HW 3 - Cason Konzer

Saturday, February 6, 2021 10:53 PM

MTH 470/570
Winter 2021
Assignment 3.

This assignment is due before class on Tuesday, February 9. Please submit your solutions via Blackboard. Solutions are required – answers must be justified.

p.62, #3.1. Show that if $f(z) = \dfrac{az+b}{cz+d}$ is a Mobius transformation, then

$$f^{-1}(z) = \frac{dz-b}{-cz+a}.$$

#3.7. Show that the Mobius transformation $f(z) = \dfrac{1+z}{1-z}$ maps the unit circle (minus the point $z = 1$) onto the imaginary axis.

#3.14. Find Mobius transformations satisfying each of the following. Write your answers in standard form, as $\dfrac{az+b}{cz+d}$.

(a) $1 \to 0$, $2 \to 1$, $3 \to \infty$.

(b) $1 \to 0$, $1+i \to 1$, $2 \to \infty$

(c) $0 \to i$, $1 \to 1$, $\infty \to -i$.

3.30. Prove that $\overline{\sin(z)} = \sin(\overline{z})$ and $\overline{\cos(z)} = \cos(\overline{z})$.

3.32. Prove that the zeros of $\sin z$ are all real numbers. Conclude that they are precisely the integer multiples of π.

Problem I. Let C be the circle with center a and radius r. Suppose that 0 does not lie on C. (In other words, 0 could be inside or outside of the circle.) Show that the image of C under the function $f(z) = 1/z$ is the circle with center $\dfrac{\overline{a}}{|a|^2 - |r|^2}$ and radius $\dfrac{r}{|\,|a|^2 - r^2\,|}$.

Hints: This is an exercise in complex algebra. Start by writing down the equations for both circles. Use the fact that, for any complex number w, $w\overline{w} = |w|^2$, to eliminate all of the complex norms in both equations. Starting from the equation of the second circle, clear all of the fractions and factor out a common term to work toward the equation of the first circle.

p.62 #3.1 let $f(z) = \dfrac{az+b}{cz+d}$ suffice $ad-bc \neq 0$

Thus, f is a möbius transform.

let $f(z) = w$ Now $w = \dfrac{az+b}{cz+d} \leftarrow z \neq \dfrac{-d}{c}$

$wcz + wd = az + b$ $z(wc - a) = b - wd$

$z = \dfrac{b-wd}{wc-a} \leftarrow w \neq \dfrac{a}{c}$ Swap w w/ z

Now $'w = \dfrac{-zd+b}{zc-a}$; $'w^{-1} = \dfrac{az+b}{cz+d}$

$z \neq \dfrac{a}{c}$

$f^{-1}(z) = 'w = \dfrac{-zd+b}{\cdots}$ | $z \neq \dfrac{a}{c}$

$z \neq \frac{a}{c}$

$$\boxed{\therefore\ f^{-1}(z) = w = \frac{-zd+b}{zc-a} \quad \Big|\ z \neq \frac{a}{c}}$$

#3.7 $f(z) = \dfrac{1+z}{1-z}$ $\quad\Big|$ $ad - bc \neq 0$ as $-z - z = -2z$

let $f(z) = w = \dfrac{1+z}{1-z}$ $\qquad$ unit circle $= e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore\ |z| = 1$

$w - wz = 1 + z \qquad w - 1 = z + wz$

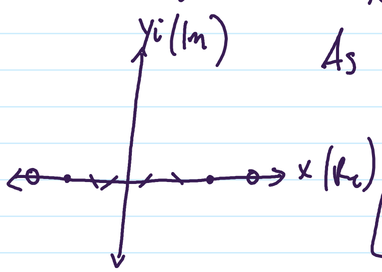
$w - 1 = z(w+1) \qquad z = \dfrac{w-1}{w+1} \Rightarrow |z| = \left|\dfrac{w-1}{w+1}\right|$

$\dfrac{|w-1|}{|w+1|} = |1| \Rightarrow |w-1| = |w+1|$ $\qquad$ $w =$ some $x + yi$ (Re, Im)

$|x + yi - 1| = |x + yi - 1| \Rightarrow |(x+1) + yi| = |(x-1) + yi|$

$(x+1)^2 + \cancel{y^2} = (x-1)^2 + \cancel{y^2} \qquad \cancel{x^2} + 2x + \cancel{1} = \cancel{x^2} - 2x + \cancel{1}$

$2x = -2x \Rightarrow x = -x \ ; \ 2x = 0$



As x cannot be $-x$ simultaneously;

$\boxed{x = 0 \ \therefore\ f(z) = \mathrm{Re}(0) + \mathrm{Im}(y)}$

Thus $f(e^{i\theta})$ is strictly imaginary.

#3.14 Recall $\dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

a) $z_1 = 1 \quad w_1 = 0$
$z_2 = 2 \quad w_2 = 1$
$z_3 = 3 \quad w_3 = \infty$

$\dfrac{(w-0)\cancel{(1-\infty)}}{\cancel{(w-\infty)}(1-0)} = w = \dfrac{(z-1)(2-3)}{(z-3)(2-1)}$

$f(z) = w \qquad w = \dfrac{-(z-1)}{(z-3)} \qquad \therefore \boxed{f(z) = \dfrac{-z+1}{z-3}}$

b) $z_1 = 1 \quad w_1 = 0$
$z_2 = 1+i \quad w_2 = 1$

$\dfrac{(w-0)\cancel{(1-\infty)}}{\ldots} = w = \dfrac{(z-1)(1+i-2)}{\ldots}$

b) $z_1 = 1 \quad w_1 = 0$
$z_2 = 1+i \quad w_2 = 1$
$z_3 = 2 \quad w_3 = \infty$

$$\frac{(w-0)\cancel{(1-\infty)}}{\cancel{(w-\infty)}(1-0)} = w = \frac{(z-1)(1+i-2)}{(z-2)(1+i-1)}$$

$$w = \frac{(z-1)(i-1)}{(z-2)(i)} = w \cdot 1 = w \cdot \frac{i}{i} = \frac{(z-1)(-i-1)}{-(z-2)} = w \cdot \frac{-1}{-1} = \frac{(z-1)(i+1)}{z-2}$$

$$\boxed{f(z) = \frac{z(i+1)-(i+1)}{z-2}}$$

c) $z_1 = 0 \quad w_1 = i$
$z_2 = 1 \quad w_2 = 1$
$z_3 = \infty \quad w_3 = -i$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{(z-0)\cancel{(1-\infty)}}{\cancel{(z-\infty)}(1-0)} = z$$

$$z = \left(\frac{w-i}{w+i}\right)\left(\frac{(1+i)^2}{1-i^2}\right) \Rightarrow z(w+i) = (w-i)\left(\frac{\cancel{1}+2i-\cancel{1}}{2}\right)$$

$$zw + iz = (w-i)\,i$$

$$zw + iz = wi + 1 \Rightarrow w(zw - wi) = 1 - iz \qquad w = \frac{1-iz}{z-i}$$

$$\boxed{\therefore f(z) = \frac{-iz+1}{z-i}}$$

#3.30 Prove $\sin(\bar{z}) = \overline{\sin(z)}$ & $\overline{\cos(z)} = \cos(\bar{z})$

$$e^{z} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y) \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

$$e^{iz} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x) \qquad \cos z := \frac{e^{iz} + e^{-iz}}{2i}$$

$$e^{-iz} = e^{y}e^{-ix} = e^{y}(\cos x + i\sin x)^{-1}$$

$$\overline{\sin z} = \overline{\left(\frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}\right)} = \overline{\left(\frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x + i\sin x)^{-1}}{2i}\right)}$$

$$\overline{\sin z} = \left(\frac{e^{-y}(\cos x - i\sin x) - e^{y}(\cos x - i\sin x)^{-1}}{2i}\right)$$

$$\overline{\sin z} = \left(\frac{e^{-y}e^{i\bar{x}} - e^{y}e^{-i\bar{x}}}{2i}\right) = \frac{e^{i\bar{x}-y} - e^{-i\bar{x}-y}}{2i} = \frac{e^{i(\bar{x}-iy)} - e^{-i(\bar{x}-iy)}}{2i}$$

$$\overline{\quad} \; e^{iz} \; -iz \qquad \qquad \overline{(e^{iz} + e^{-iz})}$$

$$\boxed{\overline{\sin z} = \frac{e^{i\bar z} - e^{-i\bar z}}{2i} = \sin \bar z}$$

$$\overline{\cos z} = \overline{\left(\frac{e^{iz} + e^{-iz}}{2}\right)}$$

$$\overline{\cos z} = \overline{\left(\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}\right)} = \overline{\left(\frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}\right)}$$

$$= \overline{\left(\frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x + i\sin x)^{-1}}{2}\right)}$$

$$= \left(\frac{e^{-y}(\cos x - i\sin x) + e^{y}(\cos x - i\sin x)^{-1}}{2}\right) = \left(\frac{e^{-y}e^{i\bar x} + e^{y}e^{i\bar x}}{2}\right)$$

$$= \left(\frac{e^{i\bar x - y} + e^{i\bar x + y}}{2}\right) = \left(\frac{e^{i(+\bar x - yi)} + e^{i(-\bar x + yi)}}{2}\right)$$

$$= \left(\frac{e^{i(\bar x - yi)} + e^{-i(\bar x - yi)}}{2}\right) = \frac{e^{i\bar z} + e^{-i\bar z}}{2}$$

$$\therefore \quad \overline{\cos z} = \frac{e^{i\bar z} + e^{-i\bar z}}{2} = \cos \bar z$$

#3.32 Consider $\sin z = 0$ $\quad \frac{e^{iz} - e^{-iz}}{2i} = 0$

To begin $2i(0) = 0 = e^{iz} - e^{-iz} \Rightarrow e^{iz} = e^{-iz}$

$$\frac{e^{iz}}{e^{-iz}} = 1 \Rightarrow e^{2iz} = 1 \qquad e^{2i(x+yi)} = 1 = e^{2ix - 2y}$$

$e^{-2y}(\cos(2x)+i\sin(2x))=1 \Rightarrow |e^{-2y}(\cos(2x)+i\sin(2x)|=1$

$\Rightarrow |e^{-2y}|=1 \Rightarrow y=0$ as $y=0$; $z=x+iy = x = Re(z)$

we are considering where $\sin(z)=0$

$\sin(z)=0$ @ $0, \pi, 2\pi \ldots$ $\therefore$ $\sin(\pi n)=0$

Again, z is purely real. as $\pi \in \mathbb{R}$ & $n \in \mathbb{R}$

Zeros of $\sin(z) = \pi n \mid n \in \mathbb{Z}$

$\therefore$ zeros are purely real & integer multiples of π

"Problem I" C = Circle w/ center @ a & radius r.

0 does not lie on C. $f(z)=\frac{1}{z}=w$; $z=\frac{1}{w}$

$f^{-1}(z)=\frac{1}{z}$; the image of C is centered @ $\frac{\bar{a}}{|a|^2-|r^2|}$ w/ radius $\frac{r}{||a|^2-r^2|}$

$a\bar{a}=|a|^2$

$0 \leq a\bar{a} \neq r^2$

$f(z)=f^{-1}(z)$ $z=re^{i\theta}+a$ $f(z)=\frac{1}{re^{i\theta}+a}$

$$f^{-1}\left(\frac{1}{\text{"}r\text{"}e^{i\theta}+\text{"}a\text{"}}\right)=\frac{1}{\left(\frac{r}{|a|^2-r^2}\right)e^{i\theta}+\left(\frac{\bar{a}}{||a|^2-r^2|}\right)}$$

$$= \frac{|a|^2 - r^2}{re^{i\theta}} + \frac{||a^2| - r^2|}{\bar{a}} = \frac{a\bar{a} - r^2}{re^{i\theta}} + \frac{|a\bar{a} - r^2|}{\bar{a}}$$

$$= \frac{a\bar{a}^2 - \bar{a}r^2 + |a\bar{a} - r^2| re^{i\theta}}{\bar{a} re^{i\theta}}$$

let $a = x + iy$

$$= \frac{\bar{a}(x^2+y^2-r^2) + |(x^2+y^2) - r^2| re^{i\theta}}{\bar{a} re^{i\theta}}$$

$$= \frac{\bar{a}(x^2+y^2-r^2) + re^{i\theta}(x^2+y^2-r^2)(x^2+y^2+r^2)}{\bar{a} re^{i\theta}}$$

$$= \frac{x - iy(x^2+y^2-r^2) + re^{i\theta}(\ldots)(\ldots)}{(x-iy) re^{i\theta}}$$

$$\mathrm{Arg}(z) = \frac{(\cos\theta + i\sin\theta) + ir^2y - ix^2y - iy^3}{(\cos\theta + i\sin\theta)}$$

$$\mathrm{Arg}(z) = 1 + \frac{iy(r^2 - x^2 - y^2)}{e^{i\theta}}$$

* Really not sure how to make the connection on this one *

image center: $\frac{\bar{a}}{|a|^2 - |r|^2} = \frac{\bar{a}}{a\bar{a} - r^2} = \frac{1}{a - r^2/\bar{a}}$

r

image radius: $\dfrac{r}{\left||a|^2-r^2\right|} = \dfrac{r}{|a\bar{a}-r^2|} = \dfrac{r}{(a\bar{a})^2-r^4}$

$$= \dfrac{1}{\dfrac{(a\bar{a})^2-r^3}{r}}$$

Center thus: $a - \dfrac{r^2}{\bar{a}} = x+iy - \dfrac{r^2}{x-iy} = x+iy - \dfrac{r^2(x+iy)}{x^2+y^2}$

$$= x+iy\left(1 - \frac{r^2}{x^2+y^2}\right) = a\left(1-\frac{r^2}{a\bar{a}}\right)$$

radius thus: $\dfrac{(a\bar{a})^2}{r} - r^3 = \dfrac{(x^2+y^2)^2}{r} - r^3 = (x^2+y^2)^2 - r^4$

$$= (a\bar{a})^2 - r^4$$