

MTH375: Mathematical Statistics - Homework #9

Cason Konzer

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Key Concepts: Uniformly Most Powerful (UMP) test of hypotheses.

1. Let X_1, \dots, X_n be $iid \sim Exponential(\theta)$ random variables, Determine ...

(a) The form of the UMP test of the hypotheses $H_0 : \theta = \theta_0$ vs. $H_A : \theta > \theta_0$.

Solution:

I will first state the general form then elaborate on the requested solution.

- $f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}$.
- $\frac{L(\theta_0)}{L(\theta_A)} = \left(\frac{\theta_A}{\theta_0}\right)^n e^{\Sigma_X \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right)} < k$.
- $\Sigma_X \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) < \ln(k) - n \ln\left(\frac{\theta_A}{\theta_0}\right)$.

Thus if $\theta_A > \theta_0$, our form is $\Sigma_X > k_\alpha$ as $\left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) < 0$.

(b) The form of the UMP test of the hypotheses $H_0 : \theta = \theta_0$ vs. $H_A : \theta < \theta_0$.

Solution:

- $\Sigma_X \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) < \ln(k) - n \ln\left(\frac{\theta_A}{\theta_0}\right)$.

Thus if $\theta_A < \theta_0$, our form is “Reject H_0 if $\Sigma_X < k_\alpha$ ” as $\left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) > 0$.

(c) The sampling distribution of the statistic defining the UMP test.

Solution:

As $X_i \sim Exponential(\theta)$, $\Sigma_X \sim Gamma(n, \theta)$.



2. Let X_1, \dots, X_{10} be $iid \sim \text{Binomial}(1, \theta)$ random variables, Determine ...

(a) The form of the UMP test of the hypotheses $H_0 : \theta = 0.6$ vs. $H_A : \theta < 0.6$.

Solution:

I will first state the general form then elaborate on the requested solution.

- $f(x; \theta) = \binom{1}{x} \theta^x (1 - \theta)^{1-x}$.
- $\frac{L(\theta_0)}{L(\theta_A)} = \left(\frac{\theta_0}{\theta_A}\right)^{\Sigma_x} \left(\frac{1 - \theta_0}{1 - \theta_A}\right)^{n - \Sigma_x} < k$.
- $\Sigma_X \ln\left(\frac{\theta_0}{\theta_A}\right) + (n - \Sigma_x) \ln\left(\frac{1 - \theta_0}{1 - \theta_A}\right) < \ln(k)$.
- $\Sigma_X \left(\ln\left(\frac{\theta_0}{\theta_A}\right) - \ln\left(\frac{1 - \theta_0}{1 - \theta_A}\right) \right) < \ln(k) - n \ln\left(\frac{1 - \theta_0}{1 - \theta_A}\right)$.

Thus if $\theta_A < \theta_0$, our form is “Reject H_0 if $\Sigma_X < k_\alpha$ ” as $\ln\left(\frac{\theta_0}{\theta_A}\right) > \ln\left(\frac{1 - \theta_0}{1 - \theta_A}\right)$.

(b) The UMP test at level of significance $\alpha = 0.05$.

Solution:

Our goal is to find k_α .

- As $X_i \sim \text{Binomial}(1, \theta)$, $\Sigma_X \sim \text{Binomial}(n, \theta)$.
- $k_\alpha = \text{qbinom}(\text{size} = n, \text{prob} = \theta, \text{p} = \alpha)$.
- $\text{qbinom}(\text{size} = 10, \text{prob} = 0.6, \text{p} = 0.05) = 3 = k_{0.05}$.

Our test is thus “Reject H_0 if $\Sigma_X < 3$.”

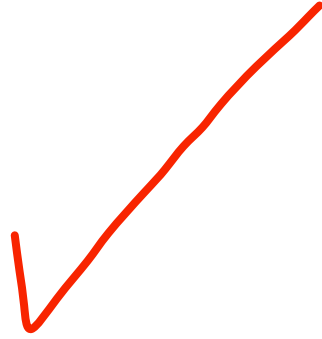
(c) If we reject H_0 given in a run of this experiment, the data came out to be, $\vec{x} = \{0, 0, 1, 0, 1, 0, 1, 0, 1, 0\}$.

Solution:

“Reject H_0 if $\Sigma_X < 3$.”

- $\Sigma_X = \text{sum}(0, 0, 1, 0, 1, 0, 1, 0, 1, 0) = 4$.

As $\Sigma_X = 4 \not< 3$, we fail to Reject H_0 .



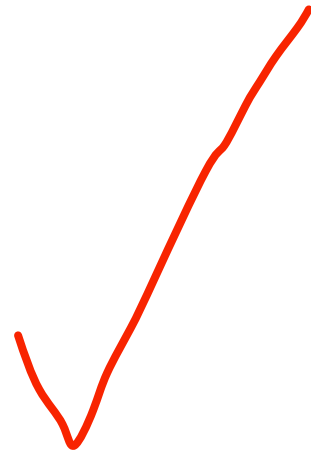
3. Let X_1, \dots, X_{10} be $iid \sim Normal(\mu = 0, \sigma^2 = \theta)$ random variables, Determine ...

(a) The form of the UMP test of the hypotheses $H_0 : \theta = 5$ vs. $H_A : \theta < 5$.

Solution:

I will first state the general form then elaborate on the requested solution.

- $f(x; \theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}.$
- $\frac{L(\theta_0)}{L(\theta_A)} = \left(\frac{\theta_A}{\theta_0}\right)^{n/2} e^{-\frac{\Sigma X^2}{2} \left(\frac{1}{\theta_0} - \frac{1}{\theta_A}\right)} < k.$
- $\frac{n}{2} \ln\left(\frac{\theta_A}{\theta_0}\right) + \frac{\Sigma X^2}{2} \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) < \ln(k).$
- $\Sigma X^2 \left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) < 2 \ln(k) - n \ln\left(\frac{\theta_A}{\theta_0}\right).$



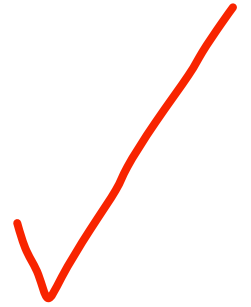
Thus if $\theta_A < \theta_0$, our form is “Reject H_0 if $\Sigma X^2 < k_\alpha$ ” as $\left(\frac{1}{\theta_A} - \frac{1}{\theta_0}\right) > 0$.

(b) The UMP test at level of significance $\alpha = 0.01$.

Solution:

Our goal is to find k_α .

- As $X_i \sim \text{Normal}(\mu = 0, \sigma^2 = \theta)$, $\frac{\Sigma X^2}{\theta} \sim \chi_n^2$.
- $k_\alpha = \theta \cdot \text{qchisq}(\text{df} = n, \text{p} = \alpha)$.
- $5 \cdot \text{qchisq}(\text{df} = 10, \text{p} = 0.01) = 12.79 = k_{0.01}$.



Our test is thus “Reject H_0 if $\Sigma_{X^2} < 12.79$.”

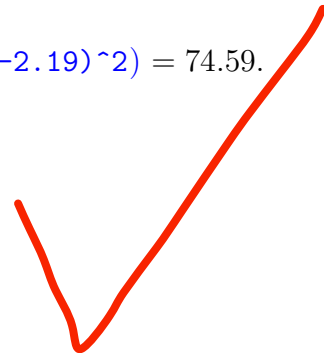
(c) If we reject H_0 given in a run of this experiment, the data came out to be,
 $\vec{x} = \{5.98, 1.94, 1.19, -3.28, -0.28, 3.43, -2.25, 0.39, 1.02, -2.19\}$.

Solution:

“Reject H_0 if $\Sigma_{X^2} < 12.79$.”

- $\Sigma_{X^2} = \text{sum}(c(5.98, 1.94, 1.19, -3.28, -0.28, 3.43, -2.25, 0.39, 1.02, -2.19)^2) = 74.59$.

As $\Sigma_{X^2} = 74.59 \not< 12.79$, we fail to Reject H_0 .



4. Let X_1, \dots, X_{10} be $iid \sim f(x; \theta) = \theta x^{\theta-1}$ random variables, Determine ...

(a) The form of the UMP test of the hypotheses $H_0 : \theta = 2$ vs. $H_A : \theta > 2$.

Solution:

I will first state the general form then elaborate on the requested solution.

- $f(x; \theta) = \theta x^{\theta-1}$.
- $\frac{L(\theta_0)}{L(\theta_A)} = \left(\frac{\theta_0}{\theta_A}\right)^n \Pi_X^{(\theta_0 - \theta_A)} < k$.
- $(\theta_0 - \theta_A) \Sigma_{\ln(X)} < \ln(k) - n \ln\left(\frac{\theta_0}{\theta_A}\right)$.



Thus if $\theta_A > \theta_0$, our form is “Reject H_0 if $-\Sigma_{\ln(X)} < k_\alpha$ ” as $(\theta_0 - \theta_A) < 0$.

(b) The UMP test at level of significance $\alpha = 0.04$; Given the statistic $T = -\sum_{k=1}^{10} \ln(X_i) \sim \text{Gamma}(10, 1/\theta)$.

Solution:

Our goal is to find k_α .

- Given $-\Sigma_{\ln(X)} \sim \text{Gamma}(10, 1/\theta)$.
- $k_\alpha = \text{qgamma}(\text{shape} = 10, \text{scale} = 1/\theta, \text{p} = \alpha)$.
- $\text{qgamma}(\text{shape} = 10, \text{scale} = 0.5, \text{p} = 0.04) = 2.60 = k_{0.04}$.



Our test is thus “Reject H_0 if $-\Sigma_{\ln(X)} < 2.60$.”

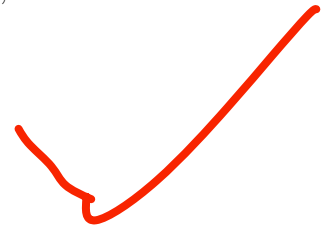
(c) If we reject H_0 given in a run of this experiment, the data came out to be,
 $\vec{x} = \{0.912, 0.839, 0.978, 0.789, 0.690, 0.502, 0.862, 0.691, 0.587, 0.557\}$.

Solution:

“Reject H_0 if $-\Sigma_{\ln(X)} < 2.60$.”

- $-\Sigma_{\ln(X)} =$
 $-\text{sum}(\log(c(0.912, 0.839, 0.978, 0.789, 0.690, 0.502, 0.862, 0.691, 0.587, 0.557)))$
 $= 3.22.$

As $-\Sigma_{\ln(X)} = 3.22 \not< 2.60$, we fail to Reject H_0 .



Extra credit: Show that, in problem 4, $T = -\sum_{k=1}^{10} \ln(X_i) \sim \text{Gamma}(10, 1/\theta)$.

Solution:

- $f(x; \theta) = \theta x^{\theta-1}$.
- Thumped ...
- $Y = -\ln(X_i) = \frac{1}{\theta} e^{-x/\theta} \sim \text{Exponential}(1/\theta)$.
- $T = -\sum_{k=1}^{10} \ln(X_i) = \Sigma_Y \sim \text{Gamma}(10, 1/\theta)$.

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