# **MTH385**: History of Mathematics - Homework #4

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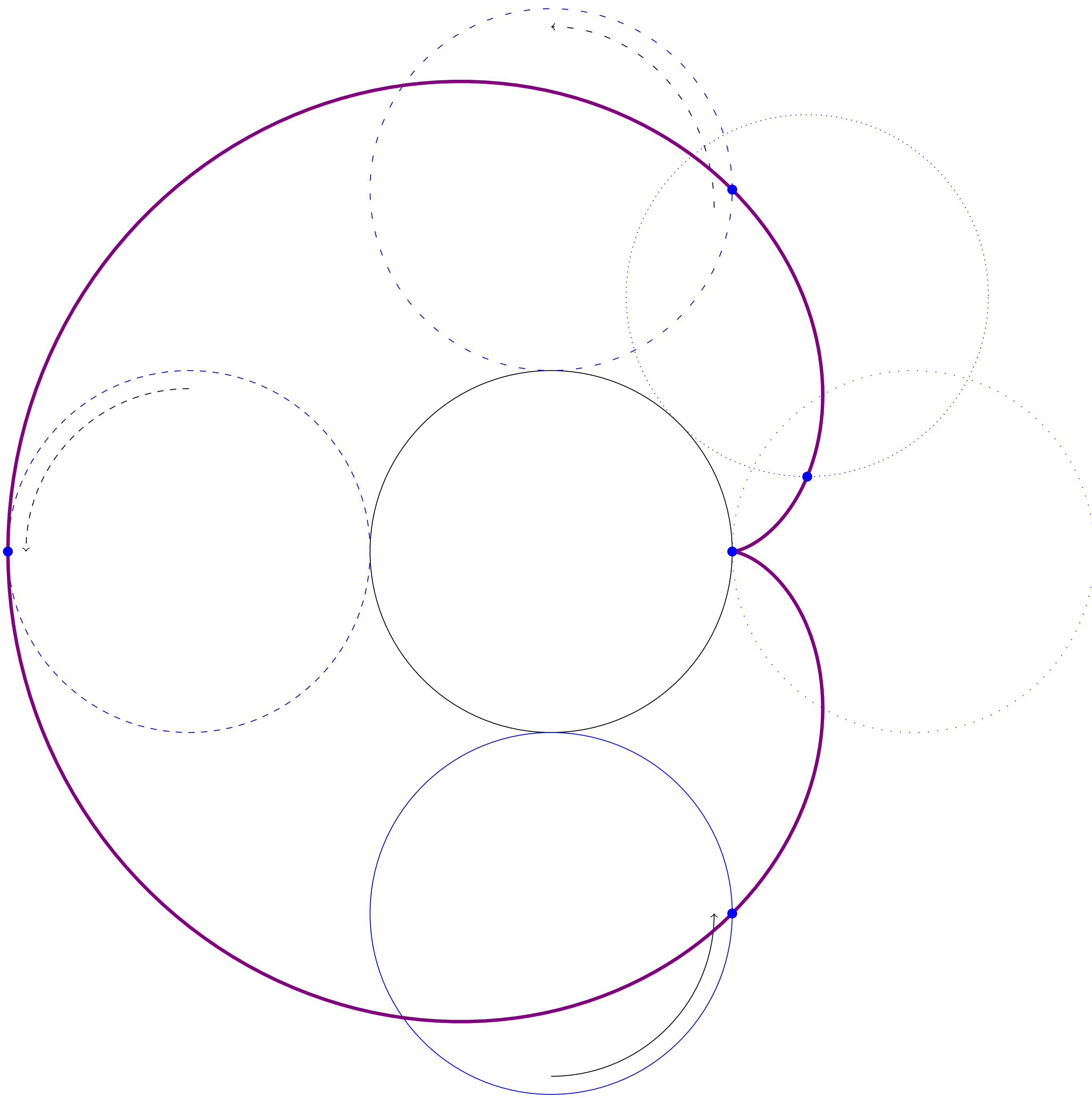
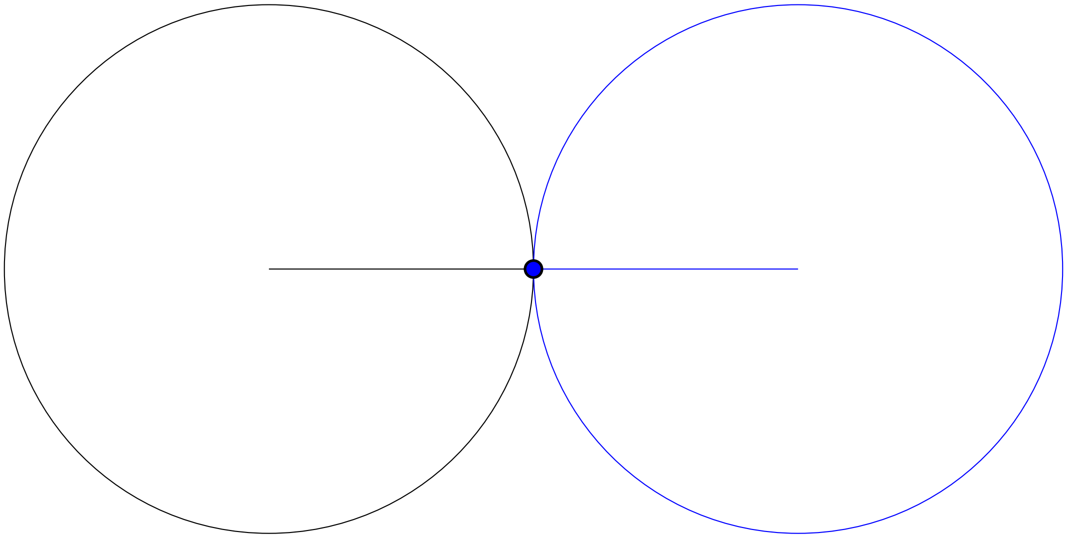
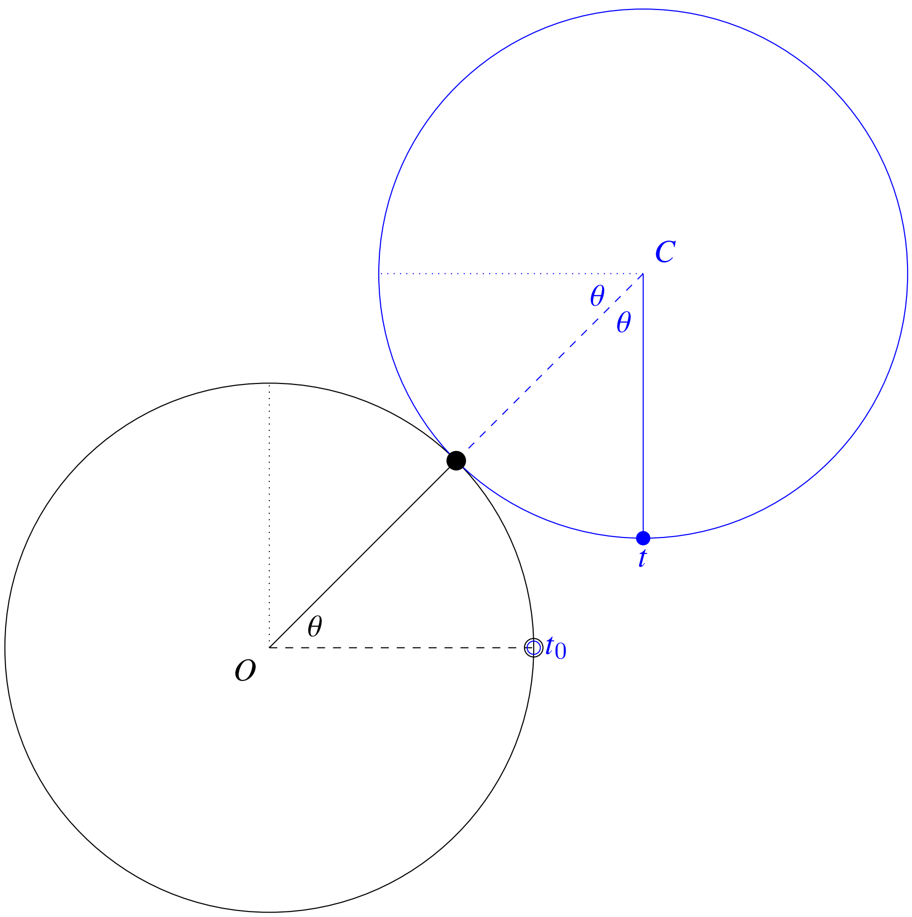
The simplest epicyclic curve is the *cardioid* (“heart-shape”), which results from a circle rolling on a fixed circle of the same size.



Cardioid



Cardioid construction $\theta = 0$




Cardioid construction $\theta = \pi/4$

**Exercise 1** (2.5.4)**.** *Show that if both circles have radius* $1$*, and we follow the point on the rolling circle initially at* $(1,0)$*, then the cardioid it traces out has parametric equations*

$$x = 2\cos\theta - \cos 2\theta,$$
$$y = 2\sin\theta - \sin 2\theta.$$

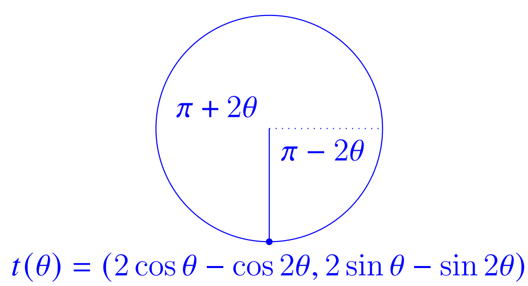
*Solution.*

From the above two examples we can see that the distance between the origin of the two circles will always be $2$ as it is the sum of their radii.

- The point $C$ will thus always be at $2(\cos(\theta), \sin(\theta))$.
- The "tracer" of the cardioid is the blue dot and will always be at $C - (\cos(\pi + 2\theta), \sin(\pi + 2\theta))$.
- We now have an equation for $t$ : $t = (2\cos(\theta) - \cos(\pi + 2\theta), 2\sin(\theta) - \sin(\pi + 2\theta))$.
- Decomposing we have $x = 2\cos(\theta) - \cos(\pi + 2\theta)$ and $y = 2\sin(\theta) - \sin(\pi + 2\theta)$.

We will now simply using sum and difference identities to arrive at the parametric equations.

- $\cos(\pi + 2\theta) = \cos(\pi)\cos(2\theta) - \sin(\pi)\sin(2\theta) = \cos(2\theta)$.
- $\sin(\pi + 2\theta) = \sin(\pi)\cos(2\theta) - \cos(\pi)\sin(2\theta) = \sin(2\theta)$.
- $x = 2\cos\theta - \cos 2\theta$.
- $y = 2\sin\theta - \sin 2\theta$.




$t(\pi/4)$

□

The cardioid is an algebraic curve. Its cartesian equation may be hard to discover, but it is easy to verify, especially if one has a computer algebra system.

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**Exercise 2** (2.5.5)**.** *Check that the point $(x, y)$ on the cardioid satisfies*

$$(x^2 + y^2 - 1)^2 = 4((x - 1)^2 + y^2).$$

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*Solution.*

We will use substitution for this exercise

- $(x^2 + y^2 - 1)^2 = 4((x - 1)^2 + y^2)$.
- $x^2 = (2\cos\theta - \cos 2\theta)^2 = 4\cos^2\theta - 4\cos\theta\cos 2\theta + \cos^2 2\theta$.
- $y^2 = (2\sin\theta - \sin 2\theta)^2 = 4\sin^2\theta - 4\sin\theta\sin 2\theta + \sin^2 2\theta$.
- $x^2 + y^2 - 1 = 4\cos^2\theta + 4\sin^2\theta - 4\cos\theta\cos 2\theta - 4\sin\theta\sin 2\theta + \cos^2 2\theta + \sin^2 2\theta - 1$.
  $= 4 - 4(\cos(\theta - 2\theta)) = 4 - 4\cos\theta$.
- $(x^2 + y^2 - 1)^2 = 16\cos^2\theta - 32\cos\theta + 16$.
- $(x - 1)^2 = (2\cos\theta - \cos 2\theta - 1)^2$.
  $= 4\cos^2\theta + \cos^2 2\theta - 4\cos\theta\cos 2\theta - 4\cos\theta + 2\cos 2\theta + 1$.
- $(x - 1)^2 + y^2 =$
  $4\cos^2\theta + 4\sin^2\theta + \cos^2 2\theta + \sin^2 2\theta + 1 - 4\cos\theta\cos 2\theta - 4\sin\theta\sin 2\theta - 4\cos\theta + 2\cos 2\theta$.
  $= 6 - 4\cos\theta - 4\cos\theta + 2\cos 2\theta$.
  $= 6 - 8\cos\theta + 2(2\cos^2\theta - 1)$.
  $= 4\cos^2\theta - 8\cos\theta + 4$.
- $4((x - 1)^2 + y^2) = 16\cos^2\theta - 32\cos\theta + 16$ .

We can now see that any point satisfying our parametric equations also satisfies the algebragic equality.

□

**Exercise 3** (3.2.3)**.** *Show that the $k^{th}$ pentagonal number is* $\dfrac{3k^2-k}{2}$.

*Solution.*

We have the $k^{th}$ pentagonal number represented as the sum : $pent_k = \sum_{i=1}^{k} 3i - 2$.

From the summation we notice that $pent_{k+1} = pent_k + 3(k+1) - 2 = pent_k + 3k + 1$.

Similarly, $pent_{k+2} = pent_{k+1} + 3(k+2) - 2 = pent_k + 3k + 1 + 3(k+2) - 2 = pent_k + 6k + 5$.

Thus if the equality holds we have show that $\dfrac{3k^2-k}{2}$ is a closed form for $pent_k$.

- For base $pent_0 + 3(0) + 1 = pent_1$.
  $\dfrac{3(0)^2-0}{2} + 3(0) + 1 = 0 + 0 + 1 = 1.$
- For base $pent_0 + 6(0) + 5 = pent_2$.
  $\dfrac{3(0)^2-0}{2} + 6(0) + 5 = 0 + 0 + 5 = 5.$
- To show $pent_k + 3k + 1 = pent_{k+1}$.
  $\dfrac{3k^2-k}{2} + 3k + 1 = \dfrac{3k^2+6k-k+2}{2} = \dfrac{(3k^2+6k-3)+(-k-1)}{2} = \dfrac{3(k+1)^2-(k+1)}{2}.$
- To show $pent_k + 6k + 5 = pent_{k+2}$.
  $\dfrac{3k^2-k}{2} + 6k + 5 = \dfrac{3k^2+12k-k+10}{2} = \dfrac{(3k^2+12k-12)+(-k-2)}{2} = \dfrac{3(k+2)^2-(k+2)}{2}.$

Thus induction shows that the $k^{th}$ pentagonal number is $\dfrac{3k^2-k}{2}$.

□

**Exercise 4** (3.2.4)**.** *Show that each square is the sum of two consecutive triangular numbers.*

*Solution.*

We have the $k^{th}$ triangular number represented as the sum : $tri_k = \sum_{i=1}^{k} i$.

The $k^{th}$ triangular number has the well know closed form : $tri_k = \frac{k(k+1)}{2}$.

We have the $l^{th}$ square number represented as the sum : $sq_l = \sum_{j=1}^{l} 2l - 1$.

The $l^{th}$ square number has the well know closed form : $sq_l = l^2$.

- We now consider two consecutive triangular numbers, $tri_k$ and $tri_{k+1}$.

  $tri_k + tri_{k+1} = \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{k^2 + k + k^2 + 2k + k + 2}{2}$.

  $= \frac{2k^2 + 4k + 2}{2} = k^2 + 2k + 1 = (k+1)^2$.

- considering $l = k + 1$

  $l^2 = tri_{l-1} + tri_l \; ; \; \forall l \geq 1$.

We have shown that each square, $sq_l$, is the sum of the coresponding consecutive triangular numbers, $tri_{l-1}$, and $tri_l$.

□

Euclid's theorem about perfect numbers depends on the prime divisor property, which will be proved in the next section. Assuming this for the moment, it follows that if $2^n - 1$ is a prime $p$, then the proper divisors of $2^{n-1}p$ (those unequal to $2^{n-1}p$ itself) are ...

$$1, 2, 2^2, \ldots, 2^{n-1} \text{ and } p, 2p, 2^2p, \ldots, 2^{n-2}p.$$

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**Exercise 5** (3.2.5)**.** *Given that the divisors of $2^{n-1}p$ are those just listed, show that $2^{n-1}p$ is perfect when $p = 2^n - 1$ is prime.*

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*Solution.*

We must show that the sum of the proper divisors of $2^{n-1}p$ is equal to $2^{n-1}p$, let $\Sigma$ denote this sum.

We will let $q = 2^{n-1}p$ ...

- $\Sigma = \Sigma_1 + \Sigma_2 = (2^0 + 2^1 + \cdots + 2^{n-1}) + p(2^0 + 2^1 + \cdots + 2^{n-2})$.
- $\Sigma + q = \Sigma_1 + \Sigma_2 + q = (2^0 + 2^1 + \cdots + 2^{n-1}) + p(2^0 + 2^1 + \cdots + 2^{n-1}) = \Sigma_1 + p\Sigma_1$.
- $\Sigma_1 + p\Sigma_1 = \Sigma_1(1 + p) = \Sigma_1(1 + 2^n - 1) = 2^n\Sigma_1$.
- As $\Sigma_1$ is a geometric progression, $\Sigma_1 = 2^n - 1 = p$.
- Thus $\Sigma + q = 2^n p = 2(2^{n-1}p) = 2q$.
- Finally $\Sigma = q$.

We have now shown that $q = 2^{n-1}p$ is perfect.

$\square$