MTH 385 Homework due 2022-04-18

The equation relating the series for $\frac{\pi}{4}$ to the continued fraction for $\frac{4}{\pi}$, namely

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\cfrac{1}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cdots}}}}}$$

follows immediately from a more general equation

$$\frac{1}{A}-\frac{1}{B}+\frac{1}{C}-\frac{1}{D}+\cdots=\cfrac{1}{A+\cfrac{A^2}{B-A+\cfrac{B^2}{C-B+\cfrac{C^2}{D-C+\cdots}}}}$$

proved by Euler (1748a), p. 311. The following exercises give a proof of Euler's result.

Exercise 1 (8.4.3)**.** *Check that*

$$\frac{1}{A}-\frac{1}{B}=\cfrac{1}{A+\cfrac{A^2}{B-A}}$$

Exercise 2 (8.4.4)**.** *When $\frac{1}{B}$ on the left side in Exercise 8.4.3 is replaced by $\frac{1}{B}-\frac{1}{C}$, which equals $\frac{1}{B-\frac{B^2}{C-B}}$ by Exercise 8.4.3, show that B on the right should be replaced by $B+\frac{B^2}{C-B}$. Hence show that*

$$\frac{1}{A}-\frac{1}{B}+\frac{1}{C}-\frac{1}{D}+\cdots=\cfrac{1}{A+\cfrac{A^2}{B-A+\cfrac{B^2}{C-B}}}$$

Thus when we modify the tail end of the series (replacing $\frac{1}{B}$ by by $\frac{1}{B}-\frac{1}{C}$), only the tail end of the continued fraction is affected. This situation continues:

Exercise 3 (8.4.5)**.** *Generalize your argument in Exercise 8.4.4 to obtain a continued fraction for a series with n terms, and hence prove Euler's equation.*

Exercise 8.5.2 shows why the inverse function $x = e^y - 1$ has a power series that begins

$$y+\frac{1}{2}y^2+\frac{1}{6}y^3+\cdots.$$

Exercise 4 (8.5.3)**.** *Show that the binomial series gives*

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots .$$

Exercise 5 (8.5.4)**.** *Use Exercise 8.5.3 and* $\sin^{-1} x = \int_0^x dt/\sqrt{1-t^2}$ *to derive Newton's series for* $\sin^{-1} x$.