MTH 385 Homework due 2022-04-04

Like the binomial theorem, the multinomial theorem can be proved combinatorially by considering the number of ways a term $a_1^{q_1}a_2^{q_2}\cdots a_n^{q_n}$ can arise from the factors of $(a_1+a_2+\cdots+a_n)^p$.

Exercise 1 (5.9.4 rewritten)**.** *Prove the formula for the multinomial coefficient*

$$\binom{p}{q_1,q_2,\ldots,q_n}=\frac{p!}{q_1!q_2!\cdots q_n!}$$

by observing that the coefficient equals the number of ways of writing a p-element set as a disjoint union of subsets of sizes $q_1,q_2,\ldots,q_n$.

As we now know, all conic sections may be given by the following standard form equations (from Section 2.4):

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ (ellipse)},\qquad y=ax^2 \text{ (parabola)},\qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ (hyperbola)}.$$

The reduction of an arbitrary quadratic equation in x and y to one of these forms depends on suitable choice of origin and axes, as Fermat and Descartes discovered. The main steps are outlined in the following exercises.

Exercise 2 (6.2.1)**.** *Show that a* quadratic form $ax^2+bxy+cy^2$ *may be converted to a form $a'x'^2+b'y'^2$ by suitable choice of θ in the substitution*

$$\begin{aligned}x&=x'\cos\theta-y'\sin\theta,\\ y&=x'\sin\theta+y'\cos\theta,\end{aligned}$$

by checking that the coefficient of $x'y'$ is $(c-a)\sin 2\theta+b\cos 2\theta$.

Exercise 3 (6.2.2)**.** *Deduce from Exercise 6.2.1 that, by suitable rotation of axes, any quadratic curve may be expressed in the form $a'x'^2+by'^2+c'x'+d'y'+e'$.*

Exercise 4 (6.2.3)**.** *If $b'=0$, but $a'\neq 0$, show that the substitution $x'=x''+f$ gives either standard-form parabola, or the "double line" $x''^2=0$.*
(Why is this called a "double line," and is it a section of a cone?)

Exercise 5 (6.2.4)**.** *If both a' and b' are nonzero, show that a shift of origin gives the standard form for either an ellipse or a hyperbola, or else a pair of lines.*