Comments Related to Binomial Coefficients, Differences, and Integer-Valued Polynomials

The question addressed by these comments is: Given a sequence of real numbers

$$(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, \ldots)$$

can we find a polynomial $f(x)$ such that $a_n = f(n)$ for all n?

[We will use a technique inspired by the study of integer-valued polynomials. So, taking $(a_n)_{n=0}^{\infty}$ to be integers is fine.]

Exercise 1. *Let $f(x) = 2x^2 - 3x + 5$. Find the first five terms of the sequence $(f(n))_{n=0}^{\infty}$.*

Let $f(x)$ be a polynomial. Define a new polynomial $\Delta f(x)$ by

$$\Delta f(x) = f(x+1) - f(x).$$

We write $\Delta^2 f(x) = \Delta(\Delta f(x))$, $\Delta^3 f(x) = \Delta(\Delta^2 f(x))$, and so on.

Exercise 2. *Find $\Delta(2x^2 - 3x + 5)$ and $\Delta^2(2x^2 - 3x + 5)$.*

Exercise 3. *Check that if $f(x)$ is a polynomial of positive degree d, $\Delta f(x)$ is a polynomial of degree $d - 1$.*

Exercise 4. *Check that if $f(x)$ is a polynomial of degree d, $\Delta^{d+1} f(x) = 0$.*

Exercise 5. *Check that if $f(x)$ and $g(x)$ are polynomials and c is a real number, then*

1. $\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x)$*;*

2. $\Delta c f(x) = c\Delta f(x)$*.*

Define the polynomials $p_k(x)$, for $k = 0, 1, 2, \ldots$ by $p_0(x) = 1$ and

$$p_k(x) = \binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$$

for $k > 1$. Notice that if n is a nonnegative integer then

$$p_k(n) = \begin{cases} 0, & \text{for } 0 \le n \le k-1 \\ \binom{n}{k}, & \text{for } n \ge k \end{cases}.$$

People sometimes write $p_k(x) = \binom{x}{k}$.

Exercise 6. *Check that if $\Delta p_k(x)$ is $p_{k-1}(x)$.*

Given a sequence $(a_n)_{n=0}^{\infty}$, define a new sequence $(b_n)_{n=0}^{\infty} = \Delta(a_n)_{n=0}^{\infty}$ by

$$b_n = a_{n+1} - a_n$$

We write $\Delta^2(a_n)_{n=0}^{\infty} = \Delta(\Delta(a_n)_{n=0}^{\infty})$, $\Delta^3(a_n)_{n=0}^{\infty} = \Delta(\Delta^2(a_n)_{n=0}^{\infty})$, and so on.

Exercise 7. *Check that $\Delta(f(n))_{n=0}^{\infty} = (\Delta f(n))_{n=0}^{\infty}$. That is the difference of a sequence given by a polynomial is given by the difference of the polynomial.*

Exercise 8. *Compute*

(a) *the first five terms of the sequence* $\Delta(0, 1, 4, 10, 20, 35, \ldots)$*;*

(b) *the first four terms of the sequence* $\Delta^2(0, 1, 4, 10, 20, 35, \ldots)$*;*

(c) *the first three terms of the sequence* $\Delta^3(0, 1, 4, 10, 20, 35, \ldots)$*;*

(d) *the first two terms of the sequence* $\Delta^4(0, 1, 4, 10, 20, 35, \ldots)$.

Assume $(0, 1, 4, 10, 20, 35, \ldots)$ is given by a polynomial $f(x)$, then the degree $f(x)$ is at least 3.

Exercise 9. *Assume $(0, 1, 4, 10, 20, 35, \ldots)$ is given by a polynomial $f(x)$. What is the minimum possible degree of $f(x)$?*

Assume $f(x)$ has that minimum degree.

Exercise 10. *Check the following.*

(a) $\Delta^3 f(x) = p_0(x)$

(b) $\Delta^2 f(x) = p_1(x) + 2p_0(x)$

(c) $\Delta f(x) = p_2(x) + 2p_1(x) + p_0(x)$

(d) $f(x) = p_3(x) + 2p_2(x) + p_1(x) + 0p_0(x)$

Exercise 11. *How are the coefficients on the $p_k(x)$ in the previous exercise determined?*

Notice that, given a sequence $(a_n)_{n=0}^{\infty}$ and a positive integer d, this procedure gives a way of finding a polynomial $f(x)$ of degree at most d such that $a_n = f(n)$ when such a polynomial exists. Furthermore, $f(x)$ is a linear combination of the $p_k(x)$ when the sequence consists of integers.