MTH 385 Homework due 2022-04-21

The equation relating the series for $\frac{\pi}{4}$ to the continued fraction for $\frac{4}{\pi}$, namely

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}}$$

follows immediately from a more general equation

$$\frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \cdots = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B + \cfrac{C^2}{D - C + \cdots}}}}$$

proved by Euler (1748a), p. 311. The following exercises give a proof of Euler's result.

Exercise 1 (8.4.3)**.** *Check that*

$$\frac{1}{A} - \frac{1}{B} = \cfrac{1}{A + \cfrac{A^2}{B - A}}$$

Solution. We will simplify the expression of the right-hand side of the equation.

$$\begin{aligned}
\cfrac{1}{A + \cfrac{A^2}{B - A}} &= \frac{B - A}{A(B - A) + A^2} \\
&= \frac{B - A}{AB - A^2 + A^2} \\
&= \frac{B - A}{AB} \\
&= \frac{B}{AB} - \frac{A}{AB} \\
&= \frac{1}{A} - \frac{1}{B}
\end{aligned}$$

□

Exercise 2 (8.4.4)**.** *When $\frac{1}{B}$ on the left side in Exercise 8.4.3 is replaced by $\frac{1}{B}-\frac{1}{C}$, which equals $\frac{1}{B-\frac{B^2}{C-B}}$ by Exercise 8.4.3, show that B on the right should be replaced by $B+\frac{B^2}{C-B}$. Hence show that*

$$\frac{1}{A}-\frac{1}{B}+\frac{1}{C}=\cfrac{1}{A+\cfrac{A^2}{B-A+\cfrac{B^2}{C-B}}}$$

Solution. Evidently,

$$\frac{1}{A}-\frac{1}{B}+\frac{1}{C}=\frac{1}{A}-\left(\frac{1}{B}-\frac{1}{C}\right).$$

Moreover,

$$\frac{1}{B}-\frac{1}{C}=\left(B+\frac{B^2}{C-B}\right)^{-1}$$

by Exercise 8.4.3. Since $\left(\frac{1}{B}-\frac{1}{C}\right)^{-1}=B+\frac{B^2}{C-B}$, B should be replaced by $B+\frac{B^2}{C-B}$ when $\frac{1}{B}$ is replaced by $\frac{1}{B}-\frac{1}{C}$. The result follows immediately.

$$\frac{1}{A}-\frac{1}{B}+\frac{1}{C}=\cfrac{1}{A+\cfrac{A^2}{B-A+\cfrac{B^2}{C-B}}}$$

□

Thus when we modify the tail end of the series (replacing $\frac{1}{B}$ by by $\frac{1}{B}-\frac{1}{C}$), only the tail end of the continued fraction is affected. This situation continues:

Exercise 3 (8.4.5)**.** *Generalize your argument in Exercise 8.4.4 to obtain a continued fraction for a series with n terms, and hence prove Euler's equation.*

Solution. I claim

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{A_k}=\cfrac{1}{A_1+\cfrac{A_1^2}{A_2-A_1+\cfrac{A_2^2}{A_3-A_2+\cfrac{A_3^2}{A_4-A_3+\cdots}}}}$$

We proceed by induction on n. Exercise 8.4.3 can be seen as the $n=2$ case. And, Exercise 8.4.4 can be seen as the induction step. That is, when $\frac{1}{A_n}$ on the left side is replaced by $\frac{1}{A_n}-\frac{1}{A_{n+1}}$, A_n on the right should be replaced by $A_n+\frac{A_n^2}{A_{n+1}-A_n}$. □

Exercise 8.5.2 shows why the inverse function $x = e^y - 1$ has a power series that begins

$$y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \cdots .$$

Exercise 4 (8.5.3)**.** *Show that the binomial series gives*

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \cdots .$$

Solution.

$$\begin{aligned}
\frac{1}{\sqrt{1-t^2}} &= (1-t^2)^{-\frac{1}{2}} \\
&= 1 - \frac{1}{2}(-t^2) + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}(-t^2)^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(-t^2)^3 + \cdots \\
&= 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \cdots
\end{aligned}$$

□

Exercise 5 (8.5.4)**.** *Use Exercise 8.5.3 and* $\sin^{-1} x = \int_0^x dt/\sqrt{1-t^2}$ *to derive Newton's series for* $\sin^{-1} x$*.*

Solution.

$$\begin{aligned}
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}} \\
&= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \cdots\right) dt \\
&= x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots
\end{aligned}$$

□