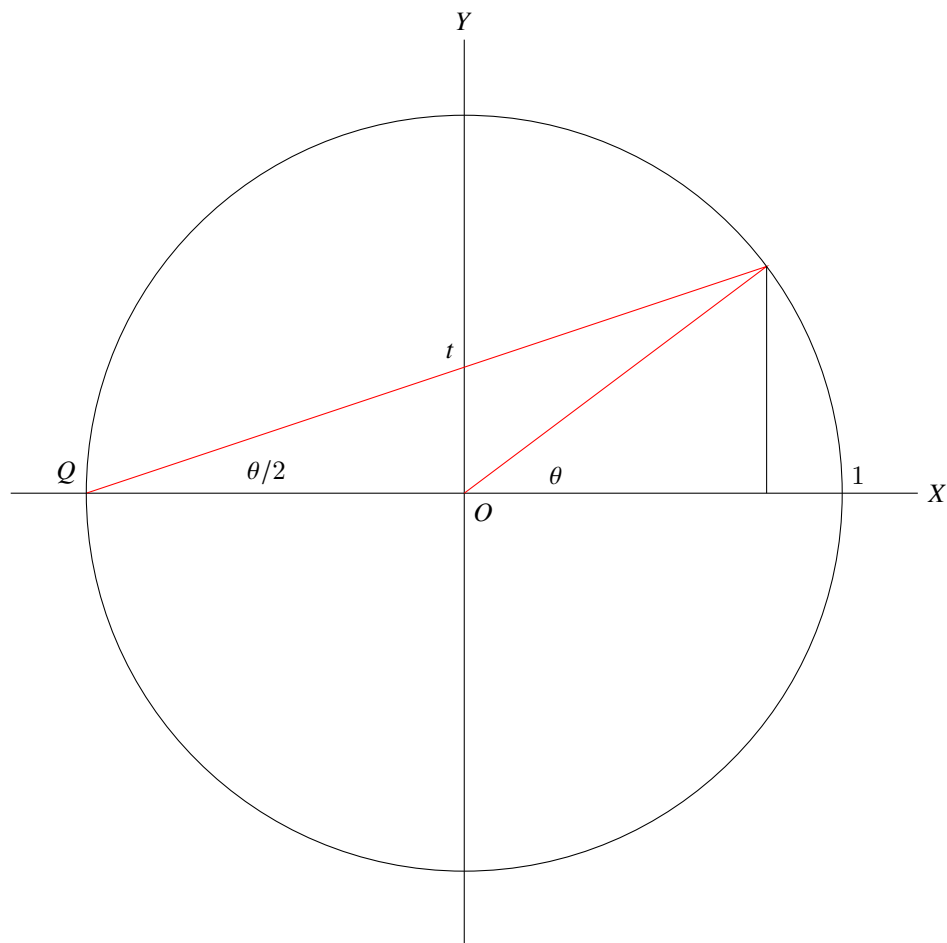


### Supplementary Material

- The parameter  $t$  in the pair  $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$  runs through all rational numbers if  $t = q/p$  and  $p, q$  run through all pairs of integers.
- Some important trigonometry may be gleaned from Diophantus's method if we compare the angle at  $O$  with the angle at  $Q$  in the following figure. High school geometry shows that the angle at  $Q$  is half the angle at  $O$ .



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**Exercises Follow**

**Exercise 1** (1.2.3). Show that any integer square leaves remainder 0 or 1 on division by 4.

*Solution.*

Integers take 2 flavors, even and odd. We will approach this by cases.

- Even numbers take the form  $2k$ ,  $k \in \mathbb{Z}$ .

$$(2k)^2 = 4k^2$$

$$4k^2 \% 4 = 0$$

- Odd numbers take the form  $2k + 1$ ,  $k \in \mathbb{Z}$ .

$$(2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

$$(4(k^2 + k) + 1) \% 4 = ((4(k^2 + k) \% 4) + (1 \% 4)) \% 4 = (0 + 1) \% 4 = 1$$

As these 2 flavors encompass all possible integers we have shown that on division by 4 the integer squares takes remainder 0, or 1. □

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**Exercise 2** (1.2.4). Deduce from Exercise 1.2.3 that if  $(a, b, c)$  is a Pythagorean triple then  $a$  and  $b$  cannot both be odd.

*Solution.*

To satisfy integer squares we must take  $a$  and  $b$  such that their sum will leave remainder 0 or 1 on division by 4.

- consider two integers of odd flavor.

$$2i + 1, 2j + 1$$

- consider their squares.

$$4(i^2 + i) + 1, 4(j^2 + j) + 1$$

- consider their sum.

$$4(i^2 + i) + 1 + 4(j^2 + j) + 1 = 4(i^2 + j^2 + i + j) + 2$$

- consider their remainder on division by 4.

$$4(i^2 + j^2 + i + j) + 2 \% 4 = ((4(i^2 + j^2 + i + j) \% 4) + (2 \% 4)) \% 4 = (0 + 2) \% 4 = 2$$

We can see their sum takes remainder 2 on division by 4. From this we know that  $c$  is not an integer and thus any  $a$  and  $b$  odd will not produce a Pythagorean triple. □

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**Exercise 3** (1.3.1). Deduce that if  $(a, b, c)$  is any Pythagorean triple then

$$\frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2}, \quad \frac{b}{c} = \frac{2pq}{p^2 + q^2}$$

for some integers  $p$  and  $q$ .

*Solution.*

We know that any Pythagorean triple,  $(a, b, c)$ , will satisfy  $a^2 + b^2 = c^2$ .

- We can divide this equation by  $c^2$  to arrive at the following.

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \text{ and } \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1, \text{ a Pythagorean triple with hypotenuse length 1.}$$

- We can now leverage our pair  $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ , as it takes the same form.

$$\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \frac{1-2t^2+t^4+4t^4}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1$$

- We can now equate our two pairs as  $1 = 1$ .

$$\frac{(1-t^2)^2}{(1+t^2)^2} + \frac{(2t)^2}{(1+t^2)^2} = 1 = \frac{(a)^2}{(c)^2} + \frac{(b)^2}{(c)^2}$$

- We thus have come to the following realization (We arbitrarily choose this pair, it could have been reversed).

$$\frac{(a)^2}{(c)^2} = \frac{(1-t^2)^2}{(1+t^2)^2}, \quad \frac{(b)^2}{(c)^2} = \frac{(2t)^2}{(1+t^2)^2}$$

- For both pairs we can take the root of both sides and substitute in  $t = \frac{q}{p}$

$$\frac{a}{c} = \frac{1-t^2}{1+t^2} = \frac{1-\frac{q^2}{p^2}}{1+\frac{q^2}{p^2}}, \quad \frac{b}{c} = \frac{2t}{1+t^2} = \frac{2\frac{q}{p}}{1+\frac{q^2}{p^2}}$$

- A multiplication by  $1 = \frac{p^2}{p^2}$  will bring us to our final form.

$$\frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2}, \quad \frac{b}{c} = \frac{2pq}{p^2 + q^2}$$

☐

**Exercise 4 (1.3.2).** Use Exercise 1.3.1 to prove Euclid's formula for Pythagorean triples, assuming  $b$  even. (Remember,  $a$  and  $b$  are not both odd.)

*Solution.*

From 1.3.1,  $a = p^2 - q^2$ ,  $b = 2pq$ ,  $c = p^2 + q^2$

- Let us verify that Euclid's formula yields Pythagorean triples.

- Consider the squares.

$$a^2 = (p^2 - q^2)^2 = p^4 - 2p^2q^2 + q^4$$

$$b^2 = (2pq)^2 = 4p^2q^2$$

$$c^2 = (p^2 + q^2)^2 = p^4 + 2p^2q^2 + q^4$$

- Now check that this holds under the Pythagorean theorem  $a^2 + b^2 = c^2$ .

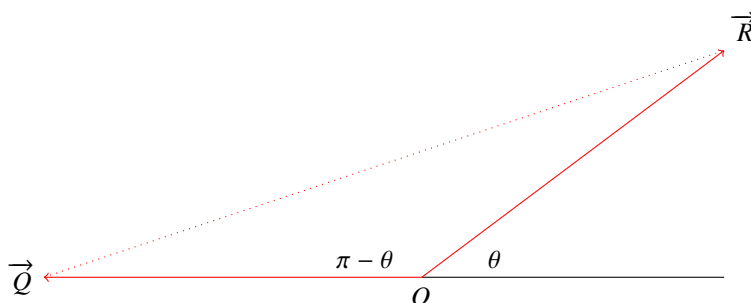
$$p^4 - 2p^2q^2 + q^4 + 4p^2q^2 = p^4 + (4p^2q^2 - 2p^2q^2) + q^4 = p^4 + 2p^2q^2 + q^4$$

☐

**Exercise 5 (1.3.4).** Why does the angle at  $Q$  equal  $\theta/2$ ? (Hint: consider angles in the red triangle.)

*Solution.*

Consider the following figure.



- Consider the length of the vectors in the figure.

$$|\vec{Q}| = |\vec{R}| = 1, \text{ as they are points on the unit circle.}$$

The Triangle  $PQR$  is isosceles.

- We thus know the angles  $\angle ORQ$  and  $\angle OQR$  are equal.

$$\text{let } \angle ORQ, \angle OQR = \omega$$

- We also know that the interior angles of a triangle sum to  $\pi$

$$\omega + \omega + \pi - \theta = \pi$$

$$2\omega = \theta$$

$$\omega = \theta/2$$

High school geometry has shown that the angle at  $Q$  is half the angle at  $O$ .

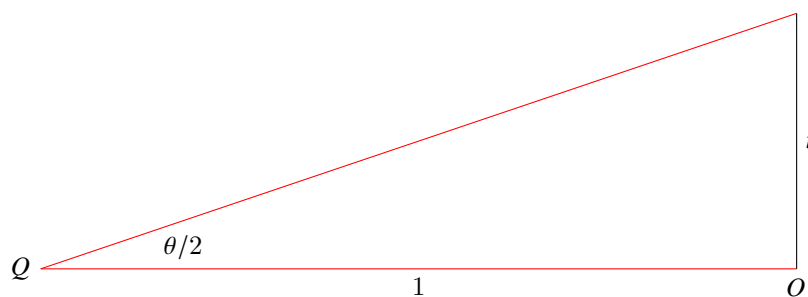
□

**Exercise 6** (1.3.5). Use Figure 1.7 to show that  $t = \tan \theta/2$  and

$$\cos \theta = \frac{1 - t^2}{1 + t^2}, \quad \sin \theta = \frac{2t}{1 + t^2}.$$

*Solution.*

Consider the following figure.



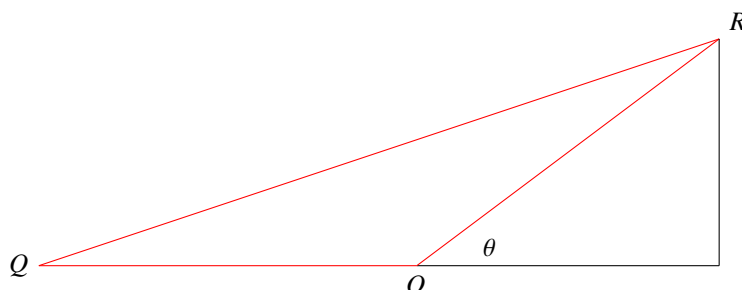
- We know that for some angle  $\omega$ ,  $\tan \omega = \frac{\sin \omega}{\cos \omega}$ .

$$\tan \theta/2 = \frac{\sin \theta/2}{\cos \theta/2} = \frac{t}{1} = t$$

- We additionally know slope is  $\frac{\text{rise}}{\text{run}}$ .

From above we have solve the slope for this triangle as  $\tan \theta/2 = t$

Consider next the following figure.



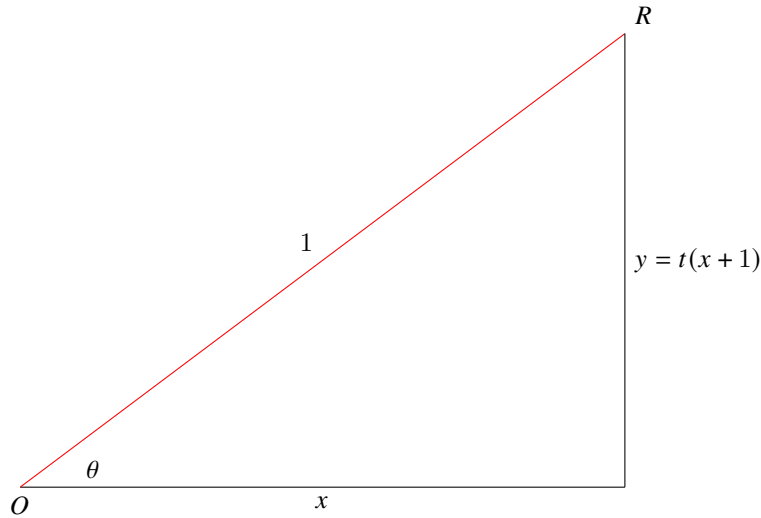
- Point-Slope form for the equation of a line is as follows:  $y - y_0 = m(x - x_0)$ .

The coordinates of  $R$  are unknown, the coordinates of  $Q$  can be seen as  $\langle -1, 0 \rangle$

- As we now have some coordinates,  $\langle x_0, y_0 \rangle$  and know our slope, we can solve for the equation of the line  $QR$ .

$$y - 0 = t(x - (-1)) ; y = t(x + 1)$$

We can now leverage the following figure.



- Let us leverage the Pythagorean theorem,  $a^2 + b^2 = c^2$

$$x^2 + (t(x + 1))^2 = 1^2$$

$$x^2 + t^2(x^2 + 2x + 1) = x^2 + t^2x^2 + 2xt^2 + t^2 = 1$$

- We are looking for the value of  $x$  as  $x = \cos \theta$

$$x^2 + t^2x^2 + t^22x + t^2 - 1 = 0 = x^2(1 + t^2) + 2xt^2 + (t^2 - 1)$$

- We will now leverage the quadratic equation to solve for  $x$ ,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .

$$x = \frac{-(2t^2) \pm \sqrt{(2t^2)^2 - 4(1 + t^2)(t^2 - 1)}}{2(1 + t^2)}$$

$$x = \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^4 - 1)}}{2(1 + t^2)} = \frac{-2t^2 \pm \sqrt{4}}{2(1 + t^2)}$$

- Consider the case of subtraction.

$$x = \frac{-2t^2 - 2}{2(t^2 + 1)} = \frac{-2(t^2 + 1)}{2(t^2 + 1)} = -1$$

This solution is trivial and can be identified as point  $Q$  from our earlier figure.

- Consider the case of addition.

$$x = \frac{-2t^2 + 2}{2(1 + t^2)} = \frac{2(1 - t^2)}{2(1 + t^2)} = \frac{1 - t^2}{1 + t^2} = \cos \theta$$

- We can now substitute our value of  $x$  into  $y = t(x + 1)$  to solve for  $y$ , a.k.a  $\sin \theta$

$$y = t\left(\frac{1 - t^2}{1 + t^2} + 1\right) = \frac{t}{1 + t^2}(1 - t^2 + 1 + t^2) = \frac{2t}{1 + t^2} = \sin \theta$$

It has been shown that  $\tan \theta/2 = t$ ,  $\cos \theta = \frac{1 - t^2}{1 + t^2}$ , and  $\sin \theta = \frac{2t}{1 + t^2}$

□