MTH 385   History of Maths   Homework 1: due 2022-01-17
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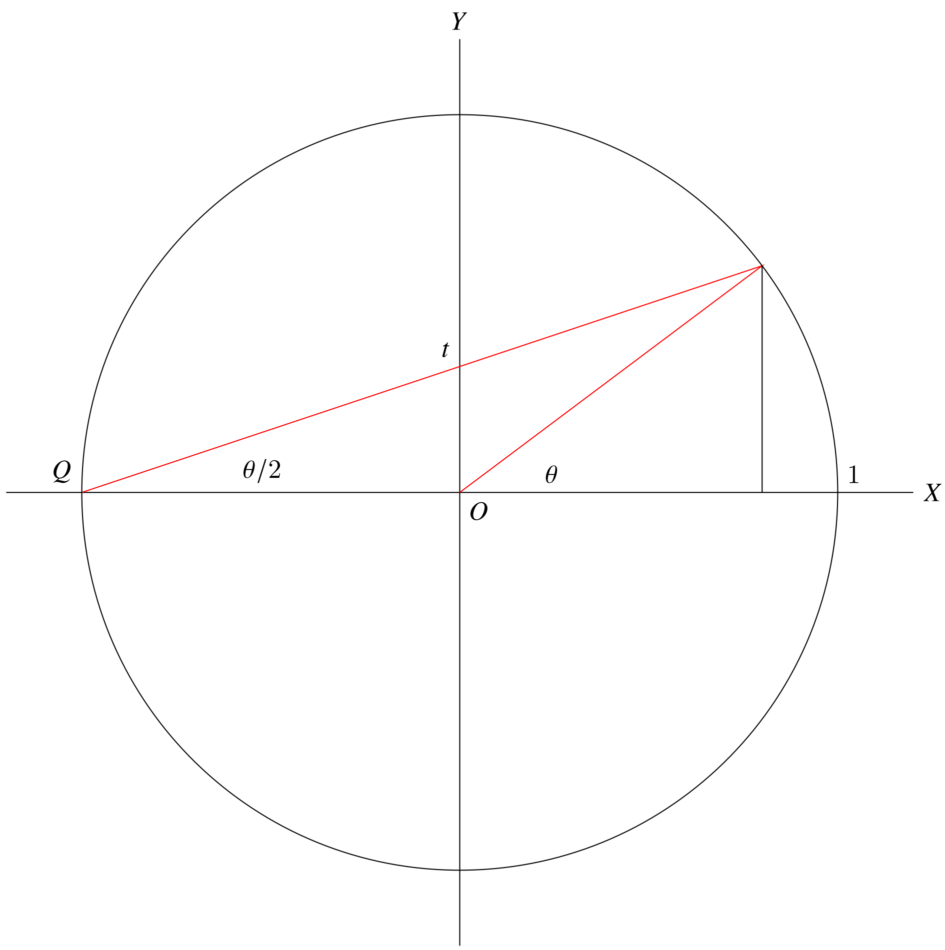
Supplementary Material

- The parameter t in the pair $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ runs through all rational numbers if $t = q/p$ and p, q run through all pairs of integers.
- Some important trigonometry may be gleaned from Diophantus's method if we compare the angle at O with the angle at Q in the following figure. High school geometry shows that the angle at Q is half the angle at O.



Exercises Follow

Exercise 1 (1.2.3)**.** *Show that any integer square leaves remainder* 0 *or* 1 *on division by* 4*.*

Solution.
Integers take 2 flavors, even and odd. We will approach this by cases.

- *Even* numbers take the form $2k$, $k \in \mathbb{Z}$.
  $(2k)^2 = 4k^2$
  $4k^2 \,\%\, 4 = 0$
- *Odd* numbers take the form $2k + 1$, $k \in \mathbb{Z}$.
  $(2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$
  $(4(k^2 + k) + 1) \,\%\, 4 = ((4k^2 \,\%\, 4) + (1 \,\%\, 4)) \,\%\, 4 = (0 + 1) \,\%\, 4 = 1$

As these 2 flavors encompass all possible integers we have shown that on division by 4 the integer squares takes remainder 0, or 1. □

Exercise 2 (1.2.4)**.** *Deduce from Exercise 1.2.3 that if* (a, b, c) *is a Pythagorean triple then* a *and* b *cannot both be odd.*

Solution.
To satisfy integer squares we must take a and b such that their sum will leave remainder 0 or 1 on division by 4.

- *consider* two integers of odd flavor.
  $2i + 1, 2j + 1$
- *consider* their squares.
  $4(i^2 + i) + 1, 4(j^2 + j) + 1$
- *consider* their sum.
  $4(i^2 + i) + 1 + 4(j^2 + j) + 1 = 4(i^2 + j^2 + i + j) + 2$
- *consider* their remainder on division by 4.
  $4(i^2 + j^2 + i + j) + 2 \,\%\, 4 = ((4(i^2 + j^2 + i + j) \,\%\, 4) + (2 \,\%\, 4)) \,\%\, 4 = (0 + 2) \,\%\, 4 = 2$

We can see their sum takes remainder 2 on division by 4. From this we know that c is not an integer and thus any a and b odd will not produce a Pythagorean triple. □

Exercise 3 (1.3.1)**.** *Deduce that if* (a, b, c) *is any Pythagorean triple then*

$$\frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2}, \qquad \frac{b}{c} = \frac{2pq}{p^2 + q^2}$$

for some integers p *and* q*.*

Solution.
We know that any Pythagorean triple, (a, b, c), will satisfy $a^2 + b^2 = c^2$.

- We can divde this equation by c^2 to arrive at the following.
  $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$ and $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, a Pythagorean triple with hypotenuse length 1.
- We can now leverage our pair $\left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right)$, as it takes the same form.

  $$\left(\frac{1 - t^2}{1 + t^2}\right)^2 + \left(\frac{2t}{1 + t^2}\right)^2 = \frac{1 - 2t^2 + t^4 + 4t^4}{(1 + t^2)^2} = \frac{(1 + t^2)^2}{(1 + t^2)^2} = 1$$

- We can now equeate our two pairs as 1 = 1.

$$\frac{(1-t^2)^2}{(1+t^2)^2} + \frac{(2t)^2}{(1+t^2)^2} = 1 = \frac{(a)^2}{(c)^2} + \frac{(b)^2}{(c)^2}$$

- We thus have come to the following realization (We arbitraritly choose this pair, it could have been reversed).

$$\frac{(a)^2}{(c)^2} = \frac{(1-t^2)^2}{(1+t^2)^2}, \frac{(b)^2}{(c)^2} = \frac{(2t)^2}{(1+t^2)^2}$$

- For both pairs we can take the root of both sides and substitute in $t = \dfrac{q}{p}$

$$\frac{a}{c} = \frac{1-t^2}{1+t^2} = \frac{1-\frac{q^2}{p^2}}{1+\frac{q^2}{p^2}}, \frac{b}{c} = \frac{2t}{1+t^2} = \frac{2\frac{q}{p}}{1+\frac{q^2}{p^2}}$$

- A multiplication by $1 = \frac{p^2}{p^2}$ will bring us to our final form.

$$\frac{a}{c} = \frac{p^2-q^2}{p^2+q^2}, \qquad \frac{b}{c} = \frac{2pq}{p^2+q^2}$$

□

Exercise 4 (1.3.2)**.** *Use Exercise 1.3.1 to prove Euclid's formula for Pythagorean triples, assuming b even. (Remember, a and b are not both odd.)*

Solution.
From 1.3.1, $a = p^2 - q^2$, $b = 2pq$, $c = p^2 + q^2$

- Let us verify that Euclid's formula yields Pythagorean triples.
- Consider the squares.

$$a^2 = (p^2 - q^2)^2 = p^4 - 2p^2q^2 + q^4$$
$$b^2 = (2pq)^2 = 4p^2q^2$$
$$c^2 = (p^2 + q^2)^2 = p^4 + 2p^2q^2 + q^4$$

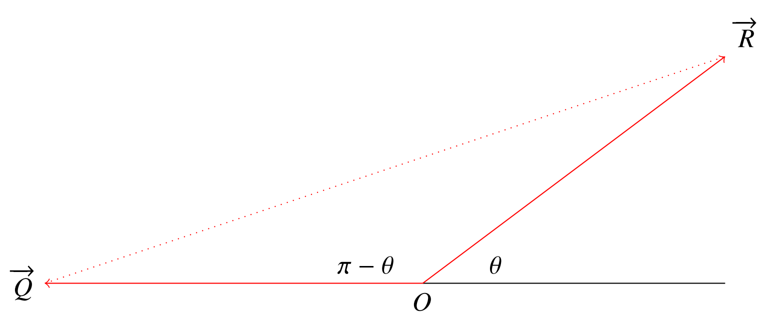
- Now check that this holds under the Pythagorean theorem $a^2 + b^2 = c^2$.

$$p^4 - 2p^2q^2 + q^4 + 4p^2q^2 = p^4 + (4p^2q^2 - 2p^2q^2) + q^4 = p^4 + 2p^2q^2 + q^4$$

□

Exercise 5 (1.3.4)**.** *Why does the angle at Q equal θ/2? (Hint: consider angles in the red triangle.)*

Solution.
Consider the following figure.



- Consider the length of the vectors in the figure.

  $|\overrightarrow{Q}| = |\overrightarrow{R}| = 1$, as they are points on the unit circle.

  The Triangle PQR is isosceles.

- We thus know the angles $\angle ORQ$ and $\angle OQR$ are equal.

  let $\angle ORQ, \angle OQR = \omega$

- We also know that the interior angles of a triangle sum to π

  $\omega + \omega + \pi - \theta = \pi$

  $2\omega = \theta$

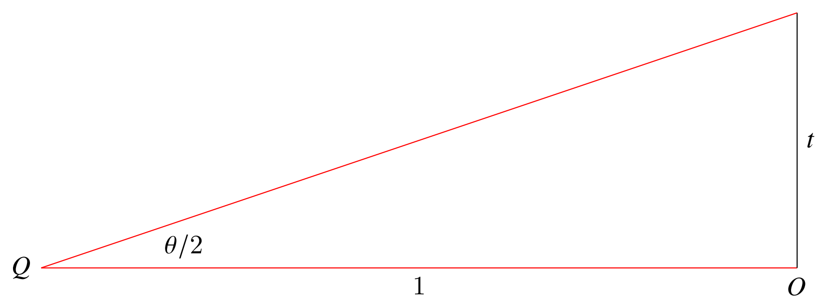
  $\omega = \theta/2$

High school geometry has shown that the angle at Q is half the angle at O. □

Exercise 6 (1.3.5)**.** *Use Figure 1.7 to show that* $t = \tan\theta/2$ *and*

$$\cos\theta = \frac{1-t^2}{1+t^2}, \qquad \sin\theta = \frac{2t}{1+t^2}.$$

Solution.
Consider the following figure.



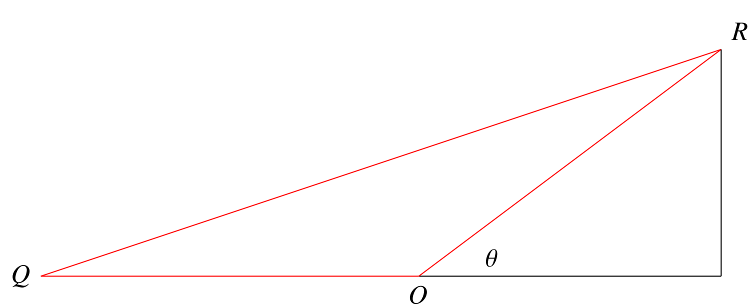
- We know that for some angle ω, $\tan\omega = \dfrac{\sin\omega}{\cos\omega}$.

  $\tan\theta/2 = \dfrac{\sin\theta/2}{\cos\theta/2} = \dfrac{t}{1} = t$

- We additionally know slope is $\dfrac{rise}{run}$.

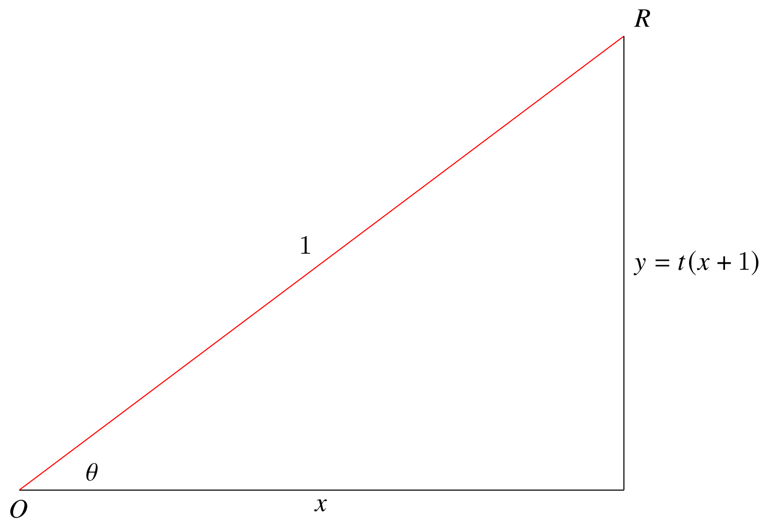
  From above we have solve the slope for this triangle as $\tan\theta/2 = t$

Consider next the following figure.



- Point-Slope form for the equation of a line is as follows: $y - y_0 = m(x - x_0)$.
  The coordinates of R are unknown, the coordinates of Q can be see as $\langle -1, 0 \rangle$
- As we now have some coordinates, $\langle x_0, y_0 \rangle$ and know our slope, we can solve for the equation of the line QR.
  $y - 0 = t(x - (-1))\ ;\ y = t(x + 1)$

We can now leverage the following figure.



- Let us leverage the Pythagorean theorem, $a^2 + b^2 = c^2$
  $x^2 + (t(x+1))^2 = 1^2$
  $x^2 + t^2(x^2 + 2x + 1)) = x^2 + t^2x^2 + 2xt^2 + t^2 = 1$
- We are looking for the value of x as $x = \cos\theta$
  $x^2 + t^2x^2 + t^2 2x + t^2 - 1 = 0 = x^2(1 + t^2) + 2xt^2 + (t^2 - 1)$
- We will now leverage the quadratic equation to solve for x, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  $$x = \frac{-(2t^2) \pm \sqrt{(2t^2)^2 - 4(1 + t^2)(t^2 - 1)}}{2(1 + t^2)}$$
  $$x = \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^4 - 1)}}{2(1 + t^2)} = \frac{-2t^2 \pm \sqrt{4}}{2(1 + t^2)}$$
- Conside the case of subtraction.
  $$x = \frac{-2t^2 - 2}{2(t^2 + 1))} = \frac{-2(t^2 + 1)}{2(t^2 + 1)} = -1$$
  This solution is trival and can be identified as point Q from our eariler figure.
- Conside the case of addition.
  $$x = \frac{-2t^2 + 2}{2(1 + t^2)} = \frac{2(1 - t^2)}{2(1 + t^2)} = \frac{1 - t^2}{1 + t^2} = \cos\theta$$
- We can now substitute our value of x into $y = t(x + 1)$ to solve for y, a.k.a $\sin\theta$
  $$y = t\left(\frac{1 - t^2}{1 + t^2} + 1\right) = \frac{t}{1 + t^2}(1 - t^2 + 1 + t^2) = \frac{2t}{1 + t^2} = \sin\theta$$

It has been shown that $\tan\theta/2 = t$, $\cos\theta = \dfrac{1 - t^2}{1 + t^2}$, and $\sin\theta = \dfrac{2t}{1 + t^2}$ $\square$

END OF ASSIGNMENT