MTH 375
Fall 2022
Hw 2 key

1. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two random samples with the same mean μ and variance σ^2.

(a) Show that $T = \frac{1}{2}\overline{X} + \frac{1}{2}\overline{Y}$ and $U = \frac{1}{3}\overline{X} + \frac{2}{3}\overline{Y}$ are both unbiased estimators of μ.
$E(T) = E(\frac{1}{2}\overline{X} + \frac{1}{2}\overline{Y}) = \frac{1}{2}E(\overline{X}) + \frac{1}{2}E(\overline{Y}) = \frac{1}{2}\mu + \frac{1}{2}\mu = \mu$, and
$E(U) = E(\frac{1}{3}\overline{X} + \frac{2}{3}\overline{Y}) = \frac{1}{3}E(\overline{X}) + \frac{2}{3}E(\overline{Y}) = \frac{1}{3}\mu + \frac{2}{3}\mu = \mu$.

(b) Evaluate MSE($T;\mu$) and MSE($U;\mu$). According to the MSE criterion, is T or U the better estimator of μ?

Since T and U are unbiased, MSE(T) $= V(\frac{1}{2}\overline{X} + \frac{1}{2}\overline{Y}) = \frac{\sigma^2}{4n} + \frac{\sigma^2}{4n} = \frac{\sigma^2}{2n}$, and
MSE(U) $= V(\frac{1}{3}\overline{X} + \frac{2}{3}\overline{Y}) = \frac{\sigma^2}{9n} + \frac{4\sigma^2}{9n} = \frac{5\sigma^2}{9n}$
According to the MSE criterion, T is a slightly better estimator of μ that U.

2. Let $X_1, \ldots, X_n$ be a random sample uniform on $[0,\theta]$.

(a) Show that $T = 2\overline{X}$ is an unbiased estimator of θ, and evaluate MSE($T;\theta$).
$E(T) = 2E(\overline{X}) = 2 \cdot \theta/2 = \theta$. Since T is unbiased, MSE($T;\theta$) $= V(T) = \frac{4}{n}V(X_i) = \frac{1}{3n}\theta^2$.

(b) Let $M = \max\{X_1, \ldots, X_n\}$. Find the pdf of M.
(Hint: Use the fact that $F_M(m) = P(M \le m) = P(X_1 \le m \,\&\cdots\&\, X_n \le m)$.)
$F_M(m) = P(X \le m)^n = (m/\theta)^n$, so $f_M(m) = nm^{n-1}/\theta^n$ for $0 < m < \theta$.

(c) Compute $E(M)$ and $V(M)$.

$$E(M) = \int_0^\theta m \cdot \frac{1}{\theta^n} nm^{n-1}\, dm = \frac{n}{n+1}\theta. \quad E(M^2) = \int_0^\theta m^2 \cdot \frac{1}{\theta^n} nm^{n-1}\, dm = \frac{n}{n+2}\theta^2,$$

so $V(M) = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{(n+1)^2(n+2)}\theta^2$.

(d) Use you answers to (c) find an unbiased estimator M^* of θ based on M, and to evaluate MSE($M^*;\theta$). According to the MSE criterion, which is a better estimator of θ, T or M^*?.

$M^* = \frac{n+1}{n}M$. Of course, $E(M^*) = \theta$. MSE(M^*) $= \left(\frac{n+1}{n}\right)^2 V(M) = \frac{1}{n(n+2)}\theta^2$.
For $n = 1$, MSE(T)=MSE(M^*), and MSE(M^*) is smaller for all $n > 1$. By the MSE criterion, M^* is the better estimator.

#3. 4.1.10 Let $X_1, \ldots, X_n$ be a sample of iid Bin(1, θ) random variables, and let $T = \overline{X}(1 - \overline{X})$ be an estimator of V(X_i) $= \theta(1-\theta)$. Determine

a) $E(T)$. $E(\overline{X}) = \theta$, and $E(\overline{X}^2) = (E(\overline{X}))^2 + V(\overline{X}) = \theta^2 + \frac{\theta(1-\theta)}{n}$ so
$E(T) = E(\overline{X}(1-\overline{X})) = \theta - \theta^2 - \frac{\theta(1-\theta)}{n} = (1 - \frac{1}{n})\theta(1-\theta)$.

b) Bias($T;\theta(1\theta)$) $= E(T) - \theta(1-\theta) = (1-\frac{1}{n})\theta(1-\theta) - \theta(1-\theta) = -\frac{1}{n}\theta(1-\theta)$.

c) the asymptotic bias of T for estimating $\theta(1-\theta)$. $\lim_{n\to\infty} \text{Bias}(T;\theta(1-\theta)) = 0$

d) an unbiased estimator of $\theta(1-\theta)$ based on T.

Since $E(T) = (\frac{n-1}{n})\theta(1-\theta)$, an unbiased estimator of $\theta(1-\theta)$ is $T^* = \frac{n}{n-1}T$.

4. Let $X_1, \ldots, X_n$ be a sample of i.i.d $N(0, \sigma^2)$ random variables. Let $T = \dfrac{1}{n}\sum_{i=1}^{n} X_i^2$.

(a) Show that T is an unbiased estimator of σ^2. Since $E(X_i) = 0$, $E(T) = E(X_1^2) = \sigma^2$, as required.

(b) Find MSE(T, σ^2). **Hint**: The mgf of X_i is $M(t) = e^{\frac{1}{2}\sigma^2 t^2}$.

MSE$(T) = \dfrac{1}{n}V(X_1^2) = \dfrac{1}{n}(E(X_1^4) - (E(X_1^2))^2 = \dfrac{1}{n}(E(X^4) - \sigma^4)$. Use the MGF to compute $E(X_1^4)$:

$M'(t) = \sigma^2 t e^{\frac{1}{2}\sigma^2 t^2}$; $M''(t) = (\sigma^2 + \sigma^4 t^2)e^{\frac{1}{2}\sigma^2 t^2}$;

$M'''(t) = (2\sigma^4 t + \sigma^4 t + \sigma^6 t^3)e^{\frac{1}{2}\sigma^2 t^2} = (3\sigma^4 t + \sigma^6 t^3)e^{\frac{1}{2}\sigma^2 t^2}$;

$M''''(t) = (3\sigma^4 + 3\sigma^6 t^2 + 3\sigma^6 t^2 + \sigma^8 t^4)e^{\frac{1}{2}\sigma^2 t^2}$, so $E(X_1^4) = M''''(0) = 3\sigma^4$. Substituting into the formula above, we have MSE$(T) = \dfrac{2}{n}\sigma^4$.

A quicker way to do 4(b): We know that X_i/σ is N(0,1), so $\dfrac{nT}{\sigma^2} = \sum_{i=1}^{n}\left(\dfrac{X_i}{\sigma}\right)^2$ is χ^2 with n df, so $V\left(\dfrac{nT}{\sigma^2}\right) = 2n$, and it follows that MSE$(T, \sigma^2) = V(T) = \dfrac{2}{n}\sigma^4$. .

5. #4.1.20. Let $X_1, \ldots X_n$ be a sample of iid Exp(β) random variables. Use the Delta Method to determine the approximate standard error of $\hat{\beta^2} = \overline{X}^2$.

Since X_i is Exp(β), $V(X_i) = \beta^2$, $V(\overline{X}) = \beta^2/n$ and so SE$(\overline{X}) = \beta/\sqrt{n}$.

Let $g(x) = x^2$, so $g'(\beta) = 2\beta$. According to the delta-method,

SE$(\overline{X}^2) \approx |g'(\beta)|$ SE$(\overline{X}) = 2\beta^2/\sqrt{n}$.