4.1 Parametric Point Estimation - Statistics

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We Need:

- probability model: $f(x, \theta)$
- paramater space: Θ | $\forall \theta \in \Theta$
- random sample: n observations

{point estimators} for θ given a sample

- properties used to evaluate the accuracy & efficiency of our estimators

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Our Random Sample $\{X_1, \dots, X_n\}$ Contains Our Available Information for Estimation

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A statistic $T = T(X_1, \dots, X_n)$ is a function of the random sample, that does not depend on any unknown paramaters or quantities.

Estimators are Statistics used to

Estimators are Statistics used to Estimate unknown quantities.

$$\sum_{i=1}^{n} X_i \quad \& \quad \sum_{i=1}^{n} X_i^2$$

are 2 basic statistics

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \quad \& \quad S^2 = \frac{\sum_{i=1}^{n}\left[X_i^2 - n\bar{X}^2\right]}{n}$$

are common estimators for mean (μ) & variance (σ^2)

Note: $S^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$

We call the probability distribution of the estimator the Sampling Distribution.

The Sampling Distribution influences the behavior & effectiveness the behavior & effectiveness of an estimator.

We denote the probability density function of the sampling distribution for an estimator T, which is estimating an unknow paramater θ

As $\boxed{f_T(t\,;\,\theta)}$

for $\bar{X}$, $T \sim N(\mu, \frac{\sigma^2}{n})$

for S^2, $T \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}$

for $X_i \sim \text{Binomial}(k, p)$

$\sum_{i=1}^{n} X_i \sim \text{Binomial}(nk, p)$

We call a statistic an unbiased estimator if $\mathbb{E}[T] = \theta$ (μ_T)

The Bias of a Statistic can be found as $\text{Bias}(T;\vartheta) = \mathbb{E}[T] - \vartheta$

For a sample of iid random variables with common mean μ,

$$\text{Bias}(\overline{X};\mu) = 0$$

For a sample of iid random variables with common variance σ^2,

$$\text{Bias}(S^2;\sigma^2) = 0$$

$$\mathbb{E}\left[\frac{n-1}{n}\cdot S^2\right] = \frac{n-1}{n}\,\mathbb{E}[S^2] = \frac{n-1}{n}\cdot\sigma^2$$

$$\text{Bias}\left(\frac{n-1}{n}\cdot S^2,\sigma^2\right) = \frac{n-1}{n}\cdot\sigma^2 - \sigma^2 = \frac{-\sigma^2}{n}$$

For iid $X_i \sim \text{Poisson}(\lambda)$, $\mathbb{E}[X_i] = \lambda$

$$\text{Bias}(\overline{X};\lambda) = 0$$

for iid $X_i \sim \text{Exponential}\left(\frac{1}{\vartheta}\right) \,\Big|\, \frac{1}{\vartheta} > 0$

$\mathbb{E}[\bar{X}] = \frac{1}{\vartheta}$, $\mathbb{E}\left[\frac{1}{\bar{X}}\right] = \frac{n\vartheta}{n-1}$

$$\text{Bias}\left(\frac{1}{\bar{X}}; \vartheta\right) = \frac{n\vartheta}{n-1} - \vartheta = \frac{n\vartheta - n\vartheta + \vartheta}{n-1} = \frac{\vartheta}{n-1}$$

$\frac{n-1}{n} \cdot \mathbb{E}\left[\frac{1}{\bar{X}}\right] = \vartheta$ thus

$$\text{Bias}\left(\frac{n-1}{n\bar{X}}; \vartheta\right) = 0$$

$$\boxed{\begin{array}{l} \underline{\text{In General}} \\ \text{Given } \mathbb{E}(T) = c\vartheta, \\ \qquad \text{Bias}\left(\frac{T}{c}; \vartheta\right) = 0 \\ \text{Given } \mathbb{E}(T) = c\vartheta + b \\ \qquad \text{Bias}\left(\frac{T-b}{c}; \vartheta\right) = 0 \end{array}}$$

If T_1 & T_2 are unbiased estimators of ϑ;

then $\alpha T_1 + (1-\alpha) T_2$ is an unbiased estimator for ϑ such that α is <u>real</u>

for n X_i with mean μ,

$T = \frac{1}{3} X_1 + \frac{2}{3} \overline{X}$, $\text{Bias}(T;\mu) = 0$

<u>Note</u>: $\mathbb{E}[X_1] = \mathbb{E}[\overline{X}] = \mu$.

for n iid $X_i \sim N(\mu, \sigma^2)$ &
m iid $Y_j \sim N(\mu, \sigma^2)$

with X_i & Y_j independent;

$T = \frac{n \overline{X}}{n+m} + \frac{m \overline{Y}}{n+m}$ $\quad \text{Bias}(T;\mu) = 0$

Some Estimators get better w/ large sample size n; If the bias approaches 0 as n approaches ∞, we call the Estimator Asymptotically Unbiased.

Eg. for T, $\lim_{n \to \infty} \text{Bias}(T_n; \theta) = 0$

We call the standard deviation of the sampling distribution of an estimator the standard error & it measures the estimator's precision.

We denote this as $SE(T)$

for n iid X_i w/ common variance σ^2,

$$SE(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

As $\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$

& $SE(\bar{X}) = \sqrt{\mathrm{Var}(\bar{X})} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$

for n $X_i \sim$ Binomial$(k=1, p)$

$$\mathrm{Bias}(\bar{X}, p) = 0$$

$$SE(\bar{X}) = \sqrt{\frac{p(1-p)}{n}}$$

for n iid $X_i \sim N(\mu, \sigma^2)$,

$$\mathrm{Bias}(S^2; \sigma^2) = 0$$

$$S^2 \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}$$

$$SE(S^2) = \sqrt{\frac{\sigma^4}{(n-1)^2} \cdot 2(n-1)} = \sigma^2 \sqrt{\frac{2}{n-1}}$$

[MSE]: MEAN SQUARED ERROR

... Squared Error

is a measure of estimator accuracy.

$$MSE(T;\vartheta) := \mathbb{E}\left[(T-\vartheta)^2\right]$$

Given $\mathrm{Var}(T) < \infty$;

$$MSE(T;\vartheta) = \mathrm{Bias}(T;\vartheta)^2 + SE(T)^2$$

Thus for unbiased estimators ;

$$MSE(T;\vartheta) = SE(T)^2 = \mathrm{Var}(T)$$

If $\lim\limits_{n\to\infty} MSE(T;\vartheta) = 0$

we say T is a MSE-Consistent Estimator for n $X_i \sim N(\mu, \sigma^2)$

S^2 & $\frac{n-1}{n} S^2$ are

MSE consitent Estimators of σ^2

the relative efficency of two estimators T_1 & T_2 is

$$RE(T_1, T_2; \vartheta) = \frac{MSE(T_1; \vartheta)}{MSE(T_2; \vartheta)}$$

the asymptotic relative efficieny of T_1 to T_2 is

$$ARE(T_1, T_2; \vartheta) = \lim_{n \to \infty} \frac{MSE(T_1; \vartheta)}{MSE(T_2; \vartheta)}$$

<u>Delta Method,</u> for approximation

$SE(g(T))$

Based on 1^{st} order Taylor Series Expansion about the RV's mean.

Assuming a RV X with μ, σ^2 & $g(x)$ be nonlinear & differentiable in an open interval about μ.

We can expand $g(x)$ such that

$$g(X) \approx g(\mu) + (X-\mu)\dot{g}(\mu)$$

Following:

$$\mathbb{E}[g(X)] \approx g(\mu)$$

$$\mathbb{V}[g(X)] \approx \dot{g}(\mu)^2 \cdot \mathbb{V}[X]$$

$$SE[g(T)] \approx |\dot{g}(\mathbb{E}[T])| \cdot SE[T]$$

for n iid $X_i \sim Exp\left(\frac{1}{\theta}\right)$, $\mathbb{E}[X_i] = \frac{1}{\theta}$

$$\mathbb{V}[X_i] = \frac{1}{\theta^2}, \quad \mathbb{E}[\bar{X}] = \frac{1}{\theta}$$

Consider $g(\bar{X}) = \frac{1}{\bar{X}}$ to estimate θ

$$SE(g(\bar{X})) \approx |\dot{g}(\mu)| \cdot SE(\bar{X})$$

$$= \left| -\frac{1}{\mu^2} \right| \cdot \frac{\sigma}{\sqrt{n}} = \frac{\sigma}{\mu^2 \sqrt{n}} = \frac{\theta}{\sqrt{n}}$$

As $\mu = \frac{1}{\theta}$ & $\sigma^2 = \frac{1}{\theta^2}$

for n iid $X_i \sim N(\mu, \sigma^2)$

Consider $g(S^2) = \sqrt{S^2} = S$ to estimate σ

$$SE(g(S^2)) \approx \left| \frac{1}{2\sqrt{\sigma^2}} \right| \cdot SE(S^2)$$

$$= \frac{1}{2\sqrt{\sigma^2}} \cdot \sqrt{\frac{2\sigma^4}{n-1}} = \sqrt{\frac{\sigma^2}{2(n-1)}}$$

As $\mathbb{V}[S^2] = \frac{2\sigma^4}{n-1}$