MTH 375
Fall 2022
Hw7 key

#1. Let $X_1, \cdots, X_{10}$ be a sample of iid N(0, θ) random variables. (θ is the variance, not the standard deviation.)

a) Show that T $= \dfrac{1}{\theta}\sum_{i=1}^{n} X_i^2$ is a pivotal quantity and find its distribution.

The random variables $X_i/\sqrt{\theta}$ are independent standard normal, so T is χ^2 with n df.

b) Determine a 99% confidence interval for θ based on T.

$P\left(2.16 < \dfrac{1}{\theta}\sum_{i=1}^{n} X_i^2 < 25.19\right) = .99$, so $P\left(\dfrac{\sum_{i=1}^{n} X_i^2}{25.19} < \theta < \dfrac{\sum_{i=1}^{n} X_i^2}{2.16}\right) = .99$.

c) Suppose that $\sum_{i=1}^{10} x_i^2 = 27.3$. What is the 99% confidence interval for θ?

Since $27.3/215.19 \approx 1.08$ and $27.3/2.16 \approx 12.66$, the interval is approximately (1.08,12.66).

d) Determine a 99% confidence interval for SD(X) $= \sqrt{\theta}$.

From (b), we have $P\left(\sqrt{\dfrac{\sum_{i=1}^{n} X_i^2}{25.19}} < \sqrt{\theta} < \sqrt{\dfrac{\sum_{i=1}^{n} X_i^2}{2.16}}\right) = .99$.

The required confidence interval is $(\sqrt{1.08}, \sqrt{12.66}) \approx (1.04, 3.56)$

#2. Let $X_1, \cdots, X_6$ be a sample of iid $U[\theta, 1]$ random variables.

a) Show that $T = \dfrac{1 - X_{(1)}}{1-\theta}$ is a pivotal quantity and find its cdf.

$P\left(T = \dfrac{1 - X_{(1)}}{1-\theta} \le t\right) = P\left(X_{(1)} \ge 1 - t(1-\theta)\right) = \left[P\left(X_i \ge 1 - t(1-\theta)\right)\right]^n$

$= \left[\dfrac{t(1-\theta))}{1-\theta}\right]^n = t^n$ (which does not depend on θ).

b) Determine a 85% confidence interval for θ based on T. Since $\sqrt[6]{.075} \approx .649$ and $\sqrt[6]{.925} \approx .987$

$P\left(.649 < \dfrac{1 - X_{(1)}}{1-\theta} < .987\right) = .85$, so

$P\left(1 - \dfrac{1 - X_{(1)}}{.649} < \theta < 1 - \dfrac{1 - X_{(1)}}{.987}\right) = .85.$

c) Suppose $\vec{x} = \{0.527, 0.803, 0.842, 0.880, 0.474, 0.558\}$. Find the 85% confidence interval for θ.

Since $x_{(1)} = .474$, $1 - \frac{1-.474}{.649} = .190$, and $1 - \frac{1-.474}{.987} = .467$, the required interval is (.190,.467).

#3. Let $X_1, \cdots, X_8$ be a sample of iid Gamma(4, θ) random variables.

a) Show that $\frac{2}{\theta}\sum_{i=1}^{n} X_i$ is a pivotal quantity and find its distribution.

Since $\sum_{i=1}^{n} X_i$ is a Gamma($4n, \theta$) random variable, $P\left(\frac{2}{\theta}\sum_{i=1}^{n} X_i \le t\right) = P\left(\sum_{i=1}^{n} X_i \le \frac{t\theta}{2}\right)$
$= \int_0^{t\theta/2} \frac{1}{\Gamma(4n)\theta^{4n}} x^{4n-1} e^{-x/\theta}\, dx = \int_0^t \frac{1}{\Gamma(4n)\theta^{4n}} \left(\frac{u\theta}{2}\right)^{4n-1} e^{-u/2} \cdot \frac{\theta}{2}\, du$
(where $u = 2x/\theta$, $du = 2/\theta\, d\theta$), $\quad = \int_0^t \frac{1}{\Gamma(4n)2^{4n}} u^{4n-1} e^{-u/2}\, du$, which does not depend on θ. Therefore $\frac{2}{\theta}\sum_{i=1}^{n} X_i$ is a pivotal quantity. Its pdf is Gamma$(4n, 2) = \chi^2$ with $8n$ df .

b) Determine a 95% confidence interval for θ based on T.

Since $P\left(43.78 < \frac{2}{\theta}\sum_{i=1}^{n} X_i < 88.00\right) = .95$, $P\left(\frac{2\sum_{i=1}^{n} X_i}{88.00} < \theta < \frac{2\sum_{i=1}^{n} X_i}{43.78}\right) = .95$.

c) Suppose that $\vec{x} = \{17.40, 15.95, 15.17, 7.92, 11.54, 10.46, 8.47, 15.54\}$. Find a 95% confidence interval for θ.

The sum is $\Sigma = 102.45$. Since $204.9/88.00 \approx 2.32$, and $204.9/43.78 \approx 4.68$, the required confidence interval for θ is (2.32,4.68).

#4. Let $X_1, \cdots, X_6$ be a sample of iid N(μ, θ) random variables. (μ, θ are both unknown. θ is the variance, not the standard deviation.)

a) Why is $T = (n-1)S^2/\theta$ a pivotal quantity? What is the distribution of $(n-1)S^2/\theta$?

$T = (n-1)S^2/\sigma^2$ is a pivotal quantity because its distribution is χ^2 with n-1 df, which does not depend on θ.

b) Determine a 93% confidence interval for θ based on T.

Since T is χ^2 with 5 df, $P\left(.965 < \frac{5S^2}{\theta} < 11.98\right) = .93$, so $P\left(\frac{5S^2}{11.98} < \theta < \frac{5S^2}{.965}\right) = .93$.

c) Suppose that $\vec{x} = \{0.71, 0.62, 2.28, 1.01, 10.01, 7.64\}$. Find a 95% confidence interval for θ.

Since T is χ^2 with 5 df, $P\left(.831 < \dfrac{5S^2}{\theta} < 12.83\right) = .95$, so $P\left(\dfrac{5S^2}{12.83} < \theta < \dfrac{5S^2}{.831}\right) = .95$.

Since $s^2 \approx 16.60$, $\dfrac{5 \cdot 16.60}{.831} \approx 99.88$ and $\dfrac{5 \cdot 16.60}{12.83} \approx 6.47$, a 95% confidence interval is $(6.47, 99.88)$.

#5. A pollster chooses 500 citizens at random, and asks whether each plans to vote 'yes' or 'no' on proposition X. In the sample, 375 indicate a 'yes' vote.

Find a 95% confidence interval for the proportion of the total population who plan to vote 'yes.'

Let Y be the number of yes's in the sample, and p the proportion of yes's in the population. Then Y is binomial(500,p), so approximately Normal$(500p, 500p(1-p))$. Therefore $P\left(-1.96 < \dfrac{Y - 500p}{\sqrt{500p(1-p)}} < 1.96\right) \approx .95$, so

$$P\left(\frac{Y}{500} - 1.96\sqrt{\frac{p(1-p)}{500}} < p < \frac{Y}{500} + 1.96\sqrt{\frac{p(1-p)}{500}}\right) \approx .95.$$

Finally, letting $Y = 375$ and replacing $p(1-p)$ by its greatest possible value, $1/4$, we obtain the confidence interval $(.706, .794)$.