

HW 7

Sunday, September 5, 2021 2:15 AM



372hw7

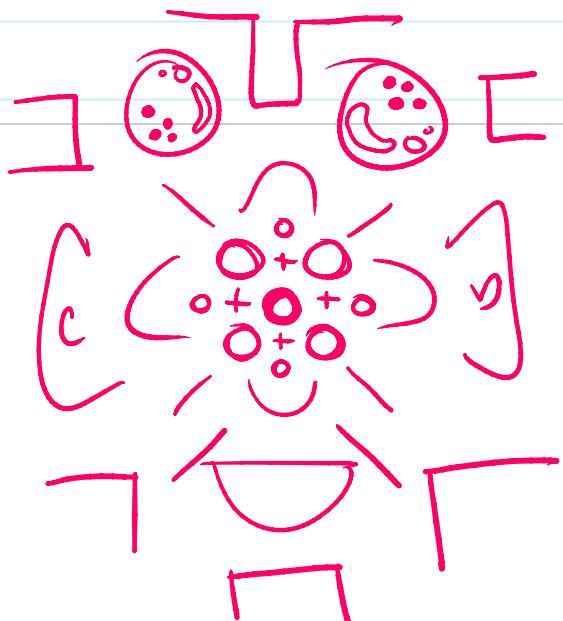
Chase Konzer

MTH 372

Hw 7

Due Thursday, 11/4/2021.

Read Chapters 14,15 of *Huber*.



A. (We began this problem in class on 10/19.) Flip 3 fair coins, A,B,C.

Let X be the total numbers of heads on coins A and B. Let Y be the total number of heads on coins B and C.

- Write out the table of the joint pmf $p_{X,Y}(x,y)$.
- Compute σ_X^2 , σ_Y^2 .
- Compute $\text{Cov}(X, Y)$ and $\rho_{X,Y}$.

p.99 #14.4 Let Y have pmf $p_Y(i) = \begin{cases} 1/4 & \text{for } i \in \{3, 5\} \\ 1/6 & \text{for } i \in \{7, 9, 11\} \end{cases}$

- Find $E[Y]$.
- Find σ_Y .

#14.8 (modified)

Suppose (X, Y) have joint density $f_{X,Y}(x, y) = (2/5)(3x + 2y)$ for $(x, y) \in [0, 1]^2$

- Compute $P(X > .5 \text{ and } Y < .6)$.
- Find the marginal pdf's $f_X(x)$ and $f_Y(y)$.
- What is $\text{Cov}(X, Y)$?

p.104 #15.1 For $(X, Y) \sim \text{Unif}(\{(0; 0); (0; 2); (1; 2)\})$, find the correlation between X and Y.

#15.2 For (X, Y) with density $f_{X,Y}(x, y) = 2 \exp(-x - 2y)$ for $x, y \geq 0$, find $\rho_{X,Y}$.
(Hint: You can solve this problem without any integration.)

#15.3 (modified) Let $f_{X,Y}(r, s) = \frac{1}{C} \cdot \frac{r^4 + s}{r^2}$ for $r \in [1, 3], s \in [0, 3]$.

- Find the value of C.
- Compute $P(X > 2)$, $P(Y > 1)$ and $P(X > 2 \text{ and } Y > 1)$. Are X, Y independent?
- Find the density of Y, $f_Y(s)$.
- Find $\rho_{X,Y}$.

A. (We began this problem in class on 10/19.) Flip 3 fair coins, A,B,C.

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- Write out the table of the joint pmf $p_{X,Y}(x,y)$.



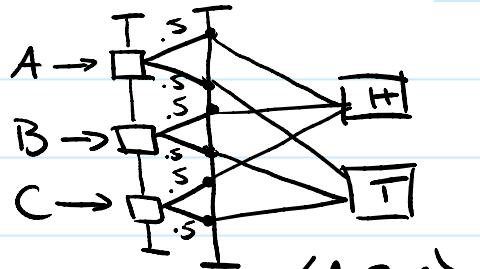
Let X be the total numbers of heads on coins A and B. Let Y be the total number of heads on coins B and C.

(a) Write out the table of the joint pmf $p_{X,Y}(x,y)$.

$$X = |A_{\text{heads}}| + |B_{\text{heads}}| \quad Y = |B_{\text{heads}}| + |C_{\text{heads}}|$$

$$2^3 = 8 \text{ outcomes}$$

X	0	1	2
0	$\frac{1}{8}$	$\frac{1}{8}$	0
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$



- (X,Y) Both 0 - $\langle 0,0,0 \rangle \rightarrow \langle A,B,C \rangle$
- Both 2 - $\langle 1,1,1 \rangle$
- $\langle 0,1 \rangle - \langle 0,0,1 \rangle$
- $\langle 1,0 \rangle - \langle 1,0,0 \rangle$
- $\langle 1,1 \rangle - \langle 1,0,1 \rangle \text{ or } \langle 0,1,0 \rangle$
- $\langle 0,2 \rangle \text{ or } \langle 2,0 \rangle - N/A$
- $\langle 1,2 \rangle - \langle 0,1,1 \rangle$
- $\langle 2,1 \rangle - \langle 1,1,0 \rangle$

(b) Compute σ_x^2, σ_y^2 .

$$\sigma_y^2 = M_{yy}^2 - M_y^2$$

$$M_x = M_y = 0(2/8) + 1(4/8) + 2(2/8) = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

$$M_{xx} = M_{yy} = 0^2(2/8) + 1^2(4/8) + 2^2(2/8) = 0 + \frac{1}{2} + 1 = 1\frac{1}{2}$$

$$\sigma_x^2 = \sigma_y^2 = 1\frac{1}{2} - 1 = \frac{1}{2}$$

By Symmetry :)

(c) Compute $\text{Cov}(X, Y)$ and $\rho_{X,Y}$.

$$\text{Cov}(X, Y) = M_{xy} - M_x M_y$$

$$M_{xy} = 0(3/8) + 1(2/8) + 2(2/8) + 4(1/8)$$

$$= 0 + \frac{2}{8} + \frac{4}{8} + \frac{4}{8} = \frac{10}{8} \quad M_x = M_y = 1$$

$$M_x M_y = 1 \therefore M_{xy} - M_x M_y = \frac{10}{8} - \frac{8}{8} = \frac{2}{8}$$

$$\text{Cov}(X, Y) = \frac{1}{4}$$

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y}$$

$$\sigma_x^2 = \sigma_y^2 = \frac{1}{2} \therefore \sigma_x = \sigma_y = \sqrt{\frac{1}{2}}$$

$$\therefore \sigma_x \sigma_y = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} = \frac{1}{2} \therefore \rho_{X,Y} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \times \frac{2}{1}$$

$$\text{so } \partial_x \partial_y = 1/2 - 2 \text{ so } P_{x,y} = \frac{1}{\frac{1}{2}} = \frac{1}{4} \times 1$$

LAST $\Rightarrow P_{x,y} = \frac{1}{2}$

p.99 #14.4 Let Y have pmf $p_Y(i) = \begin{cases} 1/4 & \text{for } i \in \{3, 5\} \\ 1/6 & \text{for } i \in \{7, 9, 11\} \end{cases}$.

(a) Find $E[Y]$.

$$M_Y = \frac{3+5}{4} + \frac{7+9+11}{6} = \frac{8}{4} + \frac{27}{6} = \frac{24}{12} + \frac{54}{12} = \frac{78}{12} = \boxed{\frac{13}{2}}$$

(b) Find σ_Y .

$$\sigma_Y = \sqrt{M_Y^2 - M_Y^2} \quad M_Y^2 = \frac{169}{4}$$

$$M_Y^2 = \frac{3^2+5^2}{4} + \frac{7^2+9^2+11^2}{6} = \frac{9+25}{4} + \frac{49+81+121}{6}$$

$$= \frac{34}{4} + \frac{251}{6} = \frac{151}{3}$$

$$\sigma_Y^2 = \frac{151}{3} - \frac{169}{4} = \frac{97}{12}$$

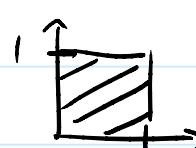
$\therefore \sigma_Y = \sqrt{97/12}$

#14.8 (modified)

Suppose (X, Y) have joint density $f_{X,Y}(x, y) = (2/5)(3x + 2y)$ for $(x, y) \in [0, 1]^2$

(a) Compute $P(X > .5 \text{ and } Y < .6)$.

$$f_{X,Y}(x, y) = \frac{2}{5}[3x + 2y] \text{ on }$$



Input:	$\frac{2}{5} \int_{0.5}^1 \int_0^{0.6} (3x + 2y) dy dx$
Result:	$0.342 \quad \boxed{\frac{342}{1000}}$

$$P(X > \frac{1}{2} \text{ and } Y < \frac{3}{5}) = \frac{2}{5} \int_{0.5}^1 \int_0^{0.6} 3x + 2y \, dy \, dx$$

$X \in [0.5, 1]$
 $Y \in [0, 0.6]$

$$\frac{2}{5} \int_0^{0.6} \left[\frac{3x^2}{2} + 2xy \Big|_0^1 \right] dy = \frac{2}{5} \int_0^{0.6} \left[\frac{3}{2} + 2y - \frac{3}{8} - y \right] dy$$

$$= \frac{2}{5} \int_0^{0.6} y + \frac{9}{8} dy = \frac{2}{5} \left[\frac{y^2}{2} + \frac{9y}{8} \Big|_0^{0.6} \right]$$

$$= \frac{2}{5} \left[\frac{9}{50} + \frac{27}{40} - 0 - 0 \right] = \frac{2}{5} \left[\frac{171}{200} \right] = \boxed{\frac{171}{500}}$$

(b) Find the marginal pdf's $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \frac{2}{5} \int_0^1 3x + 2y \, dy = \frac{2}{5} \left[3xy + y^2 \Big|_0^1 \right] = \frac{2}{5} [3x + 1]$$

$$f_Y(y) = \frac{2}{5} \int_0^1 3x + 2y \, dx = \frac{2}{5} \left[\frac{3x^2}{2} + 2xy \Big|_0^1 \right] = \frac{2}{5} \left[2y + \frac{3}{2} \right]$$

(b) What is $\text{Cov}(X, Y)$?

$$\text{Cov}(x, y) = M_{xy} - M_x M_y$$

$$M_x = \frac{2}{5} \int_0^1 3x^2 + x \, dx = \frac{2}{5} \left[x^3 + \frac{x^2}{2} \Big|_0^1 \right] = \frac{2}{5} \left[1 + \frac{1}{2} \right] = \frac{2}{5} \times \frac{3}{2} = \frac{3}{5}$$

$$M_y = \frac{2}{5} \int_0^1 2y^2 + \frac{3y^2}{2} \, dy = \frac{2}{5} \left[\frac{2y^3}{3} + \frac{3y^2}{4} \Big|_0^1 \right] = \frac{2}{5} \left[\frac{2}{3} + \frac{3}{4} \right] = \frac{2}{5} \times \frac{17}{12} = \frac{17}{30}$$

$$M_{xy} = \frac{2}{5} \iint_0^1 3x^2 y + 2y^2 x \, dx \, dy = \frac{2}{5} \int_0^1 \left[x^3 y + y^2 x^2 \Big|_0^1 \right] dy$$

$$= \frac{2}{5} \int_0^1 y + y^2 \, dy = \frac{2}{5} \left[y^2/2 + y^3/3 \Big|_0^1 \right] = \frac{2}{5} \left[\frac{1}{2} + \frac{1}{3} \right] = \frac{2}{5} \times \frac{5}{6} = \frac{1}{3}$$

$$\text{Cov}(x, y) = \frac{1}{3} - \left[\frac{3}{5} \times \frac{17}{30} \right] = \frac{1}{3} - \frac{17}{50} = \boxed{-\frac{1}{50}}$$

Input:

$$\frac{2}{5} \int_0^1 \int_0^1 (3x^2 y + 2y^2 x) \, dy \, dx - \frac{2}{5} \left(\int_0^1 (3x^2 + x) \, dx \right) \times \frac{2}{5} \int_0^1 \left(2y^2 + 3 \times \frac{y}{2} \right) \, dy$$

Result:

$$-\frac{1}{150} \approx -0.00666667$$

p.104 #15.1 For $(X, Y) \sim \text{Unif}(\{(0, 0); (0, 2); (1, 2)\})$, find the correlation between X and Y .

$$\frac{1}{3} \langle 0, 0 \rangle + \frac{1}{3} \langle 0, 2 \rangle + \frac{1}{3} \langle 1, 2 \rangle$$

$x \backslash y$	0	1	$P_Y(y)$
0	$1/3$	0	$1/3$
2	$1/3$	$1/3$	$2/3$
$P_X(x)$	$2/3$	$1/3$	

$$M_x = 0(2/3) + 1(1/3) = 1/3$$

$$M_y = 0(1/3) + 2(2/3) = 4/3$$

$$M_x^2 = 1/9 \quad M_y^2 = 16/9$$

$$M_{xy} = 0(2/3) + 2(1/3) = 1/3$$

$$M_z = -2(2/2) + 1^2(1/2) = 1/3$$

$$P(x) | 2/3 \quad 1/3$$

$$\mu_x^2 = 0^2(2/3) + 1^2(1/3) = 1/3$$

$$\text{Cov}(x, y) = \mu_{xy} - \mu_x \mu_y$$

$$\mu_y^2 = 0^2(1/3) + 2^2(2/3) = 8/3$$

$$\sigma_x^2 = \mu_2 - \mu_x^2$$

$$\mu_x \mu_y = \frac{1}{3} \times \frac{4}{3} = \frac{4}{9}$$

$$\text{Cov}(x, y) = \frac{3}{9} - \frac{4}{9} = -\frac{1}{9}$$

$$\rho_{x,y} = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y}$$

$$\sigma_x^2 = \frac{3}{9} - \frac{1}{9} = \frac{2}{9} \quad ; \quad \sigma_x = \frac{\sqrt{2}}{3}$$

$$= \frac{-\frac{1}{9}}{\frac{\sqrt{2}}{3} \times \frac{2\sqrt{2}}{3}} = \frac{-\frac{1}{9}}{\frac{4}{9}}$$

$$\sigma_y^2 = \frac{24}{9} - \frac{16}{9} = \frac{8}{9} \quad ; \quad \sigma_y = \frac{2\sqrt{2}}{3}$$

$$= -\frac{1}{9} \times \frac{9}{4} = \boxed{-\frac{1}{4}}$$

#15.2 For (X, Y) with density $f_{X,Y}(x, y) = 2 \exp(-x - 2y)$ for $x, y \geq 0$, find $\rho_{X,Y}$.
(Hint: You can solve this problem without any integration.)

$$f_{x,y} = 2e^{-x} e^{-2y} \quad ; \quad x, y \geq 0$$

$$f_x = 2e^{-x} \int_0^\infty 2e^{-2y} dy = 2e^{-x} \left[-\frac{e^{-2y}}{2} \right]_0^\infty = 2e^{-x} \left[\frac{1}{2} \right] = e^{-x}$$

$$f_y = 2e^{-2y} \int_0^\infty 2e^{-x} dx = 2e^{-2y} \left[-e^{-x} \right]_0^\infty = 2e^{-2y} [1] = 2e^{-2y}$$

* We know $X \perp\!\!\!\perp Y$ are independent \therefore

$X \perp\!\!\!\perp Y$ are uncorrelated $\therefore \boxed{\rho_{x,y} = 0}$

#15.3 (modified) Let $f_{X,Y}(r, s) = \frac{1}{C} \cdot \frac{r^4 + s}{r^2}$ for $r \in [1, 3], s \in [0, 3]$.

(a) Find the value of C .

$$1 = \frac{1}{C} \iint_{\mathbb{R}^2} \frac{r^4 + s}{r^2} ds dr = \frac{1}{C} \iint_{\mathbb{R}^2} r^2 + s r^{-2} ds dr$$

$$C = \iint_{\mathbb{R}^2} r^2 + s r^{-2} ds dr = \int_0^3 \left[r^3 + \frac{s^2 r^{-2}}{2} \right]_0^3 dr$$

$$= \int_0^3 \left[3r^2 + \frac{9r^{-2}}{2} \right] dr = \left[r^3 - \frac{9r^{-1}}{2} \right]_1^3$$

$$= 3 - \frac{9}{3+2} - 1 + \frac{9}{1+2} = 27 - \frac{3}{2} - 1 + \frac{9}{2} = 26 + \frac{6}{2} = \boxed{29}$$

Definite integral:

$$\int_1^3 \int_0^3 \left(r^2 + \frac{s}{r^2} \right) ds dr = 29$$

$x(s) \neq y(s)$

(b) Compute $P(X > 2)$, $P(Y > 1)$ and $P(X > 2 \text{ and } Y > 1)$. Are X, Y independent?

$$f_{x,y} = \frac{1}{29} [s^2 + s s^{-2}]$$

$$f_x = \frac{1}{29} \int_s^{\infty} [s^2 + s s^{-2}] ds = \frac{1}{29} \left[s^3 + \frac{s^2}{2} \right]_0^{\infty}$$

$$= \frac{1}{29} \left[3s^2 + \frac{9s^{-2}}{2} \right]$$

Definite integral:

$$\frac{1}{29} \int_0^3 \left(r^2 + \frac{s}{r^2} \right) ds = \frac{6r^4 + 9}{58r^2}$$

$$f_y = \frac{1}{29} \int_s^{\infty} [s^2 + s s^{-2}] ds = \frac{1}{29} \left[s^3/3 - s s^{-1} \right]_1^{\infty}$$

$$= \frac{1}{29} \left[\frac{27}{3} - \frac{1}{3} - \frac{5}{3} + \frac{5}{1} \right] = \frac{1}{29} \left[\frac{26}{3} + \frac{28}{3} \right]$$

Definite integral:

$$\frac{1}{29} \int_1^3 \left(r^2 + \frac{s}{r^2} \right) dr = \frac{2(s+13)}{87}$$

$$= \frac{1}{87} [26 + 28]$$

$P(X > 2)$:

Definite integral:

$$\frac{1}{29} \int_2^3 \left(\frac{9}{2r^2} + 2r^2 \right) dr = \frac{161}{348} \approx 0.46264$$

$P(Y > 1)$:

Definite integral:

$$\frac{1}{87} \int_1^3 (26 + 2s) ds = \frac{20}{29} \approx 0.68966$$

Computation result:

$P(X > 2 \text{ and } Y > 1)$:

$$\frac{1}{29} \int_1^3 \int_2^3 \left(r^2 + \frac{s}{r^2} \right) dr ds = \frac{40}{87}$$

$$\frac{161}{348} \times \frac{20}{29} = \frac{805}{2784} \neq \frac{40}{87} \quad \begin{matrix} P(X > 2) \times P(Y > 1) \\ \neq P(X > 2 \text{ and } Y > 1) \end{matrix}$$

X, Y are **DEPENDENT**

(b) Find the density of Y , $f_Y(s)$.

f_Y

(c) Find $\rho_{X,Y}$.

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\mu_X =$$

$$\text{Definite integral: } \frac{1}{29} \int_1^3 \left(\frac{9}{2r} + 3r^3 \right) dr = \frac{1}{29} \times \left(60 + \frac{9\log(3)}{2} \right) \approx 2.2394$$

$$\mu_X^2 =$$

$$\text{Definite integral: } \frac{1}{29} \int_1^3 \left(\frac{9}{2} + 3r^4 \right) dr = \frac{771}{145} \approx 5.3172$$

$$\mu_X^2 =$$

$$\text{Input: } \left(\frac{1}{29} \int_1^3 \left(3r^3 + 9 \times \frac{1}{r \times 2} \right) dr \right)^2$$

$$\text{Result: } \frac{1}{841} \left(60 + \frac{9\log(3)}{2} \right)^2 \approx 5.01509$$

$$M_y =$$

Definite integral:
 $\frac{1}{87} \int_0^3 (26s + 2s^2) ds = \frac{45}{29} \approx 1.5517$

$$M_{y^2} =$$

Definite integral:
 $\frac{1}{87} \int_0^3 (26s^2 + 2s^3) ds = \frac{183}{58} \approx 3.1552$

$$M_y^2 =$$

Input:
 $\left(\frac{1}{87} \int_0^3 (26s + 2s^2) ds \right)^2$

Result:
 $\frac{2025}{841} \approx 2.40785$

$$M_{xy} =$$

Input:

$$\frac{1}{29} \int_0^3 \int_1^3 \left(r^3 s + \frac{s^2}{r} \right) dr ds$$

Result:

$$\frac{9}{29} \times (10 + \log(3)) \approx 3.4444$$

$$M_x M_y =$$

Input:
 $\left(\frac{1}{29} \int_1^3 \left(3r^3 + 9 \times \frac{1}{r \times 2} \right) dr \right) \left(\frac{1}{87} \int_0^3 (26s + 2s^2) ds \right)$

Result:
 $\frac{45}{841} \times \left(60 + \frac{9 \log(3)}{2} \right) \approx 3.47499$

$$\text{Cov}(x, y) = M_{xy} - M_x M_y = 3.4444 - 3.47499 \\ = -0.03055$$

$$\sigma_x^2 = M_{x^2} - M_x^2 = 5.3172 - 5.01509 = 0.30211$$

$$\sigma_y^2 = M_{y^2} - M_y^2 = 3.1552 - 2.40785 = 0.74735$$

$$\rho_{x,y} = \frac{\text{Cov}(x,y)}{\sigma_x \sigma_y} = \frac{-0.03055}{\sqrt{0.30211 \cdot 0.74735}} \\ = -0.06429$$

