MTH 375
Fall 2020
Hw 5 key

#1. Let $X_1, \ldots, X_n$ be a sample of iid Negative Binomial $(4,\theta)$ random variables with $\theta \in [0,1]$. Determine the MLE and the MOM estimators of θ.

MLE: $f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} \binom{x_i - 1}{3} \theta^{4n}(1-\theta)^{\Sigma - 4n}$ $\quad$ (Σ is short for $\sum_{i=1}^{n} x_i$);

so $\log f = \log\left(\prod_{i=1}^{n} \binom{x_i-1}{3}\right) + 4n\log(\theta) + (\Sigma - 4n)\log(1-\theta)$,

so $\dfrac{d}{d\theta}(\log f) = \dfrac{4n}{\theta} - \dfrac{\Sigma - 4n}{1-\theta} = 0$ for $\hat{\theta} = \dfrac{4n}{\Sigma} = \dfrac{4}{\overline{X}}$. This is the MLE.

MOM: For Negative Binomial $(4, \theta)$ RV's, $E(X) = 4/\theta$. Solving the equation $\overline{X} = 4/\theta$, we have $\hat{\theta} = 4/\overline{X}$. This is the MOM. In this case, MLE and MOM are identical.

#2. Let $X_1, \ldots, X_n$ be a sample of iid N$(0,\theta)$ random variables with $\theta > 0$. (θ should be the variance of X_k.) Determine

a) the MLE $\hat{\theta}$ of θ.

$f(x_1, \ldots, x_n; \theta) = \dfrac{1}{(2\pi\theta)^{n/2}} e^{-\frac{1}{2\theta}(x_1^2 + \cdot + x_n^2)}$, so $\log(f) = -\dfrac{n}{2}\log(2\pi) - \dfrac{n}{2}\log(\theta) - \dfrac{x_1^2 + \cdot + x_n^2}{2\theta}$

so $\dfrac{d}{d\theta}(\log f) = -\dfrac{n}{2\theta} + \dfrac{x_1^2 + \cdot + x_n^2}{2\theta^2} = 0$ for $\hat{\theta} = \dfrac{x_1^2 + \cdot + x_n^2}{n}$. This is the MLE.

b) E$(\hat{\theta})$ and V$(\hat{\theta})$.

Since the random variable $(T = X_1^2 + \cdots + X_n^2)/\theta$ is χ^2 with n df and $\hat{\theta} = \theta T/n$,
E$(\hat{\theta}) = \theta n/n = \theta$ and V$(\hat{\theta}) = \theta^2(2n)/n^2 = 2\theta^2/n$.

c) the MLE of SD$(X_i) = \sqrt{\theta}$. (Hint: There is almost no work to do.)

By the 'Invariance principle,' $\widehat{\sqrt{\theta}} = \sqrt{\dfrac{x_1^2 + \cdot + x_n^2}{n}}$.

#3. Recall the family of distributions with pmf $p_X(x;p) = \begin{cases} p & \text{if } x = -1 \\ 2p & \text{if } x = 0 \\ 1-3p & \text{if } x = 1 \end{cases}$.

Here p is an unknown parameter, and $0 \le p \le 1/3$.

Let $X_1, X_2, \ldots, X_n$ be iid with common pmf a member of this family.

(i) Find the MOM estimator of p.

Solve the equation $E(X) = (-1)p + 0 \cdot 2p + 1 \cdot (1-3p) = 1 - 4p = \overline{X}$ to obtain $\hat{p} = \frac{1-\overline{X}}{4}$. This is the MOM estimator.

(ii) Find the MLE estimator of p. $\quad P = p^A \cdot (2p)^B \cdot (1-3p)^C \rightarrow$
$\ln P = A\ln(p) + B\ln(2 + \ln(p)) + C(\ln(1-3p) \rightarrow \dfrac{d(\ln P)}{dp} = \dfrac{A}{p} + \dfrac{B}{p} - 3\dfrac{C}{1-3p} \rightarrow$
$\dfrac{d(\ln P)}{dp} = 0$ for $\hat{p} = \dfrac{A+B}{3(A+B+C)} = \dfrac{A+B}{3n}$. This is the MLE.

To compare the two estimators, note that $\overline{X} = \dfrac{C-A}{n}$, so the MOM estimator is
$\hat{p} = \dfrac{2A+B}{4n}$.

(iii) A random sample of size 100 from this distribution produced the values { 0, -1, -1, 0, 0, -1, 1, 0, 0, 0, -1, 1, -1, 1, 0, -1, 1, -1, 1, 0, 1, -1, 0, 1, 0, -1, 0, 0, 1, -1, 0, 1, 0, 0, 0, 1, 0, 1, -1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 1, -1, 1, 0, 1, 0, 1, 1, -1, -1, 1, 1, 0, -1, -1, -1, 0, 1, 1, 0, 1, 1, 0, 1, -1, 1, 0, 1, 0, -1, 0, 1, 0, 1, 0, -1, -1, 1, 0, -1, 0, 0, 1, 1, 1, 0, -1, 1, 1, 0, 1 } (There are 23 -1's, 38 0's and 39 1's.)

Evaluate the MOM and MLE estimates of p for this data set.

$A = 23, B = 38, C = 39, n = 100$, so $\hat{p}_{\mathrm{MLE}} = \frac{61}{300} \approx .20$, and $\hat{p}_{\mathrm{MOM}} = \frac{84}{400} = .21$.

#4. (5.1.8 modified) Let $X_1, \ldots, X_n$ be a sample of iid Gamma(α, θ) random variables with α known and $\theta > 0$. Determine

a) the MLE $\hat{\theta}$ of θ.

$f(x_1, \ldots, x_n; \theta) = \dfrac{1}{(\Gamma(\alpha))^n \theta^{n\alpha}} (x_1 \cdots x_n)^{\alpha-1} e^{\Sigma/\theta}$,

so $\log f = -n \log(\Gamma(\alpha)) - n\alpha \log(\theta) + (\alpha - 1) \log(x_1 \cdot x_n) - \dfrac{\Sigma}{\theta}$

so $\dfrac{d}{d\theta}(\log f) = -\dfrac{n\alpha}{\theta} + \dfrac{\Sigma}{\theta^2} = 0$ for $\hat{\theta} = \dfrac{\Sigma}{n\alpha} = \dfrac{\overline{X}}{\alpha}$, which is the MLE.

b) $E(\hat{\theta}) = \dfrac{1}{\alpha} E(X_k) = \dfrac{1}{\alpha} \cdot \alpha\theta = \theta$.

$V(\hat{\theta}) = \dfrac{1}{\alpha^2} \cdot \dfrac{1}{n} V(X_k) = \dfrac{1}{\alpha^2 n} \cdot \alpha\theta^2 = \dfrac{\theta^2}{\alpha n}$.

e) $\hat{\theta}$ is an unbiased function of the sufficient statistic Σ, so $\hat{\theta}$ is an UMVUE for θ.

#5. Let $X_1, \ldots, X_n$ be a sample of iid random variables with pdf

$$f(x; \theta_1, \theta_2) = \frac{1}{\theta_1} e^{-(x-\theta_2)/\theta_1}, \text{ for } x > \theta_2.$$

Here $\theta_1 > 0$, and θ_2 can be any real number.

Find the MOM and MLE estimators of (θ_1, θ_2).

$L(\theta_1, \theta_2) = \dfrac{1}{\theta_1^n} e^{-(\Sigma - n\theta_2)/\theta_1}$ for $x_1, \ldots, x_n \geq \theta_2$.

To find the MLE of (θ_1, θ_2):

Since $\theta_1 e^{-(x-\theta_2)/\theta_1}$ is an increasing function of θ_2, so it maximizes when θ_2 is as large as possible, so the MLE of θ_2 is $\hat{\theta}_2 = X_{(1)}$. Now to find the MLE for θ_1:

$\log L(\theta_1, \theta_2) = -n \log(\theta_1) - \dfrac{(\Sigma - n\theta_2)}{\theta_1}$,

so $\dfrac{d}{d\theta_1}(\log L) = -\dfrac{n}{\theta_1} + \dfrac{(\Sigma - n\theta_2)}{\theta_1^2} = 0$ for $\theta_1 = \dfrac{\Sigma - n\theta_2}{n} = \overline{X} - \theta_2$.

So: The MLE of (θ_1, θ_2) is the pair $(\overline{X} - X_{(1)}, X_{(1)})$.

To find the MOM estimator of (θ_1, θ_2):

We saw in Hw4 #4 that $E(X) = \theta_1 + \theta_2$.

By a similar computation, $V(X) = \theta_1^2$, so $E(X^2) = (\theta_1 + \theta_2)^2 + \theta_1^2$.

We solve the equations (1) $\theta_1 + \theta_2 = \overline{X}$, (2) $(\theta_1 + \theta_2)^2 + \theta_1^2 = \frac{1}{n} \sum_{k=1}^n X_k^2$ for (θ_1, θ_2).

Substituting the (1) into (2), we have $\overline{X}^2 + \theta_1^2 = \frac{1}{n} \sum_{k=1}^n x_k^2$, so $\hat{\theta}_1 = \sqrt{\frac{1}{n} \sum_{k=1}^n X_k^2 - \overline{X}^2}$.

Now (1) gives us $\hat{\theta}_2 = \overline{X} - \sqrt{\frac{1}{n} \sum_{k=1}^n X_k^2 - \overline{X}^2}$

The MOM estimator for (θ_1, θ_2) is the pair $\left(\sqrt{\frac{1}{n} \sum_{k=1}^n X_k^2 - \overline{X}^2}\,,\ \overline{X} - \sqrt{\frac{1}{n} \sum_{k=1}^n X_k^2 - \overline{X}^2}\right)$.