HW7

Tuesday, November 15, 2022 6:19 PM

Assignment 7

These problems are taken from sections 8.1 and 8.2, or similar to them.

Exercise 1. Find the polynomial $p(x) = ax + b$ closest to the points $\underset{x_1,y_1}{\overset{1}{(-1,0)}}$, $\underset{x_2,y_2}{\overset{2}{(0,1)}}$ and $\underset{y_3,y_3}{\overset{3}{(1,1)}}$, in the least squares sense.

$$a = \frac{3(-1\cdot 0 + 0\cdot 1 + 1\cdot 1) - (-1+0+1)(0+1+1)}{3(-1^2+0^2+1^2) - (-1+0+1)^2} = \frac{3(0+0+1) - (0)(2)}{3(1+0+1) - (0)^2}$$

$$= \frac{3(1)}{3(2)} = \boxed{\frac{1}{2}}$$

$$b = \frac{(0)(1) - (2)(2)}{(0)^2 - (3)(2)} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

$$\boxed{p(x) = \frac{x}{2} + \frac{2}{3}}$$

Exercise 2. Find the polynomial $p(x) = ax + b$ of least squares approximation to the function $f(x) = \cos(x)$, with respect to the weight function $w(x) = 1$, on the interval $[0, 1]$. (You may use a calculator or computer to approximate integrals.)

$$\mathcal{E}(a,b) = \int_0^1 (\cos(x) - ax - b)^2 (1)\,dx = \int_0^1 (\cos(x) - ax - b)^2\,dx$$

$$\frac{\partial \mathcal{E}}{\partial a} = \int_0^1 -2x(\cos(x) - ax - b)\,dx = \frac{2a}{3} + b + 2 - 2\sin(1) - 2\cos(1) = 0$$

$$\frac{\partial \mathcal{E}}{\partial b} = \int_0^1 -2(\cos(x) - ax - b)\,dx = a + 2b - 2\sin(1) = 0$$

$$\frac{a}{2} + b - \sin(1) \Rightarrow \left(\frac{2a}{3} - \frac{a}{2}\right) + (b - b) + (\sin(1) - 2\sin(1)) - 2\cos(1) = 0$$

$$\Rightarrow \left(\frac{4a}{6} - \frac{3a}{6}\right) + 0 - \sin(1) - 2\cos(1) = 0 \Rightarrow \frac{a}{6} = \sin(1) + 2\cos(1)$$

$$\Rightarrow \boxed{a = 11.53245}$$

$$b = \sin(1) - \frac{6\sin(1) + 12\cos(1)}{2}$$

$$= \sin(1) - 3\sin(1) + 6\cos(1) = 6\cos(1) - 2\sin(1) \Rightarrow \boxed{b = 1.55887}$$

$$\boxed{p(x) = 11.53245x + 1.55887}$$

Exercise 3. Use the Gram-Schmidt process to construct the orthogonal polynomials $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$, with respect to the weight function $w(x) = 1$, on the interval $[0, 1]$. (You may use a calculator or computer to approximate integrals.)

Exercise 3. Use the Gram-Schmidt process to construct the orthogonal polynomials $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$, with respect to the weight function $w(x) = 1$, on the interval $[0, 1]$. (You may use a calculator or computer to approximate integrals.)

$\boxed{\phi_0(x) = 1}$; $\phi_1(x) = x - B_1$; $B_1 = \dfrac{\int_0^1 x\,(1)\,dx}{\int_0^1 (1)\,dx} = \dfrac{\frac{x^2}{2}\Big|_0^1}{x\Big|_0^1} = \dfrac{\frac{1}{2} - 0}{1 - 0} = \boxed{\frac{1}{2}}$

$\boxed{\phi_1 = x - \frac{1}{2}}$; $\phi_2(x) = (x - B_2)\phi_1(x) - C_2\,\phi_0(x)$

$$B_2 = \frac{\int_0^1 x\left(x - \frac{1}{2}\right)^2 (1)\,dx}{\int_0^1 \left(x - \frac{1}{2}\right)^2 (1)\,dx} = \frac{\int_0^1 x^3 - x^2 + \frac{x}{4}\,dx}{\int_0^1 x^2 - x + \frac{1}{4}\,dx} = \frac{\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{8}\Big|_0^1}{\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\Big|_0^1}$$

$$= \frac{\left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) - 0}{\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right) - 0} = \frac{\frac{6}{24} - \frac{8}{24} + \frac{3}{24}}{\frac{4}{12} - \frac{6}{12} + \frac{3}{12}} = \frac{\frac{1}{24}}{\frac{1}{12}} = \frac{1}{24} \times \frac{24}{2} = \boxed{\frac{1}{2} = B_2}$$

$$C_2 = \frac{\int_0^1 x\left(x - \frac{1}{2}\right)(1)(1)\,dx}{\int_0^1 (1)^2 (1)\,dx} = \frac{\int_0^1 x^2 - \frac{x}{2}\,dx}{\int_0^1 dx} = \frac{\frac{x^3}{3} - \frac{x^2}{4}\Big|_0^1}{x\Big|_0^1}$$

$$= \frac{\left(\frac{1}{3} - \frac{1}{4}\right) - 0}{1 - 0} = \frac{4}{12} - \frac{3}{12} = \boxed{\frac{1}{12} = C_2}$$

$$\boxed{\phi_2(x) = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right) - \frac{1}{12}(1) = \left(x - \frac{1}{2}\right)^2 - \frac{1}{12}}$$

Exercise 4. By using the results of Exercise 3, find the quadratic least squares approximation to the function $f(x) = \cos(x)$, with respect to the weight function $w(x) = 1$, on the interval $[0, 1]$. (You may use a calculator or computer to approximate integrals.)

$$p(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x)$$

$$a_0 = \frac{\int_0^1 \cos(x)(1)(1)\,dx}{\int_0^1 (1)^2(1)\,dx} = \frac{\int_0^1 \cos(x)\,dx}{\int_0^1 dx} = \frac{\sin(x)\Big|_0^1}{x\Big|_0^1} = \frac{0.84147}{1} = \boxed{0.84147 = a_0}$$

$$a_1 = \frac{\int_0^1 \cos(x)\left(x - \frac{1}{2}\right)(1)\,dx}{\int_0^1 \left(x - \frac{1}{2}\right)^2 (1)\,dx} = \frac{\int_0^1 x\cos(x) - \frac{1}{2}\cos(x)\,dx}{\int_0^1 x^2 - x + \frac{1}{4}\,dx} = \frac{-0.03896}{0.083333} = \boxed{-0.4675 = a_1}$$

$$a_2 = \frac{\int_0^1 \cos(x)\left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{12}\right)(1)\,dx}{\int_0^1 \left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{12}\right)^2 (1)\,dx} = \frac{-0.0024}{0.0055556} = \boxed{-0.431997 = a_2}$$

$$\int_0^1 \left(\left(x-\tfrac{1}{2}\right)^2 - \tfrac{1}{12}\right)^2 (1)\, dx \qquad 0.0055556$$

$$\boxed{p(x) = 0.84147 - 0.4675\left(x-\tfrac{1}{2}\right) - 0.431997\left(\left(x-\tfrac{1}{2}\right)^2 - \tfrac{1}{12}\right)}$$

Exercise 5. Use the Gram-Schmidt process to construct the orthogonal polynomials $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$, with respect to the weight function $w(x) = e^{-x}$, on the interval $(0, \infty)$. (You may use a calculator or computer to approximate integrals.)

$$\boxed{\phi_0(x) = 1} \;;\; \phi_1(x) = x - B_1 \;;\; B_1 = \frac{\int_0^\infty x e^{-x}\, dx}{\int_0^\infty e^{-x}\, dx} = \frac{1}{1} = \boxed{1 = B_1}$$

$$\boxed{\phi_1(x) = x - 1} \;;\; \phi_2(x) = (x - B_2)\phi_1(x) - C_2\phi_0(x)$$

$$B_2 = \frac{\int_0^\infty x(x-1)^2 e^{-x}\, dx}{\int_0^\infty (x-1)^2 e^{-x}\, dx} = \frac{3}{1} = \boxed{3 = B_2}$$

$$\boxed{\phi_2(x) = (x-3)(x-1) - 1}$$

$$C_2 = \frac{\int_0^\infty x(x-1)(1) e^{-x}\, dx}{\int_0^\infty (1)^2 e^{-x}\, dx} = \frac{1}{1} = \boxed{1 = C_2}$$

Exercise 6. By using the results of Exercise 5, find the quadratic least squares approximation to the function $f(x) = e^{-x}$ on the interval $(0, \infty)$, with respect to the weight function $w(x) = e^{-x}$. (You may use a calculator or computer to approximate integrals.)

$$p(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x)$$

$$a_0 = \frac{\int_0^\infty e^{-x}(1)e^{-x}\, dx}{\int_0^\infty (1)^2 e^{-x}\, dx} = \frac{\frac{1}{2}}{1} = \boxed{\tfrac{1}{2} = a_0}$$

$$a_2 = \frac{\int_0^\infty e^{-x}((x-3)(x-1)-1)e^{-x}\, dx}{\int_0^\infty ((x-3)(x-1)-1)^2 e^{-x}\, dx} = \frac{\frac{1}{4}}{4} = \boxed{\tfrac{1}{16} = a_2}$$

$$a_1 = \frac{\int_0^\infty e^{-x}(x-1)e^{-x}\, dx}{\int_0^\infty (x-1)^2 e^{-x}\, dx} = \frac{-\frac{1}{4}}{1} = \boxed{-\tfrac{1}{4} = a_1}$$

$$\boxed{p(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{16}((x-3)(x-1)-1)}$$