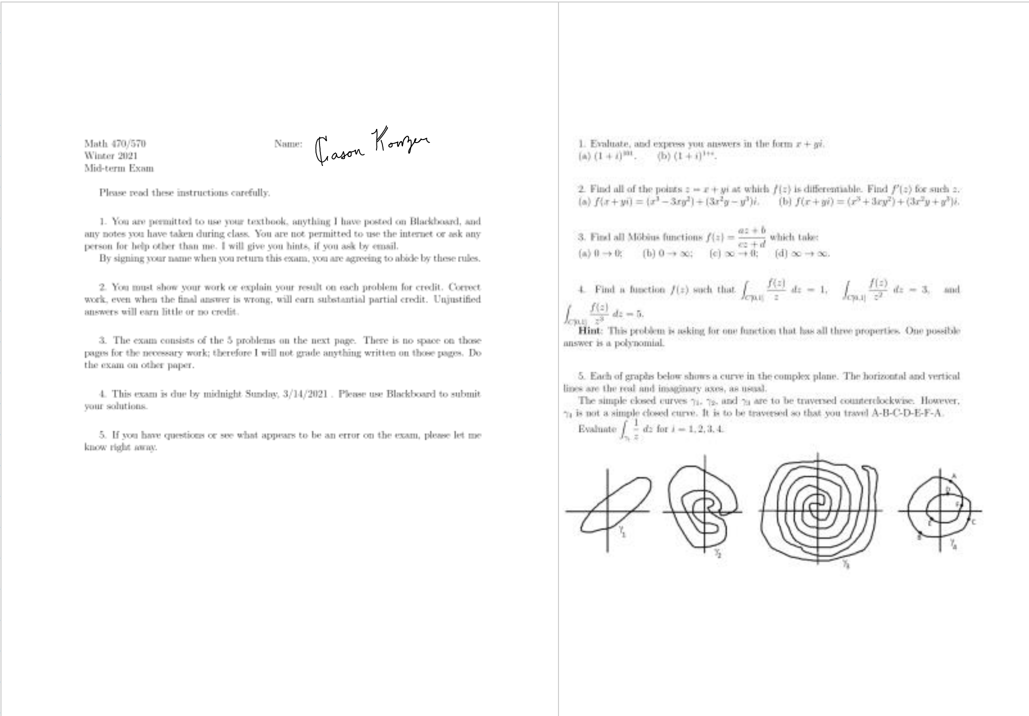
EXAM 1 - Cason Konzer

Friday, March 12, 2021 12:51 AM



Math 470/570
Winter 2021
Mid-term Exam

Name: Cason Konzer

Please read these instructions carefully.

1. You are permitted to use your textbook, anything I have posted on Blackboard, and any notes you have taken during class. You are not permitted to use the internet or ask any person for help other than me. I will give you hints, if you ask by email.
By signing your name when you return this exam, you are agreeing to abide by these rules.

2. You must show your work or explain your result on each problem for credit. Correct work, even when the final answer is wrong, will earn substantial partial credit. Unjustified answers will earn little or no credit.

3. The exam consists of the 5 problems on the next page. There is no space on those pages for the necessary work; therefore I will not grade anything written on those pages. Do the exam on other paper.

4. This exam is due by midnight Sunday, 3/14/2021 . Please use Blackboard to submit your solutions.

5. If you have questions or see what appears to be an error on the exam, please let me know right away.

1. Evaluate, and express you answers in the form $x + yi$.
(a) $(1+i)^{101}$. (b) $(1+i)^{1+i}$.

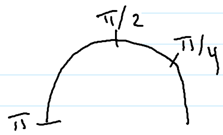
2. Find all of the points $z = x + yi$ at which $f(z)$ is differentiable. Find $f'(z)$ for such z.
(a) $f(x+yi) = (x^3 - 3xy^2) + (3x^2y - y^3)i$. (b) $f(x+yi) = (x^3 + 3xy^2) + (3x^2y + y^3)i$.

3. Find all Möbius functions $f(z) = \frac{az+b}{cz+d}$ which take:
(a) $0 \to 0$; (b) $0 \to \infty$; (c) $\infty \to 0$; (d) $\infty \to \infty$.

4. Find a function $f(z)$ such that $\int_{C[0,1]} \frac{f(z)}{z}\, dz = 1$, $\int_{C[0,1]} \frac{f(z)}{z^2}\, dz = 3$, and $\int_{C[0,1]} \frac{f(z)}{z^3}\, dz = 5$.
Hint: This problem is asking for one function that has all three properties. One possible answer is a polynomial.

5. Each of graphs below shows a curve in the complex plane. The horizontal and vertical lines are the real and imaginary axes, as usual.
The simple closed curves γ_1, γ_2, and γ_3 are to be traversed counterclockwise. However, γ_4 is not a simple closed curve. It is to be traversed so that you travel A-B-C-D-E-F-A.
Evaluate $\int_{\gamma_i} \frac{1}{z}\, dz$ for $i = 1, 2, 3, 4$.

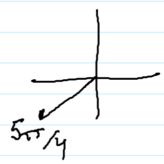
1. Evaluate, and express you answers in the form $x + yi$.
(a) $(1+i)^{101}$. (b) $(1+i)^{1+i}$.



let $z = 1+i$ thus $|z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\sqrt{2}\left(\cos(\varphi) + i\sin(\varphi)\right) = 1+i$ $\quad \varphi = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ $\quad \varphi = 45^\circ = \pi/4$

$z = |z|e^{i\varphi}$; $z = \sqrt{2}e^{i\pi/4} = \sqrt{2}(\cos(\pi/4)) + i\sqrt{2}(\sin(\pi/4))$

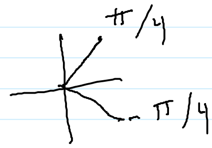


the important fact is $z^\sigma = |z|^\sigma e^{i\sigma\varphi}$ Now

a) $z^{101} = |z|^{101} e^{101 i\varphi} = 2^{\frac{101}{2}} e^{101 i \cdot \pi/4} = 2^{\frac{101}{2}} e^{\frac{101\pi i}{4}}$

$12(2\pi) = 24\pi = \frac{96\pi}{4}$ $\quad e^{\frac{101\pi i}{4}} = e^{\frac{101\pi i - 96\pi i}{4}} = e^{\frac{5\pi i}{4}}$ $\quad z^{101} = \sqrt{2}^{101} e^{\frac{5\pi i}{4}}$

$z^{101} = \sqrt{2}^{101}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$ $\quad \boxed{z^{101} = -1.126 \times 10^{15}(1+i)}$



b) $z^{1+i} = zz^i = |z|e^{i\varphi} \cdot |z|^i e^{i^2\varphi} = |z||z|^i e^{-\varphi} e^{i\varphi}$

$= \sqrt{2}\sqrt{2}^i e^{-\frac{\pi}{4}} e^{\frac{\pi i}{4}}$ $\quad z^{1+i} = \sqrt{2}^{(1+i)} \cdot e^{\frac{\pi i - \pi}{4}}$

$a^b = e^{b\ln(a)}$

$i \quad i\ln(\sqrt{2})$

$$\sqrt{2}^{\,i} = e^{i\ln(\sqrt{2})}$$
$$= \cos(\ln(\sqrt{2})) + i\sin(\ln(\sqrt{2}))$$
$$= .9405 + i(.3397)$$

$$z^{1+i} = \sqrt{2}\cdot e^{-\pi/4}\cdot \sqrt{2}^{\,i}\left(\cos(\pi/4) + i\sin(\pi/4)\right)$$

$$z^{1+i} = 0.4559\ldots\cdot\sqrt{2}^{\,i}(1+i) = (.4288 + i(.1549))(1+i)$$
$$= (.4288\ldots - .1549\ldots) + i(.4288\ldots + 1.549\ldots)$$

$$\boxed{z^{1+i} = 0.2739 + i(0.5837)}$$

$$\sqrt{2}\cdot e^{-\pi/4}\cdot(\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2}))(1+i)$$

2. Find all of the points $z = x + yi$ at which $f(z)$ is differentiable. Find $f'(z)$ for such z.
(a) $f(x+yi) = (x^3 - 3xy^2) + (3x^2y - y^3)i$. (b) $f(x+yi) = (x^3 + 3xy^2) + (3x^2y + y^3)i$.

Consider the Cauchy-Reimann Equations: $P_x = Q_y$ $P_y = -Q_x$

a) Set $P = x^3 - 3xy^2$, $Q = 3x^2y - y^3$; $P_x = 3x^2 - 3y^2$, $Q_y = 3x^2 - 3y^2$ ✓ $P_y = -6xy$, $-Q_x = -(6xy)$ ✓

Thus f satisfies the C.R. Eqs. In whole,

$\boxed{\therefore f \text{ is differentiable everywhere}}$

$z = x+yi$ $z^2 = x^2 + 2xyi - y^2$ $z^3 = x^3 + \underline{2x^2yi} - \underline{y^2x} + \underline{x^2yi} - \underline{2xy^2} - y^3 i$

$z^3 = x^3 - 3xy^2 + i(-y^3 + 3x^2y) \Rightarrow f(z) = z^3$

$\boxed{\therefore f'(z) = 3z^2}$

b) Set $P = x^3 + 3xy^2$, $Q = 3x^2y + y^3$; $P_x = 3x^2 + 3y^2$, $Q_y = 3x^2 + 3y^2$ ✓ $P_y = 6xy$, $-Q_x = -(6xy)$ $\neq$ X

To satisfy the C.R. Eqs., $\cancel{6}xy = -\cancel{6}xy$ thus as $xy = -xy$, $x = -x$ or $y = -y$, this is not possible, in general. Although, if $x = 0$ or $y = 0$ we have $P_x = Q_y \Rightarrow 3x^2 = 3x^2$ or $3y^2 = 3y^2$ and we have $P_y = -Q_x$ as $0 = 0$

$\boxed{\therefore \text{This function is differentiable only when } x=0 \vee y=0}$

$f(x+0i) = (x^3) + 0i \Rightarrow f(x) = x^3$ so $\boxed{f'(x) = 3x^2}$

$f(0+yi) = 0 + (y^3)i \Rightarrow f(yi) = y^3 i$ so $\boxed{f'(yi) = 3iy^2}$

3. Find all Möbius functions $f(z) = \dfrac{az+b}{cz+d}$ which take: $ad - bc \neq 0$

(a) $0 \to 0$; (b) $0 \to \infty$; (c) $\infty \to 0$; (d) $\infty \to \infty$. $f^{-1}(z) = \dfrac{dz-b}{a-cz}$

$f(z) = \dfrac{bc-ad}{c^2} \cdot \dfrac{1}{z + d/c} + \dfrac{a}{c}$; $f(\infty) = \dfrac{a}{c}$; $f(-\frac{d}{c}) = \infty$; $f(0) = \dfrac{b}{d}$; $f(\frac{-b}{a}) = 0$

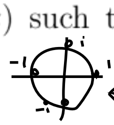
a) $\dfrac{a(0)+b}{c(0)+d} = 0$, $\dfrac{d(0)-b}{a-c(0)} = 0$ | $\dfrac{b}{d} = 0$, $\dfrac{-b}{a} = 0 \therefore b = 0 \Rightarrow ad \neq 0$ | $\boxed{f(z) = \dfrac{az}{cz+d} \mid a, d \neq 0}$ $0 \to 0$

b) $\dfrac{a(0)+b}{c(0)+d} = \infty$, $\dfrac{d(\infty)-b}{a-c(\infty)} = 0$ | $\dfrac{b}{d} = \infty$, $\dfrac{d}{-c} = 0 \therefore d = 0 \Rightarrow bc \neq 0$ | $\boxed{f(z) = \dfrac{az+b}{cz} \mid b, c \neq 0}$ $0 \to \infty$

c) $\dfrac{a(\infty)+b}{c(\infty)+d} = 0$, $\dfrac{d(0)-b}{a-c(0)} = \infty$ | $\dfrac{a}{c} = 0$, $\dfrac{-b}{a} = \infty \therefore a = 0 \Rightarrow bc \neq 0$ | $\boxed{f(z) = \dfrac{b}{cz+d} \mid b, c \neq 0}$ $\infty \to 0$

d) $\dfrac{a(\infty)+b}{c(\infty)+d} = \infty$, $\dfrac{d(\infty)-b}{a-c(\infty)} = \infty$ | $\dfrac{a}{c} = \infty$, $\dfrac{-d}{c} = \infty \therefore c = 0 \Rightarrow ad \neq 0$ | $\boxed{f(z) = \dfrac{az+b}{d} \mid a, d \neq 0}$ $\infty \to \infty$

4. Find a function $f(z)$ such that $\displaystyle\int_{C[0,1]} \frac{f(z)}{z}\, dz = 1$, $\displaystyle\int_{C[0,1]} \frac{f(z)}{z^2}\, dz = 3$, and $\displaystyle\int_{C[0,1]} \frac{f(z)}{z^3}\, dz = 5$.



$\leftarrow C[0,1]$ $a = 0$ & 0 is inside $C[0,1]$

Hint: This problem is asking for one function that has all three properties. One possible answer is a polynomial.

$\displaystyle\int_{C[0,1]} \frac{f(z)}{z-0}\, dz = 2\pi i f(0) = 1 \Rightarrow \boxed{f(0) = \frac{1}{2\pi i}}$

$\displaystyle\int_{C[0,1]} \frac{f(z)}{z^3 - 0}\, dz = \pi i f''(0) = 5$

$\boxed{f''(0) = \dfrac{5}{\pi i} = \dfrac{10}{2\pi i}}$

$\displaystyle\int_{C[0,1]} \frac{f(z)}{z^2 - 0}\, dz = 2\pi i f'(0) = 3 \Rightarrow \boxed{f'(0) = \frac{3}{2\pi i}}$

$3f(0) = f'(0)$, $10 f'(0) = 3 f''(0)$,

$2\pi i = \dfrac{1}{f(0)} = \dfrac{3}{f'(0)} = \dfrac{10}{f''(0)}$

$10 f(0) = f''(0)$, $\dfrac{f(0)}{f'(0)} = \dfrac{1}{3}$,

$\dfrac{f'(0)}{f''(0)} = \dfrac{3}{10}$, $\dfrac{f(0)}{f''(0)} = \dfrac{1}{10}$, $f(0) = = \dfrac{f''(0)}{10}$

+2 +7

1, 3, 10

x3 x3+1

$e^{3z} + \dfrac{z^2}{2}$ @ $0 = 1$

$3e^{3z} + z$ @ $0 = 3$

$9e^{3z} + 1$ @ $0 = 10$

$$\boxed{f(z) = \frac{e^{3z} + \frac{z^2}{2}}{2\pi i}}$$

— This satisfys the found Integral

$2\pi i$

— This satisfys the found Integral evaluations: $f(0) = \frac{1}{2\pi i}$; $f'(0) = \frac{3}{2\pi i}$ & $f''(0) = \frac{10}{2\pi i}$

$$\frac{1}{2\pi i}\left(\frac{-2\pi i}{-2\pi i}\right) = \frac{-2\pi i}{4\pi^2} \quad \therefore \quad f(z) = \frac{-i\left(e^{3z} + \frac{z^2}{2}\right)}{2\pi} = -i\left(\frac{e^{3z}}{2\pi} + \frac{z^2}{4\pi}\right)$$

5. Each of graphs below shows a curve in the complex plane. The horizontal and vertical lines are the real and imaginary axes, as usual.

The simple closed curves γ_1, γ_2, and γ_3 are to be traversed counterclockwise. However, γ_4 is not a simple closed curve. It is to be traversed so that you travel A-B-C-D-E-F-A.

Evaluate $\int_{\gamma_i} \frac{1}{z}\, dz$ for $i = 1, 2, 3, 4$.

$f(z) = 1$; $a = 0$ $\forall \gamma$

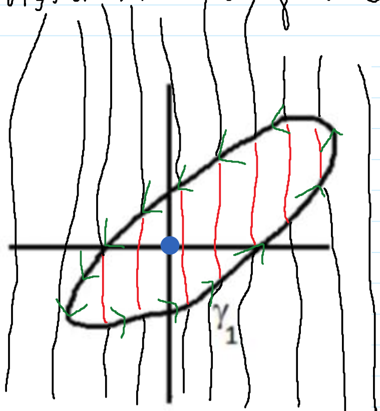
$f(z) = 1$ is holomorphic & differentiable everywhere

$$\int_\gamma \frac{f(z)\,dz}{z - 0} = 2\pi i f(0) \; ; \; f(0) = 1$$

$$\Rightarrow \int_\gamma \frac{dz}{z} = 2\pi i$$

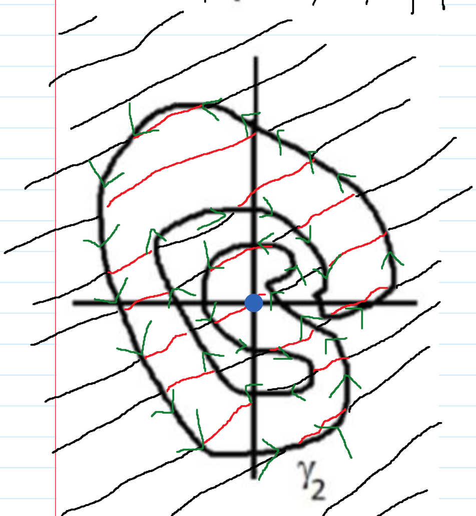
given 0 is in the interior of γ

use $\in$ to say is inside of



$0 \in \gamma_1$

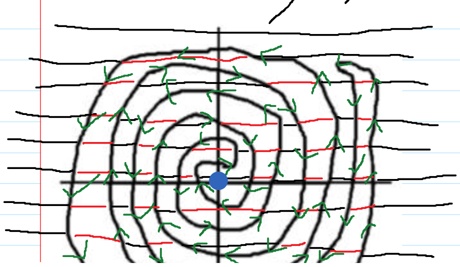
$$\boxed{\int_{\gamma_1} \frac{dz}{z} = 2\pi i}$$



$0 \in \gamma_2$

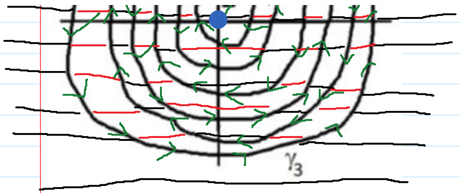
$$\boxed{\int_{\gamma_2} \frac{dz}{z} = 2\pi i}$$

$\notin$: Not inside of



$0 \notin \gamma_3$

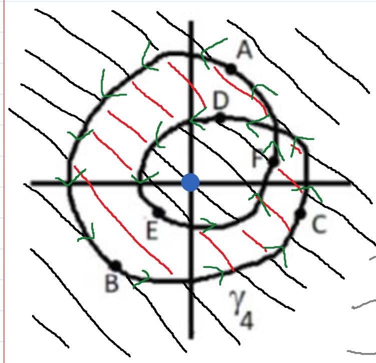
$$\boxed{\int_{\gamma_3} \frac{dz}{z} = 0}$$



$$\boxed{\int_{\gamma_3} \quad z}$$

$$\int_{\gamma_4}\frac{dz}{z} = \int_{\gamma_5}\frac{dz}{z} + 2\int_{\gamma_6}\frac{dz}{z} = 0 + 2(2\pi i)$$

$$\boxed{\int_{\gamma_4}\frac{dz}{z} = +4\pi i}$$

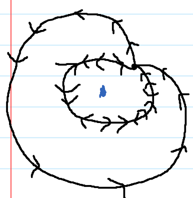


Not Simple

$\int_{\gamma_5}\frac{dz}{z} = 0$

$\int_{\gamma_6}\frac{dz}{z} = 2\pi i$

Both Simple

 =