MTH375: Mathematical Statistics - Homework #6

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Key Concepts: Bayesian statistics, prior and posterior distiributions, Bayesian estimators for squared error loss function and absolute error loss function.

1. Let $X_1, \ldots, X_n$ be a sample of *iid Binomial*$(n = 1, p = \theta)$ random variables with prior distribution $\theta \sim Beta(1,2)$, determine ...

(a) The posterior distribution $f(\theta|\overrightarrow{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x;\theta) = \binom{1}{x}\theta^x(1-\theta)^{1-x} = \theta^x(1-\theta)^{1-x}$.

- $L(\theta) = \theta^{\Sigma_x}(1-\theta)^{n-\Sigma_x}$.

- $\pi(\theta;\alpha = 1, \beta = 2) = \dfrac{\Gamma(3)}{\Gamma(1)\Gamma(2)}\theta^0(1-\theta)^1 = \dfrac{\Gamma(3)(1-\theta)}{\Gamma(1)\Gamma(2)}$.

- $f(\theta|\overrightarrow{x}) = \dfrac{\dfrac{\Gamma(3)(1-\theta)\theta^{\Sigma_x}(1-\theta)^{n-\Sigma_x}}{\Gamma(1)\Gamma(2)}}{\displaystyle\int_0^1 \dfrac{\Gamma(3)(1-\theta)\theta^x\theta^{\Sigma_x}(1-\theta)^{n-\Sigma_x}}{\Gamma(1)\Gamma(2)}\,d\theta} = \dfrac{\theta^{\Sigma_x}(1-\theta)^{n+1-\Sigma_x}}{\displaystyle\int_0^1 (1-\theta)^{n+1-\Sigma_x}\,d\theta} = k \cdot \theta^{\Sigma_x}(1-\theta)^{n+1-\Sigma_x}$.

- $f(\theta|\overrightarrow{x}) \sim Beta(1 + \Sigma_x, n + 2 - \Sigma_x)$.

(b) The Bayesian estimator of θ for the squared error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $E(Beta(1 + \Sigma_x, n + 2 - \Sigma_x)) = \dfrac{1 + \Sigma_x}{n + 3}.$

(c) Suppose that $n = 10, \sum_{k=1}^{10} x_i = 7$. Compute the Bayesian estimate of θ for square error loss, and the Bayesian estimate for θ for absolute error loss..

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.
The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $1 + \Sigma_x = 1 + 7 = 8 = \alpha.$

- $n + 2 - \Sigma_x = 10 + 2 - 7 = 5 = \beta.$

- $\hat{\theta}_{SE} = \dfrac{8}{8 + 5} = \dfrac{8}{13} = 0.615.$

- $\hat{\theta}_{AE} =$ qbeta(p = 0.5, shape1 = 8, shape2 = 5) $\approx 0.621.$

2. Let $X_1, \ldots, X_{50}$ be a sample of *iid Geometric*(θ) random variables with prior distribution $\theta \sim Beta(5, 10)$ and $\sum_{k=1}^{50} x_i = 149$, determine ...

(a) The posterior distribution $f(\theta | \vec{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x; \theta) = \theta(1 - \theta)^{x-1}$.

- $L(\theta) = \theta^{50}(1 - \theta)^{\Sigma_x - 50} = \theta^{50}(1 - \theta)^{149-50} = \theta^{50}(1 - \theta)^{99}$.

- $\pi(\theta; \alpha = 5, \beta = 10) = \dfrac{\Gamma(15)}{\Gamma(5)\Gamma(10)}\theta^4(1 - \theta)^9$.

- $f(\theta | \vec{x}) = \dfrac{\dfrac{\Gamma(15)}{\Gamma(5)\Gamma(10)}\theta^4(1 - \theta)^9\theta^{50}(1 - \theta)^{99}}{\displaystyle\int_0^1 \dfrac{\Gamma(15)}{\Gamma(5)\Gamma(10)}\theta^4(1 - \theta)^9\theta^{50}(1 - \theta)^{99}\, d\theta} = \dfrac{\theta^{54}(1 - \theta)^{108}}{\displaystyle\int_0^1 \theta^{54}(1 - \theta)^{108}\, d\theta} = k \cdot \theta^{54}(1 - \theta)^{108}$.

- $f(\theta | \vec{x}) \sim Beta(55, 109)$.

(b) The value of the Bayesian estimator of θ for the squared error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $\hat{\theta}_{SE} = \dfrac{55}{55 + 109} = \dfrac{55}{164} = 0.335.$

(c) . The value of the Bayesian estimator of θ for the absolute error loss.

Solution:

The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $\hat{\theta}_{AE} = \texttt{qbeta(p = 0.5, shape1 = 55, shape2 = 109)} = 0.335.$

3. Let $X_1, \ldots, X_{60}$ be a sample of *iid Exponential*$(1/\theta)$ random variables with prior distribution $\theta \sim Gamma(\alpha, \beta)$, determine ...

(a) The posterior distribution $f(\theta | \vec{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x; \theta) = \theta e^{-\theta x}$.

- $L(\theta) = \theta^{60} e^{-\theta \Sigma_x}$.

- $\pi(\theta; \alpha, \beta) = \dfrac{1}{\beta^\alpha \Gamma(\alpha)} \theta^{\alpha-1} e^{-\theta/\beta}$.

- $f(\theta | \vec{x}) = \dfrac{\dfrac{1}{\beta^\alpha \Gamma(\alpha)} \theta^{\alpha-1} e^{-\theta/\beta} \theta^{60} e^{-\theta \Sigma_x}}{\displaystyle\int_0^\infty \dfrac{1}{\beta^\alpha \Gamma(\alpha)} \theta^{\alpha-1} e^{-\theta/\beta} \theta^{60} e^{-\theta \Sigma_x} \, d\theta} = \dfrac{\theta^{\alpha+59} e^{-\theta/\beta - \theta \Sigma_x}}{\displaystyle\int_0^\infty \theta^{\alpha+59} e^{-\theta/\beta - \theta \Sigma_x} \, d\theta} = k \cdot \theta^{\alpha+59} e^{-\theta(1/\beta + \Sigma_x)}$.

- $f(\theta | \vec{x}) \sim Gamma(\alpha + 60, 1/\beta + \Sigma_x)$.

(b) The Bayesian estimator of θ for the squared error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $E(Gamma(\alpha + 60, 1/\beta + \Sigma_x)) = (\alpha + 60) \cdot (1/\beta + \Sigma_x)$.

(c) . The value of the Bayesian estimator of θ for the squared error loss when $\sum_{k=1}^{60} x_i = 143.1, \alpha = 3.5, \beta = 6$.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $\hat{\theta}_{SE} = (3.5 + 60) \cdot (1/6 + 143.1) = (63.5) \cdot (859.6/6) = 9097.43$.

(d) . The value of the Bayesian estimator of θ for the absolute error loss when $\sum_{k=1}^{60} x_i = 143.1, \alpha = 3.5, \beta = 6$.

Solution:

The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $\hat{\theta}_{AE} = \texttt{qgamma(p = 0.5, shape = 63.5, scale = (859.6/6))} = 9049.72$.

4. Let $X_1, \ldots, X_n$ be a sample of *iid Binomial*$(n = 2, p = \theta)$ random variables with prior distribution $\theta \sim Uniform[0, 1]$, determine ...

Determine ...

(a) The posterior distribution $f(\theta | \vec{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x; \theta) = \binom{2}{x} \theta^x (1 - \theta)^{2-x} = \dfrac{2}{x(2 - x)} \theta^x (1 - \theta)^{2-x}$.

- $L(\theta) = \dfrac{2^n}{\pi_{x(2-x)}} \theta^{\Sigma_x} (1 - \theta)^{2n - \Sigma_x}$.

- $\pi(\theta) = \dfrac{1}{1 - 0} = 1$.

- $f(\theta | \vec{x}) = \dfrac{\dfrac{2^n}{\pi_{x(2-x)}} \theta^{\Sigma_x} (1 - \theta)^{2n - \Sigma_x}}{\displaystyle\int_0^\infty \dfrac{2^n}{\pi_{x(2-x)}} \theta^{\Sigma_x} (1 - \theta)^{2n - \Sigma_x} \, d\theta} = k \cdot \theta^{\Sigma_x} (1 - \theta)^{2n - \Sigma_x}$.

- $f(\theta | \vec{x}) \sim Beta(\Sigma_x + 1, 2n + 1 - \Sigma_x)$.

(b) The Bayesian estimator of θ for the squared error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $E(Beta(\Sigma_x + 1, 2n + 1 - \Sigma_x)) = \dfrac{\Sigma_x + 1}{2(n + 1)}$.

(c) The value of the Bayesian estimator of θ for the squared error loss when $n = 10$ and $\sum_{k=1}^{10} x_i = 17$.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.

- $\hat{\theta}_{SE} = \dfrac{17 + 1}{2(10 + 1)} = \dfrac{18}{22} = 0.81\overline{81}.$

(d) The value of the Bayesian estimator of θ for the absolute error loss when $n = 10$ and $\sum_{k=1}^{10} x_i = 17$.

Solution:

The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $1 + \Sigma_x = 1 + 17 = 18 = \alpha.$

- $2n + 1 - \Sigma_x = 2(10) + 1 - 17 = 4 = \beta.$

- $\hat{\theta}_{AE} = \texttt{qbeta(p = 0.5, shape1 = 18, shape2 = 4)} \approx 0.8279.$

5. Let $X_1, \ldots, X_{10}$ be a sample of *iid Binomial*$(n = 1, p = \theta)$ random variables with prior pdf

$$\pi(\theta) = \begin{cases} 1/3 & \text{if } \theta = 0.5 \\ 2/3 & \text{if } \theta = 0.8 \end{cases}.$$

(a) The posterior distribution $f(\theta | \overrightarrow{x})$.

Solution:

Find the likelihood function, then use in conjunction with the prior to find the posterior.

- $f_X(x; \theta) = \binom{1}{x} \theta^x (1 - \theta)^{1-x} = \theta^x (1 - \theta)^{1-x}$.

- $L(\theta) = \theta^{\Sigma_x} (1 - \theta)^{10 - \Sigma_x}$.

- $\pi(\theta) = \begin{cases} 1/3 & \text{if } \theta = 0.5 \\ 2/3 & \text{if } \theta = 0.8 \end{cases}.$

- $f(\theta | \overrightarrow{x}) = \dfrac{\pi(\theta)\theta^{\Sigma_x}(1 - \theta)^{10 - \Sigma_x}}{\sum_{\Theta} \pi(\theta)\theta^{\Sigma_x}(1 - \theta)^{10 - \Sigma_x}} = \dfrac{\pi(\theta)\theta^{\Sigma_x}(1 - \theta)^{10 - \Sigma_x}}{\frac{1}{3}\left(\frac{1}{2}\right)^{\Sigma_x}\left(1 - \frac{1}{2}\right)^{10 - \Sigma_x} + \frac{2}{3}\left(\frac{4}{5}\right)^{\Sigma_x}\left(1 - \frac{4}{5}\right)^{10 - \Sigma_x}}.$

- $f(\theta | \overrightarrow{x}) = \dfrac{3\pi(\theta)\theta^{\Sigma_x}(1 - \theta)^{10 - \Sigma_x}}{\left(\frac{1}{2}\right)^{\Sigma_x}\left(\frac{1}{2}\right)^{10 - \Sigma_x} + 2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10 - \Sigma_x}} = \dfrac{3\pi(\theta)\theta^{\Sigma_x}(1 - \theta)^{10 - \Sigma_x}}{\left(\frac{1}{2}\right)^{10} + 2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10 - \Sigma_x}}.$

- $f(\theta | \overrightarrow{x}) = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{\Sigma_x}\left(\frac{1}{2}\right)^{10 - \Sigma_x}}{\left(\frac{1}{2}\right)^{10} + 2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10 - \Sigma_x}} = \dfrac{\left(\frac{1}{2}\right)^{11}}{\left(\frac{1}{2}\right)^{11} + \left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10 - \Sigma_x}} & \text{if } \theta = 0.5 \\[3em] \dfrac{2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10 - \Sigma_x}}{\left(\frac{1}{2}\right)^{10} + 2\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10 - \Sigma_x}} = \dfrac{\left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10 - \Sigma_x}}{\left(\frac{1}{2}\right)^{11} + \left(\frac{4}{5}\right)^{\Sigma_x}\left(\frac{1}{5}\right)^{10 - \Sigma_x}} & \text{if } \theta = 0.8 \end{cases}.$

(b) Suppose that $\sum_{k=1}^{10} x_i = 6$. Compute the Bayesian estimate of θ for square error loss, and the Bayesian estimate for θ for absolute error loss.

Solution:

The Bayesian estimator of θ for the squared error loss is the mean of the posterior distribution.
The Bayesian estimator of θ for the absolute error loss is the median of the posterior distribution.

- $f(\theta|\vec{x}) = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{11}}{\left(\frac{1}{2}\right)^{11} + \left(\frac{4}{5}\right)^{6}\left(\frac{1}{5}\right)^{4}} = 0.53793 & \text{if } \theta = 0.5 \\[4ex] \dfrac{\left(\frac{4}{5}\right)^{6}\left(\frac{1}{5}\right)^{4}}{\left(\frac{1}{2}\right)^{11} + \left(\frac{4}{5}\right)^{6}\left(\frac{1}{5}\right)^{4}} = 0.46207 & \text{if } \theta = 0.8 \end{cases}$.

- As our posterior distribution can only take two values, our mean and median turn out the same.

- $\hat{\theta}_{SE} = \hat{\theta}_{AE} = (0.53793 + 0.46207)/2 = 0.5$.