HW 11

Thursday, December 2, 2021 3:56 PM

372hw11

MTH 372
Hw 11
Due Thursday, 12/9/2021.

Read Chapters 23 of *Huber*.

p.153 #23.2 (Modified) Suppose (X_1, X_2, X_3) has joint density

$$f(x_1, x_2, x_3) = k(x_1 + x_2 + x_3)\mathbf{1}(x_1, x_2, x_3 \in [0, 1]).$$

(a) Evaluate k.
(b) Find the joint marginal density of X_1, X_2.
(c) Find the marginal density of X_1.
(d) Find $\text{Cov}(X_1, X_3)$.

#23.4. Suppose Z_1, Z_2, Z_3 are iid standard normal random variables. Find their joint density.

A. Let Y_1, Y_2 have joint pdf $f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} ky_1(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$

(a) Determine k.
(b) Evaluate $P(Y_1 \le \frac{3}{4}, Y_2 \ge \frac{1}{2})$.
(c) Find the marginal pdf's $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$
(d) Evaluate $P(Y_2 \le \frac{1}{2} \,|Y_1 \le \frac{3}{4})$.
(e) Find the conditional density of Y_1 given $Y_2 = 2/3$.
(f) Find the conditional density of Y_2 given Y_1.
(g) Find $P(Y_2 \ge \frac{3}{4}|Y_1 = \frac{1}{2})$.

B. Let $f_Y(y) = 2(1-y)$ for $0 \le y \le 1$.
(a) Find the pdf of $T = 2Y - 1$.
(b) Find the pdf of $U = 1 - 3Y$.

C. Let X, Y be iid Exp[1]. Let $R = \dfrac{X}{Y}$ and $S = Y$.
(a) Find the joint pdf $f_{R,S}(r, s)$.
(b) Find the marginal pdf $f_R(r)$.
Hint for (b): You can avoid integration by carefully using the fact that when $Z \sim$Exp[β], we know that $E[Z] = \dfrac{1}{\beta} = \displaystyle\int_0^\infty z \cdot \beta e^{-\beta z}\, dz$.

p.153 #23.2 (Modified) Suppose (X_1, X_2, X_3) has joint density

$$f(x_1, x_2, x_3) = k(x_1 + x_2 + x_3)\mathbf{1}(x_1, x_2, x_3 \in [0, 1]).$$

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(a) Evaluate k.

$$1 = k\iiint_{x_1,x_2,x_3=0}^{1} x_1 + x_2 + x_3 \, dx_3 dx_2 dx_1 = \frac{3k}{2} \therefore \boxed{k = \frac{2}{3}}$$

```
In[20]:= Integrate[k (x1 + x2 + x3), {x1, 0, 1}, {x2, 0, 1}, {x3, 0, 1}]
```

Out[20]= $\frac{3k}{2}$

```
In[22]:= Integrate[2/3 (x1 + x2 + x3), {x1, 0, 1}, {x2, 0, 1}, {x3, 0, 1}]
```

Out[22]= 1

(b) Find the joint marginal density of X_1, X_2.

$$f_{X_1,X_2}(x_1, x_2) = \frac{2}{3}\int_{x_3=0}^{1} x_1 + x_2 + x_3 \, dx_3 = \boxed{\frac{1 + 2(x_1 + x_2)}{3}}$$

```
In[23]:= Integrate[2/3 (x1 + x2 + x3), {x3, 0, 1}]
```

Out[23]= $\frac{1}{3} + \frac{2 x1}{3} + \frac{2 x2}{3}$

(c) Find the marginal density of X_1.

$$f_{X_1}(x_1) = \frac{2}{3}\iint_{x_3,x_2=0}^{1} x_1 + x_2 + x_3 \, dx_2 dx_3 = \boxed{\frac{2(1 + x_1)}{3}}$$

```
In[24]:= Integrate[2/3 (x1 + x2 + x3), {x2, 0, 1}, {x3, 0, 1}]
```

Out[24]= $\frac{2 \times (1 + x1)}{3}$

(d) Find $\mathrm{Cov}(X_1, X_3)$.

$$\mathrm{Cov}(X_1, X_3) = E[X_1X_3] - E[X_1]E[X_3]$$

$$= \iint_{x_1,x_3=0}^{1} x_1 x_3 f_{X_1,X_3}(x_1,x_3)\, dx_1 dx_3 - \left[\int_{x_1=0}^{1} x_1 f_{X_1}(x_1)\, dx_1 \cdot \int_{x_3=0}^{1} x_3 f_{X_3}(x_3)\, dx_3\right]$$

```
In[30]:= Integrate[x1 * x3 * Integrate[2/3 (x1 + x2 + x3), {x2, 0, 1}], {x1, 0, 1}, {x3, 0, 1}]
```

Out[30]= $\frac{11}{36}$

```
In[31]:= Integrate[x1 * Integrate[2/3 (x1 + x2 + x3), {x2, 0, 1}, {x3, 0, 1}], {x1, 0, 1}]
```

Out[31]= $\frac{5}{9}$

```
In[32]:= Integrate[x3 * Integrate[2/3 (x1 + x2 + x3), {x1, 0, 1}, {x2, 0, 1}], {x3, 0, 1}]
```

Out[32]= $\frac{5}{9}$

$$\frac{11}{36} - \left(\frac{5}{9}\right)^2 = \mathrm{Cov}(X_1, X_3)$$

$$\boxed{-\frac{1}{324}} = +$$

#23.4. Suppose Z_1, Z_2, Z_3 are iid standard normal random variables. Find their joint density.

As Z_1, Z_2, Z_3 are iid ; $f_{Z_1,Z_2,Z_3}(z_1, z_2, z_3) = f_{Z_1}(z_1) f_{Z_2}(z_2) f_{Z_3}(z_3)$

Consider an : $f(z) = e^{-\frac{(z-\mu)^2}{2\sigma^2}}$

Consider an Ambiguous Z normal r.v. ; $f_Z(z) = \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$

$$\therefore \boxed{f_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) = \frac{\exp\left(\frac{-(z_1-\mu_1)^2}{2\sigma_1^2} - \frac{(z_2-\mu_2)^2}{2\sigma_2^2} - \frac{(z_3-\mu_3)^2}{2\sigma_3^2}\right)}{\sigma_1\sigma_2\sigma_3\sqrt{2\pi}^3}}$$

If we are assuming standard normal; then each $Z_1, Z_2, Z_3 \sim N(0,1)$; $\mu_1,\mu_2,\mu_3 = 0$ & $\sigma_1,\sigma_2,\sigma_3 = 1$

$$\therefore \boxed{f_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) = \frac{\exp\left(-\left(\frac{z_1^2+z_2^2+z_3^2}{8}\right)\right)}{2\pi\sqrt{2\pi}}}$$

A. Let Y_1, Y_2 have joint pdf $f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} ky_1(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$

(a) Determine k.

here ; $1 = k\int_{y_2=0}^{1}\int_{y_1=0}^{1-y_2} y_1(1-y_2)\,\delta y_1\,\delta y_2 = \frac{k}{8}$; $\boxed{k=8}$

```
In[38]:= Integrate[Integrate[k * y1 * (1 - y2), {y1, 0, 1 - y2}], {y2, 0, 1}]
Out[38]= k/8

In[40]:= Integrate[Integrate[8 * y1 * (1 - y2), {y1, 0, 1 - y2}], {y2, 0, 1}]
Out[40]= 1
```

(b) Evaluate $P(Y_1 \le \frac{3}{4}, Y_2 \ge \frac{1}{2})$.

$P(y_1 \le \frac{3}{4}, Y_2 \ge \frac{1}{2}) = P(Y_1 \le \frac{3}{4}, Y_1 \le Y_2 \le 1) - P(0 \le Y_1 \le Y_2 \le 1, \frac{1}{2} \le Y_2 \le 1)$

$= 8\int_0^{3/4}\int_{y_1}^{1} y_1(1-y_2)\,\delta y_2\,\delta y_1 - 8\int_{1/2}^{1}\int_0^{y_2} y_1(1-y_2)\,\delta y_1\,\delta y_2 = \boxed{\frac{67}{768}}$

```
In[6]:= Integrate[Integrate[8 * y1 * (1 - y2), {y2, y1, 1}], {y1, 0, 3/4}] - Integrate[Integrate[8 * y1 * (1 - y2), {y1, 0, y2}], {y2, 1/2, 1}]
Out[6]= 67/768
```

(c) Find the marginal pdf's $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$

$f_{Y_1}(y_1) = 8\int_{y_1}^{1} y_1(1-y_2)\,\delta y_2 = \boxed{4y_1 - 8y_1^2 + 4y_1^3}$

```
In[45]:= Integrate[8 * y1 * (1 - y2), {y2, y1, 1}]
Out[45]= 4 y1 - 8 y1^2 + 4 y1^3
```

$f_{Y_2}(y_2) = 8\int_0^{y_2} y_1(1-y_2)\,\delta y_1 = \boxed{4y_2^2 - y_2^3}$

In[46]:= `Integrate[8*y1*(1 - y2), {y1, 0, y2}]`

Out[46]= $4\times(1-y2)\,y2^2$

(d) Evaluate $P(Y_2 \le \frac{1}{2} \,|Y_1 \le \frac{3}{4})$. $= \dfrac{P(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4})}{P(Y_1 \le \frac{3}{4})}$

$$\frac{8\int_0^{\frac{3}{4}}\int_{y_1}^{\frac{1}{2}} y_1(1-y_2)\,\delta y_2\,\delta y_1}{\int_0^{\frac{3}{4}} 4y_1 - 8y_1^2 + 4y_1^3\,\delta y_1} = \boxed{\frac{1}{9}}$$

In[15]:= $\dfrac{\texttt{Integrate}\left[\texttt{Integrate}\left[8*y1*(1-y2), \{y2, y1, \frac{1}{2}\}\right], \{y1, 0, \frac{3}{4}\}\right]}{\texttt{Integrate}\left[4\,y1 - 8\,y1\text{^}2 + 4\,y1\text{^}3, \{y1, 0, \frac{3}{4}\}\right]}$

Out[15]= $\frac{1}{9}$

(e) Find the conditional density of Y_1 given $Y_2 = 2/3$.

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)} = \frac{8y_1(1-y_2)}{4y_2^2 - y_2^3}$$

$$f_{Y_1|Y_2}(y_1|y_2 = \tfrac{2}{3}) = \frac{8y_1(\frac{1}{3})}{4(\frac{4}{9}) - (\frac{8}{27})} = \boxed{\frac{72}{11}y_1}$$

$; 0 \le y_1 \le y_2 \le 1$

0 else

(f) Find the conditional density of Y_2 given Y_1.

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_1}(y_1)} = \boxed{\frac{8y_1(1-y_2)}{4y_1 - 8y_1^2 + 4y_1^3}}$$

(g) Find $P(Y_2 \ge \frac{3}{4}|Y_1 = \frac{1}{2})$.

$$f_{Y_2|Y_1}(y_2|y_1 = \tfrac{1}{2}) = \frac{8(\frac{1}{2})(1-y_2)}{4(\frac{1}{2}) - 8(\frac{1}{4}) + 4(\frac{1}{8})} = 8(1-y_2)$$

$$8\int_{\frac{3}{4}}^{1} 1 - y_2\,\delta y_2 = \boxed{\frac{1}{4}}$$

In[16]:= $\texttt{Integrate}\left[8\times(1-y2), \{y2, \frac{3}{4}, 1\}\right]$

Out[16]= $\frac{1}{4}$

B. Let $f_Y(y) = 2(1-y)$ for $0 \le y \le 1$.

(a) Find the pdf of $T = 2Y - 1$. $\dfrac{T+1}{2} = Y$

$$f_T(t) = f_Y(y(t))\,Y_t$$
$$= 2\left(1 - \frac{t+1}{2}\right)\cdot\frac{1}{2} = \boxed{1 - \frac{t+1}{2}}$$

(b) Find the pdf of $U = 1 - 3Y$. $Y = \dfrac{-(u-1)}{3} = \dfrac{1-u}{3}$

$$f_U(u) = f_Y(y(u))\,Y_u$$
$$= \left(1 - 3\left(\frac{1-u}{3}\right)\right)\cdot -\frac{1}{3} = -\frac{1}{3}(1 + u - 1) = \boxed{\frac{-u}{3}}$$

C. Let X, Y be iid Exp[1]. Let $R = \frac{X}{Y}$ and $S = Y$.

(a) Find the joint pdf $f_{R,S}(r, s)$.

if $A \sim Exp(\beta)$ then $f_A(\alpha) = \beta e^{-\beta\alpha}$

$\therefore f_X(x) = e^{-x}$ & $f_Y(y) = e^{-y}$; As iid ; $f_{X,Y}(x,y) = e^{-x-y}$

$RY = X = RS$ as $S = Y$

$$J = \begin{vmatrix} X_r & X_s \\ Y_r & Y_s \end{vmatrix} = \begin{vmatrix} S & R \\ 0 & 1 \end{vmatrix} = S - 0 = S$$

$$f_{R,S}(r,s) = f_{X,Y}(x(r,s), y(r,s))\, J$$

$$= e^{-rs-s} S = \boxed{S e^{-S(r+1)}}$$

(b) Find the marginal pdf $f_R(r)$.

Hint for (b): You can avoid integration by carefully using the fact that when $Z \sim$Exp$[\beta]$, we know that $E[Z] = \frac{1}{\beta} = \int_0^\infty z \cdot \beta e^{-\beta z}\, dz$.

$$f_R(r) = \int_0^\infty S e^{-S(r+1)}\, \delta s = \boxed{\frac{1}{r+1}} = \mathbb{E}[S]$$

letting $\beta = r+1$