MTH 375
Winter 2022
Hw 8 key

#1. Let $X_1, \cdots, X_{50}$ be iid Poisson(θ) random variables, where θ has the prior distribution exponential(5). Determine a 98% Bayesian credible interval for θ when $\sum_{i=1}^{50} X_i = 170$.

The posterior distribution is $f(\theta|\vec{X}) = \dfrac{f(\vec{x}|\theta)\pi(\theta)}{\int f(\vec{x}|\theta)\pi(\theta)\ d\theta} = \dfrac{e^{-50\theta}\theta^{\sum x_k}/(x_1!\cdots x_{50}!)\cdot \frac{1}{5}e^{-\theta/5}}{\int e^{-50\theta}\theta^{\sum x_k}/(x_1!\cdots x_{50}!)\cdot \frac{1}{5}e^{-\theta/5}\ d\theta} = \dfrac{e^{-50.2\theta}\theta^{\sum x_k}}{\int e^{-50.2\theta}\theta^{\sum x_k}\ d\theta}$, so the posterior distribution of θ is gamma($\alpha = 1 + \Sigma, \beta = 1/50.2$). With the given data, this is gamma($\alpha = 171, \beta = 5/251$).

A 98% Bayesian credible interval is the 1- and 99-percentiles of this pdf, i.e.,
qgamma(shape=171,scale=5/251,c(0.01,.99)) $\to$ (2.829815, 4.041493).

#2. Let $X_1, \ldots X_{25}$ be a sample of iid Bernoulli(p) random variables, with p uniform on $[0, 1]$. Determine a 96% credible interval for p when $\sum_{i=1}^{25} X_i = 10$.

The posterior distribution is $f(\theta|\vec{X}) = \dfrac{f(\vec{x}|\theta)\pi(\theta)}{\int f(\vec{x}|\theta)\pi(\theta)\ d\theta} = \dfrac{p^{\sum x_k}(1-p)^{25-\sum x_k}}{\int p^{\sum x_k}(1-p)^{25-\sum x_k}\ dp} =$, so the posterior distribution of θ is beta($\alpha = 1 + \Sigma, \beta = 26 - \Sigma$). With the given data, this is beta($\alpha = 11, \beta = 16$).

A 96% Bayesian credible interval is the 2- and 98-percentiles of this pdf, i.e.,
qbeta(shape1=11, shape2=16, c(.02,.98)) $\to$ (0.2263560, 0.6031037).

#3. Let $X_1, \ldots, X_{25}$ be a sample of iid Binomial($1, p$) random variables. Suppose we test the hypotheses $H_0 : p = 0.6$ vs. $H_A : p > 0.6$ using the test
"Reject H_0 if $\sum_{i=1}^{25} x_i \geq 20$."

a)Level of significance, α. Under H_0, Σ is binomial($n = 25, p = .6$), so $\alpha = P(\Sigma \geq 20)$ is 1-pbinom(size=25,prob=.6,19) $\to$ 0.0293622.

b) Power of this test for p=.7, .8, .9, and 1.0.
P($\Sigma \geq 20; p$=.7,.8,.9,1.0) are, respectively
1-pbinom(size=25,prob=c(.7,.8,.9,1),19) $\to$ 0.1934884, 0.6166894, 0.9666001, 1.0000000.

#4. Let $X_1, \ldots, X_{10}$ be iid exponential(θ) random variables. For testing $H_0 : \theta = 5$ vs. $H_A : \theta = 8$, determine

a) the form of the Neyman-Pearson most powerful test.
$\dfrac{L(5)}{L(8)} = \dfrac{\prod_{i=1}^{10} \frac{1}{5}e^{-x_i/5}}{\prod_{i=1}^{10} \frac{1}{8}e^{-x_i/8}} = \left(\dfrac{8}{5}\right)^n e^{-\frac{3}{40}\sum_i x_i} \leq K$, so the test is of the form
"Reject H_0 if $\Sigma \geq K^*$."

b) the actual test at level $\alpha = .05$. Under H_0, Σ is gamma($\alpha = 10, \beta = 5$).
$P(\Sigma \geq K^*) = .05$ for K^* given by qgamma(shape=10,scale=5,.95) $\to$ 78.52608.
The test is "Reject H_0 if $\Sigma \geq 78.5$."

c) The power of this test. P($\Sigma \geq 78.5; \theta = 8$) is given by
1-pgamma(shape=10,scale=8,78.526) $\to$ 0.4811829.

#5. Let $X_1, \ldots, X_{15}$ be of iid Normal ($\mu = 10, \sigma^2 = \theta$) random variables. For testing $H_0 : \theta = 6$ vs. $H_A : \theta = 2$, determine

a) the form of the Neyman-Pearson most powerful test at level of significance α.

$\dfrac{L(6)}{L(2)} = \dfrac{\prod_{i=1}^{15} \frac{1}{\sqrt{12\pi}} e^{-\frac{1}{12}(x_i-10)^2}}{\prod_{i=1}^{15} \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}(x_i-10)^2}} = 3^{-15/2} e^{\frac{1}{6}\sum_i (x-10)^2} \leq K$, so the form of the test is "Reject H_0 if $\sum_i (X_i - 10)^2 \leq K^*$."

b) the actual test at level $\alpha = .05$.

Under H_0, the statistic $\sum_i (X_i - 10)^2/6$ is χ^2 with 15 df, so $\alpha = P(\sum_i (X_i - 10)^2/6 \leq K^*) = .05$ for K^* given by qchisq(df=15,.05) $\rightarrow$ 7.260944. The test is "Reject H_0 if $\sum_i (X_i - 10)^2/6 \leq 7.26$", or, equivalently, "if $\sum_i (X_i - 10)^2 \leq 43.57$."

c) The power of your test.

Under H_A, $\sum_i (X_i - 10)^2/2$ is χ^2 with 15 df, so the power is $P(\sum_i (X_i - 10)^2 \leq 43.57) = P(\sum_i (X_i - 10)^2/2 \leq 21.78)$, i.e., pchisq(df=15,21.78) $\rightarrow$ 0.8862963.