MTH 385 Homework due 2022-03-21 Solutions

Exercise 1 (5.5.2)**.** *Use Cardano's formula to solve* $y^3 = 2$*. Do you get the obvious solution?*

Solution. Evidently, $p = 0$ and $q = 2$, so

$$\begin{aligned} y &= \sqrt[3]{\frac{2}{2} + \sqrt{\left(\frac{2}{2}\right)^2 - \left(\frac{0}{3}\right)^3}} + \sqrt[3]{\frac{2}{2} - \sqrt{\left(\frac{2}{2}\right)^2 - \left(\frac{0}{3}\right)^3}} \\ &= \sqrt[3]{2}. \end{aligned}$$

I suspect this is the so called obvious solution. □

Exercise 2 (5.5.3)**.** *Use Cardano's formula to solve* $y^3 = 6y + 6$*, and check your answer by substitution.*

Solution. Evidently, $p = 0$ and $q = 2$.

$$\begin{aligned} y &= \sqrt[3]{\frac{6}{2} + \sqrt{\left(\frac{6}{2}\right)^2 - \left(\frac{6}{3}\right)^3}} + \sqrt[3]{\frac{6}{2} - \sqrt{\left(\frac{6}{2}\right)^2 - \left(\frac{6}{3}\right)^3}} \\ &= \sqrt[3]{4} + \sqrt[3]{2} \end{aligned}$$

$$\begin{aligned} (\sqrt[3]{4} + \sqrt[3]{2})^3 &= (\sqrt[3]{4})^3 + 3(\sqrt[3]{4})^2(\sqrt[3]{2}) + 3(\sqrt[3]{4})(\sqrt[3]{2})^2 + (\sqrt[3]{2})^3 \\ &= 4 + 3\sqrt[3]{32} + 3\sqrt[3]{16} + 2 \\ &= 3(2\sqrt[3]{4}) + 3(2\sqrt[3]{2}) + 6 \\ &= 6\sqrt[3]{4} + 6\sqrt[3]{2} + 6 \\ &= 6(\sqrt[3]{4} + \sqrt[3]{2}) + 6 \end{aligned}$$

□

Exercise 3 (5.6.2)**.** *Use (3) and* $\sin\alpha = \cos(\pi/2 - \alpha)$, $\cos\alpha = \sin(\pi/2 - \alpha)$ *to show that*

$$(\sin\theta + i\cos\theta)^n = \begin{cases} \sin n\theta + i\cos n\theta & \text{when } n = 4m+1 \\ -\sin n\theta - i\cos n\theta & \text{when } n = 4m+3. \end{cases}$$

Solution.

$$\begin{aligned} (\sin\theta + i\cos\theta)^n &= \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n \\ &= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right) \\ &= \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right) \\ &= \left(\cos\frac{n\pi}{2}\cos n\theta + \sin\frac{n\pi}{2}\sin n\theta\right) + i\left(\sin\frac{n\pi}{2}\cos n\theta - \cos\frac{n\pi}{2}\sin n\theta\right) \end{aligned}$$

When n is odd, $\cos\frac{n\pi}{2} = 0$.

$$= \sin\frac{n\pi}{2}\sin n\theta + i\sin\frac{n\pi}{2}\cos n\theta$$

Finally, when $n = 4m+1$, $\sin\frac{n\pi}{2} = 1$. And, when $n = 4m+3$, $\sin\frac{n\pi}{2} = -1$. □

For the rest of the solution set, we will pretend $\sin\theta + i\cos\theta$ **is the principal** n**th root of** $\sin\theta + i\cos\theta)^n$**. That is, we choose** $\sqrt[n]{(\sin\theta + i\cos\theta)^n} = \sin\theta + i\cos\theta$ **rather than some other** n**th root. We will also follow the textbook's assumption that** $\sqrt{\sin^2\theta - 1} = \cos\theta$ **and** $\sqrt{\cos^2\theta - 1} = \sin\theta$**, freely using the reduction from (1) to (2).**

Exercise 4 (5.6.3)**.** *Deduce from Exercise 5.6.2 that (2) is correct for* $n = 4m+1$ *and false for* $n = 4m+3$*, and hence that (1) is a correct relation between* $y = \sin n\theta$ *and* $x = \sin\theta$ *only when* $n = 4m+1$.

Solution. When $n = 4m+1$,

$$\sqrt[n]{\sin n\theta + i\cos n\theta} = \sqrt[n]{(\sin\theta + i\cos\theta)^n} = \sin\theta + i\cos\theta.$$

Since n is odd, $\sin n\pi = \sin\pi = 0$ and

$$\begin{aligned}\frac{1}{2}\sqrt[n]{\sin n\theta + i\cos n\theta} + \frac{1}{2}\sqrt[n]{\sin n\theta - i\cos n\theta} &= \frac{1}{2}\sqrt[n]{\sin n\theta + i\cos n\theta} + \frac{1}{2}\sqrt[n]{\sin n(\pi-\theta) + i\cos n(\pi-\theta)}\\ &= \frac{1}{2}(\sin\theta + i\cos\theta) + \frac{1}{2}(\sin(\pi-\theta) + i\cos(\pi-\theta))\\ &= \frac{1}{2}(\sin\theta + i\cos\theta) + \frac{1}{2}(\sin\theta - i\cos\theta)\\ &= \sin\theta.\end{aligned}$$

When $n = 4m+3$, (since n is odd, $\sqrt[n]{-1} = -1$)

$$\sqrt[n]{\sin n\theta + i\cos n\theta} = \sqrt[n]{-(\sin\theta + i\cos\theta)^n} = -(\sin\theta + i\cos\theta).$$

Thus,

$$\frac{1}{2}\sqrt[n]{\sin n\theta + i\cos n\theta} + \frac{1}{2}\sqrt[n]{\sin n\theta - i\cos n\theta} = -\sin\theta$$

when $n = 4m+3$. Hence, (1) is off by a factor of -1 when $n = 4m+3$. □

Exercise 5 (5.6.4)**.** *Show that (1) is a correct relation between* $y = \cos n\theta$ *and* $x = \cos\theta$ *for all* n *(de Moivre (1730)).*

Solution. Evidently, this reduces to

$$\cos\theta = \frac{1}{2}\sqrt[n]{\cos n\theta + i\sin n\theta} + \frac{1}{2}\sqrt[n]{\cos n\theta - i\sin n\theta}$$

Apply de Moivre's formula.

$$\sqrt[n]{\cos n\theta + i \sin n\theta} = \sqrt[n]{(\cos\theta + i\sin\theta)^n} = \cos\theta + i\sin\theta$$

And,

$$\begin{aligned}
\frac{1}{2}\sqrt[n]{\cos n\theta + i\sin n\theta} + \frac{1}{2}\sqrt[n]{\cos n\theta - i\sin n\theta} &= \frac{1}{2}\sqrt[n]{\cos n\theta + i\sin n\theta} + \frac{1}{2}\sqrt[n]{\cos n(-\theta) + i\sin n(-\theta)} \\
&= \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{2}(\cos(-\theta) + i\sin(-\theta)) \\
&= \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{2}(\cos\theta - i\sin\theta) \\
&= \cos\theta.
\end{aligned}$$

□