Midterm

Thursday, February 24, 2022 10:38 AM

1. Let $X_1, \ldots, X_n$ be beta with α unknown and $\beta = 3$. Find the Method of Moments estimator of α. From our distribution handout...

	Range	$f_X(x)$	μ	σ^2
Beta(α,β)	$(0,1)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

let $\underline{X} = X_1, \ldots, X_n$; $\mathbb{E}[X_i] = \frac{\alpha}{\alpha+\beta} = \frac{\alpha}{\alpha+3}$

Note: $\mathbb{E}[\bar{X}] = \mathbb{E}[X_i]$

Now to find the MoM estimator : $\mathbb{E}[\bar{X}] = \frac{\hat{\alpha}}{\hat{\alpha}+3} = \bar{X}$

$\therefore \quad \hat{\alpha} = \hat{\alpha}\bar{X} + 3\bar{X} \implies \hat{\alpha}(1-\bar{X}) = 3\bar{X} \implies \boxed{\hat{\alpha}_{mom} = \frac{3\bar{X}}{1-\bar{X}}}$

2. Let $Y_1, \ldots, Y_n$ have common pdf $f(y) = \begin{cases} \frac{2y}{a^2}, & 0 \le y \le a \\ 0 & \text{otherwise} \end{cases}$.

(a) Find a sufficient statistic for a.

(b) Find an UMVUE for $\frac{1}{a}$.

let $\underline{Y} = Y_1, \ldots, Y_n$

$$L(a) = \left(\frac{2}{a^2}\right)^n \times y_1 \times \cdots \times y_n = \underbrace{\frac{2^n}{a^{2n}} \prod_{i=1}^{n} y_i}_{g} \cdot \underbrace{1}_{h}$$

From the pdf, y has upper bound a ;

$\boxed{\text{Thus } T = \max\{\underline{Y}\} \text{ is a sufficient statistic for } a.}$

our cdf is $F_{Y_i} = \int_0^m \frac{2y}{a^2}\, dy = \left[\frac{y^2}{a^2}\Big|_0^m\right] = \frac{m^2}{a^2} - 0$

$$F_T = P(T \le t) = (F_{Y_i})^n = \left(\frac{m^2}{a^2}\right)^n = \frac{m^{2n}}{a^{2n}}$$

$$f_T = \frac{\delta}{\delta m}\left[\frac{m^{2n}}{a^{2n}}\right] = \frac{2n m^{2n-1}}{a^{2n}}$$

$$\mathbb{E}[T] = \int_0^a \frac{2n m^{2n}}{a^{2n}}\, \delta m = \frac{2n m^{2n+1}}{(2n+1)a^{2n}}\Big|_0^a = \frac{2n a^{2n+1}}{(2n+1)a^{2n}} - 0$$

$$\mathbb{E}[T] = \frac{2na}{2n+1}$$

$2n\,\mathbb{E}[T]$

$2n+1$

Solving for $\frac{1}{a}$: $\frac{1}{a} = \frac{2n\,\mathbb{E}[T]}{2n+1}$

as $\mathbb{E}\left[\frac{2n}{2n+1}\right] = \frac{2n}{2n+1}$ we can construct a $T^* = \left(\frac{2n}{2n+1}\right) \cdot \max\{\underline{Y}\}$ which is UMVUE for a, as $\mathbb{E}[T^*] = a$.

3. Let $Y_1, \ldots, Y_n$ have common pmf $p(y) = \begin{cases} 2p & \text{if } y=-1 \\ 1/2-p & \text{if } y=0 \\ 1/2-p & \text{if } y=1 \end{cases}$, $0 < p < \frac{1}{2}$

(a) Find a sufficient statistic for p.
(b) Find the MOM estimator of p. Is it unbiased?
(c) Find the MLE estimator of p. Is it unbiased?
(d) Find the MSE of both estimators. For which values of p is the MSE of the MLE the smaller of the two?

let $\underline{Y} = Y_1, \ldots, Y_n$

$\mathbb{E}[\underline{Y}] = \mathbb{E}[Y] = -1(2p) + 0\left(\frac{1}{2}-p\right) + 1\left(\frac{1}{2}-p\right)$

$= -2p + 0 + \frac{1}{2} - p = \frac{1}{2} - 3p$

By the MOM: $\bar{X} = \frac{1}{2} - 3\hat{p} \Rightarrow 3\hat{p} = \frac{1}{2} - \bar{X} = \frac{1-2\bar{X}}{2}$

$\therefore \boxed{\hat{p}_{MOM} = \frac{1-2\bar{X}}{6}}$

let $A = \#\, Y_i\text{'s} = -1$
$B = \#\, Y_i\text{'s} = 0$
$C = \#\, Y_i\text{'s} = 1$

$\therefore L(p) = (2p)^A \left(\frac{1}{2}-p\right)^{B+C}$

Note: $A+B+C = n$
$\therefore B+C = n-A$

$\ell(p) = A\ln(2p) + (B+C)\ln\left(\frac{1}{2}-p\right)$

$\ell_p = \frac{2A}{2p} - \frac{(B+C)}{\left(\frac{1}{2}-p\right)} = \frac{A}{p} - \frac{(n-A)}{\left(\frac{1}{2}-p\right)}$

$= \frac{A}{p} + \frac{A-n}{\frac{1}{2}-p}$

To find the MLE set $\ell_p = 0$

$\therefore \frac{A}{\hat{p}} = \frac{n-A}{\hat{p}+\frac{1}{2}} \Rightarrow A\hat{p} + \frac{A}{2} = n\hat{p} - A\hat{p}$

$\rightarrow \frac{A}{2} = n\hat{p} - 2A\hat{p} = \hat{p}(n-2A)$

$$\Rightarrow \frac{A}{2} = n\hat{p} - 2A\hat{p} = \hat{p}(n-2A)$$

$$\Rightarrow \boxed{\hat{p}_{MLE} = \frac{A}{2n-4A}}$$

We have $\hat{p}_{mom}$ unbiased ; thus $MSE[\hat{p}_{mom}; p] = \mathbb{V}[\hat{p}_{mom}]$

$$\mathbb{V}[\hat{p}_{mom}] = \mathbb{V}\left(\frac{2-2\bar{Y}}{6}\right) = \frac{\mathbb{V}(2-2\bar{Y})}{36} = \frac{4\mathbb{V}[\bar{Y}]}{36} = \frac{\mathbb{V}[\bar{Y}]}{9} = \frac{\mathbb{V}[Y_i]}{9n}$$

$$\mathbb{E}[Y_i]^2 = \left(\tfrac{1}{2} - 3p\right)^2 = \tfrac{1}{4} - 3p + 9p^2$$

$$\mathbb{E}[Y_i^2] = (-1)^2(2p) + (0)^2\left(\tfrac{1}{2} - p\right) + (1)^2\left(\tfrac{1}{2} - p\right)$$

$$= 2p + \tfrac{1}{2} - p = \tfrac{1}{2} + p$$

$$\mathbb{V}[Y_i] = \tfrac{1}{2} + p - \tfrac{1}{4} + 3p - 9p^2 = \tfrac{1}{4} + 4p - 9p^2$$

$$\boxed{\mathbb{V}[\hat{p}_{MOM}] = \frac{1}{36n} + \frac{4p}{9n} - \frac{p^2}{n} = MSE[\hat{p}_{mom}; p]}$$

$$MSE[\hat{p}_{MLE}; p] = \mathbb{V}[\hat{p}_{MLE}] = \mathbb{E}[\hat{p}^2_{MLE}] - \mathbb{E}[\hat{p}_{MLE}]^2$$

$$\hat{p}_{MLE}^2 = \left(\frac{A}{2n-4A}\right)^2 = \frac{A^2}{4n - 16nA + 16A} = \frac{A^2}{4(n-4A(n-1))}$$

$$\mathbb{V}[\hat{p}_{MLE}] = \mathbb{E}\left(\frac{A^2}{4(n-4A(n-1))}\right) - \mathbb{E}\left(\frac{A}{2n-4A}\right)^2$$

$$\mathbb{E}[A] = \tfrac{2}{3}p? \quad \mathbb{E}[A^2] = \tfrac{4}{9}p^2 \;?$$

$$\mathbb{V}[\hat{p}_{MLE}] = \frac{1}{16}\mathbb{E}\left(\frac{A^2}{n-4A(n-1)}\right) - \frac{1}{4}\mathbb{E}\left(\frac{A}{n-2A}\right)^2$$

$$= \frac{1}{16}\left(\frac{4p^2/9}{n-4p/3(n-1)}\right) - \frac{1}{4}\left(\frac{2p/3}{n-2p/3}\right)^2$$

$$= \frac{1}{\ldots}\left(\frac{p^2}{\ldots}\right) - \frac{1}{4}\left(\frac{4p^2/9}{(n-2p/3)^2}\right)$$

$$= \frac{1}{36}\left(\frac{p^2}{n - 4p/3(n-1)}\right) - \frac{1}{4}\left(\frac{4p^2/9}{(n-2p/3)^2}\right)$$

$$= \frac{p^2}{36n - 48p(n-1)} - \frac{1}{9}\left(\frac{p^2}{n^2 - 4np/3 + 4p^2/9}\right)$$

$$= \boxed{\frac{p^2}{36n - 48np + 48p} - \frac{p^2}{9n^2 - 36pn + 4p^2} = MSE[\hat{p}_{MLE}; p]}$$

$\therefore$ for small p, MSE $\hat{p}_{MLE}$ is better. ?

for large p, MSE $\hat{p}_{MoM}$ is better.

4. Let $X_1, \ldots, X_n$ be exponential with parameter β. Find UMVUE's for β, β^2 and β^3.
Hint: The exponential distribution has pdf $p(x) = \frac{1}{\beta}e^{-x/\beta}$ $(x > 0)$. The sum of n independent exponential(β) random variables has pdf Gamma($\alpha = n, \beta$).

From distributions..

	range	$f_x(x)$	μ	σ^2	$M_x(t)$
Exponential(β)	$(0, \infty)$	$\frac{1}{\beta}e^{-x/\beta}$	β	β^2	$\frac{1}{1-\beta t}$
Gamma(α, β)	$(0, \infty)$	$\frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta t}\right)^\alpha$

$$L(\beta) = \frac{1}{\beta^n} \cdot e^{\frac{\sum_{i=1} x_i}{\beta n}}$$ $\therefore \Sigma_x$ is sufficient ; $\sum_{i=1}^{n} x_i = \Sigma_x$

from hint ; $\Sigma_x \sim \text{Gamma}(n, \beta)$

$$M_{\Sigma_x}(t) = \left(\frac{1}{1-\beta t}\right)^n \; ; \; \dot{M}_{\Sigma_x}(t) = \beta n \left(\frac{1}{1-\beta t}\right)^{n+1}$$

$$\ddot{M}_{\Sigma_x}(t) = \beta^2 n(n+1)\left(\frac{1}{1-\beta t}\right)^{n+2} \; ; \; \dddot{M}_{\Sigma_x}(t) = \beta^3 n(n+1)(n+2)\left(\frac{1}{1-\beta t}\right)^{n+3}$$

* Wolfram used for mgf derivatives *

$$\mathbb{E}[\Sigma_x] = \beta n \left(\frac{1}{1}\right)^{n+1} = \beta n$$

thus $\hat{\beta} = \mathbb{E}\left[\frac{\Sigma_x}{n}\right] = \mathbb{E}[\bar{X}]$ & $\boxed{\bar{X} \text{ is UMVUE for } \beta.}$

$\wedge 2$ $\quad \Sigma_x^2$

$$\mathbb{E}\left[\Sigma_x^2\right] = \beta^2 n(n+1) \cdot 1 \Rightarrow \hat{\beta}^2 = \frac{\Sigma_x^2}{n(n+1)}$$

$$\therefore \boxed{\frac{\Sigma_x^2}{n(n+1)} \text{ is UMVUE for } \beta^2.}$$

$$\mathbb{E}\left[\Sigma_x^3\right] = \beta^3 n(n+1)(n+2) \cdot 1 \Rightarrow \hat{\beta}^3 = \frac{\Sigma_x^3}{n(n+1)(n+2)}$$

$$\therefore \boxed{\frac{\Sigma_x^3}{n(n+1)(n+2)} \text{ is UMVUE for } \beta^3.}$$

5. Let $X_1, \ldots, X_{10}$ be iid normal with unknown μ, σ^2.
Let $\overline{X} = \dfrac{X_1 + \cdots + X_{10}}{10}$ and $S^2 = \dfrac{1}{9}\sum_{i=1}^{10}(X_i - \overline{X})^2$.

here $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{10}\right)$

(a) Find a number a such that $P\left(-a < \dfrac{\overline{X} - \mu}{S} < a\right) = .95$. (ref HW1 #3)

(b) Find positive numbers a, b such that $P\left(a < \dfrac{S^2}{\sigma^2} < b\right) = .95$.

let $Y = \dfrac{\overline{X} - \mu}{S}$; $Y\sqrt{n} \sim T_{n-1} \Rightarrow Y\sqrt{10} \sim T_9$

$$\therefore P\left(-a < \frac{\overline{X} - \mu}{S} < a\right) = .95 = P\left(-a\sqrt{10} < T_9 < a\sqrt{10}\right)$$

```
> a_sqrt10 <- qt(p = c(0.025, 0.975), df = 9);
> a_sqrt10 / sqrt(10)
[1] -0.7153569  0.7153569
>
```

$\boxed{a = 0.715}$

let $Z = \dfrac{S^2}{\sigma^2}$; $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow \dfrac{9S^2}{\sigma^2} \sim \chi^2_9$

$$\therefore P\left(a < \frac{S^2}{\sigma^2} < b\right) = 0.95 = P\left(9a < \chi^2_9 < 9b\right)$$

```
> ab_9 <- qchisq(p = c(0.025, 0.975), df = 9);
> ab_9 / 9
[1] 0.3000433 2.1136409
>
```

$\boxed{\begin{aligned} a &= 0.300 \\ b &= 2.113 \end{aligned}}$