Final Exam

Thursday, December 9, 2021 4:18 PM

MTH 372-W1
Final exam
Fall 2021

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Please read these instructions carefully.

1. You are permitted to use your textbook, anything I have posted on Blackboard, any notes you have taken during class, your homework assignments, and your own calculator. You are not permitted to use the internet in any other way, or ask any person other than me for help. I will give you hints, if you ask by email.

By signing your name when you return this exam, you are agreeing to abide by these rules.

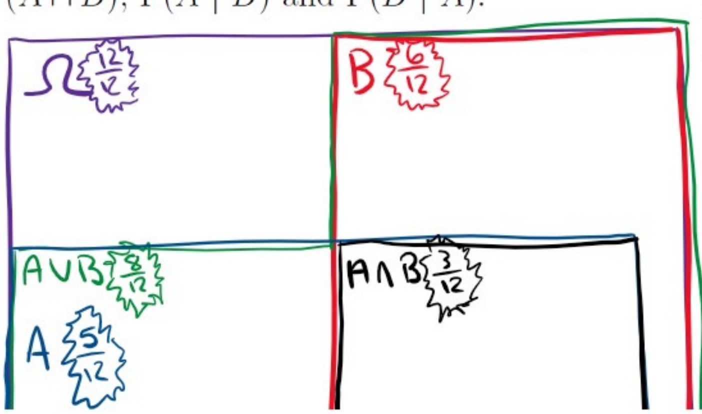
2. You must show all appropriate work on each problem for credit. Correct work, even when the final answer is wrong, will earn substantial partial credit. Unjustified answers will earn little or no credit.

3. The exam consists of the 8 problems on this page and the next. There isn't space on these pages for the necessary work; therefore I will not grade anything written on these pages. Do the exam on other paper.

4. This exam is due by midnight, Tuesday, Dec 14, 2021. That does not mean that I want you to spend all day working on it. The exam should take no longer than about 2 hours. Please use Blackboard to submit your solutions.

5. If you have questions or see what appears to be an error on the exam, please let me know right away.

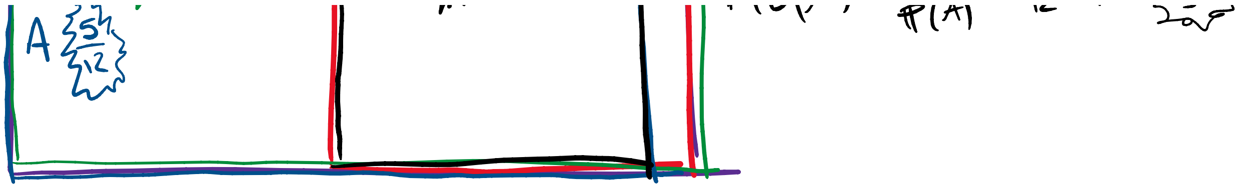
#1. . Given that $P(A) = 5/12$, $P(B) = 1/2$, and $\mathrm{P}(A \cup B) = 3/4$, evaluate $P(A \cap B)$, $\mathrm{P}(A \mid B)$ and $\mathrm{P}(B \mid A)$.



$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = \frac{5+6-8}{12} = \frac{3}{12}$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{3}{12} \div \frac{6}{12} = \frac{3}{6}$$

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{3}{12} \div \frac{5}{12} = \frac{3}{5}$$



#2. The random variable X has pdf $f_X(x) = \begin{cases} 8/x^3 & \text{for } x \geq b \\ 0 & \text{for } x < b \end{cases}$.

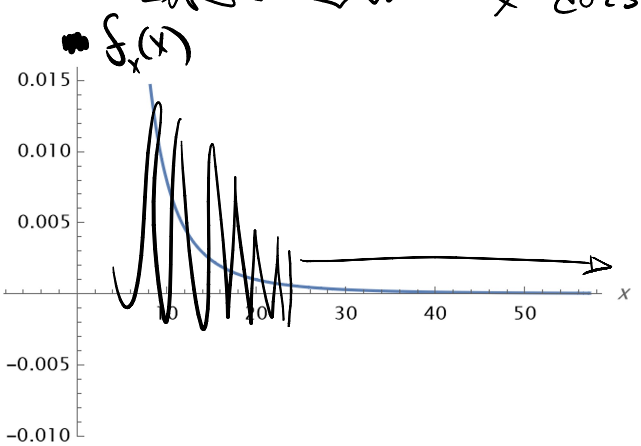
(a) What is the value of b?
(b) Evaluate $E(X)$.
(c) Evaluate $V(X)$, or explain why it doesn't exist.

$$\int_b^\infty \frac{8}{x^3}\,dx = 1 = \frac{-24}{x^2}\Big|_b^\infty = 0 + \frac{24}{b} = 1 \quad \therefore \boxed{b = 1}$$

$$\mathbb{E}[X] = \int_{24}^\infty \frac{8}{x^2}\,dx = \frac{-16}{x}\Big|_{24}^\infty = 0 + \frac{16}{24} = \boxed{\frac{16}{24}}$$

$$\mathbb{E}[x^2] = \int_{24}^\infty \frac{8}{x}\,dx = 8\log(x)\Big|_{24}^\infty = \infty - 8\log(24)$$

$\mathbb{V}(X)$ does not exist as the expected value of x^2 does not converge...



#3. A pair of (X, Y) random variables has joint pmf

	Y=−1	Y=0	Y=1
$X \backslash Y$	-1	0	1
x=1	$1/5$	$1/5$	$2/5$
x=2	$1/10$	0	$1/10$

(In case it's not clear: X takes the values 1,2, and Y takes the values $-1, 0, 1$.) Thanks ☺
a) Evaluate the marginal pmf's p_X, and p_Y, and $E(X)$, and $E(Y)$.
b) Evaluate the conditional expected values $E(Y|X=1)$ and $E(Y|X=2)$, and $E(E(Y|X))$.

p_X : 1 | $\frac{1}{5} + \frac{1}{5} + \frac{2}{5} = 8/10$

p_Y:

−1	0	1
1/5 + 1/10	1/5 + 0	2/5 + 1/10

$\mu_X = (1 \times \frac{8}{10}) + (2 \times \frac{2}{10})$
$= \frac{8}{10} + \frac{4}{10} = \boxed{\frac{12}{10}}$

p_X :

1	$\frac{1}{5}+\frac{1}{5}+\frac{2}{5}$	$= 8/10$
2	$\frac{1}{10}+0+\frac{1}{10}$	$= 2/10$

p_Y :

-1	0	1
$1/5 + 1/10$	$1/5 + 0$	$2/5 + 1/10$
$3/10$	$2/10$	$5/10$

$\mu_X = (1\times\frac{8}{10}) + (2\times\frac{2}{10}) = \frac{8}{10}+\frac{4}{10} = \frac{12}{10}$

$\mu_Y = (-1\times\frac{3}{10}) + (0\times\frac{2}{10}) + (1\times\frac{5}{10})$

$\frac{-3}{10}+\frac{0}{10}+\frac{5}{10} = \frac{2}{10}$

$\mu_{y|x=1} : (-1\times\frac{1}{5}) + (0\times\frac{1}{5}) + (1\times\frac{2}{5}) = -\frac{1}{5}+\frac{0}{5}+\frac{2}{5} = \frac{2}{10}$

$\mu_{y|x=2} : (-1\times\frac{1}{10}) + (0\times\frac{0}{10}) + (1\times\frac{1}{10}) = \frac{-1}{10}+\frac{0}{10}+\frac{1}{10} = \frac{0}{10}$

$$\mu_{\mu_{y|x}} = \left(\mu_{y|x=1} \times \mathbb{P}(x=1)\right) + \left(\mu_{y|x=2} \times \mathbb{P}(x=2)\right)$$

$$= \left(\left((-1\times\tfrac{1}{5}) + (0\times\tfrac{1}{5}) + (1\times\tfrac{2}{5})\right)\times\left(\tfrac{8}{10}\right)\right) + \left(\left((-1\times\tfrac{1}{10}) + (0\times\tfrac{0}{10}) + (1\times\tfrac{1}{10})\right)\times\left(\tfrac{2}{10}\right)\right)$$

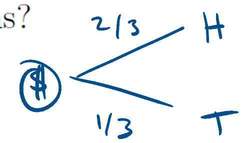
$$= \left(\left(\frac{-1+0+2}{5}\right)\times\left(\frac{8}{10}\right)\right) + \left(\left(\frac{-1+0+1}{10}\right)\times\left(\frac{2}{10}\right)\right) = \left(\frac{2}{10}\times\frac{8}{10}\right) + \left(\frac{0}{10}\times\frac{2}{10}\right) = \frac{16}{100}$$

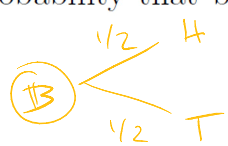
#4. An urn contains 2 fair coins, and 3 biased coins that land heads with probability 2/3.

You choose 2 of these 5 coins at random, without replacement.

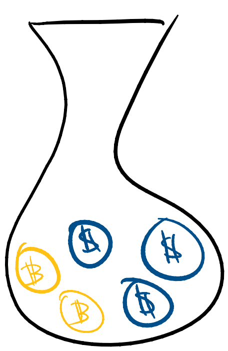
You toss the 2 chosen coins, and observe that 2 heads come up.

What is the (conditional) probability that both of the coins chosen from the urn were biased coins?





it is given we have 2 heads flipped



$\to$ HH : $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ $p(\;) = \frac{2}{20}$

$\to$ HH : $\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$ $p(\;) = \frac{12}{20}$

$\to$ HH : $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$ $p(\;) = \frac{6}{20}$

$$\mathbb{P}(\;|\,HH) = \frac{\frac{3}{10}\times\frac{4}{9}}{\left(\frac{3}{10}\times\frac{4}{9}\right) + \left(\frac{1}{10}\times\frac{1}{4}\right) + \left(\frac{6}{10}\times\frac{1}{3}\right)} = \frac{16}{43}$$

#5. The pair (X, Y) of random variables is distributed uniformly on the set of (x, y) such that $0 \le x \le 3$, $0 \le y \le x^2$.

a) Sketch the domain of $f_{X,Y}$. What is the joint pdf?

b) Evaluate the marginal pdf f_Y, and compute $E(Y)$.

c) Evaluate the conditional expected value $E(X|Y = 2)$.

$f_x = \frac{1}{3-0} = \frac{1}{3}$

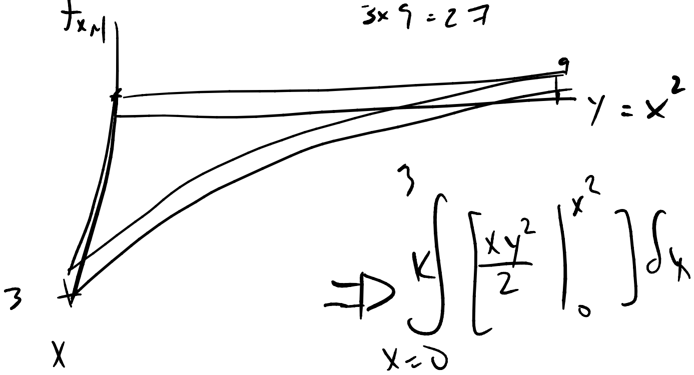
a) Sketch the domain of $f_{X,Y}$. What is the joint pdf?

b) Evaluate the marginal pdf f_Y, and compute $E(Y)$.

c) Evaluate the conditional expected value $E(X|Y=2)$.

$J_x = 3-0 = 3$

$3\times 9 = 27$



$$K\int_{x=0}^{3}\int_{0}^{x^2} x\cdot y\, dy\, dx = 1$$

$$\Rightarrow K\int_{x=0}^{3}\left[\frac{xy^2}{2}\Big|_0^{x^2}\right]dx = 1 \;;\; K\int_{x=0}^{3}\frac{x^4}{2}\,dx = K\frac{x^5}{8}\Big|_0^3 = K\frac{3^5}{8} = 1$$

$$K = \frac{8}{3^5} = \frac{8}{243} \;;\; f_Y = \int_{y=0}^{x^2}\int_0^3 xy\, dx\, dy = \int_{y=0}^{x^2}\left[\frac{x^2y}{2}\Big|_0^3\right]dy$$

$$\therefore f_Y = \int_{y=0}^{x^2}\frac{9y}{2} - 0 \quad \left\{ f_Y = \int_{y=0}^{x^2}\frac{9y}{2} \right\}$$

$$f_{X|Y=2} = \int_{x=0}^{3} 2x\, dy = \left[x^2\Big|_0^3\right] = 3^2 - 0 = \{9\}$$

#6. Suppose X, Y are random variables with $V(X) = 4$, $V(Y) = 9$, and $V(X+Y) = 16$. Compute the covariance $Cov(X,Y)$ and the correlation coefficient $\rho_{X,Y}$.

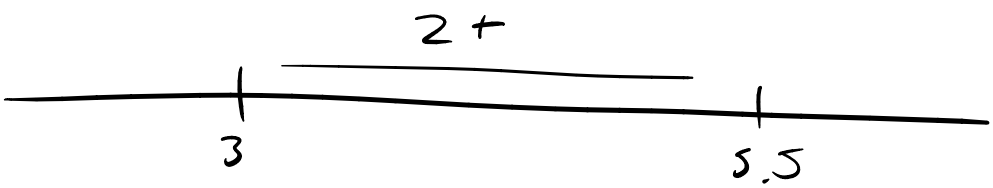
$$4 = \mu_{x^2} - \mu_x^2 \quad 9 = \mu_{y^2} - \mu_y^2 \;;\; 16 = \mu_{(x+y)^2} + \mu_{(x+y)}^2$$

$$Cov(X,Y) = \mu_{xy} - \mu_x\mu_y =$$

#7. Starting from time 0, the occurrence of potholes you see on the highway is a Poisson process with parameter $\lambda = 2$ potholes/mile.

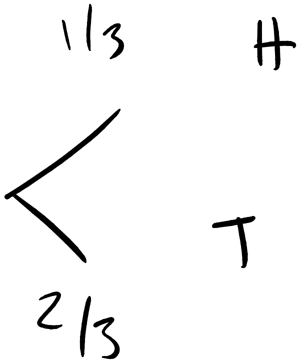
a) Between the 3 mile marker and the 3.5 mile marker, what is the probability that you see two or more potholes?

b) When you begin driving, what is the probability that you don't see a pothole within the first mile?



#8. One hundred people independently perform the following experiment: Each takes a coin that lands heads 1/3 of the time, flips the coin until the first time it lands heads, and records the number of times that he or she flipped the coin.

What is the probability that the sum of these 100 recorded numbers lies between 280 and 315?



100 people

$\int_{280}^{315}$

Gave 2 hours But ran out of time.

↳ long day at office

→ No time to study