MTH 375
Fall 2022
Exam key

1. Let $X_1, \ldots, X_n$ be beta with α unknown and $\beta = 3$. Find the Method of Moments estimator of α.

Since $E(X) = \dfrac{\alpha}{\alpha+3}$ (from the table of distributions), solve the equation $\overline{X} = \dfrac{\alpha}{\alpha+3}$ for α to obtain the estimator $\hat{\alpha} = \dfrac{3\overline{X}}{1-\overline{X}}$.

2. Let $Y_1, \ldots, Y_n$ have common pdf $f(y) = \begin{cases} \dfrac{2y}{a^2}, & 0 \le y \le a \\ 0 & \text{otherwise} \end{cases}$.

(a) Find a sufficient statistic for a.

$L(a) = \left(\dfrac{1}{a^{2n}} \text{ for } 0 \le y_{(n)} \le a\right) \cdot (2^n y_1 \cdots y_n) = g(y_{(n)}, a) \cdot h(y_1, \ldots, y_n)$,
so $Y_{(n)}$ is sufficient for a.

(b) Find an UMVUE for $\dfrac{1}{a}$.

$P(Y_{(n)} \le t) = (P(Y_i \le t)^n = \dfrac{t^{2n}}{a^{2n}}$, so $f_{Y_{(n)}}(t) = \dfrac{2nt^{2n-1}}{a^{2n}}$,

so $E\left(\dfrac{1}{Y_{(n)}}\right) = \displaystyle\int_0^a \frac{1}{t} \cdot \frac{2nt^{2n-1}}{a^{2n}}\, dt = \frac{2n}{(2n-1)a}$. The UMVUE is $\hat{a} = \dfrac{2n-1}{2nY_{(n)}}$.

3. Let $Y_1, \ldots, Y_n$ have common pmf $p(y) = \begin{cases} 2p & \text{if } y = -1 \\ 1/2 - p & \text{if } y = 0 \\ 1/2 - p & \text{if } y = 1 \end{cases}$, $0 < p < \frac{1}{2}$

(a) Find a sufficient statistic for p.

Let A, B, C be the number of (-1)'s, 0's, and 1's respectively, in our sample. Then $L(p) = (2p)^A(\dfrac{1}{2} - p)^{B+C} = (2p)^A(\dfrac{1}{2} - p)^{n-A}$, so A is a sufficient statistic.

(b) Find the MOM estimator of p. Is it unbiased?

$E(Y_i) = -1 \cdot 2p + 0 \cdot (\frac{1}{2} - p) + 1 \cdot (\frac{1}{2} - p) = \frac{1}{2} - 3p$. Solve the equation $\overline{Y} = \frac{1}{2} - 3p$ for p to obtain the MOM estimator $\hat{p} = \dfrac{1}{6} - \dfrac{1}{3}\overline{Y}$. Since $E(\hat{p}) = \frac{1}{6} - \frac{1}{3}E(\overline{Y}) = \frac{1}{6} - \frac{1}{3}(\frac{1}{2} - 3p) = p$, $\hat{p}$ is an unbiased estimator of p.

(c) Find the MLE estimator of p. Is it unbiased?

$\ln(L(p)) = A\ln(2p) + (n-A)\ln(\frac{1}{2} - p)$, so $\dfrac{d}{dp}(\ln(L(p)) = \dfrac{A}{p} - \dfrac{n-A}{\frac{1}{2} - p} = 0$. Solve this equation for p to obtain $\tilde{p} = \dfrac{A}{2n}$. Since A is a binomial(n,2p) random variable, $E(\tilde{p}) = n \cdot 2p/(2n) = p$; $\tilde{p}$ is an unbiased estimator of p.

(d) Find the MSE of both estimators. For which values of p is the MSE of the MLE the smaller of the two?

Both estimators are unbiased, so in both cases MSE=variance. Now

MOM: $V(\hat{p}) = V\left(\dfrac{1}{6} - \dfrac{1}{3}\overline{Y}\right) = \dfrac{1}{9n}V(Y_i) = \dfrac{\frac{1}{2} + p - (\frac{1}{2} - 3p)^2}{9n}$.

MLE: Since A is binomial$(2p, n)$, $V(\tilde{p}) = \dfrac{1}{4n^2}V(A) = \dfrac{2np(1-2p)}{4n^2} = \dfrac{p(1-2p)}{2n}$.

MLE is has the lower MSE whenever $0 < p < 1/2$. This can be proved by an algebraic computation, as follows:

Factored, MSE(MOM)$= \dfrac{(1-2p)(1+18p)}{36n} = (1-2p)(\dfrac{1}{36n}+\dfrac{p}{2n})$, which is slightly greater than MSE(MLE)$= (1-2p)\dfrac{p}{2n}$.

A quicker (and better) way is by observing (using (a) and (c)) that the MLE is an unbiased function of the sufficient statistic A, so it is an UMVUE.

4. Let $X_1, \ldots, X_n$ be exponential with parameter β. Find UMVUE's for β, β^2 and β^3.

Hint: The exponential distribution has pdf $p(x) = \frac{1}{\beta}e^{-x/\beta}$ $(x > 0)$. The sum of n independent exponential(β) random variables has pdf Gamma($\alpha = n, \beta$).

We know from previous work that $\overline{X}$ is a sufficient statistic for β and that $E(\overline{X}) = \beta$, so an UMVUE for β is $\hat{\beta} = \overline{X}$.

Since $E(\overline{X}^2) = V(\overline{X})+E(\overline{X})^2 = \dfrac{\beta^2}{n}+\beta^2 = \dfrac{n+1}{n}\beta^2$, the UMVUE for β^2 is $\hat{\beta^2} = \dfrac{n}{n+1}\overline{X}^2$.

The sum $\Sigma = X_1+\cdots+X_n$ is Gamma with $\alpha = n$ and the same β, so $M_\Sigma(t) = (1-\beta t)^{-n}$.

Therefore $M'_\Sigma = n\beta\,(1-\beta t)^{-n-1}$, $M''_\Sigma = n(n+1)\beta^2\,(1-\beta t)^{-n-2}$, and $M'''_\Sigma = n(n+1)(n+2)\beta^3\,(1-\beta t)^{-n-3}$, so $E(\Sigma^3) = n(n+1)(n+2)\beta^3$.

Therefore $E(\overline{X}^3) = \dfrac{n(n+1)(n+2)}{n^3}\beta^3$, so an UMVUE for β^3 is $\hat{\beta^3} = \dfrac{n^3}{n(n+1)(n+2)}\overline{X}^3$.

5. Let $X_1, \ldots, X_{10}$ be iid normal with unknown μ, σ^2.

Let $\overline{X} = \dfrac{X_1+\cdots+X_{10}}{10}$ and $S^2 = \dfrac{1}{9}\displaystyle\sum_{i=1}^{10}(X_i-\overline{X})^2$.

(a) Find a number a such that $P\left(-a < \dfrac{\overline{X}-\mu}{S} < a\right) = .95$.

We know that $\dfrac{\sqrt{10}(\overline{X}-\mu)}{S}$ has a t-distribution with 9 df, so, using the R command qt(df=9,.975) we obtain

$P\left(-2.262 < \dfrac{\sqrt{10}(\overline{X}-\mu)}{S} < 2.262\right) = .95$ Therefore $a \approx \dfrac{2.262}{\sqrt{10}} \approx .715$.

(b) Find positive numbers a, b such that $P\left(a < \dfrac{S^2}{\sigma^2} < b\right) = .95$.

We know that $\dfrac{9S^2}{\sigma^2}$ has χ^2 distribution with 9 df, so using the R commands qchisq(df=9,.025) and qchisq(df=9,.975), we obtain $P\left(2.70 < \dfrac{9S^2}{\sigma^2} < 19.02\right) = .95$ Therefore we may take $a \approx \dfrac{2.70}{9} = .30$ and $b \approx \dfrac{19.02}{9} \approx 2.11$.