

Midterm

Thursday, February 24, 2022 10:38 AM

1. Let X_1, \dots, X_n be beta with α unknown and $\beta = 3$. Find the Method of Moments estimator of α . *From our distribution handout...*

Range	$f_X(x)$	μ	σ^2
Beta(α, β) $(0, 1)$	$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$

$$\text{let } \bar{X} = \bar{x}_1, \dots, \bar{x}_n \quad ; \quad \mathbb{E}[X_i] = \frac{\alpha}{\alpha + \beta} = \frac{\alpha}{\alpha + 3}$$

$$\text{Note: } \mathbb{E}[X] = \mathbb{E}[x_i]$$

$$\text{Now to find the MOM estimator : } \mathbb{E}[\bar{X}] = \frac{\hat{\alpha}}{\hat{\alpha} + 3} = \bar{x}$$

$$\therefore \hat{\alpha} = \hat{\alpha}\bar{x} + 3\bar{x} \Rightarrow \hat{\alpha}(1 - \bar{x}) = 3\bar{x} \Rightarrow \boxed{\hat{\alpha}_{\text{mom}} = \frac{3\bar{x}}{1 - \bar{x}}}$$

$$2. \text{ Let } Y_1, \dots, Y_n \text{ have common pdf } f(y) = \begin{cases} \frac{2y}{a^2}, & 0 \leq y \leq a \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find a sufficient statistic for a .

(b) Find an UMVUE for $\frac{1}{a}$.

$$\text{let } \bar{Y} = \bar{y}_1, \dots, \bar{y}_n$$

$$L(a) = \left(\frac{2}{a^2}\right)^n \times y_1 \times \dots \times y_n = \frac{2^n}{a^{2n}} \prod_{i=1}^n y_i \cdot 1$$

\underbrace{g}_{g} \underbrace{h}_{h}

From the pdf, y has upper bound a :

Thus $T = \max\{Y_i\}$ is a sufficient statistic for a .

$$\text{our cdf is } F_{Y_i} = \int_0^y \frac{2y}{a^2} dy = \left[\frac{y^2}{a^2} \right]_0^m = \frac{m^2}{a^2} - 0$$

$$F_T = P(T \leq t) = (F_{Y_i})^n = \left(\frac{m^2}{a^2} \right)^n = \frac{m^{2n}}{a^{2n}}$$

$$f_T = \frac{d}{dm} \left[\frac{m^{2n}}{a^{2n}} \right] = \frac{2nm^{2n-1}}{a^{2n}}$$

$$\mathbb{E}[T] = \int_0^a \frac{2nm^{2n}}{a^{2n}} dm = \frac{2nm^{2n+1}}{(2n+1)a^{2n}} \Big|_0^a = \frac{2na^{2n+1}}{(2n+1)a^{2n}} - 0$$

$$\mathbb{E}[T] = \frac{2na}{2n+1}$$

$$< 1 \dots r \dots 1 \dots \perp 2n \mathbb{E}[T]$$

Solving for $\frac{1}{a}$: $\frac{1}{a} = \frac{2n \mathbb{E}[T]}{2n+1}$

as $\mathbb{E}\left[\frac{2n}{2n+1}\right] = \frac{2n}{2n+1}$ we can construct a
 $T^* = \left(\frac{2n}{2n+1}\right) \cdot \max\{\bar{Y}^3\}$ which is UMVUE for
 a , as $\mathbb{E}[T^*] = a$.

3. Let Y_1, \dots, Y_n have common pmf $p(y) = \begin{cases} 2p & \text{if } y = -1 \\ 1/2 - p & \text{if } y = 0 \\ 1/2 - p & \text{if } y = 1 \end{cases}, 0 < p < \frac{1}{2}$

- (a) Find a sufficient statistic for p .
- (b) Find the MOM estimator of p . Is it unbiased?
- (c) Find the MLE estimator of p . Is it unbiased?
- (d) Find the MSE of both estimators. For which values of p is the MSE of the MLE the smaller of the two?

$$\mathbb{E}[Y] = \mathbb{E}[Y] = -1(2p) + 0\left(\frac{1}{2} - p\right) + 1\left(\frac{1}{2} - p\right) \\ = -2p + 0 + \frac{1}{2} - p = \frac{1}{2} - 3p$$

By the MOM: $\bar{X} = \frac{1}{2} - 3\hat{p} \Rightarrow 3\hat{p} = \frac{1}{2} - \bar{X} = \frac{1 - 2\bar{X}}{2}$

$$\therefore \hat{p}_{MOM} = \frac{1 - 2\bar{X}}{6}$$

let $A = \# Y_i's = -1$ $\therefore L(p) = (2p)^A \left(\frac{1}{2} - p\right)^{B+C}$
 $B = + = 0$
 $C = + = 1$ Note: $A+B+C = n$
 $\therefore B+C = n-A$

$$\ell(p) = A \ln(2p) + (B+C) \ln\left(\frac{1}{2} - p\right)$$

$$\ell_p = \frac{2A}{2p} - \frac{(B+C)}{\left(\frac{1}{2} - p\right)} = \frac{A}{p} - \frac{(n-A)}{\left(\frac{1}{2} - p\right)}$$

$$= \frac{A}{p} + \frac{n-A}{\frac{1}{2} - p}$$

To find the MLE set $\ell_p = 0$

$$\therefore \frac{A}{p} = \frac{n-A}{\hat{p} + \frac{1}{2}} \Rightarrow A\hat{p} + \frac{A}{2} = n\hat{p} - A\hat{p}$$

should still be $(\hat{p} - 1/2)$, so $\hat{p} = A/(2n)$

$$\therefore \hat{p} = n\hat{p} - 2A\hat{p} = \hat{p}(n-2A)$$

$$\Rightarrow \frac{A}{2} = n\hat{p} - 2A\hat{\gamma} = \hat{p}(n-2A)$$

$$\Rightarrow \hat{p}_{MLE} = \frac{A}{2n-4A}$$

Can't be right. When $A=n$, this comes out negative.

We have \hat{p}_{mom} unbiased if thus $MSE[\hat{p}_{mom} - p] = V[\hat{p}_{mom}]$

$$V[\hat{p}_{mom}] = V\left(\frac{2-2\bar{Y}}{6}\right) = \frac{V(2-2\bar{Y})}{36} = \frac{4V[\bar{Y}]}{36} = \frac{V[\bar{Y}]}{9} = \frac{V[Y_i]}{9n}$$

$$\mathbb{E}[Y_i]^2 = (\frac{1}{2} - 3p)^2 = \frac{1}{4} - 3p + 9p^2$$

$$\begin{aligned} \mathbb{E}[Y_i^2] &= (-1)^2(2p) + (0)(\frac{1}{2} - p) + (1)(\frac{1}{2} - p) \\ &= 2p + \frac{1}{2} - p = \frac{1}{2} + p \end{aligned}$$

$$V[Y_i] = \frac{1}{2} + p - \frac{1}{4} + 3p - 9p^2 = \frac{1}{4} + 4p - 9p^2$$

$$V[\hat{p}_{mom}] = \frac{1}{36n} + \frac{4p}{9n} - \frac{p^2}{n} = MSE[\hat{p}_{mom} - p]$$

$$MSE[\hat{p}_{MLE} - p] = V[\hat{p}_{MLE}] = \mathbb{E}[\hat{p}_{MLE}^2] - \mathbb{E}[\hat{p}_{MLE}]^2$$

$$\hat{p}_{MLE}^2 = \left(\frac{A}{2n-4A}\right)^2 = \frac{A^2}{4n-16nA+16A} = \frac{A^2}{4(n-4A(n-1))}$$

$$V[\hat{p}_{MLE}] = \mathbb{E}\left(\frac{A^2}{4(n-4A(n-1))}\right) - \mathbb{E}\left(\frac{A}{2n-4A}\right)^2$$

These are expectations
are very difficult to
evaluate.

$$\mathbb{E}[A] = \frac{2}{3}p? \quad \mathbb{E}[A^2] = \frac{4}{9}p^2?$$

$$\begin{aligned} V[\hat{p}_{MLE}] &= \frac{1}{16} \mathbb{E}\left(\frac{A^2}{n-4A(n-1)}\right) - \frac{1}{4} \mathbb{E}\left(\frac{A}{n-2A}\right)^2 \\ &= \frac{1}{16} \left(\frac{4p^2/9}{n-4p/3(n-1)}\right) - \frac{1}{4} \left(\frac{2p/3}{n-2p/3}\right)^2 \\ &= \frac{1}{16} \left(\frac{p^2}{n-2p/3}\right) - \frac{1}{4} \left(\frac{4p^2/9}{n-2p/3}\right)^2 \end{aligned}$$

$$= \frac{1}{36} \left(\frac{\gamma^2}{n - 4\gamma/3(n-1)} \right) - \frac{1}{4} \left(\frac{4\gamma^2/9}{(n-2\gamma/3)^2} \right)$$

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$$= \frac{\gamma^2}{36n - 48\gamma(n-1)} - \frac{1}{9} \left(\frac{\gamma^2}{n^2 - 4\gamma/3 + 4\gamma^2/9} \right)$$

$$= \boxed{\frac{\gamma^2}{36n - 48\gamma n + 48\gamma^2} - \frac{\gamma^2}{9n^2 - 36\gamma n + 4\gamma^2} = \text{MSE}[\hat{\beta}_{MLE}; \beta]}$$

for small γ , $\text{MSE}[\hat{\beta}_{MLE}]$ is better.

?

for large γ , $\text{MSE}[\hat{\beta}_{mom}]$ is better.

4. Let X_1, \dots, X_n be exponential with parameter β . Find UMVUE's for β , β^2 and β^3 .

Hint: The exponential distribution has pdf $p(x) = \frac{1}{\beta} e^{-x/\beta}$ ($x > 0$). The sum of n independent exponential(β) random variables has pdf Gamma($\alpha = n, \beta$).

From distributions..

	range	$f_x(x)$	μ	σ^2	$M_x(t)$
Exponential(β)	$(0, \infty)$	$\frac{1}{\beta} e^{-x/\beta}$	β	β^2	$\frac{1}{1-\beta t}$
Gamma(α, β)	$(0, \infty)$	$\frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta t}\right)^\alpha$

$$L(\beta) = \frac{1}{\beta^n} \cdot e^{\frac{\sum x_i}{\beta n}} \quad \text{so } \sum x_i \text{ is sufficient} \quad ; \quad \sum_{i=1}^n x_i = \sum x$$

from hint : $\sum x \sim \text{Gamma}(n, \beta)$

$$M_{\sum x}(t) = \left(\frac{1}{1-\beta t}\right)^n \quad ; \quad M'_{\sum x}(t) = \beta^n \left(\frac{1}{1-\beta t}\right)^{n+1}$$

$$M''_{\sum x}(t) = \beta^n (n+1) \left(\frac{1}{1-\beta t}\right)^{n+2} \quad ; \quad M'''_{\sum x}(t) = \beta^n (n+1)(n+2) \left(\frac{1}{1-\beta t}\right)^{n+3}$$

* Wolfram used for mgf derivatives *

$$\mathbb{E}[\sum x] = \beta^n \left(\frac{1}{1}\right)^{n+1} = \beta n$$

thus $\hat{\beta} = \mathbb{E}\left[\frac{\sum x}{n}\right] = \mathbb{E}[\bar{x}]$ \notin \bar{x} is UMVUE for β .

$$\lambda > \sum x^2$$

$$\mathbb{E}[\sum_x^2] = \beta^2 n(n+1) \cdot 1 \Rightarrow \hat{\beta}^2 = \frac{\sum_x^2}{n(n+1)}$$

$\boxed{\frac{\sum_x^2}{n(n+1)} \text{ is UMVUE for } \beta^2.}$

~~$$\mathbb{E}[\sum_x^3] = \beta^3 n(n+1)(n+2) \cdot 1 \Rightarrow \hat{\beta}^3 = \frac{\sum_x^3}{n(n+1)(n+2)}$$~~

$\boxed{\frac{\sum_x^3}{n(n+1)(n+2)} \text{ is UMVUE for } \beta^3.}$

5. Let X_1, \dots, X_{10} be iid normal with unknown μ, σ^2 .

Let $\bar{X} = \frac{X_1 + \dots + X_{10}}{10}$ and $S^2 = \frac{1}{9} \sum_{i=1}^{10} (X_i - \bar{X})^2$.

here $\bar{X} \sim N(\mu, \frac{\sigma^2}{10})$

(a) Find a number a such that $P\left(-a < \frac{\bar{X} - \mu}{S} < a\right) = .95$. (ref HWI #3)

(b) Find positive numbers a, b such that $P\left(a < \frac{S^2}{\sigma^2} < b\right) = .95$.

let $Y = \frac{\bar{X} - \mu}{S}$; $Y\sqrt{n} \sim T_{n-1} \Rightarrow Y\sqrt{10} \sim T_9$

$\therefore P\left(-a < \frac{\bar{X} - \mu}{S} < a\right) = .95 = P\left(-a\sqrt{10} < T_9 < a\sqrt{10}\right)$

```
> a_sqrt10 <- qt(p = c(0.025, 0.975), df = 9);
> a_sqrt10 / sqrt(10)
[1] -0.7153569  0.7153569
> |
```

$\boxed{a = 0.715}$

let $Z = \frac{S^2}{\sigma^2} \sim \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow \frac{9S^2}{\sigma^2} \sim \chi^2_9$

$\therefore P\left(a < \frac{S^2}{\sigma^2} < b\right) = 0.95 = P(9a < \chi^2_9 < 9b)$

```
> ab_9 <- qchisq(p = c(0.025, 0.975), df = 9);
> ab_9 / 9
[1] 0.3000433 2.1136409
> |
```

$\boxed{a = 0.300}$
 $\boxed{b = 2.113}$