MTH375: History of Mathematics - Homework #2

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Exercise 1 (1.5.2)**.** *Show that the square of* $2q + 1$ *is in fact of the form* $4s + 1$*, and hence explain why every integer square leaves remainder* 0 *or* 1 *on division by* 4*.*

Solution.
Note: $\%$ is used as the modulus operator.

- Taking the square of $2q + 1$ we have the following . . .
  $(2q + 1)^2 = 4q^2 + 4q + 1 = 4(q^2 + q) + 1$
- Letting $s = q^2 + q$ we have the form $4s + 1$.
- Considering q integer, we have two cases to equate.
- For q $even$ we have $q = 2k$, $k \in \mathbb{Z}$.
  $q^2 = (2k)^2 = 4k^2$.
  $q^2 \ \% \ 4 = 4k^2 \ \% \ 4 = 0$.
- For q odd we have $q = 2k + 1$, $k \in \mathbb{Z}$.
  $q^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.
  $q^2 \ \% \ 4 = (4(k^2 + k) + 1) \ \% \ 4 = ((4(k^2 + k) \ \% \ 4) + (1 \ \% \ 4)) \ \% \ 4 = (0 + 1) \ \% \ 4 = 1$.
- Hence every integer square leaves remainder 0 or 1 on division by 4.

□

Exercise 2 (2.1.1)**.** *Explain how Common Notions 1 and 4 may be interpreted as the transitive and reflexive properties. Note that the natural way to write Common Notion 1 symbolically is slightly different from the statement of transitivity above.*

Solution.

- *Common Notions* 1 says : Things which are equal to the same thing are also equal to one another.

  From this notion we will take two things equal to the same thing, $a = b$ and $c = b$.

  It can be seen that from the notion as both a and c are equal to the same thing, b, then a and c are equal to one another.

  This can be interpreted as the *transitive* property: $a \cong b,\ b \cong c \Rightarrow a \cong c$.

- *Common Notions* 4 says : Things which coincide with one another are equal to one another.

  Having two things coincide can be thought of as two points taking the same coordinates, or two lines/angles laid on oneanother.

  We conclude from this notion that these two objects are equal to one another, of which they are then the same base object.

  Taking the points as an example, we have $p_1 \cong p_2 \cong p$ and $p_2 \cong p_1 \cong p$ where p is the base object.

  The reflexive property follows nicely as $p \cong p$.

$\square$

Exercise 3 (2.1.2)**.** *Show that the symmetric property follows from Euclid's Common Notions 1 and 4.*

Solution.

- From above we have $a \cong b$ and $a \cong a$

  By the *reflexive* property, $a \cong a \cong b$

  As $a \cong b$, $a \cong b \cong a \cong b$

  Since $b \cong b$ and $a \cong a$, we see that $b \cong b \cong a \cong a$

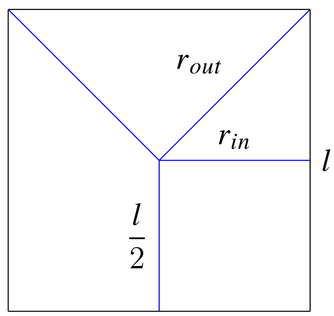
  It then follows that $b \cong a$.

- We have thus shown that the *symmetric* property, $a \cong b \Rightarrow b \cong a$, follows as as result of *Euclid's Common Notions* 1 and 4.

$\square$

Exercise 4 (2.2.1)**.** *Show that* $\dfrac{circumradius}{inradius} = \sqrt{3}$ *for both the cube and the octahedron.*

Solution.
Consider the following figure.



- We will utilize the cross sectional square of the cube, with side length l, shown above.

  From basic geometry we can see that $r_{in} = \dfrac{l}{2}$.

- We can also see that r_{out} is the hypotenuse of the tiangle with base and height r_{in}.

  From the Pythagorean Theorem, $r_{out}^2 = r_{in}^2 + r_{in}^2$

  Thus $r_{out}^2 = \dfrac{l^2}{4} + \dfrac{l^2}{4} = \dfrac{l^2}{2}$

  We can now see that $r_{out} = \dfrac{l}{\sqrt{2}}$

- Consider the unit cube as a means to find the ratio between r_{circum} and r_{out}.

- r_{out} is the distance from $(0, 0, 0)$ to $(1, 1, 0)$.

  This distance d_{out} is simply $\sqrt{(1-0)^2 + (1-0)^2 + (0-0)^2} = \sqrt{2}$.

- r_{circum} is the distance from $(0, 0, 0)$ to $(1, 1, 1)$.

  This distance d_{circum} is simply $\sqrt{(1-0)^2 + (1-0)^2 + (1-0)^2} = \sqrt{3}$.

- Thus we have $d_{circum} = \dfrac{d_{out}\sqrt{3}}{\sqrt{2}}$.

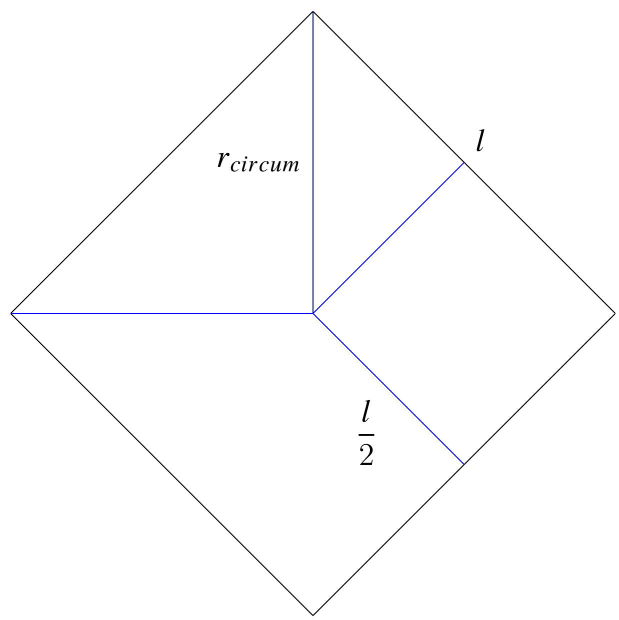
  It follows that $r_{circum} = \dfrac{r_{out}\sqrt{3}}{\sqrt{2}} = \dfrac{\frac{l}{\sqrt{2}}\sqrt{3}}{\sqrt{2}} = \dfrac{l\sqrt{3}}{2}$.

- We now have acquired the relevant information to compute $\dfrac{\text{circumradius}}{\text{inradius}}$.

  $\dfrac{r_{circum}}{r_{in}} = \dfrac{\frac{l}{2}\sqrt{3}}{\frac{l}{2}} = \sqrt{3}$.

- We have show that $\dfrac{\text{circumradius}}{\text{inradius}} = \sqrt{3}$ for the cube.

Consider next the following figure.



- We will utilize the cross sectional square of the octahedron, with side length l, shown above.

- We can see that l is the hypotenuse of the triangle with base and height r_{circum}.

  From the Pythagorean Theorem $l^2 = r_{circum}^2 + r_{circum}^2 = 2r_{circum}^2$.

  It follows that $r_{circum}^2 = \dfrac{l^2}{2}$ and $r_{circum} = \dfrac{l}{\sqrt{2}}$.

- We know that r_{in} can be constructed as the length of a line from the origin to the center of a face with respect to the octahedron.
- This line can be represented by the vector from $(0, 0, 0)$ to $\left(\frac{r_{circum}}{3}, \frac{r_{circum}}{3}, \frac{r_{circum}}{3}\right)$.
- We thus have the length as the distance of this vector ...

  $$d_{in} = \sqrt{\left(0 - \frac{r_{circum}}{3}\right)^2 + \left(0 - \frac{r_{circum}}{3}\right)^2 + \left(0 - \frac{r_{circum}}{3}\right)^2} = \sqrt{3\left(\frac{r_{circum}}{3}\right)^2} = \frac{r_{circum}\sqrt{3}}{3}.$$

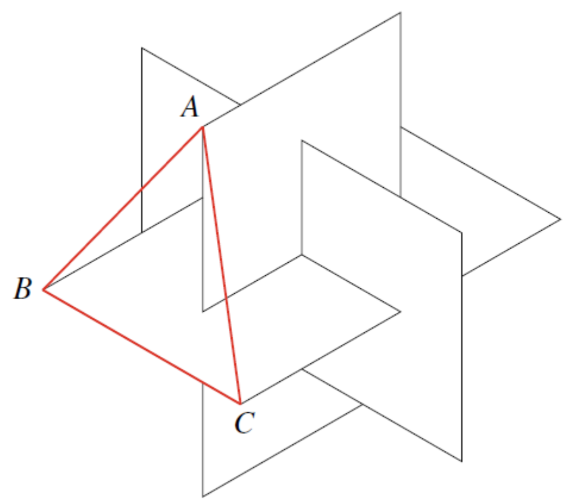
  Thus we can see that $r_{in} = \dfrac{r_{circum}}{\sqrt{3}} = \dfrac{l}{\sqrt{2}\sqrt{3}}$.
- We now have acquired the relevant information to compute $\dfrac{\text{circumradius}}{\text{inradius}}$.

  $$\frac{r_{circum}}{r_{in}} = \frac{\frac{l}{\sqrt{2}}}{\frac{l}{\sqrt{2}\sqrt{3}}} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}.$$
- We have show that $\dfrac{\text{circumradius}}{\text{inradius}} = \sqrt{3}$ for the octahedron.

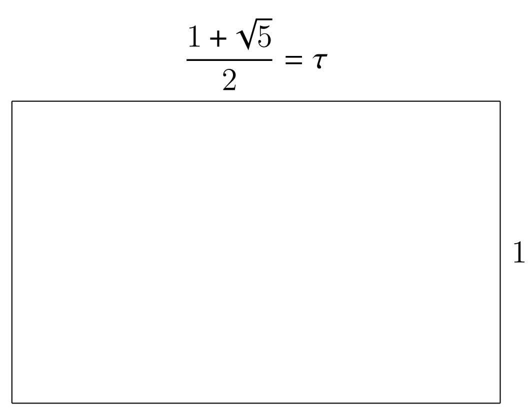
$\square$

Exercise 5 (2.2.2). *Check Pacioli's construction: use the Pythagorean theorem to show that $AB = BC = CA$ in Figure 2.2. (It may help to use the additional fact that $\tau = (1+\sqrt{5})/2$ satisfies $\tau^2 = \tau + 1$.)*

Solution.
Consider first *Pacioli's* construction of the icosahedron.



Consider the next *golden rectangle*.



- We will let the z coordinate of B and C be 0 while the y coordinant of A be 0.
- We can thus find the position of the points A, B, and C in this coordinant system.

$$A = \left(\frac{1}{2}, 0, \frac{\tau}{2}\right), \quad B = \left(\frac{\tau}{2}, -\frac{1}{2}, 0\right), \quad C = \left(\frac{\tau}{2}, \frac{1}{2}, 0\right).$$

- Thus we can now solve for AB, BC, and CA by basic distance measures.
- $$AB = \sqrt{\left(\frac{\tau}{2} - \frac{1}{2}\right)^2 + \left(-\frac{1}{2} - 0\right)^2 + \left(0 - \frac{\tau}{2}\right)^2} = \sqrt{\left(\frac{\tau - 1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{\tau}{2}\right)^2}.$$
$$= \sqrt{\left(\frac{\tau^2 - 2\tau + 1}{4}\right) + \left(\frac{1}{4}\right) + \left(\frac{\tau^2}{4}\right)} = \sqrt{\frac{\tau + 1 - 2\tau + 1}{4} + \frac{1}{4} + \frac{\tau + 1}{4}}.$$
$$= \sqrt{\frac{2 - \tau + 1 + \tau + 1}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1.$$
- $BC = 1$, by the given length of the short side of the *golden rectangle*.
- $$CA = \sqrt{\left(\frac{1}{2} - \frac{\tau}{2}\right)^2 + \left(0 - \frac{1}{2}\right)^2 + \left(\frac{\tau}{2} - 0\right)^2} = \sqrt{\left(\frac{1 - \tau}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{\tau}{2}\right)^2}.$$
$$= \sqrt{\left(\frac{\tau^2 - 2\tau + 1}{4}\right) + \left(\frac{1}{4}\right) + \left(\frac{\tau^2}{4}\right)} = \sqrt{\frac{\tau + 1 - 2\tau + 1}{4} + \frac{1}{4} + \frac{\tau + 1}{4}}.$$
$$= \sqrt{\frac{2 - \tau + 1 + \tau + 1}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1.$$
- We can now see that $AB = BC = CA = 1$ and that *Pacioli's* construction of the icosahedron holds.

$\square$