MTH375: Mathematical Statistics - Homework #9

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Key Concepts: Uniformly Most Powerful (UMP) test of hypotheses.

1. Let $X_1, \ldots, X_n$ be $iid \sim Exponential(\theta)$ random variables, Determine ...

(a) The form of the UMP test of the hypotheses $H_0 : \theta = \theta_0$ vs. $H_A : \theta > \theta_0$.

Solution:

I will first state the general form then elaborate on the requested solution.

- $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$.
- $\frac{L(\theta_0)}{L(\theta_A)} = \left(\frac{\theta_A}{\theta_0}\right)^n e^{\Sigma_X \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right)} < k$.
- $\Sigma_X \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) < \ln(k) - n \ln\left(\frac{\theta_A}{\theta_0}\right)$.

Thus if $\theta_A > \theta_0$, our form is $\Sigma_X > k_\alpha$ as $\left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) < 0$.

(b) The form of the UMP test of the hypotheses $H_0 : \theta = \theta_0$ vs. $H_A : \theta < \theta_0$.

Solution:

- $\Sigma_X \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) < \ln(k) - n \ln\left(\frac{\theta_A}{\theta_0}\right)$.

Thus if $\theta_A < \theta_0$, our form is "Reject H_0 if $\Sigma_X < k_\alpha$" as $\left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) > 0$.

(c) The sampling distribution of the statistic defining the UMP test.

Solution:

As $X_i \sim Exponential(\theta)$, $\Sigma_X \sim Gamma(n, \theta)$.

2. Let $X_1, \dots, X_{10}$ be *iid* $\sim Binomial(1, \theta)$ random variables, Determine ...

(a) The form of the UMP test of the hypotheses $H_0 : \theta = 0.6$ vs. $H_A : \theta < 0.6$.

Solution:

I will first state the general form then elaborate on the requested solution.

- $f(x;\theta) = \binom{1}{x}\theta^x(1-\theta)^{1-x}$.
- $\dfrac{L(\theta_0)}{L(\theta_A)} = \Big(\dfrac{\theta_0}{\theta_A}\Big)^{\Sigma_x}\Big(\dfrac{1-\theta_0}{1-\theta_A}\Big)^{n\,-\,\Sigma_x} < k$.
- $\Sigma_X \ln\Big(\dfrac{\theta_0}{\theta_A}\Big) + (n - \Sigma_x)\ln\Big(\dfrac{1-\theta_0}{1-\theta_A}\Big) < \ln(k)$.
- $\Sigma_X\left(\ln\Big(\dfrac{\theta_0}{\theta_A}\Big) - \ln\Big(\dfrac{1-\theta_0}{1-\theta_A}\Big)\right) < \ln(k) - n\ln\Big(\dfrac{1-\theta_0}{1-\theta_A}\Big)$.

Thus if $\theta_A < \theta_0$, our form is "Reject H_0 if $\Sigma_X < k_\alpha$" as $\ln\Big(\dfrac{\theta_0}{\theta_A}\Big) > \ln\Big(\dfrac{1-\theta_0}{1-\theta_A}\Big)$.

(b) The UMP test at level of significance $\alpha = 0.05$.

Solution:

Our goal is to find k_α.

- As $X_i \sim Binomial(1,\theta)$, $\Sigma_X \sim Binomial(n,\theta)$.
- $k_\alpha =$ `qbinom(size = ` n`, prob = ` θ`, p = ` α`)`.
- `qbinom(size = 10, prob = 0.6, p = 0.05)` $= 3 = k_{0.05}$.

Our test is thus "Reject H_0 if $\Sigma_X < 3$."

(c) If we reject H_0 given in a run of this experiment, the data came out to be, $\overrightarrow{x} = \{0, 0, 1, 0, 1, 0, 1, 0, 1, 0\}$.

Solution:

"Reject H_0 if $\Sigma_X < 3$."

- $\Sigma_X =$ `sum(0, 0, 1, 0, 1, 0, 1, 0, 1, 0)` $= 4$.

As $\Sigma_X = 4 \not< 3$, we fail to Reject H_0.

3. Let $X_1, \ldots, X_{10}$ be *iid* $\sim Normal(\mu = 0, \sigma^2 = \theta)$ random variables, Determine ...

(a) The form of the UMP test of the hypotheses $H_0 : \theta = 5$ vs. $H_A : \theta < 5$.

Solution:

I will first state the general form then elaborate on the requested solution.

- $f(x;\theta) = \dfrac{1}{\sqrt{2\pi\theta}} e^{-\dfrac{x^2}{2\theta}}$.
- $\dfrac{L(\theta_0)}{L(\theta_A)} = \left(\dfrac{\theta_A}{\theta_0}\right)^{n/2} e^{-\dfrac{\Sigma_{X^2}}{2}\left(\dfrac{1}{\theta_0} - \dfrac{1}{\theta_A}\right)} < k$.
- $\dfrac{n}{2} \ln\left(\dfrac{\theta_A}{\theta_0}\right) + \dfrac{\Sigma_{X^2}}{2}\left(\dfrac{1}{\theta_A} - \dfrac{1}{\theta_0}\right) < \ln(k)$.
- $\Sigma_{X^2}\left(\dfrac{1}{\theta_A} - \dfrac{1}{\theta_0}\right) < 2\ln(k) - n\ln\left(\dfrac{\theta_A}{\theta_0}\right)$.

Thus if $\theta_A < \theta_0$, our form is "Reject H_0 if $\Sigma_{X^2} < k_\alpha$" as $\left(\dfrac{1}{\theta_A} - \dfrac{1}{\theta_0}\right) > 0$.

(b) The UMP test at level of significance $\alpha = 0.01$.

Solution:

Our goal is to find k_α.

- As $X_i \sim Normal(\mu = 0, \sigma^2 = \theta)$, $\dfrac{\Sigma_{X^2}}{\theta} \sim \chi^2_n$.
- $k_\alpha = \theta \cdot$ `qchisq(df =` n`, p =` α`)`.
- `5*qchisq(df = 10, p = 0.01)` $= 12.79 = k_{0.01}$.

Our test is thus "Reject H_0 if $\Sigma_{X^2} < 12.79$."

(c) If we reject H_0 given in a run of this experiment, the data came out to be, $\overrightarrow{x} = \{5.98, 1.94, 1.19, -3.28, -0.28, 3.43, -2.25, 0.39, 1.02, -2.19\}$.

Solution:

"Reject H_0 if $\Sigma_{X^2} < 12.79$."

- $\Sigma_{X^2} =$ `sum(c(5.98, 1.94, 1.19, -3.28, -0.28, 3.43, -2.25, 0.39, 1.02, -2.19)^2)` $= 74.59$.

As $\Sigma_{X^2} = 74.59 \not< 12.79$, we fail to Reject H_0.

4. Let $X_1, \ldots, X_{10}$ be *iid* $\sim f(x;\theta) = \theta x^{\theta-1}$ random variables, Determine ...

(a) The form of the UMP test of the hypotheses $H_0 : \theta = 2$ vs. $H_A : \theta > 2$.

Solution:

I will first state the general form then elaborate on the requested solution.

- $f(x;\theta) = \theta x^{\theta-1}$.
- $\dfrac{L(\theta_0)}{L(\theta_A)} = \left(\dfrac{\theta_0}{\theta_A}\right)^n \Pi_X^{(\theta_0-\theta_A)} < k$.
- $(\theta_0 - \theta_A)\Sigma_{\ln(X)} < \ln(k) - n\ln\left(\dfrac{\theta_0}{\theta_A}\right)$.

Thus if $\theta_A > \theta_0$, our form is "Reject H_0 if $-\Sigma_{\ln(X)} < k_\alpha$" as $(\theta_0 - \theta_A) < 0$.

(b) The UMP test at level of significance $\alpha = 0.04$; Given the statistic $T = -\sum_{k=1}^{10} \ln(X_i) \sim Gamma(10, 1/\theta)$.

Solution:

Our goal is to find k_α.

- Given $-\Sigma_{\ln(X)} \sim Gamma(10, 1/\theta)$.
- $k_\alpha =$ `qgamma(shape = 10, scale =` $1/\theta$`, p =` α`)`.
- `qgamma(shape = 10, scale = 0.5, p = 0.04)` $= 2.60 = k_{0.04}$.

Our test is thus "Reject H_0 if $-\Sigma_{\ln(X)} < 2.60$."

(c) If we reject H_0 given in a run of this experiment, the data came out to be, $\overrightarrow{x} = \{0.912, 0.839, 0.978, 0.789, 0.690, 0.502, 0.862, 0.691, 0.587, 0.557\}$.

Solution:

"Reject H_0 if $-\Sigma_{\ln(X)} < 2.60$."

- $-\Sigma_{\ln(X)} =$

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-sum(log(c(0.912, 0.839, 0.978, 0.789, 0.690, 0.502, 0.862, 0.691, 0.587, 0.557)))
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$= 3.22.$

As $-\Sigma_{\ln(X)} = 3.22 \not< 2.60$, we fail to Reject H_0.

Extra credit: Show that, in problem 4, $T = -\sum_{k=1}^{10} \ln(X_i) \sim Gamma(10, 1/\theta)$.

Solution:

- $f(x;\theta) = \theta x^{\theta-1}$.
- Thumped ...
- $Y = -\ln(X_i) = \frac{1}{\theta} e^{-x/\theta} \sim Exponential(1/\theta)$.
- $T = -\sum_{k=1}^{10} \ln(X_i) = \Sigma_Y \sim Gamma(10, 1/\theta)$.

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