Advanced ℂalculus

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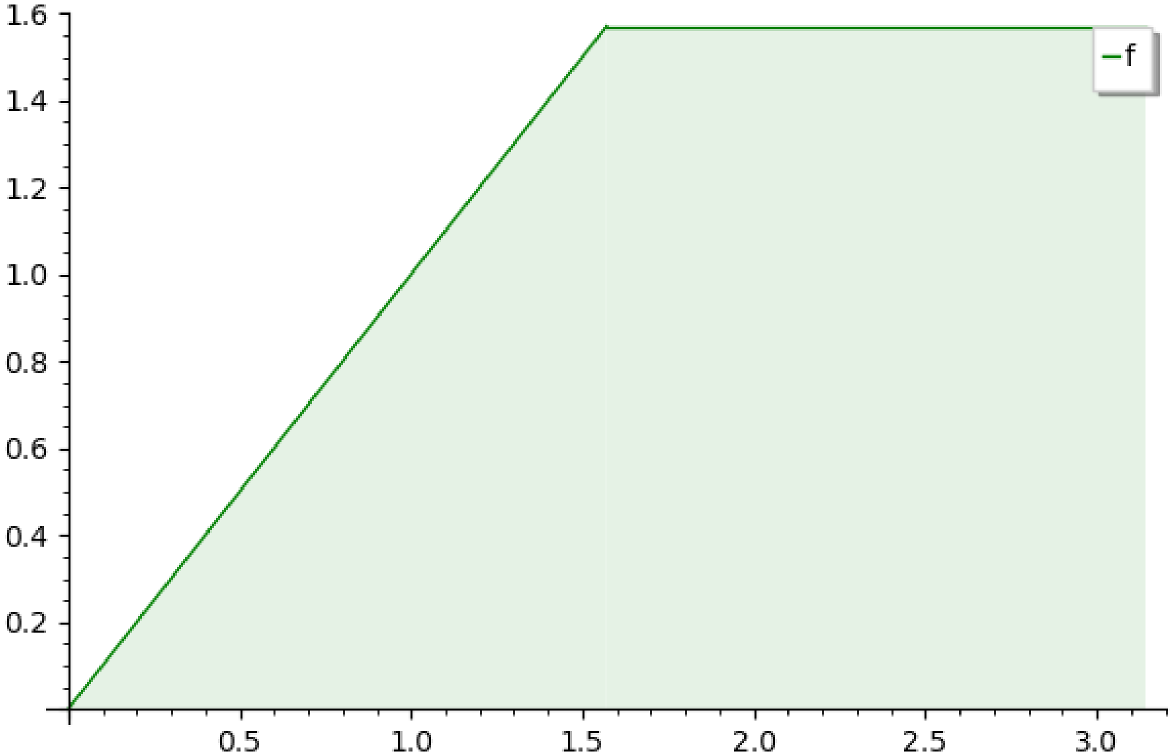
Advanced $\mathbb{C}$alculus

Problem 1

Find the Fourier cosine and sine seriers of the even and odd half range expansions of;



$$f = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} < x < \pi \end{cases}$$

Euler's Formulas

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos(\tfrac{n\pi x}{L})\, dx$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin(\tfrac{n\pi x}{L})\, dx$$

Period

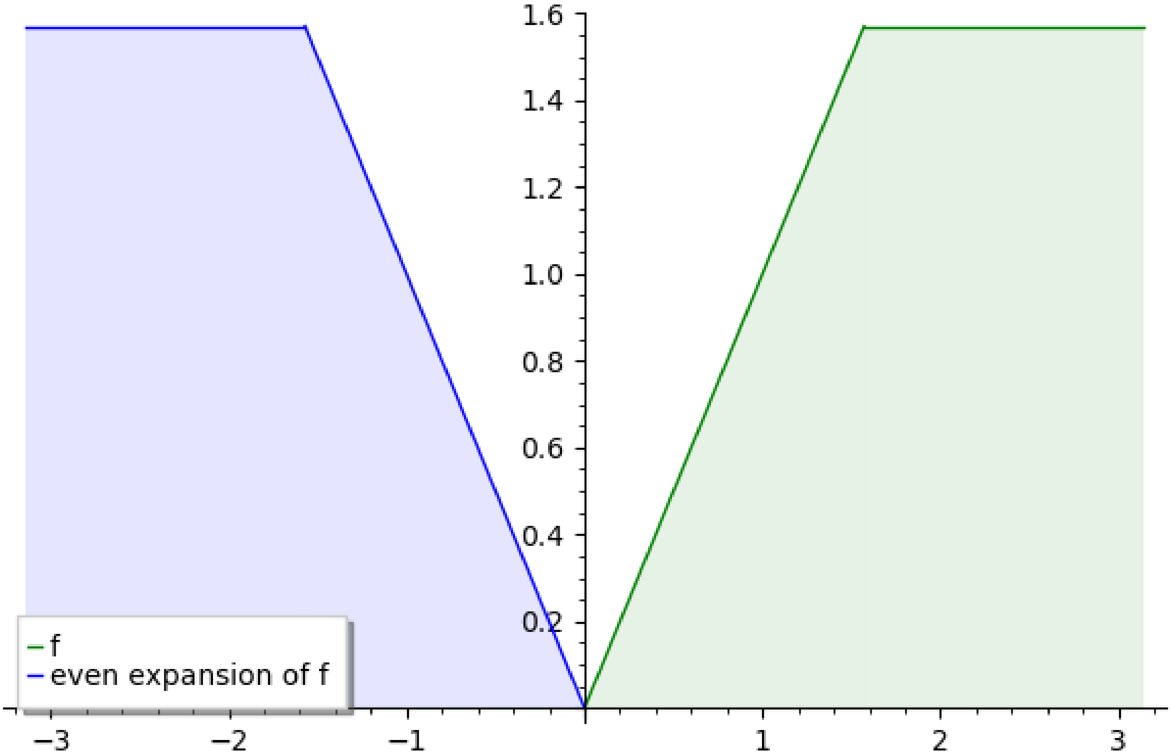
$$p = 2\pi \ ; \ L = p/2 \ ; \ L = 2\tfrac{\pi}{2} \ ; \ L = \pi$$

Fourier Series

$$f(x) = a_0 + \textstyle\sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)$$

$$f(x) = a_0 + \textstyle\sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Even Expansion



$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}[\int_0^{\pi} x\,dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\,dx]$$

$$a_0 = \frac{1}{\pi}[[\frac{x^2}{2}\Big|_0^{\frac{\pi}{2}}] + [\frac{\pi x}{2}\Big|_{\frac{\pi}{2}}^{\pi}]] = \frac{1}{\pi}[[\frac{\pi^2}{8}] + [\frac{\pi^2}{2} - \frac{\pi^2}{4}]] = \frac{1}{\pi}[\frac{\pi^2}{8} + \frac{4\pi^2}{8} - \frac{2\pi^2}{8}]$$

$$a_0 = [\frac{\pi}{8} + \frac{4\pi}{8} - \frac{2\pi}{8}] = \frac{3\pi}{8}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(\frac{n\pi x}{\pi})\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\cos(nx)\,dx$$

$$a_n = \frac{2}{\pi}[\frac{x\sin(nx)}{n}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin(nx)}{n}\,dx] + \int_{\frac{\pi}{2}}^{\pi} \cos(nx)\,dx$$

$$a_n = \frac{2}{\pi}[\frac{\pi\sin(\frac{n\pi}{2})}{2n} - [\frac{-\cos(nx)}{n^2}\Big|_0^{\frac{\pi}{2}}]] + [\frac{sin(nx)}{n}\Big|_{\frac{\pi}{2}}^{\pi}]$$

$$a_n = \frac{2}{\pi}[\frac{\pi\sin(\frac{n\pi}{2})}{2n} - [\frac{-\cos(\frac{n\pi}{2})}{n^2} + \frac{1}{n^2}]] + [\frac{sin(n\pi)}{n} - \frac{sin(\frac{n\pi}{2})}{n}]$$

$$note : \sin(n\pi) = 0 \;;\; \forall n \in \mathbb{N}$$

$$a_n = \frac{2}{\pi}[\frac{\pi\sin(\frac{n\pi}{2})}{2n} + \frac{\cos(\frac{n\pi}{2})}{n^2} - \frac{1}{n^2}] - [\frac{sin(\frac{n\pi}{2})}{n}] = [\frac{2n\pi\sin(\frac{n\pi}{2})}{2n^2\pi} + \frac{2\cos(\frac{n\pi}{2})}{n^2\pi} - \frac{2}{n^2\pi}] - [\frac{sin(\frac{n\pi}{2})}{n}]$$

$$a_n = [\frac{n\pi\sin(\frac{n\pi}{2})+2\cos(\frac{n\pi}{2})-2}{n^2\pi}] - [\frac{n\pi\sin(\frac{n\pi}{2})}{n^2\pi}] = \frac{2\cos(\frac{n\pi}{2})-2}{n^2\pi}$$

$$note : \cos(\frac{n\pi}{2}) = 0 \;;\; \forall n \in \{1,3,5\dots\},$$

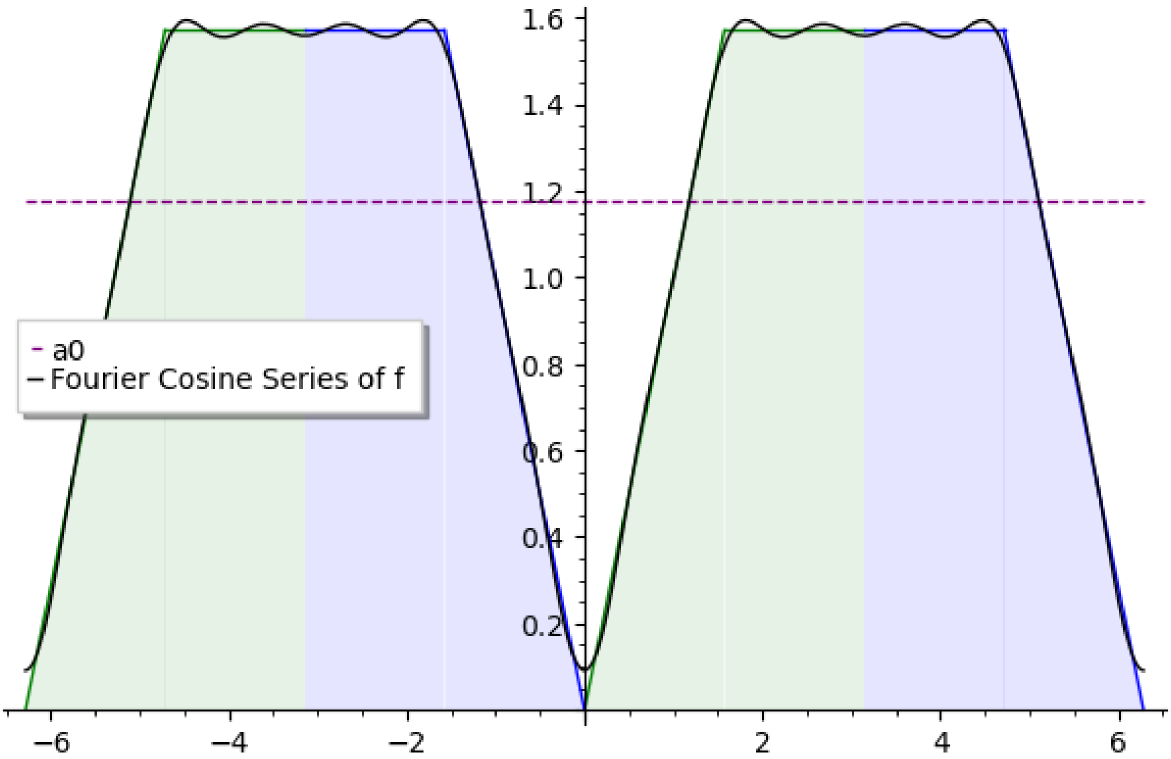
$$\cos(\frac{n\pi}{2}) = -1 \;;\; \forall n \in \{2,6,10\dots\},\ \cos(\frac{n\pi}{2}) = 1 \;;\; \forall n \in \{4,8,12\dots\}$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi} & n = 1, 3, 5 \dots \\ -\frac{4}{n^2\pi} & n = 2, 6, 10 \dots \\ 0 & n = 4, 8, 12 \dots \end{cases}$$

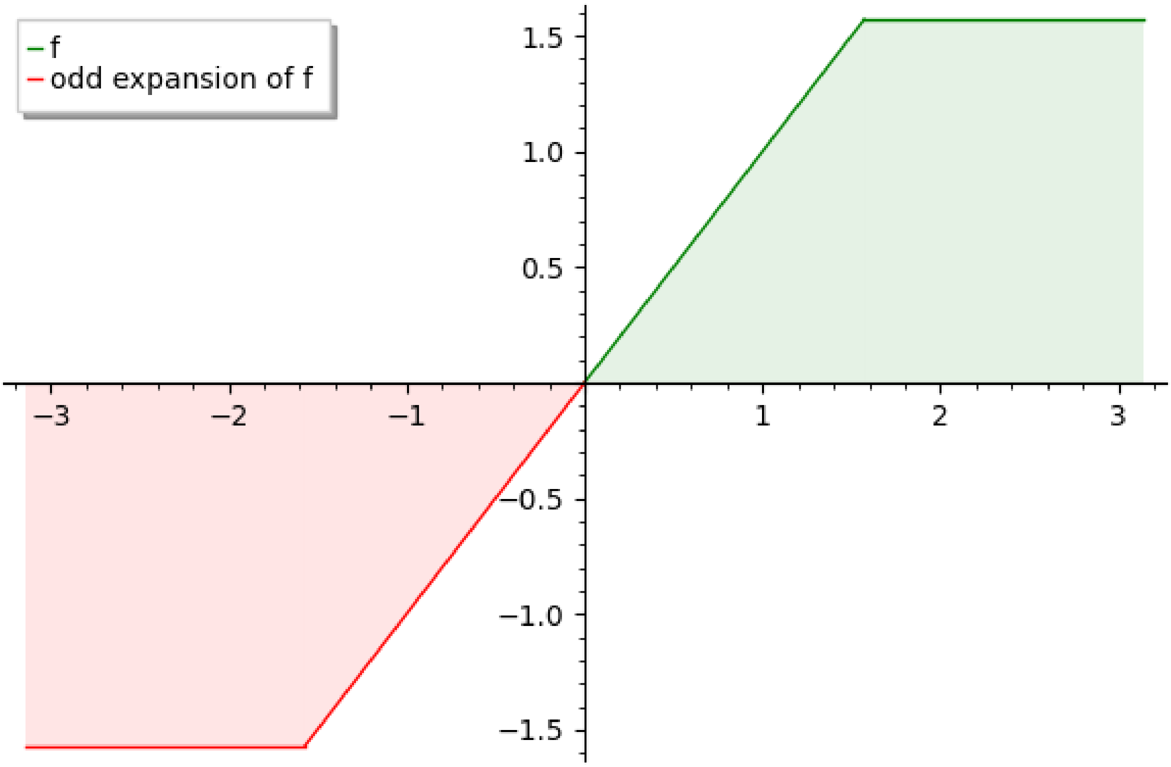
Cosine Fourier Series

$$f(x) = \frac{3\pi}{8} - \sum_{n=1,3,5\dots}^{\infty} \frac{2}{n^2\pi} \cos(nx) - \sum_{n=2,6,10\dots}^{\infty} \frac{4}{n^2\pi} \cos(nx)$$

$$f(x) = \frac{3\pi}{8} - [\frac{2\cos(x)}{\pi} + \frac{4\cos(2x)}{4\pi} + \frac{2\cos(3x)}{9\pi} + \frac{2\cos(5x)}{25\pi} + \frac{4\cos(6x)}{36\pi} \dots]$$



Odd Expansion



$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(\tfrac{n\pi x}{\pi})\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\Big[\frac{-x\cos(nx)}{n}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-\cos(nx)}{n}\Big] + \int_{\frac{\pi}{2}}^{\pi}\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\Big[\Big[\frac{-\pi\cos(\frac{n\pi}{2})}{2n}\Big] - \Big[\frac{-\sin(nx)}{n^2}\Big|_0^{\frac{\pi}{2}}\Big]\Big] + \Big[\frac{-\cos(nx)}{n}\Big|_{\frac{\pi}{2}}^{\pi}\Big]$$

$$b_n = \frac{2}{\pi}\Big[\Big[\frac{-\pi\cos(\frac{n\pi}{2})}{2n}\Big] - \Big[\frac{-\sin(\frac{n\pi}{2})}{n^2}\Big]\Big] + \Big[\frac{-\cos(n\pi)}{n} - \frac{-\cos(\frac{n\pi}{2})}{n}\Big]$$

$$b_n = \Big[\frac{-2\pi\cos(\frac{n\pi}{2})}{2n\pi} + \frac{2\sin(\frac{n\pi}{2})}{n^2\pi}\Big] + \Big[\frac{-\cos(n\pi)}{n} + \frac{\cos(\frac{n\pi}{2})}{n}\Big]$$

$$b_n = \Big[\frac{-n\pi\cos(\frac{n\pi}{2})}{n^2\pi} + \frac{2\sin(\frac{n\pi}{2})}{n^2\pi} + \frac{-n\pi\cos(n\pi)}{n^2\pi} + \frac{n\pi\cos(\frac{n\pi}{2})}{n^2\pi}\Big]$$

$$b_n = \Big[\frac{2\sin(\frac{n\pi}{2}) + n\pi\cos(\frac{n\pi}{2}) - n\pi\cos(\frac{n\pi}{2}) - n\pi\cos(n\pi)}{n^2\pi}\Big] = \frac{2\sin(\frac{n\pi}{2}) - n\pi\cos(n\pi)}{n^2\pi}$$

$note : \sin(\frac{n\pi}{2}) = 1 \ ; \forall n \in \{1, 5, 9 \dots\}$, $\sin(\frac{n\pi}{2}) = -1 \ ; \forall n \in \{3, 7, 11 \dots\}$,

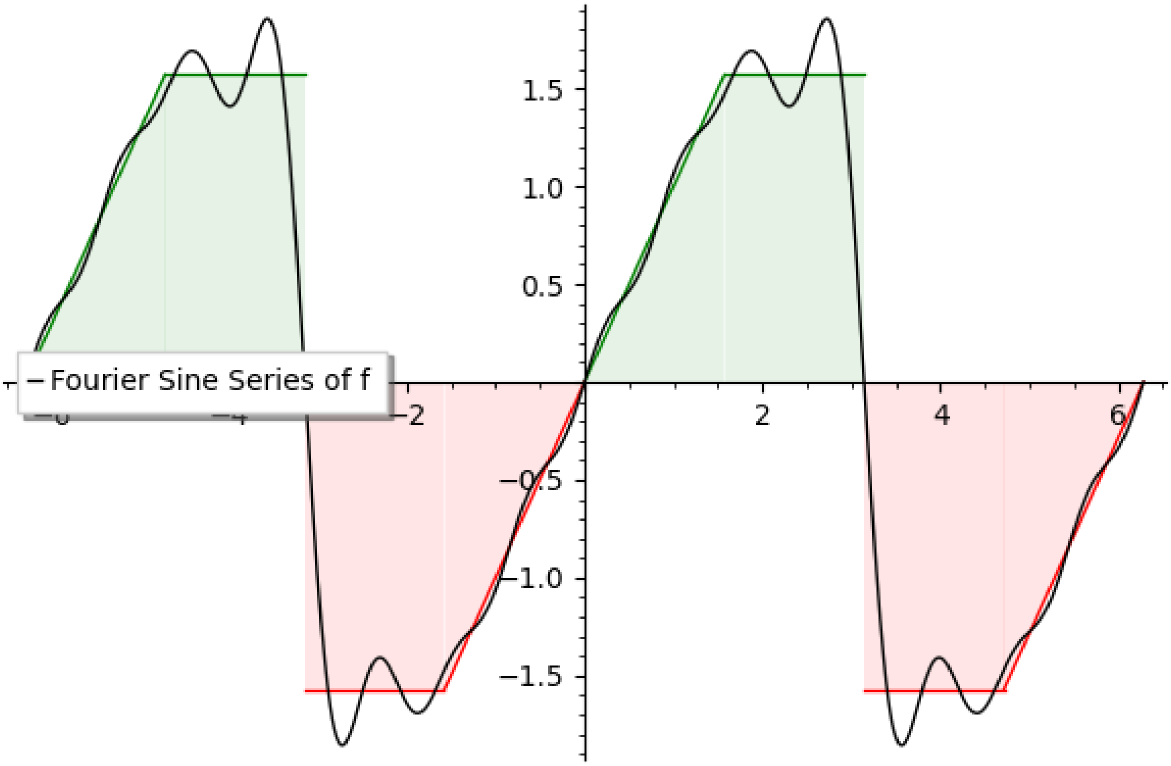
$\cos(n\pi) = -1 \ ; \forall n \in \{1, 3, 5 \dots\}$, $\cos(n\pi) = 1$ & $\sin(\frac{n\pi}{2}) = 0; \forall n \in \{2, 4, 6 \dots\}$

$$b_n = \begin{cases} \frac{n\pi+2}{n^2\pi} & n = 1, 5, 9 \dots \\ -\frac{1}{n} & n = 2, 4, 6 \dots \\ \frac{n\pi-2}{n^2\pi} & n = 3, 7, 11 \dots \end{cases}$$

Sine Fourier Series

$$f(x) = \sum_{n=1,5,9\ldots}^{\infty} \frac{n\pi+2}{n^2\pi}\sin(nx) - \sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n}\sin(nx) + \sum_{n=3,7,11\ldots}^{\infty} \frac{n\pi-2}{n^2\pi}\sin(nx)$$

$$f(x) = \frac{(\pi+2)\sin(x)}{\pi} - \frac{\sin(2x)}{2} + \frac{(3\pi-2)\sin(3x)}{9\pi} - \frac{\sin(4x)}{4} + \frac{(5\pi+2)\sin(5x)}{25\pi} - \frac{sin(6x)}{6} + \frac{(7\pi-2)\sin(7x)}{49\pi}\ldots$$



Average of Fourier Cosine and Sine Series

$$a(x) = \Big[\ \frac{3\pi}{8} - \sum_{n=1,3,5\ldots}^{\infty} \frac{2}{n^2\pi}\cos(nx)\ - \sum_{n=2,6,10\ldots}^{\infty} \frac{4}{n^2\pi}\cos(nx)$$

$$+ \sum_{n=1,5,9\ldots}^{\infty} \frac{n\pi+2}{n^2\pi}\sin(nx) - \sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n}\sin(nx) + \sum_{n=3,7,11\ldots}^{\infty} \frac{n\pi-2}{n^2\pi}\sin(nx)\ \Big]/2$$