MTH 375
Fall 2022
Hw1 – key

1. Suppose that Z_1, Z_2 are independent standard normal random variables. Let $Y_1 = Z_1 - 2Z_2$, $Y_2 = Z_1 - Z_2$.

(a) Find the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$.

$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$, so $\Sigma = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, so $\Sigma^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$. and $\det(\Sigma) = 1$. Therefore $\mathbf{y}^T \Sigma^{-1} \mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2y_1^2 - 6y_1y_2 + 5y_2^2$, and $f_{Y_1,Y_2}(y_1, y_2) = \dfrac{1}{2\pi\sqrt{1}} e^{-\frac{1}{2}(2y_1^2 - 6y_1y_2 + 5y_2^2)} = \dfrac{1}{2\pi} e^{-y_1^2 + 3y_1y_2 - \frac{5}{2}y_2^2}$.

(b) Find the marginal pdf $f_{Y_2}(y_2)$.

Since $Y_2 = Z_1 - Z_2$, Y_2 is normally distributed with $\mu_{Y_2} = E(Z_1) - E(Z_2) = 0$ and $\sigma_{Y_2}^2 = V(Z_1) + V(Z_2) = 2$. $f_{Y_2}(y_2) = \dfrac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}y_2^2}$.

(c) Find the conditional pdf of Y_1 given $Y_2 = 1$, that is, $f_{Y_1|Y_2}(y_1|1)$. Use (a) and (b), and do the necessary division. By completing the square, identify by name the required conditional pdf.

$$f_{Y_1|Y_2}(y_1|1) = \frac{f_{Y_1,Y_2}(y_1, 1)}{f_{Y_2}(1)} = \frac{\frac{1}{2\pi} e^{-y_1^2 + 3y_1 - \frac{5}{2}}}{\frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}}} = \frac{1}{\sqrt{\pi}} e^{-y_1^2 + 3y_1 - \frac{9}{4}}$$

$= \dfrac{1}{\sqrt{\pi}} e^{-(y_1 - \frac{3}{2})^2}$. In words, the conditional distribution of Y_1 given $Y_2 = 1$ is normal with mean 3/2 and variance 1/2.

2. Suppose $Z_1, Z_2, \ldots, Z_6$ are independent standard normal random variables.

(a) $P(-a < 3Z_1 + 2Z_2 - 4Z_3 < a) = .99$.

The random variable $Y = 3Z_1 + 2Z_2 - 4Z_3$ is normal with $\mu = 0$ and $\sigma^2 = 9 + 4 + 16 = 29$. R says

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> qnorm(mean=0,sd=sqrt(29),.995)
[1] 13.87127
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so $a \approx 13.87$.

(b) $P(a < Z_1^2 + Z_2^2 + \cdots + Z_6^2 < b) = .99$.

The random variable $Y = Z_1^2 + Z_2^2 + \cdots + Z_6^2$ is χ^2 with 6 df. R says

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>qchisq(df=6,.995)
[1] 18.54758
> qchisq(df=6,.005)
[1] 0.6757268
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so we may take $a \approx .68$, $b \approx 18.55$.

(c) $P(-a < \dfrac{Z_1}{\sqrt{Z_2^2 + Z_3^2 + \cdots + Z_6^2}} < a) = .99$.

The random variable $T = \dfrac{Z_1}{\sqrt{(Z_2^2 + Z_3^2 + \cdots + Z_6^2)/5}}$ has the t-distribution with 5 df, so $P(-a < \dfrac{Z_1}{\sqrt{Z_2^2 + Z_3^2 + \cdots + Z_6^2}} < a) = P(-\sqrt{5}a < T < \sqrt{5}a)$.

R says

```
> qt(df=5,.995)
[1] 4.032143
```

Therefore $\sqrt{5}a \approx 4.03$, so $a \approx 1.80$.

(d) $P(a < \dfrac{Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2}{Z_5^2 + Z_6^2} < b) = .99$.

The random variable $F = \dfrac{(Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2)/4}{(Z_5^2 + Z_6^2)/2}$ has the F-distribution with 4,2 df, so $P(a < \dfrac{Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2}{Z_5^2 + Z_6^2} < b) = P(a/2 < F < b/2)$. R says

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> qf(df1=4,df2=2,.995)
[1] 199.2497
> qf(df1=4,df2=2,.005)
[1] 0.03804557
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Therefore we may take $a/2 \approx .038$, $b/2 \approx 199.2$, so $a \approx .076$, $b \approx 398$.

3. Let $X_1, X_2, \ldots, X_8$ be independent normal(μ, σ^2) random variables, $\overline{X} = \frac{1}{8}(X_1 + X_2 + \ldots + X_8)$, and $S^2 = \frac{1}{7}\sum_{i=1}^{8}(X_i - \overline{X})^2$.

Find a number a such that $P\left(-a < \dfrac{\overline{X} - \mu}{S} < a\right) = .99$.

The random variable $T = \dfrac{\overline{X} - \mu}{S/\sqrt{8}}$ has the t-distribution with 7 df, so $P\left(-a < \dfrac{\overline{X} - \mu}{S} < a\right) = P(-\sqrt{8}a < T < \sqrt{8}a)$. R says

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> qt(df=7,.995)
[1] 3.499483
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Therefore $\sqrt{8}a \approx 3.499$, so $a \approx 1.24$.