Cason Konzer **202210-MTH357-01**
Assignment hw4 due 09/28/2021 at 11:59pm EDT

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1. (1 point)

Evaluate $\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (x^2 + y, z^2, e^y - z)$ and W is the solid rectangular box whose sides are bounded by the coordinate planes, and the planes $x = 7,\ y = 1,\ z = 8$.

____

Answer(s) submitted:

- `336`

(correct)

Correct Answers:

- `336`

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2. (1 point)

Use the divergence theorem to calculate the flux of the vector field $\vec{F}(x,y,z) = -4xy\vec{i} + 4yz\vec{j} + 3xz\vec{k}$ through the sphere S of radius 4 centered at the origin and oriented outward.

$\iint_S \vec{F} \cdot d\vec{A} =$ ____________

Answer(s) submitted:

- `0`

(correct)

Correct Answers:

- `0`

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3. (1 point)

Use the divergence theorem to calculate the flux of the vector field $\vec{F}(x,y,z) = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ out of the closed, outward-oriented surface S bounding the solid $x^2 + y^2 \leq 9,\ 0 \leq z \leq 5$.

$\iint_S \vec{F} \cdot d\vec{A} =$ ____________

Answer(s) submitted:

- `1732.5pi`

(correct)

Correct Answers:

- `1732.5*pi`

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4. (1 point) Use Stokes' Theorem to find the circulation of $\vec{F} = \langle xy, yz, xz \rangle$ around the boundary of the surface S given by $z = 4 - x^2$ for $0 \leq x \leq 2$ and $-5 \leq y \leq 5$, oriented upward. Sketch both S and its boundary C.

Circulation = $\int_C \vec{F} \cdot d\vec{r} =$ ____________

Answer(s) submitted:

- `-20`

(correct)

Correct Answers:

- `-2^2*5`

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5. (1 point) Use Stokes' Theorem to find the circulation of $\vec{F} = 5y\vec{i} + 7z\vec{j} + 7x\vec{k}$ around the triangle obtained by tracing out the path $(6,0,0)$ to $(6,0,3)$, to $(6,3,3)$ back to $(6,0,0)$.

Circulation = $\int_C \vec{F} \cdot d\vec{r} =$ ____________

Answer(s) submitted:

- `63/2`

(correct)

Correct Answers:

- `7*3*3/2`

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6. (1 point)

Verify Stokes' Theorem for the given vector field and surface, oriented with an upward-pointing normal:

$\mathbf{F} = \langle e^{y-z}, 0, 0 \rangle$, the square with vertices $(6,0,7)$, $(6,6,7)$, $(0,6,7)$, and $(0,0,7)$.

$\int_C \mathbf{F} \cdot d\mathbf{s} =$ ________

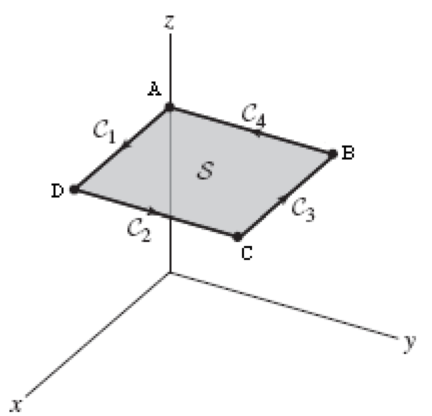
$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} =$ ________

Solution:

Solution:

Step 1. Compute the integral around the boundary curve.

The boundary consists of four segments $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ shown in the figure:



$A = (0,0,7)$, $B = (0,6,7)$, $C = (6,6,7)$, $D = (6,0,7)$

We parametrize the segments by

$$\mathcal{C}_1 : \gamma_1(t) = (t, 0, 7), \quad 0 \leq t \leq 6$$

$$\mathcal{C}_2 : \gamma_2(t) = (6, t, 7), \quad 0 \leq t \leq 6$$

$$\mathcal{C}_3 : \gamma_3(t) = (6 - t, 6, 7), \quad 0 \leq t \leq 6$$

$$\mathcal{C}_4 : \gamma_4(t) = (0, 6 - t, 7), \quad 0 \leq t \leq 6$$

We compute the following values:

$$\mathbf{F}(\gamma_1(t)) = \left\langle e^{y-z}, 0, 0 \right\rangle = \left\langle e^{-7}, 0, 0 \right\rangle$$

$$\mathbf{F}(\gamma_2(t)) = \langle e^{y-z}, 0, 0\rangle = \langle e^{t-7}, 0, 0\rangle$$
$$\mathbf{F}(\gamma_3(t)) = \langle e^{y-z}, 0, 0\rangle = \langle e^{-1}, 0, 0\rangle$$
$$\mathbf{F}(\gamma_4(t)) = \langle e^{y-z}, 0, 0\rangle = \langle e^{-t-1}, 0, 0\rangle$$

Hence,

$$\mathbf{F}(\gamma_1(t)) \cdot \gamma_1'(t) = \langle e^{-7}, 0, 0\rangle \cdot \langle 1, 0, 0\rangle = e^{-7}$$
$$\mathbf{F}(\gamma_2(t)) \cdot \gamma_2'(t) = \langle e^{t-7}, 0, 0\rangle \cdot \langle 0, 1, 0\rangle = 0$$
$$\mathbf{F}(\gamma_3(t)) \cdot \gamma_3'(t) = \langle e^{-1}, 0, 0\rangle \cdot \langle -1, 0, 0\rangle = -e^{-1}$$
$$\mathbf{F}(\gamma_4(t)) \cdot \gamma_4'(t) = \langle e^{-t-1}, 0, 0\rangle \cdot \langle 0, -1, 0\rangle = 0$$

We obtain the following integral:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{4} \int_{C_i} \mathbf{F} \cdot d\mathbf{s} =$$

$$\int_0^6 e^{-7}\, dt + 0 + \int_0^6 -e^{-1}\, dt + 0 = 6\left(e^{-7} - e^{-1}\right) \quad (\mathbf{1})$$

Step 2. Compute the curl.

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{y-z} & 0 & 0 \end{vmatrix} =$$

$$-e^{y-z}\mathbf{j} - e^{y-z}\mathbf{k} = \langle 0, -e^{y-z}, -e^{y-z}\rangle$$

Step 3. Compute the flux of the curl through the surface.

We parametrize the surface by

$$\Phi(x,y) = (x, y, 7), \quad 0 \le x, y \le 6$$

The upward pointing normal is $\mathbf{n} = \langle 0, 0, 1\rangle$. We express curl($\mathbf{F}$) in terms of the parameters x and y:

$$\text{curl}(\mathbf{F})(\Phi(x,y)) = \langle 0, -e^{y-7}, -e^{y-7}\rangle$$

Hence,

$$\text{curl}(\mathbf{F}) \cdot \mathbf{n} = \langle 0, -e^{y-7}, -e^{y-7}\rangle \cdot \langle 0, 0, 1\rangle = -e^{y-7}$$

The surface integral is thus

$$\iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{D}} -e^{y-7}\, dA =$$

$$\int_0^6 \int_0^6 -e^{y-7}\, dy\, dx = 6 \int_0^6 -e^{y-7}\, dy =$$

$$6\left(-e^{y-7}\Big|_0^6\right) = 6\left(-e^{6-7} + e^{-7}\right) =$$

$$6\left(e^{-7} - e^{-1}\right) \quad (\mathbf{2})$$

We see that the integrals in ($\mathbf{1}$) and ($\mathbf{2}$) are equal.

Answer(s) submitted:

- `-2.202`
- `-2.202`

(correct)

Correct Answers:

- `-2.20181`
- `-2.20181`

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7. (1 point)

Use the Divergence Theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$\mathbf{F} = \langle 2x + y, z, 10z - x\rangle$, $\mathcal{S}$ is the boundary of the region between the paraboloid $z = 49 - x^2 - y^2$ and the xy-plane.

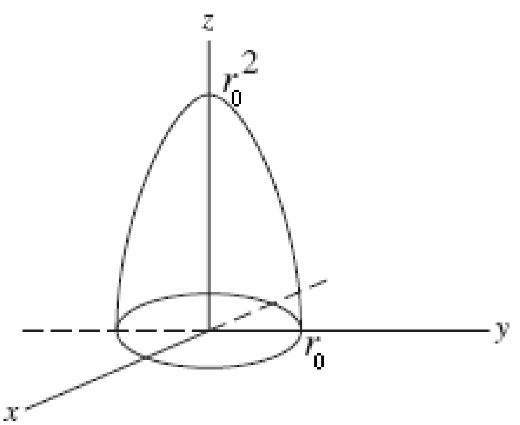
$\iint_S \mathbf{F} \cdot d\mathbf{S} =$ __________

Solution:

Solution: We compute the divergence of $\mathbf{F} = \langle 2x + y, z, 10z - x\rangle$,

$$\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}(2x + y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(10z - x) =$$

$$2 + 0 + 10 = 12.$$



With $r_0 = 7$

Using the Divergence Theorem we have

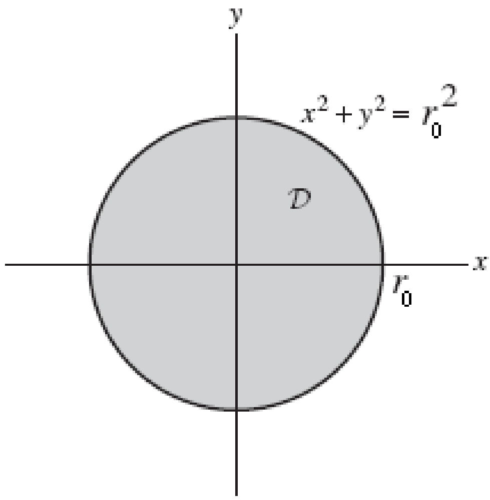
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \text{div}(\mathbf{F})\, dV = \iiint_W 12\, dV$$

We compute the triple integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W 12\, dV =$$

$$\iint_{\mathcal{D}} \int_0^{49-x^2-y^2} 12\, dz\, dx\, dy = \iint_{\mathcal{D}} 12z\Big|_0^{49-x^2-y^2} dx\, dy =$$

$$\iint_{\mathcal{D}} 12(49 - x^2 - y^2)\, dx\, dy$$



With $r_0 = 7$

We convert the integral to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \le r \le 7, \quad 0 \le \theta \le 2\pi$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^7 12\left(49 - r^2\right) r\,dr\,d\theta =$$

$$24\pi\int_0^7 (49r - r^3)\,dr = 24\pi\left(\frac{49r^2}{2} - \frac{r^4}{4}\bigg|_0^7\right) = 14406\pi$$

Answer(s) submitted:

- 14406pi

(correct)

Correct Answers:

- 45257.8

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8. (1 point)

A smooth vector field $\vec{F}$ has div $\vec{F}(3,3,2) = 2$. Estimate the flux of $\vec{F}$ out of a small sphere of radius 0.05 centered at the point $(3,3,2)$.

flux $\approx$ ________________

Solution:

SOLUTION

Since div $F(3,3,2)$ is the flux density out of a small region surrounding the point $(3,3,2)$, we have

$$\text{div}\,\vec{F}(1,2,3) \approx \frac{\text{Flux out of small region around }(3,3,2)}{\text{Volume of region.}}$$

So

$$\text{Flux out of region } \approx (\text{div}\,\vec{F}(3,3,2))\cdot\text{Volume of region}$$

$$= 2\cdot\frac{4}{3}\pi(0.05)^3 = 0.0010472.$$

Answer(s) submitted:

- 2(4/3)pi(.05^3)

(correct)

Correct Answers:

- 2*4*pi*0.05^3/3

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9. (1 point)

Let $\vec{F} = (8z+9)\vec{i} + 4z\vec{j} + (6z+2)\vec{k}$, and let the point $P = (a,b,c)$, where a, b and c are constants. In this problem we will calculate div $\vec{F}$ in two different ways, first by using the geometric definition and second by using partial derivatives.

(a) Consider a (three-dimensional) box with four of its corners at (a,b,c), $(a+w,b,c)$, $(a,b+w,c)$ and $(a,b,c+w)$, where w is a constant edge length. Find the flux through the box.

flux = ________________

Thus, we have

div $\vec{F}(x,y,z) = \lim_{w\to 0}$ (_____ / _____) = _____

(b) Next, find the divergence using partial derivatives:

div $\vec{F}(x,y,z) =$ ________________

Solution:

SOLUTION

(a) To calculate the flux, we find the value of the vector field on each face of the box and take the dot product with $d\vec{A}$ on the face. For example, on the face $z = c$, the outward normal is in the negative z-direction, so that $d\vec{A} = -\vec{k}\,dx\,dy$. Thus

$$\vec{F}\cdot d\vec{A} = ((8c+9)\vec{i} + 4c\vec{j} + (6c+2)\vec{k})\cdot(-\vec{k}\,dx\,dy) = -(6c+2)\,dx\,dy.$$

Thus, integrating over this face S of the box to find the flux, we have

$$\int_S \vec{F}\cdot d\vec{A} = \int_S -(6c+2)\,dx\,dy = -(6c+2)(\text{Area of } S) = -(6c+2)\,w^2.$$

Similarly, on the face $z = c+w$, the outward normal is in the positive z-direction so that $d\vec{A} = \vec{k}\,dx\,dy$, and we get

$$\vec{F}\cdot d\vec{A} = (6(c+w)+2)\,dx\,dy.$$

Thus, integrating to find the flux, we get

$$\text{flux} = (6(c+w)+2)w^2.$$

Next, on the two pairs of faces $x = a$ and $x = a+w$, and $y = b$ and $y = b+w$, note that the flux through the front and back face in the pairs are equal and opposite. Thus they will exactly cancel, and we get a total flux of

$$\text{Total flux} = -(6c+2)w^2 + (6(c+w)+2)w^2 = 6w^3.$$

Using the geometric definition of the divergence, we therefore have

$$\text{div}\,\vec{F}(x,y,z) = \lim_{w\to 0}\left(\frac{6w^3}{w^3}\right) = 6.$$

(b) Using partial derivatives, we have

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(8z+9) + \frac{\partial}{\partial y}(4z) + \frac{\partial}{\partial z}(6z+2) = 6.$$

Answer(s) submitted:

- 6w^3
- 6w^3
- w^3
- 6
- 6

(correct)

Correct Answers:

- 6*w^3
- 6*w^3
- w^3
- 6
- 6

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10. (2 points)

As a result of radioactive decay, heat is generated uniformly throughout the interior of the earth at a rate of around 30 watts per cubic kilometer. (A watt is a rate of heat production.) The heat then flows to the earth's surface where it is lost to space. Let $\vec{F}(x,y,z)$ denote the rate of flow of heat measured in watts per square kilometer. By definition, the flux of $\vec{F}$ across a surface is the quantity of heat flowing through the surface per unit of time.

(a) Suppose that the actual heat generation is $30\mathrm{W}/\mathrm{km}^3$ What is the value of div $\vec{F}$?

div $\vec{F} =$ ____________

(Include **units**.).

(b) Assume the heat flows outward symmetrically. Verify that $\vec{F} = \alpha\vec{r}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and α is a suitable constant, satisfies the given conditions. Find α.

$\alpha =$ ____________

(Include **units**.).

(c) Let $T(x,y,z)$ denote the temperature inside the earth. Heat flows according to the equation $\vec{F} = -k \operatorname{grad} T$, where k is a constant. If T is in °C, then $k = 30{,}000°\mathrm{C}/\mathrm{km}$. Assuming the earth is a sphere with radius 6400 km and surface temperature 20°C, what is the temperature at the center?

$T =$ ____________ (degrees C)

Solution:

SOLUTION

(a) The rate at which heat is generated at any point in the earth is div $\vec{F}$ at that point. So div $\vec{F} = 30\ \mathrm{W}/\mathrm{km}^3$.

(b) Differentiating gives div $(\alpha(x\vec{i} + y\vec{j} + z\vec{k})) = \alpha(1+1+1) = 3\alpha$, so $\alpha = 10\mathrm{W}/\mathrm{km}^3$. Thus, $\vec{F} = \alpha\vec{r}$ has constant divergence. Note that $\vec{F} = \alpha\vec{r}$ has flow lines going radially outward, and symmetric about the origin.

(c) The vector grad T gives the direction of greatest increase in temperature. Thus, $-$grad T gives the direction of greatest decrease in temperature. The equation $\vec{F} = -k \operatorname{grad} T$ says that heat will flow in the direction of greatest decrease in temperature (i.e. from hot regions to cold), and at a rate proportional to the temperature gradient.

We assume that $\vec{F}$ is given by the answer to part (b). Then, using this idea,

$$\vec{F} = 10(x\vec{i} + y\vec{j} + z\vec{k}) = -30{,}000 \operatorname{grad} T,$$

so

$$\operatorname{grad} T = -\frac{10}{30{,}000}(x\vec{i} + y\vec{j} + z\vec{k}).$$

Integrating we get

$$T = \frac{-10}{2(30{,}000)}(x^2 + y^2 + z^2) + C.$$

At the surface of the earth, $x^2 + y^2 + z^2 = 6400^2$, and $T = 20°\mathrm{C}$, so

$$T = \frac{-1}{6000}(6400^2) + C = 20.$$

Thus,

$$C = 20 + \frac{6400^2}{6000} \approx 6847.$$

At the center of the earth, $x^2 + y^2 + z^2 = 0$, so

$$T \approx 6847°\ \mathrm{C}.$$

Answer(s) submitted:

- 30W/km^3
- 10W/km^3
- 6846.666666

(correct)

Correct Answers:

- 30 W/km^3
- 10 W/km^3
- 20+6400^2/6000

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11. (1 point)

Use Stokes' Theorem to find the circulation of the vector field $\vec{F} = 6xz\vec{i} + (6x + 6yz)\vec{j} + 6x^2\vec{k}$ around the paths. C, is the circle $x^2 + y^2 = 9$, $z = 3$, oriented counterclockwise when viewed from above.

circulation = ________________

Solution:

SOLUTION

The circulation is the line integral $\int_C \vec{F} \cdot d\vec{r}$ which can be evaluated directly by parameterizing the circle, C. Or, since C is the boundary of a flat disk S, we can use Stokes' Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \operatorname{curl} \vec{F} \cdot d\vec{A}$$

where S is the disk $x^2 + y^2 \le 9$, $z = 3$ and is oriented upward (using the right hand rule). Then curl $\vec{F} = -6y\vec{i} + 6x\vec{j} + 6\vec{k}$ and the unit normal to S is $\vec{k}$. So

$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_S (-6y\vec{i} + 6x\vec{j} + 6\vec{k}) \cdot \vec{k}\, dxdy = \int_S 6\, dxdy = 54\pi.$$

Answer(s) submitted:

- 54pi

(correct)

Correct Answers:

- 6*pi*9