HW 6

Sunday, September 5, 2021 2:15 AM

372hw6

MTH 372
Hw 6
Due Thursday, 10/28/2021.

Read Chapters 12,13 of *Huber*.

p.83 #12.2 Suppose I choose $N \sim \text{Unif}(\{1,2\})$. Then I roll N dice independently and identically distributed as $\text{Unif}(\{1,2,3,4,5,6\})$ and sum them to get S. That is, $[S|N=1] = X_1$, $[S|N=2] = X_1 + X_2$. Or more compactly, $[S|N] = \sum_{i=1}^{N} X_i$.

(a) What is the probability $S = 4$?
(b) What is the probability $S = 7$?
(c) Find the pmf of S, $p_S(i)$ for $i \in \{1,2,\ldots,12\}$.
(d) Find $E[S]$ from $p_S(i)$.
(e) Find $E[S]$ from the Fundamental Theorem of Probability.

#12.4 Lisa and Bart go spelunking in a cave, and unfortunately, soon get lost. Each time they try to find the exit, they have a 20% chance of finding the exit in an hour, a 45% of returning back to where they started after an hour, and a 35% of returning back to where they started after three hours.

(a) What is the chance that they find their way out after exactly four hours?
(b) What is the chance that they find their way out after exactly eight hours?
(c) What is the expected amount of time they spend in the cave? (Hint: Use the "Fundamental Theorem".)

#12.6 The probability p of success for an experiment is modeled as uniform over $[0.4, 0.5]$. Then 27 independent trials of the experiment are run. What is the expected number of successes?

p.90 #13.2 (modified) Suppose (X,Y) has pmf $p_{X,Y}(x,y) = (x^3+y^2)/150$ for $X \in \{1,2,3\}$ and $Y \in \{1,2,3\}$.

(a) Find the marginal pmf's $p_X(x)$ and $p_Y(y)$.
(b) Find E[XY].
(c) True or false?: $E(XY) = E(X) \cdot E(Y)$.

13.3 (modified) Suppose (X,Y) has joint pmf

$$p_{X,Y}(x,y) = \frac{x}{9\sqrt{y}} \mathbf{1}(x \in \{1,2,3\}) \mathbf{1}(y \in \{1,4\})$$

(a) Prove that X and Y are independent.
(b) What is $P(X=2)$? What is $P(Y=4)$?

13.K Let $X \sim \text{Unif}(\{1,2,3,4\})$ and $Y \sim \text{Unif}(\{1,..,X\})$.
(In words, X is chosen uniformly from 1,2,3,4, then Y is chosen uniformly from the numbers from 1 up to Y).

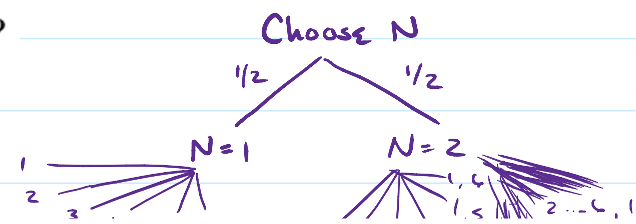
(a) Write in table form the joint pmf $p_{X,Y}(x,y)$.
(b) Find the marginal pmf $p_Y(y)$, and use it to compute $E(Y)$.
(c) Find $E(Y)$ by another method, using the "Fundamental Theorem of Probablity."

p.83 #12.2 Suppose I choose $N \sim \text{Unif}(\{1,2\})$. Then I roll N dice independently and identically distributed as $\text{Unif}(\{1,2,3,4,5,6\})$ and sum them to get S. That is, $[S|N=1] = X_1$, $[S|N=2] = X_1 + X_2$. Or more compactly, $[S|N] = \sum_{i=1}^{N} X_i$.

(a) What is the probability $S = 4$?

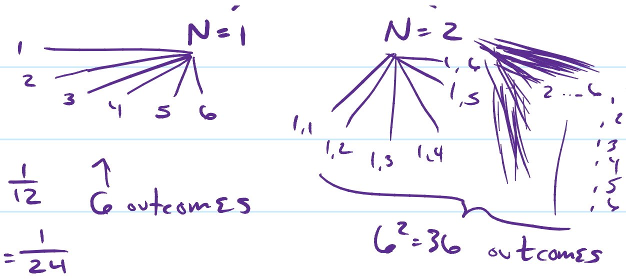
$S=4$? $N=1$; roll $=4$

$N=2$; roll $=1,3$ or $3,1$



$N=2$; roll $= 1,3$ or $3,1$

$N=2$; roll $= 2,2$



For $N=1$ we have $P_1(4) = (1/2)(1/6) = \frac{1}{12}$

For $N=2$ we have $P_2(4) = (1/2)(3/36) = \frac{1}{24}$

$\therefore P(4) = \frac{2}{24} + \frac{1}{24} = \boxed{3/24}$

(b) What is the probability $S = 7$?

$\exists = 6$ ways to roll 7

For $N=1$; $P_1(7) = (1/2)(0) = 0$

For $N=2$; we can roll 7 as ... $\{(6,1),(5,2),(4,3),(1,6),(2,5),(3,4)\}$

Thus $P_2(7) = (1/2)(6/36) = \frac{1}{12}$

$\therefore P(7) = 0 + \frac{1}{12} = \boxed{\frac{1}{12}}$

* Note that $1/12 \neq 6/42$ due to weighted branches *

(c) Find the pmf of S, $p_S(i)$ for $i \in \{1, 2, \ldots, 12\}$.

$$P(i) = P_1(i) + P_2(i)$$

$$P_1(i) = 1/2 \left\{ 1/6 \mid i \in \{1,2,3,4,5,6\} \right. \qquad 1/6 = 6/36$$

$$P_2(i) = 1/2 \begin{cases} 0 & i = 1 \\ 1/36 & i \in \{2,12\} \\ 2/36 & i \in \{3,11\} \\ 3/36 & i \in \{4,10\} \\ 4/36 & i \in \{5,9\} \\ 5/36 & i \in \{6,8\} \\ 6/36 & i = 7 \end{cases}$$

$$P(i) = 1/2 \begin{cases} 6/36 & \{1\} \\ \frac{6+1}{36} = 7/36 & \{2\} \\ \frac{6+2}{36} = 8/36 & \{3\} \\ \frac{6+3}{36} = 9/36 & \{4\} \\ \frac{6+4}{36} = 10/36 & \{5\} \\ \frac{6+5}{36} = 11/36 & \{6\} \\ 6/36 & \{7\} \\ 5/36 & \{8\} \\ 4/36 & \{9\} \\ 3/36 & \{10\} \\ 2/36 & \{11\} \\ 1/36 & \{12\} \end{cases}$$

$$\boxed{P_S(i) = \begin{cases} \frac{5+i}{72} & i \in \{1,2,3,4,5,6\} \\ \frac{13-i}{72} & i \in \{7,8,9,10,11,12\} \end{cases}}$$

(d) Find $E[S]$ from $p_S(i)$.

$$E(S) = \frac{6(1)+7(2)+8(3)+9(4)+10(5)+11(6) + 6(7)+5(8)+4(9)+3(10)+2(11)+1(12)}{72}$$

$(6+14+24+36+50+66+42+40+36+30+22+12)/72$

$= 378/72 = \boxed{5.25}$

(e) Find $E[S]$ from the Fundamental Theorem of Probability.

$$E[E[S|N]] = E[S] = \frac{E[S|N=1]}{2} + \frac{E[S|N=2]}{2}$$

$$E[S|N=1] = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

$E[S|N=2] = 2E[S|N=1] = 7$

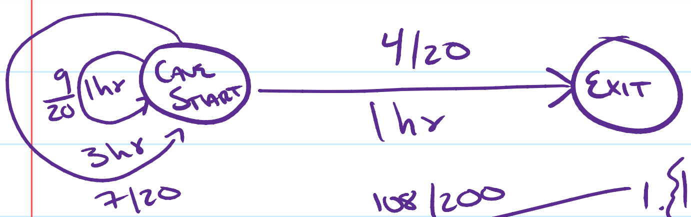
$$\mathbb{E}[S|N=1] = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

$$\mathbb{E}[S|N=2] = 2\,\mathbb{E}[S|N=1] = 7$$

$$\mathbb{E}[S] = \frac{1}{2}(3.5+7) = \frac{1}{2}(10.5) = \boxed{5.25}$$

#12.4 Lisa and Bart go spelunking in a cave, and unfortunately, soon get lost. Each time they try to find the exit, they have a 20% chance of finding the exit in an hour, a 45% of returning back to where they started after an hour, and a 35% of returning back to where they started after three hours.

(a) What is the chance that they find their way out after exactly four hours?



To Exit After Exactly 4 Hours? 2 ways

1. {1hr, 1hr, 1hr} cave {1hr} exit

2. {3hr} cave {1hr} exit

$3(9/20)(4/20) \xrightarrow{108/200}$

$(7/20)(4/20) \xrightarrow{28/200}$

$\Rightarrow$ $\boxed{\frac{136}{200} \text{ They Exit at exactly 4 hrs}}$

(b) What is the chance that they find their way out after exactly eight hours?

Similarly

1 way {1hr, 1hr, 1hr, 1hr, 1hr, 1hr, 1hr} cave {1hr} exit

5 way {1hr, 1hr, 1hr, 1hr, 3hr} cave {1hr} exit

3 way {1hr, 3hr, 3hr} cave {1hr} exit

Thus $(\frac{20}{20})7(9/20)(4/20) \longrightarrow \frac{5040}{8000}$

$5(4(9/20)(7/20)(4/20)) \longrightarrow \frac{1008}{8000}$

$3(2(7/20)(9/20)(4/20)) \longrightarrow \frac{216}{8000}$

$\Rightarrow$ $\boxed{\frac{6264}{8000} \text{ They Exit After Exactly 8 hrs}}$

(c) What is the expected amount of time they spend in the cave? (Hint: Use the "Fundamental Theorem".)

$$\mathbb{E}[\mathbb{E}[\text{Exit}|\text{Event}]] = \mathbb{E}[\text{Exit}]$$

$$= \mathbb{E}[\text{Exit}|\text{Exit}]\left(\frac{4}{20}\right) + \mathbb{E}[\text{Exit}|1\text{hr}]\left(\frac{9}{20}\right) + \mathbb{E}[\text{Exit}|3\text{hr}]\left(\frac{7}{20}\right)$$

$$\mathbb{E}[\text{Exit}|\text{Exit}] = 1$$

$$\mathbb{E}[\text{Exit}|1\text{hr}] = 1 + \mathbb{E}[\text{Exit}]$$

$$\mathbb{E}[\text{Exit}|3\text{hr}] = 3 + \mathbb{E}[\text{Exit}]$$

$$\mathbb{E}[\text{Exit}] = (1)\left(\frac{4}{20}\right) + (1+\mathbb{E}[\text{Exit}])\left(\frac{9}{20}\right) + (3+\mathbb{E}[\text{Exit}])\left(\frac{7}{20}\right)$$

$$+\left(3+\mathbb{E}[\text{Exit}]\right)\left(\tfrac{7}{20}\right)$$

$$\mathbb{E}[\text{Exit}] = \frac{4+9+21}{20} + \frac{(9+7)\mathbb{E}[\text{Exit}]}{20}$$

$$20\,\mathbb{E}[\text{Exit}] - 16\,\mathbb{E}[\text{Exit}] = 34 = 4\,\mathbb{E}[\text{Exit}]$$

$$\mathbb{E}[\text{Exit}] = \frac{34}{4} = \boxed{8.5 \text{ hrs}}$$

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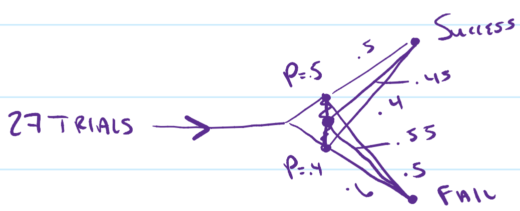


$$\mathbb{E}[S] = \frac{.5+.4}{2} = .45$$

$\mu_S \times$ # trials $\longrightarrow$ $.45 \times 27 = \boxed{12.15 \text{ SUCCESSES}}$

p.90 #13.2 (modified) Suppose (X, Y) has pmf $p_{X,Y}(x, y) = (x^3+y^2)/150$ for $X \in \{1, 2, 3\}$ and $Y \in \{1, 2, 3\}$.

(a) Find the marginal pmf's $p_X(x)$ and $p_Y(y)$.

	$x \backslash y$	1 (①)	2 (④)	3 (⑨)	$p_X(x)$
①	1	2/150	5/150	10/150	17/150
⑧	2	9/150	12/150	17/150	38/150
㉗	3	28/150	31/150	36/150	95/150
	$p_Y(y)$	39/150	48/150	63/150	

$$p_X(x) = \begin{cases} 17/150 & x=1 \\ 38/150 & x=2 \\ 95/150 & x=3 \end{cases}$$

$$p_Y(y) = \begin{cases} 39/150 & y=1 \\ 48/150 & y=2 \\ 63/150 & y=3 \end{cases}$$

$$\boxed{p_X(x)} = \sum_{y\in\{1,2,3\}} (1/150)(x^3+y^2) = (1/150)\left[(x^3+1)+(x^3+4)+(x^3+9)\right] = \boxed{\frac{3x^3+14}{150}}$$

$$\boxed{p_Y(y)} = \sum_{x\in\{1,2,3\}} (1/150)(x^3+y^2) = (1/150)\left[(1+y^2)+(8+y^2)+(27+y^2)\right] = \boxed{\frac{36+3y^2}{150}}$$

(b) Find E[XY].

$$\mathbb{E}[XY] = \sum_{x,y} xy\,p_{X,Y}(x,y) = 1\left(\frac{2}{150}\right) + 2\left(\frac{5+9}{150}\right) + 3\left(\frac{10+28}{150}\right)$$

$$\quad 4\left(\frac{12}{}\right) \quad 6\left(\frac{17+31}{}\right) \quad 9\left(\frac{36}{}\right)$$

$$+4\left(\frac{12}{150}\right)+6\left(\frac{17+31}{150}\right)+9\left(\frac{36}{150}\right)$$

$$=\frac{2+28+114+48+288+324}{150}=\frac{804}{150}=\boxed{5.36}$$

(c) True or false?: $E(XY) = E(X) \cdot E(Y)$.

$$\mathbb{E}[X]=\sum_{x} x\, p_X(x) = 1\left({}^{17}/_{150}\right)+2\left({}^{38}/_{150}\right)+3\left({}^{95}/_{150}\right)$$

$$=\frac{17+76+285}{150}=\frac{378}{150}\ldots$$

$$\mathbb{E}[Y]=\sum_{y} y\, p_Y(y) : 1\left({}^{39}/_{150}\right)+2\left({}^{31}/_{150}\right)+3\left({}^{63}/_{150}\right)$$

$$=\frac{39+62+189}{150}=\frac{290}{150}\ldots$$

$\boxed{5.36 \therefore \text{False!}}$

$$\mathbb{E}[X]\mathbb{E}[Y]=\frac{378\times 290}{150^2}=\frac{109620}{22500}=\boxed{4.872}$$

13.3 (modified) Suppose (X, Y) has joint pmf

$$p_{X,Y}(x,y) = \frac{x}{9\sqrt{y}}\mathbf{1}(x \in \{1,2,3\})\mathbf{1}(y \in \{1,4\})$$

(a) Prove that X and Y are independent.

$$p_X(x) = \frac{1}{9}\sum_{y\in\{1,4\}}\frac{x}{\sqrt{y}} = \frac{1}{9}\left[\frac{x}{1}+\frac{x}{2}\right]=\frac{1}{9}\left[\frac{3x}{2}\right]=\frac{x}{6}$$

$$p_Y(y) = \frac{1}{9}\sum_{x\in\{1,2,3\}}\frac{x}{\sqrt{y}} = \frac{1}{9}\left[\frac{1}{\sqrt{y}}+\frac{2}{\sqrt{y}}+\frac{3}{\sqrt{y}}\right]=\frac{1}{9}\left[\frac{6}{\sqrt{y}}\right]=\frac{2}{3\sqrt{y}}$$

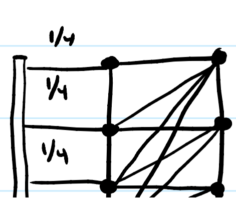
$$\boxed{p_X(x)\cdot p_Y(y)=\frac{x}{6}\cdot\frac{2}{3\sqrt{y}}=\frac{x}{9\sqrt{y}}=p_{X,Y}(x,y)\ \blacksquare}$$

(b) What is $P(X = 2)$? What is $P(Y = 4)$?

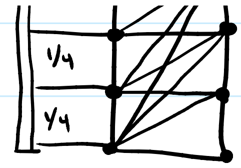
$$p_X(2)=\frac{2}{6}=\boxed{\frac{1}{3}}\ ;\ p_Y(4)=\frac{2}{3\sqrt{4}}=\boxed{\frac{1}{3}}$$

13.K Let X ~Unif($\{1, 2, 3, 4\}$) and Y ~Unif($\{1, .., X\}$).
(In words, X is chosen uniformly from 1,2,3,4, then Y is chosen uniformly from the numbers from 1 up to Y). ← X

(a) Write in table form the joint pmf $p_{X,Y}(x, y)$.



$\backslash\ x\ |$



$y \backslash x$	1	2	3	4	$p_Y(y)$
1	1/4	1/8	1/12	1/16	25/48
2	0	1/8	1/12	1/16	13/48
3	0	0	1/12	1/16	7/48
4	0	0	0	1/16	3/48
$p_X(x)$	1/4	1/4	1/4	1/4	Σ 1

(b) Find the marginal pmf $p_Y(y)$, and use it to compute $E(Y)$.

$p_Y(y)$ is above

$$\mathbb{E}[Y] = 1\cdot\left(\frac{25}{48}\right) + 2\cdot\left(\frac{13}{48}\right) + 3\cdot\left(\frac{7}{48}\right) + 4\cdot\left(\frac{3}{48}\right)$$

$$= \frac{25+26+21+12}{48} = \frac{84}{48} = 1.75$$

(c) Find $E(Y)$ by another method, using the "Fundamental Theorem of Probablity."

$$\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y] = \frac{\mathbb{E}[Y|1] + \mathbb{E}[Y|2] + \mathbb{E}[Y|3] + \mathbb{E}[Y|4]}{4}$$

$$\mathbb{E}[Y] = \left(1(1 + 1/2 + 1/3 + 1/4) + 2(0 + 1/2 + 1/3 + 1/4) + 3(0+0+1/3+1/4) + 4(0+0+0+1/4)\right)\left(\frac{1}{4}\right)$$

$$= \left(1(25/12) + 2(13/12) + 3(7/12) + 4(3/12)\right)\left(\frac{1}{4}\right)$$

$$= \frac{84}{48} = 1.75$$