MTH385: History of Mathematics - Homework #12

Cason Konzer

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The equation relating the series for $\frac{\pi}{4}$ to the continued fraction for $\frac{4}{\pi}$, namely

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}}$$

follows immediately from a more general equation

$$\frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \cdots = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B + \cfrac{C^2}{D - C + \cdots}}}}$$

proved by Euler (1748a), p. 311. The following exercises give a proof of Euler's result.

Exercise 1 (8.4.3)**.** *Check that*

$$\frac{1}{A} - \frac{1}{B} = \cfrac{1}{A + \cfrac{A^2}{B - A}}$$

Solution.

- $$\frac{1}{A} - \frac{1}{B} = \frac{B - A}{AB} = \frac{1}{\frac{AB}{B - A}} = \frac{1}{\frac{AB - A^2 + A^2}{B - A}} = \frac{1}{\frac{A(B - A) + A^2}{B - A}} = \cfrac{1}{A + \cfrac{A^2}{B - A}}.$$

□

Exercise 2 (8.4.4)**.** *When* $\frac{1}{B}$ *on the left side in Exercise 8.4.3 is replaced by* $\frac{1}{B} - \frac{1}{C}$*, which equals* $\frac{1}{B - \frac{B^2}{C-B}}$ *by Exercise 8.4.3, show that B on the right should be replaced by* $B + \frac{B^2}{C-B}$*. Hence show that*

$$\frac{1}{A} - \frac{1}{B} + \frac{1}{C} = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B}}}$$

Solution.

- *Let* $\quad B - \dfrac{B^2}{C - B} = X.$
- $\dfrac{1}{A} - \dfrac{1}{B} + \dfrac{1}{C} = \dfrac{1}{A} - \left(\dfrac{1}{B} - \dfrac{1}{C}\right) = \dfrac{1}{A} - \cfrac{1}{B - \cfrac{B^2}{C - B}} = \dfrac{1}{A} - \dfrac{1}{X} = \cfrac{1}{A + \cfrac{A^2}{X - A}} = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B}}}.$

$\square$

Thus when we modify the tail end of the series (replacing $\frac{1}{B}$ by by $\frac{1}{B}-\frac{1}{C}$), only the tail end of the continued fraction is affected. This situation continues:

Exercise 3 (8.4.5)**.** *Generalize your argument in Exercise 8.4.4 to obtain a continued fraction for a series with n terms, and hence prove Euler's equation.*

Solution.

Consider ...

- $\frac{1}{A_0}-\frac{1}{A_1}+\frac{1}{A_2}-\frac{1}{A_3}+-\cdots+-\frac{1}{A_{n-3}}+\frac{1}{A_{n-2}}-\frac{1}{A_{n-1}}+\frac{1}{A_n}.$
- $=\frac{1}{A_0}-\left(\frac{1}{A_1}-\frac{1}{A_2}+\frac{1}{A_3}-+\cdots-+\frac{1}{A_{n-3}}-\frac{1}{A_{n-2}}+\frac{1}{A_{n-1}}-\frac{1}{A_n}\right).$
- $=\cdots.$
- $=\frac{1}{A_0}-\left(\frac{1}{A_1}-\left(\frac{1}{A_2}-\left(\frac{1}{A_3}-\left(+\cdots+-\left(\frac{1}{A_{n-3}}-\left(\frac{1}{A_{n-2}}-\frac{1}{A_{n-1}}+\frac{1}{A_n}\right)\right)\right)\right)\right)\right).$
- $=\frac{1}{A_0}-\left(\frac{1}{A_1}-\left(\frac{1}{A_2}-\left(\frac{1}{A_3}-\left(+\cdots+-\left(\frac{1}{A_{n-3}}-\left(\frac{1}{A_{n-2}}-\left(\frac{1}{A_{n-1}}-\frac{1}{A_n}\right)\right)\right)\right)\right)\right)\right).$

Now making similar substitutions ...

- $=\frac{1}{A_0}-\left(\frac{1}{A_1}-\left(\frac{1}{A_2}-\left(\frac{1}{A_3}-\left(+\cdots+-\left(\frac{1}{A_{n-3}}-\left(\frac{1}{A_{n-2}}-\frac{1}{X_n}\right)\right)\right)\right)\right)\right).$
- $=\frac{1}{A_0}-\left(\frac{1}{A_1}-\left(\frac{1}{A_2}-\left(\frac{1}{A_3}-\left(+\cdots+-\left(\frac{1}{A_{n-3}}-\frac{1}{X_{n-1}}\right)\right)\right)\right)\right).$
- $=\cdots.$
- $=\frac{1}{A_0}-\left(\frac{1}{A_1}-\frac{1}{X_2}\right).$
- $=\frac{1}{A_0}-\frac{1}{X_1}.$

Which by the same method arrives at Euler's equation ...

- $\dfrac{1}{A_0} - \dfrac{1}{X_1} = \cfrac{1}{A_0 + \cfrac{A_0^2}{X_1 - A_0}}.$

- $= \dfrac{1}{A_0} - \left(\dfrac{1}{A_1} - \dfrac{1}{X_2}\right) = \cfrac{1}{A_0 + \cfrac{A_0^2}{A_1 - A_0 + \cfrac{A_1^2}{X_2 - A_1}}}.$

- $= \dfrac{1}{A_0} - \left(\dfrac{1}{A_1} - \left(\dfrac{1}{A_2} - \dfrac{1}{X_3}\right)\right) = \cfrac{1}{A_0 + \cfrac{A_0^2}{A_1 - A_0 + \cfrac{A_1^2}{A_2 - A_1 + \cfrac{A_2^2}{X_3 - A_2}}}}.$

- $= \cdots.$

- $= \dfrac{1}{A_0} - \left(\dfrac{1}{A_1} - \left(\dfrac{1}{A_2} - \left(\dfrac{1}{A_3} - \left(+ \cdots + - \left(\dfrac{1}{A_{n-3}} - \left(\dfrac{1}{A_{n-2}} - \left(\dfrac{1}{A_{n-1}} - \dfrac{1}{A_n}\right)\right)\right)\right)\right)\right)\right).$

- $= \cfrac{1}{A_0 + \cfrac{A_0^2}{A_1 - A_0 + \cfrac{A_1^2}{A_2 - A_1 + \cfrac{A_2^2}{A_3 - A_2 + \cfrac{A_{\dots}^2}{A_{\dots} - A_{n-3} + \cfrac{A_{n-2}^2}{A_{n-1} - A_{n-2} + \cfrac{A_{n-1}^2}{A_n - A_{n-1}}}}}}}}.$

$\square$

Exercise 8.5.2 shows why the inverse function $x = e^y - 1$ has a power series that begins

$$y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \cdots .$$

Exercise 4 (8.5.3)**.** *Show that the binomial series gives*

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \cdots .$$

Solution.

- $(1+a)^p = 1 + pa + \frac{p(p-1)}{2}a^2 + \frac{p(p-1)(p-2)}{6}a^3 + \cdots .$
- $\frac{1}{\sqrt{1-t^2}} = (1+(-t^2))^{-1/2} = 1 + -\frac{1}{2}(-t^2) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-t^2)^2 + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-t^2)^3 + \cdots .$
- $= 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \cdots .$

□

Exercise 5 (8.5.4)**.** *Use Exercise 8.5.3 and* $\sin^{-1}(x) = \int_0^x dt/\sqrt{1-t^2}$ *to derive Newton's series for* $\sin^{-1}(x)$.

Solution.

- $\sin^{-1}(x) = \displaystyle\int_0^x 1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots\, dt.$
- $= \displaystyle\int_0^x dt + \int_0^x \frac{1}{2}t^2\, dt + \int_0^x \frac{3}{8}t^4\, dt + \int_0^x \frac{5}{16}t^6\, dt + \int_0^x \cdots\, dt.$
- $= t\Big|_0^x + \dfrac{1}{6}t^3\Big|_0^x + \dfrac{3}{40}t^5\Big|_0^x + \dfrac{5}{112}t^7\Big|_0^x + \cdots.$
- $= x + \dfrac{1}{6}x^3 + \dfrac{3}{40}x^5 + \dfrac{5}{112}x^7 + \cdots.$

$\square$