MTH 470/570
Winter 2021
Assigment 6 key

Problem Z. Find the value of the constant C that makes the theorem below true, and prove the theorem.

Theorem 5.1. (continued) Suppose f is holomorphic in the region G and γ is a positively oriented, simple, closed, piecewise smooth, G-contractible path. If w is inside γ then $f'''(w)$ exists, and

$$f'''(w) = C \int_\gamma \frac{f(z)}{(z-w)^4} dz.$$

Hint: Follow the argument from class for the cases of f' and f''. (The proof from class is simpler than the proof in the textbook.) The algebraic identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ is likely to be helpful.

$$\begin{aligned}
\frac{f''(w+h) - f''(w)}{h} &= \frac{1}{h \cdot \pi i} \int_\gamma \frac{f(z)}{(z-w-h)^3} - \frac{f(z)}{(z-w)^3}\ dz \\
&= \frac{1}{h \cdot \pi i} \int_\gamma \frac{f(z) \cdot (\ (z-w)^3 - (z-w-h)^3\ )}{(z-w-h)^3 (z-w)^3}\ dz \\
&= \frac{1}{h \cdot \pi i} \int_\gamma \frac{f(z) \cdot h \cdot (\ (z-w)^2 + (z-w)(z-w-h) + (z-w-h)^2\ )}{(z-w-h)^2(z-w)^2}\ dz.
\end{aligned}$$

Now by letting $h \to 0$, we have $f'''(w) = \dfrac{1}{\pi i} \displaystyle\int_\gamma \frac{f(z) \cdot 3(z-w)^2}{(z-w)^6}\ dz = \frac{3}{\pi i} \int_\gamma \frac{f(z)}{(z-w)^4}\ dz.$
The proof is complete. We see that $C = 3/(\pi i)$.

5.1. Compute the following integrals, where S is the boundary of the square with vertices $-4-4i$, $4-4i$, $4+4i$,$-4+4i$, (oriented counterclockwise).

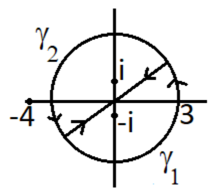
(a) $\displaystyle\int_S \frac{\exp(z^2)}{z^3}\ dz.$ $\quad \dfrac{d^2}{dz^2}(\exp(z^2)) = \dfrac{d}{dz}(2z\exp(z^2)) = (4z^2+2)\exp(z^2)$, so
$\displaystyle\int_S \frac{\exp(z^2)}{z^3}\ dz = \pi i \frac{d^2}{dz^2}(\exp(z^2))|_{z=0} = 2\pi i.$

(b) $\displaystyle\int_S \frac{\exp(3z)}{(z-\pi i)^2}\ dz = 2\pi i \cdot 3\exp(3z)|_{z=\pi i} = -6\pi i.$

(e) $\displaystyle\int_S \frac{\sin(z/3)}{(z-\pi)^4}\ dz = \frac{\pi i}{3}\frac{d^3}{dz^3}(\sin(z/3))|_{z=\pi} = \frac{\pi i}{3} \cdot \frac{1}{27}(-\cos(\pi/3)) = -\frac{\pi i}{162}.$

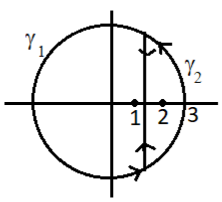
5.3. Integrate the following functions over the circle C[0, 3]:

5.3 h



$$\int_{C[0,3]} \frac{dz}{(z+4)(z^2+1)} = \int_{\gamma_1} \frac{\frac{1}{(z+4)(z-i)}}{z+i}\, dz + \int_{\gamma_2} \frac{\frac{1}{(z+4)(z+i)}}{z-i} = \left.\frac{2\pi i}{(z+4)(z-i)}\right|_{z=-i} + \left.\frac{2\pi i}{(z+4)(z+i)}\right|_{z=i} =$$
$$-\frac{\pi}{4-i} + \frac{\pi}{4+i} = -\frac{2\pi i}{17}.$$

5.3i



$$\int_{C[0,3]} \frac{\exp(2z)}{(z-1)^2(z-2)}\, dz = \int_{\gamma_1} \frac{\exp(2z)/(z-2)}{(z-1)^2} + \int_{\gamma_2} \frac{\exp(2z)/(z-1)^2}{z-2}$$
$$= 2\pi i \cdot \frac{d}{dz}\left.\left(\frac{\exp(2z)}{z-2}\right)\right|_{z=1} + 2\pi i \cdot \left.\frac{\exp(2z)}{z-1}\right|_{z=2} = 2\pi i(e^4 - 3e^2)$$

5.15. Suppose f is entire with bounded real part, i.e., writing $f(z) = u(z) + iv(z)$, there exists $M > 0$ such that $|u(z)| \leq M$ for all $z \in \mathbf{C}$. Prove that f is constant. (Hint: Consider the function $\exp(f(z))$.)

Proof: Let $g(z) = \exp(f(z))$. Then $|g(z)| = \exp(u(z)) \leq e^M$, that is, $g(z)$ is bounded. Since g is a bounded entire function, by Liouville's Theorem, g is constant. Say $g(z) = K$ for all $z \in \mathbf{C}$. There are infinitely many different possible values for $f(z)$, but they all differ by $2\pi i$. Since f is continuous, f can take only one of these values, so f is constant.