1/11/22

Tuesday, January 11, 2022 4:04 PM

Lets Remember the Normal Distribution : $N(\mu, \sigma^2)$

μ : mean

σ^2 : variance

σ : standard deviation

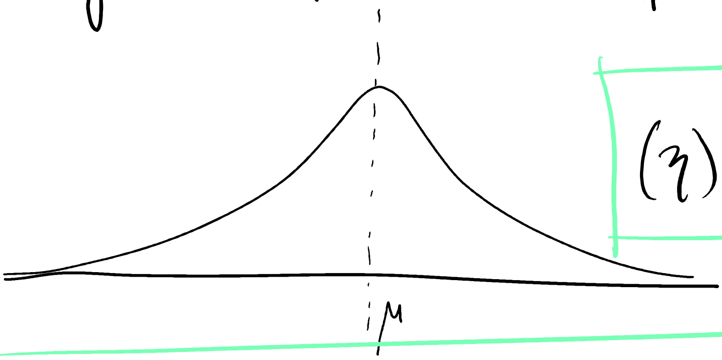
& In 'R' use <>norm(μ,σ)

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2} = \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

For some $X_1, X_2, \dots X_n$, with $X_i \sim N(\mu_i, \sigma_i^2)$

Then $(c_1X_1 + c_2X_2 + \dots + c_nX_n) \sim N(c_1\mu_1 + \dots + c_n\mu_n, c_1^2\sigma_1^2 + \dots c_n^2\sigma_n^2)$

given that these X_i are all independant



(z) Standard Normal : $N(\mu=0, \sigma^2=1)$

If $z \sim N(0,1)$, then $az + b \sim (b, a^2)$

If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$

Multivariate Normal : Say $z_1, z_2, \dots, z_n$ are iid and $\sim N(0,1)$

then $y_1 = a_{11}z_1 + a_{12}z_2 + \dots + a_{1n}z_n + \mu_1$

$$y_2 = a_{21}\eta_1 + a_{22}\eta_2 + \cdots + a_{2n}\eta_n + \mu_2$$

$$\vdots$$

$$y_n = a_{n1}\eta_1 + a_{n2}\eta_2 + \cdots + a_{nn}\eta_n + \mu_n$$

$\langle y_1, \cdots, y_n \rangle$ is multivariate normal

$$\boxed{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}}$$

we require A to be invertible

$$\therefore \boxed{\vec{y} = A\vec{\eta} + \vec{\mu}}$$

$$\therefore \boxed{\vec{\eta} = A^{-1}(\vec{y} - \vec{\mu})}$$

(cov)

$$\text{Covariance}(y_i, y_j) = \text{Cov}(a_{i1}\eta_1 + \cdots + a_{in}\eta_n, a_{j1}\eta_1 + \cdots + a_{jn}\eta_n)$$

We end up with ; As all other terms equal 0.

$$\boxed{\text{Cov}(y_i, y_j) = a_{i1}a_{j1} + a_{i2}a_{j2} + \cdots + a_{in}a_{jn}}$$

The covariance matrix of $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \Big[\text{Cov}(y_i, y_j)\Big]_{i,j}$

This is really $= \boxed{\boxed{AA^T} = \text{cov}(\vec{y})}$ denoted as $\boxed{\Sigma}$

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Thus $\boxed{\vec{y} \sim N(\vec{\mu}, \Sigma)}$

$$\Sigma = \begin{bmatrix} a_{11}a_{21} & a_{12}a_{22} & \cdots & a_{1n-1}a_{2n-1} & a_{1n}a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1\,1}a_{n1} & a_{n-1\,2}a_{n2} & \cdots & a_{n-1\,n-1}a_{nn-1} & a_{n-1\,n}a_{nn} \end{bmatrix}$$

We want to <u>find the joint pdf</u> of $N(\vec{\mu}, \Sigma)$

Recall; $\vec{y} = A\vec{z} + \vec{\mu}$ & $\Sigma = A^TA$

$\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ where z_i are iid. on $N(0,1)$

$$f_{\vec{z}}(\vec{z}) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(z_1^2)}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-(z_2^2)}{2}} \cdot \cdots \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-(z_n^2)}{2}}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\frac{-(z_1^2 + \cdots z_n^2)}{2}}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\left[[z_1 \cdots z_n]\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}\right]}$$

$$\boxed{f_{\vec{z}}(\vec{z}) = \frac{1}{\ldots} e^{-\frac{1}{2} \cdot \vec{z}^T \cdot \vec{z}}}$$

$\vec{z} = A^{-1}(\vec{y} - \vec{\mu})$

$$\boxed{f_{\vec{z}}(\vec{z}) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\cdot \vec{z}^T \cdot \vec{z}}}$$

$$\vec{z} = A^{-1}(\vec{y} - \vec{\mu})$$

Now we will use the change of variables theorem

$$f_{\vec{y}}(\vec{y}) = f_{\vec{z}}(\vec{z}(\vec{y})) \cdot \left| \text{determinant}\left(\frac{\partial \vec{z}}{\partial \vec{y}}\right) \right|$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2}(A^{-1}(\vec{y}-\vec{\mu}))^T \cdot (A^{-1}(\vec{y}-\vec{\mu}))} \cdot \left|\det(A^{-1})\right|$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2}(\vec{y}-\vec{\mu})^T (A^{-1})^T A^{-1} (\vec{y}-\vec{\mu})} \cdot \frac{1}{|\det(A)|}$$

$$\boxed{f_{\vec{y}}(\vec{y}) = \frac{1}{\sqrt{|\det(\Sigma)|}\,(2\pi)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2}(\vec{y}-\vec{\mu})^T \cdot \Sigma^{-1} \cdot (\vec{y}-\vec{\mu})}}$$

Consider just the case for $n=2$: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim N(\vec{\mu}, \Sigma)$

$$y_1 = a z_1 + b z_2 + \mu_1 \qquad z_1 = p(y_1 - \mu_1) + q(y_2 - \mu_2)$$

$$y_2 = c z_1 + d z_2 + \mu_2 \qquad z_2 = r(y_1 - \mu_1) + s(y_2 - \mu_2)$$

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi\sqrt{|ps-qr|}} e^{-\frac{1}{2}\left(\left(p(y_1-\mu_1)+q(y_2-\mu_2)\right)^2+\left(r(y_1-\mu_1)+s(y_2-\mu_2)\right)^2\right)}$$

This is on HW #1 P # 1

This is the "joint distribution"

here $\Sigma = AA^T$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2 & ac+bd \\ ca+bd & c^2+d^2 \end{bmatrix}$$

We now will start to think about Sampling Distributions

χ^2 ; T ; F

Recall: $\mathcal{E}xp(\text{mean } \beta)$: $f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$, $x>0$ $\quad \mu=\beta$, $\sigma^2=\beta^2$

$\text{Gamma}(\alpha,\beta)$: $f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot y^{\alpha-1} e^{-\frac{x}{\beta}}$, $y>0$, $\alpha,\beta>0$

when $\alpha = 1,2,3,\ldots$, think $Y = X_1+X_2+\cdots+X_\alpha$; X_i i.i.d. $\sim \mathcal{E}xp(\beta)$

$\therefore$ $\mu=\alpha\beta$ & $\sigma^2=\alpha\beta^2$

Problem: $Y = Z^2 \mid Z \sim N(0,1)$; find $f_Y(y) \mid y > 0$

first find the CDF

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(-\sqrt{y} \le Z \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

By Symmetry

$$= 2 \cdot \int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \qquad \& \qquad f_Y(y) = F_Y'(y)$$

$$f_Y(y) = 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \cdot \frac{1}{2} y^{-\frac{1}{2}} \quad , \quad y > 0$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \cdot y^{-\frac{1}{2}}} : \boxed{Z^2 \sim \text{Gamma}\left(\lambda = \tfrac{1}{2}, \beta = 2\right)}$$

$\boxed{\text{definition}}$ $W \sim \left[\chi^2 \text{ with } n \text{ degrees of freedom (df)}\right]$

if $W = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \mid Z_i \text{ iid } \sim N(0,1)$

if $\quad W = \xi_1 + \xi_2 + \cdots + \xi_n \quad \Big| \; \xi_i \overset{iid}{\sim} N(0,1)$

$\therefore \; \chi^2(1\,df) = \text{Gamma}\left(\alpha = \frac{1}{2}, \beta = 2\right)$

$\therefore \; \chi^2(n\,df) = \text{Gamma}\left(\alpha = \frac{n}{2}, \beta = 2\right)$

Note; we know the pdf of χ^2 as we know the pdf of Gamma

$\mathbb{E}[W] = n \quad \therefore \; \mathbb{V}[W] = 2n = \sigma_w^2$