Hilbert's Axioms for Euclidean Plane Geometry

Undefined Terms

point, line, incidence, betweenness, congruence

Axioms

I. ***Axioms of Incidence***

Postulate I.1.
For every point P and for every point Q not equal to P, there exists a unique line ℓ incident with the points P and Q.

Postulate I.2.
For every line ℓ there exist at least two distinct points P and Q incident with ℓ.

Postulate I.3.
There exist at least three distinct points with the property that no line is incident with all three of them.

II. ***Axioms of Betweenness***

Postulate B.1.
If a point B lies between a point A and a point C then the points A, B, C are three distinct points of a line, and B then also lies between C and A.

Postulate B.2.
For two distinct points B and D, there exist points A, C, and E on the line BD such that B lies between A and D, C lies between B and D, and D lies between B and E.

Postulate B.3.
If A, B and C are three distinct points lying on the same line, then one and only one of the points lies between the other two.

Postulate B.4. (Plane Separation Axiom)
For every line ℓ and for any three points A, B, and C that do not lie on ℓ:

a) If A and B are on the same side of ℓ and B and C are on the same side of ℓ, then A and C are on the same side of ℓ.
b) If A and B are on opposite sides of ℓ and B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ.

III. ***Axioms of Congruence***

Postulate C.1.
If A and B are distinct points and A' is a point, then for each ray r emanating from A' there is a *unique* point B' on r such that $B' \neq A'$ and such that AB and $A'B'$ are congruent.

Postulate C.2.
If a segment $A'B'$ and a segment $A''B''$ are congruent to the same segment AB, then segments $A'B'$ and $A''B''$ are congruent to each other.

Postulate C.3.
On a line m, let AB and BC be two segments which, except for B, have no points in common. Furthermore, on the same or another line m', let $A'B'$ and $B'C'$ be two

segments which, except for B', have no points in common. In that case if $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.

Postulate C.4.

If $\angle BAC$ is an angle and if $\overrightarrow{B'C'}$ is a ray, then there is exactly one ray $\overrightarrow{B'A'}$ on each side of line $\overleftrightarrow{B'C'}$ such that $\angle B'A'C' \cong \angle BAC$.

Postulate C.5.

If $\angle A \cong \angle B$ and $\angle A \cong \angle C$, then $\angle B \cong \angle C$. Moreover, every angle is congruent to itself.

Postulate C.6. (*SAS*)

If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

IV. ***Axioms of Continuity***

Archimedes Axiom

If AB and CD are any segments, then there exists a number n such if segment CD is laid off n times on the ray $\overrightarrow{AB}$ emanating from A, then a point E is reached where $n \cdot CD \cong AE$ and B is between A and E.

Dedekind's Axiom

Suppose that the set of all points on a line ℓ is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and *vice versa*. Then there is a unique point, O, lying on ℓ such that O liest between P_1 and P_2 if and only if $P_1 \in \Sigma_1$ and $P_2 \in \Sigma_2$ and $O \neq P_1,\ P_2$.

The following two Principles follow from Dedekind's Axiom, yet are at times more useful.

> **Elementary Continuity Principle:**
> If one endpoint of a segment is inside a circle and the other is outside, then the segment intersects the circle.
> **Circular Continuity Principle:**
> If one circle has one point inside and one point outside another circle, then the two circles intersect in two points.

V. ***Axiom of Parallels***

Let ℓ be any line and P a point not on it. Then there is at most one line that passes through P and does not intersect ℓ.