$\mathbb{A}$dvanced $\mathbb{C}$alculus

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1

The velocity field of a fluid with density $\rho = 400\ \frac{kg}{m^3}$ is given by $\mathbf{F}(x, y, z) = \langle 1-x, -y, -z-1\rangle\ \frac{m}{s}$. Let T be the polyhedral region with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. S is the boundary of T.

Compute $\iint\limits_S \rho\mathbf{F}\cdot\mathbf{n}\,dA$ by both using the divergence theorem and directly by considering the flux on each face.

First we will first solve via the divergence theorem.

- We know by the divergence theorem that $\rho\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dA = \rho\iiint\limits_T \nabla\cdot\mathbf{F}\,dV$
- We will first solve for $\nabla\cdot\mathbf{F}$.

  $$\nabla\cdot\mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle\cdot\langle 1-x, -y, -z-1\rangle\ \frac{m}{s} = [(1-x)_x + (-y)_y + (-1-z)_z]\frac{m}{s} = [-1-1-1]\frac{m}{s}$$

  $$\nabla\cdot\mathbf{F} = -3\frac{m}{s}$$

- The volume the region T is $V = \dfrac{1}{6}m^3$.
- We can now put everything together ...

  $$\rho\cdot\nabla\cdot\mathbf{F}\cdot V = 400\ \frac{kg}{m^3}\cdot -3s^{-1}\cdot\frac{1}{6}m^3 = -200\frac{kg}{s}$$

  The region is a sink with $200kg$ flowing in per second.

Second we will first solve by considering the flux through each face of T.

- Consider the face with verticies $(0,0,0), (1,0,0), (0,1,0)$.

  The unit normal vector at this point is strictly in the z direction.

  $\mathbf{n} = \langle 0, 0, -1\rangle$

  $\mathbf{F}\cdot\mathbf{n} = \langle 1-x, -y, -z-1\rangle\cdot\langle 0,0,-1\rangle = z+1$

  Across this face z is a constant of 0 $\therefore$ $\mathbf{F}\cdot\mathbf{n} = 1$

  This face is a triangle with area $\dfrac{1}{2}m^2$

  The flux through this face is $\rho\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dA = 400\frac{kg}{s}\cdot 1\frac{m}{s}\cdot\frac{1}{2}m^2 = 200\frac{kg}{s}$

- Consider the face with verticies $(0,0,0), (1,0,0), (0,0,1)$.

  The unit normal vector at this point is strictly in the z direction.

  $\mathbf{n} = \langle 0, -1, 0\rangle$

$\mathbf{F} \cdot \mathbf{n} = \langle 1-x, -y, -z-1 \rangle \cdot \langle 0, -1, 0 \rangle = y$

Across this face y is a constant of $0 \therefore \mathbf{F} \cdot \mathbf{n} = 0$

This face is a triangle with area $\frac{1}{2}m^2$

The flux through this face is $\rho \iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dA = 400\frac{kg}{s} \cdot 0\frac{m}{s} \cdot \frac{1}{2}m^2 = 0\frac{kg}{s}$

- Consider the face with verticies $(0,0,0), (0,1,0), (0,0,1)$.

  The unit normal vector at this point is strictly in the z direction.

  $\mathbf{n} = \langle -1, 0, 0 \rangle$

  $\mathbf{F} \cdot \mathbf{n} = \langle 1-x, -y, -z-1 \rangle \cdot \langle 0, 0, -1 \rangle = x - 1$

  Across this face x is a constant of $0 \therefore \mathbf{F} \cdot \mathbf{n} = -1$

  This face is a triangle with area $\frac{1}{2}m^2$

  The flux through this face is $\rho \iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dA = 400\frac{kg}{s} \cdot -1\frac{m}{s} \cdot \frac{1}{2}m^2 = -200\frac{kg}{s}$

- Consider the face with verticies $(1,0,0), (0,1,0), (0,0,1)$.

  The unit normal vector at this point is equal in the x, y, and z direction.

  $\mathbf{n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

  $\mathbf{F} \cdot \mathbf{n} = \langle 1-x, -y, -z-1 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \frac{1}{\sqrt{3}}\Big[1 - x + -y + -z - 1\Big] = \frac{1}{\sqrt{3}}\Big[-x + -y + -z\Big]$

  Across this face $x + y + z$ is a constant of $1 \therefore \mathbf{F} \cdot \mathbf{n} = -\frac{1}{\sqrt{3}}\frac{m}{s}$

  This face is a triangle with area $\frac{\sqrt{3}}{2}m^2$

  The flux through this face is $\rho \iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dA = 400\frac{kg}{m^3} \cdot -\frac{1}{\sqrt{3}}\frac{m}{s} \cdot \frac{\sqrt{3}}{2}m^2 = -200\frac{kg}{s}$

- Last we can find the flux through this surface by summing the flux through each face.

  $\text{flux} = \Big[200 + 0 - 200 - 200\Big]\frac{kg}{s} = -200\frac{kg}{s}$

  As we oriented the normals directed out of the surface, the negative sign in the flux indicates flow into the surface.

  Again we find that the region is a sink with $200kg$ flowing in per second.

2

Let $f(x) = -x$ for $0 < x < 1$. Find the Fourier series for an **odd** periodic extension of f, listing the first 4 non-zero terms. Then, find the general solution to the ODE $y'' + y = f(x)$.

First we will solve for the Fourier series.

- We know the form of a Fourier series is $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)$.

  Here $L = 1$ and the function is treated as odd, we now have $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$.

- We also know $b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx$.

  Here $b_n = 2\int_0^1 -x\sin(n\pi x)\,dx$.

- We can solve the itergral to simplify b_n.

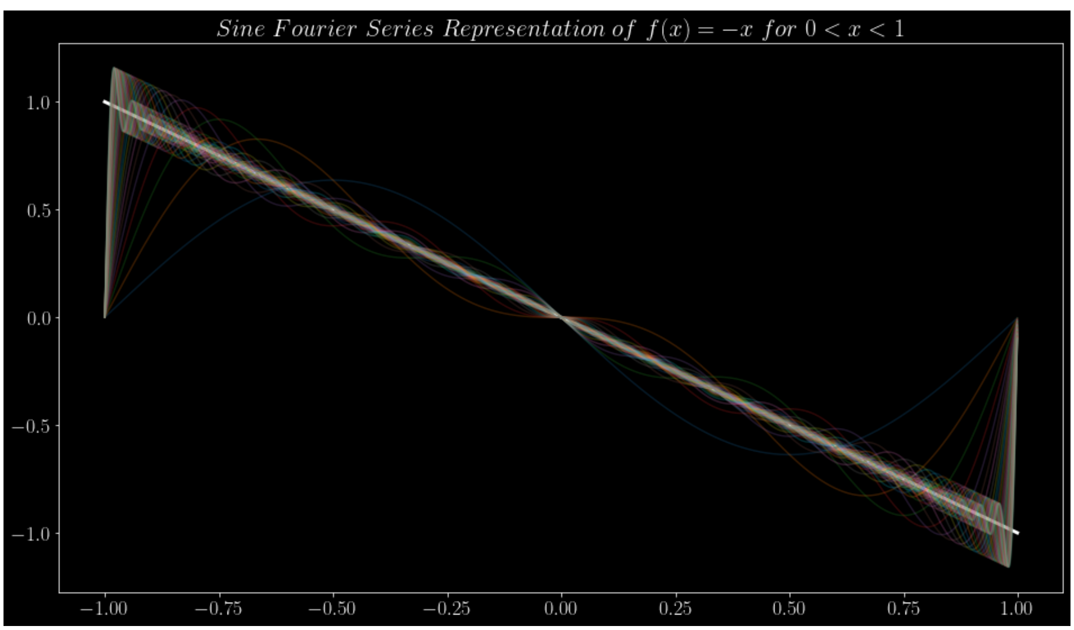
  $$b_n = \frac{\pi n\cos(\pi n) - \sin(\pi n)}{\pi^2 n^2}$$

  Furthermore; $\sin(\pi n)$ is 0 for all integer multiples of n. Thus ...

  $$b_n = \frac{\cos(\pi n)}{\pi n}.$$

- The Fourier series of f is then $f(x) = \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\pi n}\sin(n\pi x)$.

  The first 4 terms are $-\frac{\sin(\pi x)}{\pi}, \frac{\sin(2\pi x)}{2\pi}, -\frac{\sin(3\pi x)}{3\pi}, \frac{\sin(4\pi x)}{4\pi}$.



Second we will solve the ODE.

- To solve the ODE, we must first solve for a homogeneous solution ; $y_h \mid y'' + y = 0$.

  $h^2 + 1 = 0$; $h^2 = -1$; $h = \pm i$.

  $y_h = c_1 cos(x) + c_2 sin(x)$.

- Second, we must solve for a particular solution ; $y_p \mid y'' + y = f(x)$.

  $y = B\sin(n\pi x)$; $y' = Bn\pi\cos(n\pi x)$; $y'' = -Bn^2\pi^2\sin(n\pi x)$.

- We can now substitute into the origional equation.

  $$-Bn^2\pi^2\sin(n\pi x) + B\sin(n\pi x) = \frac{\cos(\pi n)}{\pi n}\sin(n\pi x).$$

  $B(1 - n^2\pi^2) = \frac{\cos(\pi n)}{\pi n}$; $B = \frac{\cos(\pi n)}{\pi n(1 - n^2\pi^2)}$; We can notice here that $\cos(n\pi) = (-1)^n$.

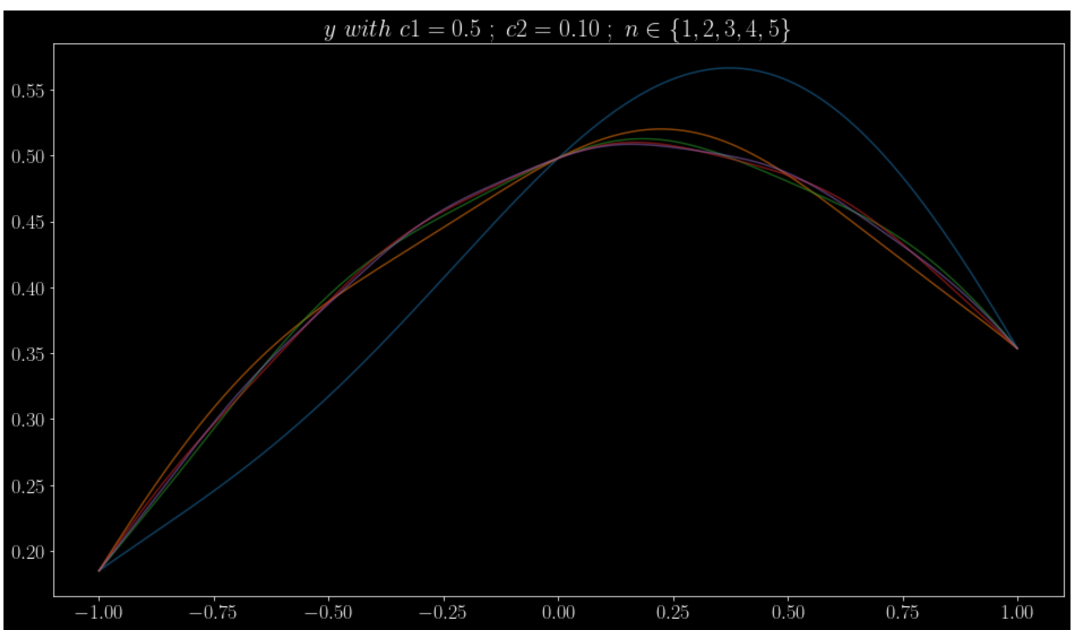
  $$B = \frac{(-1)^n}{n\pi - n^3\pi^3}.$$

- From here we can solve for the particular solution $y_p = \sum_{n=1}^{\infty} B\sin(n\pi x)$.

  $$y_p = \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n\pi - n^3\pi^3}.$$

- Combining the homogeneous and particular solution, we have the general solution $y = y_h + y_p$.

  $$y = c_1 cos(x) + c_2 sin(x) + \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n\pi - n^3\pi^3}.$$



3

Use the Fourier sine transform to derive the solution formula for the heat equation $u_t = c^2 u_{xx}$ on the half-infinite bar $(0 \le x < \infty)$ with Dirichlet boundary condition $u(0,t) = a$, for some constant a, and initial condition $u(x,0) = f(x)$.

First transfrom the PDE to an ODE by the Fourier sine transform.

- $\mathcal{F}_s\{u_t\} = \mathcal{F}_s\{c^2 u_{tt}\}$.

$$\dot{\hat{u}}_s = c^2\left[-w^2\hat{u}_s + \sqrt{\frac{2}{\pi}}wu(0,t)\right]$$

$$\dot{\hat{u}}_s = -w^2c^2\hat{u}_s + awc^2\sqrt{\frac{2}{\pi}}$$

$$\dot{\hat{u}}_s + w^2c^2\hat{u}_s = awc^2\sqrt{\frac{2}{\pi}}$$

- Solve for the homogeneous solution $(\hat{u}_s)_h \mid \dot{\hat{u}} + w^2c^2\hat{u}_s = 0$

$$\lambda = -w^2c^2$$

$$(\hat{u}_s)_h = \kappa(w)e^{\lambda t}$$

$$(\hat{u}_s)_h(w,0) = \mathcal{F}_s\{u(x,0)\} = \mathcal{F}_s\{f(x)\} = \hat{f}_s(w)$$

$$(\hat{u}_s)_h = \hat{f}_s(w)e^{-w^2c^2t}$$

- Solve for the particular solution $(\hat{u}_s)_p \mid \dot{\hat{u}} + w^2c^2\hat{u}_s = awc^2\sqrt{\frac{2}{\pi}}$

$$(\hat{u}_s)_p = \kappa(w)$$

$$(\dot{\hat{u}}_s)_p = 0$$

$$0 + w^2c^2\kappa(w) = awc^2\sqrt{\frac{2}{\pi}}$$

$$\kappa(w) = \frac{awc^2\sqrt{\frac{2}{\pi}}}{w^2c^2} = \frac{a}{w}\sqrt{\frac{2}{\pi}}$$

$$(\hat{u}_s)_p = \frac{a}{w}\sqrt{\frac{2}{\pi}}$$

- Combining the homogeneous and particular solution, we have the general solution $\hat{u}_s = (\hat{u}_s)_h + (\hat{u}_s)_p$.

$$\hat{u}_s = \hat{f}_s e^{-w^2c^2t} + \frac{a}{w}\sqrt{\frac{2}{\pi}}$$

- We know the form of $\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(p) \sin{(wp)}\, dp$

$$\hat{u}_s(w,t) = \sqrt{\frac{2}{\pi}} \Big[\int\limits_0^\infty f(p) \sin{(wp)}\, dp\Big] e^{-w^2c^2t} + \frac{a}{w}\sqrt{\frac{2}{\pi}}$$

$$\hat{u}_s(w,t) = \sqrt{\frac{2}{\pi}} \Big[e^{-w^2c^2t} \int\limits_0^\infty f(p) \sin{(wp)}\, dp + \frac{a}{w}\Big]$$

- We can now transfer back into $u(x,t)$ as $u(x,t) = \mathcal{F}_s^{-1}\{\hat{u}_s(w,t)\}$; $f(x) = \mathcal{F}_s^{-1}\{f(w)\} = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(w) \sin{(wx)}\, dw$

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \Bigg[\sqrt{\frac{2}{\pi}} \Big[e^{-w^2c^2t} \int\limits_0^\infty f(p) \sin{(wp)}\, dp + \frac{a}{w}\Big]\Bigg] \sin{(wx)}\, dw$$

$$u(x,t) = \frac{2}{\pi} \int\limits_0^\infty \Bigg[\Big[e^{-w^2c^2t} \int\limits_0^\infty f(p) \sin{(wp)}\, dp + \frac{a}{w}\Big]\Bigg] \sin{(wx)}\, dw$$

4

Once the temperature in an object reaches a steady state, the heat equation becomes the Laplace equation. Use separation of variables to derive the steady-state solution to the heat equation on the rectangle $R = [0,1] \times [0,1]$ with the following Dirichlet boundary conditions: $u = 0$ on the left and right sides; $u = f(x)$ on the bottom; $u = g(x)$ on the top. That is, solve $u_{xx} + u_{yy} = 0$ subject to $u(0,y) = u(1,y) = 0$, $u(x,0) = f(x)$, and $u(x,1) = g(x)$. You may assume the separation constant is negative: $F''/F = -k$, for $k > 0$. Finally, plot $u(x,y)$ when $f(x) = \sin(\pi x)$ and $g(x) = \sin(3\pi x)$.

First we will use seperation of variable to obtain 2 ODEs.

- Assume $u(x,y) = F(x)G(y)$ solve for the partial derivates ...

$$u_x = F'G$$

$$u_{xx} = F''G$$

$$u_y = FG'$$

$$u_{yy} = FG''$$

- Substitute into the know equation $u_{xx} + u_{yy} = 0$.

$$F''G + FG'' = 0$$

$$F''G = -FG''$$

$$\frac{F''}{F} = -\frac{G''}{G} = -k$$

$$F'' = -kF \ ; \ G'' = kG$$

- Solve the ODEs...

$$\lambda_f^2 = -k \ ; \ \lambda_g^2 = k$$

$$\lambda_f = \pm i\sqrt{k} \ ; \ \lambda_g = \pm\sqrt{k}$$

$$F(x) = A_f \cos(x\sqrt{k}) + B_f \sin(x\sqrt{k}) \ ; \ G(y) = A_g e^{y\sqrt{k}} + B_g e^{-y\sqrt{k}}$$

- Inspect the initial conditions ...

$$u(0,y) = 0 \Rightarrow F(0)G(y) = 0 \Rightarrow F(0) = 0$$

$$u(1,y) = 0 \Rightarrow F(1)G(y) = 0 \Rightarrow F(1) = 0$$

$$u(x,0) = f(x) \Rightarrow F(x)G(0) = f(x)$$

$$u(x,1) = g(x) \Rightarrow F(x)G(1) = g(x)$$

- Use the initial conditions to simplify F and G.

$$F(0) = A_f \cos(0) + B_f \sin(0) = A_f = 0 \Rightarrow F(x) = B_f \sin(x\sqrt{k})$$

$$F(1) = B_f \sin(\sqrt{k}) = 0 \Rightarrow \sqrt{k}_n = n\pi \Rightarrow F_n(x) = B_{f_n} \sin(n\pi x)$$

$$G(0) = A_g + B_g = f(x)$$

$$G(1) = A_g e^{n\pi} + B_g e^{-n\pi} = g(x)$$

$$G_n(y) = A_{g_n} e^{yn\pi} + B_{g_n} e^{-yn\pi}$$

- Find the form of u.

$$u_n(x,y) = F_n(x)G_n(y) = B_{f_n} \sin(n\pi x)\Big[A_{g_n} e^{yn\pi} + B_{g_n} e^{-yn\pi}\Big]$$

$$u_n(x,y) = \sin(n\pi x)\Big[A_n e^{yn\pi} + B_n e^{-yn\pi}\Big]$$

$$u(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x)\Big[A_n e^{yn\pi} + B_n e^{-yn\pi}\Big]$$

- Use the initial conditions to find the constants A_n and B_n.

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \sin(n\pi x)\Big[A_n + B_n\Big]$$

$$A_n + B_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx - B_n$$

$$u(x,1) = g(x) = \sum_{n=1}^{\infty} \sin(n\pi x)\Big[A_n e^{n\pi} + B_n e^{-n\pi}\Big]$$

$$A_n e^{n\pi} + B_n e^{-n\pi} = 2\int_0^1 g(x)\sin(n\pi x)\,dx$$

$$2\int_0^1 f(x)\sin(n\pi x)e^{n\pi}\,dx - B_n e^{n\pi} + B_n e^{-n\pi} = 2\int_0^1 g(x)\sin(n\pi x)\,dx$$

$$B_n\Big[e^{-n\pi} - e^{n\pi}\Big] = 2\int_0^1 g(x)\sin(n\pi x)\,dx - 2\int_0^1 f(x)\sin(n\pi x)e^{n\pi}\,dx$$

$$B_n = \frac{2\int_0^1 [g(x) - f(x)e^{n\pi}]\sin(n\pi x)\,dx}{e^{-n\pi} - e^{n\pi}}$$

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx - \frac{2\int_0^1 [g(x) - f(x)e^{n\pi}]\sin(n\pi x)\,dx}{e^{-n\pi} - e^{n\pi}}$$

$$A_n = \frac{2\int_0^1 [f(x)[e^{-n\pi} - e^{n\pi}] - g(x) - f(x)e^{n\pi}] \sin(n\pi x)\, dx}{e^{-n\pi} - e^{n\pi}}$$

$$A_n = \frac{2\int_0^1 [f(x)e^{-n\pi} - 2f(x)e^{n\pi} - g(x)] \sin(n\pi x)\, dx}{e^{-n\pi} - e^{n\pi}}$$

- Substitue A_n and B_n into $u(x,t)$.

$$u(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x)\Big[\frac{2e^{yn\pi}\int_0^1 [f(x)e^{-n\pi} - 2f(x)e^{n\pi} - g(x)] \sin(n\pi x)\, dx}{e^{-n\pi} - e^{n\pi}} + \frac{2e^{-yn\pi}\int_0^1 [g(x) - f(x)e^{n\pi}] \sin(n\pi x)\, dx}{e^{-n\pi} - e^{n\pi}}\Big]$$

$$u(x,y) = 2\sum_{n=1}^{\infty} \sin(n\pi x)\Big[\frac{e^{yn\pi}\int_0^1 [f(x)e^{-n\pi} - 2f(x)e^{n\pi} - g(x)] \sin(n\pi x)\, dx + e^{-yn\pi}\int_0^1 [g(x) - f(x)e^{n\pi}] \sin(n\pi x)\, dx}{e^{-n\pi} - e^{n\pi}}\Big]$$

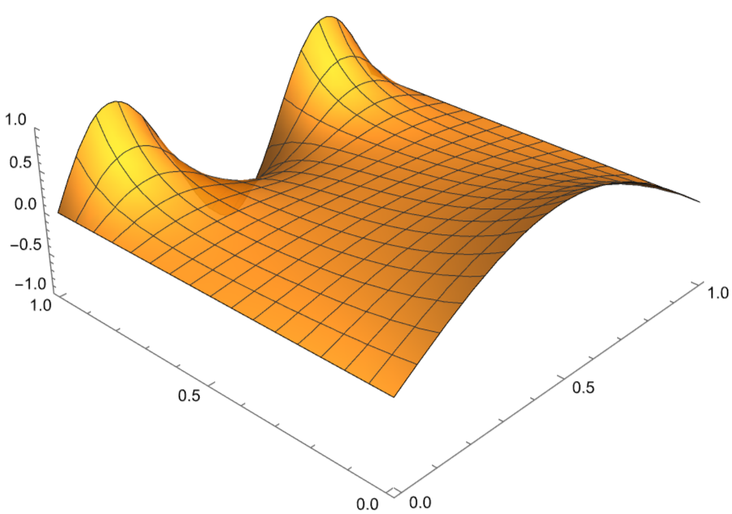
- Solving for $u(x,y)$ when $f(x) = \sin(\pi x)$ and $g(x) = \sin(3\pi x)\ldots$

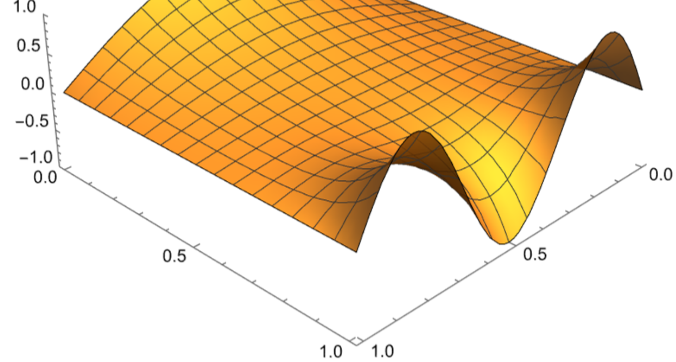
$$B_n = \frac{2\int_0^1 [\sin(3\pi x) - \sin(\pi x)e^{n\pi}] \sin(n\pi x)\, dx}{e^{-n\pi} - e^{n\pi}} = \frac{2\left(\frac{3}{9\pi - \pi n^2} + \frac{e^{n\pi}}{\pi(n^2-1)}\right)\sin(\pi n)}{e^{-n\pi} - e^{n\pi}}$$

$$A_n = 2\int_0^1 \sin(\pi x)\sin(n\pi x)\, dx - B_n = \frac{2\sin(\pi n)}{\pi - \pi n^2} - B_n = 2\sin(n\pi)\left[\frac{1}{\pi - \pi n^2} - \frac{\left(\frac{3}{9\pi - \pi n^2} + \frac{e^{n\pi}}{\pi(n^2-1)}\right)}{e^{-n\pi} - e^{n\pi}}\right]$$

$$A_n = 2\sin(n\pi)\left[\frac{(\pi - \pi n^2) - \left(\frac{3}{9\pi - \pi n^2} + \frac{e^{\pi n}}{\pi(n^2-1)}\right)}{\pi(1-n^2)(e^{-n\pi} - e^{n\pi})}\right]$$

$$u(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x)\Big[A_n e^{yn\pi} + B_n e^{-yn\pi}\Big]$$

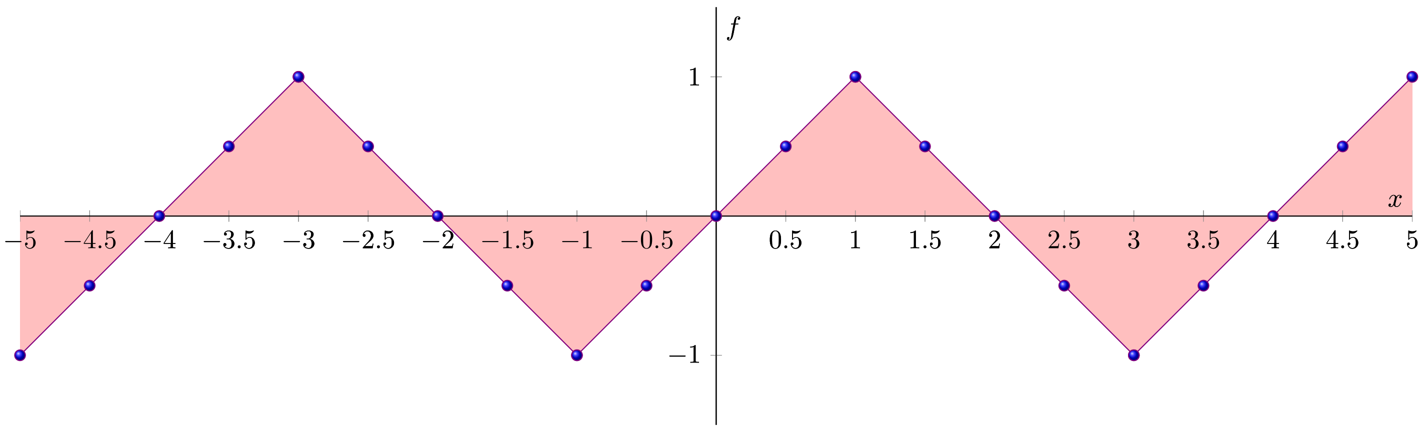




5

Let $u(x,t)$ be the solution to $u_{tt} = 16u_{xx}$ for $0 \leq x \leq 2$ and $t \geq 0$, where: $u(0,t) = 0$, $u(2,t) = 0$, and $u(x,0) = f(x) = 1 - |x-1|$ for $0 \leq x \leq 2$. Use D'Alembert's solution to find $u(1, 0.1)$ and $u(1, 0.6)$. Be careful to consider that D'Alembert's solution uses the odd periodic extension of $f(x)$.

First consider the form of the $f(x)$ extended as an odd function.



Second we can evaluate given the D'Alembert's solution : $2u(x,t) = f(x-ct) + f(x+ct)$.

- Note that from the equation given, $c^2 = 16$, Thus $c = 4$.

- Evaluate $u(1, 0.1)$.

  $$2u(1,0.1) = f(1-4(0.1)) + f(1+4(0.1)) = f(1-0.4) + f(1+0.4)$$

  $$2u(1,0.1) = f(0.6) + f(1.4) = 0.6 + 0.6 = 2(0.6)$$

  $$u(1,0.1) = 0.6$$

- Evaluate $u(1, 0.6)$.

  $$2u(1,0.6) = f(1-4(0.6)) + f(1+4(0.6)) = f(1-2.4) + f(1+2.4)$$

  $$2u(1,0.6) = 0.6 - 0.6 = 0$$

  $$u(1,0.6) = 0$$

6

Find the general solution of $u_{xx} - 3u_{xy} + 2u_{yy} = 0$ using the the method of characteristics: let $v = y + 2x$ and $w = y + x$; define $U(v, w)$ to be $U(v, w) = U(y+2x, y+x) = u(x, y)$; derive and solve a PDE for $U(v, w)$; convert back to $u(x, y)$. Use your solution to provide a non-trivial example of a solution.

Solve for u.

- We will classify based on the characteristic classifier; $4AC - B^2$.

  $A = 1\ ;\ B = -3\ ;\ C = 2$

  $4AC - B^2 = 4(1)(2) - (-3)^2 = 8 - 9 = -1$

  The PDE is Hyperbolic. $\Rightarrow$ Target $U_{vw} = 0$.

- Solve for x and y in terms of u and v.

  $v - w = y + 2x - (y + x) = y + 2x - y - x$

  $x = v - w$

  $y = v - 2x = v - 2(v - w) = v - 2v + 2w$

  $y = 2w - v$

- Solve for the partials of x and y with respect to v and w.

  $x_v = 1$

  $x_w = -1$

  $y_v = -1$

  $y_w = 2$

- Solve for the partials of U.

  $U_v = U_x x_v + U_y y_v = U_x - U_y$

  $U_{vv} = (U_x - U_y)_x x_v + (U_x - U_y)_y y_v = (U_x - U_y)_x - (U_x - U_y)_y = U_{xx} - U_{yx} - U_{xy} + U_{yy} = U_{xx} - 2U_{xy} + U_{yy}$

  $U_{vw} = (U_x - U_y)_x x_w + (U_x - U_y)_y y_w = 2(U_x - U_y)_y - (U_x - U_y)_x = 2U_{xy} - 2U_{yy} - U_{xx} + U_{yx} = -U_{xx} + 3U_{xy} - 2U_{yy}$

  $U_w = U_x x_w + U_y y_w = 2U_y - U_x$

  $U_{ww} = (2U_y - U_x)_x x_w + (2U_y - U_x)_y y_w = 2(2U_y - U_x)_y - (2U_y - U_x)_x = 4U_{yy} - U_{xy} - 2U_{yx} - U_{xx} = -U_{xx} - 3U_{xy} + 4U_{yy}$

- At this point we leverage the form of the PDE.

$$-U_{vw} = u_{xx} - 3u_{xy} + 2u_{yy} = 0 = U_{vw}$$

- Integrate to get back to U

$$U_v = \phi(v)$$

$$U(v,w) = v\phi(v) + \psi(w)$$

- Substitue $v = y + 2x$; $w = y + x$ into U to obtain u.

$$U(y+2x, y+x) = (y+2x)\phi(y+2x) + \psi(y+x)$$

$$u(x,y) = y\phi(y+2x) + 2x\phi(y+2x) + \psi(y+x)$$

Find a non-trivial example of u.

- $u(x,y) = y\sin(y+2x) + 2x\sin(y+2x) - (y+x)^2$.