MTH 375
Fall 2022

Introduction to the multivariate normal distribution

Let $\mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix}$ be iid standard normal random variables, and let $\mathbf{Y} = A\mathbf{Z} + \boldsymbol{\mu}$, where A is an $n \times n$ invertible matrix and $\boldsymbol{\mu} \in \mathbf{R}^n$. Then $\mathbf{Y}$ is a multivariate normal ($\mathbf{R}^n$-valued) random variable with mean $\boldsymbol{\mu}$ and covariance matrix $\Sigma = AA^T$.

We use the Change of Variables Theorem to find the joint pdf of $\mathbf{Y}$. Note that $\mathbf{Z} = A^{-1}(\mathbf{Y} - \boldsymbol{\mu})$, and that the joint pdf of $\mathbf{Z}$ is $f_{\mathbf{Z}}(\mathbf{z}) = \dfrac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(z_1^2 + \cdots + z_n^2)} = \dfrac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\mathbf{z}^T\mathbf{z}}$. Therefore

$$
\begin{aligned}
f_{\mathbf{Y}}(\mathbf{y}) &= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}(\boldsymbol{y}-\mu))^T A^{-1}(\boldsymbol{y}-\mu)} |\det(A^{-1})| = \tfrac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}((\boldsymbol{y}-\mu))^T (A^{-1})^T A^{-1}(\boldsymbol{y}-\mu)} |\det(A^{-1})| \\
&= \frac{1}{(2\pi)^{n/2}\sqrt{|\det(\Sigma)|}} e^{-\frac{1}{2}((\boldsymbol{y}-\mu))^T \Sigma^{-1}(\boldsymbol{y}-\mu)}.
\end{aligned}
$$