MTH385: History of Mathematics - Homework #5

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Exercise 1 (3.3.2)**.** *Show that, for any integers a and b, there are integers m and n such that*

$$\gcd(a, b) = ma + nb$$

Solution.

Let $G = \gcd(a, b)$. We know that $G|a$ and $G|b$.

Thus we have some $x = a/G$ and $y = b/G$, where $x, y \in \mathbb{Z}$.

This system provides that $y|b$, as $Gy = b$ and $y|bx$, as $Gx = bx/y$.

We now consider $\gcd(x, y) = 1$.

As x and y are relatively prime, $mx + ny = 1$, $m, n \in \mathbb{Z}$.

For proof: $mbx + nyb = b$, where $y|bx$; $y|y$; $y|b$.

Now . . .

- $ma/G + nb/G = 1$.
- $ma + nb = G$.

And $\gcd(a, b) = ma + nb$. ◻

This in turn gives a general way to find integer solutions of linear equations.

Exercise 2 (3.3.3)**.** *Deduce from Exercise 3.3.2 that the equation $ax + by = c$ with integer coefficients a, b, and c has an integer solution x, y if $\gcd(a, b)$ divides c.*

Solution.

Let $G = \gcd(a, b)$, Thus $G = ma + nb$, from **Ex. 1**.

Assuming $G|c$, $Gi = c$ for some $i \in \mathbb{Z}$.

Now $Gi = mai + nbi = c$

Letting $mi = x$ and $bi = y$, $ax + by = c$.

As $m, n, i \in \mathbb{Z}$, x, y are an integer solution.

□

Exercise 3 (3.3.5)**.** *(Solution of linear Diophantine equations) Give a test to decide, for any given integers a, b, c, whether there are integers x, y such that*

$$ax + by = c.$$

Solution.

From **Ex. 1** & **Ex. 2**, if $\gcd(a, b)$ divides c, there are integers x, y that satisfy $ax + by = c$.

- $G = ma + nb$; For any integers a and b.
- $Gi = mai + nbi$
- $c = ax + by$; Where $c, a, x, b, y \in \mathbb{Z}$.
- Thus if $G|c$; we can finx $x, y \in \mathbb{Z}$.
- Else; we contradict that $a, b \in \mathbb{Z}$.

□

Exercise 4 (3.4.3)**.** *Show that*

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

Solution.

The continued fraction of a real number $\alpha_0 > 0$ is written

$$\alpha_0 = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \cfrac{1}{\ddots}}}}$$

Where, $n_1 = \lfloor \alpha_0 \rfloor$; $\alpha_1 = 1/(\alpha_0 - n_1) > 1$; $n_k = \lfloor \alpha_{k-1} \rfloor$; $\alpha_k = 1/(\alpha_{k-1} - n_k) > 1, \forall k \geq 1$

We have, $\alpha_0 = \sqrt{2}$; $n_1 = \lfloor \sqrt{2} \rfloor = 1$; $\alpha_1 = 1/(\sqrt{2} - 1)$.

It follows that, $\alpha_1 = 1 + \sqrt{2}$ as $(1 + \sqrt{2})(\sqrt{2} - 1) = 1$.

Thus we have, $n_2 = \lfloor 1 + \sqrt{2} \rfloor = 2$ and $\alpha_2 = 1/(1 + \sqrt{2} - 2) = 1/(\sqrt{2} - 1) = \alpha_1$.

Similarly, $n_3 = \lfloor 1 + \sqrt{2} \rfloor = 2 = n_2$.

We can now see, $\forall i \geq 1;\ \alpha_i = 1/(\sqrt{2} - 1)$, and $\forall j \geq 2; n_j = 2$.

By substitution we arrive at the requested . . .

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

□

Exercise 5 (3.4.4)**.** *Show that* $\sqrt{3}+1$ *also has a periodic continued fraction, and hence derive the continued fraction for* $\sqrt{3}$*.*

Solution.

Consider, $\alpha_0 = \sqrt{3}+1$; $n_1 = \lfloor \sqrt{3}+1 \rfloor = 2$; $\alpha_1 = 1/(\sqrt{3}+1-2) = 1/(\sqrt{3}-1)$.

It follows that, $\alpha_1 = (\sqrt{3}+1)/2$ as $(\sqrt{3}+1)(\sqrt{3}-1)/2 = 1$.

Thus we have, $[2 < (\sqrt{3}+1) < 3]$; $[1 < (\sqrt{3}+1)/2 < 2]$; $n_2 = \lfloor (\sqrt{3}+1)/2 \rfloor = 1$.

Now, $(\sqrt{3}+1)/2 - 1 = (\sqrt{3}+1-2)/2 = (\sqrt{3}-1)/2$ and $\alpha_2 = 2/(\sqrt{3}-1)$.

Following again, $\alpha_2 = \sqrt{3}+1$ as $(\sqrt{3}+1)(\sqrt{3}-1) = 2$.

We can now see the following reccurance relations for $even$ and odd values of $i \ldots$

- $\forall\ even\ i;\ \alpha_i = \sqrt{3}+1$, and $\forall\ odd\ i;\ \alpha_i = (\sqrt{3}+1)/2$.
- $\forall\ even\ i;\ n_i = 2$, and $\forall\ odd\ i;\ n_i = 1$.

Now . . .

$$\sqrt{3}+1 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

And . . .

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

□