MTH385: History of Mathematics - Homework #8

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Cardano's Fromula : $y^3 = py + q \Rightarrow y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}.$

Exercise 1 (5.5.2)**.** *Use Cardano's formula to solve $y^3 = 2$. Do you get the obvious solution?*

Solution.

Using Cardano's Fromula ...

- $p = 0; \quad q = 2.$
- $y = \sqrt[3]{\frac{2}{2} + \sqrt{\left(\frac{2}{2}\right)^2 - \left(\frac{0}{3}\right)^3}} + \sqrt[3]{\frac{2}{2} - \sqrt{\left(\frac{2}{2}\right)^2 - \left(\frac{0}{3}\right)^3}}.$
- $y = \sqrt[3]{1 + \sqrt{1}} + \sqrt[3]{1 - \sqrt{1}}.$
- $y = \sqrt[3]{2}.$

Yes we find the obvious solution.

□

Exercise 2 (5.5.3)**.** *Use Cardano's formula to solve* $y^3 = 6y + 6$*, and check your answer by substitution.*

Solution.

Using Cardano's Fromula . . .

- $p = 6; \quad q = 6.$
- $y = \sqrt[3]{\frac{6}{2} + \sqrt{\left(\frac{6}{2}\right)^2 - \left(\frac{6}{3}\right)^3}} + \sqrt[3]{\frac{6}{2} - \sqrt{\left(\frac{6}{2}\right)^2 - \left(\frac{6}{3}\right)^3}}.$
- $y = \sqrt[3]{3 + \sqrt{(3)^2 - (2)^3}} + \sqrt[3]{3 - \sqrt{(3)^2 - (2)^3}}.$
- $y = \sqrt[3]{3 + \sqrt{9-8}} + \sqrt[3]{3 - \sqrt{9-8}} = \sqrt[3]{3 + \sqrt{1}} + \sqrt[3]{3 - \sqrt{1}}.$
- $y = \sqrt[3]{4} + \sqrt[3]{2}.$

Checking by substitution.

- $y^3 = (\sqrt[3]{4} + \sqrt[3]{2})^3 = 4 + 3(\sqrt[3]{4})^2(\sqrt[3]{2}) + 3(\sqrt[3]{4})(\sqrt[3]{2})^2 + 2.$
- $y^3 = 6 + 3(\sqrt[3]{4})(\sqrt[3]{2})(\sqrt[3]{4} + \sqrt[3]{2}) = 6 + 3(\sqrt[3]{8})(\sqrt[3]{4} + \sqrt[3]{2}).$
- $y^3 = 6 + 6(\sqrt[3]{4} + \sqrt[3]{2}) = 6 + 6y.$

Substitution checks out.

□

(1) $x = \frac{1}{2}\sqrt[n]{y + \sqrt{y^2 - 1}} + \frac{1}{2}\sqrt[n]{y - \sqrt{y^2 - 1}}.$

(2) $\sin\theta = \frac{1}{2}\sqrt[n]{\sin n\theta + i\cos n\theta} + \frac{1}{2}\sqrt[n]{\sin n\theta - i\cos n\theta}.$

(3) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

Exercise 3 (5.6.2)**.** *Use (3) and* $\sin(\alpha) = \cos(\pi/2 - \alpha)$*,* $\cos(\alpha) = \sin(\pi/2 - \alpha)$ *to show that*

$$(\sin\theta + i\cos\theta)^n = \begin{cases} \sin(n\theta) + i\cos(n\theta) & \textit{when } n = 4m+1 \\ -\sin(n\theta) - i\cos(n\theta) & \textit{when } n = 4m+3. \end{cases}$$

Solution.

- $(\sin(\theta) + i\cos(\theta))^n = (\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta))^n.$
- $= \cos n(\pi/2 - \theta) + i\sin n(\pi/2 - \theta).$
- $= \cos(n\pi/2)\cos(-n\theta) - \sin(n\pi/2)\sin(-n\theta) +$
  $i(\sin(n\pi/2)\cos(-n\theta) + \cos(n\pi/2)\sin(-n\theta)).$
- $= \cos(n\pi/2)\cos(n\theta) + \sin(n\pi/2)\sin(n\theta) +$
  $i(\sin(n\pi/2)\cos(n\theta) - \cos(n\pi/2)\sin(n\theta)).$

Just to be clear with the proceeding, note that . . .

- $\cos(2m\pi + \pi/2) = \cos(2m\pi + 3\pi/2) = 0\ ;\ \forall m \in \mathbb{Z}$
- $\sin(2m\pi + \pi/2) = -\sin(2m\pi + 3\pi/2) = 1\ ;\ \forall m \in \mathbb{Z}$

When $n = 4m + 1$

- $\cos(2m\pi + \pi/2)\cos(n\theta) + \sin(2m\pi + \pi/2)\sin(n\theta) +$
  $i(\sin(2m\pi + \pi/2)\cos(n\theta) - \cos(2m\pi + \pi/2)\sin(n\theta)).$
- $(\sin(\theta) + i\cos(\theta))^{4m+1} = \sin(n\theta) + i\cos(n\theta).$

When $n = 4m + 3$

- $\cos(2m\pi + 3\pi/2)\cos(n\theta) + \sin(2m\pi + 3\pi/2)\sin(n\theta) + i(\sin(2m\pi + 3\pi/2)\cos(n\theta) - \cos(2m\pi + 3\pi/2)\sin(n\theta))$.
- $(\sin(\theta) + i\cos(\theta))^{4m+3} = -\sin(n\theta) - i\cos(n\theta)$.

Thus shown.

$\square$

Exercise 4 (5.6.3)**.** *Deduce from Exercise 5.6.2 that (2) is correct for* $n = 4m + 1$ *and false for* $n = 4m + 3$*, and hence that (1) is a correct relation between* $y = \sin(n\theta)$ *and* $x = \sin(\theta)$ *only when* $n = 4m + 1$*.*

Solution.

When $n = 4m + 1 \ldots$

- $\frac{1}{2}\sqrt[n]{\sin(n\theta) + i\cos(n\theta)} + \frac{1}{2}\sqrt[n]{\sin(n\theta) - i\cos(n\theta)}.$
$$= \frac{1}{2}\big(\sin(n\theta) + i\cos(n\theta)\big)^{1/n} + \frac{1}{2}\big(\sin(n\theta) - i\cos(n\theta)\big)^{1/n}.$$
$$= \frac{1}{2}\Big(\big(\sin(\theta) + i\cos(\theta)\big)^{n/n} + \big(\sin(n\theta) + i\cos(n\theta)\big)^{1/-n}\Big).$$
$$= \frac{1}{2}\Big(\sin(\theta) + i\cos(\theta) + \big(\sin(\theta) + i\cos(\theta)\big)^{n/-n}\Big).$$
$$= \frac{1}{2}\big(\sin(\theta) + i\cos(\theta) + \sin(\theta) - i\cos(\theta)\big) = \frac{1}{2}\big(2\sin(\theta)\big).$$
$$= \sin(\theta).$$

When $n = 4m + 3 \ldots$

- $\frac{1}{2}\sqrt[n]{\sin(n\theta) + i\cos(n\theta)} + \frac{1}{2}\sqrt[n]{\sin(n\theta) - i\cos(n\theta)}.$
$$= \frac{1}{2}\big(\sin(n\theta) + i\cos(n\theta)\big)^{1/n} + \frac{1}{2}\big(\sin(n\theta) - i\cos(n\theta)\big)^{1/n}.$$
$$= -\frac{1}{2}\Big(\big(-\sin(n\theta) - i\cos(n\theta)\big)^{1/n} + \big(-\sin(n\theta) - i\cos(n\theta)\big)^{1/-n}\Big).$$
$$= -\frac{1}{2}\Big(\big(\sin(\theta) + i\cos(\theta)\big)^{n/n} + \big(\sin(\theta) + i\cos(\theta)\big)^{n/-n}\Big).$$
$$= -\frac{1}{2}\big(\sin(\theta) + i\cos(\theta) + \sin(\theta) - i\cos(\theta)\big) = -\frac{1}{2}\big(2\sin(\theta)\big).$$
$$= -\sin(\theta).$$

When $y = \sin(n\theta)$ and $x = \sin(\theta)$ we have . . .

- $\sin(\theta) = \frac{1}{2}\sqrt[n]{\sin(n\theta) + \sqrt{\sin^2(n\theta) - 1}} + \frac{1}{2}\sqrt[n]{\sin(n\theta) - \sqrt{\sin^2(n\theta) - 1}}.$

$$= \frac{1}{2}\sqrt[n]{\sin(n\theta) + \sqrt{-\left(1 - \sin^2(n\theta)\right)}} + \frac{1}{2}\sqrt[n]{\sin(n\theta) - \sqrt{-\left(1 - \sin^2(n\theta)\right)}}.$$

$$= \frac{1}{2}\sqrt[n]{\sin(n\theta) + \sqrt{-\cos^2(n\theta)}} + \frac{1}{2}\sqrt[n]{\sin(n\theta) - \sqrt{-\cos^2(n\theta)}}.$$

$$= \frac{1}{2}\sqrt[n]{\sin(n\theta) - i\cos(n\theta)} + \frac{1}{2}\sqrt[n]{\sin(n\theta) - i\cos(n\theta)}.$$

From above this is only $\sin(\theta)$ when $n = 4m + 1$ and thus *(1)* is a correct relation between

$y = \sin(n\theta)$ and $x = \sin(\theta)$ only when $n = 4m + 1$.

$\square$

Exercise 5 (5.6.4)**.** *Show that (1) is a correct relation between* $y = \cos(n\theta)$ *and* $x = \cos(\theta)$ *for all* n *(de Moivre (1730)).*

Solution.

When $y = \cos(n\theta)$ and $x = \cos(\theta)$ we have . . .

- $$\begin{aligned}\cos(\theta) &= \frac{1}{2}\sqrt[n]{\cos(n\theta) + \sqrt{\cos^2(n\theta) - 1}} + \frac{1}{2}\sqrt[n]{\cos(n\theta) - \sqrt{\cos^2(n\theta) - 1}}. \\ &= \frac{1}{2}\sqrt[n]{\cos(n\theta) + \sqrt{-\sin^2(n\theta)}} + \frac{1}{2}\sqrt[n]{\cos(n\theta) - \sqrt{-\sin^2(n\theta)}}. \\ &= \frac{1}{2}\sqrt[n]{\cos(n\theta) + i\sin(n\theta)} + \frac{1}{2}\sqrt[n]{\cos(n\theta) - i\sin(n\theta)}. \\ &= \frac{1}{2}\Big(\big(\cos(n\theta) + i\sin(n\theta)\big)^{1/n} + \big(\cos(n\theta) - i\sin(n\theta)\big)^{1/n}\Big). \\ &= \frac{1}{2}\Big(\big(\cos(\theta) + i\sin(\theta)\big)^{n/n} + \big(\cos(n\theta) + i\sin(n\theta)\big)^{n/-n}\Big). \\ &= \frac{1}{2}\big(\cos(\theta) + i\sin(\theta) + \cos(n\theta) - i\sin(n\theta)\big) = \frac{1}{2}\big(2\cos(\theta)\big). \\ &= \cos(\theta).\end{aligned}$$

This is sufficient for all n by *de Moivre's formula*.

$\square$