Introduction to Community Detection in Networks via Modularity Maximization

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  - for example references, and transit paths.
  - the connections may also have attributes such as hyperlink, road, and shipping lane.

Example: Web Pages

Contact

Mailing list
Issue tracker
Source

Releases

Stable (notes)

2.8.8 — November 2022
download | doc | pdf

Latest (notes)

3.0 development
github | doc | pdf

Archive

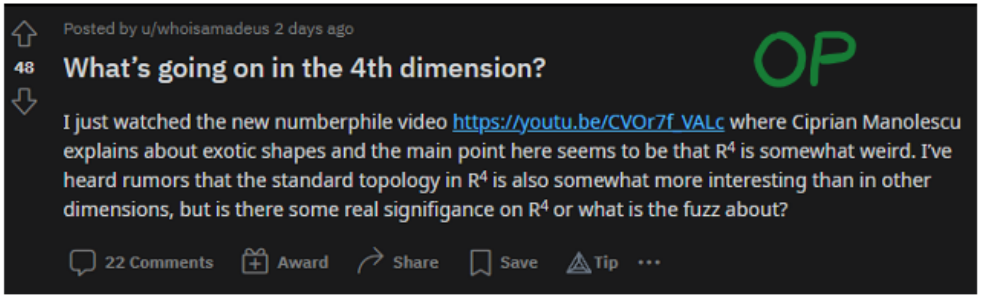
NetworkX
Network Analysis in Python

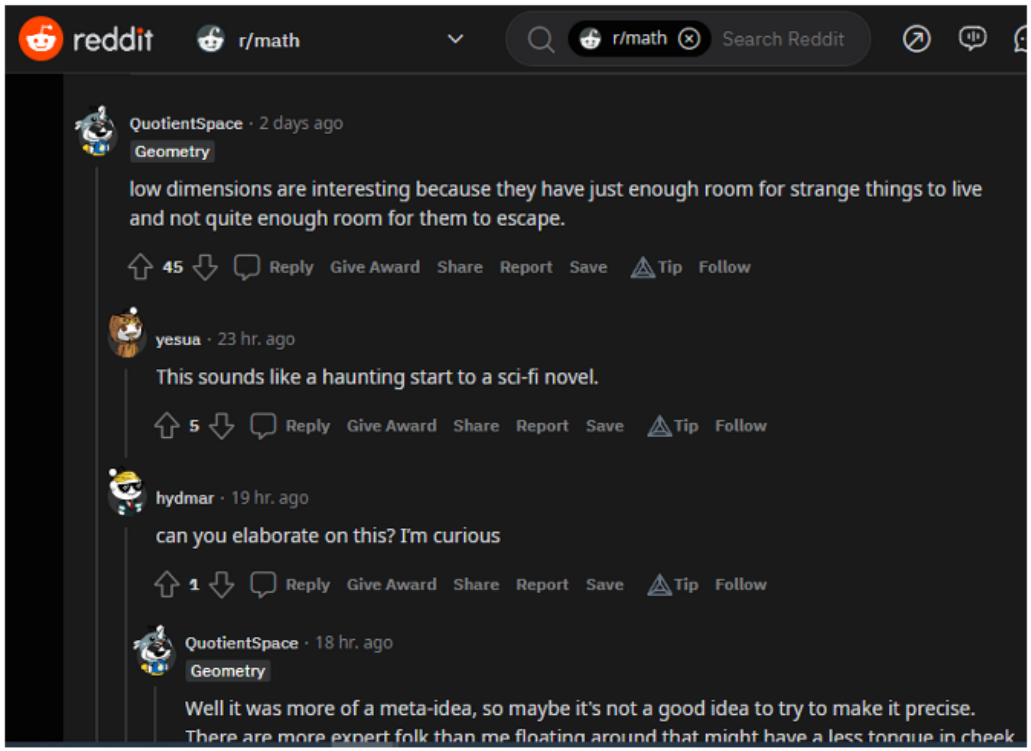
NetworkX is a Python package for the creation, manipulation, and study of the structure, dynamics, and functions of complex networks.

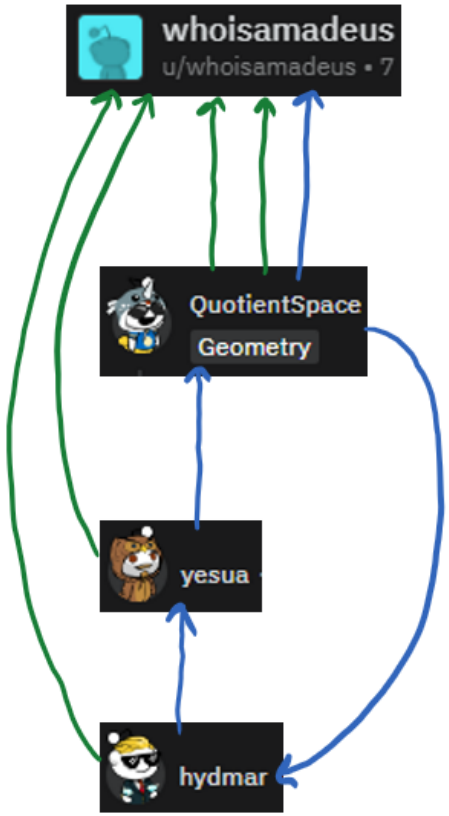
Software for complex networks

- Data structures for graphs, digraphs, and multigraphs
- Many standard graph algorithms
- Network structure and analysis measures

Example: Social Media







Example: Transit

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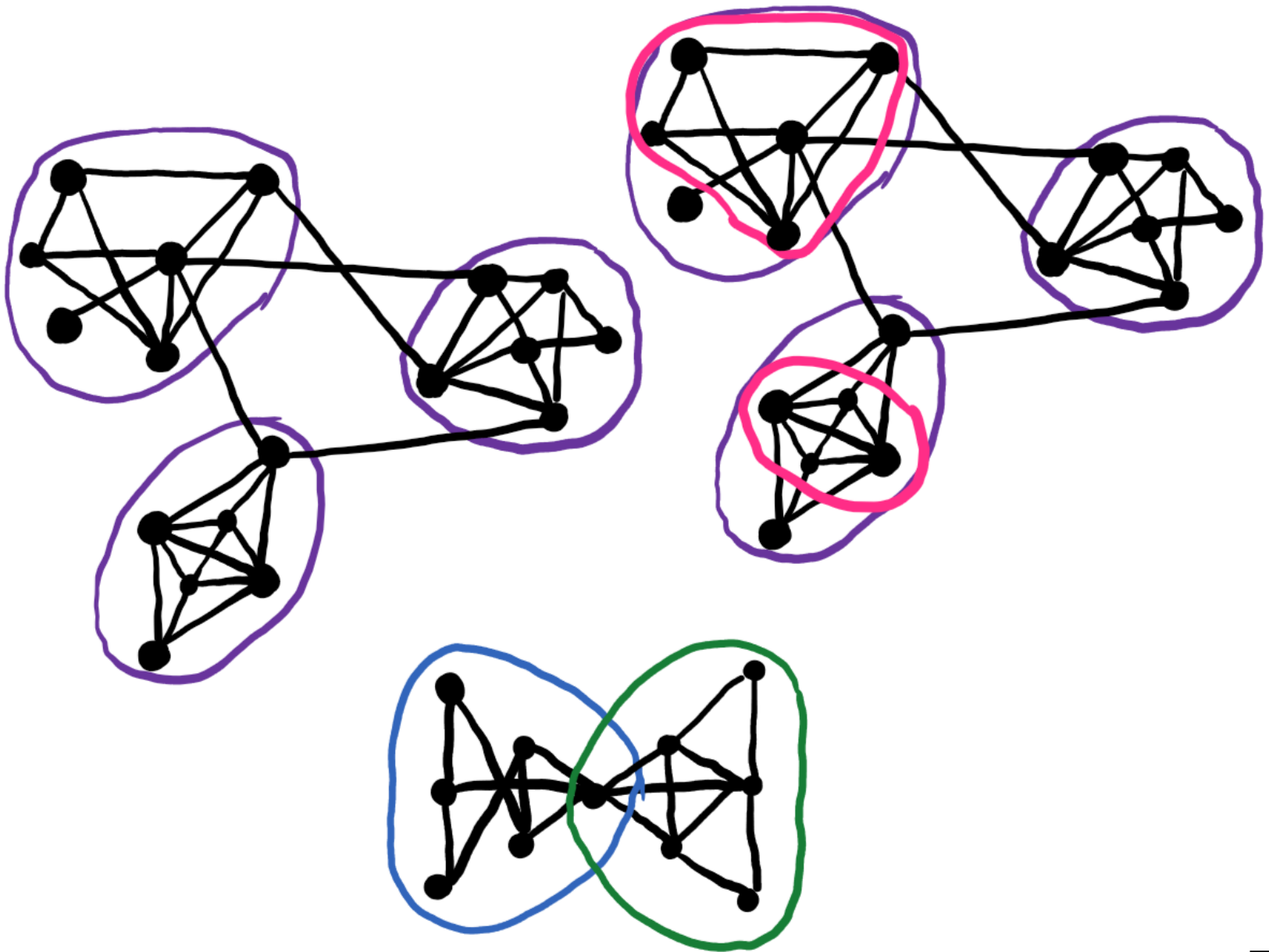
  - often communities are considered to have a high density of internal edges compared to that of external edges.

Community Examples



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  - etc.

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Today we will focus on Modularity Maximization methods.

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- Let $\vec{A}$ be a directed adjacency matrix with entries $\vec{A}_{i,j}$ representing an edge from node i to j.

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In general different models generate graphs based on an underlying theory of how edges are formed between nodes. Modularity Maximization leverages the configuration model.

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Proof of Spoke Expectation (Undirected)

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$$Q = \frac{1}{s}\left[\sum_{i,j\neq i}^{n}\left(A_{i,j} - \frac{k_i k_j}{s-1}\right)\delta(c_i, c_j) + \sum_{i}^{n}\left(A_{i,i} - \frac{k_i(k_i-1)}{s-1}\right)\right].$$

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$$Q = \frac{1}{s}\left[\sum_{i,j}^{n}\left(A_{i,j} - \frac{k_i k_j}{s}\right)\delta(c_i, c_j)\right].$$

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In an iterative fashion, we can performed the reassignment with the greatest increase in modularity.

One note on modularity maximization methods is that they detect disjoint communities, so it has a decreased accuracy for networks with underlying hierarchical or overlapping mechanics.

Complexity Optimization in Modularity Maximization [1/2]

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- How we calculate modularity changes.
- When we stop iterating.

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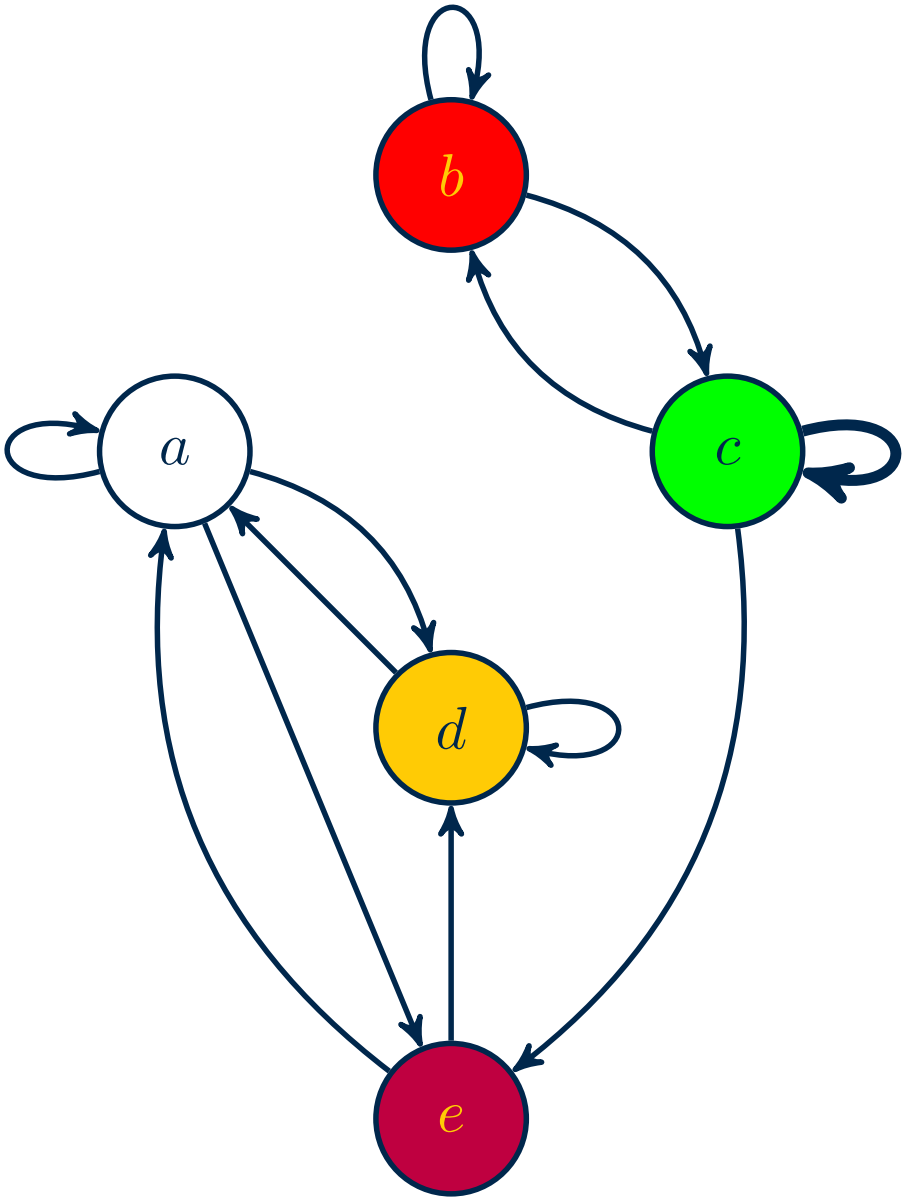
- Consider reassignments only within node neighborhoods (the neighbors of a node are all nodes in which it shares an edge).
- Cache relevant values (node degrees, expectations) rather than computing during runtime.
- Halt the algorithm once the highest possible change in modularity is below a set threshold.

Example: Network



Example: Adjacency Matrix

$\vec{A}$	a	b	c	d	e
a	1	0	0	1	1
b	0	1	1	0	0
c	0	1	2	0	1
d	1	0	0	1	0
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The adjacency matrix is the expected form computers will store networks in.

Example: Out-Degree Calculations

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$k_a^{in} = 1 + 0 + 0 + 1 + 1 = 3.$

Example: All Degree Calculations

i	a	b	c	d	e
k_i^{out}	3	2	4	2	2
k_i^{in}	3	2	3	3	2
k_i	6	4	7	5	4

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$$\vec{Q} = \frac{1}{m}\Big[(\vec{A}_{a,a} - \vec{\mathbb{E}}_{a,a}) + (\vec{A}_{b,b} - \vec{\mathbb{E}}_{b,b}) + (\vec{A}_{c,c} - \vec{\mathbb{E}}_{c,c}) + (\vec{A}_{d,d} - \vec{\mathbb{E}}_{d,d}) + (\vec{A}_{e,e} - \vec{\mathbb{E}}_{e,e})\Big].$$

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$$\vec{Q} = \frac{1}{13}\Big[(1 - \frac{3 \cdot 3}{13}) + (1 - \frac{2 \cdot 2}{13}) + (2 - \frac{4 \cdot 3}{13}) + (1 - \frac{2 \cdot 3}{13}) + (0 - \frac{2 \cdot 2}{13})\Big].$$

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By doing so we can treat merged nodes as meta nodes, and form a new adjacency matrix with them.

As a result we need only update the cached values which change.

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Similarly, for recalculating k_i^{out} and k_i^{in} we need only one addition.

Last, we will only consider merges for nodes which are neighbors.

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The merges we need consider should be much less that the $\binom{n}{2}$ node pairs.

Example: $\Delta\vec{Q}$ Calculation [1/2]

We will choose the node merge with the best change $\Delta\vec{Q}$.

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$\{a, d\}$	1	1	1	1	4
$\{a, e\}$	1	0	1	1	3
$\{b, c\}$	1	2	1	1	5
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$merge$	k_i^{out}	k_j^{out}	$k_{i'}^{out}$	k_i^{in}	k_j^{in}	$k_{i'}^{in}$
$\{a,d\}$	3	2	5	3	3	6
$\{a,e\}$	3	2	5	3	2	5
$\{b,c\}$	2	4	6	2	3	5
$\{c,e\}$	4	2	6	3	2	5
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Where i' is the new meta node formed from merging nodes i and j.

Example: $\Delta\vec{Q}$ Calculation [2/2]

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$\{a, d\}$	$4/13$	$7/13$	$22/13$	$11/169$
$\{a, e\}$	$4/13$	$-4/13$	$14/13$	$14/169$
$\{b, c\}$	$9/13$	$14/13$	$35/13$	$12/169$
$\{c, e\}$	$14/13$	$-4/13$	$9/13$	$-1/169$
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The process should continue until the best change in modularity is lower than a set threshold.

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$\{a, d\}$	$4/13$	$7/13$	$22/13$	$11/169$
$\{a, e\}$	$4/13$	$-4/13$	$14/13$	$14/169$
$\{b, c\}$	$9/13$	$14/13$	$35/13$	$12/169$
$\{c, e\}$	$14/13$	$-4/13$	$9/13$	$-1/169$
$\{d, e\}$	$7/13$	$-4/13$	$6/13$	$3/169$

Example: Multiple Merges [1/2]

To minimize complexity, one may perform many merges in a single iteration.

As is, we already compute modularity gain for each considered merge.

Let us perform the best merge for each node.

$merge$	$\partial\vec{Q}_i$	$\partial\vec{Q}_j$	$\partial\vec{Q}_{i'}$	$\Delta\vec{Q}$
$\{a, d\}$	$4/13$	$7/13$	$22/13$	$11/169$
$\{a, e\}$	$4/13$	$-4/13$	$14/13$	$14/169$
$\{b, c\}$	$9/13$	$14/13$	$35/13$	$12/169$
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The best merge for nodes a, and e, is to merge them together, similarly, the best for b, and c, is to merge them together, last, the best merge for d is to merge it with a.

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For a small network as in the example, we detect the underlying community structure in one iteration.

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Thus nodes a, d, e form one community, and nodes b, c another.

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    - $\mathcal{O}((n-1)(mn^2)) = \mathcal{O}(mn^3)$.

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QUESTIONS?

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