A Gentle Introduction to Hyperbolic Geometry

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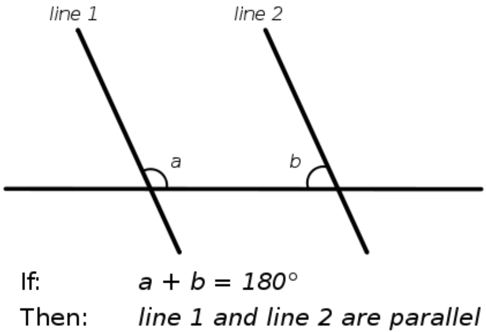
The Parallel Postulate For any given line ℓ and a point P not on ℓ, there is exactly one line through P that does not intersect ℓ.

Here's how we usually look at it:



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Hence, **non-Euclidean** geometries were born.

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2. Given ℓ and P, there exist multiple lines through P that do not intersect ℓ.

Of course, this means that we have to decide what a "line" is.

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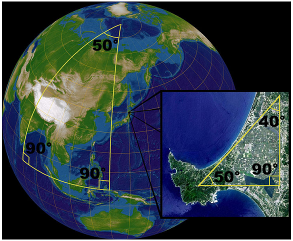
In the usual two-dimensional plane, this is exactly what we think of, but in other contexts it might be something else.

Example:

On the surface of a sphere (like our planet), the shortest path between two points isn't a straight line, but rather an arc of a longitude through the two points.



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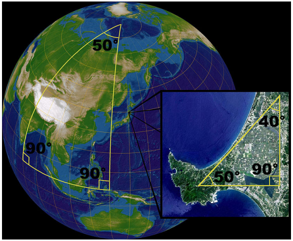

This is an example of **elliptic** or **spherical** geometry. In this case, every line through a point not on a given line intersects the line.

Notice also that the sum of the angles of a triangle add up to more than 180° in this case. Since we are so small relative to the size of the earth, we don't really notice this, and we generally observe that the shortest distance between points is a straight line in the usual sense.

What about the other case–where more than one line can exist?

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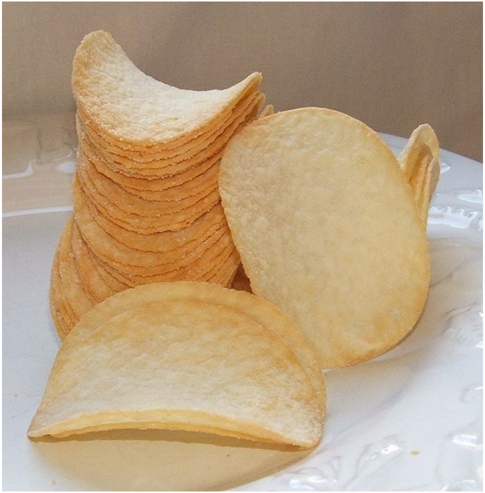
This leads to **hyperbolic geometry**, and examples exist in nature.

Coral reefs:

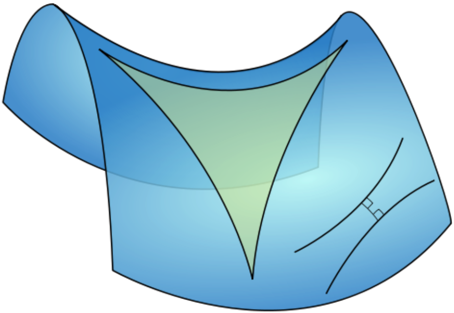
Lettuce:

Our
ROMAINE
Lettuce
GREEN
BEANS
$3.00/lb
Our
ARUGULA
Our
RED LEAF
Lettuce
OUR
BOSTON
Lettuce

Pringles:



The Pringle is a realization of a **hyperbolic paraboloid**:



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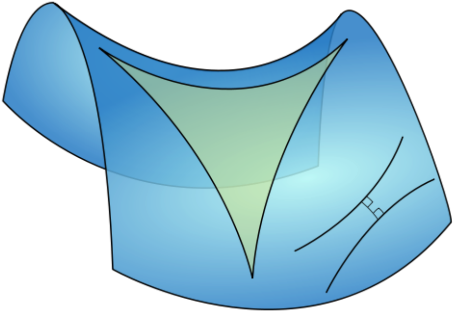

Note that if we draw a triangle on this surface, the angles add up to less than 180°.

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1. diameters of the circle

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1. diameters of the circle
2. circles perpendicular to the boundary circle

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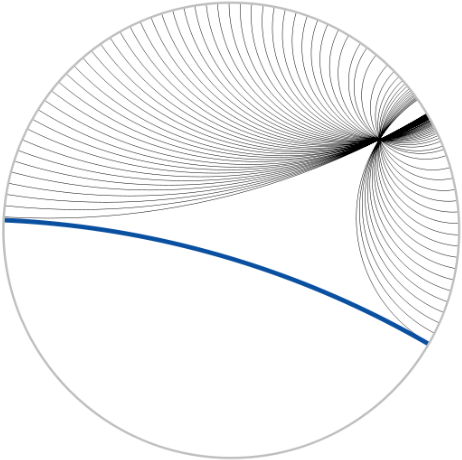

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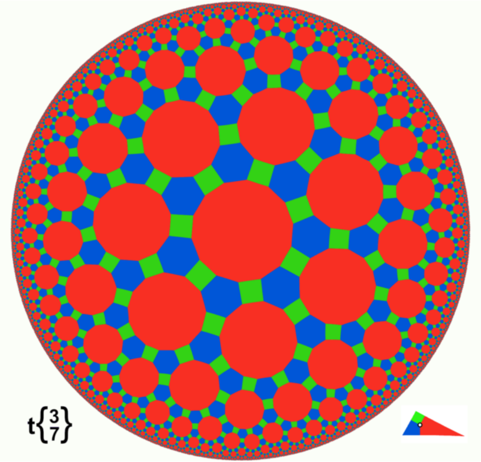
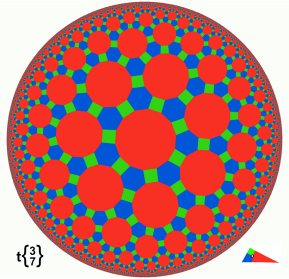


Every red region has the same area in hyperbolic space. Note that they look "smaller" as you go out towards the boundary circle. What this means is that the boundary circle is infinitely far away in this space.



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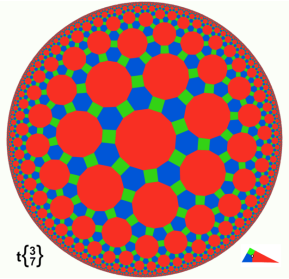

Note also that the number of red (or greeen or blue) regions increases exponentially as you head toward the boundary circle.

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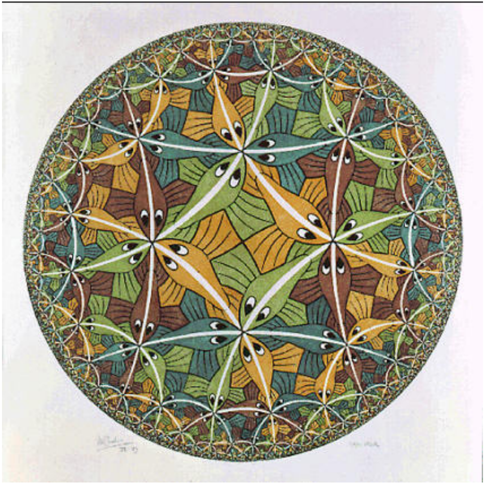

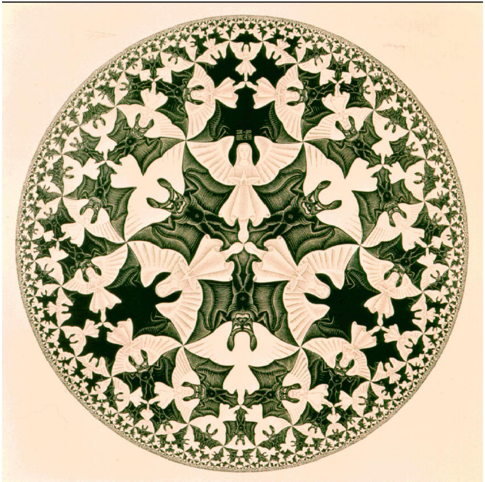

It's this notion that the regions really all have the same area that creates the unusual structures we observe in lettuce and coral. If we try to embed this geometry into ordinary Euclidean space, we run into trouble. So the space is forced to curve to preserve the areas and parallel lines.

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But I'll leave it to the biologists to explain why this is advantageous for the organisms.

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