HW2

Friday, September 9, 2022 12:58 AM

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Assignment 2

(Please do **all parts** of each multiple-part problem.)

1. Apply the Intermediate Value Theorem (see Chapter 1) to show that the function

$$f(x) = 1 - x - \tan x$$

has a zero in the interval $[0, \frac{\pi}{4}]$, without graph.

Consider our function $f(x)$ evaluated at the bounds given ...

$$f(0) = 1 - 0 - \tan(0) = 1 - \frac{\sin(0)}{\cos(0)} = 1 - \frac{0}{1} = 1$$

$$f(\tfrac{\pi}{4}) = 1 - \tfrac{\pi}{4} - \tan(\tfrac{\pi}{4}) = \frac{3\pi}{4} - \frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} = \frac{3\pi}{4} - \frac{\sqrt{.5}}{\sqrt{.5}} = -\frac{\pi}{4}$$

As 1, x, & tan(x) are all continuous, f(x) is continuous

Last as $f(\frac{\pi}{4}) < 0 < f(0)$, there exists a number c in $[0, \frac{\pi}{4}]$ for which $f(c) = 0$

2. Apply the bisection method to find just the first and second approximations to a zero of f in the previous problem. (Use a calculator.)

1st Approximation : $p_1 = \dfrac{\frac{\pi}{4} - 0}{2} = \boxed{\frac{\pi}{8}}$

2nd Approximation : $f(\frac{\pi}{8}) = 1 - \frac{\pi}{8} - \tan(\frac{\pi}{8}) \approx 0.19308$

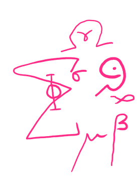
$\Rightarrow$ $f(\frac{\pi}{4}) < 0 < f(\frac{\pi}{8})$ $\Rightarrow$ $p_2 = \dfrac{\frac{2\pi}{8} + \frac{\pi}{8}}{2} = \boxed{\frac{3\pi}{16}}$

3. Consider this sequence:

$$a_1 = 1 + \frac{1}{2}, \quad a_2 = 1 + \frac{1}{1 + 1 + \frac{1}{2}}, \quad a_3 = 1 + \frac{1}{1 + 1 + \frac{1}{1+1+\frac{1}{2}}}, \quad \cdots.$$

(a) Express a_2 in terms of a_1.

$$a_2 = 1 + \frac{1}{1 + a_1}$$



(b) Express a_3 in terms of a_2.

$$a_3 = 1 + \frac{1}{1 + a_2}$$

(c) Express a_{n+1} in terms of a_n. (So, this sequence has a one-term recursive formula.)

$a_{n+1} = 1 + \frac{1}{\ }$

(c) Express a_{n+1} in terms of a_n. (So, this sequence has a one-term recursive formula.)

$$a_{n+1} = 1 + \frac{1}{1+a_n}$$

4. (Do all parts. This is one of the two problems **to be graded**. See the syllabus.) Consider this function:

$$g(x) = 1 + \frac{1}{1+x}.$$

(a) Show that g maps the interval $[1, 2]$ into the same interval $[1, 2]$, which means to show that $1 \leq g(x) \leq 2$, for all $x \in [1, 2]$.

$$g(1) = 1 + \frac{1}{1+1} = 1 + \frac{1}{2} = \frac{3}{2} \quad ; \quad g(2) = 1 + \frac{1}{1+2} = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\frac{1}{2} \leq \frac{1}{1+x} \leq \frac{1}{3} \Big| \ 1 \leq x \leq 2 \quad \Rightarrow \quad \frac{3}{2} \leq g(x) \leq \frac{4}{3} \Big| \ 1 \leq x \leq 2$$

$$\Rightarrow 1 \leq g(x) \leq 2 \ \forall \ x \in [1,2]$$

(b) Find a theorem, or part of a theorem from the book (or the class notebook) concluding that g has **at least one** fixed point in the interval $[1, 2]$. Give the theorem number, theorem part (Part (i), or (ii), or ...) and book's edition (or show how to find it in the notebook) or simply state the part that matters to this question. And, clearly indicate how the hypotheses of that theorem are met. (Hint: g is continuous everywhere, except at $x = -1$.)

Theorem 2.3 ; Edition 10 , pg. 56

Part (i)

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

Theorem 2.3 (i) If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least one fixed point in $[a, b]$.

As g is continuous on $[1,2]$ & $1 \leq g(x) \leq 2 \ \forall \ x \in [1,2]$

g has at least one fixed point in $[1,2]$

(c) Find the best possible value of k, such that

$$|g'(x)| \leq k, \ \text{ for all } x \in [1, 2].$$

Show your reasoning clearly.

$$g'(x) = \frac{\partial}{\partial x}\left(1 + \frac{1}{1+x}\right) = 0 + \frac{-1}{(1+x)^2} = \frac{-1}{(1+x)^2}$$

$$g'(1) = \frac{-1}{(1+1)^2} = \frac{-1}{2^2} = \frac{-1}{4} \qquad \text{As } \left|\frac{-1}{(1+x)^2}\right| < \frac{1}{4} \ \forall \ x > 1,$$

$\boxed{\text{The best possible } k \text{ is } \frac{1}{4}}$

(d) Find a theorem, or part of a theorem from the book (or the class notebook) concluding that g has **exactly one** fixed point in the interval $[1, 2]$. Give the theorem number, theorem part (Part (i), or (ii), or ...) and book's edition (or show how to find it in the notebook) or simply state the part that matters to this question. And, clearly indicate how the hypotheses of that theorem are met. (Hint: Does g' exist on the interval? Is $k < 1$?)

Theorem 2.3 ; Edition 10 , pg. 56

Part (ii)

(ii) If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b),$$

then there is exactly one fixed point in $[a, b]$. (See Figure 2.3.)

g' exists everywhere but when $x = -1$, thus g' exists on $[1,2]$.

from previous work we know the prior conditions exist & a constant exists such that $|g'| \le \frac{1}{4} \ \forall \ x \in [1,2]$, the hypotheses are met

(e) Find the fixed point of g in the interval $[1, 2]$. Show your work.

Consider the fixed point $g(p) = p$

We have $p = 1 + \frac{1}{1+p} \Rightarrow \frac{p-1}{1} = \frac{1}{1+p} \Rightarrow p + p^2 - 1 - p = 1$

Thus $p^2 = 2$ & $\boxed{p = \sqrt{2}}$ & $1 \le \sqrt{2} \le 2$:)

(f) Find a theorem from the book (or the class notebook) concluding that the fixed-point iteration

$$x_{n+1} = g(x_n)$$

converges to the unique fixed point (found above) in the interval $[1, 2]$. Give the theorem name, or number and book's edition (or show how to find it in the notebook) or simply state the theorem. And, clearly indicate how the hypotheses of that theorem are met.

Theorem 2.4; Edition 10, pg. 56

Theorem 2.4 **(Fixed-Point Theorem)**

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \le k, \quad \text{for all } x \in (a, b).$$

Then, for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \ge 1,$$

converges to the unique fixed point p in $[a, b]$.

We Know $g \in C[1,2] \mid g(x) \in [1,2] \ \forall x \in [1,2]$

Additionally $g' \in C[1,2]$ & $|g'| \le \frac{1}{4} \ \forall x \in [1,2]$

Thus for any number $p_0 \in [1,2]$, $p_n = g(p_{n-1}) \mid n \ge 1$ converges to p.

5. (This problem is from the book, Section 2.2. In the tenth edition its solution is in the back of the book, which I have included below. This problem is not one of the two to be graded, but I would like you to at least read and try understanding the whole or part of the method used to prove it. Jot something down (anything) to indicate that you have read the proof. Sorry about the font! In the eighth edition it is Problem 24. You can also find it in the ninth edition. The necessary calculus is discussed in Chapter 1.)

19. Let $g \in C^1[a, b]$ and p be in (a, b) with $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

Solution:

$\epsilon > 0$

19. Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that $|g'(x) - g'(p)| < |g'(p)| - 1$ whenever $0 < |x - p| < \delta$. Hence, for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \ge |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

$|g'(p)| - |g'(p)| + 1 = 1$

let $g' = f(x)$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

let $p = x_0$

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

let $f(p) = L$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.

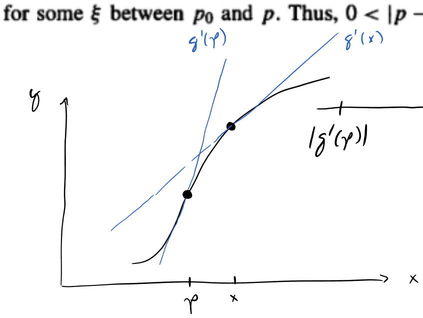
let $\epsilon = L - 1$

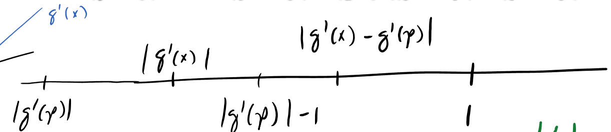
$g'(p)$ $g'(x)$ $|g'(x) - g'(p)|$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.

let $f(p) = L$

let $\epsilon = L - 1$





$|f(x) - L| < \epsilon$

$|L| - |f(x) - L| > |L| - |\epsilon|$

$|g'(\xi)| = \dfrac{|g(p_0) - g(p)|}{|p_0 - p|}$ $\xi \neq p \Rightarrow |p - \xi| > 0$

$|g'(\xi)| > 1 \Rightarrow |g'(\xi)||p_0 - p| > |p_0 - p|$

6. (This problem is from the book, Section 2.2. In the tenth and eighth editions its solution is in the back of the book, which I have included below. This problem is not one of the two to be graded, but I would like you to at least read and try understanding the whole or part of the method used to prove it. Jot something down for each part (anything) to indicate that you have read the proof. Sorry about the font! In the eighth edition it is Problem 19. You can also find it in the ninth edition. Theorem 2.4 refers to the Fixed-Point Theorem. It is Theorem 2.3 in the eighth edition. The necessary calculus is discussed in Chapter 1.)

23. **a.** Use Theorem 2.4 to show that the sequence defined by

$x_n = g(x_{n-1})$

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \geq 1,$$

$* \; g(x) = \frac{x}{2} + \frac{1}{x}$

converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

b. Use the fact that $0 < (x_0 - \sqrt{2})^2$ whenever $x_0 \neq \sqrt{2}$ to show that if $0 < x_0 < \sqrt{2}$, then $x_1 > \sqrt{2}$.

c. Use the results of parts (a) and (b) to show that the sequence in (a) converges to $\sqrt{2}$ whenever $x_0 > 0$.

Theorem 2.4 **(Fixed-Point Theorem)**

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

Then, for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a, b]$.

Proof Theorem 2.3 implies that a unique point p exists in $[a, b]$ with $g(p) = p$. Since g maps $[a, b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \geq 0$, and $p_n \in [a, b]$ for all n. Using the fact that $|g'(x)| \leq k$ and the Mean Value Theorem 1.8, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)||p_{n-1} - p| \leq k|p_{n-1} - p|,$$

where $\xi_n \in (a, b)$. Applying this inequality inductively gives

$$|p_n - p| \leq k|p_{n-1} - p| \leq k^2|p_{n-2} - p| \leq \cdots \leq k^n|p_0 - p|. \tag{2.4}$$

Since $0 < k < 1$, we have $\lim_{n\to\infty} k^n = 0$ and

$$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n|p_0 - p| = 0.$$

Hence, $\{p_n\}_{n=0}^{\infty}$ converges to p. ∎

$x_1 = \frac{x_0}{2} + \frac{1}{x_0}$; $\frac{1}{x_0} < \frac{1}{\sqrt{2}}$ | $x_0 > \sqrt{2}$

Solution:

23. a. Suppose that $x_0 > \sqrt{2}$. Then

$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$

$$x_1 - \sqrt{2} = g(x_0) - g\left(\sqrt{2}\right) = g'(\xi)\left(x_0 - \sqrt{2}\right),$$

where $\sqrt{2} < \xi < x$. Thus, $x_1 - \sqrt{2} > 0$ and $x_1 > \sqrt{2}$. Further,

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2}$$

$x_0 > \frac{x_0 + \sqrt{2}}{2} > \sqrt{2}$ | $x_0 > \sqrt{2}$

$\Rightarrow x_1 > \sqrt{2}$

and $\sqrt{2} < x_1 < x_0$. By an inductive argument,

$$\sqrt{2} < x_{m+1} < x_m < \ldots < x_0.$$

Thus, $\{x_m\}$ is a decreasing sequence which has a lower bound and must converge.

Suppose $p = \lim_{m\to\infty} x_m$. Then

$$p = \lim_{m\to\infty}\left(\frac{x_{m-1}}{2} + \frac{1}{x_{m-1}}\right) = \frac{p}{2} + \frac{1}{p}. \quad \text{Thus,} \quad p = \frac{p}{2} + \frac{1}{p},$$

which implies that $p = \pm\sqrt{2}$. Since $x_m > \sqrt{2}$ for all m, we have $\lim_{m\to\infty} x_m = \sqrt{2}$.

b. We have

$2 \cdot 1\sqrt{2} = 2\sqrt{2} < 3 = 1^2 + 2$

$$0 < (x_0 - \sqrt{2})^2 = x_0^2 - 2x_0\sqrt{2} + 2,$$

$2\sqrt{2}\sqrt{2} = 4 = \sqrt{2}^2 + 2$

so $2x_0\sqrt{2} < x_0^2 + 2$ and $\sqrt{2} < \frac{x_0}{2} + \frac{1}{x_0} = x_1$.

c. Case 1: $0 < x_0 < \sqrt{2}$, which implies that $\sqrt{2} < x_1$ by part (b). Thus, ← by a & b

$\frac{x_0^2 + 2}{2x_0} = \frac{x_0}{2} + \frac{1}{x_0} = x_1$

$$0 < x_0 < \sqrt{2} < x_{m+1} < x_m < \ldots < x_1 \quad \text{and} \quad \lim_{m\to\infty} x_m = \sqrt{2}.$$

Case 2: $x_0 = \sqrt{2}$, which implies that $x_m = \sqrt{2}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{2}$. ← trivial

Case 3: $x_0 > \sqrt{2}$, which by part (a) implies that $\lim_{m\to\infty} x_m = \sqrt{2}$. ← by a alone

7. Try applying Newton's method with initial approximation $x_0 = -0.677651$, to find just x_1 and x_2 (no more) for the equation

$$x^3 - 3x + 1 = 0.$$

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

Show your work. What happened? Explain algebraically. Explain graphically.

$f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

q7.m

```
function [output] = q7(input)
%Q7
f  = input^3 - (3*input) + 1;
f_ = 3*(input^2) - 3;
output = input - (f / f_);
end
```

$X_1 = X_0 - \frac{f(X_0)}{f'(X_0)} =$

```
>> q7(-0.677651)
ans =
   1.0000
```

$\boxed{1.0000}$

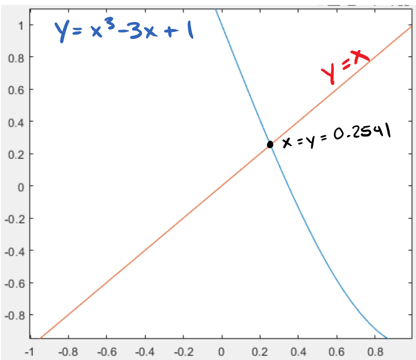
$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)} =$

```
>> q7(1.0000)
ans =
   Inf
```

$\boxed{\text{Inf}}$

The method diverged

$\frac{f(x)}{f'(x)} = \frac{x^3 - 3x + 1}{3x^2 - 3}$



$\frac{f(1)}{f'(1)} = \frac{1-3+1}{3-3} = \frac{-1}{0}$

Note: $x = x^3 - 3x + 1 \Rightarrow x^3 - 4x + 1 = 0 \Rightarrow x \approx -2.1149, 0.2541, 1.8608$

$f'(0.2541) = -2.8063$

Thus $|f'(0.2541)| > 1$ ∴ The fixed point itteration does not converge as shown Algebraically in Q5

★ Graphically f does not map $C[-1,1]$ into $C[-1,1]$ E.g. $f(-0.677651) = 2.7217$

8. (This problem is from the book, Section 2.4.)

$5 \times 10^{-2} = \frac{5}{100} = \frac{1}{20}$

6. Show that the following sequences converge ^sub^ linearly to $p = 0$. How large must n be before $|p_n - p| \leq 5 \times 10^{-2}$?

a. $p_n = \frac{1}{n}, \quad n \geq 1$

b. $p_n = \frac{1}{n^2}, \quad n \geq 1$

a.

$$\frac{|p_{n+1} - p|}{|p_n - p|} = \left|\frac{1}{n+1}\right| \cdot \left|\frac{n}{1}\right| = \left|\frac{n}{n+1}\right|$$

$$\boxed{\lim_{n \to \infty} \left|\frac{n}{n+1}\right| = 1}$$

$$|p - p_n| = p_n = \frac{1}{n}$$

$$\boxed{\frac{1}{n} \leq \frac{1}{20} \quad n \geq 20}$$

b.

$$\frac{|p_{n+1} - p|}{|p_n - p|} = \left|\frac{1}{n^2+1}\right| \cdot \left|\frac{n^2}{1}\right| = \left|\frac{n^2}{n^2+1}\right|$$

$$\boxed{\lim_{n \to \infty} \left|\frac{n^2}{n^2+1}\right| = 1}$$

$$|p - p_n| = p_n = \frac{1}{n^2}$$

$$\boxed{\frac{1}{n^2} \leq \frac{1}{20} \quad n \geq 5}$$