MTH385: History of Mathematics - Homework #10

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April 4, 2022

Like the binomial theorem, the multinomial theorem can be proved combinatorially by considering the number of ways a term $a_1^{q_1} a_2^{q_2} \cdots a_n^{q_n}$ can arise from the factors of $(a_1 + a_2 + \cdots + a_n)^p$.

Exercise 1 (5.9.4 rewritten)**.** *Prove the formula for the multinomial coefficient*

$$\binom{p}{q_1, q_2, \ldots, q_n} = \frac{p!}{q_1! q_2! \cdots q_n!}$$

by observing that the coefficient equals the number of ways of writing a p-element set as a disjoint union of subsets of sizes $q_1, q_2, \ldots, q_n$.

Solution.

- We know a single combination takes the form $\binom{p}{q} = \dfrac{p!}{(p-q)!q!}$.
- If we are to then take then remaining $p-q$ items and choose again we have $\binom{p-q}{r} = \dfrac{(p-q)!}{(p-q-r)!r!}$.
- Extending now to the general case ...
  - $\binom{p}{q_1, q_2, \ldots, q_n} = \binom{p}{q_1} \cdot \binom{p}{q_2} \cdots \binom{p}{q_n} = \dfrac{p!(p-q_1)!(p-q_1-q_2)! \cdots (p-q_1-q_2-\cdots-q_{n+1})!}{(p-q_1)!q_1!(p-q_1-q_2)!q_2! \cdots (p-q_1-q_2-\cdots-q_n)!q_n!}$.
  - In the numerator all but the $p!$ cancels with the corresponding term in the denemonator.
  - In the denemonator the $(p-q_1-q_2-\cdots-q_n)! = 0! = 1$ as $q_1 + q_2 + \cdots + q_n = n$.
  - Thus the denemonator contains only the $q_i!'s$.
- As a result we are left with the requested formula.

$\square$

As we now know, all conic sections may be given by the following standard form equations (from Section 2.4):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ (ellipse)}, \qquad y = ax^2 \text{ (parabola)}, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ (hyperbola)}.$$

The reduction of an arbitrary quadratic equation in x and y to one of these forms depends on suitable choice of origin and axes, as Fermat and Descartes discovered. The main steps are outlined in the following exercises.

Exercise 2 (6.2.1)**.** *Show that a* quadratic form $ax^2 + bxy + cy^2$ *may be converted to a form* $a'x'^2 + b'y'^2$ *by suitable choice of* θ *in the substitution*

$$x = x' \cdot \cos(\theta) - y' \cdot \sin(\theta),$$
$$y = x' \cdot \sin(\theta) + y' \cdot \cos(\theta),$$

by checking that the coefficient of $x'y'$ *is* $(c-a) \cdot \sin(2\theta) + b \cdot \cos(2\theta)$.

Solution.

- The substitution leaves us with the following . . .

  $x^2 = x'^2 \cdot \cos^2(\theta) - 2x'y' \cdot \sin(\theta)\cos(\theta) + y'^2 \cdot \sin^2(\theta)$.

  $xy = x'^2 \cdot \sin(\theta)\cos(\theta) + x'y' \cdot \cos^2(\theta) - x'y' \cdot \sin^2(\theta) - y'^2 \cdot \sin(\theta)\cos(\theta)$.

  $y^2 = x'^2 \cdot \sin^2(\theta) + 2x'y' \cdot \sin(\theta)\cos(\theta) + y'^2 \cdot \cos^2(\theta)$.

- Our equation thus becomes . . .

  $ax^2 + bxy + cy^2$

  $=$

  $a(x'^2 \cdot \cos^2(\theta) - 2x'y' \cdot \sin(\theta)\cos(\theta) + y'^2 \cdot \sin^2(\theta))+$

  $b(x'^2 \cdot \sin(\theta)\cos(\theta) + x'y' \cdot \cos^2(\theta) - x'y' \cdot \sin^2(\theta) - y'^2 \cdot \sin(\theta)\cos(\theta))+$

  $c(x'^2 \cdot \sin^2(\theta) + 2x'y' \cdot \sin(\theta)\cos(\theta) + y'^2 \cdot \cos^2(\theta))$

  $=$

  $x'^2(a \cdot \cos^2(\theta) + b \cdot \sin(\theta)\cos(\theta) + c \cdot \sin^2(\theta))+$

  $x'y'(2c \cdot \sin(\theta)\cos(\theta) - 2a \cdot \sin(\theta)\cos(\theta) + b \cdot \cos^2(\theta) - b \cdot \sin^2(\theta))+$

  $y'^2(a \cdot \sin^2(\theta) - b \cdot \sin(\theta)\cos(\theta) + c \cdot \cos^2(\theta))$

- Notice . . .

  $2c \cdot \sin(\theta)\cos(\theta) - 2a \cdot \sin(\theta)\cos(\theta) = (c-a) \cdot 2 \cdot \sin(\theta)\cos(\theta) = (c-a) \cdot \sin(2\theta)$.

  $b \cdot \cos^2(\theta) - b \cdot \sin^2(\theta) = b \cdot (\cos^2(\theta) - \sin^2(\theta)) = b \cdot \cos(2\theta)$.

- If $ax^2 + bxy + cy^2 = a'x'^2 + b'y'^2$, the coefficient on $x'y'$ must be 0. Thus . . .

  $(c-a) \cdot \sin(2\theta) + b \cdot \cos(2\theta) = 0$.

  $(c-a) \cdot \sin(2\theta) = -b \cdot \cos(2\theta)$.

  $\tan(2\theta) = -b/(c-a)$.

  $2\theta = \tan^{-1}(b/(a-c))$.

  $\theta = \tan^{-1}(b/(a-c))/2$.

- Letting . . .

  $a' = a \cdot \cos^2(\theta) + b \cdot \sin(\theta)\cos(\theta) + c \cdot \sin^2(\theta)$.

  $b' = a \cdot \sin^2(\theta) - b \cdot \sin(\theta)\cos(\theta) + c \cdot \cos^2(\theta)$.

  $\theta = \tan^{-1}(b/(a-c))/2$.

- We arrive at the requested conversion.

$\square$

Exercise 3 (6.2.2)**.** *Deduce from Exercise 6.2.1 that, by suitable rotation of axes, any quadratic curve may be expressed in the form $a'x'^2 + by'^2 + c'x' + d'y' + e'$.*

Solution.

- Assuming we are dealing with a generic quadratic curve, we have ...

  $$(ax^2 + bxy + cy^2) + (dx + ey + f).$$

- Under the prior substitution, with same assumptions for θ, a', b', we have the following ...

  $$(a'x'^2 + b'y'^2) + (d(x' \cdot \cos(\theta) - y' \cdot \sin(\theta)) + e(x' \cdot \sin(\theta) + y' \cdot \cos(\theta)) + f)$$
  $$= a'x'^2 + b'y'^2 + (d \cdot \cos(\theta) + e \cdot \sin(\theta)) \cdot x' + (e \cdot \cos(\theta) - d \cdot \sin(\theta)) \cdot y' + f.$$

- Letting ...

  $$c' = d \cdot \cos(\theta) + e \cdot \sin(\theta).$$
  $$d' = e \cdot \cos(\theta) - d \cdot \sin(\theta).$$
  $$e' = f.$$

- We deduce the expression through rotation via θ.

$\square$

Exercise 4 (6.2.3)**.** *If $b' = 0$, but $a' \neq 0$, show that the substitution $x' = x'' + f$ gives either standard-form parabola, or the "double line" $x''^2 = 0$.*
(Why is this called a "double line," and is it a section of a cone?)

Solution.

- If $b' = 0$, but $a' \neq 0$, we have . . .
  $$a'x'^2 + c'x' + d'y' + e' = 0.$$
- By substitution . . .
  $$a'(x'' + f)^2 + c'(x'' + f) + d'y' + e' = 0.$$
  $$a'x''^2 + 2a'x''f + a'f^2 + c'x'' + c'f + e' = -d'y'.$$
  $$a'x''^2 + (2a'f + c') \cdot x'' + (a'f^2 + c'f + e') = -d'y'.$$
- Division by $-d'$, and letting $-a'/d' = a''$ leaves . . .
  $$y' = a''x''^2 + (2a''f + c') \cdot x'' + (a''f^2 + c'f + e').$$
- Given d' is non-zero we have thus the *"standard-form parabola."*
- Under close choice of f . . .
  $$a'' = -c'/2f = -e',$$
- We find the familar $y' = a''x''^2$.
- Else if d' is zero we have the *"double line,"* of form
  $$a'x''^2 + (2a'f + c') \cdot x'' + (a'f^2 + c'f + e') = 0.$$
- Such called as we arrive at two vertical lines as the solution . . .
  $$x'' = \frac{-(2a'f + c') \pm \sqrt{(2a'f + c')^2 - 4(a')(a'f^2 + c'f + e')}}{2a'}.$$
  $$x'' = \frac{-2a'f - c' \pm \sqrt{c'^2 - 4f^2 - 4a'e'}}{2}.$$
  $$x'' = -a'f - c'/2 \pm \sqrt{c'^2/4 - f^2 - a'e'}.$$

$\square$

Exercise 5 (6.2.4)**.** *If both a' and b' are nonzero, show that a shift of origin gives the standard form for either an ellipse or a hyperbola, or else a pair of lines.*

Solution.

- If $b' \neq 0$, but $a' \neq 0$, we have ...

  $a'x'^2 + b'y'^2 + c'x' + d'y' + e' = 0.$

- Shifting the origin by the substitutions $x' = x'' + f$ and $y' = y'' + g$ ...

  $a'(x'' + f)^2 + b'(y'' + g)^2 + c'(x'' + f) + d'(y'' + g) + e' = 0.$

  $a'x''^2 + 2a'x''f + a'f^2 + b'y''^2 + 2b'y''g + b'g^2 + c'x'' + c'f + d'y'' + d'g + e' = 0.$

  $a'x''^2 + (2a'f + c') \cdot x'' + b'y''^2 + (2b'g + d') \cdot y'' = -(a'f^2 + b'g^2 + c'f + d'g + e').$

- Division by the constant leaves ...

  $a''x''^2 + c''x'' + b''y''^2 + d''y'' = 1.$

- Where ...

  $k'' = -a'f^2 + b'g^2 + c'f + d'g + e'.$

  $a'' = a'/k''.$

  $b'' = b'/k''.$

  $c'' = (2a'f + c')/k''.$

  $d'' = (2b'g + d')/k''.$

- Letting $c'' = 0 = d''$ by appropriate choices of f and g ...

∗ $a''x''^2 + b''y''^2 = 1$

  If the signs are the same for a'' and b'', our equation is an ellipse given $k'' \neq 0$.

  If the signs are opposite for a'' and b'', our equation is a hyperbola given $k'' \neq 0$.

  If $k'' = 0$, $a'x''^2 - b'y''^2 = 0$, leaving our two lines $a'x'' - b'y''$ and $a'x'' + b'y''$.

□