1/20/22

Thursday, January 20, 2022 5:02 AM

<u>Usual Situation</u>: $X_1, X_2, \cdots, X_n$ iid RVs := ⟨Random Sample⟩

with common pdf := $f_X(x;\theta)$; $\theta \in \Theta$

A statistic : $Y = h(X_1, X_2, \cdots, X_n)$

A estimator : A statistic designed to estimate θ

$$\theta \approx g(X_1, X_2, \cdots, X_n) = Y$$

An Unbiased Estimator : $\mathbb{E}[Y] = \theta \quad \forall\, \theta \in \Theta$

Some Examples

$$\mathbb{E}[\bar{X}] = \mu \qquad \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$\mathbb{E}[S^2] = \sigma^2 \qquad S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$$

<u>Example 1</u> Suppose $X_1, \cdots, X_n$ iid $\sim$ Poisson(λ) $\therefore \mathbb{E}[X_i] = \mathbb{V}[X_i] = \lambda$

- We initially have 2 unbiased estimators $(\bar{X}, S^2)$ of λ.
- Question, what estimator is better?

Mean Square Error [MSE]

If T is an estimator of θ, on average, how far away are T & θ?

$$MSE[T;\theta] = \mathbb{E}[(T-\theta)^2] = \mathbb{E}[T^2 - 2T\theta + \theta^2]$$

$$= \mathbb{E}[T^2] - 2\theta\,\mathbb{E}[T] + \theta^2$$

Notice: If T is an unbiased estimator of ϑ, then $\mathbb{E}[T]=\vartheta$, & $MSE[T;\vartheta]=\mathbb{V}[T]$

$= \mathbb{E}[T^2]-2\vartheta\mathbb{E}[T]+\vartheta^2$

$= \mathbb{E}[T^2]-2\vartheta^2+\vartheta^2 = \mathbb{E}[T^2]-\mathbb{E}[T]^2$

Examples: ① If $X_1,\dots,X_n$ are a random sample from $f_{X_i}(x;\vartheta)$ with $\mathbb{E}[X]=\mu$ & $\mathbb{V}[X]=\sigma^2$

then $MSE[\bar{X},\mu]=\mathbb{V}[\bar{X}]=\frac{\sigma^2}{n}$ (so $\lim\limits_{n\to\infty} MSE[\bar{X},\mu]=0$)

② If $X_1,\dots,X_n$ is a Random Sample from $N(\mu,\sigma^2)$

then $\mathbb{E}\left[\frac{(n-1)S^2}{\sigma^2}\right]$ is χ^2_{n-1} $\therefore$

$$\underbrace{\mathbb{E}\left[\frac{(n-1)S^2}{\sigma^2}\right]=n-1}_{\mathbb{E}[S^2]=\sigma^2} \quad \& \quad \mathbb{V}\left[\frac{(n-1)S^2}{\sigma^2}\right]=2(n-1)$$

$$\mathbb{V}[S^2]=2\cdot\frac{(n-1)}{(n-1)^2}\cdot\sigma^4=\frac{2\sigma^4}{n-1}$$

$\therefore$ S^2 is unbiased

so $MSE[S^2,\sigma^2]=\mathbb{V}[S^2]=\frac{2\sigma^4}{n-1}$ (So $\lim\limits_{n\to\infty} MSE[S^2,\sigma^2]=0$)

What we see is that as we take more samples our MSE decreases, thus we get better & better estimations

Definition: If Y is an estimator of ϑ; we say that Y is a consistent estimator of ϑ if

$$\lim_{n\to\infty} MSE[Y(X_1,\dots,X_n),\vartheta]=0$$

Example: Recall: $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{E}xp(\beta)$

has $f_{X_i}(x_i, \beta) = \frac{1}{\beta} e^{\frac{-x}{\beta}}$, $x > 0$ w/ $\mathbb{E}[X_i] = \beta$ & $\mathbb{V}[X_i] = \beta^2$

To estimate $\frac{1}{\beta}$

Recall:	$\Sigma = X_1 + \dots + X_n$	$\Sigma \sim \text{Gamma}(\alpha = n, \beta = \beta)$	$\mathbb{V}[\Sigma] = n\beta^2$
$\overline{X} = \frac{\Sigma}{n}$	$f_\Sigma(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{\frac{-x}{\beta}}$	$\mathbb{E}[\Sigma] = n\beta$	$\int_0^\infty x^{n-1} e^{\frac{-x}{\beta}} dx = (n-1)!\beta^n$
$\mathbb{E}[\frac{1}{\Sigma}] = \frac{1}{(n-1)\beta}$	$T = \frac{n-1}{\Sigma}$ is an unbiased estimator of $\frac{1}{\beta}$		

Compute $MSE\left[T = \frac{n-1}{\Sigma}; \frac{1}{\beta}\right] = \mathbb{V}[T] = \mathbb{V}\left[\frac{n-1}{\Sigma}\right] = (n-1)^2 \cdot \mathbb{V}\left[\frac{1}{\Sigma}\right]$

$$\mathbb{V}\left[\frac{1}{\Sigma}\right] = \mathbb{E}\left[\frac{1}{\Sigma^2}\right] - \mathbb{E}\left[\frac{1}{\Sigma}\right]^2 \quad ; \quad \mathbb{E}\left[\frac{1}{\Sigma}\right]^2 = \left(\frac{1}{(n-1)\beta}\right)^2 = \frac{1}{(n-1)^2\beta^2}$$

$$\mathbb{E}\left[\frac{1}{\Sigma^2}\right] = \int_0^\infty \frac{1}{x^2} \cdot f_\Sigma(x)\, dx = \int_0^\infty \frac{1}{x^2} \cdot \frac{1}{(n-1)!\beta^n} \cdot x^{n-1} \cdot e^{-\frac{x}{\beta}}\, dx$$

$$= \frac{1}{(n-1)!\beta^n} \int_0^\infty x^{n-3} \cdot e^{-\frac{x}{\beta}}\, dx$$

$$= \frac{1}{(n-1)!\beta^n} \cdot (n-3)!\beta^{n-2} = \frac{1}{(n-1)(n-2)\beta^2}$$

$$\mathbb{V}\left[\frac{1}{\Sigma}\right] = \frac{1}{(n-1)(n-2)\beta^2} - \frac{1}{(n-1)^2\beta^2}$$

$$= \frac{(n-1) - (n-2)}{(n-1)^2(n-2)\beta^2} = \frac{1}{(n-1)^2(n-2)\beta^2}$$

$$MSE[T] = \mathbb{V}[T] = (n-1)^2\,\mathbb{V}\left[\frac{1}{\Sigma}\right] = \frac{(n-1)^2}{(n-1)^2(n-2)\beta^2}$$

$$= MSE\left[\frac{n-1}{\Sigma}\right] = MSE\left[\frac{n-1}{X_1+\cdots+X_n}\right] = \frac{1}{(n-2)\beta^2}$$

$\left(\text{Again as } \lim_{n\to\infty} MSE[T] = 0\right)$ $\therefore$ T is a <u>CONSISTENT</u> <u>Estimator</u> of $\frac{1}{\beta}$

<u>Example #2)</u> let $X_1,\dots,X_n$ iid $\sim Exp(\beta)$ $\therefore$ $X_1+\cdots+X_n = \Sigma \sim Gamma(n,\beta)$

The common pdf $f_{X_i}(x_i) = \frac{1}{\beta}e^{-\frac{x}{\beta}}\,\mathbb{1}(x>0)$; $\mathbb{E}[X_i]=\beta$; $\mathbb{V}[X_i]=\beta^2$

Let $Y = \min\{X_1,\dots,X_n\}$

The cdf: $F_Y(y) = \mathbb{P}(\min\{X_1,\dots,X_n\} \le y) = 1 - \mathbb{P}(\min\{X_1,\dots,X_n\} > y)$

$$= 1 - [\mathbb{P}(X_i>y)]^n$$

$$= 1 - \left(e^{-\frac{x}{\beta}}\right)^n$$

$$= 1 - e^{-\frac{nx}{\beta}}$$

We know $f_Y(y) = \frac{d}{dx}[F_Y(y)]$

$$f_Y(y) = \frac{d}{dx}\left[1 - e^{-\frac{nx}{\beta}}\right]$$

$$= \frac{n}{\beta}e^{-\frac{nx}{\beta}}$$

$\therefore$ $Y \sim Exp\left(\frac{\beta}{n}\right)$

$\therefore$ $\mathbb{E}[Y] = \frac{\beta}{n}$ & $\mathbb{V}[Y] = \frac{\beta^2}{n^2}$

We now can see that $Y^* = nY = n\cdot\min\{X_1,\dots,X_n\}$ is

We now can see that $Y^* = nY = n \cdot \min\{X_1, \cdots, X_n\}$ is an unbiased estimator of β

$$\text{as } \mathbb{E}[Y^*] = \mathbb{E}[nY] = n\mathbb{E}[Y] = n \cdot \frac{\beta}{n} = \beta$$

We now have 2 unbiased estimators of β, being $\bar{X}$ & Y^*

So what estimator is better?

We know $MSE[\bar{X} ; \beta] = \frac{\sigma^2}{n} = \frac{\beta^2}{n}$

For $MSE[Y^* ; \beta] = MSE[n \cdot \min\{X_1, \cdots, X_n\} ; \beta]$

$= \mathbb{V}[n \cdot \min\{X_1, \cdots, X_n\}] = n^2 \cdot \mathbb{V}[\min\{X_1, \cdots, X_n] = n^2 \mathbb{V}[Y] = n^2 \cdot \frac{\beta^2}{n^2} = \beta^2$

$\therefore$ We can now see that for any $n > 1$; $\bar{X}$ is a better estimator than Y^*

* We can see here that Y^* is <u>not</u> a <u>consistent estimator</u> of β.