Comments Related to Exercise 3.5.2

Question 1. *Suppose we wanted to find the tangent line to the curve*

$$x^3 - y^3 = a^3 - b^3$$

at the point (a, b) *without using calculus. What might we do?*

Answer. We will proceed in four steps:

Step 1 Translate the point (a, b) to the origin with the substitution $x = w + a$, $y = z + b$ to make the rest of the calculation simpler.

$$\begin{aligned} (w+a)^3 - (z+b)^3 &= a^3 - b^3 \\ w^3 + 3aw^2 + 3a^2w + a^3 - (z^3 + 3bz^2 + 3b^2z + b^3) - a^3 + b^3 &= 0 \\ w^3 + 3aw^2 + 3a^2w - z^3 - 3bz^2 - 3b^2z &= 0 \end{aligned}$$

Step 2 Make the substitution $z = mw$ to represent intersecting the translated curve with the line through the origin $z = mw$.

$$\begin{aligned} w^3 + 3aw^2 + 3a^2w - (mw)^3 - 3b(mw)^2 - 3b^2mw &= 0 \\ w^3 + 3aw^2 + 3a^2w - m^3w^3 - 3bm^2w^2 - 3b^2mw &= 0 \end{aligned}$$

Step 3 Choose the slope m so that the intersection multiplicity (at the origin) of the line with the curve is at least two. That is, choose m so that the polynomial

$$w^3 + 3aw^2 + 3a^2w - m^3w^3 - 3bm^2w^2 - 3b^2mw$$

is divisible by w^2. In other words, set the coefficient on w to zero.

$$\begin{aligned} 3a^2 - 3b^2m &= 0 \\ m &= \frac{a^2}{b^2} \end{aligned}$$

Step 4 Use point-slope form.

$$y = \frac{a^2}{b^2}(x - a) + b$$

□

Question 2. *Suppose we also want to know if and where this curve intersects the tangebt line away from* (a, b)*. What might we do?*

Answer. We can divide $w^3 + 3aw^2 - \left(\frac{a^2}{b^2}\right)^3 w^3 - 3b\left(\frac{a^2}{b^2}\right)^2 w^2$ by w^2 and look at the other factor.

$$
\begin{aligned}
w + 3a - \left(\frac{a^2}{b^2}\right)^3 w - 3b\left(\frac{a^2}{b^2}\right)^2 &= 0 \\
\left(\frac{b^6 - a^6}{b^6}\right) w + 3a\left(\frac{b^3 - a^3}{b^3}\right) &= 0 \\
w &= -3a\left(\frac{b^3 - a^3}{b^3}\right)\left(\frac{b^6}{b^6 - a^6}\right) \\
&= -3a\left(\frac{b^3}{a^3 + b^3}\right)
\end{aligned}
$$

The x-coordinate of the other intersection point is

$$
a - 3a\left(\frac{b^3}{a^3 + b^3}\right) = a\frac{a^3 + b^3 - 3b^3}{a^3 + b^3} = a\frac{a^3 - 2b^3}{a^3 + b^3}.
$$

Now, we find the y-coordinate by reflecting across the line $x = y$, switching the roles of x and y as well as switching the roles of a and b. The y-coordinate of the other intersection point is $b\frac{b^3 - 2a^3}{a^3 + b^3}$. □