

MTH375: Mathematical Statistics - Homework #1

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Key Concepts: Multivariate normal($\boldsymbol{\mu}, \Sigma$), χ^2 with n df, t with n df, F with m, n df

1. Suppose that Z_1, Z_2 are independent standard normal random variables.

Let $Y_1 = Z_1 - 2Z_2$, $Y_2 = Z_1 - Z_2$.

(a) Find the joint pdf $f_{Y_1, Y_2}(y_1, y_2)$.

You may do this either of two ways. Either (i) use the change of variables theorem from MTH 372, OR (ii) evaluate the matrices Σ and Σ^{-1} from class, then multiply the necessary matrices and vectors to obtain a formula for $f_{Y_1, Y_2}(y_1, y_2)$. In either case, obtain a formula containing no matrices and no vectors.

Solution:

We have a system of equations, $\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A\mathbf{Z} + \boldsymbol{\mu} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We can solve for $\Sigma = A A^T = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \Sigma$.

We can now evaluate Σ^{-1} .

$$\begin{bmatrix} 5 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 2 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 3 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 2 & -6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 5 \end{bmatrix}.$$

Our solution yields $\Sigma^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$.

The last piece we need to evaluate before solving for the joint pdf is $||\Sigma||$.

$$||\Sigma|| = \left\| \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \right\| = |10 - 9| = |1| = 1.$$

We can now solve for $f_{Y_1, Y_2}(y_1, y_2) = \frac{e^{-\frac{(\mathbf{y}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{y}-\boldsymbol{\mu})}{2}}}{(2\pi)^{\frac{n}{2}} \sqrt{||\Sigma||}} = \frac{e^{-\frac{(\mathbf{y})^T \Sigma^{-1} (\mathbf{y})}{2}}}{(2\pi)^{\frac{2}{2}} \sqrt{1}} = \frac{e^{-\frac{(\mathbf{y})^T \Sigma^{-1} (\mathbf{y})}{2}}}{2\pi}$.

$$\mathbf{y}^T \Sigma^{-1} \mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 - 3y_2 & -3y_1 + 5y_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

$$= 2y_1^2 - 3y_2y_1 - 3y_1y_2 + 5y_2^2 = 2y_1^2 - 6y_1y_2 + 5y_2^2.$$

By Substitution, $f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2\pi} \exp\left(\frac{-2y_1^2 + 6y_1y_2 - 5y_2^2}{2}\right)$.

(b) Find the marginal pdf $f_{Y_2}(y_2)$. Don't use integration – you can derive the needed pdf doing only the simplest arithmetic using some facts from MTH 372.

Solution:

We have one equation, $\mathbf{Y} = Y_2 = \mathbf{A}\mathbf{Z} + \boldsymbol{\mu} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Z_1 - Z_2$.

We can solve for $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^T = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 = \boldsymbol{\Sigma}$.

We can now evaluate $\boldsymbol{\Sigma}^{-1} = \frac{1}{2}$.

The last piece we need to evaluate before solving for the joint pdf is $\|\boldsymbol{\Sigma}\| = 2$.

We can now solve for $f_{Y_2}(y_2) = \frac{e^{-\frac{(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})}{2}}}{(2\pi)^{\frac{n}{2}} \sqrt{\|\boldsymbol{\Sigma}\|}} = \frac{e^{-\frac{(\mathbf{y})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y})}{2}}}{(2\pi)^{\frac{1}{2}} \sqrt{2}} = \frac{e^{-\frac{(\mathbf{y})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y})}{2}}}{\sqrt{2}\sqrt{2\pi}}$.

$$\mathbf{y}^T \boldsymbol{\Sigma}^{-1} \mathbf{y} = y_2 \frac{1}{2} y_2 = \frac{y_2^2}{2}.$$

By Substitution, $f_{Y_2}(y_2) = \frac{1}{\sqrt{2}\sqrt{2\pi}} \exp\left(\frac{-y_2^2}{4}\right)$.

(c) Find the conditional pdf of Y_1 given $Y_2 = 1$, that is, $f_{Y_1|Y_2}(y_1|1)$. Use (a) and (b), and do the necessary division. By completing the square, identify by name – *including parameters* – the required conditional pdf.

Solution:

$$\begin{aligned}
 f_{Y_1|Y_2}(y_1|y_2) &= \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{\frac{1}{2\pi} \exp\left(\frac{-2y_1^2 + 6y_1y_2 - 5y_2^2}{2}\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y_2^2}{4}\right)} = \frac{\sqrt{2} \exp\left(\frac{-2y_1^2 + 6y_1y_2 - 5y_2^2}{2}\right)}{\sqrt{2\pi} \exp\left(\frac{-y_2^2}{4}\right)}. \\
 &= \frac{\sqrt{2} \exp\left(\frac{-4y_1^2 + 12y_1y_2 - 10y_2^2}{4} - \frac{-y_2^2}{4}\right)}{\sqrt{2\pi}} = \frac{\sqrt{2} \exp\left(\frac{-4y_1^2 + 12y_1y_2 - 9y_2^2}{4}\right)}{\sqrt{2\pi}}. \\
 f_{Y_1|Y_2}(y_1|y_2 = 1) &= \frac{f_{Y_1,Y_2}(y_1, y_2 = 1)}{f_{Y_2}(y_2 = 1)} = \frac{\sqrt{2} \exp\left(\frac{-4y_1^2 + 12y_1(1) - 9(1)^2}{4}\right)}{\sqrt{2\pi}} = \frac{\sqrt{2} \exp\left(\frac{-4y_1^2 + 12y_1 - 9}{4}\right)}{\sqrt{2\pi}}. \\
 &= \frac{\sqrt{2} \exp\left(-\left(\frac{1}{2}\right)\left(\frac{4y_1^2 - 12y_1 + 9}{2}\right)\right)}{\sqrt{2\pi}} = \frac{\sqrt{2} \exp\left(-\left(\frac{1}{2}\right)(2y_1^2 - 6y_1 + 4.5)\right)}{\sqrt{2\pi}} = \frac{\sqrt{2} \exp\left(-(y_1^2 - 3y_1 + 2.25)\right)}{\sqrt{2\pi}}. \\
 &= \frac{\sqrt{2} \exp\left(-(y_1 - 1.5)^2\right)}{\sqrt{2\pi}}.
 \end{aligned}$$

It is now worthwhile to consider the pdf of Random Variable $X \sim N(\mu, \sigma^2)$.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right).$$

Under close inspection, we can see that our above distribution for $f_{Y_1|Y_2}(y_1|y_2 = 1)$ is normal, taking $\sigma = \frac{1}{\sqrt{2}}$, $\sigma^2 = \frac{1}{2}$, and $\mu = 1.5$.

2. Suppose Z_1, Z_2, \dots, Z_6 are independent standard normal random variables.

(a) Find a number a such that $P(-a < 3Z_1 + 2Z_2 - 4Z_3 < a) = .99$.

Solution:

Let $A = 3Z_1 + 2Z_2 - 4Z_3$.

$A \sim N(0, (3^2 + 2^2 + 4^2)) \sim N(0, (9 + 4 + 16)) \sim N(0, 29)$.

As $V(A) = 29$, $\sigma_A = \sqrt{29}$.

We now wish to have probabilities for $-a$ corresponding to 0.005 and a corresponding to 0.995.

```
qnorm(p = c(0.005, 0.995), mean = 0, sd = sqrt(29))
```

```
## [1] -13.87127 13.87127
```

We can see here that $a = 13.87127$ is the solution.

(b) Find numbers a, b such that $P(a < Z_1^2 + Z_2^2 + \dots + Z_6^2 < b) = .99$.

Solution:

Let $B = Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 + Z_5^2 + Z_6^2$

$B \sim \chi_6^2$.

Let us make the generalizing assumption $a = 0$.

We now wish to have probabilities for a corresponding to 0.0 and b corresponding to 0.99.

```
qchisq(p = c(0.0, 0.99), df = 6)
```

```
## [1] 0.00000 16.81189
```

We can see here that $a = 0.0$ and $b = 16.81189$ is one solution.

(c) Find a numbers a such that $P\left(-a < \frac{Z_1}{\sqrt{Z_2^2 + Z_3^2 + \dots + Z_6^2}} < a\right) = .99$.

Solution:

$$\text{Let } C = \frac{Z_1}{\sqrt{Z_2^2 + Z_3^2 + \dots + Z_6^2}}.$$

We will leverage the following two points to solve this problem:

- $C\sqrt{5} = \frac{Z_1\sqrt{5}}{\sqrt{Z_2^2 + Z_3^2 + \dots + Z_6^2}} \sim T_5.$
- $P\left(-a < \frac{Z_1}{\sqrt{Z_2^2 + Z_3^2 + \dots + Z_6^2}} < a\right) = P\left(-a\sqrt{5} < \frac{Z_1\sqrt{5}}{\sqrt{Z_2^2 + Z_3^2 + \dots + Z_6^2}} < a\sqrt{5}\right).$

We now wish to have probabilities for $-a\sqrt{5}$ corresponding to 0.005 and $a\sqrt{5}$ corresponding to 0.995.

In particular, we will solve $P\left(-a\sqrt{5} < T_5 < a\sqrt{5}\right) = 0.99$.

```
a_sqrt_5 <- qt(p = c(0.005, 0.995), df = 5)
a_sqrt_5
```

```
## [1] -4.032143  4.032143
```

```
a_sqrt_5 / sqrt(5)
```

```
## [1] -1.803229  1.803229
```

We can see here that $a = 1.803229$ is the solution.

(d) Find numbers a, b such that: $P\left(a < \frac{Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2}{Z_5^2 + Z_6^2} < b\right) = .99$.

Solution:

$$\text{Let } D = \frac{Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2}{Z_5^2 + Z_6^2}.$$

We will leverage the following two points to solve this problem:

- $\frac{D}{2} = \frac{1}{2} \cdot \frac{Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2}{Z_5^2 + Z_6^2} = \frac{2\chi_4^2}{4\chi_2^2} \sim F_{4,2}.$
- $P\left(a < \frac{Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2}{Z_5^2 + Z_6^2} < b\right) = P\left(\frac{a}{2} < \frac{1}{2} \cdot \frac{Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2}{Z_5^2 + Z_6^2} < \frac{b}{2}\right).$

Let us make the generalizing assumption $a = 0$.

We now wish to have probabilities for a corresponding to 0.0 and b corresponding to 0.99.

In particular, we will solve $P\left(0 < F_{4,2} < \frac{b}{2}\right) = 0.99$.

```
ab_halved <- qf(p = c(0.0, 0.99), df1 = 4, df2 = 2)
ab_halved
```

```
## [1] 0.00000 99.24937
```

```
ab_halved * 2
```

```
## [1] 0.00000 198.4987
```

We can see here that $a = 0.0$ and $b = 198.4987$ is one solution.

3. Let X_1, X_2, \dots, X_8 be independent $\text{normal}(\mu, \sigma^2)$ random variables,

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_8}{8}, \text{ and } S^2 = \frac{1}{7} \sum_{i=1}^8 (X_i - \bar{X})^2.$$

Find a number a such that $P\left(-a < \frac{\bar{X} - \mu}{S} < a\right) = .99$.

Solution:

We will leverage the following from class in this solution...

- $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.
- $\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi_{n-1}^2$.
- \bar{X} and S are independent.

Letting $Y = \frac{\bar{X} - \mu}{S}$.

Under close inspection, for arbitrary n , we can see the following...

$$Y \cdot \sqrt{n} = \frac{\bar{X} - \mu}{S} \cdot \sqrt{n} \cdot \frac{\sigma}{\sigma} \cdot \frac{\sqrt{n-1}}{\sqrt{n-1}} = \frac{(\bar{X} - \mu)\sqrt{n}}{\sigma} \cdot \frac{\sqrt{\sigma^2}}{\sqrt{S^2}\sqrt{n-1}} \cdot \frac{\sqrt{n-1}}{1} \sim Z \cdot \frac{\sqrt{n-1}}{\sqrt{\chi_{n-1}^2}} \sim T_{n-1}.$$

We will thus leverage the following two points to solve this problem:

- $Y\sqrt{8} = \frac{(\bar{X} - \mu)\sqrt{8}}{\sigma} \cdot \frac{\sqrt{\sigma^2}}{\sqrt{S^2}\sqrt{7}} \cdot \frac{\sqrt{7}}{1} \sim Z \cdot \frac{\sqrt{7}}{\sqrt{\chi_7^2}} \sim T_7$.
- $P\left(-a < \frac{\bar{X} - \mu}{S} < a\right) = P\left(-a\sqrt{8} < \frac{(\bar{X} - \mu)\sqrt{8}}{\sigma} \cdot \frac{\sqrt{\sigma^2}}{\sqrt{S^2}\sqrt{7}} \cdot \frac{\sqrt{7}}{1} < a\sqrt{8}\right).$

We now wish to have probabilities for $-a\sqrt{8}$ corresponding to 0.005 and $a\sqrt{8}$ corresponding to 0.995.

In particular, we will solve $P\left(-a\sqrt{8} < T_7 < a\sqrt{8}\right) = 0.99$.


```
a_sqrt_8 <- qt(p = c(0.005, 0.995), df = 7)
a_sqrt_8
```

```
## [1] -3.499483  3.499483
```

```
a_sqrt_8 / sqrt(8)
```

```
## [1] -1.237254  1.237254
```

We can see here that $a = 1.237254$ is the solution.
