# **MTH375**: Mathematical Statistics - Homework #8

*Cason Konzer*

March 30, 2022

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Key Concepts: Bayesian credible intervals, hypothesis testing, size = level of significance, power curve, likelihood ratio, Neyman-Pearson most powerful test.

1. Let $X_1, \dots X_{50}$ be a sample of $iid \sim Poisson(\theta)$ random variables, where $\theta$ has the prior distribution $\pi(\theta) \sim Exponential(5)$. Determine a 98% Bayesian credible interval for $\theta$ when $\sum_{i=1}^{n=50} X_i = 170$.

*Solution*:

Recall the posterior for $n$ $iid$ samples from $f(x|\theta) \sim Poisson(\theta)$, $\pi(\theta) \sim Exponential(\lambda)$ is ...

$$f(\theta|\overrightarrow{X}) \sim Gamma(\Sigma_X + 1, \frac{\lambda}{n\lambda + 1}).$$

For our question ...

- $f(\theta|\overrightarrow{X}) \sim Gamma(170 + 1, \frac{5}{50 \cdot 5 + 1}) \sim Gamma(171, \frac{5}{251})$.
- `qgamma(p = c(0.01, 0.99), shape = 171, scale = (5/251))` $= (2.830, 4.041) = (L(\overrightarrow{X}), U(\overrightarrow{X}))$.

Our 98% Bayesian credible interval for $\theta$ when $\sum_{i=1}^{n=50} X_i = 170$ is $(2.830, 4.041)$.

2. Let $X_1, \ldots X_{25}$ be a sample of $iid \sim Bernoulli(p)$ random variables, where $p$ has the prior distribution $\pi(p) \sim Uniform[0,1]$. Determine a 96% Bayesian credible interval for $p$ when $\sum_{i=1}^{n=25} X_i = 10$.

*Solution*:

Recall the posterior for $n$ $iid$ samples from $f(x|\theta) \sim Bernoulli(p)$, $\pi(\theta) \sim Uniform[0,1]$ is ...

$f(\theta|\overrightarrow{X}) \sim Beta(\Sigma_X + 1, n - \Sigma_X + 1)$.

For our question ...

- $f(\theta|\overrightarrow{X}) \sim Beta(10 + 1, 25 - 10 + 1) \sim Beta(11, 16)$.
- `qbeta(p = c(0.02, 0.98), shape1 = 11, shape2 = 16)` $= (0.226, 0.603) = (L(\overrightarrow{X}), U(\overrightarrow{X}))$.

Our 96% Bayesian credible interval for $p$ when $\sum_{i=1}^{n=25} X_i = 10$ is $(0.226, 0.603)$.

3. Let $X_1, \ldots X_{25}$ be a sample of $iid \sim Bernoulli(p)$ random variables,

$$\text{Suppose we test the hypotheses:} \begin{cases} H_0 : p = 0.6 \\ H_A : p > 0.6 \end{cases}.$$

Using the test, "Reject $H_0$ if $\displaystyle\sum_{i=1}^{n=25} X_i \geq 20$," Determine ...

(a) The level of significance, $\alpha$.

*Solution*:

$H_0$ is true if $p = 0.6$.

- `1 - pbinom(q = 20, size = 25, prob = 0.6)` $= 0.0095 = \alpha$.

The level of significance of this test is 0.0095 .

(b) The power of this test for $p = 0.7, 0.8, 0.9, 1.0$.

*Solution*:

- `1 - pbinom(q = 20, size = 25, prob = c(0.7,0.8,0.9,1.0))`
  $= (0.090, 0.421, 0.902, 1.000) = Power(p = 0.7, 0.8, 0.9, 1.0)$.

We can see that our power is low when $p$ is close to the null hypothesis.

4. Let $X_1, \ldots X_{10}$ be a sample of $iid \sim Exponential(\theta)$ random variables,

$$\text{Suppose we test the hypotheses: } \begin{cases} H_0 : \theta = 5 \\ H_A : \theta = 8 \end{cases}.$$

Determine ...

(a) The form of the Neyman-Pearson most powerful test.

*Solution*:

- $f(x|\theta) = \dfrac{1}{\theta} e^{-x/\theta}, \ x > 0.$
- $L(\theta) = \dfrac{e^{-\Sigma_X/\theta}}{\theta^n}.$
- $\dfrac{L(5)}{L(8)} = \dfrac{8^n \cdot e^{-\Sigma_X/5}}{5^n \cdot e^{-\Sigma_X/8}} = 1.6^n e^{(\Sigma_X/8)-(\Sigma_X/5)} = 1.6^n e^{-3\Sigma_X/40}.$
- $1.6^n e^{-3\Sigma_X/40} < k \Rightarrow n \cdot \ln(1.6) - 3\Sigma_X/40 < k.$
- $-3\Sigma_X/40 < k - n \cdot \ln(1.6) \Rightarrow \Sigma_X > (-40/3)(k - n \cdot \ln(1.6)) \Rightarrow \Sigma_X > k_\alpha.$

Our form is "Reject $H_0$ if $\Sigma_X > k_\alpha$."

(b) The actual test at level $\alpha = 0.05$ .

*Solution*:

- $\Sigma_X \sim Gamma(10, 5).$
- $Power(H_0 : \theta = 5) = P\Big(\Sigma_X > k_{0.05} \mid \theta = 5\Big) = 0.05 \Rightarrow P\Big(\Sigma_X < k_{0.05} \mid \theta = 5\Big) = 1 - 0.05$
- `qgamma(shape = 10, scale = 5, p = (1 - 0.05))` $= 78.53 = k_{0.05}.$

Our test is thus "Reject $H_0$ if $\Sigma_X > 78.53$."

(c) The power of this test.

*Solution*:

- $Power(H_A : \theta = 8) = P\Big(\Sigma_X > 78.53 \mid \theta = 8\Big) = 1 - P\Big(\Sigma_X < 78.53 \mid \theta = 8\Big) = \beta_{0.05}$
- `1 - pgamma(shape = 10, scale = 8, q = 78.53)` $= 0.481 = \beta_{0.05}$.

Our power is thus $\beta_{0.05} = 0.481$.

5. Let $X_1, \ldots X_{15}$ be a sample of $iid \sim Normal(\mu = 10, \sigma^2 = \theta)$ random variables,

$$\text{Suppose we test the hypotheses: } \begin{cases} H_0 : \theta = 6 \\ H_A : \theta = 2 \end{cases}.$$

Determine ...

(a) The form of the Neyman-Pearson most powerful test at significance level $\alpha$.

*Solution*:

- $f(x|\theta) = \dfrac{1}{\sqrt{2\pi\theta}} e^{-(x-10)^2/2\theta}$.
- $L(\theta) = \dfrac{1}{(2\pi\theta)^{15/2}} e^{-\frac{1}{2\theta}\sum_{i=1}^{15}(x_i-10)^2}$.
- $\dfrac{L(6)}{L(2)} = \dfrac{\dfrac{e^{-\frac{1}{12}\sum_{i=1}^{15}(x_i-10)^2}}{(12\pi)^{15/2}}}{\dfrac{e^{-\frac{1}{4}\sum_{i=1}^{15}(x_i-10)^2}}{(4\pi)^{15/2}}} = \dfrac{e^{\left(\frac{3}{12}-\frac{1}{12}\right)\sum_{i=1}^{15}(x_i-10)^2}}{(3)^{15/2}} = \dfrac{e^{\frac{1}{6}\sum_{i=1}^{15}(x_i-10)^2}}{(3)^{15/2}}$.
- $\dfrac{e^{\frac{1}{6}\sum_{i=1}^{15}(x_i-10)^2}}{(3)^{15/2}} < k \Rightarrow e^{\frac{1}{6}\sum_{i=1}^{15}(x_i-10)^2} < (3)^{15/2} \cdot k$.
- $\dfrac{1}{6}\sum_{i=1}^{15}(x_i-10)^2 < \ln\big((3)^{15/2} \cdot k\big) \Rightarrow \sum_{i=1}^{15}(x_i-10)^2 < 6 \cdot \ln\big((3)^{15/2} \cdot k\big)$.

Our form is "Reject $H_0$ if $\sum_{i=1}^{15}(x_i-10)^2 < k_\alpha$."

(b) The actual test at level $\alpha = 0.05$ . (Your answer will be "Reject $H_0$ if ...")

*Solution*:

- $\dfrac{1}{\sigma^2}\displaystyle\sum_{i=1}^{n}(x_i - 10)^2 \sim \chi^2_n$.
- $\displaystyle\sum_{i=1}^{15}(x_i - 10)^2 \sim \theta \cdot \chi^2_{15}$.
- $Power(H_0 : \theta = 6) = P\Big(\displaystyle\sum_{i=1}^{15}(x_i - 10)^2 < k_{0.05} \mid \theta = 6\Big) = 0.05$
- `6 * qchisq(df = 15, p = 0.05)` $= 43.566 = k_{0.05}$.

Our test is thus "Reject $H_0$ if $\displaystyle\sum_{i=1}^{15}(x_i - 10)^2 < 43.566$."

(c) The power of this test.

*Solution*:

- $Power(H_A : \theta = 2) = P\Big(\displaystyle\sum_{i=1}^{15}(x_i - 10)^2 < 43.566 \mid \theta = 2\Big) = \beta_{0.05}$
- `pchisq(df = 15, q = (43.566 / 2))` $= 0.886 = \beta_{0.05}$.

Our power is thus $\beta_{0.05} = 0.886$.