HW 2

372hw2

Sunday, September 5, 2021 2:14 AM

Jason Konzer

MTH 372
Hw 2
Due Thursday, 9/16/2021.

Solutions are required, not just answers. Unsupported answers will recieve little or no credit.

Read Chapters 4,5 of *Huber*.

p.28 #4.2 Suppose $Y \sim \text{Unif}[0, 10]$.
(a) Find $\text{P}(Y \in [3, 7])$.
(b) Find $\text{P}(Y \in [6, 12])$.

#4.4 Suppose that $U = (U_1, U_2)$ is uniformly chosen over the region
$\{(x; y) : x \geq 2, 0 \leq y \leq 1/x^2\}$.
(a) What is $\text{P}(U_1 \leq 5)$?
(b) What is $\text{P}(U_2 \geq .01)$?

#4.5 (revised) Let U_1 and U_2 be independent uniform random variables over $[0, 1]$. What is the chance that $5\,U_2 < U_1$?

#4.8 Suppose that (U_1, U_2) is uniform over the quadrilateral region with vertices (0, 0); (0, 1); (2, 2); (2, 0). Prove that U_1 and U_2 are not independent. (Hint: Start by drawing a picture.

#5.2 Suppose $U \sim \text{Unif}([0, 1])$ and $W = 1/U$.
(a) Find $\text{P}(W \geq 2)$.
(b) Find $\text{P}(W \geq -2)$.

#5.4 Let $U \sim \text{Unif}([-1, 1])$. Find the cdf of U^3.

#5.10 Suppose that (U_1, U_2) is uniform over the quadrilateral region with vertices $(0, 0)$; $(0, 1)$; $(2, 2)$; and $(2, 0)$. Find the cdf of U_1.

#5.12 Suppose $T \sim \text{Exp}(2)$. Find and graph $F_T(t)$.

p.28 #4.2 Suppose $Y \sim \text{Unif}[0, 10]$.

(a) Find $\text{P}(Y \in [3, 7])$.

$P(Y \in [3,7]) = \frac{7-3}{10-0} = \boxed{\frac{4}{10}} = \boxed{\frac{2}{5}}$

(b) Find $\text{P}(Y \in [6, 12])$.

$P(Y \in [6,12]) = P(Y \in [6,10] \cup Y \in (10,12])$

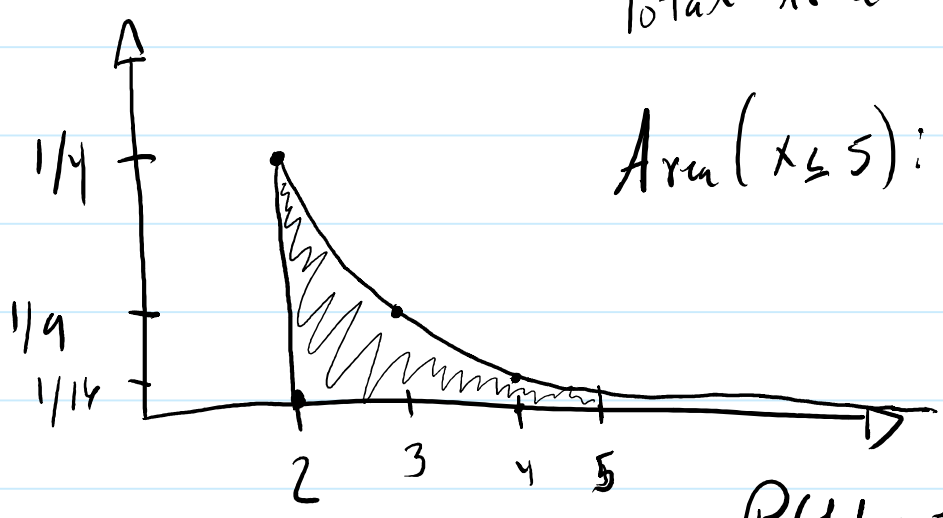
$= P(Y \in [6,10]) + P(Y \in (10,12])$ As $(10,12] \notin \Omega$, $P(Y \in (10,12]) = 0$

$\therefore P(Y \in [6,12]) = P(Y \in [6,10]) = \frac{10-6}{10-0} = \boxed{\frac{4}{10}} = \boxed{\frac{2}{5}}$

∴ $P(Y \in [6,12]) = P(Y \in [6,10]) = \frac{4}{10-0} = \frac{4}{10} = \frac{2}{5}$

#4.4 Suppose that $U = (U_1, U_2)$ is uniformly chosen over the region $\{(x; y) : x \geq 2, 0 \leq y \leq 1/x^2\}$.

(a) What is $P(U_1 \leq 5)$?



Total area: $\int_2^\infty x^{-2}\,dx = -x^{-1}\Big|_2^\infty = \frac{-1}{\infty} + \frac{+1}{2} = \frac{1}{2}$

Area $(x \leq 5)$: $\int_2^5 x^{-2}\,dx = -x^{-1}\Big|_2^5 = \frac{-1}{5} + \frac{1}{2} = \frac{3}{10}$

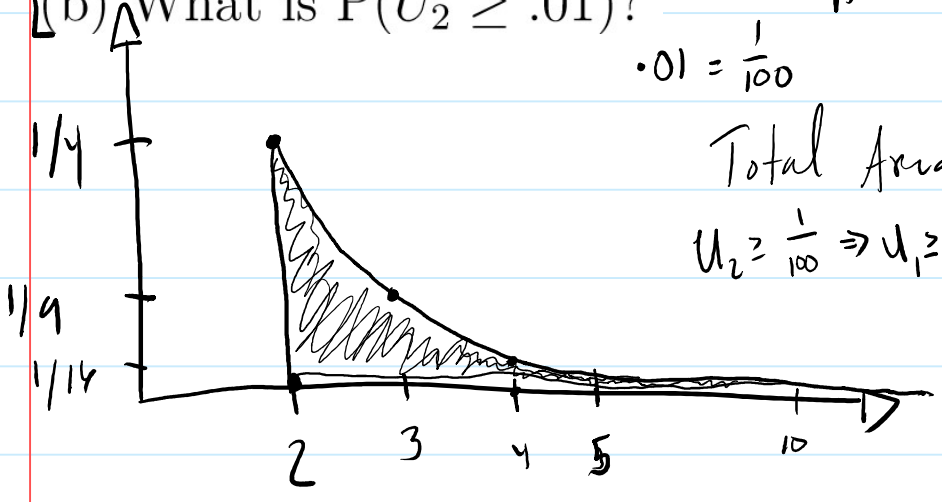
$\frac{5}{10} - \frac{2}{10}$

$\frac{3}{10} * \frac{1}{2}$

$5/10^*$

$P(U_1 \leq 5) = 3/10 \,/\, 5/10 = \boxed{3/5}$

(b) What is $P(U_2 \geq .01)$?



$.01 = \frac{1}{100}$

Total Area $= \int_2^\infty \frac{dx}{x^2} = \frac{-1}{x}\Big|_2^\infty = 0 + \frac{1}{2} = \frac{1}{2} = \frac{5}{10}^*$

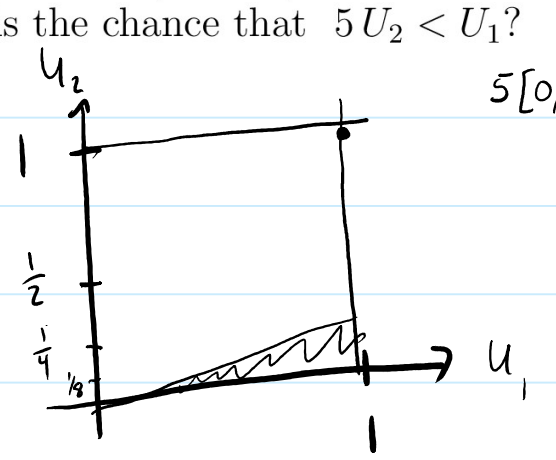
$U_2 \geq \frac{1}{100} \Rightarrow U_1 \geq 10$

Area $(x \geq 10) = \int_{10}^\infty \frac{dx}{x^2} = \frac{-1}{x}\Big|_{10}^\infty = 0 + \frac{1}{10} = \frac{1}{10}^*$

$*\,/\,*$

Thus $P(U_2 \geq \frac{1}{100}) = 1/10 \,/\, 5/10 = \boxed{1/5}$

#4.5 (revised) Let U_1 and U_2 be independent uniform random variables over $[0, 1]$. What is the chance that $5\,U_2 < U_1$?

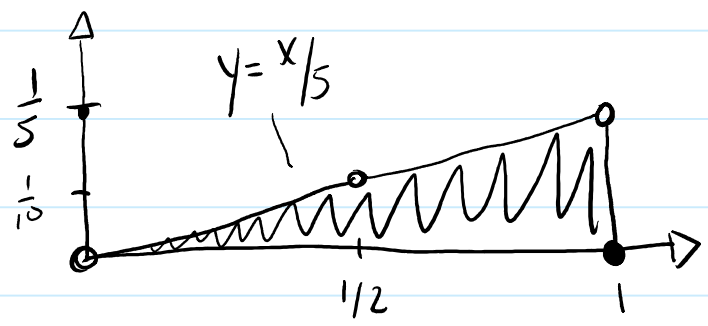


$5[0,1] < [0,1]$

$5 \times 1/5 = 1 \not< 1$ $\quad \frac{1}{10} \times 5 = \frac{1}{2}$

Total Area is $|X| = 1$

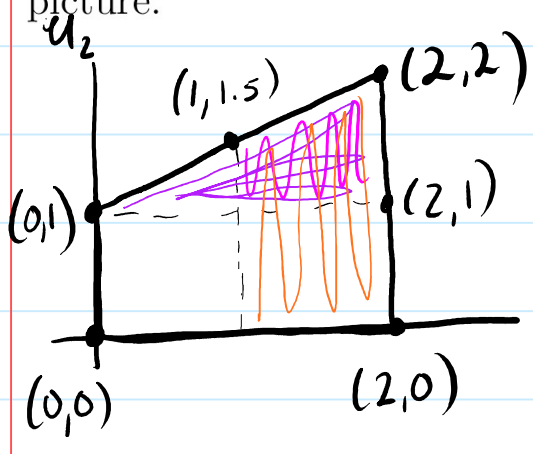
$\Rightarrow$



$\frac{1}{5}\int x = \frac{x^2}{10}$; $\frac{x^2}{10}\Big|_0^1 = 1/10 - 0 = 1/10$

$P(5U_2 < U_1) = 1/10 \,/\, 1 = \boxed{1/10}$

#4.8 Suppose that (U_1, U_2) is uniform over the quadrilateral region with vertices (0, 0); (0, 1); (2, 2); (2, 0). Prove that U_1 and U_2 are not independent. (Hint: Start by drawing a picture.



Total Area $= 2\times1 + \frac{2\times1}{2} = 2 + 1 = 3$

- $P(U_1 > 1) = \frac{1\times1.5 + \frac{1\times.5}{2}}{3} = \frac{1.5 + \frac{.5}{2}}{3} = \frac{1.75}{3} = \frac{1.75}{3}$
- $P(U_2 > 1) = \frac{\frac{2\times1}{2}}{3} = \frac{1}{3} = \frac{1/3}{2}$
- $P(U_1 > 1 \,\&\, U_2 > 1) = \frac{1\times.5 + \frac{1\times.5}{2}}{3} = \frac{.5 + \frac{.5}{2}}{3} = \frac{.75}{3} = \frac{.25}{2}$

$P(U_1 > 1) \cdot P(U_2 > 1) = \frac{1.75}{3} \cdot \frac{1}{3} = \frac{1.75}{9} = .1944 \neq P(U_1 > 1 \,\&\, U_2 > 1) = .25$

#5.2 Suppose $U \sim \text{Unif}([0, 1])$ and $W = 1/U$.

(a) Find $P(W \geq 2)$. $\quad W \geq 2 \Leftrightarrow U \leq .5$

as $\frac{1}{1} \cdot \frac{1}{2} = ?$

(a) Find $P(W \geq 2)$. $W \leq 2 \Leftrightarrow U \leq \cdot$

$\therefore P(W \geq 2) = P(U \leq .5)$ as $\frac{1}{\frac{1}{2}} = \frac{1}{1} \cdot \frac{2}{1} = 2$

$P(U \leq .5) = P(U \in [0, 0.5]) = \frac{0.5-0}{1-0} = \frac{.5}{1} = \underline{0.5}$

Thus $P(W \geq 2) = \boxed{.5}$

(b) Find $P(W \geq -2)$.

w/ $\lim_{U \to 0} \frac{1}{U} = \infty$; $\lim_{U \to 1} \frac{1}{U} = 1$; Thus $W \sim Unif([0, \infty])$

As all outcomes for W are bigger than or equal to 0 ; $\boxed{P(W \geq -2) = 1}$

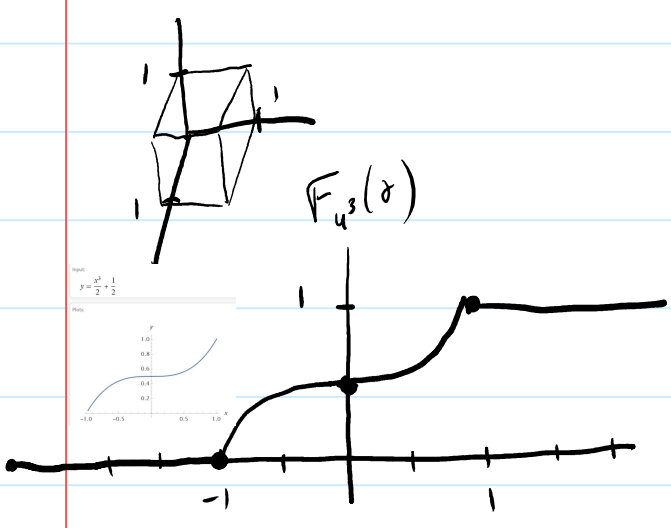
#5.4 Let $U \sim \text{Unif}([-1, 1])$. Find the cdf of U^3.

$-1^3 = -1$; $1^3 = 1$

$P(U^3 \leq 1) = 1$ ← $\frac{1^3 - (-1)^3}{1-(-1)} = \frac{2}{2}$

$P(U^3 \leq 0) = 1/2$ $\frac{0^3 - (-1)^3}{1-(-1)} = \frac{+1}{2}$

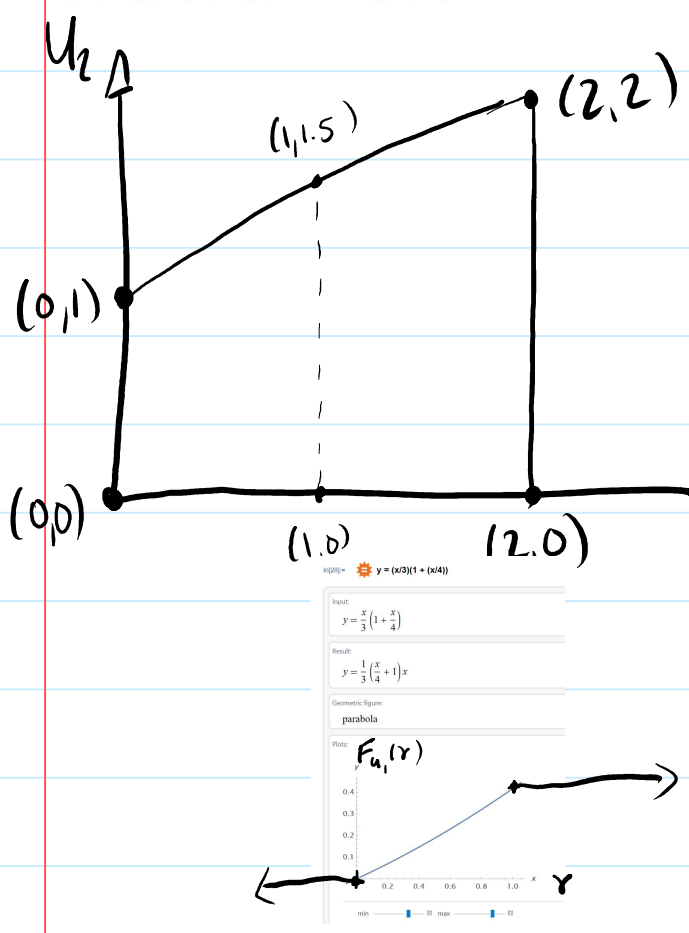
$P(U^3 \leq -1) = 0$ $\frac{(-1)^3 - (-1)^3}{1-(-1)}$



$\frac{y^3+1}{2}$

$$F_{U^3}(y) = \begin{cases} \frac{y^3}{2} + \frac{1}{2} & -1 \leq y \leq 1 \\ 1 & y > 1 \\ 0 & y < -1 \end{cases}$$

#5.10 Suppose that (U_1, U_2) is uniform over the quadrilateral region with vertices $(0, 0)$; $(0, 1)$; $(2, 2)$; and $(2, 0)$. Find the cdf of U_1. Total Area $= 2 \times 1 + \frac{2 \times 1}{2} = 2 + 1 = 3$



$P(U_1 \leq 2) = 3/3 = 1$

$P(U_1 \leq 1) = \frac{1 \cdot 1 + \frac{1 \cdot .5}{2}}{3} = \frac{1 + \frac{.5}{2}}{3} = \frac{1.25}{3}$

$\frac{y + \frac{y \cdot \frac{y}{2}}{2}}{3} = \frac{y + y^2/4}{3} = y/3 + y^2/12$

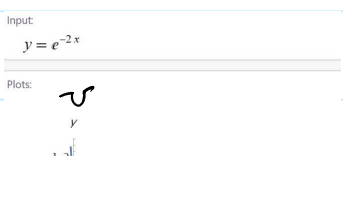
$$F_{U_1}(y) = \begin{cases} y/3\,(1 + y/4) & 0 \leq y \leq 2 \\ 1 & y > 2 \\ 0 & y < 0 \end{cases}$$

#5.12 Suppose $T \sim \text{Exp}(2)$. Find and graph $F_T(t)$.

$T = \frac{-\ln(U)}{2}$ | $U \sim unif([0,1])$

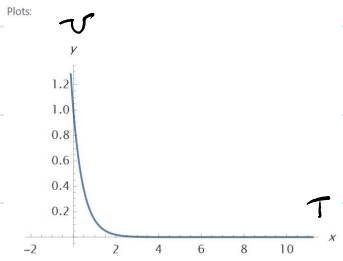
$-2T = +\ln(U)$

$U = e^{-2T}$



Total Area $= \int_0^\infty e^{-2x} = e^{-2x}/-2 \Big|_0^\infty = 0 - -1/2 = \frac{1}{2}$

$P(T \leq \infty) = 1$



$$P(T \le \infty) = 1$$

$$P(T \le 10) = 2 \cdot \left[\frac{e^{-2x}}{-2} \Big|_0^{10} \right] = -e^{-20} - -1 = .999\ldots$$

$$P(T \le 1) = -\left[e^{-2x} \Big|_0^{1} \right] = -e^{-2} + 1 = 0.864\ldots$$

$$P(T \le 0) = 0$$

$$F_T(\gamma) = \begin{cases} 1 - e^{-2\gamma} & | \ 0 \le \gamma < \infty \\ 1 & | \ \gamma \ge \infty \\ 0 & | \ \gamma < 0 \end{cases}$$

$F_T(\gamma)$