MTH 375
Fall 2022
Hw 9 key .

#1. Let $X_1, \cdots X_n$ be of iid Exponential(θ) random variables. (The common pdf is $f(x;\theta) = (1/\theta)e^{-x/\theta}, x > 0$.) Determine

a) the form of the UMP test of the hypotheses $H_0 : \theta = \theta_0$ vs. $H_A : \theta > \theta_0$.

$\dfrac{L(\theta_0)}{L(\theta)} = \dfrac{\dfrac{1}{\theta_0^n} e^{-\sum_i x_i/\theta_0}}{\dfrac{1}{\theta^n} e^{-\sum_i x_i/\theta}} = \left(\dfrac{\theta}{\theta_0}\right)^n e^{(\sum_i x_i)(\frac{1}{\theta} - \frac{1}{\theta_0})} < K$. Since $\theta > \theta_0$, $\frac{1}{\theta} - \frac{1}{\theta_0} < 0$, and so the test is "Reject H_0 when $\sum_i x_i > K^*$".

b) the form of the UMP test of the hypotheses $H_0 : \theta = \theta_0$ vs. $H_A : \theta < \theta_0$.

This is the same as (a), except that since $\theta < \theta_0$, $\frac{1}{\theta} - \frac{1}{\theta_0} > 0$, and so the test is "Reject H_0 when $\sum_i x_i < K^*$".

c) the sampling distribution (i.e., the pdf) of the statistic defining the UMP test.

The test statistic is $\sum_i x_i$. Under H_0, it is the sum of n independent exponential(θ_0) random variables, and so its pdf is gamma($\alpha = n, \beta = \theta_0$).

#2. Let $X_1, \cdots, X_{10}$ be iid binomial($1, \theta$) random variables.

a) Determine the form of the UMP test of the hypotheses $H_0 : \theta = .6$ vs $H_A : \theta < .6$.

$\dfrac{L(\theta_0)}{L(\theta)} = \dfrac{(.6)^{\sum_i x_i}(.4)^{10-\sum_i x_i}}{\theta^{\sum_i x_i}(1-\theta)^{10-\sum_i x_i}} = \left(\dfrac{.6}{\theta} \cdot \dfrac{1-\theta}{.4}\right)^{\sum_i x_i} \cdot \left(\dfrac{.4}{1-\theta}\right)^{10} < K.$

Since $\theta < .6$ and $1 - \theta > .4$, we can see that $\frac{.6}{\theta} \cdot \frac{1-\theta}{.4} > 1$, , and so the test is "Reject H_0 when $\sum_i x_i < K^*$".

b) Find the UMP test at level of significance $\alpha = .05$.

Since under H_0 each X_i is binomial(1,.6), $\sum_i X_i$ is binomial(10,.6). R tells us that qbinom(size=10,prob=.6,.05) $\rightarrow$ 3; "Reject H_0 when $\sum_i x_i \leq 3$."

c) In one run of this experiment, the data came out to be $\vec{x} = \{0, 0, 1, 0, 1, 0, 1, 0, 1, 0\}$. $\sum_i x_i = 4$; do not reject H_0.

Do we reject H_0? $\sum_i x_i = 4$; do not reject H_0.

#3. Let $X_1, \cdots, X_{10}$ be iid normal($\mu = 0, \sigma^2 = \theta$) random variables.

a) Determine the form of the UMP test of the hypotheses $H_0 : \theta = 5$ vs $H_A : \theta < 5$.

$\dfrac{L(5)}{L(\theta)} = \dfrac{\dfrac{1}{(\sqrt{10\pi})^{10}} e^{-\frac{1}{10}\sum_i x_i^2}}{\dfrac{1}{(\sqrt{2\theta\pi})^{10}} e^{-\frac{1}{2\theta}\sum_i x_i^2}} = (\theta/5)^5 e^{\sum_i x_i^2(\frac{1}{2\theta} - \frac{1}{10})} < K$. Since $\theta < 5$, $\frac{1}{2\theta} - \frac{1}{10} > 0$, and so the test is "Reject H_0 when $\sum_i x_i^2 < K^*$".

b) Find the UMP test at level of significance $\alpha = .01$.

Since under H_0 each X_i is normal(0,5), $\sum_i X_i^2/5$ is χ^2 with 10 df. R tells us that qchisq(df=10,.01) $\to$ 2.558; "Reject H_0 when $\sum_i x_i^2/5 < 2.558$," or "when $\sum_i x_i^2 < 12.79$."

c) In one run of this experiment, the data came out to be

$\vec{x} = \{5.98, 1.94, 1.19, -3.28, -0.28, 3.43, -2.25, 0.39, 1.02, -2.19\}$.

Do we reject H_0? Since $\sum_i x_i^2 = 74.59$, we do not reject H_0.

#4. Let $X_1, \cdots, X_{10}$ be iid random variables with pdf $f(x;\theta) = \theta x^{\theta-1}$ for $x \in (0,1)$.

a) Determine the form of the UMP test of the hypotheses $H_0 : \theta = 2$ vs $H_A : \theta > 2$.

$\dfrac{L(2)}{L(\theta)} = \dfrac{2^{10}\prod_i x_i}{\theta^{10}\prod_i x_i^{\theta-1}} = (2/\theta)^{10}\left(\prod_i x_i\right)^{2-\theta} < K$. Since $2 - \theta < 0$, the test is "Reject H_0 when $\prod_i x_i > K^*$."

b) Given that the statistic $T = -\sum_{k=1}^{10} \ln(X_i)$ has pdf gamma$(10, 1/\theta)$, determine the UMP test of the hypotheses in part(a) at level of significance $\alpha = 0.04$.

$\prod_i X_i > K^* \iff -\ln(\prod_i X_i) < K^{**} \iff -\sum_i \ln(X_i) < K^{**}$ so we will reject H_0 if $T < K^{**}$. To find K^{**} using R: qgamma(shape=10,scale=1/2,.04)$\to$ 2.603841. The test is: "Reject H_0 if $T = -\sum_i \ln(X_i) < 2.60$."

c) In one run of this experiment, the data came out to be

$\vec{x} = \{0.912, 0.839, 0.978, 0.789, 0.690, 0.502, 0.862, 0.691, 0.587, 0.557\}$.

Do we reject H_0? Since $-\sum_{i=1}^{10} \ln(x_i) = 3.22$, we do not reject H_0.

Extra credit: Show that, in problem 4, the pdf of T really is gamma$(10, 1/\theta)$.

$$P(-\ln(X_i) \le t) = P(\ln(X_i) \ge -t) = P(X_i \ge e^{-t}) = \int_{e^{-t}}^{1} \theta x^{\theta-1}\, dx = x^\theta \Big|_{e^{-t}}^{1} = 1 - e^{-\theta t}.$$

This is the cdf of an exponential$(1/\theta)$ random variable; for $i = 1, 2, \ldots 10$, the random variable $-\ln(X_i)$ is exponential$(1/\theta)$. Therefore $T = \sum_{i=1}^{10} \ln(X_i)$, the independent sum of 10 of them, is gamma(10,$1/\theta$).