HW4

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Assignment 4

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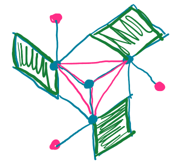


This assignment has three parts with different subjects.

Part A: Divided Differences

Let

$$a \le z_0 \le z_1 \le z_2 \le \cdots \le z_n \le b.$$

(Notice that each z_i could be repeated a few or more times.) Suppose $f(x)$ is a function defined on the interval $[a, b]$. For any z_i that appears more than once, we assume $f'(z_i)$ exists. More generally, if z_i is repeated k times, we assume $f^{(k-1)}(z_i)$ exists.

We are looking for a polynomial $P(x)$ of degree $\le n$, which has the highest possible contact with f at the given nodes. It means that for a node z_i appearing k times, we want

$$\begin{aligned} P(z_i) &= f(z_i), \\ P'(z_i) &= f'(z_i), \\ &\vdots \\ P^{(k-1)}(z_i) &= f^{(k-1)}(z_i). \end{aligned}$$

(Note that if all the nodes are the same node, then $P(x)$ is the n-th Taylor polynomial. In the case where all the nodes are different, we have Lagrange interpolation. And if each node appears twice, then we have Hermite interpolation.)

We can express $P(x)$ by using *Newton's interpolation formula*:

$$\begin{aligned} P(x) = f[z_0] &+ \\ & f[z_0, z_1](x - z_0) + \\ & f[z_0, z_1, z_2](x - z_0)(x - z_1) + \\ & \vdots \\ & f[z_0, z_1, z_2, \ldots, z_n](x - z_0)(x - z_1) \cdots (x - z_{n-1}), \end{aligned}$$

where the coefficients are the divided differences as defined below. (If you are wondering what happened to z_n, it will appear in the remainder term.)

The *zeroth divided differences*:

$$f[z_i] = f(z_i), \text{ for } i = 0, \ldots, n.$$

$$\frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}$$

The *first divided differences*:

$$f[z_i,\ z_{i+1}] = \begin{cases} \dfrac{f[z_{i+1}] - f[z_i]}{z_{i+1} - z_i}, & \text{if } z_i \neq z_{i+1} \\ f'(z_i), & \text{if } z_i = z_{i+1} \end{cases}, \text{ for } i = 0, \ldots, n-1.$$

$$\frac{\dfrac{f(z_{i+2}) - f(z_{i+1})}{z_{i+2} - z_{i+1}} - \dfrac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}}{z_{i+2} - z_i}$$

The *second divided differences*:

$$f[z_i,\ z_{i+1},\ z_{i+2}] = \begin{cases} \dfrac{f[z_{i+1}, z_{i+2}] - f[z_i, z_{i+1}]}{z_{i+2} - z_i}, & \text{if } z_i \neq z_{i+2} \\ \frac{1}{2} f''(z_i), & \text{if } z_i = z_{i+2} \end{cases}$$

for $i = 0, \ldots, n-2$.

$$\frac{f(z_{i+2}) - f(z_{i+1})}{\ldots} - \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}$$

$$\left\{ \tfrac{1}{2} f''(z_i), \text{ if } z_i = z_{i+2} \right.$$

for $i = 0, \ldots, n-2$.

$$\frac{\dfrac{\dfrac{f(z_{i+3})-f(z_{i+2})}{z_{i+3}-z_{i+2}} - \dfrac{f(z_{i+2})-f(z_{i+1})}{z_{i+2}-z_{i+1}}}{z_{i+3}-z_{i+1}} - \dfrac{\dfrac{f(z_{i+2})-f(z_{i+1})}{z_{i+2}-z_{i+1}} - \dfrac{f(z_{i+1})-f(z_i)}{z_{i+1}-z_i}}{z_{i+2}-z_i}}{z_{i+3}-z_i}$$

The *third divided differences*:

$$f[z_i,\ z_{i+1},\ z_{i+2},\ z_{i+3}] = \begin{cases} \dfrac{f[z_{i+1}, z_{i+2}, z_{i+3}] - f[z_i, z_{i+1}, z_{i+2}]}{z_{i+3} - z_i}, & \text{if } z_i \neq z_{i+3} \\ \tfrac{1}{6} f'''(z_i), & \text{if } z_i = z_{i+3} \end{cases}$$

for $i = 0, \ldots, n-3$.

The *j-th divided differences* $(1 \leq j \leq n)$:

$$f[z_i,\ z_{i+1},\ \ldots,\ z_{i+j-1},\ z_{i+j}] = \begin{cases} \dfrac{f[z_{i+1}, z_{i+2}, \ldots, z_{i+j}] - f[z_i, z_{i+1}, \ldots, z_{i+j-1}]}{z_{i+j} - z_i}, & \text{if } z_i \neq z_{i+j} \\ \tfrac{1}{j!} f^{(j)}(z_i), & \text{if } z_i = z_{i+j} \end{cases}$$

for $i = 0, \ldots, n-j$.

Theorem 1 (The Remainder Theorerm). *If $f^{(n+1)}(x)$ is continuous on $[a, b]$, then*

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - z_0) \cdots (x - z_n),$$

for each $x \in [a, b]$ and some $\xi_x \in [a, b]$.

Exercise 1. In this problem, by using divided differences, we find the polynomial $P(x)$, interpolating the function $f(x) = e^x$, such that $P(0) = f(0)$, $P'(0) = f'(0)$, $P''(0) = f''(0)$, $P(1) = f(1)$, and $P'(1) = f'(1)$.

z_0 $\quad z_1 \quad z_2 \quad z_3 \quad z_4$

(a) For this problem, setup the values of the nodes $z_0, \ldots z_n$, with repetition. (Read the beginning of this document to find the correct answer.)

$a \leq z_0 \leq \cdots \leq z_n \leq b$; $a \leq 0 \leq 0 \leq 0 \leq 1 \leq 1 \leq b$

$z_0 = 0$, $z_1 = 0$, $z_2 = 0$, $z_3 = 1$, $z_4 = 1$

(b) Make a table of divided differences for this problem.

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f[0]$				
	$f[0,0]$			
$f[0]$		$f[0,0,0]$		
	$f[0,0]$		$f[0,0,0,1]$	
$f[0]$		$f[0,0,1]$		$f[0,0,0,1,1]$
	$f[0,1]$		$f[0,0,1,1]$	
$f[1]$		$f[0,1,1]$		
	$f[1,1]$			
$f[1]$				

$e^0 = 1$	$e^0 = 1$	$e^0 = 1$	$\frac{e-2-1}{1-0} = e-3$	$\frac{-e-1-e+3}{1-0} = 2-2e$
$e^0 = 1$	$e^0 = 1$	$\frac{e-1-1}{1-0} = e-2$	$\frac{1-e-2}{1-0} = -e-1$	
$e^0 = 1$	$\frac{e-1}{1-0} = e-1$	$\frac{e-e+1}{1-0} = 1$		
$e^1 = e$	$e^1 = e$			
$e^1 = e$				

(c) Express $P(x)$ in Newton's interpolation form by using the divided-differences table that you made in the previous part.

$$P(x) = 1 + 1(x-0) + 1(x-0)(x-0) + (e-3)(x-0)(x-0)(x-0)$$
$$+ (2-2e)(x-0)(x-0)(x-0)(x-1)$$

$$P(x) = 1 + x + x^2 + (e-3)x^3 + (2-2e)x^3(x-1)$$

(d) Based on the error term provided by the Remainder Theorem (above) find the best upper bound for the maximum absolute error on the interval $[0, 1]$. (You may need to use a calculus to find the maximum value of the remainder term.)

$$R(x) = \frac{f^{(6)}(\xi_x)}{6!} \cdot x^3 \cdot (x-1)^2 \qquad f^{(6)}(\xi_x) = e^{\xi_x} \;;\; \xi_x \in [0,1]$$

As e^{ξ_x} is strictly increasing: $\max\left(f^{(6)}(\xi_x)\right) = e^1 = e$

$\max\left(x^3(x-1)^2\right)$ happen when $\frac{\partial}{\partial x}\left(x^3(x-1)^2\right) = 0$ or @ $x=0$, or $x=1$

$$\frac{\partial}{\partial x}\left(x^3(x-1)^2\right) = 3x^2(x-1)^2 + 2x^3(x-1) = 3x^4 - 6x^3 + 3x^2 + 2x^4 - 2x^3$$
$$= x^2(5x^2 - 8x + 3) \;;\; x = \frac{8 \pm \sqrt{64-60}}{10} = \frac{8 \pm 2}{10} = \frac{3}{5}, 1$$

$x^3(x-1)^2$ @ $0 = 0$, $x^3(x-1)^2$ @ $1 = 0$, $x^3(x-1)^2$ @ $\frac{3}{5} = \frac{27}{125} \cdot \frac{4}{25} = \frac{108}{3125}$

$\max\left(x^3(x-1)^2\right) = \frac{108}{3125}$; $6! = 720$

$$\max(R(x)) = \frac{\max\left(f^{(6)}(\xi_x)\right)}{6!} \cdot \max\left(x^3(x-1)^2\right) = \frac{e}{6!} \cdot \frac{108}{3125} \approx 0.0001305$$

Exercise 2. This is a problem from the book (in the Divided Differences Section). We assume that the nodes $x_0, x_1, \ldots,$ are distinct.

21. Given

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1)$$

$$+ a_3(x - x_0)(x - x_1)(x - x_2) + \cdots$$

$$+ a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}),$$

use $P_n(x_2)$ to show that $a_2 = f[x_0, x_1, x_2]$.

$P_n(x_0) = f[x_0] = f(x_0)$

$P_n(x_1) = f[x_0] + f[x_0, x_1](x_1 - x_0) = f(x_1)$

$P_n(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$

$f(x_2) - f(x_1) = f[x_0, x_1](x_2 - x_0 - x_1 + x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

$f(x_2) - f(x_1) = f[x_0, x_1](x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1)$

$\dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} - f[x_0, x_1] = a_2(x_2 - x_0)$

$\dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = a_2 = f[x_0, x_1, x_2]$

Note: $R(x_0) = 0$ as $x_0 - x_0 = 0$, $R(x_1) = 0$ as $x_1 - x_1 = 0$, & $R(x_2) = 0$ as $x_2 - x_2 = 0$ $\Rightarrow$ $P(x_0) = f(x_0)$, $P(x_1) = f(x_1)$, $P(x_2) = f(x_2)$

Exercise 3. For this exercise, we assume that the nodes $\{x_0, x_1, \ldots, x_n\}$ are distinct. We know that $f[x_0, \ldots, x_n]$ is the coefficient of x^n in the unique polynomial of degree $\leq n$ that interpolates f in the nodes $x_0, \ldots, x_n$. And, any permutation of the nodes should give us the same coefficient of the same polynomial. Therefore, regardless of what permutation we choose for the nodes, the divided differences must remain the same. In the following parts, we will verify this fact directly for just three cases.

(a) Assuming that x_0 and x_1 are distinct, verify directly that

$$f[x_0, x_1] = f[x_1, x_0].$$

$f[x_0] = f(x_0)$, $f[x_1] = f(x_1)$

$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$(x_1 - x_0)(x_1 - x_0) = x_1^2 - 2x_1x_0 + x_0^2 = (x_0 - x_1)(x_0 - x_1)$

& $(f(x_1) - f(x_0))(f(x_1) - f(x_0)) = f(x_1)^2 - 2f(x_0)f(x_1) + f(x_0)^2 = (f(x_0) - f(x_1))(f(x_0) - f(x_1))$

Thus $f[x_0, x_1] \cdot \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{f(x_1)^2 - 2f(x_0)f(x_1) + f(x_0)^2}{x_1^2 - 2x_1x_0 + x_0^2} = \dfrac{f(x_0) - f(x_1)}{(x_0 - x_1)}$ $f[x_1, x_0]$

Last $\dfrac{f(x_1) - f(x_0)}{\quad} \cdot \dfrac{x_0 - x_1}{f(x_0) - f(x_1)} = \dfrac{x_0 f(x_1) - x_0 f(x_0) - x_1 f(x_1) + x_1 f(x_0)}{x_0 f(x_1) - x_0 f(x_0) - x_1 f(x_1) + x_1 f(x_0)} = 1$

Last $\dfrac{f(x_1)-f(x_0)}{x_1-x_0} \cdot \dfrac{x_0-x_1}{f(x_0)-f(x_1)} = \dfrac{x_0 f(x_1) - x_0 f(x_0) - x_1 f(x_1) + x_1 f(x_0)}{x_0 f(x_1) - x_0 f(x_0) - x_1 f(x_1) + x_1 f(x_0)} = 1$

And we have $f[x_0, x_1] = f[x_1, x_0]$

(b) Assuming that x_0, x_1, and x_2 are distinct, verify directly that

$$f[x_0, x_1, x_2] = f[x_2, x_1, x_0].$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

We will follow similar to that of part (a)

$$f[x_0, x_1, x_2] \cdot \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{f[x_2, x_1]^2 - 2f[x_2, x_1]f[x_1, x_0] + f[x_1, x_0]^2}{x_2^2 - 2x_2x_0 + x_0^2}$$

$$= \frac{f[x_1, x_0] - f[x_2, x_1]}{x_0 - x_2} \; f[x_2, x_1, x_0]$$

$$\frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} \cdot \frac{x_0 - x_2}{f[x_1, x_0] - f[x_2, x_1]} = \frac{f[x_2, x_1]x_0 - f[x_2, x_1]x_2 - f[x_1, x_0]x_0 + f[x_1, x_0]x_2}{f[x_2, x_1]x_0 - f[x_2, x_1]x_2 - f[x_1, x_0]x_0 + f[x_1, x_0]x_2} = 1$$

Thus $f[x_0, x_1, x_2] = f[x_2, x_1, x_0]$

(c) (A bit harder!) Assuming that x_0, x_1, and x_2 are distinct, verify directly that

$$f[x_0, x_1, x_2] = f[x_1, x_0, x_2].$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{f[x_1, x_2] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2)-f(x_1)}{x_2 - x_1} - \dfrac{f(x_0)-f(x_1)}{x_0 - x_1}}{x_2 - x_0} = \frac{f(x_2)-f(x_1)}{(x_2-x_1)(x_2-x_0)} - \frac{f(x_0)-f(x_1)}{(x_0-x_1)(x_2-x_0)}$$

$$= \frac{(f(x_2)-f(x_1))(x_0-x_1) - (f(x_0)-f(x_1))(x_2-x_1)}{(x_2-x_1)(x_2-x_0)(x_0-x_1)}$$

$$f[x_1, x_0, x_2] = \frac{f[x_0, x_2] - f[x_1, x_0]}{x_2 - x_1} = \frac{\dfrac{f(x_2)-f(x_0)}{x_2-x_0} - \dfrac{f(x_0)-f(x_1)}{x_0-x_1}}{x_2 - x_1}$$

$$= \frac{f(x_2)-f(x_0)}{(x_2-x_0)(x_2-x_1)} - \frac{f(x_0)-f(x_1)}{(x_0-x_1)(x_2-x_1)}$$

$$= \frac{(f(x_2)-f(x_0))(x_0-x_1) - (f(x_0)-f(x_1))(x_2-x_0)}{(x_2-x_0)(x_2-x_1)(x_0-x_1)}$$

$f(x)x + \cancel{f(x_1)}x_1$

$$= \frac{\qquad}{(x_2-x_0)(x_2-x_1)(x_0-x_1)}$$

$$(f(x_2)-f(x_1))(x_0-x_1) - (f(x_0)-f(x_1))(x_2-x_1) = f(x_2)x_0 - f(x_2)x_1 - f(x_1)x_0 + \cancel{f(x_1)x_1} - f(x_0)x_2 + f(x_0)x_1 + f(x_1)x_2 - \cancel{f(x_1)x_1}$$

$$(f(x_2)-f(x_0))(x_0-x_1) - (f(x_0)-f(x_1))(x_2-x_0) = f(x_2)x_0 - f(x_2)x_1 - \cancel{f(x_0)x_0} + f(x_0)x_1 - f(x_0)x_2 + \cancel{f(x_0)x_0} + f(x_1)x_2 - f(x_1)x_0$$

As both our numerators & denominators are equal,

$$f[x_0, x_1, x_2] = f[x_1, x_0, x_2] \quad ;)$$

Exercise 4. If $p(x)$ is a polynomial of degree at most $n-1$, how much is $p[x_0,\ldots,x_n]$?

$$p(x) = P(x) + R(x)$$

$$= p[x_0] + p[x_0,x_1](x-x_0) + \cdots + p[x_0,\ldots,x_n]\prod_{i=0}^{n-1}(x-x_i) + p[x_0,\ldots,x_n,x]\prod_{i=0}^{n}(x-x_i)$$

So

$$p[x_0,\ldots,x_n] = \frac{p(x) - p[x_0] - p[x_0,x_1](x-x_1) - \cdots - p[x_0,\ldots,x_{n-1}]\prod_{i=0}^{n-2}(x-x_i) - p[x_0,\ldots,x_n,x]\prod_{i=0}^{n}(x-x_i)}{\prod_{i=0}^{n-1}(x-x_i)}$$

Part B: Inverse Interpolation

Exercise 5. This problem is from the book.

INVERSE INTERPOLATION Suppose $f \in C^1[a,b]$, $f'(x) \neq 0$ on $[a,b]$ and f has one zero p in $[a,b]$. Let $x_0,\ldots,x_n$ be $n+1$ distinct numbers in $[a,b]$ with $f(x_k) = y_k$, for each $k = 0,1,\ldots,n$. To approximate p, construct the interpolating polynomial of degree n on the nodes $y_0,\ldots,y_n$ for f^{-1}. Since $y_k = f(x_k)$ and $0 = f(p)$, it follows that $f^{-1}(y_k) = x_k$ and $p = f^{-1}(0)$. Using iterated interpolation to approximate $f^{-1}(0)$ is called *iterated inverse interpolation.*

12. Use iterated inverse interpolation to find an approximation to the solution of $x - e^{-x} = 0$, using the data

	x_0	x_1	x_2	x_3
x	**0.3**	**0.4**	**0.5**	**0.6**
e^{-x}	**0.740818**	**0.670320**	**0.606531**	**0.548812**
$f(x) = x - e^{-x}$	-0.440818	-0.270320	-0.106531	0.051188
	y_0	y_1	y_2	y_3

0.3

$$\frac{0.4-0.3}{-0.27032+0.440818} = 0.5865171439$$

$$\frac{0.6105\ldots - 0.5865\ldots}{-0.106531+0.44081} = 0.0718695091$$

$$\frac{0.073185\ldots - 0.071869\ldots}{0.051188 + 0.440818}$$

0.4

$$\frac{0.5-0.4}{-0.106531+0.27032} = 0.6105416115$$

$$0.63407\ldots - 0.61054\ldots$$

$$= 0.002673765$$

0.4

0.5

0.6

$$\frac{0.5-0.4}{-0.106531+0.27032} = 0.6105416115$$

$$\frac{0.6-0.5}{0.051188+0.106531} = 0.6340711808$$

$$\frac{0.63407\ldots - 0.61054\ldots}{0.051188+0.27032} = 0.0731850197$$

$$\frac{}{0.051188+0.440818} = 0.0026737695$$

$$I(y) = 0.3 + 0.58651\ldots(y+0.440818) + 0.07186\ldots(y+0.440818)(y+0.27032)$$
$$+ 0.00267\ldots(y+0.440818)(y+0.27032)(y+0.106531)$$

$$I(0) \approx 0.5671453652$$

$$f(0.5671453652) = 0.00000325$$

Part C: Cubic Splines

Exercise 6. Find cubic polynomials $S_0(x)$ and $S_1(x)$, such that

$$S_0(0) = 0,$$
$$S_0(1) = 1,$$
$$S_1(1) = 1,$$
$$S_1(2) = 1,$$
$$S_0'(1) = S_1'(1),$$
$$S_0''(1) = S_1''(1),$$

let,
$x_0 = 0$
$x_1 = 1$
$x_2 = 2$

plus two boundary conditions for each of the following spline types. For each type, **graph the spline** (on the interval $[0,2]$) by a computer. (Graph S_0 on $[0,1]$ and S_1 on $[1,2]$, in the same coordinate plane. So, you are producing two graphs, one for Part a, and one for Part b.)

$[0,1]$ $S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3$

$[1,2]$ $S_1(x) = a_1 + b_1(x-x_1) + c_0(x-x_1)^2 + d_0(x-x_1)^3$

$S_0(x_0) = S_0(0) = 0 \Rightarrow \boxed{a_0 = 0}$

$S_0(x_1) = S_0(1) = \boxed{1 = b_0 + c_0 + d_0}$

$S_1(x_1) = S_1(1) = 1 \Rightarrow \boxed{a_1 = 1}$ $\quad \boxed{0 = b_1 + c_1 + d_1}$

$S_1(x_2) = S_1(2) = 1 = 1 + b_1 + c_1 + d_1$

$S_0'(x) = b_0 + 2c_0(x-x_0) + 3d_0(x-x_0)^2$

$S_1'(x) = b_1 + 2c_1(x-x_1) + 3d_1(x-x_1)^2$

$S_0'(x_1) = S_0'(1) = b_0 + 2c_0(x_1-x_0) + 3d_0(x_1-x_0)^2 = b_0 + 2c_0 + 3d_0$

$S_1'(x_1) = S_1'(1) = b_1$

$\boxed{b_1 = b_0 + 2c_0 + 3d_0}$

$S_0''(x) = 2c_0 + 6d_0(x-x_0)$

$S_1''(x) = 2c_1 + 6d_1(x-x_1)$

$S_0''(x_1) = 2c_0 + 6d_0(x_1-x_0) = 2c_0 + 6d_0$

$S_0''(x_1) = 2c_1$

$\boxed{2c_1 = 2c_0 + 6d_0}$

(a) Natural:

$$\begin{array}{cccccccc|c} a_0 & a_1 & b_0 & b_1 & c_0 & c_1 & d_0 & d_1 & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

(a) Natural:

$$S_0''(0) = 0,$$
$$S_1''(2) = 0.$$

$S_0''(x_0) = S_0''(0) = 2c_0 = 0 \Rightarrow \boxed{c_0 = 0}$

$S_1''(x_2) = S_1''(2) = 2c_1 + 6d_1(x_2 - x_1) = \boxed{2c_1 + 6d_1 = 0}$

a_0	a_1	b_0	b_1	c_0	c_1	d_0	d_1	
1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	1
0	0	1	0	0	0	1	0	1
0	0	0	1	0	1	0	1	0
0	0	1	-1	0	0	3	0	0
0	0	0	0	0	-2	6	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	2	0	6	0

In[7]:= b_0 + d_0 = 1, b_1 + c_1 + d_1 = 0, b_0 - b_1 + 3d_0 = 0, -2c_1 + 6d_0 = 0, 2c_1 + 6d_1 =0

Input

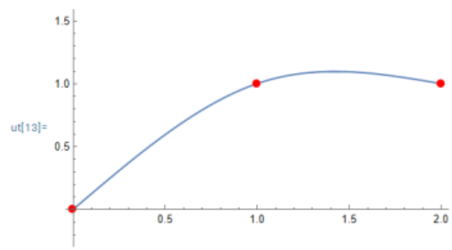
$\{b_0 + d_0 = 1, b_1 + c_1 + d_1 = 0, b_0 - b_1 + 3d_0 = 0, -2c_1 + 6d_0 = 0, 2c_1 + 6d_1 = 0\}$

Solution: Decimal form

$b_0 = \frac{5}{4}, \quad b_1 = \frac{1}{2}, \quad c_1 = -\frac{3}{4}, \quad d_0 = -\frac{1}{4}, \quad d_1 = \frac{1}{4}$, $a_0 = 0$, $a_1 = 1$, $c_0 = 0$

```
In[8]:= s0[x_] := (5/4)*x - (1/4)*x^3 (* Replace the right side by the formula for S0. *)
s1[x_] := 1 + (1/2)*(x-1) - (3/4)*(x-1)^2 + (1/4)*(x-1)^3
(* Replace the right side by the formula for S1. *)
plotS0 = Plot[s0[x], {x, 0, 1}];
plotS1 = Plot[s1[x], {x, 1, 2}];
plotPoints = ListPlot[{{0, 0}, {1, 1}, {2, 1}}, PlotStyle → {Red, PointSize[Large]}];
Show[plotS0, plotS1, plotPoints, PlotRange → {{0, 2}, {-.2, 1.5}}]
```

Out[13]=



$S_0(x) = \frac{5}{4}(x - x_0) - \frac{1}{4}(x - x_0)^3$

$S_1(x) = 1 + \frac{1}{2}(x - x_1) - \frac{3}{4}(x - x_1)^2 + \frac{1}{4}(x - x_1)^3$

(b) Clamped:

$$S_0'(0) = 1,$$
$$S_1'(2) = 0.$$

$S_0'(x_0) = S_0'(0) = \boxed{b_0 = 1}$

$S_1'(x_2) = S_1'(2) = \boxed{b_1 + 2c_1 + 3d_1 = 0}$

a_0	a_1	b_0	b_1	c_0	c_1	d_0	d_1	
1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	1
0	0	0	0	1	0	1	0	0
0	0	0	1	0	1	0	1	0
0	0	0	-1	2	0	3	0	-1
0	0	0	0	2	-2	6	0	0
0	0	1	0	0	0	0	0	1
0	0	0	1	0	2	0	3	0

In[14]:= c_0 + d_0 = 0, b_1 + c_1 + d_1 = 0, - b_1 + 2c_0 + 3d_0 = -1, 2c_0 -2c_1 + 6d_0 = 0, b_1 + 2c_1 + 3d_1 =0

Input

$\{c_0 + d_0 = 0, b_1 + c_1 + d_1 = 0, -b_1 + 2c_0 + 3d_0 = -1, 2c_0 - 2c_1 + 6d_0 = 0, b_1 + 2c_1 + 3d_1 = 0\}$

Solution: Decimal form Step-by-step s

$b_1 = \frac{1}{2}, \quad c_0 = \frac{1}{2}, \quad c_1 = -1, \quad d_0 = -\frac{1}{2}, \quad d_1 = \frac{1}{2}$, $a_0 = 0$, $a_1 = 1$, $b_0 = 1$

$S_0(x) = x + \frac{1}{2}x^2 - \frac{1}{2}x^3$

$S_1(x) = 1 + \frac{1}{2}(x-1) - (x-1)^2 + \frac{1}{2}(x-1)^3$

```
In[15]:= s0[x_] := x + (1/2) * x^2 - (1/2) * x^3 (* Replace the right side by the formula for S0. *)
         s1[x_] := 1 + (1/2) * (x - 1) - (x - 1)^2 + (1/2) * (x - 1)^3
         (* Replace the right side by the formula for S1. *)
         plotS0 = Plot[s0[x], {x, 0, 1}];
         plotS1 = Plot[s1[x], {x, 1, 2}];
         plotPoints = ListPlot[{{0, 0}, {1, 1}, {2, 1}}, PlotStyle → {Red, PointSize[Large]}];
         Show[plotS0, plotS1, plotPoints, PlotRange → {{0, 2}, {-.2, 1.5}}]
```

Out[20]=