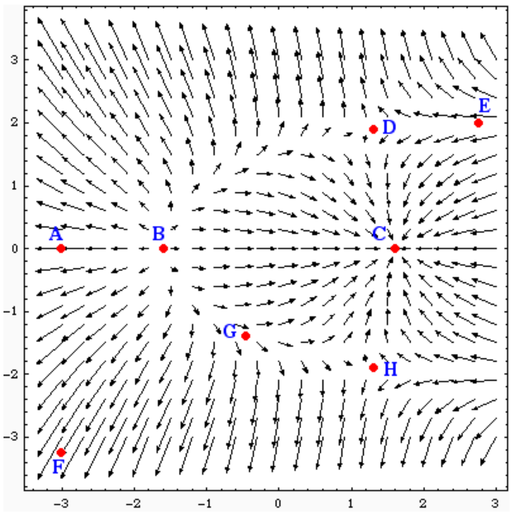
1. (1 point)



The gradient vector field for a function $f: \mathbf{R}^2 \to \mathbf{R}$ is given at the left.

f has a saddle at [?].

f has a relative maximum at [?].

f is steepest at [?].

f has a relative minimum at [?].

Answer(s) submitted:

- `D and H`
- `C`
- `F`
- `B`

(correct)

Correct Answers:

- `D and H`
- `C`
- `F`
- `B`

2. (1 point)

Find the directional derivative of $f(x,y,z) = z^3 - x^2 y$ at the point (-5, 3, 2) in the direction of the vector $\mathbf{v} = \langle 3, -3, 5 \rangle$.

______

Answer(s) submitted:

- `34.3122`

(correct)

Correct Answers:

- `34.312178324836`

3. (1 point)

Find the maximum rate of change of $f(x,y) = \ln(x^2 + y^2)$ at the point (5, -3) and the direction in which it occurs.

Maximum rate of change: ______

Direction (unit vector) in which it occurs: $\langle$ ______, ______ $\rangle$

Answer(s) submitted:

- `0.343`
- `0.8575`
- `-0.5145`

(correct)

Correct Answers:

- `0.342997170285018`
- `0.857492925712544`
- `-0.514495755427526`

4. (1 point)

Apply the Laplace operator to the function $h(x,y,z) = e^x \sin(-4y)$.

$\nabla^2 h =$ ______.

Answer(s) submitted:

- `15e^xsin(4y)`

(correct)

Correct Answers:

- `exp(1 * x) * sin(-4 * y) * (1*1 - -4*-4)`

5. (1 point)

I.

Let $\mathbf{F} = 6x\mathbf{i} + 8y\mathbf{j} + 2z\mathbf{k}$. Compute the divergence and the curl.

A. div $\mathbf{F} =$ ______

B. curl $\mathbf{F} =$ ______ $\mathbf{i}+$ ______ $\mathbf{j}+$ ______ $\mathbf{k}$

II.

Let $\mathbf{F} = (3xy, 4y, 5z)$.

The curl of $\mathbf{F} =$ (______, ______, ______).

Is there a function f such that $\mathbf{F} = \nabla f$? ______ (yes/no)

Answer(s) submitted:

- `16`
- `0`
- `0`
- `0`
- `0`
- `0`
- `-3x`
- `no`

(correct)

Correct Answers:

- `16`
- `0`

- `0`
- `0`
- `0`
- `0`
- `-3*x`
- `NO`

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6. (1 point) Let $\mathbf{F} = (7yz)\,\mathbf{i} + (10xz)\,\mathbf{j} + (8xy)\,\mathbf{k}$. Compute the following:

A. div $\mathbf{F} =$ ____________

B. curl $\mathbf{F} =$ ____ $\mathbf{i}+$ ____ $\mathbf{j}+$ ____ $\mathbf{k}$

C. div curl $\mathbf{F} =$ ____________

Note: Your answers should be expressions of x, y and/or z; e.g. "3xy" or "z" or "5"

Answer(s) submitted:

- `0`
- `-2x`
- `-y`
- `3z`
- `0`

(correct)

Correct Answers:

- `0`
- `(8 - 10)*x`
- `(7 - 8)*y`
- `(10 - 7)*z`
- `0`

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7. (1 point) The temperature at any point in the plane is given by $T(x,y) = \dfrac{120}{x^2+y^2+1}$.

(a) What shape are the level curves of T?

- A. lines
- B. hyperbolas
- C. ellipses
- D. parabolas
- E. circles
- F. none of the above

(b) At what point on the plane is it hottest? ____________

What is the maximum temperature? ____________

(c) Find the direction of the greatest increase in temperature at the point $(-2,-3)$. ____________

What is the value of this maximum rate of change, that is, the maximum value of the directional derivative at $(-2,-3)$? ____________

(d) Find the direction of the greatest decrease in temperature at the point $(-2,-3)$. ____________

What is the value of this most negative rate of change, that is, the minimum value of the directional derivative at $(-2,-3)$? ____________

Solution:

SOLUTION

(a) Let $k>0$ be constant. The level curves have equation $k = \frac{120}{x^2+y^2+1}$ or $x^2+y^2 = \frac{120}{k} - 1$. If $\frac{120}{k} - 1 > 0$, these equations represent circles in the xy- plane.

(b) The largest value of the temperature T is attained at $(0,0)$. The value of the temperature at this point is $T(0,0) = \frac{120}{1} = 120$

(c) The greatest increase in temperature at $(-2,-3)$ is in the direction of the gradient at the point.

We have $\nabla f(x,y) = \left\langle -\frac{240x}{\left(x^2+y^2+1\right)^2}, -\frac{240y}{\left(x^2+y^2+1\right)^2} \right\rangle \Rightarrow$ $\nabla f(-2,-3) = \left\langle \frac{120}{49}, \frac{180}{49} \right\rangle \approx \langle 2.44898, 3.67347 \rangle$

The maximum value of the directional derivative at $(-2,-3)$ is given by the magnitude of the gradient at $(-2,-3)$: $|\nabla f(-2,-3)| = \sqrt{19.4919}$

(c) The greatest decrease in temperature at $(-2,-3)$ is in the opposite direction of the gradient at the point: $-\nabla f(-2,-3) = -\langle 2.44898, 3.67347 \rangle$

The minimum value of the directional derivative at $(-2,-3)$ is given by $-|\nabla f(-2,-3)| = -\sqrt{19.4919}$

Answer(s) submitted:

- `E`
- `(0,0)`
- `120`
- `2.449i+3.6735j`
- `4.415`
- `-2.449i-3.6735j`
- `-4.415`

(correct)

Correct Answers:

- `E`
- `(0,0)`
- `120`
- `<2.44898,3.67347>`
- `4.41496`
- `<-2.44898,-3.67347>`
- `-4.41496`

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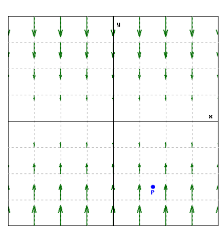
8. (1 point) Determine whether the divergence of each vector field (in green) at the indicated point P (in blue) is positive, negative, or zero.



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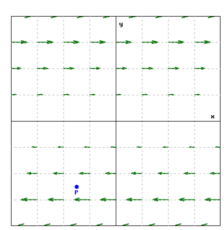

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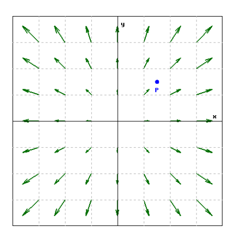

[?]

(Click on a graph to enlarge it)

Solution:

SOLUTION

If we draw a small circle around the point P we can see that the flow rate out is smaller than the flow rate in (the vectors exiting the circle are shorter than the vectors entering the circle). Thus the divergence is negative.

If we draw a small circle around the point P we can see that the flow rate out equals the flow rate in. Thus the divergence is zero. The vector field is incompressible at the point P.

If we draw a small circle around the point P we can see that the flow rate out is larger than the flow rate in (the vectors exiting the circle are longer than the vectors entering the circle). Thus the divergence is positive.

Answer(s) submitted:

- `Negative`
- `Zero`
- `Positive`

(correct)

Correct Answers:

- `NEGATIVE`
- `ZERO`
- `POSITIVE`

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9. (1 point)

Determine whether the vector field is conservative and, if so, find the general potential function.

$$\mathbf{F} = \langle \cos z, 2y^3, -x\sin z \rangle$$

$\varphi =$ __________ $+c$

Note: if the vector field is not conservative, write "DNE".

Solution:

Solution: We examine whether $\mathbf{F}$ satisfies the cross partials condition:

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}(\cos z) = 0 \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}(2y^3) = 0 \end{aligned} \quad \Rightarrow \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\begin{aligned} \frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial z}(2y^3) = 0 \\ \frac{\partial F_3}{\partial y} &= \frac{\partial}{\partial y}(-x\sin z) = 0 \end{aligned} \quad \Rightarrow \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$$\begin{aligned} \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial x}(-x\sin z) = -\sin z \\ \frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z}(\cos z) = -\sin z \end{aligned} \quad \Rightarrow \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

We see that the conditions are satisfied, therefore $\mathbf{F}$ is conservative. We find the general potential function for $\mathbf{F}$.

Step 1. Use the condition $\frac{\partial \varphi}{\partial x} = F_1$.

$\varphi(x,y,z)$ is an antiderivative of $F_1 = \cos z$ when y and z are fixed, therefore:

$$\varphi(x,y,z) = \int \cos z\, dx = x\cos z + g(y,z) \quad (\mathbf{1})$$

Step 2. Use the condition $\frac{\partial \varphi}{\partial y} = F_2$.

Using $(\mathbf{1})$ we get:

$$\frac{\partial}{\partial y}(x\cos z + g(y,z)) = 2y^3$$

$$g_y(y,z) = 2y^3$$

We integrate with respect to y, holding z fixed:

$$g(y,z) = \int 2y^3\, dy = \frac{1}{2} \cdot y^4 + g(z)$$

Substituting in $(\mathbf{1})$ gives

$$\varphi(x,y,z) = x\cos z + \frac{y^4}{2} + g(z) \quad (\mathbf{2})$$

Step 3. Use the condition $\frac{\partial \varphi}{\partial z} = F_3$.

By $(\mathbf{2})$ we have

$$\frac{\partial}{\partial z}\left(x\cos z + \frac{y^4}{2} + g(z)\right) = -x\sin z$$

$$-x\sin z + g'(z) = -x\sin z$$

$$g'(z) = 0 \quad \Rightarrow \quad g(z) = c$$

Substituting in $(\mathbf{2})$ we obtain the general potential function:

$$\varphi(x,y,z) = x\cos z + \frac{y^4}{2} + c$$

Answer(s) submitted:

- `xcos(z) + (y^4/2)`

(correct)

Correct Answers:

- `x*cos(z)+y^4/2`

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10. (1 point) Compute the curl of the vector field $\vec{F} = \langle xy + z^2, x^2, xz - 2 \rangle$.

curl$(\vec{F}(x,y,z))$ = ____________

What is the curl at the point $(0,-1,0)$?

curl$(\vec{F}(0,-1,0))$ = ____________

Is this vector field irrotational or not?

- Choose
- irrotational
- not irrotational
- cannot be determined

Answer(s) submitted:

- `0i + zj + xk`
- `0i + 0j +0k`
- `not irrotational`

(correct)

Correct Answers:

- `<0,2*z-z,2*x-x>`
- `<0,0,0>`
- `not irrotational`