_0 + A\cos(nt)$; $y' = -An\sin(nt)$; $y'' = -An^2\cos(nt)$.

- $-An^2\cos(nt) + 9A\cos(nt) + 9A_0 = \frac{1}{2} - \frac{1}{n^2-1}\cos(nt)$.

$$A_0 = \frac{1}{18}.$$

$$A(9-n^2) = \frac{-1}{n^2-1} \ ; \ A = \frac{-1}{(n^2-1)(9-n^2)}.$$

- $y_p = A_0 + \sum_{n=2 \ |n \ even}^{\infty} Acos(nt) = \frac{1}{18} - \sum_{n=2 \ |n \ even}^{\infty} \frac{cos(nt)}{(n^2-1)(9-n^2)}$.

$$y_p = \frac{1}{18} - \frac{cos(2t)}{(2^2-1)(9-2^2)} - \frac{cos(4t)}{(4^2-1)(9-4^2)} - \frac{cos(6t)}{(n^2-1)(9-6^2)} \cdots$$

$$y_p = \frac{1}{18} - \frac{cos(2t)}{(3)(5)} - \frac{cos(4t)}{(15)(-7)} - \frac{cos(6t)}{(35)(-27)} - \frac{cos(8t)}{(63)(-54)} \cdots$$

$$y_p = \frac{1}{18} - \frac{cos(2t)}{15} + \frac{cos(4t)}{105} + \frac{cos(6t)}{945} + \frac{cos(8t)}{3402} \cdots$$

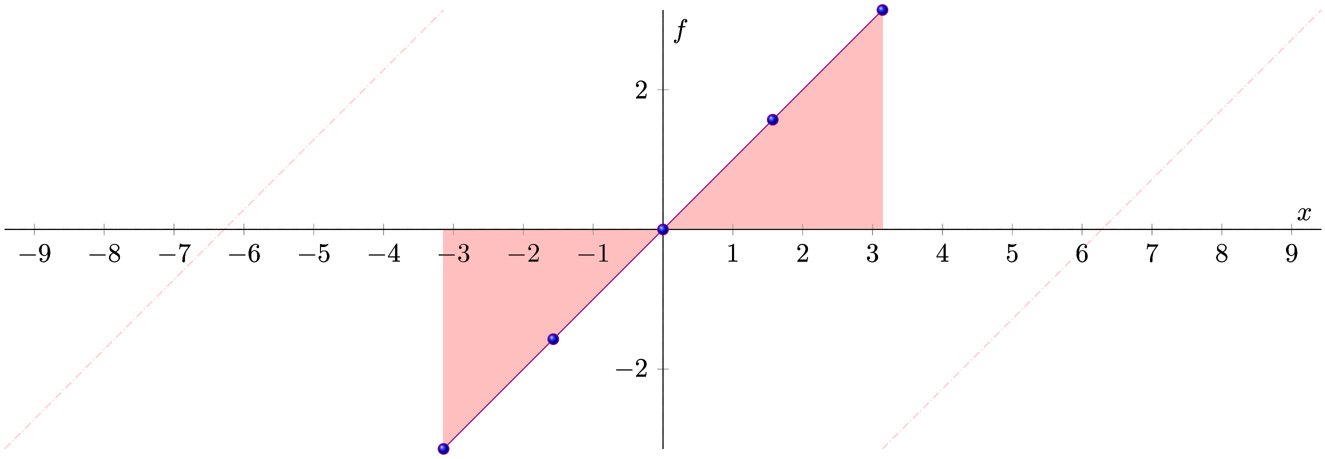
Last combine to form the general solution $y = y_h + y_p$.

- $y = c_1cos(3t) + c_2sin(3t) + \frac{1}{18} - \sum_{n=2 \ |n \ even}^{\infty} \frac{cos(nt)}{(n^2-1)(9-n^2)}$.

2

For $f(x) = x$ on $-\pi < x < \pi$, find the trigonometric polynomial $F(x) = A_0 + \sum_{n=1}^{N}(A_n \cos(nx) + B_n \sin(nx))$ that minimizes $||f - F||_2$ on $(-\pi, \pi)$, for $N = 1, 3, 5$.

Find the Fourier Series $|$ $p = 2\pi$; $L = \pi$



- Notice r is odd thus the *Euler* coefficents $a_0, a_n = 0$. Solve for b_n.

- $b_n = \frac{2}{\pi}\int_0^{\pi} x \sin(\frac{n\pi x}{\pi})\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\,dx$. † Invoke Mathematica ...

  $b_n = \frac{2}{\pi}[\frac{sin(nx) - nxcos(nx)}{n^2}\Big|_0^{\pi}] = \frac{2}{\pi}[\frac{(sin(n\pi) - n\pi cos(n\pi)) - (sin(0) - 0)}{n^2}]$.

  $b_n = \frac{2}{\pi}[\frac{-n\pi cos(n\pi)}{n^2}] = \frac{-2cos(n\pi)}{n}$; note $cos(n\pi) = 1 \mid n\ even$; $cos(n\pi) = -1 \mid n\ odd$.

  $b_n = B_n = -\frac{2}{n} \mid n\ even$; $\frac{2}{n} \mid n\ odd$.

- $f = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) = F = A_0 + \sum_{n=1}^{\infty}(A_n \cos(nx) + B_n \sin(nx))$.

  $F_N = \sum_{n=1\ \mid\ n\ odd}^{N} +\frac{2}{n}\sin(nx) - \sum_{n=2\ \mid\ n\ even}^{N} +\frac{2}{n}\sin(nx)$.

  $F_{N=1} = 2\sin(x)$.

  $F_{N=3} = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x)$.

  $F_{N=5} = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x)$.

3

Refer to problem 2 to complete problem 3. Use software to compute $||f-F||_2$ for $N = 1, 3, 5$. Then use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to show $\lim_{N\to\infty} (||f-F||_2)^2 = \lim_{N\to\infty} E^*(N) = 0$.

Recall our solutions from 2. **Recall** $||f-F||_2 = \sqrt{\int_R (f-F)^2\, dx}$.

- $f = x \mid -\pi < x < \pi$; $F_N = \sum_{n=1 \mid n\ odd}^{N} +\frac{2}{n}\sin(nx) - \sum_{n=2 \mid n\ even}^{N} +\frac{2}{n}\sin(nx)$.

- $F_{N=1} = 2\sin(x)$.

$$f - F_{N=1} = x - 2\sin(x)$$
$$||f-F_{N=1}||_2 = \sqrt{\int_{-\pi}^{\pi} (x - 2\sin(x))^2\, dx} = 2.84684$$

- $F_{N=3} = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x)$.

$$f - F_{N=3} = x - 2\sin(x) + \sin(2x) - \frac{2}{3}\sin(3x)$$
$$||f-F_{N=3}||_2 = \sqrt{\int_{-\pi}^{\pi} (x - 2\sin(x) + \sin(2x) - \frac{2}{3}\sin(3x))^2\, dx} = 1.88855$$

- $F_{N=5} = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x)$.

$$f - F_{N=5} = x - 2\sin(x) + \sin(2x) - \frac{2}{3}\sin(3x) + \frac{1}{2}\sin(4x) - \frac{2}{5}\sin(5x)$$
$$||f-F_{N=5}||_2 = \sqrt{\int_{-\pi}^{\pi} (x - 2\sin(x) + \sin(2x) - \frac{2}{3}\sin(3x) + \frac{1}{2}\sin(4x) - \frac{2}{5}\sin(5x))^2\, dx} = 1.50949$$

Recall $||f-F||_2 = E$ and as $A_0 = a_0,\ A_n = a_n,\ B_n = b_n$; $E = E^*$. **Thus** $\lim_{N\to\infty} (||f-F||_2)^2 = \lim_{N\to\infty} E(N) = \lim_{N\to\infty} E^*(N)\ldots$

Find $E^* = \int_{-\pi}^{\pi} f^2\, dx - \pi[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)]$.

- $E^* = \int_{-\pi}^{\pi} f^2\, dx - \pi\sum_{n=1}^{N} b_n^2$.

$$\int_{-\pi}^{\pi} f^2\, dx = \int_{-\pi}^{\pi} x^2\, dx = \frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^3}{3} + \frac{\pi^3}{3} = \frac{2\pi^3}{3}$$
$$\pi\sum_{n=1}^{N} b_n^2 = \pi\sum_{n=1}^{N}((-1)^{n-1}\frac{2}{n})^2 = \pi\sum_{n=1}^{N}(-1)^{2(n-1)}\frac{4}{n^2} = \pi\sum_{n=1}^{N}\frac{4}{n^2} = 4\pi\sum_{n=1}^{N}\frac{1}{n^2}$$

Find $\lim_{N\to\infty} E^*(N)$.

- $\lim_{N\to\infty} E^*(N) = \int_{-\pi}^{\pi} f^2\, dx - \pi\sum_{n=1}^{\infty} b_n^2 = \frac{2\pi^3}{3} - 4\pi\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^3}{3} - 4\pi(\frac{\pi^2}{6}) = \frac{2\pi^3}{3} - \frac{2\pi^3}{3} = 0$