HW6

Wednesday, November 2, 2022 1:37 PM

Assignment 6

Exercise 1. Find w_1, w_2, and w_3 so that the quadrature rule

$$\int_{-1}^{2} f(x)\,dx \approx w_1 f(-1) + w_2 f(0) + w_3 f(2)$$

has the highest possible degree of precision. What is the $\underbrace{\text{degree of precision}}_{r}$?

$\int_{-1}^{2} dx = 2+1 = 3 = \omega_1 + \omega_2 + \omega_3$ $\boxed{r \geq 0}$

$\int_{-1}^{2} x\,dx = \frac{2^2-(-1)^2}{2} = \frac{3}{2} = -\omega_1 + 2\omega_3$ $\boxed{r \geq 1}$

$\int_{-1}^{2} x^2\,dx = \frac{2^3-(-1)^3}{3} = \frac{9}{3} = 3 = \omega_1 + 4\omega_3$ $\boxed{r \geq 2}$

$\int_{-1}^{2} x^3\,dx = \frac{16-1}{4} = \frac{15}{4} = \frac{3}{4} \times 2^3 + Err$

$3 \times 8 \neq 15 \Rightarrow Error \neq 0 \Rightarrow \boxed{r=2}$

$\frac{3}{2} + 3 = -\omega_1 + \omega_1 + 2\omega_3 + 4\omega_3$

$\frac{9}{2} = 6\omega_3 \Rightarrow \frac{9}{12} = \boxed{\omega_3 = \frac{3}{4}}$

$\omega_1 = 3 - 4 \cdot \frac{3}{4} = 3-3 = \boxed{0 = \omega_1}$

$3 - \frac{3}{4} - 0 = \boxed{\omega_2 = \frac{9}{4}}$

Exercise 2. Find x_1, x_2, and x_3 so that the quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx \frac{2}{3} f(x_1) + \frac{2}{3} f(x_2) + \frac{2}{3} f(x_3)$$

has the highest possible degree of precision. What is the degree of precision?

$\int_{-1}^{1} dx = 1+1 = 2 = 3\left(\frac{2}{3}\right)$ ✓ $\boxed{r \geq 0}$

$x_1 = -x_2 - x_3$

$x_1^2 = x_2^2 + 2x_2x_3 + x_3^2$

$\int_{-1}^{1} x\,dx = \frac{1-1}{2} = 0 = \frac{2}{3}(x_1 + x_2 + x_3) \Rightarrow 0 = x_1 + x_2 + x_3$

$\int_{-1}^{1} x^2\,dx = \frac{1+1}{3} = \frac{2}{3} = \frac{2}{3}(x_1^2 + x_2^2 + x_3^2) \Rightarrow 1 = x_1^2 + x_2^2 + x_3^2$

$\int_{-1}^{1} x^3\,dx = \frac{1-1}{4} = 0 = \frac{2}{3}(x_1^3 + x_2^3 + x_3^3) \Rightarrow 0 = x_1^3 + x_2^3 + x_3^3 = x_1 + x_2 + x_3$

Symmetry prompts $x_1 = -x_3$ & $x_2 = 0 \Rightarrow \frac{1}{2} = x_3^2$ & $x_3 = \sqrt{\frac{1}{2}}$, $x_1 = -\sqrt{\frac{1}{2}}$

Consider

$\int_{-1}^{1} x^4\,dx = \frac{1+1}{5} = \frac{2}{5} = \frac{2}{3}(x_1^4 + x_2^4 + x_3^4) \Rightarrow \frac{3}{5} = 2x_1^4$ $(\Rightarrow\Leftarrow)$ Thus $\boxed{r=3}$

$\int_{-1}^{1} x^4\,dx = \frac{1+1}{5} = \frac{2}{5} = \frac{2}{3}(x_1 + x_2 + \dots)$

Exercise 3. Write the following matrix equation as a system of equations:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

$x - y + 0 = 3$

$0 - y + 0 = 2$

$0 - y + z = 1$

$y = -2,\ x = 1,\ z = -1$

Exercise 4. Write the following linear system of equations as a matrix equation:

$$\begin{aligned} x + y + 2z &= 3 \\ 5x + 8y &= 13 \end{aligned}.$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 5 & 8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$$

$2 \times 3 \cdot 3 \times 1 = 2 \times 1$

Exercise 5. Perform the following matrix operation:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \left(3 \begin{bmatrix} -1 & 2 \\ 2 & 0 \\ 0 & 5 \end{bmatrix} - 4 \begin{bmatrix} 1 & 1 \\ 3 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} -3 & 6 \\ 6 & 0 \\ 0 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 12 & -4 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -7 & 2 \\ -6 & 4 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} -19 & 6 \\ 1 & -9 \\ -12 & 19 \end{bmatrix}$$

$3 \times 3 \cdot 3 \times 2 = 3 \times 2$

Exercise 6. For each of the following statements just say whether it is true or false.

(a)

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$\boxed{\text{FALSE}}$

(b) If A is a 3×4 matrix, and B is a 3×4 matrix, then AB is defined, and is a 3×4 matrix.

$\boxed{\text{FALSE}}$

(c) If A is a 3×3 matrix, and B is a 3×5 matrix, then AB is defined, and is a 3×5 matrix.

$\boxed{\text{TRUE}}$

(d) The identity matrix of order 5 is a square matrix. $\boxed{\text{TRUE}}$

(e) The identity matrix of order 5 is a diagonal matrix. $\boxed{\text{TRUE}}$

Exercise 7. For each of the following statements just say whether it is true or false.

(a) The following matrices are inverses of one another.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$\boxed{\text{TRUE}}$

(b) The following matrices are inverses of one another.

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

$\boxed{\text{FALSE}}$

(c) The following matrices are inverses of one another.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$\boxed{\text{FALSE}}$

(d) The following matrices are transposes of one another.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$\boxed{\text{TRUE}}$

(e) A is nonsingular if and only if A^{-1} is nonsingular.

$\boxed{\text{TRUE}}$

Exercise 8. Prove that if A is nonsingular, then A^{-1} is unique. (Hint. Assume B and C are inverses of A. Then show that $B = C$.

Given A^{-1} exists :

$$A \cdot A^{-1} = I = A \cdot B = A \cdot C \Rightarrow B = C = A^{-1}$$

Exercise 9. For each of the following statements just say whether it is true or false.

(a) The set of all real numbers is a vector space. $\boxed{\text{TRUE}}$

(b) The xy-plane is a vector space. $\boxed{\text{TRUE}}$

(c) The set of all polynomials of degree ≤ 3 is a vector space. $\boxed{\text{TRUE}}$

(d) The set of all polynomials of degree exactly $= 5$ is a vector space. $\boxed{\text{FALSE}}$

(e) The set of all real-valued functions of a real variable (defined on $\mathbb{R}$) is a vector space.

$\boxed{\text{TRUE}}$

Exercise 10. Let V be the vector space of all functions f, infinitely many times differentiable on $(-\infty, \infty)$. Consider the operator

$$L(f) = f' + cf$$

for $f \in V$, where c is a constant. By using the following definition, show that L is a linear operator on V.

Definition. A *linear operator* $L(v)$ on a vector space V is a function which maps elements of V into V $(L : V \to V)$ with the following property:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v).$$

$$L(\alpha u + \beta v) = (\alpha u' + \beta v') + c(\alpha u + \beta v) = \alpha(u' + cu) + \beta(v' + cv) = \alpha L(u) + \beta L(v)$$