8½

MTH385: History of Mathematics - Homework #11

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+1

Exercise 1 (8.2.3)**.** *Show that the volume of the solid obtained by rotating the portion of $y = 1/x$ from $x = 1$ to ∞ about the x-axis is finite. Show, on the other hand, that its surface area is infinite.*

Solution.

- $A = \pi r^2 \Rightarrow V = \pi \int_1^\infty (1/x)^2 \, dx = \pi[-(1/\infty) + (1/1)] = \pi.$

  ∞ is not a real number!

- $C = 2\pi r \Rightarrow SA = 2\pi \int_1^\infty (1/x) \, dx = 2\pi[\ln(\infty) - \ln(1)] = 2\pi \ln(\infty) = \infty.$

  $SA = 2\pi \int_1^\infty \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$ where $y = \frac{1}{x}$.

$\square$

Cavalieri's most elegant application of his method of indivisibles was to prove Archimedes' formula for the volume of a sphere. His argument is simpler than that of Archimedes, and it goes as follows.

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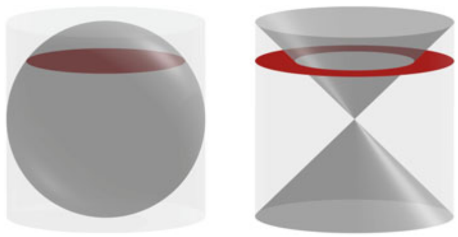
x2 **Exercise 2** (8.2.4)**.** *Show that the slice $z = c$ of the sphere $x^2 + y^2 + z^2 = 1$ has the same area as the slice $z = c$ of the cylinder $x^2 + y^2 = 1$ outside the cone $x^2 + y^2 = z^2$ (Figure 8.2).*



Solution.

- $x^2 + y^2 = r_{circle}^2 = 1 - z^2 \Rightarrow A_{circle} = \pi(1 - z^2)$.
- $x^2 + y^2 = r_{cylinder}^2 = 1$, $x^2 + y^2 = r_{cone}^2 = z^2 \Rightarrow A_{ring} = \pi(r_{cylinder}^2 - r_{cone}^2) = \pi(1 - z^2)$.

□

+2 **Exercise 3** (8.2.5). *Deduce from Exercise 8.2.4, and the known volume of the cone, that the volume of the sphere is* $2/3$ *the volume of the circumscribing cylinder.*

Solution.

- $V_{cylinder} = 2z\pi r^2$.
- $V_{cone} = z\pi r^2/3$. $= \frac{1}{3} V_{cylinder}$
- $V_{sphere} = V_{cylinder} - 2V_{cone} = (6-2)z\pi r^2/3 = 4z\pi r^2/3$.
- $V_{sphere}/V_{cylinder} = (4z\pi r^2/3)/2z\pi r^2 = 2/3$.

Notice in all cases we have $z = r$ such that we see the common formula $V_{sphere} = 4\pi r^3/3$. □

The examples in Exercise 8.3.1 and Exercise 8.3.2 show how tangents can be found by looking for double roots, though it requires some foresight to make the right substitution. With calculus, the process is more mechanical.

+2 **Exercise 4** (8.3.3)**.** *Derive the formula of Hudde and Sluse by differentiating* $\sum a_{ij}x^i y^j = 0$ *with respect to* x*.*

Solution.

- $\dfrac{d}{dx}\left(\sum a_{ij}x^i y^j\right) = \sum\left(a_{ij}ix^{i-1}y^j + a_{ij}jx^i y^{j-1}\dfrac{dy}{dx}\right) = 0.$
- $\sum a_{ij}ix^{i-1}y^j = -\sum a_{ij}jx^i y^{j-1}\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = -\dfrac{\sum a_{ij}ix^{i-1}y^j}{\sum a_{ij}jx^i y^{j-1}}.$

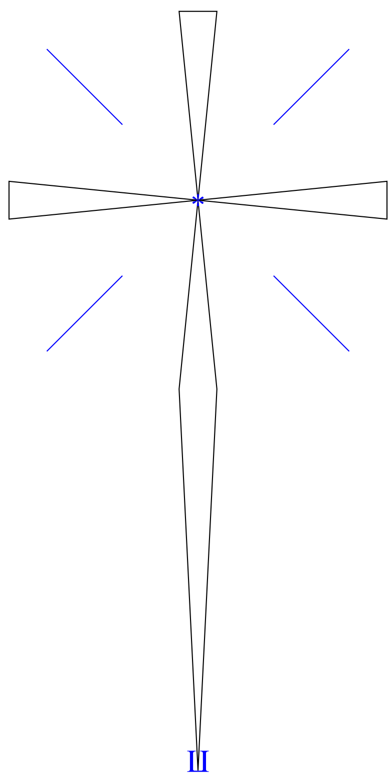
□

+1½ **Exercise 5** (8.3.4)**.** *Use differentiation to find the tangent to the folium $x^3 + y^3 = 3axy$ at the point (b, c).*

(not the slope of the tangent)

Solution.

- $3x^2 + 3y^2 \dfrac{dy}{dx} = 3ay + 3ax\dfrac{dy}{dx}$.
- $(y^2 - ax)\dfrac{dy}{dx} = ay - x^2$.
- $\dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$.
- $\dfrac{dy}{dx}(b, c) = \dfrac{ac - b^2}{c^2 - ab}$.



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