MTH375: Mathematical Statistics - Homework #6

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Key Concepts: Bayesian statistics, prior and posterior distiributions, Bayesian estimators for squared error loss function and absolute error loss function.

1. Let $X_1, \ldots, X_n$ be a sample of *iid* $Binomial(n = 1, p = \theta)$ random variables with prior distribution $\theta \sim Beta(1, 2)$, determine ...

(a) The posterior distribution $f(\theta|\overrightarrow{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x;\theta) = \binom{1}{x}\theta^x(1-\theta)^{1-x} = \theta^x(1-\theta)^{1-x}$.
- $L(\theta) = \theta^{\Sigma_x}(1-\theta)^{n-\Sigma_x}$.
- $\pi(\theta;\alpha = 1, \beta = 2) = \dfrac{\Gamma(3)}{\Gamma(1)\Gamma(2)}\theta^0(1-\theta)^1 = \dfrac{\Gamma(3)(1-\theta)}{\Gamma(1)\Gamma(2)}$.
- $f(\theta|\overrightarrow{x}) = \dfrac{\dfrac{\Gamma(3)(1-\theta)\theta^{\Sigma_x}(1-\theta)^{n-\Sigma_x}}{\Gamma(1)\Gamma(2)}}{\displaystyle\int_0^1 \dfrac{\Gamma(3)(1-\theta)\theta^x\theta^{\Sigma_x}(1-\theta)^{n-\Sigma_x}}{\Gamma(1)\Gamma(2)}\,d\theta} = \dfrac{\theta^{\Sigma_x}(1-\theta)^{n+1-\Sigma_x}}{\displaystyle\int_0^1 (1-\theta)^{n+1-\Sigma_x}\,d\theta} = k\cdot\theta^{\Sigma_x}(1-\theta)^{n+1-\Sigma_x}$.
- $f(\theta|\overrightarrow{x}) \sim Beta(1+\Sigma_x, n+2-\Sigma_x)$.

(b) The Bayesian estimator of θ for the squared error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $E(Beta(1 + \Sigma_x, n + 2 - \Sigma_x)) = \dfrac{1 + \Sigma_x}{n + 3}$.

(c) Suppose that $n = 10, \sum_{k=1}^{10} x_i = 7$. Compute the Bayesian estimate of θ for square error loss, and the Bayesian estimate for θ for absolute error loss..

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.
The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $1 + \Sigma_x = 1 + 7 = 8 = \alpha$.
- $n + 2 - \Sigma_x = 10 + 2 - 7 = 5 = \beta$.
- $\hat{\theta}_{SE} = \dfrac{8}{8+5} = \dfrac{8}{13} = 0.615$.
- $\hat{\theta}_{AE} =$ `qbeta(p = 0.5, shape1 = 8, shape2 = 5)` ≈ 0.621.

2. Let $X_1, \ldots, X_{50}$ be a sample of *iid Geometric*(θ) random variables with prior distribution $\theta \sim Beta(5, 10)$ and $\sum_{k=1}^{50} x_i = 149$, determine ...

(a) The posterior distribution $f(\theta|\overrightarrow{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x;\theta) = \theta(1-\theta)^{x-1}$.

- $L(\theta) = \theta^{50}(1-\theta)^{\Sigma_x - 50} = \theta^{50}(1-\theta)^{149-50} = \theta^{50}(1-\theta)^{99}$.

- $\pi(\theta; \alpha = 5, \beta = 10) = \dfrac{\Gamma(15)}{\Gamma(5)\Gamma(10)}\theta^4(1-\theta)^9$.

- $f(\theta|\overrightarrow{x}) = \dfrac{\dfrac{\Gamma(15)}{\Gamma(5)\Gamma(10)}\theta^4(1-\theta)^9\theta^{50}(1-\theta)^{99}}{\displaystyle\int_0^1 \dfrac{\Gamma(15)}{\Gamma(5)\Gamma(10)}\theta^4(1-\theta)^9\theta^{50}(1-\theta)^{99}\,d\theta} = \dfrac{\theta^{54}(1-\theta)^{108}}{\displaystyle\int_0^1 \theta^{54}(1-\theta)^{108}\,d\theta} = k \cdot \theta^{54}(1-\theta)^{108}$.

- $f(\theta|\overrightarrow{x}) \sim Beta(55, 109)$.

(b) The value of the Bayesian estimator of θ for the squared error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $\hat{\theta}_{SE} = \dfrac{55}{55+109} = \dfrac{55}{164} = 0.335.$

(c) . The value of the Bayesian estimator of θ for the absolute error loss.

Solution:

The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $\hat{\theta}_{AE} =$ `qbeta(p = 0.5, shape1 = 55, shape2 = 109)` $= 0.335.$

3. Let $X_1, \ldots, X_{60}$ be a sample of *iid* $Exponential(1/\theta)$ random variables with prior distribution $\theta \sim Gamma(\alpha, \beta)$, determine ...

(a) The posterior distribution $f(\theta|\overrightarrow{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x;\theta) = \theta e^{-\theta x}$.
- $L(\theta) = \theta^{60} e^{-\theta \Sigma_x}$.
- $\pi(\theta;\alpha,\beta) = \dfrac{1}{\beta^\alpha \Gamma(\alpha)} \theta^{\alpha-1} e^{-\theta/\beta}$. You are using one form of exp(theta) here
- $f(\theta|\overrightarrow{x}) = \dfrac{\frac{1}{\beta^\alpha\Gamma(\alpha)}\theta^{\alpha-1}e^{-\theta/\beta}\theta^{60}e^{-\theta\Sigma_x}}{\int_0^\infty \frac{1}{\beta^\alpha\Gamma(\alpha)}\theta^{\alpha-1}e^{-\theta/\beta}\theta^{60}e^{-\theta\Sigma_x}\,d\theta} = \dfrac{\theta^{\alpha+59}e^{-\theta/\beta-\theta\Sigma_x}}{\int_0^\infty \theta^{\alpha+59}e^{-\theta/\beta-\theta\Sigma_x}\,d\theta} = k \cdot \theta^{\alpha+59}e^{-\theta(1/\beta+\Sigma_x)}$. ✓
- $f(\theta|\overrightarrow{x}) \sim Gamma(\alpha+60, 1/\beta+\Sigma_x)$ but the other form here.

(so the values below are wrong).

(b) The Bayesian estimator of θ for the squared error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $E(Gamma(\alpha + 60, 1/\beta + \Sigma_x)) = (\alpha + 60) \cdot (1/\beta + \Sigma_x)$.

(c) . The value of the Bayesian estimator of θ for the squared error loss when $\sum_{k=1}^{60} x_i = 143.1, \alpha = 3.5, \beta = 6$.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $\hat{\theta}_{SE} = (3.5 + 60) \cdot (1/6 + 143.1) = (63.5) \cdot (859.6/6) = 9097.43$.

Conclusion.

(d) . The value of the Bayesian estimator of θ for the absolute error loss when $\sum_{k=1}^{60} x_i = 143.1, \alpha = 3.5, \beta = 6$.

Solution:

The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $\hat{\theta}_{AE} =$ `qgamma(p = 0.5, shape = 63.5, scale = (859.6/6))` $= 9049.72$.

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Conclusion.

4. Let $X_1, \ldots, X_n$ be a sample of *iid* $Binomial(n = 2, p = \theta)$ random variables with prior distribution $\theta \sim Uniform[0, 1]$, determine ...

Determine ...

(a) The posterior distribution $f(\theta|\overrightarrow{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x;\theta) = \binom{2}{x}\theta^x(1-\theta)^{2-x} = \dfrac{2}{x(2-x)}\theta^x(1-\theta)^{2-x}$.
- $L(\theta) = \dfrac{2^n}{\pi_{x(2-x)}}\theta^{\Sigma_x}(1-\theta)^{2n-\Sigma_x}$.
- $\pi(\theta) = \dfrac{1}{1-0} = 1$.
- $f(\theta|\overrightarrow{x}) = \dfrac{\dfrac{2^n}{\pi_{x(2-x)}}\theta^{\Sigma_x}(1-\theta)^{2n-\Sigma_x}}{\displaystyle\int_0^\infty \dfrac{2^n}{\pi_{x(2-x)}}\theta^{\Sigma_x}(1-\theta)^{2n-\Sigma_x}\,d\theta} = k \cdot \theta^{\Sigma_x}(1-\theta)^{2n-\Sigma_x}$.
- $f(\theta|\overrightarrow{x}) \sim Beta(\Sigma_x + 1, 2n + 1 - \Sigma_x)$.

(b) The Bayesian estimator of θ for the squared error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $E(Beta(\Sigma_x + 1, 2n + 1 - \Sigma_x)) = \dfrac{\Sigma_x + 1}{2(n+1)}$.

(c) The value of the Bayesian estimator of θ for the squared error loss when $n = 10$ and $\sum_{k=1}^{10} x_i = 17$.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $\hat{\theta}_{SE} = \dfrac{17+1}{2(10+1)} = \dfrac{18}{22} = 0.81\overline{81}$.

Conclusion.

(d) The value of the Bayesian estimator of θ for the absolute error loss when $n = 10$ and $\sum_{k=1}^{10} x_i = 17$.

Solution:

The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $1 + \Sigma_x = 1 + 17 = 18 = \alpha$.
- $2n + 1 - \Sigma_x = 2(10) + 1 - 17 = 4 = \beta$.
- $\hat{\theta}_{AE} =$ `qbeta(p = 0.5, shape1 = 18, shape2 = 4)` ≈ 0.8279.

5. Let $X_1, \ldots, X_{10}$ be a sample of *iid Binomial*$(n = 1, p = \theta)$ random variables with prior pdf

$$\pi(\theta) = \begin{cases} 1/3 & \text{if } \theta = 0.5 \\ 2/3 & \text{if } \theta = 0.8 \end{cases}.$$

(a) The posterior distribution $f(\theta|\overrightarrow{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x;\theta) = \binom{1}{x}\theta^x(1-\theta)^{1-x} = \theta^x(1-\theta)^{1-x}$.

- $L(\theta) = \theta^{\Sigma_x}(1-\theta)^{10-\Sigma_x}$.

- $\pi(\theta) = \begin{cases} 1/3 & \text{if } \theta = 0.5 \\ 2/3 & \text{if } \theta = 0.8 \end{cases}.$

- $f(\theta|\overrightarrow{x}) = \dfrac{\pi(\theta)\theta^{\Sigma_x}(1-\theta)^{10-\Sigma_x}}{\sum\limits_{\Theta}\pi(\theta)\theta^{\Sigma_x}(1-\theta)^{10-\Sigma_x}} = \dfrac{\pi(\theta)\theta^{\Sigma_x}(1-\theta)^{10-\Sigma_x}}{\frac{1}{3}\left(\frac{1}{2}\right)^{\Sigma_x}\left(1-\frac{1}{2}\right)^{10-\Sigma_x} + \frac{2}{3}\left(\frac{4}{5}\right)^{\Sigma_x}\left(1-\frac{4}{5}\right)^{10-\Sigma_x}}.$

- $f(\theta|\overrightarrow{x}) = \dfrac{3\pi(\theta)\theta^{\Sigma_x}(1-\theta)^{10-\Sigma_x}}{\left(\frac{1}{2}\right)^{\Sigma_x}\left(\frac{1}{2}\right)^{10-\Sigma_x} + 2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10-\Sigma_x}} = \dfrac{3\pi(\theta)\theta^{\Sigma_x}(1-\theta)^{10-\Sigma_x}}{\left(\frac{1}{2}\right)^{10} + 2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10-\Sigma_x}}.$

- $f(\theta|\overrightarrow{x}) = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{\Sigma_x}\left(\frac{1}{2}\right)^{10-\Sigma_x}}{\left(\frac{1}{2}\right)^{10} + 2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10-\Sigma_x}} = \dfrac{\left(\frac{1}{2}\right)^{11}}{\left(\frac{1}{2}\right)^{11} + \left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10-\Sigma_x}} & \text{if } \theta = 0.5 \\[2ex] \dfrac{2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10-\Sigma_x}}{\left(\frac{1}{2}\right)^{10} + 2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10-\Sigma_x}} = \dfrac{\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10-\Sigma_x}}{\left(\frac{1}{2}\right)^{11} + \left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10-\Sigma_x}} & \text{if } \theta = 0.8 \end{cases}.$

(b) Suppose that $\sum_{k=1}^{10} x_i = 6$. Compute the Bayesian estimate of θ for square error loss, and the Bayesian estimate for θ for absolute error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.
The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $$f(\theta|\overrightarrow{x}) = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{11}}{\left(\frac{1}{2}\right)^{11} + \left(\frac{4}{5}\right)^{6}\left(\frac{1}{5}\right)^{4}} = 0.53793 & \text{if } \theta = 0.5 \\ \\ \dfrac{\left(\frac{4}{5}\right)^{6}\left(\frac{1}{5}\right)^{4}}{\left(\frac{1}{2}\right)^{11} + \left(\frac{4}{5}\right)^{6}\left(\frac{1}{5}\right)^{4}} = 0.46207 & \text{if } \theta = 0.8 \end{cases}.$$
- As our posterior distribution can only take two values, our mean and median turn out the same. No.
- $\hat{\theta}_{SE} = \hat{\theta}_{AE} = (0.53793 + 0.46207)/2 = 0.5.$

No. This computation just says that (p + (1-p))/ 2 = .5.

E(theta | x1,...,xn) is not .5. Use the definition of E(theta|x) to compute it correctly. The absolute error estimator is in fact .5, but not for this reason.

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