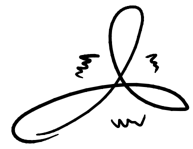
Exam 1

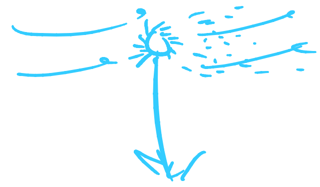
~~Friday, October 15, 2021 1:01 AM~~

10/14/21

372ex1



Cason Konzer



MTH 372-W1
Midterm exam
Fall 2021

Please read these instructions carefully.

1. You are permitted to use your textbook, anything I have posted on Blackboard, any notes you have taken during class, your homework assignments, and your own calculator. You are not permitted to use the internet in any other way, or ask any person for help other than me. I will give you hints, if you ask by email.

By signing your name when you return this exam, you are agreeing to abide by these rules.

2. You must show all appropriate work on each problem for credit. Correct work, even when the final answer is wrong, will earn substantial partial credit. Unjustified answers will earn little or no credit.

3. The exam consists of the 6 problems on the next page. There is no space on that page for the necessary work; therefore I will not grade anything written on that page. Do the exam on other paper.

4. This exam is due by midnight, Thursday, Oct 14, 2021. That does not mean that I want you to spend all day working on it. The exam should take no longer that $1\frac{1}{2}$ hours. Please use Blackboard to submit your solutions.

5. If you have questions or see what appears to be an error on the exam, please let me know right away.

1. San Francisco Giants shortstop Brandon Crawford hits a home run in any given game with probability 1/8. When Crawford hits a home run, the Giants win that game 2/3 of the time. What is the probability that in tomorrow's game, Crawford hits a home run and the Giants win?

2. Let $X \sim$Unif($\{3,4,5\}$) and $Y \sim$Unif($\{4,5,6\}$) be independent random variables. Find the pmf of $Z = X + Y$.

3. Suppose that $U = (U_1, U_2)$ is uniform over the triangular region with vertices $(0,0)$, $(4,0)$, $(4,2)$.
(a) Evaluate the probabilities $P(U_1 < 2)$, $P(U_2 > 1)$, and $P(U_1 < 2 \text{ and } U_2 > 1)$.
(b) Are U_1, U_2 independent random variables? Carefully explain why or why not.

4. You have a coin that lands 'heads' with probability 1/3. You flip the coin 6 times. What is the probability that 2 or fewer of the coin flips land 'heads?'

5. Urn 1 contains 1 red and 1 green marbles.
Urn 2 contains 3 red and 1 green marbles.
Urn 3 contains 1 red and 5 green marbles.
You roll an ordinary 6-sided fair die.
If you roll a 1, you choose a marble at random from urn 1.
If you roll a 2 or a 3, you choose a marble at random from urn 2.
If you roll a 4, 5 or 6, you choose a marble at random from urn 3.
(a) What is the probability that you choose a red marble?
(b) Suppose you chose a red marble. What is the probability that it came from urn 1? Urn 2? Urn 3?

6. The random variable T has pdf $f_T(t) = \begin{cases} 6t^5 & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$.
(a) What is the pdf of $Z = T^2$?
(b) Use your pdf from (a) to evaluate $E(Z)$.
(c) Now use the Rule of the Unconscious Statistician to evaluate $E(Z)$.

1. San Francisco Giants shortstop Brandon Crawford hits a home run in any given game with probability 1/8. When Crawford hits a home run, the Giants win that game 2/3 of the time. What is the probability that in tomorrow's game, Crawford hits a home run and the Giants win?

$P(H) = 1/8$; $P(W|H) = 2/3 = \dfrac{P(W \cap H)}{P(H)}$

$P(W \cap H) = P(H)P(W|H) = \frac{1}{8} \cdot \frac{2}{3} = \frac{2}{24}$

$P(W \cap H) = P(H)\,P(W|H) = \ldots 8 \ldots 3 = 24$

2. Let $X \sim$Unif$(\{3,4,5\})$ and $Y \sim$Unif$(\{4,5,6\})$ be independent random variables. Find the pmf of $Z = X + Y$.

$P(X=3) = P(X=4) = P(X=5) = P(Y=4) = P(Y=5) = P(Y=6)$

$3+4=7$; $3+5=8$; $3+6=9$
(1)
$4+4=8$; $4+5=9$; $4+6=10$
(2)
$5+4=9$; $5+5=10$; $5+6=10$
(3) (2) (1)

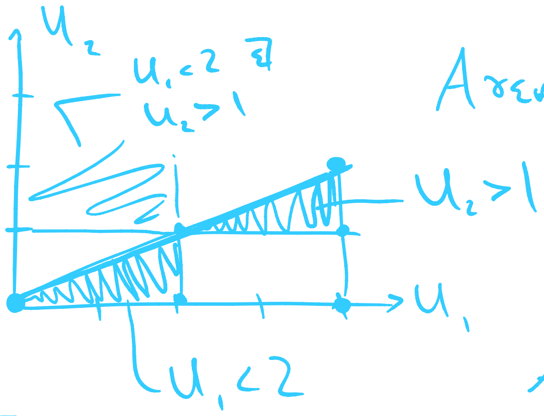
$\sum_i = \cancel{1+2+3+2+1}\ 3+3+3 = 9$

$P(Z=7) = P(Z=11) = 1/9 \quad P(Z=9) = 3/9$

$P(Z=8) = P(Z=10) = 2/9$

$$\text{pmf} = \left\{ \begin{array}{l|l} 1/9 & z = 7, 11 \\ 2/9 & z = 8, 10 \\ 3/9 & z = 11 \end{array} \right. \ldots 0 \text{ ELSE}$$

3. Suppose that $U = (U_1, U_2)$ is uniform over the triangular region with vertices $(0,0)$, $(4,0)$, $(4,2)$.
(a) Evaluate the probabilities $P(U_1 < 2)$, $P(U_2 > 1)$, and $P(U_1 < 2 \text{ and } U_2 > 1)$.
(b) Are U_1, U_2 independent random variables? Carefully explain why or why not.



$\text{Area} = \frac{BH}{2} = \frac{4\times 2}{2} = 4 \ldots$

$A(U_1<2) = \frac{2\times 1}{2} = 1$

$A(U_2>1) = \frac{2\times 1}{2} = 1$

$A(U_1<2 \cap U_2>1) = 0$

$\boxed{P(U_1<2) = P(U_2>2) = 1/4 \ ; P(U_1<2 \ \& \ U_2>1) = 0}$

U_1 & U_2 are Not independent as knowing the Distribution of one tells you about the other

$$\underbrace{* \text{ELSE } P(U_1<2 \ \& \ U_2>1) = P(U_1<2)\cdot P(U_2>2) = 1/4^2 = 1/16 *}_{\text{THIS FAILS THUS DEPENDENT}}$$

4. You have a coin that lands 'heads' with probability 1/3. You flip the coin 6 times. What is the probability that 2 or fewer of the coin flips land 'heads?'

$P(H) = 1/3 \ \langle p \rangle \ ; P(T) = 2/3 \ \langle q \rangle$

Distribution is Binomial

Distribution is Binomial

$\hookrightarrow Bin(n=6, p=1/3)$ $\quad$ USE! $\binom{n}{i}p^i(1-p)^{n-1}\mathbb{1}(i\in\{0,\ldots,n\})$

$${}_6C_2 = \frac{6\cdot5\cdot\cancel{4\cdot3\cdot2}}{(\cancel{4\cdot3\cdot2})(2)} = \frac{30}{2} = 15$$ WAYS TO CHOOSE 2 COINS

$${}_6C_1 = 6$$ ... 6 WAYS TO CHOOSE 1 COIN

$$P(H=2) = 15(1/3)^2(2/3)^4 = \frac{15\times2^4}{3^6} = \frac{240}{729}$$
$$P(H=1) = 6(1/3)(2/3)^5 = \frac{6\times2^5}{3^6} = \frac{192}{729}$$
$$P(H=0) = 1(1/3)^0(2/3)^6 = \frac{1\times2}{3^6} = \frac{2}{729}$$

Thus $P(H\le2) = P(H=2)+P(H=1)+P(H=0)$

$$= \frac{240+192+2}{729} = \boxed{\frac{434}{729}}$$

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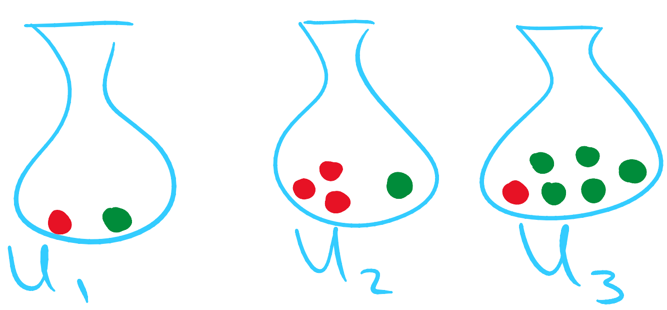
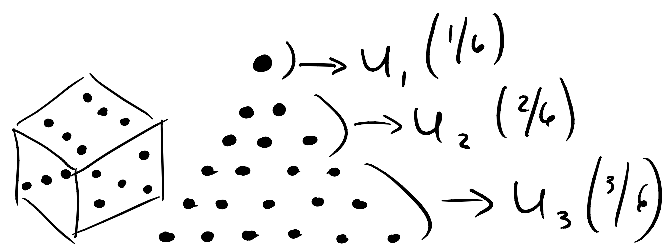
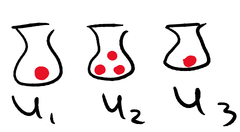




$$\underline{P(R)} = P(R\cap U_1) + P(R\cap U_2) + P(R\cap U_3)$$
$$= \left(\frac{1}{2}\cdot\frac{1}{6}\right) + \left(\frac{3}{\cancel{4}(2)}\cdot\frac{\cancel{2}}{6}\right) + \left(\frac{1}{\cancel{6}(2)}\cdot\frac{\cancel{3}}{6}\right)$$
$$= \frac{1}{12} + \frac{3}{12} + \frac{1}{12} = \boxed{\frac{5}{12}} \ldots$$

GIVEN RED WAS CHOSEN ...



Σ of 5 RED MARBLES

$$\boxed{P(U_1|R) = 1/5 = P(U_3|R) \;\&\; P(U_2|R) = 3/5}$$

$P(U_1|R) = 1/5 = P(U_3|R)$; $P(U_2|R) = 3/5$

6. The random variable T has pdf $f_T(t) = \begin{cases} 6t^5 & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$.

(a) What is the pdf of $Z = T^2$?
(b) Use your pdf from (a) to evaluate $E(Z)$.
(c) Now use the Rule of the Unconscious Statistician to evaluate $E(Z)$.

$$f_z = f_t T_z = 6t^5 \cdot \frac{1}{2z^{1/2}} = \frac{6z^{5/2} \cdot z^{-1/2}}{2} = 3z^{4/2} = 3z^2$$

$0 < z < 1$

$T = z^{1/2} \quad T_z = \frac{1}{2z^{1/2}}$

$$E(z) = \int_0^1 z \, 3z^2 \, dz = \int_0^1 3z^3 \, dz = \left. \frac{3z^4}{4} \right|_0^1 = 3/4$$

$$E(Z) = E(T^2) = \int_0^1 t^2 \cdot 6t^5 \, dt = \int_0^1 6t^7 \, dt$$

$$= \left. \frac{6t^8}{8} \right|_0^1 = 6/8 = 3/4$$