MTH 385 Homework due 2022-04-11

Exercise 1 (8.2.3)**.** *Show that the volume of the solid obtained by rotating the portion of $y = 1/x$ from $x = 1$ to ∞ about the x-axis is finite. Show, on the other hand, that its surface area is infinite.*

Solution. Let V be the volume and let A be the surface area.

$$\begin{aligned}
V &= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 \, dx \\
&= \pi \lim_{t\to\infty} \left(\int_1^t \frac{dx}{x^2} \right) \\
&= \pi \lim_{t\to\infty} \left[-\frac{1}{x} \right]_1^t \\
&= \pi \lim_{t\to\infty} \left[1 - \frac{1}{t} \right] \\
&= \pi
\end{aligned}$$

$$\begin{aligned}
A &= \int_1^\infty 2\pi \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \, dx \\
&= 2\pi \lim_{t\to\infty} \left(\int_1^t \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \, dx \right) \\
&\geq 2\pi \lim_{t\to\infty} \left(\int_1^t \frac{dx}{x} \right) \\
&= 2\pi \lim_{t\to\infty} \left[\ln x \right]_1^t \\
&= 2\pi \lim_{t\to\infty} [\ln t - 1] \\
&= \infty
\end{aligned}$$

□

Cavalieri's most elegant application of his method of indivisibles was to prove Archimedes' formula for the volume of a sphere. His argument is simpler than that of Archimedes, and it goes as follows.

Exercise 2 (8.2.4)**.** *Show that the slice $z = c$ of the sphere $x^2 + y^2 + z^2 = 1$ has the same area as the slice $z = c$ of the cylinder $x^2 + y^2 = 1$ outside the cone $x^2 + y^2 = z^2$ (Figure 8.2).*

Solution. Out of laziness I have chosen not to replicate Figure 8.2.

Let's compute the area of the slice $z = c$ of the sphere $x^2+y^2+z^2 = 1$. It is the circle $x^2+y^2 = 1-c^2$ of radius $\sqrt{1-c^2}$ and area $\pi(1-c^2)$. The slice $z = c$ of the cylinder $x^2+y^2 = 1$ is the circle $x^2+y^2 = 1$ of radius 1 and area π. The slice $z = c$ of the cone $x^2+y^2 = 1$ is the circle $x^2+y^2 = c^2$ of radius c and area πc^2. Evidently, $\pi(1-c^2) = \pi - \pi c^2$. □

Exercise 3 (8.2.5)**.** *Deduce from Exercise 8.2.4, and the known volume of the cone, that the volume of the sphere is $2/3$ the volume of the circumscribing cylinder.*

Solution. We know the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$. This is $1/3$ the volume of the enclosing cylinder. Thus, the volume of the sphere is $2/3$ that of the circumscribing cylinder. □

The examples in Exercise 8.3.1 and Exercise 8.3.2 show how tangents can be found by looking for double roots, though it requires some foresight to make the right substitution. With calculus, the process is more mechanical.

Exercise 4 (8.3.3)**.** *Derive the formula of Hudde and Sluse by differentiating $\sum a_{ij}x^i y^j = 0$ with respect to x.*

Solution.

$$\begin{aligned}
\frac{d}{dx}\left(\sum a_{ij}x^i y^j\right) &= 0 \\
\sum i a_{ij}x^{i-1}y^j + \sum j a_{ij}x^i y^{j-1}\frac{dy}{dx} &= 0 \\
\sum i a_{ij}x^{i-1}y^j + \left(\sum j a_{ij}x^i y^{j-1}\right)\frac{dy}{dx} &= 0 \\
\left(\sum j a_{ij}x^i y^{j-1}\right)\frac{dy}{dx} &= -\sum i a_{ij}x^{i-1}y^j \\
\frac{dy}{dx} &= -\frac{\sum i a_{ij}x^{i-1}y^j}{\sum j a_{ij}x^i y^{j-1}}
\end{aligned}$$

□

Exercise 5 (8.3.4)**.** *Use differentiation to find the tangent to the folium $x^3+y^3 = 3axy$ at the point (b, c).*

Solution. First, determine the slope of the tangent line.

$$\begin{aligned}
\frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(3axy) \\
3x^2 + 3y^2\frac{dy}{dx} &= 3ay + 3ax\frac{dy}{dx} \\
x^2 + y^2\frac{dy}{dx} &= ay + ax\frac{dy}{dx} \\
(y^2 - ax)\frac{dy}{dx} &= ay - x^2 \\
\frac{dy}{dx} &= \frac{ay - x^2}{y^2 - ax} \\
\frac{dy}{dx} &= \frac{ac - b^2}{c^2 - ab}
\end{aligned}$$

Then, use point-slope form.

$$y = c + \frac{ac - b^2}{c^2 - ab}(x - b)$$

$\square$