MTH385: History of Mathematics - Homework #9

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Exercise 1 (5.7.2). *If $p(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$, use Exercise 5.7.1 to show that $p(x) - p(a)$ has a factor $x - a$.*

Solution.

- $p(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$.
- $p(a) = a_k a^k + a_{k-1} a^{k-1} + \cdots + a_1 a + a_0$.
- $p(x) - p(a) = a_k(x^k - a^k) + a_{k-1}(x^{k-1} - a^{k-1}) + \cdots + a_2(x^2 - a^2) + a_1(x - a)$.

- $(x^k - a^k)/(x - a) = (x^{k-1} + ax^{k-2} + \cdots + a^{k-2}x + a^{k-1}) = C$.
- $(x^{k-1} - a^{k-1})/(x - a) = (x^{k-2} + ax^{k-3} + \cdots + a^{k-3}x + a^{k-2}) = A$.
- $\ldots/(x - a) = \cdots = S$.
- $(x^2 - a^2)/(x - a) = (x + a) = O$.
- $(x - a)/(x - a) = 1 = N$.

- $(p(x) - p(a))/(x - a) = C + A + S + O + N$.

$\square$

Exercise 2 (5.7.3)**.** *Deduce Descartes's theorem from Exercise 5.7.2.*

Solution.

If $p(a) = 0$ Then ...

- $(p(x) - p(a))/(x - a) = p(x)/(x - a) = C + A + S + O + N = K$.

Thus $p(x)$, with value 0 when $x = a$, has a factor $(x - a)$.

As the largest degree present in K, found in C, is $k - 1$, we are left with a polynomial of degree $k - 1$ when dividing the polynomial $(p(x) - p(a)) = p(x)$ of degree k by $(x - a)$. □

Recall

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This property gives an easy way to calculate Pascal's triangle to any depth, and hence compute a fair division of stakes in a game that has to be called off with n plays remaining. We suppose that players I and II have an equal chance of winning each play, and that I needs to win k of the remaining n plays to carry off the stakes.

Exercise 3 (5.8.2)**.** *Show that the ratio of I's winning the stakes to that of II's winning is*

$$\binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{k} : \binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0}.$$

Solution.

We must assume that if player I does not win, then Player II wins . . .

We can then think of all results in terms of Player I's outcome.

If Player I wins i times, where $k \leq i \leq n$, then Player I wins the stakes.

If Player I wins j times, where $0 \leq j < k$, then Player II wins the stakes.

Thus there is $\displaystyle\sum_{i=k}^{n} \binom{n}{i}$ ways for Player I to win the stakes.

Similarly there is $\displaystyle\sum_{j=0}^{k-1} \binom{n}{j}$ ways for Player II to win the stakes.

Our ratio, $P_{I_{wins\ stakes}} : P_{II_{wins\ stakes}}$, is then $\displaystyle\sum_{i=k}^{n} \binom{n}{i} : \sum_{j=0}^{k-1} \binom{n}{j}$, as asked to be shown. □

The sum property of the binomial coefficients also explains the presence of some interesting numbers in Pascal's triangle.

Exercise 4 (5.8.3)**.** *Explain why the third diagonal from the left in the triangle, namely* $1, 3, 6, 10, 15, 21, \ldots$*, consists of the triangular numbers.*

Solution.

Note first that the triangular numbers take the form, $T_t = \sum_{i=1}^{t} i$.

Now note that this diagonal referenced represents the binomial coefficients $\binom{n}{2}$ such that $n \geq 2$.

Considering an arbitrany n, we have $\binom{n}{2} = \binom{n-1}{1} + \binom{n-1}{2}$.

This process is then repeated until we arrive at $\binom{n}{2} = \binom{n-1}{1} + \binom{n-2}{1} + \cdots + \binom{3}{1} + \binom{2}{1} + \binom{2}{2}$.

These combinations take the integer values : $n-1, n-2, \ldots, 3, 2, 1$.

Thus we have that $\binom{n}{2} = \sum_{i=1}^{n-1} i = T_{n-1}$, hence why the diagonal consists of the triangular numbers. $\square$

Exercise 5 (5.8.4)**.** *The numbers on the next diagonal, namely* $1, 4, 10, 20, 35 \ldots$*, can be called* tetrahedral numbers*. Why is this an apt description?*

Solution.

This description is apt as these numbers take the from $TT_t = \sum_{i=1}^{t} T_i$.

e.g. $1 = 1;\ 4 = 1 + 3;\ 10 = 1 + 3 + 6;\ 20 = 1 + 3 + 6 + 10;\ 35 = 1 + 3 + 6 + 10 + 15 \ldots$

The numbers can be visualized in a similar manner to that of the triangular numbers, such that TT_t is a triangular pyramid with base T_t and each incremental level above the base, or the level below it, is the triangluar number which is index as one below, until finally reaching T_1, the single tip of the triangular pyramid.

$\square$