MTH 385 Homework due 2022-03-14

Consider the intersection of two circles. Fortunately, it is easy to reduce these two quadratic equations to the case handled in Exercise 5.3.4.

Exercise 1 (5.3.5)**.** *The equations of any two circles can be written in the form*

$$\begin{aligned}(x-a)^2+(y-b)^2&=r^2\\(x-c)^2+(y-d)^2&=s^2\end{aligned}$$

Explain why. Now subtract one of these equations from the other, and hence show that their common solutions can be found by rational operations and square roots.

When a sequence of quadratic equations is solved, the solution may involve *nested* square roots, such as $\sqrt{(5+\sqrt{5})/2}$. This very number, in fact, occurs in the icosahedron, as one sees from Pacioli's construction in Section 2.2.

Exercise 2 (5.3.6)**.** *Show that the diagonal of a golden rectangle (which is also the diameter of an icosahedron of edge length* 1*) is* $\sqrt{(5+\sqrt{5})/2}$*.*

We know from Exercise 5.4.1 that $\sqrt[3]{2}$ is not in F_0, but if it is constructible it will occur in some F_{k+1}. A contradiction now ensues by considering (hypothetically) the first such F_{k+1}.

Exercise 3 (5.4.3)**.** *Show that if* $a, b, c \in F_k$ *but* $\sqrt{c} \notin F_k$*, then* $a + b\sqrt{c} = 0 \Leftrightarrow a = b = 0$*. (For* $k = 0$ *this is in the* Elements*, Book X, Prop. 79.)*

Exercise 4 (5.4.4)**.** *Suppose* $\sqrt[3]{2} = a + b\sqrt{c}$*, where* $a, b, c \in F_k$*, but that* $\sqrt[3]{2} \notin F_k$*. (We know that* $\sqrt[3]{2} \notin F_0$ *from Exercise 5.4.1.) Cube both sides and deduce from Exercise 5.4.3 that*

$$2 = a^3 + 3ab^2c \quad \text{and} \quad 0 = 3a^2b + b^3c.$$

Exercise 5 (5.4.5)**.** *Deduce from Exercise 5.4.4 that* $\sqrt[3]{2} = a - b\sqrt{c}$ *also, and explain why this is a contradiction.*