MTH 375
Fall 2022
Assignment 6 key

#1. Let $X_1, \ldots, X_n$ be iid binomial(1, θ) random variables.
If the prior pdf of θ is Beta(1, 2), determine
a) the posterior distribution of $\theta|\vec{x}$.

$$f(\theta|\vec{x}) = \frac{f(\vec{x}|\theta)\pi(\theta)}{\int_\theta f(\vec{x}|\theta)\pi(\theta)\ d\theta} = \frac{\theta^{x_1+\cdots+x_n}(1-\theta)^{n-(x_1+\cdots+x_n)}\cdot\Gamma(3)/(\Gamma(1)\Gamma(2))\cdot\theta^0(1-\theta)^1}{\int_\theta \theta^{x_1+\cdots+x_n}(1-\theta)^{n-(x_1+\cdots+x_n)}\cdot\Gamma(3)/(\Gamma(1)\Gamma(2))\cdot\theta^0(1-\theta)^1\ d\theta}$$

$$= \frac{\theta^{x_1+\cdots+x_n}(1-\theta)^{n+1-(x_1+\cdots+x_n)}}{\int_\theta \theta^{x_1+\cdots+x_n}(1-\theta)^{n+1-(x_1+\cdots+x_n)}\ d\theta}.$$

Thus the posterior distribution of $\theta|\vec{x}$ is Beta with parameters $\alpha = \sum_{i=1}^n x_i + 1$ and $\beta = n + 2 - \sum_{i=1}^n x_i$.

b) the Bayesian estimator of θ for the squared error loss function is

$$E(\theta|\vec{x}) = \frac{\alpha}{\alpha+\beta} = \frac{\sum_{i=1}^n x_i + 1}{n+3}.$$

c) Suppose that $n = 10$, $\sum_{k=1}^{10} x_i = 7$. The Bayesian estimate of θ for square error loss is $\mathrm{E}(\theta|\vec{x}) = \frac{8}{13} \approx .615$.

The Bayesian estimate for θ for absolute error loss is the median, which is (via R) .6215.

#2. Let $X_1, \ldots, X_{50}$ be iid geometric(θ), and suppose that θ has prior pdf Beta(5,10). Say $\sum_{i=1}^{50} x_i = 149$.

a) Find the posterior pdf of θ. I am using the geometric pmf formula $f(x|\theta) = \theta(1-\theta)^{x-1}$.

$$f(\theta|\vec{x}) = \frac{f(\vec{x}|\theta)\pi(\theta)}{\int_\theta f(\vec{x}|\theta)\pi(\theta)\ d\theta} = \frac{\theta^{50}(1-\theta)^{\sum_{i=1}^{50} x_i - 50}\Gamma(15)/(\Gamma(5)\Gamma(10))\theta^4(1-\theta)^9}{\int_\theta \theta^{50}(1-\theta)^{\sum_{i=1}^{50} x_i - 50}\Gamma(15)/(\Gamma(5)\Gamma(10))\theta^4(1-\theta)^9\ d\theta} =$$

$$\frac{\theta^{54}(1-\theta)^{108}}{\int_\theta \theta^{54}(1-\theta)^{108}\ d\theta}.$$

The posterior pdf of θ is beta($\alpha = 55, \beta = 109$).

b) The Bayesian estimator of θ for squared error loss is $E(\theta|\vec{x}) = \frac{55}{164} \approx .3354$.

c) The Bayesian estimator of θ for the absolute error loss is the median $\approx .3347$.

#3. Let $X_1, \ldots, X_{60}$ be iid random variables with pdf $f(x|\theta) = \theta e^{-\theta x}$. If the prior pdf of θ is Gamma(α, β), determine

a) the posterior distribution of $\theta|x_1, \ldots, x_n$ is .

$$f(\theta|\vec{x}) = \frac{f(\vec{x}|\theta)\pi(\theta)}{\int_\theta f(\vec{x}|\theta)\pi(\theta)\ d\theta} = \frac{\theta^n e^{-\theta\sum_{i=1}^{60} x_i}\cdot 1/(\Gamma(\alpha)\beta^\alpha)\cdot\theta^{\alpha-1}e^{-\theta/\beta}}{\int_\theta \theta^n e^{-\theta\sum_{i=1}^{60} x_i}\cdot 1/(\Gamma(\alpha)\beta^\alpha)\cdot\theta^{\alpha-1}e^{-\theta/\beta}\ dx}$$

$$= \frac{\theta^{n+\alpha-1}e^{-\theta(\sum_{i=1}^{60} x_i + 1/\beta)}}{\int_\theta \theta^{n+\alpha-1}e^{-\theta(\sum_{i=1}^{60} x_i + 1/\beta)}\ d\theta}.$$

Thus the posterior distribution of $\theta|\vec{x}$ is

Gamma($60+\alpha, 1/(\ \sum_{i=1}^{60} x_i + 1/\beta$), or Gamma$\left(60+\alpha, \frac{\beta}{\beta\sum_{i=1}^{60} x_i + 1}\right)$.

b) The Bayesian estimator of θ for squared error loss is $E(\theta|\vec{x}) = \frac{\beta(60+\alpha)}{\beta\sum_{i=1}^{60} x_i + 1}$

c) If $\sum_{i=1}^{60} x_i = 143.1$, $\alpha = 3.5$, and $\beta = 6$ the posterior distribution of $\theta|\vec{x}$ is Gamma(63.5, .00698), so $E(\theta|\vec{x}) = .4432$.

d)The Bayesian estimator of θ for absolute error loss is .4409.

#4. Let $X_1, \ldots, X_n$ be iid binomial(2,θ) random variables. If the prior pdf of θ is Uniform[0, 1], find

a) The posterior distribution of $\theta|\vec{x}$ is

$$f(\theta|\vec{x}) = \frac{f(\vec{x}|\theta)\pi(\theta)}{\int_\theta f(\vec{x}|\theta)\pi(\theta)\ d\theta} = \frac{\prod_{i=1}^n \binom{2}{x_i}\theta^{\sum_{i=1}^n x_i}(1-\theta)^{2n-\sum_{i=1}^n x_i}}{\int_\theta \prod_{i=1}^n \binom{2}{x_i}\theta^{\sum_{i=1}^n x_i}(1-\theta)^{2n-\sum_{i=1}^n x_i}\ d\theta}$$

$$= \frac{\theta^{\sum_{i=1}^n x_i}(1-\theta)^{2n-\sum_{i=1}^n x_i}}{\int_\theta \theta^{\sum_{i=1}^n x_i}(1-\theta)^{2n-\sum_{i=1}^n x_i}\ d\theta},$$ that is, Beta$(\sum_{i=1}^n x_i + 1, 2n - \sum_{i=1}^n x_i + 1)$.

b) The Bayesian estimator of θ for squared error loss is

$$E(\theta|\vec{x}) = \frac{\sum_{i=1}^n x_i + 1}{2n+2}.$$

c) When when $n = 10$ and $\sum_{i=1}^{10} x_i = 17$, the posterior distribution of θ is Beta(18,4), so the Bayesian estimator for the squared error loss function is $E(\theta|\vec{x}) = \frac{18}{22} \approx .818$, and

d) the Bayesian estimator of θ for absolute loss is the median of Beta(18,4) $\approx .828$.

#5. Let $X_1, \ldots, X_{10}$ be iid binomial(1,θ) random variables, with prior pdf

$$\pi(\theta) = \begin{cases} 1/3 & \text{if } \theta = .5 \\ 2/3 & \text{if } \theta = .8 \end{cases}.$$

(a) Find the posterior distribution of $\theta|x_1, \ldots, x_{10}$.

(Hint: Since θ can take only two values, you need only compute $p(\theta = .5 \mid x_1, \ldots, x_{10}$ and $p(\theta = .8 \mid x_1, \ldots, x_{10})$.)

$$P(\theta = .5|\vec{x}) = \frac{p(\vec{x}|\theta = .5)\pi(.5)}{\sum_\theta p(\vec{x}|\theta)\pi(\theta)} = \frac{.5^{10}\cdot\frac{1}{3}}{.5^{10}\cdot\frac{1}{3} + .8^{\sum_{i=1}^{10} x_i}\cdot .2^{10-\sum_{i=1}^{10} x_i}\cdot\frac{2}{3}}, \quad \text{and}$$

$$P(\theta = .8|\vec{x}) = \frac{p(\vec{x}|\theta = .8)\pi(.8)}{\sum_\theta p(\vec{x}|\theta)\pi(\theta)} = \frac{.8^{\sum_{i=1}^{10} x_i}\cdot .2^{10-\sum_{i=1}^{10} x_i}\cdot\frac{2}{3}}{.5^{10}\cdot\frac{1}{3} + .8^{\sum_{i=1}^{10} x_i}\cdot .2^{10-\sum_{i=1}^{10} x_i}\cdot\frac{2}{3}}.$$

(b) Suppose $\sum_{k=1}^{10} x_i = 6$. Find the Bayesian estimate of θ for the squared error loss function, and for the absolute error loss function.

Plugging in $\sum_{k=1}^{10} x_i = 6$, we have

$P(\theta = .5|\vec{x}) \approx .538$, and $P(\theta = .8|\vec{x}) \approx .462$.

The Bayesian estimate for squared error loss function is $E(\theta|\vec{x}) \approx .5\cdot .538 + .8\cdot .462 \approx .639$.

The Bayesian estimate for absolute error loss function is harder to find, because the median of a discrete distribution has been defined by different statisticians in different ways. In fact, E$(|a - \theta| \mid \vec{x}) = .538|a - .5| + .639|a - .8|$ is minimized at $a = .5$.