MTH375: Mathematical Statistics - Homework #5

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Key Concepts: MOM and MLE estimators.

1. Let $X_1, \dots, X_n$ be a sample of *iid Negative Binomial*$(r = 4, p = \theta)$ random variables with $\theta \in [0, 1]$.

Determine the MLE and the MOM estimators of θ.

Solution:

We will first find the likelihood and log-likelihood function then sent the derivative to zero to find the MLE.

- $p_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} = \binom{x-1}{3} \theta^4 (1-\theta)^{x-4} = \dfrac{(x-1)(x-2)}{3!} \theta^4 (1-\theta)^{x-4}$.
- $L(\theta) = \dfrac{(x_1-1)(x_1-2)}{6} \cdot \dfrac{(x_2-1)(x_2-2)}{6} \cdots \dfrac{(x_n-1)(x_n-2)}{6} \theta^{4n} (1-\theta)^{(x_1-4)+(x_2-4)+\cdots+(x_n-4)}$.
- $L(\theta) = \dfrac{\prod_{i=1}^n (x_i-1)(x_i-2)}{6^n} \theta^{4n} (1-\theta)^{\sum_{i=1}^n x_i - 4}$; *Let* $\displaystyle\prod_{i=1}^n (x_i-1)(x_i-2) = \Pi_x$ & $\displaystyle\sum_{i=1}^n x_i = \Sigma_x$.
- $L(\theta) = \dfrac{\Pi_x}{6^n} \theta^{4n} (1-\theta)^{\Sigma_x - 4n}$.
- $\ell(\theta) = \ln(L(\theta)) = \ln(\Pi_x) - n\ln(6) + 4n\ln(\theta) + (\Sigma_x - 4n)\ln(1-\theta)$.
- $\ell_\theta = \dfrac{d\ell}{d\theta} = \dfrac{4n}{\theta} - \dfrac{\Sigma_x - 4n}{1-\theta}$.
- $\ell_\theta = 0 \Rightarrow \dfrac{4n}{\hat{\theta}} = \dfrac{\Sigma_x - 4n}{1-\hat{\theta}} \Rightarrow 4n - 4n\hat{\theta} = \Sigma_x\hat{\theta} - 4n\hat{\theta} \Rightarrow 4n = \Sigma_x\hat{\theta} \Rightarrow 4 = \overline{X}\hat{\theta}$.
- $\hat{\theta}_{MLE} = \dfrac{4}{\overline{X}}$.

We will now find the expected value, $E(X_i)$ then set equal to $\overline{X}$ to find the MOM.

- $E(X_i) = \dfrac{r}{p} = \dfrac{4}{\hat{\theta}} = \overline{X} \Rightarrow 4 = \overline{X}\hat{\theta}$.
- $\hat{\theta}_{MOM} = \dfrac{4}{\overline{X}}$.

In this problem the MLE and MOM estimators of θ are the same.

2. Let $X_1, \ldots, X_n$ be a sample of $iid\ Normal(\mu = 0, \sigma^2 = \theta)$ random variables with $\theta > 0$. Determine ...

(a) The MLE $\hat{\theta}$ of θ.

Solution:

We will first find the likelihood and log-likelihood function then sent the derivative to zero to find the MLE.

- $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} = \dfrac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}$.
- $L(\theta) = \dfrac{1}{(2\pi\theta)^{n/2}} e^{-(x_1^2+x_2^2+\cdots+x_n^2)/2\theta} = \dfrac{1}{(2\pi\theta)^{n/2}} e^{-\Sigma_{x^2}/2\theta}\ ;\ Where\ \Sigma_{x^2} = \displaystyle\sum_{i=1}^{n} x_i^2$.
- $\ell(\theta) = -\dfrac{n}{2}\ln(2\pi\theta) - \dfrac{\Sigma_{x^2}}{2\theta}$.
- $\ell_\theta = -\dfrac{2n\pi}{4\pi\theta} + \dfrac{\Sigma_{x^2}}{2\theta^2} = \dfrac{\Sigma_{x^2}}{2\theta^2} - \dfrac{n}{2\theta} = \dfrac{1}{2\theta^2}\Big[\Sigma_{x^2} - n\theta\Big]$.
- $\ell_\theta = 0 \Rightarrow \Sigma_{x^2} = n\hat{\theta}$.
- $\hat{\theta}_{MLE} = \dfrac{\Sigma_{x^2}}{n}$.

We now have the MLE $\hat{\theta}$ of θ is $\dfrac{\Sigma_{x^2}}{n}$.

(b) $E(\hat{\theta})$ and $V(\hat{\theta})$.

Solution:

First find $E(X_i)$, then $E(X_i)^2$, $V(X_i)$ and $E(X_i^2)$ to last find $E(\hat{\theta})$.

- $E(X_i) = 0\ ;\ E(X_i)^2 = 0^2\ ;\ V(X_i) = \theta$.
- $E(X_i^2) = V(X_i) + E(X_i)^2 = \theta$.
- $E(\hat{\theta}) = E\Big(\dfrac{\Sigma_{x^2}}{n}\Big) = E(X_i^2) = \theta$.

Next find the mgf of X_i, then find $E(X_i^4)$ and $V(X_i^2)$, to last find $V(\hat{\theta})$.

- $E(X_i^2)^2 = \theta^2$.
- $M_{X_i}(t) = e^{\theta t^2/2}$.
- $M'_{X_i}(t) = \theta t e^{\theta t^2/2}$.
- $M''_{X_i}(t) = \theta e^{\theta t^2/2} + \theta^2 t^2 e^{\theta t^2/2}$.
- $M^{(3)}_{X_i}(t) = \theta^2 t e^{\theta t^2/2} + 2\theta^2 t e^{\theta t^2/2} + \theta^3 t^3 e^{\theta t^2/2}$.
- $M^{(4)}_{X_i}(t) = \theta^2 e^{\theta t^2/2} + \theta^3 t^2 e^{\theta t^2/2} + 2\theta^2 e^{\theta t^2/2} + 2\theta^3 t^2 e^{\theta t^2/2} + 3\theta^3 t^2 e^{\theta t^2/2} + \theta^4 t^4 e^{\theta t^2/2}$.
- $M^{(4)}_{X_i}(0) = \theta^2 e^0 + 2\theta^2 e^0 = 3\theta^2 = E(X_i^4)$.
- $V(\hat{\theta}) = V\left(\frac{\Sigma_{x^2}}{n}\right) = \frac{1}{n^2} \cdot V(\Sigma_{x^2}) = \frac{1}{n^2} \cdot nV(X_i^2) = \frac{V(X_i^2)}{n}$.
- $V(X_i^2) = E(X_i^4) - E(X_i^2)^2 = 3\theta^2 - \theta^2 = 2\theta^2 \Rightarrow V(\hat{\theta}) = \frac{2\theta^2}{n}$.

Thus we have $E(\hat{\theta}) = \theta$ and $V(\hat{\theta}) = \frac{2\theta^2}{n}$.

(c) The MLE of $SD(X_i) = \sqrt{\theta}$.

Solution:

By the invariance principal, as $\hat{\theta}$ is the MLE of θ, $\tau(\hat{\theta})$ is the MLE of $\tau(\theta)$.

- $\hat{\theta}_{MLE} = \frac{\Sigma_{x^2}}{n}$.
- $\sqrt{\hat{\theta}} = \sqrt{\frac{\Sigma_{x^2}}{n}}$

Thus $\sqrt{\frac{\Sigma_{x^2}}{n}}$ is the MLE of $SD(X_i) = \sqrt{\theta}$.

3. Recall the family of distributions with pmf: $p_X(x) = \begin{cases} p & \text{if } x = -1 \\ 2p & \text{if } x = 0 \\ 1-3p & \text{if } x = 1 \end{cases}$

Here p is an unknown paramater and $0 \leq p \leq \dfrac{1}{3}$.

Let $X_1, \ldots, X_n$ be iid with common pmf be a member of this family.

$A =$ the number of i with $X_i = -1$, $B =$ the number of i with $X_i = 0$, $C =$ the number of i with $X_i = 1$.

(i) Find the MOM estimator of p.

Solution:

Find the expected value, $E(X_i)$ then set equal to $\overline{X}$ to find the MOM.

- $E(X_i) = p \cdot -1 + 2p \cdot 0 + (1-3p) \cdot 1 = -p + 1 - 3p = 1 - 4p$
- $1 - 4\hat{p} = \overline{X} \Rightarrow \hat{p} = \dfrac{\overline{X} - 1}{-4}$
- $\hat{p}_{MOM} = \dfrac{1 - \overline{X}}{4}$

Thus $\dfrac{1 - \overline{X}}{4}$ is the MOM estimator of p.

(ii) Find the MLE estimator of p.

Solution:

Find the likelihood and log-likelihood function then sent the derivative to zero to find the MLE.

- $L(p) = 2^2 \cdot p^{A+B} \cdot (1-3p)^C$.
- $\ell(p) = 2\ln(2) + (A+B)\ln(p) + C\ln(1-3p)$.
- $\ell_p = \dfrac{A+B}{p} - \dfrac{3C}{1-3p}$.
- $\ell_p = 0 \Rightarrow \dfrac{A+B}{\hat{p}} = \dfrac{3C}{1-3\hat{p}} \Rightarrow A+B-3A\hat{p}-3B\hat{p} = 3C\hat{p} \Rightarrow A+B = 3\hat{p}(A+B+C) = 3\hat{p}n$.
- $\hat{p}_{MLE} = \dfrac{A+B}{3n}$.

Thus $\dfrac{A+B}{3n}$ is the MLE estimator of p.

(iii) A random sample of size 100 from this distribution produced 23: -1's, 38: 0's and 39: 1's. Evaluate the MOM and MLE estimates of p for this data set.

Solution:

This problem is plug and play ...

- $\hat{p}_{MOM} = \dfrac{1-\overline{X}}{4} = \dfrac{1-\frac{39+0-23}{100}}{4} = \dfrac{1}{4} - \dfrac{16}{400} = \dfrac{84}{400} = 0.21$.
- $\hat{p}_{MLE} = \dfrac{A+B}{3n} = \dfrac{23+38}{3\cdot 100} = \dfrac{61}{300} \approx 0.2033$.

The MOM estimator of p evaluates to 0.21 while the MLE estimator of p evaluates to ≈ 0.2033.

We can see they estimate similar values of p for this dataset.

4. Let $X_1, \ldots, X_n$ be a sample of *iid* $Gamma(\alpha = \alpha, \beta = \theta)$ random variables with α know and $\theta > 0$.

Determine ...

(a) The MLE $\hat{\theta}$ of θ.

Solution:

We will first find the likelihood and log-likelihood function then sent the derivative to zero to find the MLE.

- $f_X(x) = \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} = \dfrac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta}$.
- $L(\theta) = \dfrac{1}{\theta^{n\alpha} \Gamma(\alpha)^n} x_1^{\alpha-1} \cdot x_2^{\alpha-1} \cdots x_n^{\alpha-1} e^{-(x_1+x_2+\cdots+x_n)/\theta}$.
- $L(\theta) = \dfrac{1}{\theta^{n\alpha} \Gamma(\alpha)^n} \Pi_x^{\alpha-1} e^{-\Sigma_x/\theta}$; $Where\ \Pi_x = \displaystyle\prod_{i=1}^n x_i$ & $\Sigma_x = \displaystyle\sum_{i=1}^n x_i$.
- $\ell(\theta) = -n\alpha \ln(\theta) - n \ln(\Gamma(\alpha)) + (\alpha - 1) \ln(\Pi_x) - \dfrac{\Sigma_x}{\theta}$.
- $\ell_\theta = -\dfrac{n\alpha}{\theta} + \dfrac{\Sigma_x}{\theta^2} = \dfrac{n}{\theta^2}\left[\overline{X} - \alpha\theta\right]$.
- $\ell_\theta = 0 \Rightarrow \overline{X} = \alpha\hat{\theta}$.
- $\hat{\theta} = \dfrac{\overline{X}}{\alpha}$.

Thus $\dfrac{\overline{X}}{\alpha}$ is the MLE estimator of θ.

(b) $E(\hat{\theta})$.

Solution:

First find $E(X_i)$ then use to find $E(\hat{\theta})$.

- $E(X_i) = \alpha\beta = \alpha\theta$
- $E(\hat{\theta}) = E\Big(\dfrac{\overline{X}}{\alpha}\Big) = E\Big(\dfrac{\Sigma_x}{n\alpha}\Big) = \dfrac{1}{\alpha}E(X_i) = \dfrac{\alpha\theta}{\alpha}$
- $E(\hat{\theta}) = \theta$

We can see the expected value of the MLE estimator of θ is θ.

(c) If $\hat{\theta}$ is UMVUE for θ.

Solution:

We will first ensure $\hat{\theta}$ is a sufficient statistic for θ.

- $L(\theta) = \dfrac{1}{\theta^{n\alpha}\Gamma(\alpha)^n}\Pi_x^{\alpha-1}e^{-\Sigma_x/\theta} = c(\theta) \cdot h(x_1, \ldots, x_n) \cdot e^{q(\theta)t(x_1,\ldots,x_n)}$.
- $c(\theta) = \dfrac{1}{\theta^{n\alpha}\Gamma(\alpha)^n}$.
- $h(x_1, \ldots, x_n) = \Pi_x^{\alpha-1}$.
- $q(\theta) = \dfrac{1}{\theta}$.
- $t(x_1, \ldots, x_n) = -\Sigma_x$.

Thus Σ_x being sufficient implies $\hat{\theta}$ is sufficient.

Then as $E(\hat{\theta}) = \theta$, $\hat{\theta}$ is UMVUE for θ.

5. Let $X_1, \ldots, X_n$ be a sample of *iid* random variables with common pdf $f(x_i; \theta_1, \theta_2)$.

$$f(x_i; \theta_1, \theta_2) = \frac{1}{\theta_1} e^{-(x_i - \theta_2)/\theta_1} \quad \text{for} \quad x_i > \theta_2.$$

Here $\theta_1 > 0$, and θ_2 can be any real number.
Determine the MLE and the MOM estimators of (θ_1, θ_2).

Solution:

We will first find the likelihood and log-likelihood function, then sent the parital derivatives to zero, last solve the system of equations to find the MLE.

- $L(\theta_1, \theta_2) = \dfrac{1}{\theta_1^n} e^{n\theta_2/\theta_1 - \Sigma_x/\theta_1} \quad \text{for} \quad x_i > \theta_2, \quad Where\ \Sigma_x = \sum_{k=1}^n x_k.$
- $\ell(\theta_1, \theta_2) = -n \ln(\theta_1) + \dfrac{n\theta_2}{\theta_1} - \dfrac{\Sigma_x}{\theta_1}.$
- $\ell_{\theta_1} = -\dfrac{n}{\theta_1} - \dfrac{n\theta_2}{\theta_1^2} + \dfrac{\Sigma_x}{\theta_1^2} = \dfrac{-n}{\theta_1^2}\Big[\theta_1 + \theta_2 - \overline{X}\Big].$
- $\ell_{\theta_1} = 0 \Rightarrow \hat{\theta}_1 + \hat{\theta}_2 = \overline{X} \Rightarrow \hat{\theta}_1 = \overline{X} - \hat{\theta}_2.$
- $\ell(\theta_2) = -n \ln(\overline{X} - \theta_2) + \dfrac{n\theta_2}{\overline{X} - \theta_2} - \dfrac{\Sigma_x}{\overline{X} - \theta_2}.$
- $\ell_{\theta_2} = \dfrac{n}{\overline{X} - \theta_2} + \dfrac{n\overline{X}}{(\overline{X} - \theta_2)^2} - \dfrac{\Sigma_x}{(\overline{X} - \theta_2)^2} = \dfrac{n}{(\overline{X} - \theta_2)^2}\Big[\overline{X} - \theta_2 + \overline{X} - \overline{X}\Big].$
- $\ell_{\theta_2} = 0 \Rightarrow \overline{X} - \hat{\theta}_2 = 0.$
- As we have the constraint $x_i > \theta_2$ we know that $\overline{X} - \theta_2 > 0$.
- We minimize $\overline{X} - \hat{\theta}_2$ such that $x_i > \hat{\theta}_2$ when $\hat{\theta}_2 = \min\{X_1, \ldots, X_n\} = X_{(1)}$.
- Thus $\hat{\theta}_2 = X_{(1)}$ & $\hat{\theta}_1 = \overline{X} - X_{(1)}$.

We have $\overrightarrow{\theta}_{MLE} = \big(\overline{X} - X_{(1)}, X_{(1)}\big)$.

Find the expected value, $E(X_i)$ then set equal to $\overline{X}$, and find the expected squared value, $E(X_i^2)$ then set equal to $\frac{\Sigma_{x^2}}{n}$, last solve the system of equations to find the MOM.

- $E(X_i) = \hat{\theta}_2 + \hat{\theta}_1 = \overline{X} \Rightarrow \hat{\theta}_2 = \overline{X} - \hat{\theta}_1$
- $E(X_i^2) = \int_{\theta_2}^{\infty} \frac{x_i^2}{\theta_1} e^{\theta_2/\theta_1 - x_i/\theta_1}\,dx = -(x_i^2 + 2\theta_1 x_i + 2\theta_1^2)e^{\theta_2/\theta_1 - x_i/\theta_1}\Big|_{\theta_2}^{\infty}.$
- $E(X_i^2) = (\theta_2^2 + 2\theta_1\theta_2 + 2\theta_1^2)e^{\theta_2/\theta_1 - \theta_2/\theta_1} - (\infty^2 + 2\theta_1\infty + 2\theta_1^2)e^{\theta_2/\theta_1 - \infty/\theta_1}.$
- $E(X_i^2) = (\theta_2^2 + 2\theta_1\theta_2 + 2\theta_1^2)e^{0} - (\infty)e^{-\infty} = \theta_2^2 + 2\theta_1\theta_2 + 2\theta_1^2 - \frac{\infty}{e^{\infty}}\ ;\ Note: \lim_{x_i \to \infty} \frac{x_i}{e^{x_i}} = 0.$
- $E(X_i^2) = \hat{\theta}_2 + 2\hat{\theta}_1\hat{\theta}_2 + 2\hat{\theta}_1^2 = (\hat{\theta}_2 + \hat{\theta}_1)^2 + \hat{\theta}_1^2 = \overline{X}^2 + \hat{\theta}_1^2 = \frac{\Sigma_{x^2}}{n} \Rightarrow \hat{\theta}_1^2 = \frac{\Sigma_{x^2}}{n} - \overline{X}^2.$
- $\hat{\theta}_1 = \sqrt{\frac{\Sigma_{x^2}}{n} - \overline{X}^2} \Rightarrow \hat{\theta}_2 = \overline{X} - \sqrt{\frac{\Sigma_{x^2}}{n} - \overline{X}^2}.$

We have $\overrightarrow{\theta}_{MOM} = \left(\sqrt{\frac{\Sigma_{x^2}}{n} - \overline{X}^2}, \overline{X} - \sqrt{\frac{\Sigma_{x^2}}{n} - \overline{X}^2}\right).$