MTH 375
Fall 2022
Hw 4 key

#1. Let $X_1, \ldots, X_n$ be iid Binomial$(1, p)$ random variables.

(a) $L(p) = p^{x_1}(1-p)^{1-x_1} \cdots p^{x_n}(1-p)^{1-x_n} = p^{x_1+\cdots x_n}(1-p)^{n-(x_1+\cdots+x_n)}$ so a sufficient statistic for p is $\Sigma = X_1 + \cdots + X_n$.

(b) $E(\Sigma) = nE(X_k) = np$, so $\mathrm{E}(\Sigma/n) = p$. Therefore the statistic Σ/n, better known as $\overline{X}$, is an unbiased estimator of p which is a function of the complete sufficient statistic Σ, and is therefore an UMVUE.

(c) $\overline{X}$ has mean p and variance $p(1-p)/n$, so $E(\overline{X}^2) = \dfrac{p(1-p)}{n} + p^2 = \dfrac{n-1}{n}p^2 + \dfrac{p}{n}$.

(d) Solving for the equation in (c) for p^2,
$p^2 = \frac{n}{n-1}E(\overline{X}^2) - \frac{1}{n-1}p = \frac{n}{n-1}E(\overline{X}^2) - \frac{1}{n-1}E(\overline{X}) = E(\frac{n}{n-1}\overline{X}^2 - \frac{1}{n-1}\overline{X})$.
The statistic $T = \dfrac{n}{n-1}\overline{X}^2 - \dfrac{1}{n-1}\overline{X}$ is an unbiased estimator of p^2 and a function of the sufficient statistic Σ, T is an UMVUE for p^2.

Another way of writing the same statistic is $T = \dfrac{\Sigma^2 - \Sigma}{n(n-1)}$.

#2. Let $X_1, \ldots, X_n$ be iid Normal$(\mu, 1)$. We know that $\overline{X}$ is an UMVUE for μ.

(a) $E(\overline{X}^2) = V((\overline{X}) + (E(\overline{X}))^2 = \frac{1}{n} + \mu^2$, so the UMVUE for is $T = \overline{X}^2 - \frac{1}{n}$.

(b) The RV $(\overline{X})$ is $\mathrm{N}(\mu, \frac{1}{n})$, so moment-generating function of $\overline{X}$ is $M(t) = e^{\mu t + \frac{1}{2n}t^2}$.

$M'(t) = (\mu + \dfrac{t}{n})e^{\mu t + \frac{1}{2n}t^2}$.

$M''(t) = (\dfrac{1}{n} + (\mu + \dfrac{t}{n})^2)e^{\mu t + \frac{1}{2n}t^2}$.

$M'''(t) = (\dfrac{3}{n}(\mu + \dfrac{t}{n}) + (\mu + \dfrac{t}{n})^3)e^{\mu t + \frac{1}{2n}t^2}$.

Therefore $E(\overline{X}^3) = M'''(0) = \dfrac{3}{n}\mu + \mu^3$

(c) Solve for μ^3: $\mu^3 = E(\overline{X}^3) - \frac{3}{n}\mu = \; E(\overline{X}^3) - \frac{3}{n}E(\overline{X})$.

An UMVUE for μ^3 is $U = \overline{X}^3 - \frac{3}{n}\overline{X}$.

#3. Let $X_1 \ldots, X_n$ be iid with common pdf

$$f(x; \theta) = \frac{3x^2}{\theta^3} \text{ for } 0 \le x \le \theta.$$

(a) $L(\theta) = \left(\; 3^n(x_1 \cdots x_n)^2 \;\right) \cdot \left(\dfrac{1}{\theta^{3n}} \text{ for } 0 \le \max\{x_1, \ldots, x_n\} \le \theta \right)$.
The first factor is $h(x_1, \ldots, x_n)$. The second factor is $f(\max\{x_1, \ldots, x_n\}, \theta)$. The required UMVUE is $T = \max\{X_1, \ldots, X_n\} = X_{(n)}$.

(b) $F_{X_{(n)}}(x) = (P(X_i \le x))^n = (x/\theta)^{3n}$ for $x \in (0, \theta)$.

$f_{X_{(n)}}(x) = F'_{(n)}(x) = 3nx^{3n-1}/\theta^{3n}$ for $x \in (0, \theta)$.

$E[X_{(n)}] = \displaystyle\int_0^\theta x \cdot 3nx^{3n-1}/\theta^{3n}\, dx = 3n\theta/(3n+1)$.

The required UMVUE of θ is $T = (3n+1)X_{(n)}/(3n)$.

#4. Let $X_1 \ldots, X_n$ be iid with common pdf

$$f(x; \theta_1, \theta_2) = \frac{1}{\theta_1} e^{-(x-\theta_2)/\theta_1} \quad \text{for } x > \theta_2.$$

(a) Show that the pair $(\sum_{k=1}^n X_k \ , \ X_{(1)})$ is sufficient for (θ_1, θ_2). We use the notation $X_{(1)} = \min\{X_1, \ldots, X_n\}$.

This follows from the computation $L(\theta_1, \theta_2) = \dfrac{1}{\theta_1^n} e^{-(x_1+\cdots+x_n-n\theta_2)/\theta_1}$ for $x_{(1)} > \theta_2$, which shows that $L(\theta_1, \theta_2)$ is a function of $(\sum_{i=1}^n x_i, x_{(1)}, \theta_1, \theta_2)$.

(b) Find a pair (T_1, T_2) which is an UMVUE for (θ_1, θ_2).

$E(\overline{X}) = E(X_k) = \displaystyle\int_{\theta_2}^{\infty} x \cdot \frac{1}{\theta_1} e^{-(x-\theta_2)/\theta_1} \, dx = \int_0^{\infty} (u + \theta_2) \cdot \frac{1}{\theta_1} e^{-u/\theta_1} \, du = \theta_1 + \theta_2.$ (*)

[Using the substitution $u = x - \theta_2$, $du = dx$.]

To compute $E(X_{(1)})$:

$P(X_k > t) = \displaystyle\int_t^{\infty} \frac{1}{\theta_1} e^{-(x-\theta_2)/\theta_1} \, dx = e^{-(t-\theta_2)/\theta_1},$

so $P(X_{(1)} > t) = P(X_1 > t \ \& \ \cdots \ \& \ X_n > t) = e^{n(t-\theta_2)/\theta_1} = e^{(t-\theta_2)/(\theta_1/n)}$.

In words, $X_{(1)}$ has the pdf $f(x; \theta_1/n, \theta_2)$, where f is the pdf defined at the beginning of this problem. From the computation (*), we conclude that $E(X_{(1)}) = \dfrac{\theta_1}{n} + \theta_2$.

Finally, solve the system of equations

$$E(\overline{X}) = \theta_1 + \theta_2, \qquad E(X_{(1)}) = \frac{\theta_1}{n} + \theta_2$$

for (θ_1, θ_2) to obtain

$$\theta_1 = \frac{n}{n-1}\left(\ E(\overline{X}) - E(X_{(1)}) \ \right), \qquad \theta_2 = -\frac{1}{n-1} E(\overline{X}) + \frac{n}{n-1} E(X_{(1)})$$

An UMVUE for (θ_1, θ_2) is $\left(\ T_1 = \dfrac{n}{n-1}(\overline{X} - X_{(1)}), \ \ T_2 = -\dfrac{1}{n-1}\overline{X} + \dfrac{n}{n-1} X_{(1)} \ \right)$.