HW 2

Sunday, September 5, 2021 2:14 AM

372hw2

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MTH 372
Hw 2
Due Thursday, 9/16/2021.

Solutions are required, not just answers. Unsupported answers will recieve little or no credit.

Read Chapters 4,5 of *Huber*.

p.28 #4.2 Suppose $Y \sim$ Unif$[0, 10]$.
(a) Find $P(Y \in [3, 7])$.
(b) Find $P(Y \in [6, 12])$.

#4.4 Suppose that $U = (U_1, U_2)$ is uniformly chosen over the region $\{(x; y) : x \geq 2, 0 \leq y \leq 1/x^2\}$.
(a) What is $P(U_1 \leq 5)$?
(b) What is $P(U_2 \geq .01)$?

#4.5 (revised) Let U_1 and U_2 be independent uniform random variables over $[0, 1]$. What is the chance that $5U_2 < U_1$?

#4.8 Suppose that (U_1, U_2) is uniform over the quadrilateral region with vertices (0, 0); (0, 1); (2, 2); (2, 0). Prove that U_1 and U_2 are not independent. (Hint: Start by drawing a picture.

#5.2 Suppose $U \sim$ Unif$([0, 1])$ and $W = 1/U$.
(a) Find $P(W \geq 2)$.
(b) Find $P(W \geq -2)$.

#5.4 Let $U \sim$ Unif$([-1, 1])$. Find the cdf of U^3.

#5.10 Suppose that (U_1, U_2) is uniform over the quadrilateral region with vertices $(0, 0)$; $(0, 1)$; $(2, 2)$; and $(2, 0)$. Find the cdf of U_1.

#5.12 Suppose $T \sim$ Exp(2). Find and graph $F_T(t)$.

p.28 #4.2 Suppose $Y \sim$ Unif$[0, 10]$.

(a) Find $P(Y \in [3, 7])$.

$$P(Y \in [3,7]) = \frac{7-3}{10-0} = \boxed{\frac{4}{10}} = \boxed{\frac{2}{5}}$$

(b) Find $P(Y \in [6, 12])$.

$$P(Y \in [6,12]) = P(Y \in [6,10] \cup Y \in (10,12])$$

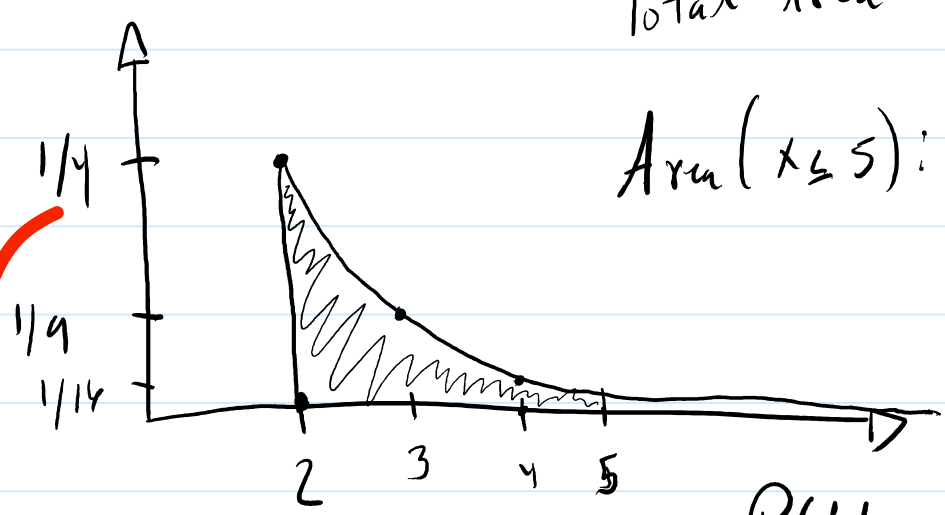
$$= P(Y \in [6,10]) + P(Y \in (10,12])$$ As $(10,12] \notin \Omega$, $P(Y \in (10,12]) = 0$

$$\therefore P(Y \in [6,12]) = P(Y \in [6,10]) = \frac{10-6}{10-0} = \boxed{\frac{4}{10}} = \boxed{\frac{2}{5}}$$

$\therefore P(Y \in [6,12]) = P(Y \in [6,10]) = \frac{\;}{10-0} = \boxed{\frac{\;}{10}} = \boxed{\frac{\;}{5}}$

#4.4 Suppose that $U = (U_1, U_2)$ is uniformly chosen over the region $\{(x; y) : x \geq 2, 0 \leq y \leq 1/x^2\}$.

(a) What is $P(U_1 \leq 5)$?



Total Area: $\int_2^\infty x^{-2}\,dx = -x^{-1}\Big|_2^\infty = \frac{-1}{\infty} + \frac{+1}{2} = \frac{1}{2}$ $\rightarrow 5/10^*$

Area $(x \leq 5)$: $\int_2^5 x^{-2}\,dx = -x^{-1}\Big|_2^5 = \frac{-1}{5} + \frac{1}{2} = \frac{3^*}{10}$

$\frac{5}{10} - \frac{2}{10}$

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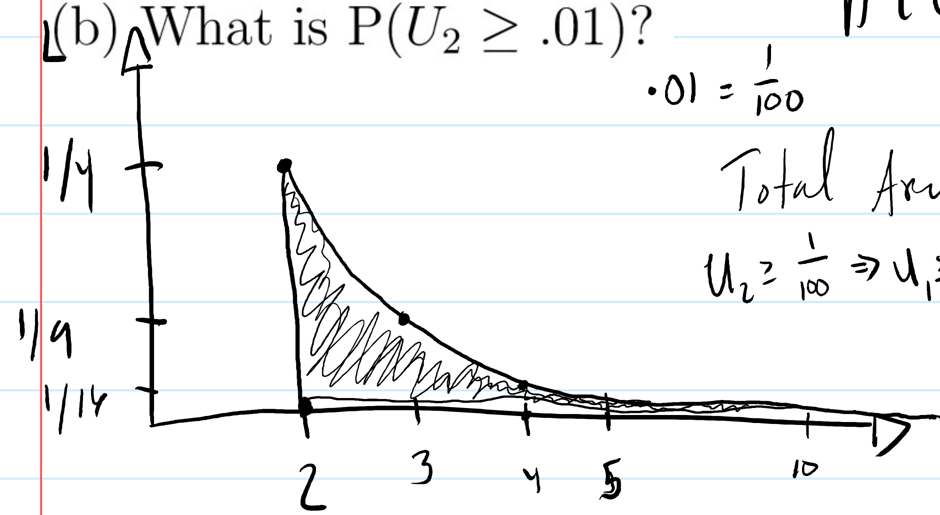
$P(U_1 \leq 5) = 3/10 \,/\, 5/10 = \boxed{3/5}$

1.5 / 2

(b) What is $P(U_2 \geq .01)$?



$.01 = \frac{1}{100}$

Total Area $= \int_2^\infty \frac{dx}{x^2} = \frac{-1}{x}\Big|_2^\infty = 0 + \frac{1}{2} = \frac{1}{2} = \frac{5^*}{10}$

$U_2 \geq \frac{1}{100} \Rightarrow U_1 \geq 10$

Area$(x \geq 10) = \int_{10}^\infty \frac{dx}{x^2} = \frac{-1}{x}\Big|_{10}^\infty = 0 + \frac{1}{10} = \frac{1^*}{10}$

No.

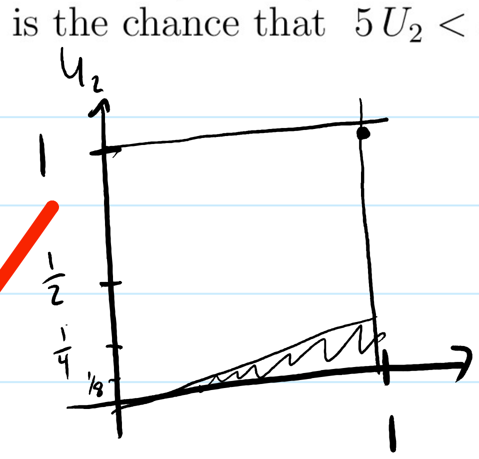
$* / *$

Thus $P(U_2 \geq \frac{1}{100}) = 1/10 \,/\, 5/10 = \boxed{1/5}$

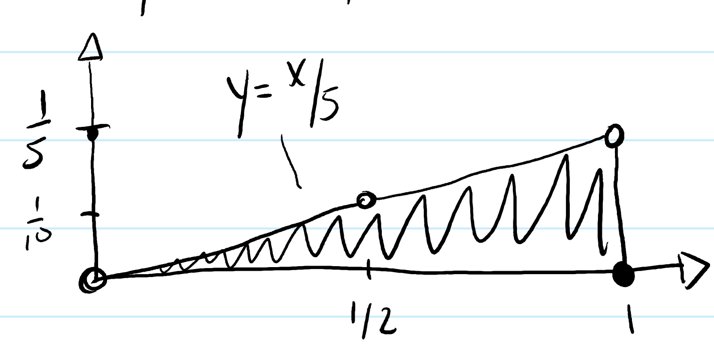
#4.5 (revised) Let U_1 and U_2 be independent uniform random variables over $[0, 1]$. What is the chance that $5U_2 < U_1$?

$5 \times 1/5 = 1 \not< 1$ $\qquad \frac{1}{10} \times 5 = \frac{1}{2}$

$5[0,1] < [0,1]$



$\Rightarrow$

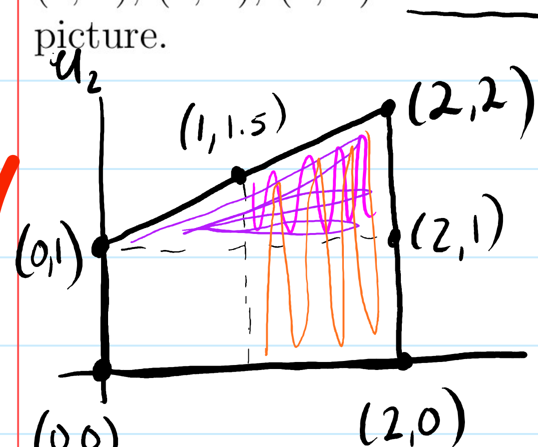


Total Area is $|x| = 1$

$\frac{1}{5}\int x = \frac{x^2}{10}$; $\frac{x^2}{10}\Big|_0^1 = 1/10 - 0 = 1/10$

$P(5U_2 < U_1) = 1/10 \,/\, 1 = \boxed{1/10}$

#4.8 Suppose that (U_1, U_2) is uniform over the quadrilateral region with vertices (0, 0); (0, 1); (2, 2); (2, 0). Prove that U_1 and U_2 are not independent. (Hint: Start by drawing a picture.



Total Area $= 2 \times 1 + \frac{2 \times 1}{2} = 2 + 1 = 3$

$P(U_1 > 1) = \frac{1 \times 1.5 + \frac{1 \times .5}{2}}{3} = \frac{1.5 + \frac{.5}{2}}{3} = \frac{1.75}{3}$

$P(U_2 > 1) = \frac{\frac{2 \times 1}{2}}{3} = \frac{1}{3}$

$P(U_1 > 1 \;\&\; U_2 > 1) = \frac{1 \times .5 + \frac{1 \times .5}{2}}{3} = \frac{.5 + \frac{.5}{2}}{3} = \frac{.75}{3} = \frac{.25}{\,}$

$\boxed{P(U_1 > 1) \cdot P(U_2 > 1) = \frac{1.75}{3} \cdot \frac{1}{3} = \frac{1.75}{9} = .1944 \neq P(U_1 > 1 \;\&\; U_2 > 1) = .25}$

#5.2 Suppose $U \sim \text{Unif}([0, 1])$ and $W = 1/U$.

(a) Find $P(W \geq 2)$. $\quad W \geq 2 \Leftrightarrow U \leq .5$

(a) Find P($W \geq 2$). $W \geq 2 \iff U \leq .5$

as $\frac{1}{\frac{1}{2}} = \frac{1 \cdot 2}{1} = 2$

$\therefore P(W \geq 2) = P(U \leq .5)$

$P(U \leq .5) = P(U \in [0, 0.5]) = \frac{0.5 - 0}{1 - 0} = \frac{.5}{1} = \underline{0.5}$

Thus $P(W \geq 2) = \boxed{.5}$

(b) Find P($W \geq -2$).

w/ $\lim_{U \to 0} \frac{1}{U} = \infty$; $\lim_{U \to 1} \frac{1}{U} = 1$; Thus $W \sim Unif([0, \infty])$

As all outcomes for W are bigger than or equal to 0 ; $\boxed{P(W \geq -2) = 1}$

#5.4 Let $U \sim$ Unif($[-1, 1]$). Find the cdf of U^3.

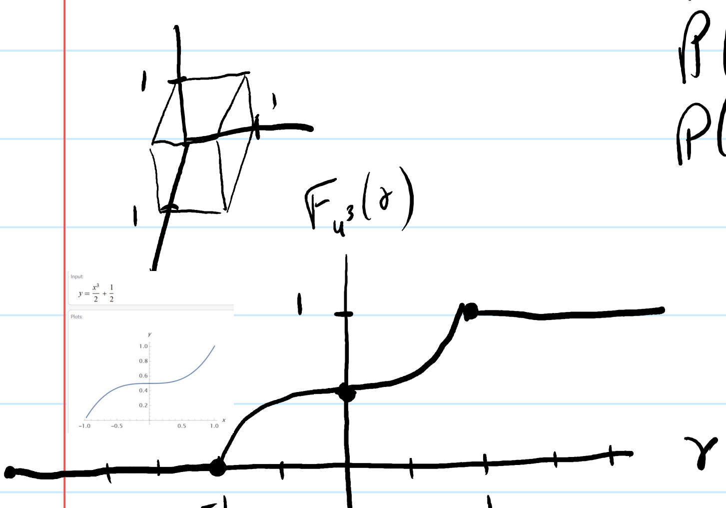
$-1^3 = -1$; $1^3 = 1$

$P(U^3 \leq 1) = 1$ ← $\frac{1^3 - (-1)^3}{1 - (-1)} = \frac{2}{2}$

$P(U^3 \leq 0) = 1/2$ $\quad \frac{0^3 - (-1)^3}{1 - (-1)} = \frac{1}{2}$

$P(U^3 \leq -1) = 0$ $\quad \frac{(-1)^3 - (-1)^3}{1 - (-1)}$

No. P(U^3 <= t) = P(U <= t^(1/3)) = (t^(1/3) + 1)/2 for -1 <=t<=1. The exponent is 1/3, not 3.

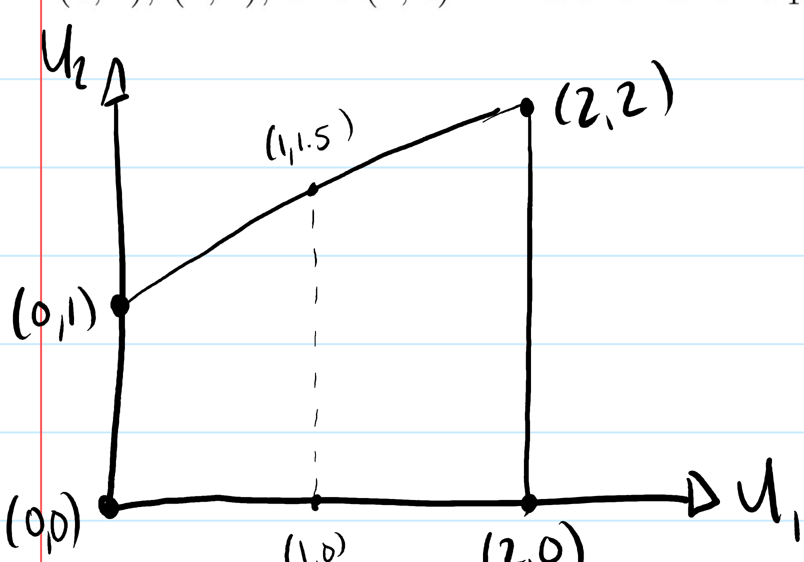


$\frac{r^3 + 1}{2}$

$$F_{U^3}(r) \begin{cases} \frac{r^3}{2} + \frac{1}{2} & -1 \leq r \leq 1 \\ 1 & r > 1 \\ 0 & r < -1 \end{cases}$$

1.5 / 2

#5.10 Suppose that (U_1, U_2) is uniform over the quadrilateral region with vertices (0, 0); (0, 1); (2, 2); and (2, 0). Find the cdf of U_1. Total Area = $2 \times 1 + \frac{2 \times 1}{2} = 2 + 1 = 3$

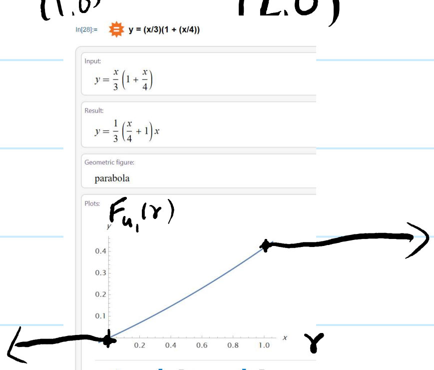


$P(U_1 \leq 2) = 3/3 = 1$

$P(U_1 \leq 1) = 1 \cdot 1 + \frac{1 \cdot .5}{2} = 1 + \frac{.5}{2} = 1.25 = \frac{1.25}{3}$

$\frac{r + \frac{r \cdot \frac{r}{2}}{2}}{3} = \frac{r + r^2/4}{3} = r/3 + r^2/12$

$$F_{U_1}(r) = \begin{cases} r/3\,(1 + r/4) & 0 \leq r \leq 2 \\ 1 & r > 2 \\ 0 & r < 0 \end{cases}$$

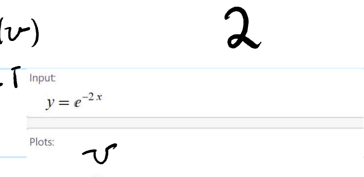


#5.12 Suppose $T \sim$ Exp(2). Find and graph $F_T(t)$.

$T = \frac{-\ln(U)}{2}$ | $U \sim unif([0,1])$

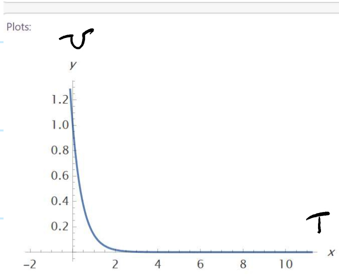
$-2T = +\ln(U)$

$U = e^{-2T}$



Total Area = $\int_0^\infty e^{-2x} = e^{-2x}/-2 \Big|_0^\infty = 0 - -1/2 = 1/2$

$P(T \leq \infty) = 1$



$\mathbb{P}(T \leq \infty) = 1$

$\mathbb{P}(T \leq 10) = 2 \cdot \left[\frac{e^{-2x}}{-2} \Big|_0^{10} \right] = -e^{-20} - -1 = .999\ldots$

$\mathbb{P}(T \leq 1) = -\left[e^{-2x} \Big|_0^{1} \right] = -e^{-2} + 1 = 0.864\ldots$

$\mathbb{P}(T \leq 0) = 0$

$$F_T(\gamma) = \begin{cases} 1 - e^{-2\gamma} & 0 \leq \gamma < \infty \\ 1 & \gamma \geq \infty \\ 0 & \gamma < 0 \end{cases}$$