MTH385: History of Mathematics - Homework #3

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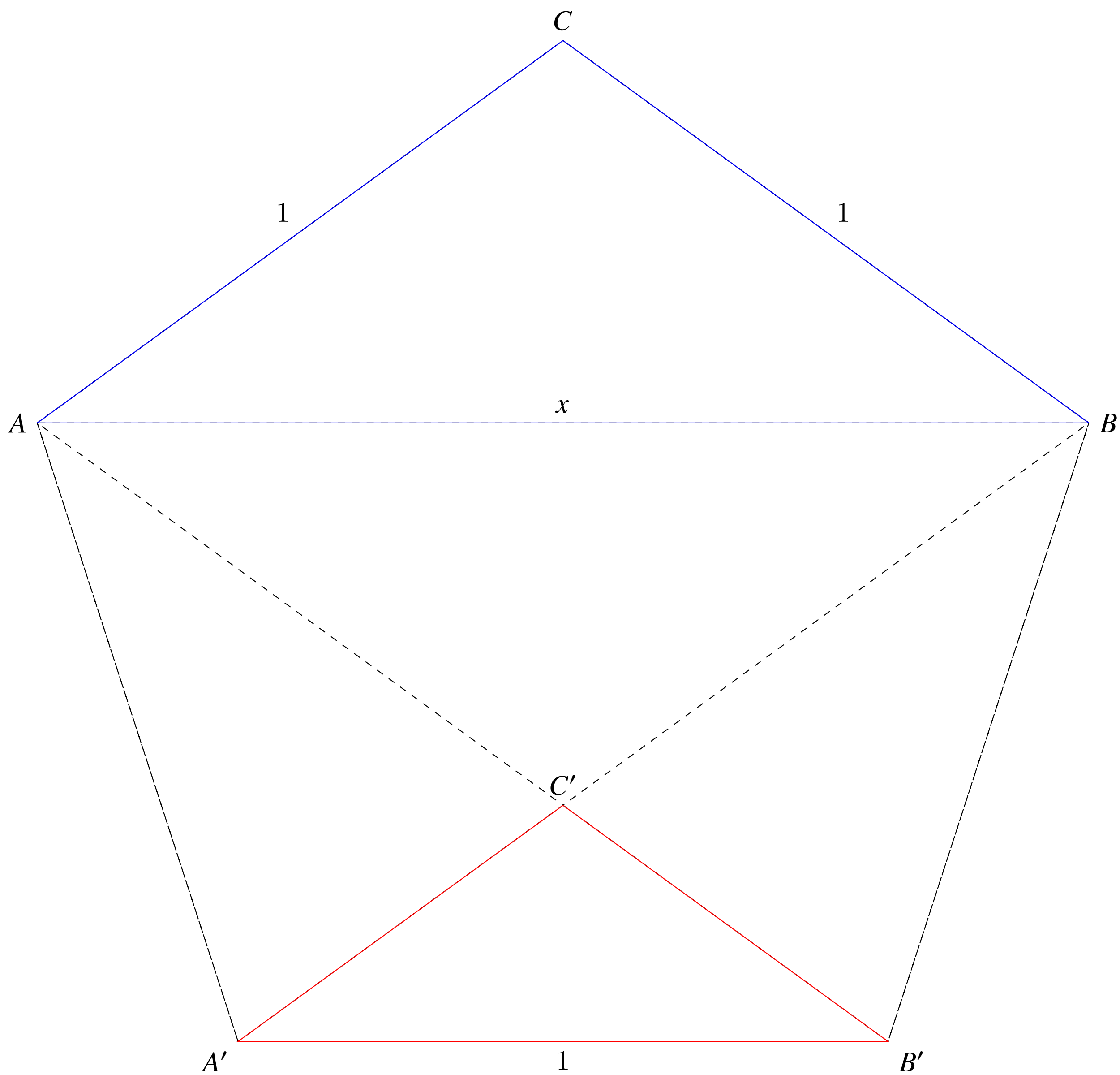
March 12, 2022

Exercise 1 (2.3.3)**.** *By finding some parallels and similar triangles in Figure 2.5, show that the diagonal x of the regular pentagon of side* 1 *satisfies* $x/1 = 1/(x-1)$.



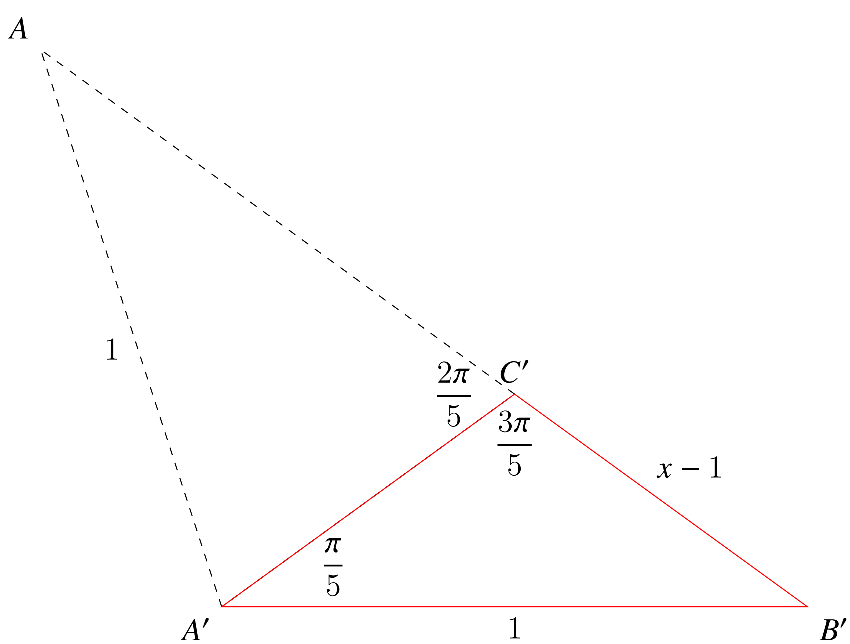
Regular pentagon

Solution.

From basic principals, the interior angle at each vertex in the regular pentagon is $\theta = \dfrac{(5-2)\pi}{5} = \dfrac{3\pi}{5}$. Notice the following . . .

- $\triangle ACB$ is isosceles.
- $\angle ACB = \theta = \dfrac{3\pi}{5}$.
- $\angle ABC = \angle BAC = \dfrac{\pi}{5}$.
- $\triangle AA'B'$ and $\triangle BB'A'$ are similar to $\triangle ACB$.
- $\angle ABC = \angle A'B'C' = \angle BAC = \angle B'A'C' = \dfrac{\pi}{5}$.
- $\angle ACB = \angle A'C'B' = \dfrac{3\pi}{5}$.
- $\triangle A'C'B'$ is isosceles.
- $\triangle ACB$ is similar to $\triangle A'C'B'$.

With this established we will now move forward . . .



Truncated pentagon

The above truncated pentagon will be useful to reference when making the following realizations . . .

- $|\overline{AA'}| = 1$ by definition.
- $|\overline{AB'}| = x$ by symmetry.
- $\angle A'C'B' + \angle A'C'A = \pi \Rightarrow \angle A'C'A = \pi - \angle A'C'B' = \pi - \frac{3\pi}{5} = \frac{2\pi}{5}$.
- $\angle AA'C' + \angle B'A'C' = \frac{3\pi}{5} \Rightarrow \angle AA'C' = \frac{3\pi}{5} - \angle B'A'C' = \frac{3\pi}{5} - \frac{\pi}{5} = \frac{2\pi}{5}$.
- $\triangle AA'C$ is isosceles.
- $|\overline{AC'}| = 1$
- $|\overline{AB'}| = |\overline{AC'}| + |\overline{C'B'}| \Rightarrow |\overline{C'B'}| = |\overline{AB'}| - |\overline{AC'}| = x - 1$

It is now clear that as $\triangle ACB$ is similar to $\triangle A'C'B'$, the ratios $\frac{|\overline{BA}|}{|\overline{BC}|}$ and $\frac{|\overline{B'A'}|}{|\overline{B'C'}|}$ are equal.

Thus as $|\overline{BA}| = x$, $|\overline{BC}| = 1$, $|\overline{B'A'}| = 1$, and $|\overline{B'C'}| = x - 1$; We can see that $\frac{x}{1} = \frac{1}{x-1}$. $\square$

Exercise 2 (2.3.4)**.** *Deduce from Exercise 2.3.3 that the diagonal of the pentagon is* $(1+\sqrt{5})/2$ *and hence that the regular pentagon is constructible.*

Solution.

This proof will leverage the ratio derived above and the quadratic equation . . .

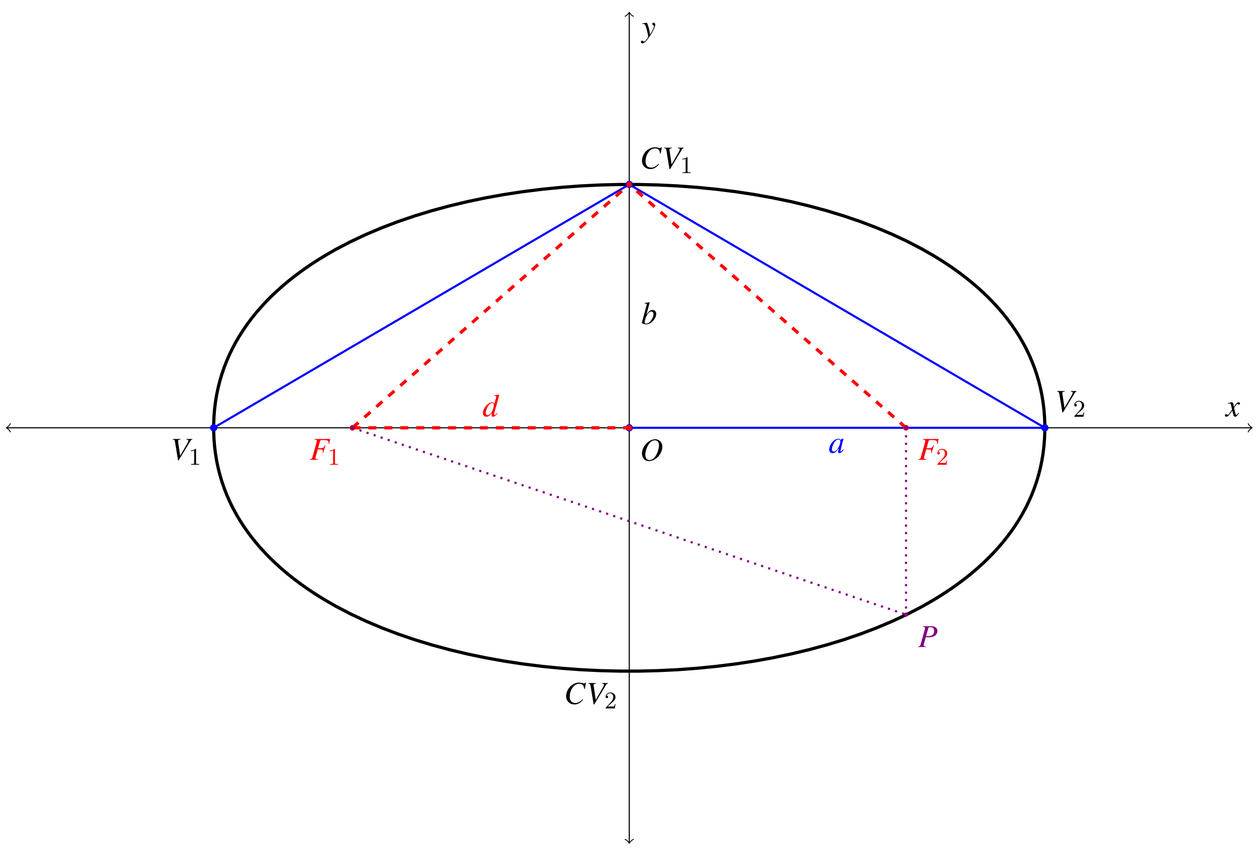
- $\dfrac{x}{1} = \dfrac{1}{x-1}$.
- $x(x-1) = x^2 - x = 1 \Rightarrow x^2 - x - 1 = 0$.
- $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$.
- $\dfrac{1-\sqrt{5}}{2} < 0$

Thus we have the diagonal of the pentagon is $x = \dfrac{1+\sqrt{5}}{2}$ and hence that the regular pentagon is constructible. $\square$

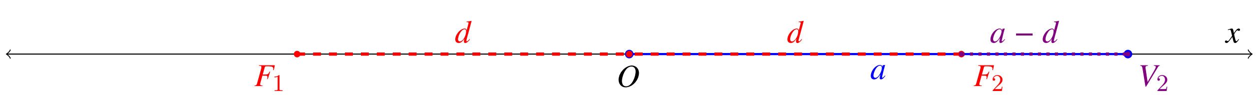
Exercise 3 (2.4.2)**.** *By introducing suitable coordinate axes, show that a curve with the above "constant sum" property indeed has an equation of the form*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(It is a good idea to start with the two square root terms, representing the distances F_1P and F_2P, on opposite sides of the equation.) Show also that any equation of this form is obtainable by suitable choice of F_1, F_2, and $F_1P + F_2P$.



Ellipse



Constant sum

Solution.

Notes:

- Variables are taken from the Ellipse and Constant sum figures shown above.
- F_1 and F_2 are the foci of the ellipse.
- V_1 and V_2 are the verticies of the ellipse.
- CV_1 and CV_2 are the co-verticies of the ellipse.
- O is the origin of our cartesian axis and the center of the ellipse.
- d is the length from the center of the ellipse to either of the two foci.
- a is the length from the center of the ellipse to either of the two verticies.
- b is the length from the center of the ellipse to either of the two co-verticies.

We will first define the length l of the path $\overline{F_1PF_2}$. As this path has constant sum for all points P, we will solve for l from the trivial case; $P = V_2$.

- $\overline{F_1V_2F_2} = \overline{F_1V_2} + \overline{V_2F_2}$.
- $\overline{F_1V_2} = d + a$.
- $\overline{V_2F_2} = a - d$.
- $\overline{F_1V_2F_2} = d + a + a - d = 2a = l$.

We now know that for any P, the path $\overline{F_1PF_2}$ will have constant length $l = 2a$. Letting $P = (x, y)$ we can leverage the distance equation to create the following equality . . .

- $l = 2a = \overline{F_1P} + \overline{F_2P} = \sqrt{(x-(-d))^2 + (y)^2} + \sqrt{(x-d)^2 + (y)^2}$.
- $2a = \sqrt{(x+d)^2 + y^2} + \sqrt{(x-d)^2 + y^2} = \sqrt{x^2 + 2dx + d^2 + y^2} + \sqrt{x^2 - 2dx + d^2 + y^2}$.

Some rearrangment allows us to bring the roots to opposite sides and simplfy our equality . . .

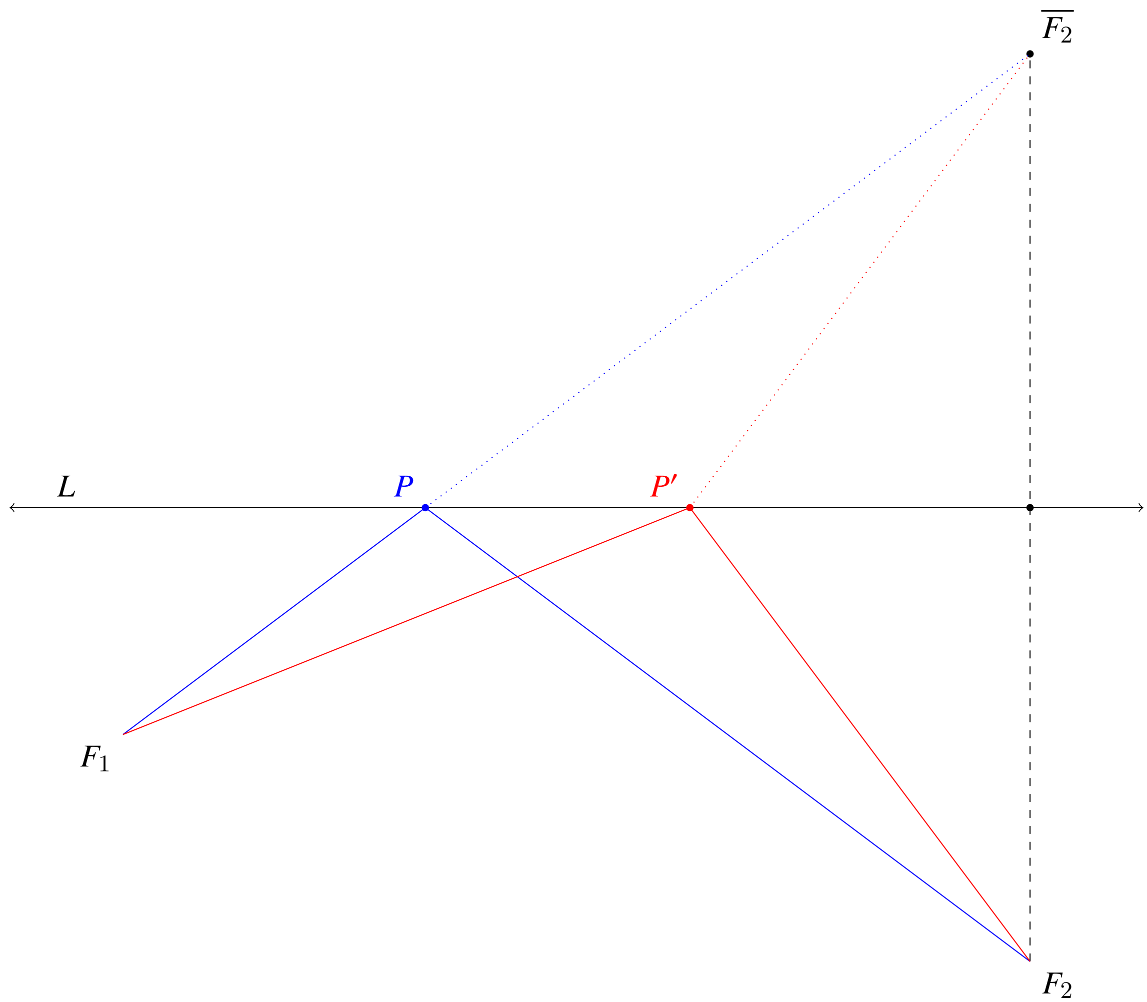
- $2a - \sqrt{x^2 - 2dx + d^2 + y^2} = \sqrt{x^2 + 2dx + d^2 + y^2}$.
- $4a^2 - 4a\sqrt{x^2 - 2dx + d^2 + y^2} + x^2 - 2dx + d^2 + y^2 = x^2 + 2dx + d^2 + y^2$.
- $4a^2 - 4a\sqrt{x^2 - 2dx + d^2 + y^2} = 4dx$.
- $a^2 - dx = a\sqrt{x^2 - 2dx + d^2 + y^2}$.
- $a^4 - 2dxa^2 + d^2x^2 = a^2(x^2 - 2dx + d^2 + y^2) = x^2a^2 - 2dxa^2 + d^2a^2 + y^2a^2$.
- $a^4 + d^2x^2 = x^2a^2 + d^2a^2 + y^2a^2$.
- $d^2x^2 - x^2a^2 = d^2a^2 + y^2a^2 - a^4$.
- $x^2(d^2 - a^2) = a^2(d^2 + y^2 - a^2)$.
- $\dfrac{x^2}{a^2} = \dfrac{d^2 + y^2 - a^2}{d^2 - a^2} = \dfrac{y^2}{d^2 - a^2} + \dfrac{d^2 - a^2}{d^2 - a^2} = \dfrac{y^2}{d^2 - a^2} + 1$.
- $\dfrac{x^2}{a^2} - \dfrac{y^2}{d^2 - a^2} = 1 = \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - d^2}$.

At this step we now consider the trivial point $P = CV_1 = (0, b)$. By the equation above we find . . .

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - d^2} = \dfrac{0^2}{a^2} + \dfrac{b^2}{a^2 - d^2} = \dfrac{b^2}{a^2 - d^2} = 1$
- $b^2 = a^2 - d^2$

It is now evident that a curve with the above “constant sum” property indeed has an equation of the aformentioned form: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. □

Another interesting property of the lines from the foci to a point P on the ellipse is that they make equal angles with the tangent at P. It follows that a light ray from F_1 to P is reflected through F_2. A simple proof of this can be based on the shortest-path property of reflection, shown in Figure 2.7 and discovered by the Greek scientist Heron around 100 CE.



Shortest-path property

Shortest-path property. The path F_1PF_2 of reflection in the line L from F_1 to F_2 is shorter than any other path $F_1P'F_2$ from F_1 to L to F_2.

Exercise 4 (2.4.3)**.** *Prove the shortest-path property, by considering the two paths F_1PF_2 and $F_1P'F_2$, where $\overline{F_2}$ is the reflection of the point F_2 in the line L.*

Solution.

It is straightforward to see that by symmetry we have similar triangles $\triangle F_1PF_2$ and $\triangle F_1P\overline{F_2}$, reguardless of where on the line L it may be that P lies. It follows directly that $|\overline{PF_2}| = |\overline{P\overline{F_2}}|$.

We wish to minimze the path length $l = |\overline{F_1PF_2}|$.

Utilizing the above information we can see . . .

- $l = |\overline{F_1PF_2}| = |\overline{F_1P}| + |\overline{PF_2}| = |\overline{F_1P}| + |\overline{P\overline{F_2}}| = |\overline{F_1P\overline{F_2}}|$.

It is now evident that the choice for P we should choose to minimze the pathlength $l = |\overline{F_1PF_2}|$ is the P that lies at the intersection of the line L and the path $|\overline{F_1\overline{F_2}}|$. Under this circumstance $|\overline{F_1P\overline{F_2}}| = |\overline{F_1\overline{F_2}}| = |\overline{F_1F_2}|$. To chose any other point P would increase the path length l as it is well known that the shortest path between any two points is the line segment that connects them.

$\square$

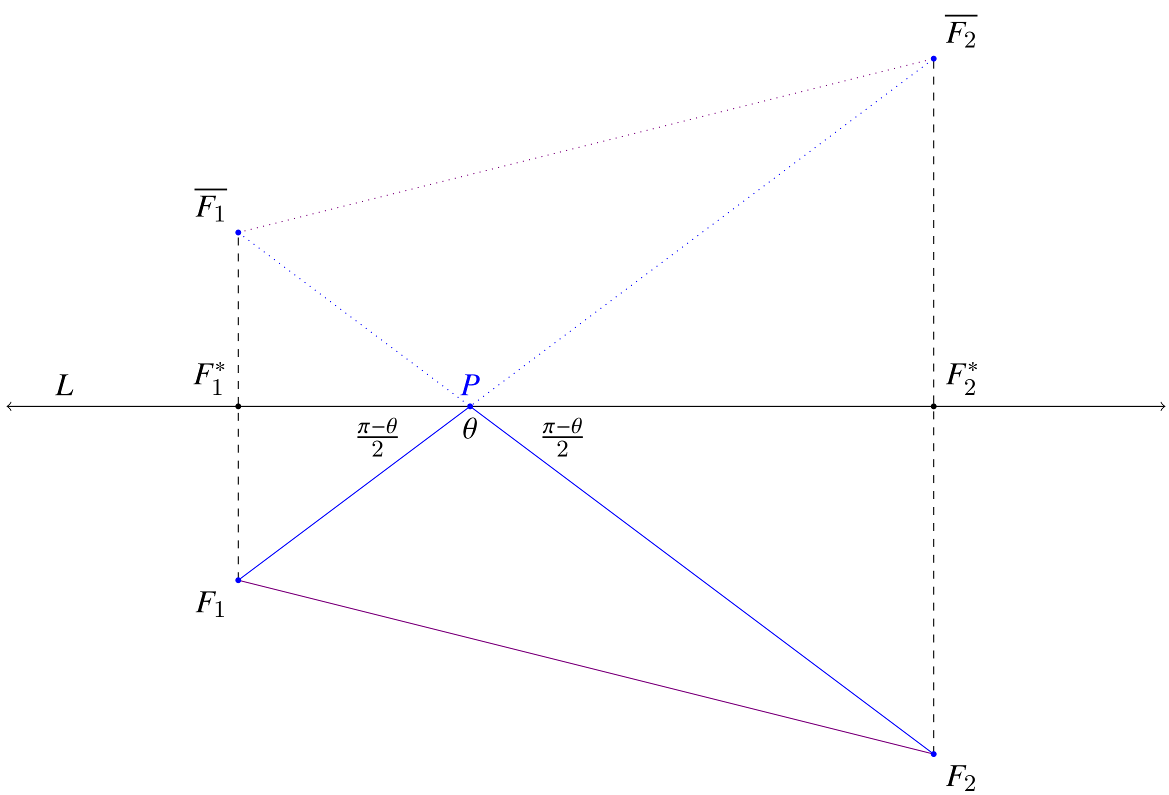
Thus to prove that the lines F_1P and F_2P make equal angles with the tangent, it is enough to show that F_1PF_2 is shorter than $F_1P'F_2$ for any other point P' on the tangent at P.

Exercise 5 (2.4.4)**.** *Prove this, using the fact that F_1PF_2 has the same length for all points P on the ellipse.*

Solution.

In excersie 3 we made uses of the trivial point $P = V_2$ to compute the constant sum path length $l = 2a$. It is relevant once again as we can clearly see that the path $\overline{F_1PF_2}$ takes the shortest possible length as all points lie on the same line. We followed to show that this path length l is the constant for any point P on the ellipse. In exercise 4 we proved that the shortest path is that which chooses P at the intersection of the line L and the line segment $\overline{F_1\overline{F_2}}$. We will now continue to show that the line L must be tangent to the shortest path at the turning point P.

Consider now a revised depiction of the shortest-path property.



Revised shortest-path property

From the above graphic we can see three pairs of similar triangles, all of which are reflections across the line L.

- $\triangle \overline{F_1} P F_1^*$ is the reflection over L of $\triangle F_1 P F_1^*$.
- $\triangle \overline{F_2} P F_2^*$ is the reflection over L of $\triangle F_2 P F_2^*$.
- $\triangle \overline{F_1} P \overline{F_2}$ is the reflection over L of $\triangle F_1 P F_2$.

Due to this symmetry we have the following equivalent angles.

- $\angle F_1 P F_1^* = \angle \overline{F_1} P F_1^*$.
- $\angle F_2 P F_2^* = \angle \overline{F_2} P F_2^*$.
- $\angle F_1 P F_1^* = \angle \overline{F_2} P F_2^* = \angle F_2 P F_2^*$.
- $\angle F_2 P F_2^* = \angle \overline{F_1} P F_1^* = \angle F_1 P F_1^*$.
- $\angle F_1 P F_2 = \angle \overline{F_1} P \overline{F_2}$.

Letting $\angle F_1 P F_2 = \theta$, we can solve for $\angle F_1 P \overline{F_1}$ and $\angle F_1 P F_1^*$.

- $\angle F_1 P \overline{F_1} = \pi - \angle F_1 P F_2 = \pi - \theta$.
- $\angle F_1 P \overline{F_1} = \angle F_1 P F_1^* + \angle \overline{F_1} P F_1^* = 2 \cdot \angle F_1 P F_1^*$.
- $2 \cdot \angle F_1 P F_1^* = \pi - \theta$.
- $\angle F_1 P F_1^* = \dfrac{\pi - \theta}{2}$.

As $\angle F_1 P F_1^* = \angle F_2 P F_2^* = \dfrac{\pi - \theta}{2}$, the line L at point P is a tangent line and will be so for all values taken on by θ.

For good measure it is worthwile to show that indeed $\angle F_1 P F_1^* + \angle F_1 P F_2 + \angle F_2 P F_2^* = \pi$.

- $\angle F_1 P F_1^* + \angle F_1 P F_2 + \angle F_2 P F_2^* = 2\angle F_1 P F_1^* + \angle F_1 P F_2$.
- $2\angle F_1 P F_1^* + \angle F_1 P F_2 = 2 \cdot \dfrac{\pi - \theta}{2} + \theta = \pi - \theta + \theta = \pi$.

We have shown that the lines $F_1 P$ and $F_2 P$ make equal angles with the line L, L is the tangent, and as only this P lies on the line segment $\overline{F_1 \overline{F_2}}$; Thus $F_1 P F_2$ is shorter than $F_1 P' F_2$ for any other point P' on the tangent at P. $\square$