HW5

Saturday, October 22, 2022 5:17 AM

Assignment 5

1. Consider the integral

$$\int_1^e \ln(x)\, dx.$$

(a) Find the exact value of the integral.
(Hint: from the table of integrals: $\int \ln(x)\, dx = x\ln(x) - x$.)

$$\int_1^e \ln(x)\,\delta x = x\ln(x) - x \Big|_1^e = (e\ln(e) - e) - (1\cdot \ln(1) - 1)$$

$$= e\ln(e) - \ln(1) + 1 - e$$

$$= e(1) - 0 + 1 - e = \boxed{1}$$

(b) Approximate the integral using the trapezoidal rule, with three significant digits.

$x_1 = 1$; $1 + h = e \Rightarrow h = e - 1$ $\quad f' = \frac{1}{x}$; $f'' = \frac{-1}{x^2}$

$$\int_1^e \ln(x)\,dx = \frac{e-1}{2}\cdot \ln(1) + \frac{e-1}{2}\cdot \ln(e) - \frac{(e-1)^3}{12}\cdot -\frac{1}{\xi^2}$$

$$\int_1^e \ln(x)\,dx \approx \frac{e-1}{2} \approx \boxed{0.859}$$

(c) Find an upper bound for the absolute error using the error formula of the trapezoidal rule, and compare it to the actual absolute error.

$\max\left\{ \left| \frac{-1}{x^2} \right| \right\} \Big| \; 1 \le x \le e$ happens @ $x = 1 \Rightarrow |f''(\xi)| \le 1$

$$\Rightarrow \boxed{|\text{Error}| \le \frac{(e-1)^3}{12} \approx 0.423}$$

Absolute Error $= |1 - 0.859| = 0.141 < 0.423$

2. Repeat Exercise 1 using Simpson's rule. You may skip Part a.

(b) Approximate the integral using the Simpson's rule, with three significant digits.

$x_1 = 1$; $1 + 2h = e \Rightarrow h = \frac{e-1}{2}$ $\quad f''' = \frac{2}{x^3}$; $f^{(4)} = \frac{-6}{x^4}$

$$\int_1^e \ln(x)\,dx \approx \frac{e-1}{6} \cdot \ln(1) + \frac{2(e-1)}{3} \ln\left(1 + \frac{e-1}{2}\right) + \frac{e-1}{6} \ln(e)$$

$$\int_1^e \ln(x)\,dx \approx \frac{2(e-1)}{3} \ln\left(\frac{e+1}{2}\right) + \frac{e-1}{6} \approx \boxed{0.997}$$

(c) Find an upper bound for the absolute error using the error formula of the Simpson's rule, and compare it to the actual absolute error.

$\max\left\{\left|\frac{-6}{x^4}\right|\right\} \mid 1 \le x \le e$ happens @ $x = 1 \Rightarrow |f^{(4)}(\xi)| \le 6$

$$\Rightarrow \boxed{|\text{Error}| \le \frac{6}{90} \cdot \left(\frac{e-1}{2}\right)^5 \approx 0.031}$$

Absolute Error $= |1 - 0.997| = 0.003 < 0.031$

3. Repeat Exercise 1 using the midpoint rule. You may skip Part a.

(b) Approximate the integral using the midpoint rule, with three significant digits.

$x_1 - h = 1$; $x_1 = \frac{1+e}{2}$; $h = \frac{1+e}{2} - \frac{2}{2} = \frac{e-1}{2}$ $\quad f'' = \frac{-1}{x^2}$

$$\int_1^e \ln(x)\,dx \approx (e-1) \cdot \ln\left(\frac{1+e}{2}\right) \approx \boxed{1.066}$$

(c) Find an upper bound for the absolute error using the error formula of the midpoint rule, and compare it to the actual absolute error.

From 1.c, $|f''(\xi)| \le 1 \mid 1 \le x \le e$

$$\Rightarrow \boxed{|\text{Error}| \le \left|\frac{\left(\frac{e-1}{2}\right)^3}{3}\right| \approx 0.211}$$

$$\Rightarrow \boxed{|\text{Error}| \le \left| \frac{\left(\frac{e-1}{2}\right)^3}{3} \right| \approx 0.211}$$

$$\text{Absolute Error} = |1 - 1.0661| = 0.066 < 0.211$$

4. (From the book)

13. The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value 4, and Simpson's rule gives the value 2. What is $f(1)$?

$$\frac{h}{2}\left(f(x_0) + f(x_1)\right) = 4 = \frac{2}{2}\left(f(0) + f(2)\right) = f(0) + f(2)$$

$$\frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) = 2 = \frac{2}{3}\left(f(0) + f(1) + f(2)\right)$$

$$\Rightarrow 3 = 4 + f(1) \Rightarrow \boxed{-1 = f(1)}$$

5. (From the book)

14. The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value 5, and the Midpoint rule gives the value 4. What value does Simpson's rule give?

$$\frac{h}{2}\left(f(x_0) + f(x_1)\right) = 5 = \frac{2}{2}\left(f(0) + f(2)\right) = f(0) + f(2)$$

$$h \cdot f\left(\tfrac{h}{2}\right) = 4 = 2 \cdot f\left(\tfrac{2}{2}\right) = 2f(1) \Rightarrow f(1) = \frac{4}{2} = 2$$

$$\frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) = \frac{2}{3}\left(f(0) + f(1) + f(2)\right)$$

$$= \frac{2}{3}(5 + 2) = \frac{2}{3} \cdot 7 = \boxed{\frac{14}{3}}$$

6. (From the book)

20. Let $h = (b-a)/3$, $x_0 = a$, $x_1 = a + h$, and $x_2 = b$. Find the degree of precision of the quadrature formula $a+h = \frac{3a}{3} + \frac{b-a}{3} = \frac{b+2a}{3}$

$$\int_a^b f(x)\,dx = \frac{9}{4}hf(x_1) + \frac{3}{4}hf(x_2).$$

$\int_a^b dx = x\Big|_a^b = b-a = \frac{9}{12}(b-a) + \frac{3}{12}(b-a)$ ✓ $\boxed{r \geq 0}$

$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2} = \frac{9}{12}(b-a)\left(\frac{b+2a}{3}\right) + \frac{3}{12}(b-a)(b)$

$\Rightarrow 6(b^2-a^2) = 3(b^2+2ab-ab-2a^2) + 3(b^2-ab)$

$\Rightarrow 2b^2-2a^2 = (b^2+b^2) + (ab-ab) - (2a^2)$ ✓ $\boxed{r \geq 1}$

$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3-a^3}{3} = \frac{9}{12}(b-a)\left(\frac{b+2a}{3}\right)^2 + \frac{3}{12}(b-a)(b)^2$

$\Rightarrow 4(b^3-a^3) = (b-a)(b^2+4ab+4a^2) + 3(b-a)(b^2)$

$\Rightarrow 4b^3-4a^3 = (b^3+4ab^2+4a^2b) - (b^2a+4a^2b+4a^3) + 3(b^3-ab^2)$ $\boxed{r \geq 2}$

$\Rightarrow 4b^3-4a^3 = (b^3+3b^3) + (4ab^2-ab^2-3ab^2) + (4a^2b-4a^2b) - (4a^3)$ ✓

$\int_a^b x^3\,dx = \frac{x^4}{4}\Big|_a^b = \frac{b^4-a^4}{4} = \frac{9}{12}(b-a)\left(\frac{b+2a}{3}\right)^3 + \frac{3}{12}(b-a)(b^3)$

$\Rightarrow 3(b^4-a^4) = \frac{(b-a)(8a^3+12a^2b+6ab^2+b^3)}{3} + 3(b^4-ab^3)$

$\Rightarrow 9b^4-9a^4 = (8a^3b+12a^2b^2+6ab^3+b^4) - (8a^4+12a^3b+6a^2b^2+ab^3) + (9b^4-9ab^3)$

$\Rightarrow 9b^4-9a^4 = (b^4+9b^4) + (6ab^3-ab^3-9ab^3) + (12a^2b^2-6a^2b^2) + (8a^3b-12a^3b) - (8a^4)$

$$9b^4 - 9a^4 = 10b^4 - 4ab^3 + 6a^2b^2 - 4a^3b - 8a^4$$ As $a \neq b$, this is a contradiction, thus error exists and our degree of precision is $\boxed{r=2}$

7. (From the book)

22. The quadrature formula $\int_0^2 f(x)\,dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for all polynomials of degree less than or equal to two. Determine c_0, c_1, and c_2.

$$\int_0^2 dx = x\Big|_0^2 = 2 - 0 = 2 = C_0 + C_1 + C_2$$

$$\int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = 2 = C_0(0) + C_1(1) + C_2(2) = C_1 + 2C_2$$

$$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3} = C_0(0^2) + C_1(1^2) + C_2(2^2) = C_1 + 4C_2$$

$$\cancel{C_1 - C_1} + 4C_2 - 2C_2 = 2C_2 = \frac{8}{3} - 2 = \frac{2}{3} \Rightarrow \boxed{C_2 = \frac{1}{3}}$$

$$2 = C_1 + \frac{2}{3} \Rightarrow \boxed{C_1 = \frac{4}{3}} \qquad 2 = C_0 + \frac{5}{3} \Rightarrow \boxed{C_0 = \frac{1}{3}}$$

8. (From the book)

24. Find the constants x_0, x_1, and c_1 so that the quadrature formula

$$\int_0^1 f(x)\,dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$$

has the highest possible degree of precision.

$$\int_0^1 dx = x\Big|_0^1 = 1 - 0 = 1 = \frac{1}{2} + C_1 \Rightarrow \boxed{C_1 = \frac{1}{2}} \qquad \boxed{r \geq 0}$$

$\boxed{r \geq 1}$

$$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} = \frac{1}{2}x_0 + C_1 x_1 = \frac{1}{2}(x_0 + x_1) \Rightarrow 1 = x_0 + x_1$$

$\boxed{r \geq 2}$

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} = \frac{1}{2}x_0^2 + C_1 x_1^2 = \frac{1}{2}(x_0^2 + x_1^2) \Rightarrow \frac{2}{3} = x_0^2 + x_1^2$$

$$x_0^2 = 1 - 2x_1 + x_1^2 \Rightarrow \frac{2}{3} = 1 - 2x_1 + 2x_1^2$$

$$x_0 = 1 - x_1 \Rightarrow x_0^2 = 1 - 2x_1 + x_1^2 \Rightarrow \frac{2}{3} = 1 - 2x_1 + 2x_1^2$$

$$0 = \frac{1}{3} - 2x_1 + x_1^2 \ ; \ x_1 = \frac{2 \pm \sqrt{4 - 4(\frac{1}{3})(1)}}{2(1)} = 1 \pm \sqrt{\frac{4(1-\frac{1}{3})}{4}} = 1 \pm \sqrt{\frac{2}{3}}$$

As $0 \le x_1 \le 1$; $\boxed{x_1 = 1 - \sqrt{\frac{2}{3}}}$; $\boxed{x_0 = 1 - x_1 = \sqrt{\frac{2}{3}}}$

$$\int_0^1 x^3\,dx = \frac{1}{4} = \frac{1}{2}(x_0^3 + x_1^3) \Rightarrow \frac{1}{2} \neq x_0^3 + x_1^3 \quad (\Rightarrow\Leftarrow) \quad \text{Thus } \boxed{r=2}$$

9. (From the book)

26. Derive Simpson's rule with error term by using

$$\int_{x_0}^{x_2} f(x)\,dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi).$$

Find a_0, a_1, and a_2 from the fact that Simpson's rule is exact for $f(x) = x^n$ when $n = 1, 2,$ and 3. Then find k by applying the integration formula with $f(x) = x^4$.

$$\hbar = 2h = x_2 - x_0 = (x_2 - x_1) + (x_1 - x_0)$$

$$u = x_2 - x_0$$

$$\int_{x_0}^{x_2} dx = x_2 - x_0 = \hbar = a_0 + a_1 + a_2$$

$$\int_{x_0}^{x_2} (x - x_0)\,dx = \int_0^{\hbar} u\,du = \frac{u^2}{2}\Big|_0^{\hbar} = \frac{\hbar^2}{2} = a_1\left(\frac{\hbar}{2}\right) + a_2\hbar \Rightarrow \hbar = a_1 + 2a_2$$

$$\int_{x_0}^{x_2} (x - x_0)^2\,dx = \int_0^{\hbar} u^2\,du = \frac{u^3}{3} = \frac{\hbar^3}{3} = a_1\frac{\hbar^2}{4} + a_2\hbar^2 \Rightarrow \hbar = \frac{3}{4}a_1 + 3a_2$$

$$\int_{x_0}^{x_2} (x - x_0)^3\,dx = \int_0^{\hbar} u^3\,du = \frac{u^4}{4} = \frac{\hbar^4}{4} = a_1\frac{\hbar^3}{8} + a_2\hbar^3 \Rightarrow \hbar = \frac{1}{2}a_1 + 4a_2$$

$$2a_2 = \hbar - a_1 \Rightarrow \hbar = \frac{1}{2}a_1 + 2\hbar - 2a_1 \Rightarrow -\hbar = -\frac{3}{2}a_1 \Rightarrow a_1 = \frac{2\hbar}{3} \Rightarrow \boxed{a_1 = \frac{4h}{3}}$$

$$\hbar = \frac{2\hbar}{3} + 2a_2 \Rightarrow \frac{\hbar}{3} = 2a_2 \Rightarrow a_2 = \frac{\hbar}{6} = \boxed{a_2 = \frac{h}{3}}$$

$$\hbar = a_0 + \frac{4\hbar}{6} + \frac{\hbar}{6} \Rightarrow a_0 = \frac{\hbar}{6} \Rightarrow \boxed{a_0 = \frac{h}{3}}$$

$$\ldots 12(x - x_0)^2 \ ; \ g^{(3)} = 24(x - x_0) \ ; \ g^{(4)} = 24$$

$g=(x-x_0)^4$; $g'=4(x-x_0)^3$; $g''=12(x-x_0)^2$; $g^{(3)}=24(x-x_0)$; $g^{(4)}=24$

$$\int_{x_0}^{x_2}(x-x_0)^4 = \int_0^{2h} u^4\,du = \frac{u^5}{5}\Big|_0^{2h} = \frac{(2h)^5}{5} = \frac{32h^5}{5} = \frac{h}{3}\left(0+4h^4+16h^4\right)+24K$$

$$\frac{32h^5}{5} - \frac{4h^5}{3} - \frac{16h^5}{3} = 24K$$

$$h^5\left(\frac{96}{15} - \frac{20}{15} - \frac{80}{15}\right) = 24K \; ; \; \frac{-4h^5}{360} = K \Rightarrow \boxed{K = -\frac{h^5}{90}}$$

Thus we have $$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right) - \frac{h^5}{90}f^{(4)}(\xi)$$