

HW 11

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MTH 372
Hw 11
Due Thursday, 12/9/2021.

Read Chapters 23 of *Huber*.

p.153 #23.2 (Modified) Suppose (X_1, X_2, X_3) has joint density

$$f(x_1, x_2, x_3) = k(x_1 + x_2 + x_3)\mathbf{1}(x_1, x_2, x_3 \in [0, 1]).$$

- (a) Evaluate k .
- (b) Find the joint marginal density of X_1, X_2 .
- (c) Find the marginal density of X_1 .
- (d) Find $\text{Cov}(X_1, X_3)$.

#23.4. Suppose Z_1, Z_2, Z_3 are iid standard normal random variables. Find their joint density.

A. Let Y_1, Y_2 have joint pdf $f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} ky_1(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

- (a) Determine k .
- (b) Evaluate $P(Y_1 \leq \frac{3}{4}, Y_2 \geq \frac{1}{2})$.
- (c) Find the marginal pdf's $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$.
- (d) Evaluate $P(Y_2 \leq \frac{1}{2} | Y_1 \leq \frac{3}{4})$.
- (e) Find the conditional density of Y_1 given $Y_2 = 2/3$.
- (f) Find the conditional density of Y_2 given Y_1 .
- (g) Find $P(Y_2 \geq \frac{3}{4} | Y_1 = \frac{1}{2})$.

B. Let $f_Y(y) = 2(1 - y)$ for $0 \leq y \leq 1$.

- (a) Find the pdf of $T = 2Y - 1$.
- (b) Find the pdf of $U = 1 - 3Y$.

C. Let X, Y be iid $\text{Exp}[1]$. Let $R = \frac{X}{Y}$ and $S = Y$.

- (a) Find the joint pdf $f_{R,S}(r, s)$.
- (b) Find the marginal pdf $f_R(r)$.

Hint for (b): You can avoid integration by carefully using the fact that when $Z \sim \text{Exp}[\beta]$, we know that $E[Z] = \frac{1}{\beta} = \int_0^\infty z \cdot \beta e^{-\beta z} dz$.

p.153 #23.2 (Modified) Suppose (X_1, X_2, X_3) has joint density

$$f(x_1, x_2, x_3) = k(x_1 + x_2 + x_3)\mathbf{1}(x_1, x_2, x_3 \in [0, 1]).$$

$$f(x_1, x_2, x_3) = k(x_1 + x_2 + x_3) \mathbf{1}(x_1, x_2, x_3 \in [0, 1]).$$

(a) Evaluate k .

$$\int_{x_1, x_2, x_3=0}^1 x_1 + x_2 + x_3 \delta_{x_3} \delta_{x_2} \delta_{x_1} = \frac{3k}{2} \Rightarrow k = \frac{2}{3}$$

In[20]:= Integrate[k (x1 + x2 + x3), {x1, 0, 1}, {x2, 0, 1}, {x3, 0, 1}]

$$\text{Out}[20] = \frac{3k}{2}$$

In[22]:= Integrate[$\frac{2}{3} (x_1 + x_2 + x_3)$, {x1, 0, 1}, {x2, 0, 1}, {x3, 0, 1}]

$$\text{Out}[22] = 1$$

(b) Find the joint marginal density of X_1, X_2 .

$$f_{X_1, X_2}(x_1, x_2) = \frac{2}{3} \int_{x_3=0}^1 x_1 + x_2 + x_3 \delta_{x_3} = \boxed{\frac{1+2(x_1+x_2)}{3}}$$

In[23]:= Integrate[$\frac{2}{3} (x_1 + x_2 + x_3)$, {x3, 0, 1}]

$$\text{Out}[23] = \frac{1}{3} + \frac{2x_1}{3} + \frac{2x_2}{3}$$

(c) Find the marginal density of X_1 .

$$f_{X_1}(x_1) = \frac{2}{3} \int_{x_3, x_2=0}^1 x_1 + x_2 + x_3 \delta_{x_2} \delta_{x_3} = \boxed{\frac{2(1+x_1)}{3}}$$

In[24]:= Integrate[$\frac{2}{3} (x_1 + x_2 + x_3)$, {x2, 0, 1}, {x3, 0, 1}]

$$\text{Out}[24] = \frac{2 \times (1+x_1)}{3}$$

(d) Find $\text{Cov}(X_1, X_3)$.

$$\text{Cov}(X_1, X_3) = E[X_1 X_3] - E[X_1] E[X_3]$$

$$= \int_{x_1, x_3=0}^1 x_1 x_3 f_{X_1, X_3}(x_1, x_3) \delta_{x_1} \delta_{x_3} - \left[\int_{x_1=0}^1 x_1 f_{X_1}(x_1) dx_1 \cdot \int_{x_3=0}^1 x_3 f_{X_3}(x_3) \delta_{x_3} \right]$$

In[30]:= Integrate[x1 * x3 * Integrate[$\frac{2}{3} (x_1 + x_2 + x_3)$, {x2, 0, 1}], {x1, 0, 1}, {x3, 0, 1}]

$$\text{Out}[30] = \frac{11}{36}$$

In[31]:= Integrate[x1 * Integrate[$\frac{2}{3} (x_1 + x_2 + x_3)$, {x2, 0, 1}, {x3, 0, 1}], {x1, 0, 1}]

$$\text{Out}[31] = \frac{5}{9}$$

In[32]:= Integrate[x3 * Integrate[$\frac{2}{3} (x_1 + x_2 + x_3)$, {x1, 0, 1}, {x2, 0, 1}], {x3, 0, 1}]

$$\text{Out}[32] = \frac{5}{9}$$

$$\frac{11}{36} - \left(\frac{5}{9}\right)^2 = \text{Cov}(X_1, X_3)$$

$$\boxed{-\frac{1}{324}} = +$$

#23.4. Suppose Z_1, Z_2, Z_3 are iid standard normal random variables. Find their joint density.

$$\text{As } Z_1, Z_2, Z_3 \text{ are iid; } f_{Z_1, Z_2, Z_3}(z_1, z_2, z_3) = f_{Z_1}(z_1) f_{Z_2}(z_2) f_{Z_3}(z_3)$$

Consider an

$$\therefore f(z) = e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

Consider an ambiguous Z normal r.v. ; $f_Z(z) = \frac{e^{-\frac{|z|}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$

$$f_{Z_1, Z_2, Z_3}(z_1, z_2, z_3) = \frac{\exp\left(-\frac{(z_1-\mu_1)^2}{2\sigma_1^2} - \frac{(z_2-\mu_2)^2}{2\sigma_2^2} - \frac{(z_3-\mu_3)^2}{2\sigma_3^2}\right)}{\sigma_1\sigma_2\sigma_3\sqrt{2\pi^3}}$$

If we are assuming standard normal; then each $Z_1, Z_2, Z_3 \sim N(0, 1)$ & $\mu_1, \mu_2, \mu_3 = 0$ & $\sigma_1, \sigma_2, \sigma_3 = 1$

$$f_{Z_1, Z_2, Z_3}(z_1, z_2, z_3) = \frac{\exp\left(-\left(\frac{z_1^2 + z_2^2 + z_3^2}{8}\right)\right)}{2\pi\sqrt{2\pi}} \quad 2/2$$

A. Let Y_1, Y_2 have joint pdf $f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} ky_1(1-y_2), & 0 \leq y_1 \leq y_2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

(a) Determine k . 1-y₂ No - should be y₂ here. $k = 24$.

here is $1 = k \int_{y_2=0}^{1-y_2} \int_{y_1=0}^{y_2} y_1(1-y_2) dy_1 dy_2 = \frac{k}{8}$; k=8

In[38]:= Integrate[Integrate[k * y1 * (1 - y2), {y1, 0, 1 - y2}], {y2, 0, 1}]

Out[38]= $\frac{k}{8}$

In[40]:= Integrate[Integrate[8 * y1 * (1 - y2), {y1, 0, 1 - y2}], {y2, 0, 1}]

Out[40]= 1

(b) Evaluate $P(Y_1 \leq \frac{3}{4}, Y_2 \geq \frac{1}{2})$.

$$\begin{aligned} P(Y_1 \leq \frac{3}{4}, Y_2 \geq \frac{1}{2}) &= P(Y_1 \leq \frac{3}{4}, Y_1 \leq Y_2 \leq 1) - P(0 \leq Y_1 \leq Y_2 \leq 1, \frac{1}{2} \leq Y_2 \leq 1) \\ &= 8 \int_0^{\frac{3}{4}} \int_{y_1}^1 y_1(1-y_2) dy_2 dy_1 - 8 \int_{\frac{1}{2}}^1 \int_0^{y_2} y_1(1-y_2) dy_1 dy_2 = \boxed{\frac{67}{768}} \end{aligned}$$

In[6]:= Integrate[Integrate[8 * y1 * (1 - y2), {y2, y1, 1}], {y1, 0, \frac{3}{4}}] - Integrate[Integrate[8 * y1 * (1 - y2), {y1, 0, y2}], {y2, \frac{1}{2}, 1}]

Out[6]= $\frac{67}{768}$

(c) Find the marginal pdf's $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$

$$f_{Y_1}(y_1) = 8 \int_{y_1}^1 y_1(1-y_2) dy_2 = \boxed{4y_1 - 8y_1^2 + 4y_1^3}$$

In[45]:= Integrate[8 * y1 * (1 - y2), {y2, y1, 1}]

$$f_{Y_2}(y_2) = 8 \int_0^{y_2} y_1(1-y_2) dy_1 = \boxed{4y_2^2 - y_2^3}$$

Out[45]= $4y_2^2 - y_2^3$

In[46]:= Integrate[8 * y1 * (1 - y2), {y1, 0, y2}]

$$\text{Out[46]}= 4 \times (1 - y_2) y_2^2$$

$$(d) \text{ Evaluate } P(Y_2 \leq \frac{1}{2} | Y_1 \leq \frac{3}{4}) = \frac{P(Y_2 \leq \frac{1}{2}, Y_1 \leq \frac{3}{4})}{P(Y_1 \leq \frac{3}{4})}$$

$$8 \int_0^{\frac{3}{4}} \int_{Y_1}^{\frac{1}{2}} y_1(1-y_2) dy_2 dy_1 = \boxed{\frac{1}{9}}$$

$$\text{In[15]}= \frac{\text{Integrate}[\text{Integrate}[8 * y1 * (1 - y2), \{y2, y1, \frac{1}{2}\}], \{y1, 0, \frac{3}{4}\}]}{\text{Integrate}[4 y1 - 8 y1^2 + 4 y1^3, \{y1, 0, \frac{3}{4}\}]}$$

$$\text{Out[15]}= \frac{1}{9}$$

(e) Find the conditional density of Y_1 given $Y_2 = 2/3$.

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1, Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{8 y_1(1-y_2)}{4 y_2^2 - y_2^3}$$

$$f_{Y_1|Y_2}(y_1|y_2 = \frac{2}{3}) = \frac{8 y_1(\frac{1}{3})}{4(\frac{1}{3}) - (\frac{1}{27})} = \boxed{\frac{72}{11} y_1} > ; 0 \leq y_1 \leq y_2 \leq 1$$

(f) Find the conditional density of Y_2 given Y_1 .

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1, Y_2}(y_1, y_2)}{f_{Y_1}(y_1)} = \boxed{\frac{8 y_1(1-y_2)}{4 y_1 - 8 y_1^2 + 4 y_1^3}}$$

(g) Find $P(Y_2 \geq \frac{3}{4} | Y_1 = \frac{1}{2})$.

$$f_{Y_2|Y_1}(y_2|y_1 = \frac{1}{2}) = \frac{8(\frac{1}{2})(1-y_2)}{4(\frac{1}{2}) - 8(\frac{1}{4}) + 4(\frac{1}{8})} = 8(1-y_2)$$

Right ideas otherwise.

$$8 \int_{\frac{3}{4}}^1 1-y_2 dy_2 = \boxed{\frac{1}{4}}$$

In[16]:= Integrate[8 * (1 - y2), {y2, 3/4, 1}]

$$\text{Out[16]}= \frac{1}{4}$$

1.5 / 2

B. Let $f_Y(y) = 2(1-y)$ for $0 \leq y \leq 1$.

(a) Find the pdf of $T = 2Y - 1$. $\frac{T+1}{2} = y$

$$f_T(t) = f_Y(y(t)) y'_t$$

$$= 2 \left(1 - \frac{t+1}{2}\right) \cdot \frac{1}{2} = \boxed{1 - \frac{t+1}{2}}$$

For what values of t?

(b) Find the pdf of $U = 1 - 3Y$. $y = \frac{-(u-1)}{3} = \frac{1-u}{3}$

$$f_U(u) = f_Y(y(u)) y'_u$$

$$= \left(1 - 3\left(\frac{1-u}{3}\right)\right) \cdot \frac{-1}{3}$$

Must be $| -1/3 |$

$$= -\frac{1}{3} (1+u-1) = \boxed{\frac{-u}{3}}$$

Negative pdf?

No, this is not $f_Y(-1-u/3)$

1 / 2

C. Let X, Y be iid $\text{Exp}[1]$. Let $R = \frac{X}{Y}$ and $S = Y$.

(a) Find the joint pdf $f_{R,S}(r,s)$.

$$\text{if } X \sim \text{Exp}(\beta) \text{ then } f_X(x) = \beta e^{-\beta x}$$

$$\therefore f_X(x) = e^{-x} \text{ and } f_Y(y) = e^{-y} \quad \text{as iid if } f_{X,Y}(x,y) = e^{-x-y}$$

$$RY = X = RS \quad \text{as } S = Y$$

$$J = \begin{vmatrix} X_r & X_s \\ Y_r & Y_s \end{vmatrix} = \begin{vmatrix} S & R \\ 0 & 1 \end{vmatrix} = S - 0 = S$$

$$\begin{aligned} f_{R,S}(r,s) &= f_{X,Y}(x(r,s), y(r,s)) J \\ &= e^{-rs-s} S = \boxed{S e^{-s(r+1)}} \end{aligned}$$

(b) Find the marginal pdf $f_R(r)$.

1.5 / 2

Hint for (b): You can avoid integration by carefully using the fact that when $Z \sim \text{Exp}[\beta]$, we know that $E[Z] = \frac{1}{\beta} = \int_0^\infty z \cdot \beta e^{-\beta z} dz$.

$$f_R(r) = \int_0^\infty S e^{-s(r+1)} ds = \boxed{\frac{1}{r+1}} = \mathbb{E}[S]$$

$$\text{letting } \beta = r+1$$

1 / (1+r)^2, actually.