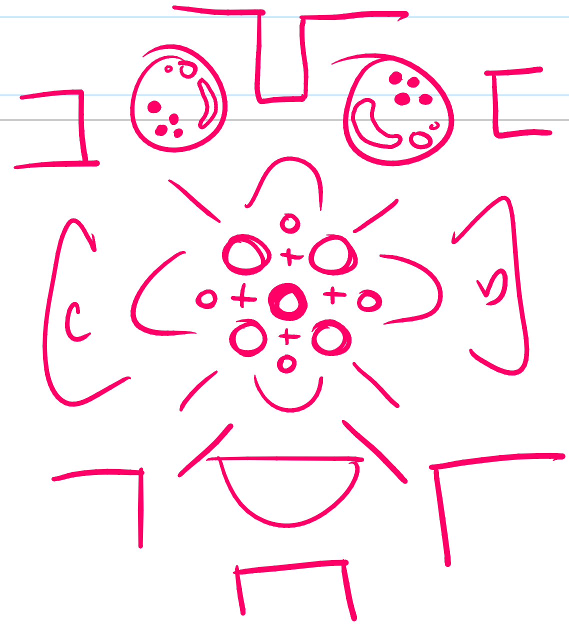
HW 7

Sunday, September 5, 2021 2:15 AM

372hw7



Jason Konzer

MTH 372
Hw 7
Due Thursday, 11/4/2021.

Read Chapters 14,15 of *Huber*.

A. (We began this problem in class on 10/19.) Flip 3 fair coins, A,B,C.
Let X be the total numbers of heads on coins A and B. Let Y be the total number of heads on coins B and C.
(a) Write out the table of the joint pmf $p_{X,Y}(x,y)$.
(b) Compute σ_X^2, σ_Y^2.
(c) Compute $\mathrm{Cov}(X,Y)$ and $\rho_{X,Y}$.

p.99 #14.4 Let Y have pmf $p_Y(i) = \begin{cases} 1/4 & \text{for } i \in \{3,5\} \\ 1/6 & \text{for } i \in \{7,9;11\} \end{cases}$.
(a) Find $E[Y]$.
(b) Find σ_Y .

#14.8 (modified)
Suppose (X,Y) have joint density $f_{X,Y}(x,y) = (2/5)(3x+2y)$ for $(x,y) \in [0,1]^2$
(a) Compute $P(X > .5 \text{ and } Y < .6)$.
(b) Find the marginal pdf's $f_X(x)$ and $f_Y(y)$.
(b) What is $\mathrm{Cov}(X,Y)$?

p.104 #15.1 For $(X,Y) \sim \mathrm{Unif}(\{(0;0);(0;2);(1;2)\})$, find the correlation between X and Y .

#15.2 For (X,Y) with density $f_{X,Y}(x,y) = 2\exp(-x-2y)$ for $x,y \geq 0$, find $\rho_{X,Y}$. (Hint: You can solve this problem without any integration.)

#15.3 (modified) Let $f_{X,Y}(r,s) = \dfrac{1}{C} \cdot \dfrac{r^4+s}{r^2}$ for $r \in [1,3], s \in [0,3]$.
(a) Find the value of C.
(b) Compute $P(X > 2), P(Y > 1)$ and $P(X > 2 \text{ and } Y > 1)$. Are X,Y independent?
(b) Find the density of Y, $f_Y(s)$. .
(c) Find $\rho_{X,Y}$.

A. (We began this problem in class on 10/19.) Flip 3 fair coins, A,B,C.
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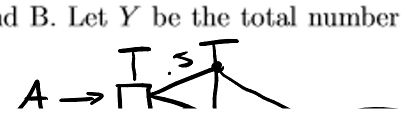

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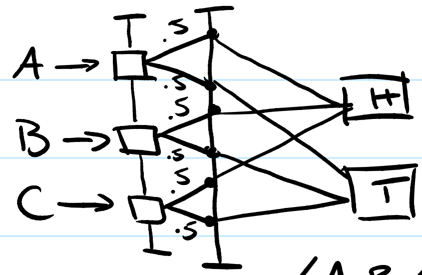


$X = |A_H| + |B_H|$

$Y = |B_H| + |C_H|$

$\dashv$ $\langle 0,0\rangle$ $\langle 0,1\rangle$ $\langle 1,0\rangle$ $\langle 1,1\rangle$

$2^3 = 8$ outcomes

Y \ X	0	1	2
0	1/8	1/8	0
1	1/8	2/8	1/8
2	0	1/8	1/8

$\langle A,B,C\rangle$

$\langle X,Y\rangle$ Both 0 — $\langle 0,0,0\rangle$ ①

Both 2 — $\langle 1,1,1\rangle$ ②

$\langle 0,1\rangle$ — $\langle 0,0,1\rangle$ ③

$\langle 1,0\rangle$ — $\langle 1,0,0\rangle$ ④

$\langle 1,1\rangle$ — $\langle 1,0,1\rangle$ ⑤ or $\langle 0,1,0\rangle$ ⑥

$\langle 0,2\rangle$ or $\langle 2,0\rangle$ — N/A

$\langle 1,2\rangle$ — $\langle 0,1,1\rangle$ ⑦

$\langle 2,1\rangle$ — $\langle 1,1,0\rangle$ ⑧

(b) Compute σ_X^2, σ_Y^2.

$$\sigma_Y^2 = \mu_{Y^2} - \mu_Y^2$$

$$\mu_x = \mu_y = 0(2/8) + 1(4/8) + 2(2/8) = 0 + \tfrac{1}{2} + \tfrac{1}{2} = 1$$

$$\mu_{x^2} = \mu_{y^2} = 0^2(2/8) + 1^2(4/8) + 2^2(2/8) = 0 + \tfrac{1}{2} + 1 = 1\tfrac{1}{2}$$

$$\sigma_x^2 = \sigma_y^2 = 1\tfrac{1}{2} - 1 = \tfrac{1}{2}$$

By Symmetry :)

(c) Compute $\mathrm{Cov}(X,Y)$ and $\rho_{X,Y}$.

$$\mathrm{Cov}(x,y) = \mu_{xy} - \mu_x \mu_y$$

$$\mu_{xy} = 0(3/8) + 1(2/8) + 2(2/8) + 4(1/8)$$

$$= 0 + \tfrac{2}{8} + \tfrac{4}{8} + \tfrac{4}{8} = \tfrac{10}{8} \qquad \mu_x = \mu_y = 1$$

$$\mu_x \mu_y = 1 \therefore \mu_{xy} - \mu_x \mu_y = \tfrac{10}{8} - \tfrac{8}{8} = \tfrac{2}{8}$$

$$\mathrm{Cov}(x,y) = 1/4$$

$$\rho_{x,y} = \frac{\mathrm{Cov}(x,y)}{\sigma_x \sigma_y}$$

$$\sigma_x^2 = \sigma_y^2 = \tfrac{1}{2} \therefore \sigma_x = \sigma_y = \sqrt{\tfrac{1}{2}}$$

$$\therefore \sigma_x \sigma_y = \sqrt{1/2}^{\,2} = \tfrac{1}{2} \therefore \rho_{x,y} = \frac{\tfrac{1}{4}}{\tfrac{1}{2}} = \tfrac{1}{4} \times \tfrac{2}{1}$$

2 / 2

$\therefore \sigma_x \sigma_y = \sqrt{1/2} \cdot 2 \quad \therefore \rho_{x,y} = \frac{\cdot}{\frac{1}{2}} = \frac{\cdot}{4} \times \frac{\cdot}{1}$

$= \frac{2}{4}$

LAST: $\rho_{x,y} = \frac{1}{2}$

p.99 #14.4 Let Y have pmf $p_Y(i) = \begin{cases} 1/4 & \text{for } i \in \{3,5\} \\ 1/6 & \text{for } i \in \{7,9;11\} \end{cases}$.

(a) Find $E[Y]$.

$\mu_y = \frac{3+5}{4} + \frac{7+9+11}{6} = \frac{8}{4} + \frac{27}{6} = \frac{24}{12} + \frac{54}{12} = \frac{78}{12} = \boxed{\frac{13}{2}}$

(b) Find σ_Y.

$\sigma_y = \sqrt{\mu_{y^2} - \mu_y^2} \qquad \mu_y^2 = \frac{169}{4}$

$\mu_{y^2} = \frac{3^2+5^2}{4} + \frac{7^2+9^2+11^2}{6} = \frac{9+25}{4} + \frac{49+81+121}{6}$

$= \frac{34}{4} + \frac{251}{6} = \frac{151}{3}$

2 / 2

$\sigma_y^2 = \frac{151}{3} - \frac{169}{4} = \frac{97}{12} \quad \therefore \sigma_y = \sqrt{97/12}$

#14.8 (modified)

Suppose (X, Y) have joint density $f_{X,Y}(x, y) = (2/5)(3x + 2y)$ for $(x, y) \in [0, 1]^2$

(a) Compute $P(X > .5 \text{ and } Y < .6)$.

Input:

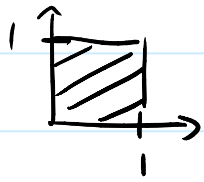
$\frac{2}{5} \int_{0.5}^{1} \int_{0}^{0.6} (3x + 2y)\, dy\, dx$

Result:

0.342

$342/1000$

$f_{X,Y}(x,y) = \frac{2}{5}[3x+2y]$ ON



$\mathbb{P}(X > \frac{1}{2} \;\&\; Y < \frac{3}{5}) = \frac{2}{5} \int\limits_{\substack{X \in (.5,1] \\ Y \in [0,.6)}} 3x+2y \, dx\, dy$

$\frac{2}{5} \int_0^{.6} \left[\frac{3x^2}{2} + 2yx \Big|_{.5}^{1} \right] dy = \frac{2}{5} \int_0^{.6} \left[\frac{3}{2} + 2y - \frac{3}{8} - y \right] dy$

$= \frac{2}{5} \int_0^{.6} y + \frac{9}{8}\, dy = \frac{2}{5} \left[\frac{y^2}{2} + \frac{9y}{8} \Big|_0^{.6} \right]$

$= \frac{2}{5} \left[\frac{9}{50} + \frac{27}{40} - 0 - 0 \right] = \frac{2}{5} \left[\frac{171}{200} \right] = \boxed{\frac{171}{500}}$

(b) Find the marginal pdf's $f_X(x)$ and $f_Y(y)$.

$$f_x(x) = \frac{2}{5}\int_0^1 3x+2y\,dy = \frac{2}{5}\left[3xy+y^2\Big|_0^1\right]$$

$$= \boxed{\frac{2}{5}[3x+1]}$$

$$f_y(y) = \frac{2}{5}\int_0^1 3x+2y\,dx = \frac{2}{5}\left[\frac{3x^2}{2}+2xy\Big|_0^1\right]$$

$$\boxed{= \frac{2}{5}\left[2y+\frac{3}{2}\right]}$$

(b) What is Cov(X, Y)?

$$\mathrm{Cov}(x,y) = \mu_{xy} - \mu_x\mu_y$$

$$\mu_x = \frac{2}{5}\int_0^1 3x^2+x\,dx = \frac{2}{5}\left[x^3+\frac{x^2}{2}\Big|_0^1\right] = \frac{2}{5}\left[1+\frac{1}{2}\right] = \frac{2}{5}\times\frac{3}{2} = \frac{3}{5}$$

$$\mu_y = \frac{2}{5}\int_0^1 2y^2+\frac{3y}{2}\,dy = \frac{2}{5}\left[\frac{2y^3}{3}+\frac{3y^2}{4}\Big|_0^1\right] = \frac{2}{5}\left[\frac{2}{3}+\frac{3}{4}\right] = \frac{2}{5}\times\frac{17}{12} = \frac{17}{30}$$

$$\mu_{xy} = \frac{2}{5}\int_0^1\int_0^1 3x^2y+2y^2x\,dx\,dy = \frac{2}{5}\int_0^1\left[x^3y+y^2x^2\Big|_0^1\right]dy$$

$$= \frac{2}{5}\int_0^1 y+y^2\,dy = \frac{2}{5}\left[y^2/2+y^3/3\Big|_0^1\right] = \frac{2}{5}\left[\frac{1}{2}+\frac{1}{3}\right] = \frac{2}{5}\times\frac{5}{6} = \frac{1}{3}$$

2 / 2

$$\mathrm{Cov}(x,y) = \frac{1}{3} - \left[\frac{3}{5}\times\frac{17}{30}\right] = \frac{1}{3} - \frac{17}{50} = \boxed{\frac{-1}{150}}$$

Input:

$$\frac{2}{5}\int_0^1\int_0^1 (3x^2y+2y^2x)\,dy\,dx - \frac{2}{5}\left(\int_0^1 (3x^2+x)\,dx\right)\times\frac{2}{5}\int_0^1\left(2y^2+3\times\frac{y}{2}\right)dy$$

Result:

$$-\frac{1}{150} \approx -0.00666667$$

p.104 #15.1 For $(X, Y) \sim \mathrm{Unif}(\{(0;0);(0;2);(1;2)\})$, find the correlation between X and Y.

$$\frac{1}{3}\langle 0,0\rangle;\ \frac{1}{3}\langle 0,2\rangle;\ \frac{1}{3}\langle 1,2\rangle$$

Y \ X	0	1	$P_Y(y)$
0	1/3	0	1/3
2	1/3	1/3	2/3
$P_X(x)$	2/3	1/3	

$$\mu_x = 0(2/3)+1(1/3) = 1/3$$

$$\mu_y = 0(1/3)+2(2/3) = 4/3$$

$$\mu_x^2 = 1/9 \quad \mu_y^2 = 16/9$$

$$\mu_{xy} = 0(2/3)+2(1/3) = 1/3$$

$$\mu_{x^2} = 0^2(2/3)+1^2(1/3) = 1/3$$

$P_X(x)$ | $2/3$ | $1/3$

$\mu_{X^2} = 0^2(2/3) + 1^2(1/3) = 1/3$

$Cov(X,Y) = \mu_{XY} - \mu_X \mu_Y$

$\mu_{Y^2} = 0^2(1/3) + 2^2(2/3) = 8/3$

$\sigma_X^2 = \mu_{X^2} - \mu_X^2$

$\mu_X \mu_Y = \frac{1}{3} \times \frac{4}{3} = \frac{4}{9}$

E(XY)=2/3

$Cov(X,Y) = \frac{3}{9} - \frac{4}{9} = -\frac{1}{9}$

$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$\sigma_X^2 = \frac{3}{9} - \frac{1}{9} = \frac{2}{9}$; $\sigma_X = \frac{\sqrt{2}}{3}$

$= \frac{-\frac{1}{9}}{\frac{\sqrt{2}}{3} \times \frac{2\sqrt{2}}{3}} = \frac{-\frac{1}{9}}{\frac{4}{9}}$

$\sigma_Y^2 = \frac{24}{9} - \frac{16}{9} = \frac{8}{9}$; $\sigma_Y = \frac{2\sqrt{2}}{3}$

$= -\frac{1}{9} \times \frac{9}{4} = \boxed{\frac{-1}{4}}$

Cov<0 can't be right. When X moves from 0 to 1, Y gets larger. 1.5 / 2

#15.2 For (X,Y) with density $f_{X,Y}(x,y) = 2\exp(-x-2y)$ for $x, y \geq 0$, find $\rho_{X,Y}$. (Hint: You can solve this problem without any integration.)

$f_{X,Y} = 2e^{-x}e^{-2y}$; $x,y \geq 0$

$f_X = 2e^{-x}\int_0^\infty e^{-2y}\,dy = 2e^{-x}\left[-\frac{e^{-2y}}{2}\Big|_0^\infty\right] = 2e^{-x}\left[\frac{1}{2}\right] = e^{-x}$

$f_Y = 2e^{-2y}\int_0^\infty e^{-x}\,dx = 2e^{-2y}\left[-e^{-x}\Big|_0^\infty\right] = 2e^{-2y}[1] = 2e^{-2y}$

2 / 2

* WE KNOW X & Y ARE INDEPENDENT ∴ X & Y ARE UNCORRELATED & $\boxed{\rho_{X,Y} = 0}$

#15.3 (modified) Let $f_{X,Y}(r,s) = \frac{1}{C} \cdot \frac{r^4 + s}{r^2}$ for $r \in [1,3], s \in [0,3]$.

(a) Find the value of C.

$1 = \frac{1}{C}\iint_{r\,s} \frac{r^4+s}{r^2}\,ds\,dr = \frac{1}{C}\iint_{r\,s} r^2 + sr^{-2}\,ds\,dr$

$C = \iint_{r\,s} r^2 + sr^{-2}\,ds\,dr = \int_r \left[r^2 s + \frac{s^2 r^{-2}}{2}\Big|_0^3\right] dr$

$= \int_r 3r^2 + \frac{9r^{-2}}{2}\,dr = \left[r^3 - \frac{9r^{-1}}{2}\Big|_1^3\right]$

$= 3^3 - \frac{9}{3\times2} - 1^3 + \frac{9}{1\times2} = 27 - \frac{3}{2} - 1 + \frac{9}{2} = 26 + \frac{6}{2} = 29$

Definite integral:

$$\int_1^3 \int_0^3 \left(r^2 + \frac{s}{r^2}\right) ds\,dr = 29$$

$X(r)$; $Y(s)$

$X(r)$; $Y(s)$

(b) Compute $P(X>2)$, $P(Y>1)$ and $P(X>2 \text{ and } Y>1)$. Are X, Y independent?

$$f_{X,Y} = \frac{1}{29}\left[r^2 + s r^{-2}\right]$$

$$f_X = \frac{1}{29}\int_s r^2 + s r^{-2}\, ds = \frac{1}{29}\left[r^2 s + \frac{s^2 r^{-2}}{2}\Big|_0^3\right]$$

$$= \frac{1}{29}\left[3r^2 + \frac{9r^{-2}}{2}\right]$$

Definite integral:

$$\frac{1}{29}\int_0^3\left(r^2 + \frac{s}{r^2}\right)ds = \frac{6r^4+9}{58r^2}$$

$$f_Y = \frac{1}{29}\int_r r^2 + s r^{-2}\, dr = \frac{1}{29}\left[r^3/3 - s r^{-1}\Big|_1^3\right]$$

$$= \frac{1}{29}\left[\frac{27}{3} - \frac{1}{3} - \frac{s}{3} + \frac{s}{1}\right] = \frac{1}{29}\left[\frac{26}{3} + \frac{2s}{3}\right]$$

Definite integral:

$$\frac{1}{29}\int_1^3\left(r^2 + \frac{s}{r^2}\right)dr = \frac{2(s+13)}{87}$$

$$= \frac{1}{87}\left[26 + 2s\right]$$

$P(X>2)$:

Definite integral: Miscopied.

$$\frac{1}{29}\int_2^3\left(\frac{9}{2r^2} + 2r\right)dr = \frac{161}{348} \approx 0.46264$$

NO

$P(Y>1)$:

Definite integral:

$$\frac{1}{87}\int_1^3 (26 + 2s)\, ds = \frac{20}{29} \approx 0.68966$$

$P(X>2 \;\&\; Y>1)$:

Computation result:

$$\frac{1}{29}\int_1^3\int_2^3\left(r^2 + \frac{s}{r^2}\right)dr\, ds = \frac{40}{87}$$

$$\frac{161}{348} \times \frac{20}{29} = \frac{805}{2784} \neq \frac{40}{87}$$

$P(X>2) \times P(Y>1) \neq P(X>2 \;\&\; Y>1)$

Right idea

X, Y ARE DEPENDENT

(b) Find the density of Y, $f_Y(s)$.

$\boxed{f_Y}$

(c) Find $\rho_{X,Y}$.

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

$\mu_X =$ Definite integral:

$$\frac{1}{29}\int_1^3\left(\frac{9}{2r} + 3r^3\right)dr = \frac{1}{29} \times \left(60 + \frac{9\log(3)}{2}\right) \approx 2.2394$$

$\mu_{X^2} =$ Definite integral:

$$\frac{1}{29}\int_1^3\left(\frac{9}{2} + 3r^4\right)dr = \frac{771}{145} \approx 5.3172$$

$\mu_X^2 =$ Input:

$$\left(\frac{1}{29}\int_1^3\left(3r^3 + 9 \times \frac{1}{r \times 2}\right)dr\right)^2$$

Result:

$$\frac{1}{841}\left(60 + \frac{9\log(3)}{2}\right)^2 \approx 5.01509$$

This is fine, but in the future could you make this print large so it is easier to read?

$\mu_y =$ Definite integral: $\frac{1}{87}\int_0^3 (26s + 2s^2)\,ds = \frac{45}{29} \approx 1.5517$

$\mu_{y^2} =$ Definite integral: $\frac{1}{87}\int_0^3 (26s^2 + 2s^3)\,ds = \frac{183}{58} \approx 3.1552$

$\mu_y^2 =$ Input: $\left(\frac{1}{87}\int_0^3 (26s + 2s^2)\,ds\right)^2$

Result: $\frac{2025}{841} \approx 2.40785$

$\mu_{xy} =$ Input: $\frac{1}{29}\int_0^3 \int_1^3 \left(r^3 s + \frac{s^2}{r}\right) dr\,ds$

Result: $\frac{9}{29} \times (10 + \log(3)) \approx 3.4444$

$\mu_x \mu_y =$ Input: $\left(\frac{1}{29}\int_1^3 \left(3r^3 + 9 \times \frac{1}{r \times 2}\right) dr\right)\left(\frac{1}{87}\int_0^3 (26s + 2s^2)\,ds\right)$

Result: $\frac{45}{841} \times \left(60 + \frac{9\log(3)}{2}\right) \approx 3.47499$

$$\mathrm{Cov}(x,y) = \mu_{xy} - \mu_x \mu_y = 3.4444 - 3.47499$$
$$= -0.03055$$

2 / 2

$$\sigma_x^2 = \mu_{x^2} - \mu_x^2 = 5.3172 - 5.01509 = 0.30211$$

$$\sigma_y^2 = \mu_{y^2} - \mu_y^2 = 3.1552 - 2.40785 = 0.74735$$

$$\rho_{x,y} = \frac{\mathrm{Cov}(x,y)}{\sigma_x \sigma_y} = \frac{-0.03055}{\sqrt{0.30211}\sqrt{0.74735}}$$

$$= -0.06429$$