HW3

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Assignment 3: Polynomial Interpolation

Polynomial interpolation is about finding a polynomial $P(x)$ of degree $\leq n$, for a certain n, that goes through a given set of $n+1$ data points: (x_0, y_0), (x_1, y_1), (x_2, y_2), $\dots$, (x_n, y_n). Hence, it satisfies a system of equations:

$$\begin{aligned} P(x_0) &= y_0, \\ P(x_1) &= y_1, \\ &\vdots \\ P(x_n) &= y_n. \end{aligned}$$

We can express a polynomial algebraically in different ways. In the standard form,

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where the constant coefficients $a_0, \dots, a_n$ are the unknowns to be found. The number of coefficients should match the number of equations, which is the case here. There are $n+1$ equations and $n+1$ unknown coefficients.

Here's how the system of equations would look like:

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n &= y_0, \\ a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n &= y_1, \\ &\vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n &= y_n. \end{aligned} \tag{1}$$

In this system, the a_js are the unknowns, while the powers of x_js are known coefficients for the system. This is a square system, since the number of equations is the same as the number of unknowns. It is also a linear system. For example, when $n = 1$, we only have two equations with two unknowns a_0 and a_1, and if we treat them as variables, then each equation is a line. For $n = 2$, each equation is a plane in $\mathbb{R}^3$, and in general, each equation is a hyperplane in $\mathbb{R}^{n+1}$. In linear algebra, we use matrices to represent linear systems, and we learn that a linear system either has no solution, a unique solution, or infinitely many solutions. We will see a theorem about it below.

Exercise 1. To interpolate a set of 5 data points, the interpolating polynomial of lowest degree has degree at most what?

We have data points x_0, x_1, x_2, x_3, x_4.

Our degree is at most 4 as our system of equations is of the form

$a_0 + a_1 x_i + a_2 x_i^2 + a_3 x_i^3 + a_4 x_i^4$ for $i = 0, 1, 2, 3, 4$.

Exercise 2. Is it possible for two of the x_js to be equal? Would there be an interpolating polynomial in that case? (Just use something you learned in precalculus or college algebra to answer this question.)

If 2 x_js are equal, but the corresponding y_js are not, the data we are dealing with is not a function as it fails the verticle line test, and thus we are not able to find an interpolating polynomial.

Exercise 3. Now let's find an interpolating polynomial.

(a) Setup a linear system for the polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, interpolating the points $(-2, 1)$, $(-1, 2)$, $(1, 1)$, and $(2, 3)$.

$P(x_0) = P(-2) = a_0 - 2a_1 + 4a_2 - 8a_3 = 1$

$P(x_1) = P(-1) = a_0 - a_1 + a_2 - a_3 = 2$

$P(x_2) = P(1) = a_0 + a_1 + a_2 + a_3 = 1$

$P(x_3) = P(2) = a_0 + 2a_1 + 4a_2 + 8a_3 = 3$

(b) Use a calculator or computer to solve this 4×4 linear system for the coefficients a_0, a_1, a_2, and a_3. Then plot the given points and the graph of the interpolating polynomial in the same coordinate system. You don't have to use Mathematica for this problem, but if you choose to use it, try this program:

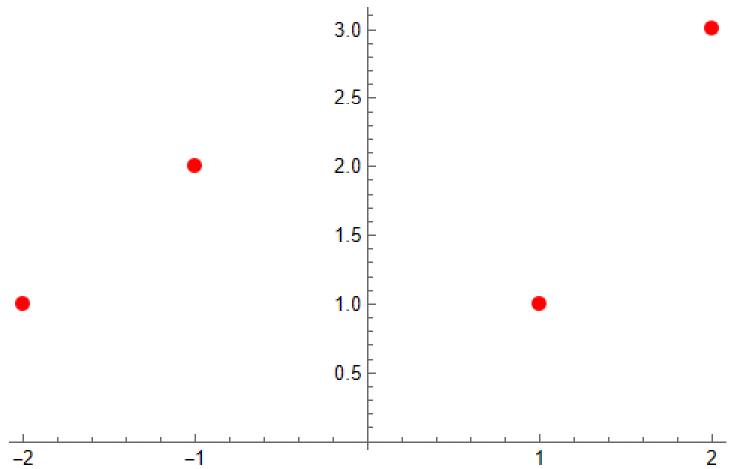
```
x = {-2, -1, 1, 2};
y = {1, 2, 1, 3};
m = {x^0, x, x^2, x^3};
m = Transpose[m];
a = LinearSolve[m, y]
pts = ListPlot[Transpose[{x, y}], PlotStyle → {Red, PointSize[Large]}]
poly = Plot[a.{1, x, x^2, x^3}, {x, -3, 3}];
Show[poly, pts, PlotRange -> {{-2, 2}, {0, 4}}]
```

Mathematica is available through LabAnywhere at UM-Flint. It is case sensitive. To execute a program, place the cursor anywhere in the program, hold the shift key, and then press the return (or enter) key. A semicolon at the end of a line will suppress printing the result of that line.

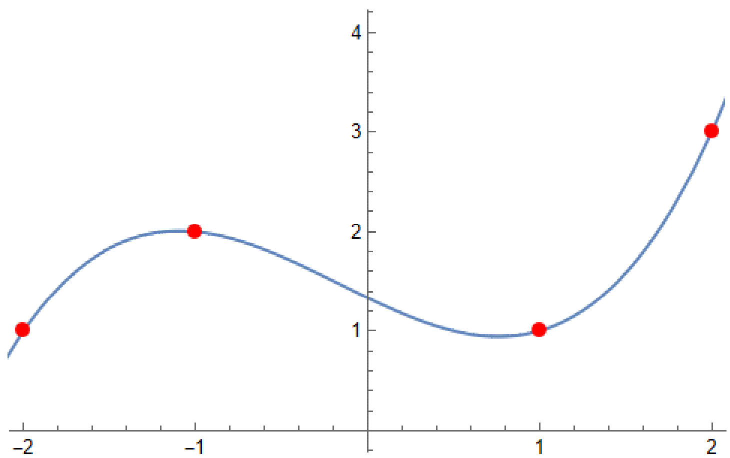
```
In[1]:= x = {-2, -1, 1, 2};
        y = {1, 2, 1, 3};
        m = {x^0, x^1, x^2, x^3};
        m = Transpose[m];
        a = LinearSolve[m, y]
        pts = ListPlot[Transpose[{x, y}], PlotStyle → {Red, PointSize[Large]}]
        poly = Plot[a.{1, x, x^2, x^3}, {x, -3, 3}];
        Show[poly, pts, PlotRange → {{-2, 2}, {0, 4}}]
```

Out[5]= $\left\{\frac{4}{3}, -\frac{5}{6}, \frac{1}{6}, \frac{1}{3}\right\}$ $a_0 = \frac{4}{3}$, $a_1 = -\frac{5}{6}$, $a_2 = \frac{1}{6}$, $a_3 = \frac{1}{3}$

Out[6]=



Out[8]=



Here's a theorem on the existence and uniqueness of interpolating polynomials:

Theorem 1. *For any set of $n+1$ points with distinct x-coordinates, there exists a unique interpolating polynomial of degree $\leq n$.*

In the following exercise we consider using a different form of a polynomial, called *Lagrange interpolating polynomial.*

Exercise 4. Assuming x_0, x_1, and x_2 are distinct, consider the polynomial

$$P(x) = y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

Verify that $P(x_0) = y_0$, $P(x_1) = y_1$, and $P(x_2) = y_2$.

Verify that $P(x_0) = y_0$, $P(x_1) = y_1$, and $P(x_2) = y_2$.

$$P(x_0) = y_0\overbrace{\frac{(x_0 - x_1)(x_0 - x_2)}{(x_0 - x_1)(x_0 - x_2)}}^{1} + y_1\frac{\overset{0}{(x_0 - x_0)}(x_0 - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2\frac{\overset{0}{(x_0 - x_0)}(x_0 - x_1)}{(x_2 - x_0)(x_2 - x_1)}. \checkmark$$

$$P(x_1) = y_0\frac{\overset{0}{(x_1 - x_1)}(x_1 - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1\overbrace{\frac{(x_1 - x_0)(x_1 - x_2)}{(x_1 - x_0)(x_1 - x_2)}}^{1} + y_2\frac{(x_1 - x_0)\overset{0}{(x_1 - x_1)}}{(x_2 - x_0)(x_2 - x_1)}. \checkmark$$

$$P(x_2) = y_0\frac{(x_2 - x_1)\overset{0}{(x_2 - x_2)}}{(x_0 - x_1)(x_0 - x_2)} + y_1\frac{(x_2 - x_0)\overset{0}{(x_2 - x_2)}}{(x_1 - x_0)(x_1 - x_2)} + y_2\overbrace{\frac{(x_2 - x_0)(x_2 - x_1)}{(x_2 - x_0)(x_2 - x_1)}}^{1}. \checkmark$$

$P(x_0) = y_0 \cdot 1 + 0 + 0$

$P(x_1) = 0 + y_1 \cdot 1 + 0$

$P(x_2) = 0 + 0 + y_2 \cdot 1$

$X = [-2, -1, 1, 2]$

$Y = [1, 2, 1, 3]$

Exercise 5. Find the Lagrange interpolating polynomial for the data points in Exercise 3.

$$P(x) = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$P(x) = 1 \cdot \frac{(x+1)(x-1)(x-2)}{(-2+1)(-2-1)(-2-2)} + 2 \cdot \frac{(x+2)(x-1)(x-2)}{(-1+2)(-1-1)(-1-2)} + 1 \cdot \frac{(x+2)(x+1)(x-2)}{(1+2)(1+1)(1-2)} + 3 \cdot \frac{(x+2)(x+1)(x-1)}{(2+2)(2+1)(2-1)}$$

$$P(x) = \frac{(x+1)(x-1)(x-2)}{(-1)(-3)(-4)} + 2 \cdot \frac{(x+2)(x-1)(x-2)}{(1)(-2)(-3)} + \frac{(x+2)(x+1)(x-2)}{(3)(2)(-1)} + 3 \cdot \frac{(x+2)(x+1)(x-1)}{(4)(3)(1)}$$

$$P(x) = -\frac{1}{12}(x+1)(x-1)(x-2) + \frac{1}{3}(x+2)(x-1)(x-2) - \frac{1}{6}(x+2)(x+1)(x-2) + \frac{1}{4}(x+2)(x+1)(x-1)$$

The data points could come from a known or unknown function. Here is a theorem about it:

Theorem 2. *Let $a \le x_0 < x_1 < x_2 < \cdots < x_n \le b$. Suppose $f(x)$ is a function whose $(n+1)$-st derivative $f^{(n+1)}(x)$ is continuous on the interval $[a,b]$. Let $P(x)$ be the polynomial of degree at most n that interpolates f at the nodes x_0 through x_n. Then we have*

$$f(x) = P(x) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n), \tag{2}$$

for each $x \in [a,b]$, and some $c_x \in [a,b]$, where c_x depends on x.

Exercise 6. Consider the function

$$f(x) = \ln(x),$$

and the nodes $x_0 = 1$ and $x_1 = 2$.

1. Find the polynomial $P(x)$ of degree at most 1 that interpolates f at the given nodes.

$$P(x) = y_0 \frac{(x-x_1)}{(x_0-x_1)} + y_1 \frac{(x-x_0)}{(x_1-x_0)} = \ln(1)\frac{(x-2)}{(1-2)} + \ln(2)\frac{(x-1)}{(2-1)}$$

$$P(x) = \ln(2)(x-1)$$

2. By using Theorem 2, find an upper bound for the absolute value of the remainder in Equation 2, on the interval $[1,2]$.

$$f(x) = \ln(2)(x-1) + \frac{f''(\xi_x)}{2!}(x-x_0)(x-x_1) \quad ; \quad R(x) = -\frac{1}{2\xi_x^2}(x-1)(x-2)$$

$$f(x) = \ln(x) \; ; \; f'(x) = \frac{1}{x} \; ; \; f''(x) = -\frac{1}{x^2}$$

$$\text{ArgMax}\{|f''(x)|\} \mid x \in [1,2] = \left|-\frac{1}{1^2}\right| = |-1| \quad @ \; x = 1$$

$$\text{ArgMax}\{|(x-1)(x-2)|\} \mid x \in [1,2] = |(1.5-1)(1.5-2)| = |(.5)(-.5)| = |-.25| = \frac{1}{4} \; @ \; x = \frac{3}{2}$$

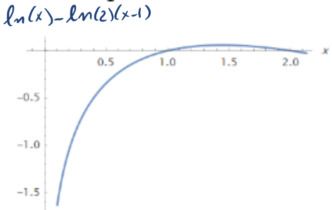
$$|R(x)| \le 1 \cdot \frac{3}{2} = \frac{3}{2}$$

3. By using a calculator or computer, but not Theorem 2, find the maximum value of $|f(x) - P(x)|$ on the interval $[1,2]$. Compare this value with that of the previous part.

$$\frac{\partial}{\partial x}\left(\ln(x) - \ln(2)(x-1)\right) = \frac{1}{x} - \ln(2)$$

$$\frac{1}{x} - \ln(2) = 0 \; ; \; \frac{1}{x} = \ln(2) \; ; \; x = \frac{1}{\ln(2)}$$

$y = \ln(x) - \ln(2)(x-1)$



$$\text{ArgMax}\{|\ln(x) - \ln(2)(x-1)|\} = \frac{1}{\ln(2)} \approx 1.44$$

[illegible] ...mation : $1.5 > 1.44$

Arg MAX $\{ | \ln(x) - \ln(2)(x-1) | \} = \ln(2)$ ...

Our upperbound is slightly larger but a decent approximation : $1.5 > 1.44$

4. Draw the graphs of f and P over the interval $[1, 2]$ in the same coordinate system. Then, on the graph locate where the maximum error occurs.