MTH 385 Homework due 2022-03-28 Solutions

Exercise 1 (5.7.2)**.** *If $p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, use Exercise 5.7.1 to show that $p(x) - p(a)$ has a factor $x - a$.*

Solution. In class, we did Exercise 5.7.1. So, we know that, for all positive integers m,

$$\begin{aligned}&(x-a)(x^{m-1} + ax^{m-2} + \cdots + a^{m-2}x + a^{m-1})\\ &\qquad = (x^m - ax^{m-1}) + (ax^{m-1} - a^2x^{m-2}) + \cdots + (a^{m-2}x^2 - a^{m-1}x) + (a^{m-1}x - a^m)\\ &\qquad = x^m - a^m.\end{aligned}$$

Evidently,

$$\begin{aligned}p(x) - p(a) &= (a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0) - (a_k a^k + a_{k-1}a^{k-1} + \cdots + a_1 a + a_0)\\ &= a_k(x^k - a^k) + a_{k-1}(x^{k-1} - a^{k-1}) + \cdots + a_1(x-a)\end{aligned}$$

is divisible by $x - a$. □

Exercise 2 (5.7.3)**.** *Deduce Descartes's theorem from Exercise 5.7.2.*

Solution. Look at Exercise 5.7.2. If we suppose $p(a) = 0$, then $x - a$ is a factor of

$$p(x) - p(a) = p(x).$$

□

Recall

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This property gives an easy way to calculate Pascal's triangle to any depth, and hence compute a fair division of stakes in a game that has to be called off with n plays remaining. We suppose that players I and II have an equal chance of winning each play, and that I needs to win k of the remaining n plays to carry off the stakes.

Exercise 3 (5.8.2)**.** *Show that the ratio of I's winning the stakes to that of II's winning is*

$$\binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{k} : \binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0}.$$

Solution. Encode the possible sequences of the remaining plays as strings of the length n consisting of 1s and 2s, where the kth entry in a given string is 1 if player I won the kth play and 2 if player II won the kth play. Since each player has an equal chance of winning each play, each of the sequences of n plays is equally likely. So,the ratio of I's winning the stakes to that of II's winning is the number of our strings that contain at least k 1s divided by the number of our strings that contain fewer than k 1s. Moreover, the number of our strings that contain exactly m 1s ones is $\binom{n}{m}$ since constructing such a string is equivalent to choosing the m locations in the string to place the 1s from the possible n locations in the string. □

The sum property of the binomial coefficients also explains the presence of some interesting numbers in Pascal's triangle.

Exercise 4 (5.8.3)**.** *Explain why the third diagonal from the left in the triangle, namely* $1, 3, 6, 10, 15, 21, \ldots$, *consists of the triangular numbers.*

Solution. For the next two exercises, let the nth triangular number be

$$T_n = \left| \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{Z}^3 \,\middle|\, x, y, z \geq 0,\ x + y + z = n - 1 \right\} \right|.$$

That is,

$$T_1 = \left| \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \right| = 1, \quad T_2 = \left| \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right| = 3, \quad T_3 = \left| \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\} \right| = 6, \ldots$$

This is the number of lattice (integer coordinate) points in the triangle

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x, y, z \geq 0,\ x + y + z = n - 1 \right\}.$$

When we group the points in the triangles by decreasing z-coordinate, we see

$$T_n = \sum_{k=1}^{n} k.$$

I claim $T_n = \binom{n+1}{2}$. We will prove this by induction. We showed above that $T_1 = 1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Now, suppose $T_{m-1} = \begin{pmatrix} n \\ 2 \end{pmatrix}$ and consider T_m

$$T_m = m + T_{m-1} = \begin{pmatrix} m \\ 1 \end{pmatrix} + \begin{pmatrix} m \\ 2 \end{pmatrix} = \begin{pmatrix} m+1 \\ 2 \end{pmatrix} \qquad \left(\text{Recall } \begin{pmatrix} m \\ 1 \end{pmatrix} = m \text{ for all } m. \right)$$

□

Exercise 5 (5.8.4)**.** *The numbers on the next diagonal, namely* $1, 4, 10, 20, 35 \ldots$, *can be called* tetrahedral numbers. *Why is this an apt description?*

Solution. Consider $t_n = \binom{n+2}{3}$. Notice that when $n > 1$,

$$\begin{aligned} t_n &= \binom{n+2}{3} \\ &= \binom{n+1}{2} + \binom{n+1}{3} \\ &= T_n + t_{n-1} \\ &= \left| \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{Z}^3 \,\middle|\, x, y, z \geq 0,\ x + y + z \leq n - 1 \right\} \right|. \end{aligned}$$

This is the number of lattice points in the tetrahedron

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x, y, z \geq 0,\ x + y + z \leq n - 1 \right\}.$$

$\square$