MTH 385 Homework due 2022-02-14

The simplest epicyclic curve is the *cardioid* (“heart-shape”), which results from a circle rolling on a fixed circle of the same size.

Exercise 1 (2.5.4)**.** *Show that if both circles have radius* 1*, and we follow the point on the rolling circle initially at* $(1, 0)$*, then the cardioid it traces out has parametric equations*

$$\begin{aligned} x &= 2\cos\theta - \cos 2\theta, \\ y &= 2\sin\theta - \sin 2\theta. \end{aligned}$$

The cardioid is an algebraic curve. Its cartesian equation may be hard to dis- cover, but it is easy to verify, especially if one has a computer algebra system.

Exercise 2 (2.5.5)**.** *Check that the point* (x, y) *on the cardioid satisfies*

$$(x^2 + y^2 - 1)^2 = 4((x - 1)^2 + y^2).$$

Exercise 3 (3.2.3)**.** *Show that the* k*th pentagonal number is* $(3k^2 - k)/2$*.*

Exercise 4 (3.2.4)**.** *Show that each square is the sum of two consecutive triangular numbers.*

Euclid’s theorem about perfect numbers depends on the prime divisor property, which will be proved in the next section. Assuming this for the moment, it follows that if $2^n - 1$ is a prime p, then the proper divisors of $2^{n-1}p$ (those unequal to $2^{n-1}p$ itself) are

$$1, 2, 2^2, \ldots, 2^{n-1} \text{ and } p, 2p, 2^2p, \ldots, 2^{n-2}p.$$

Exercise 5 (3.2.5)**.** *Given that the divisors of* $2^{n-1}p$ *are those just listed, show that* $2^{n-1}p$ *is perfect when* $p = 2^n - 1$ *is prime.*