MTH385: History of Mathematics - Homework #7

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Consider the intersection of two circles. Fortunately, it is easy to reduce these two quadratic equations to the case handled in Exercise 5.3.4.

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Exercise 1 (5.3.5)**.** *The equations of any two circles can be written in the form*

$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-c)^2 + (y-d)^2 = s^2$$

Explain why. Now subtract one of these equations from the other, and hence show that their common solutions can be found by rational operations and square roots.

Solution.

The form given describes circles with center (a, b), (c, d) and radius r, s, as all circles have a center and radius, they can be written in this form.

We will first expand the squares and then subtract our two equations . . .

- $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$.
- $x^2 - 2cx + c^2 + y^2 - 2dy + d^2 = s^2$.
- $2x(c-a) + 2y(c-b) + (c^2 + d^2 - a^2 - b^2) = r^2 - s^2$.

Invoking the quadratic equation . . .

- $2x(c-a) + 2y(c-b) + (c^2 + d^2 + s^2 - a^2 - b^2 - r^2) = 0$.

If the circles are not concentric, this is the equation of a line. So, we've reduced to Exercise 5.3.4

As subtracting these equations is equivalent to setting them equal, we have found a solution to their intersection which consists of only rational operations and square roots, as we only perform rational operations and the square roots of the coefficents must exist to have pur centers and radai.

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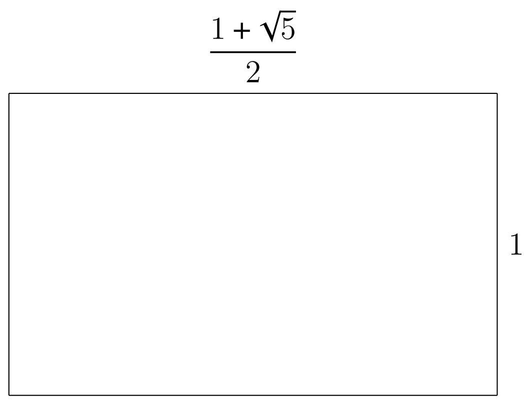
When a sequence of quadratic equations is solved, the solution may involve *nested* square roots, such as $\sqrt{(5+\sqrt{5})/2}$. This very number, in fact, occurs in the icosahedron, as one sees from Pacioli's construction in Section 2.2.

+2 **Exercise 2** (5.3.6)**.** *Show that the diagonal of a golden rectangle (which is also the diameter of an icosahedron of edge length* 1*) is* $\sqrt{(5+\sqrt{5})/2}$*.*

Solution.

Consider the *golden rectangle*.



The diagonal is the hypotenus of a right triangle thus we will invoke the Pythagorean theorem.

- $D^2 = 1^2 + \left(\frac{1+\sqrt{5}}{2}\right)^2 = 1 + \frac{1+2\sqrt{5}+5}{4} = \frac{5+\sqrt{5}}{2}$.
- $D = \sqrt{(5+\sqrt{5})/2}$.

As the *golden rectangle* is used to construct the edges of the icosahedron of edge length 1 the diagonals are equivalent.

$\square$

We know from Exercise 5.4.1 that $\sqrt[3]{2}$ is not in F_0, but if it is constructible it will occur in some F_{k+1}. A contradiction now ensues by considering (hypothetically) the first such F_{k+1}.

+1 **Exercise 3** (5.4.3)**.** *Show that if $a, b, c \in F_k$ but $\sqrt{c} \notin F_k$, then $a + b\sqrt{c} = 0 \Leftrightarrow a = b = 0$. (For $k = 0$ this is in the* Elements*, Book X, Prop. 79.)*

Solution.

For arbitrary k . . .

- $a + b\sqrt{c} = 0 \Rightarrow a = -b\sqrt{c} \Rightarrow a \notin F_k$, as $\sqrt{c} \notin F_k \Rightarrow a = b = 0$.
- $a = b = 0 \Rightarrow a + b\sqrt{c} = 0 + 0\sqrt{c} = 0$.

The first direction leverages that $\sqrt{c}$ would need exist in the same field as a for $a = -b\sqrt{c}$ to hold. Why?
The second direction follows elementary algebra.

If $a+b\sqrt{c} = 0$ and $b \neq 0$, then $\sqrt{c} = -\frac{a}{b} \in F_k$.

So, if $\sqrt{c} \notin F_k$ and $a+b\sqrt{c} = 0$, then $b = 0$.

□

+2 **Exercise 4** (5.4.4)**.** *Suppose* $\sqrt[3]{2} = a + b\sqrt{c}$, *where* $a, b, c \in F_k$, *but that* $\sqrt[3]{2} \notin F_k$. *(We know that* $\sqrt[3]{2} \notin F_0$ *from Exercise 5.4.1.) Cube both sides and deduce from Exercise 5.4.3 that*

$$2 = a^3 + 3ab^2c \quad \textit{and} \quad 0 = 3a^2b + b^3c.$$

Solution.

We will first expand and then group like terms.

- $(a + b\sqrt{c})^3 = (a^2 + 2ab\sqrt{c} + b^2c)(a + b\sqrt{c}) = a^3 + 2a^2b\sqrt{c} + ab^2c + a^2b\sqrt{c} + 2ab^2c + b^3c\sqrt{c}$.
- $2 = (\sqrt[3]{2})^3 = (a^3 + 3ab^2c) + \sqrt{c}(3a^2b + b^3c)$. You can reduce to the previous exercise by subtracting 2.
- $2 \in F_k \Rightarrow (a^3 + 3ab^2c) + \sqrt{c}(3a^2b + b^3c) \in F_k$.
- $\sqrt{c} \notin F_k \Rightarrow (3a^2b + b^3c) = 0$.

As 2 is integer, we know 2 is in all F_k. Thus our right hand side must be in the taken F_k. But if the coefficents on $\sqrt{c}$ is non zero, then it is implied that $\sqrt{c} \in F_k$, which is a contradiction. Thus the coefficent on $\sqrt{c}$ must be zero, and $2 = a^3 + 3ab^2c$ alone.

□

+2 **Exercise 5** (5.4.5)**.** *Deduce from Exercise 5.4.4 that* $\sqrt[3]{2} = a - b\sqrt{c}$ *also, and explain why this is a contradiction.*

Solution.

We will first expand and then group like terms.

- $(a - b\sqrt{c})^3 = (a^2 - 2ab\sqrt{c} + b^2c)(a - b\sqrt{c}) = a^3 - 2a^2b\sqrt{c} + ab^2c - a^2b\sqrt{c} + 2ab^2c - b^3c\sqrt{c}$.
- $2 = (\sqrt[3]{2})^3 = (a^3 + 3ab^2c) - \sqrt{c}(3a^2b + b^3c)$.

  By the previous exercises, $-(3a^2b+b^3c)=0$ and $a^3+3ab^2c=2$.
- $2 \in F_k \Rightarrow (a^3 + 3ab^2c) - \sqrt{c}(3a^2b + b^3c) \in F_k$.
- $\sqrt{c} \notin F_k \Rightarrow (3a^2b + b^3c) = 0$.

Adding the two results we will arrive at our contraditcion . . .

- $\sqrt[3]{2} + \sqrt[3]{2} = a - b\sqrt{c} + a + b\sqrt{c} = 2a \Rightarrow \sqrt[3]{2} = a \Rightarrow \sqrt[3]{2} \in F_k$.

As a is taken in F_k, and $\sqrt[3]{2}$ is given not in F_k, our equations implying that $\sqrt[3]{2}$ is in F_k is the contraditcion.

$\square$