MTH375: Mathematical Statistics - Homework #10

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Key Concepts: χ^2 test for goodness of fit, linear models, estimators of β, σ^2 and their distributions.

1. A six-sided die is rolled 100 times. The numbers that come up are ...

number	1	2	3	4	5	6
frequency	20	15	17	11	18	19

We want to use the χ^2 test to decide whether there is suffcient evidence at level of significance $\alpha = 0.05$ to conclude that the die is unfair, i.e., that some numbers are more likely than others.

State H_0 and H_A in statistical language, describe the test you will use, carry out the test, and state the conclusion.

Solution:

We want to test for a (un)fair die, and thus that all sides have equal probablility, are hypotheses are thus ...

$$H_0 : p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1/6 \text{ vs. } H_A : \neg H_0.$$

Our test is know as a χ^2 test for categorical data, as we have 6 categories we test on a χ^2 statistic with 5 degrees of freedom.

Now to carry out the test in **R** ...

- `c(20, 15, 17, 11, 18, 19) -> Oi` .
- `c(100, 100, 100, 100, 100, 100) / 6 -> Ei` .
- `sum( (Oi - Ei)^2 / Ei ) -> T` .
- `qchisq(p = (1 - 0.05), df = 5) -> t` .

Our test is of the form "Reject H_0 if $T > t$".

Our results are $T = 3.2$ and $t = 11.0705$
As $3.2 < 11.0705$ we fail to reject H_0 with a 95% confidence.

2. Use the χ^2 test to decide, at level of significance $\alpha = 0.05$, whether the data supports or refutes the claim that the 60 data points below came from an exponentially distributed population.

0.105, 0.183, 0.219, 0.313, 0.326, 0.345, 0.454, 0.461, 0.467, 0.551,
0.603, 0.757, 0.802, 0.824, 0.826, 0.844, 0.987, 1.087, 1.159, 1.180,
1.249, 1.252, 1.317, 1.326, 1.390, 1.398, 1.580, 1.618, 1.653, 1.660,
1.759, 1.850, 1.875, 2.638, 2.691, 2.811, 2.823, 2.828, 2.924, 3.108,
3.323, 3.671, 3.792, 3.797, 4.574, 4.855, 4.924, 5.098, 5.287, 5.346,
6.000, 6.335, 6.491, 6.625, 7.125, 7.586, 8.028, 9.071, 10.783, 11.034

The data was put in order from least to greatest and sorted into 6 groups of 10 only to make it easier to read. The sum of these numbers is 176.0.

Divide the positive real numbers into 5 intervals in such a way that the probability of landing in a given interval is 0.2, and use those intervals to perform the test.

Since you will be estimating a parameter, you will substract one additional degree to freedom when performing the χ^2 test.

Solution:

We first need the MLE for a random variable distibuted on $Exponential(\lambda)$.

This is $\hat{\lambda}_{MLE} = \overline{X} = \Sigma_X/n = 176/60$.

We thus are testing the following hypotheses ...

$$H_0 : X_i \sim Exponential(\lambda = 176/60) \text{ vs. } H_A : \neg H_0.$$

Our test is know as a χ^2 test for categorical data, as we have 5 categories we test on a χ^2 statistic with 4 degrees of freedom.

We can now use **R** to compute our data categories of equal probability interval 0.2, and then carry out the test, leveraging the expected outcomes from our interval assumption ...

- `pexp(q = c(0.2, 0.4, 0.6, 0.8, 1.0), rate = 60 / 176) * 60 -> cuts` .

No; pexp gives you the cdf. We need qexp to get the quantiles.

So these are the wrong numbers.

We find the following cuts, $(0, 3.955), (3.955, 7.648), (7.648, 11.099), (11.099, 14.322), (14.322, 17.333)$, from which we can then count our observations for each category.

- `c(44, 12, 4, 0, 0) -> Oi` .
- `c(60, 60, 60, 60, 60) / 5 -> Ei`.
- `sum( (Oi - Ei)^2 / Ei ) -> T` .
- `qchisq(p = (1 - 0.05), df = 4) -> t` .

So these are the wrong numbers.

The data really is exponential. 1.5/2

Our test is of the form "Reject H_0 if $T > t$".

Our results are $T = 114.6667$ and $t = 9.487729$
As $114.666 > 9.487729$ we reject H_0 with a 95% confidence.

3. Find the equation of the line that best fits the points …

$$\{(-4, 5), (-4, 4), (-3, 2), (-2, 3), (-2, 1), (0, 1), (0, 2), (1, 3), (1, 0), (2, 0), (2, -1), (3, 1)\}$$

(in the sense of least squares). Sketch a graph containing these points and your line.

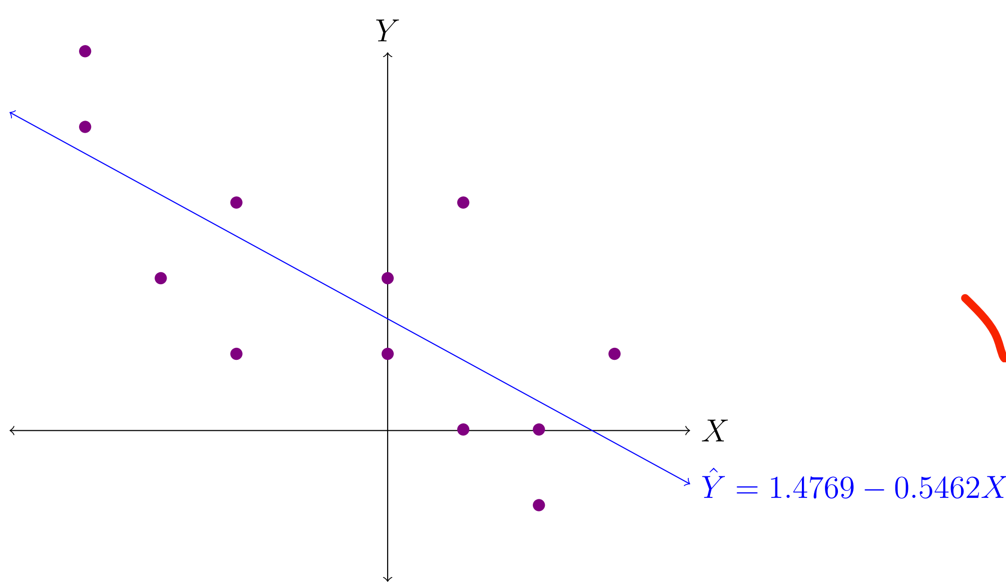
Solution:

We can use **R** to compute our regression.

- `c(-4, -4, -3, -2, -2, 0, 0, 1, 1, 2, 2, 3) -> X` .
- `c(5, 4, 2, 3, 1, 1, 2, 3, 0, 0, -1, 1) -> Y` .
- `lm(Y~X) -> reg` .
- `plot(reg)` .

R outputs our coefficients $\hat{\beta}_0 = 1.4769$ and $\hat{\beta}_1 = -0.5462$, thus our equation is $\hat{Y} = 1.4769 - 0.5462X$.

Sketching the situation …



4. Suppose the data in #3 came from a population satisfying the model in which $Y_1, \ldots, Y_{12}$ are independent, $\sim Normal(\mu_k = \beta_0 + \beta_1 x_k, \sigma^2)$. Test the hypotheses at level of significance $\alpha = 0.01 \ldots$

$$H_0 : \beta_1 = 0 \text{ vs. } H_A : \beta_1 \neq 0.$$

Solution:

From the previous problem we can conduct this test easily in **R** with the command `summary(reg)` .

From the command we can see that the p-value given when conducting a two tailed t-test against the null hypothesis is 0.00213. Thus the hypothesis is rejected for $\alpha > 0.00123$, hence we reject the H_0 at $\alpha = 0.01$.