

# MTH375: Mathematical Statistics - Homework #8

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Key Concepts: Bayesian credible intervals, hypothesis testing, size = level of significance, power curve, likelihood ratio, Neyman-Pearson most powerful test.

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1. Let  $X_1, \dots, X_{50}$  be a sample of  $iid \sim \text{Poisson}(\theta)$  random variables, where  $\theta$  has the prior distribution  $\pi(\theta) \sim \text{Exponential}(5)$ . Determine a 98% Bayesian credible interval for  $\theta$  when  $\sum_{i=1}^{n=50} X_i = 170$ .

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*Solution:*

Recall the posterior for  $n$  iid samples from  $f(x|\theta) \sim \text{Poisson}(\theta)$ ,  $\pi(\theta) \sim \text{Exponential}(\lambda)$  is ...

$$f(\theta|\vec{X}) \sim \text{Gamma}(\Sigma_X + 1, \frac{\lambda}{n\lambda + 1}).$$

For our question ...

- $f(\theta|\vec{X}) \sim \text{Gamma}(170 + 1, \frac{5}{50 \cdot 5 + 1}) \sim \text{Gamma}(171, \frac{5}{251})$ .
- `qgamma(p = c(0.01, 0.99), shape = 171, scale = (5/251)) = (2.830, 4.041) = (L( $\vec{X}$ ), U( $\vec{X}$ )).`

Our 98% Bayesian credible interval for  $\theta$  when  $\sum_{i=1}^{n=50} X_i = 170$  is (2.830, 4.041).

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2. Let  $X_1, \dots, X_{25}$  be a sample of  $iid \sim \text{Bernoulli}(p)$  random variables, where  $p$  has the prior distribution  $\pi(p) \sim \text{Uniform}[0, 1]$ . Determine a 96% Bayesian credible interval for  $p$  when  $\sum_{i=1}^{n=25} X_i = 10$ .

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*Solution:*

Recall the posterior for  $n$  iid samples from  $f(x|\theta) \sim \text{Bernoulli}(p)$ ,  $\pi(\theta) \sim \text{Uniform}[0, 1]$  is ...

$$f(\theta|\vec{X}) \sim \text{Beta}(\Sigma_X + 1, n - \Sigma_X + 1).$$

For our question ...

- $f(\theta|\vec{X}) \sim \text{Beta}(10 + 1, 25 - 10 + 1) \sim \text{Beta}(11, 16)$ .
- `qbeta(p = c(0.02, 0.98), shape1 = 11, shape2 = 16)` = (0.226, 0.603) =  $(L(\vec{X}), U(\vec{X}))$ .

Our 96% Bayesian credible interval for  $p$  when  $\sum_{i=1}^{n=25} X_i = 10$  is (0.226, 0.603).

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3. Let  $X_1, \dots, X_{25}$  be a sample of  $iid \sim \text{Bernoulli}(p)$  random variables,

Suppose we test the hypotheses:  $\begin{cases} H_0 : p = 0.6 \\ H_A : p > 0.6 \end{cases}$ .

Using the test, "Reject  $H_0$  if  $\sum_{i=1}^{n=25} X_i \geq 20$ ," Determine ...

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(a) The level of significance,  $\alpha$ .

*Solution:*

$H_0$  is true if  $p = 0.6$ .

- **q=19, actually, because sum = 20 is in the rejection region.**  
`1 - pbinom(q = 20, size = 25, prob = 0.6) = 0.0095 =  $\alpha$ .`

The level of significance of this test is 0.0095 .

(b) The power of this test for  $p = 0.7, 0.8, 0.9, 1.0$ .

**1.5/2.0**

*Solution:*

**Likewise here.**

- `1 - pbinom(q = 20, size = 25, prob = c(0.7, 0.8, 0.9, 1.0))`  
`= (0.090, 0.421, 0.902, 1.000) = Power( $p = 0.7, 0.8, 0.9, 1.0$ ).`

We can see that our power is low when  $p$  is close to the null hypothesis.

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4. Let  $X_1, \dots, X_{10}$  be a sample of  $iid \sim Exponential(\theta)$  random variables,

Suppose we test the hypotheses:  $\begin{cases} H_0 : \theta = 5 \\ H_A : \theta = 8 \end{cases}$ .

Determine ...

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(a) The form of the Neyman-Pearson most powerful test.

*Solution:*

- $f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}, x > 0.$
- $L(\theta) = \frac{e^{-\Sigma_X/\theta}}{\theta^n}.$
- $\frac{L(5)}{L(8)} = \frac{8^n \cdot e^{-\Sigma_X/5}}{5^n \cdot e^{-\Sigma_X/8}} = 1.6^n e^{(\Sigma_X/8) - (\Sigma_X/5)} = 1.6^n e^{-3\Sigma_X/40}.$
- $1.6^n e^{-3\Sigma_X/40} < k \Rightarrow n \cdot \ln(1.6) - 3\Sigma_X/40 < k.$
- $-3\Sigma_X/40 < k - n \cdot \ln(1.6) \Rightarrow \Sigma_X > (-40/3)(k - n \cdot \ln(1.6)) \Rightarrow \Sigma_X > k_\alpha.$

Our form is “Reject  $H_0$  if  $\Sigma_X > k_\alpha$ .”

(b) The actual test at level  $\alpha = 0.05$ .

*Solution:*

- $\Sigma_X \sim Gamma(10, 5).$
- $Power(H_0 : \theta = 5) = P(\Sigma_X > k_{0.05} | \theta = 5) = 0.05 \Rightarrow P(\Sigma_X < k_{0.05} | \theta = 5) = 1 - 0.05$
- `qgamma(shape = 10, scale = 5, p = (1 - 0.05)) = 78.53 =  $k_{0.05}$ .`

Our test is thus “Reject  $H_0$  if  $\Sigma_X > 78.53$ .”

(c) The power of this test.

*Solution:*

- $Power(H_A : \theta = 8) = P(\Sigma_X > 78.53 \mid \theta = 8) = 1 - P(\Sigma_X < 78.53 \mid \theta = 8) = \beta_{0.05}$
- `1 - pgamma(shape = 10, scale = 8, q = 78.53)` = 0.481 =  $\beta_{0.05}$ .

Our power is thus  $\beta_{0.05} = 0.481$ .

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5. Let  $X_1, \dots, X_{15}$  be a sample of  $iid \sim Normal(\mu = 10, \sigma^2 = \theta)$  random variables,

Suppose we test the hypotheses:  $\begin{cases} H_0 : \theta = 6 \\ H_A : \theta = 2 \end{cases}$ .

Determine ...

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(a) The form of the Neyman-Pearson most powerful test at significance level  $\alpha$ .

*Solution:*

- $f(x|\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-(x-10)^2/2\theta}$ .

- $L(\theta) = \frac{1}{(2\pi\theta)^{15/2}} e^{-\frac{1}{2\theta} \sum_{i=1}^{15} (x_i - 10)^2}$ .

- $\frac{L(6)}{L(2)} = \frac{\frac{e^{-\frac{1}{12} \sum_{i=1}^{15} (x_i - 10)^2}}{(12\pi)^{15/2}}}{\frac{e^{-\frac{1}{4} \sum_{i=1}^{15} (x_i - 10)^2}}{(4\pi)^{15/2}}} = \frac{e^{\left(\frac{3}{12} - \frac{1}{12}\right) \sum_{i=1}^{15} (x_i - 10)^2}}{(3)^{15/2}} = \frac{e^{\frac{1}{6} \sum_{i=1}^{15} (x_i - 10)^2}}{(3)^{15/2}}.$

- $\frac{e^{\frac{1}{6} \sum_{i=1}^{15} (x_i - 10)^2}}{(3)^{15/2}} < k \Rightarrow e^{\frac{1}{6} \sum_{i=1}^{15} (x_i - 10)^2} < (3)^{15/2} \cdot k.$

- $\frac{1}{6} \sum_{i=1}^{15} (x_i - 10)^2 < \ln((3)^{15/2} \cdot k) \Rightarrow \sum_{i=1}^{15} (x_i - 10)^2 < 6 \cdot \ln((3)^{15/2} \cdot k).$

Our form is “Reject  $H_0$  if  $\sum_{i=1}^{15} (x_i - 10)^2 < k_\alpha$ .”

(b) The actual test at level  $\alpha = 0.05$  . (Your answer will be “Reject  $H_0$  if ...”)

*Solution:*

- $\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - 10)^2 \sim \chi_n^2$ .
- $\sum_{i=1}^{15} (x_i - 10)^2 \sim \theta \cdot \chi_{15}^2$ .
- $Power(H_0 : \theta = 6) = P\left(\sum_{i=1}^{15} (x_i - 10)^2 < k_{0.05} \mid \theta = 6\right) = 0.05$
- `6 * qchisq(df = 15, p = 0.05) = 43.566 = k_{0.05}`.

Our test is thus “Reject  $H_0$  if  $\sum_{i=1}^{15} (x_i - 10)^2 < 43.566$ .”

(c) The power of this test.

*Solution:*

- $Power(H_A : \theta = 2) = P\left(\sum_{i=1}^{15} (x_i - 10)^2 < 43.566 \mid \theta = 2\right) = \beta_{0.05}$
- `pchisq(df = 15, q = (43.566 / 2)) = 0.886 = \beta_{0.05}`.

Our power is thus  $\beta_{0.05} = 0.886$ .