HW 7 - Cason Konzer

Thursday, April 1, 2021 7:43 AM

Due 4/6/21 @ 5:30 pm

470hw7

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Taylor & Laurent Series

All Final Answers are Underlined in Black

MTH 470/570
Winter 2021
Assignment 7

This assignment is due before class on Tuesday, April 6.

Please submit your solutions via Blackboard. Solutions are required – answers must be justified.

p.123, #8.5. Find the terms through third order and the radius of convergence of the power series for each of the following functions, centered at z_0 . (Do not find the general form for the coefficients.)

(c) $f(z) = (1 + z)^{1/2}$, $z_0 = 0$. (d) $f(z) = \exp(z^2)$, $z_0 = i$.

#8.22.

(a) Find the power series representation for $\exp(az)$ centered at 0, where $a \in \mathbf{C}$ is any constant. (Hint: Substitution!)

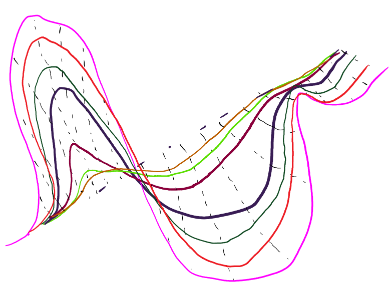
(b) Show that $\exp(z)\cos(z) = \frac{1}{2}(\exp((1 + i)z) + \exp((1 - i)z))$.

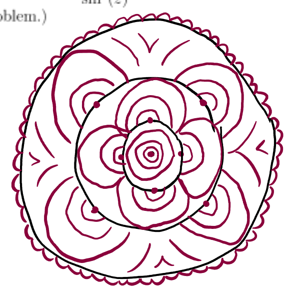
(c) Find the power series expansion for $\exp(z)\cos(z)$ centered at 0. (The complete expansion, that is, not just the first few terms.)

#8.32. Find the three Laurent series of $f(z) = \dfrac{3}{(1 - z)(z + 2)}$, centered at 0, defined on the three regions $|z| < 1$, $1 < |z| < 2$, and $2 < |z|$, respectively. (Hint: Use a partial fraction decomposition.)

Problem H: Find the Laurent series for the function $g(z) = \dfrac{1}{1 - \cos z}$ about 0, up through the term $c_6 z^6$. (Hint: This is similar to Example 8.23 in the textbook, which we did by another method in class. Choose your favorite method.)

#8.20. Find the terms $c_n z^n$ in the Laurent series for $\dfrac{1}{\sin^2(z)}$ centered at $z = 0$, for $-4 \le n \le 4$. (Same hint as for the previous problem.)





p. 123 #8.5

[f(z) is holomorphic]

(c) $f(z) = (1+z)^{1/2}, z_0 = 0$ $\left\{ f(z_0) = \sqrt{1+z_0} = \sqrt{1} = 1 \right\}$

$f'(z) = (1/2)(1+z)^{-1/2}$; $f''(z) = (-1/4)(1+z)^{-3/2}$; $f'''(z) = (3/8)(1+z)^{-5/2}$

$D(z_0, r)$ $r = \sqrt{0^2 + (-1)^2} = \sqrt{1}$; $\underline{r = 1}$

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)(z-z_0)^2}{2!} + \frac{f'''(z_0)(z-z_0)^3}{3!}$$

$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)(z-z_0)^2}{2!} + \frac{f'''(z_0)(z-z_0)^3}{3!}$

$$f(z) = (1)^{1/2} + (1/2)(1)^{-1/2}(z) + \frac{(-1/4)(1)^{-3/2}(z)^2}{2!} + \frac{(3/8)(1)^{-5/2}(z)^3}{3!}$$

$$\boxed{f(z) = 1 + \frac{z}{2} - \frac{z^2}{8} + \frac{z^3}{16} - \dots + \dots} \qquad \boxed{D(0,1)}$$

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d) [$f(z)$ is holomorphic]

$$f(z) = e^{z^2}, \; z_0 = i \qquad \left\{ f(z_0) = e^{z_0^2} = e^{i^2} = e^{-1} = \frac{1}{e} \right\}$$

$$f'(z) = 2z \cdot e^{z^2} \; ; \; f''(z) = 2e^{z^2} + 4z^2 e^{z^2} \; ; \; f'''(z) = 4z e^{z^2} + 8z e^{z^2} + 8z^3 e^{z^2}$$
$$= 8z^3 e^{z^2} + 12z e^{z^2}$$

$$f(z) = e^{i^2} + 2ie^{i^2}(z-i) + \frac{(2e^{i^2} + 4i^2 e^{i^2})(z-i)^2}{\cancel{2}} + \frac{(8i^3 e^{i^2} + \cancel{12} i e^{i^2})(z-i)^3}{3 \, \cancel{6}}$$

$$f(z) = e^{-1}\left[1 + 2i(z-i) + (1 + 2i^2)(z-i)^2 + \frac{(\cancel{4i^3} + 6i)(z-i)^3}{3}\right]$$

(with -2 written above $2i^2$ and $-4i$ above $4i^3$)

$$\boxed{f(z) = \frac{1}{e}\left[1 + 2i(z-i) - (z-i)^2 + \frac{2i(z-i)^3}{3} - + \dots\right]}$$

$r = \infty$; $\boxed{D(i, \infty)}$ $\therefore f(z)$ is Analytic Everywhere in $\mathbb{C}$

8.22 (a) $f(z) = e^{az} \mid a \in \mathbb{C}$; let $\omega = az$

$$f(\omega) = e^{\omega} \; ; \; f(\omega) = 1 + \omega + \frac{\omega^2}{2!} + \frac{\omega^3}{3!} + \frac{\omega^4}{4!} + \dots\dots$$

$$f(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{k!} \qquad \omega = az \; ; \; f(az) = \sum_{k=0}^{\infty} \frac{(az)^k}{k!}$$

$$\boxed{f(z) = \sum_{k=0} \frac{a^k z^k}{k!} = e^{az}}$$

(b) show $e^z \cos z = \frac{e^{z+iz} + e^{z-iz}}{2} = \frac{e^{(1+i)z} + e^{(1-i)z}}{2}$

$\quad (\;)(\;iz \quad -iz\;) \; .$

We Know $\cos(z) = \left(\frac{1}{2}\right)\left(e^{iz} + e^{-iz}\right) \therefore$

$e^{z}\cos(z) = \left(\frac{1}{2}\right)\left(e^{z}\right)\left(e^{iz} + e^{-iz}\right)$. Now Distributing e^{z},

$e^{z}\cos(z) = \left(\frac{1}{2}\right)\left(e^{z}e^{iz} + e^{z}e^{-iz}\right) = \left(\frac{1}{2}\right)\left(e^{z+iz} + e^{z-iz}\right)$

indeed ; $\boxed{e^{z}\cos(z) = \dfrac{e^{(1+i)z} + e^{(1-i)z}}{2}}$ $\blacksquare$

(c) $f(z) = e^{z}\cos(z)$, $z_0 = 0$ $\left\{ f(z_0) = e^{0}\cos(0) = 1 \right\}$

$e^{az} = \sum_{k=0}^{\infty} \frac{a^k z^k}{k!}$; $a_1 = (1+i)$, $a_2 = (1-i)$

$e^{a_{1,2}z} = \sum_{k_{1,2}=0}^{\infty} \frac{a_{1,2}^{k_{1,2}} z^{k_{1,2}}}{k_{1,2}!}$ $\qquad e^{z}\cos(z) = \dfrac{e^{a_1 z} + e^{a_2 z}}{2}$

$$\boxed{f(z) = e^{z}\cos(z) = \frac{\sum_{k=0}^{\infty} \frac{(1+i)^k z^k}{k!} + \sum_{\lambda=0}^{\infty} \frac{(1-i)^{\lambda} z^{\lambda}}{\lambda!}}{2}}$$

$r = \infty$

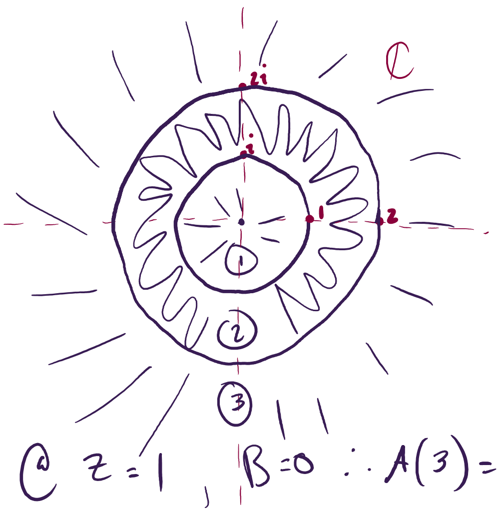
$\boxed{D(0, \infty)}$

$$\boxed{f(z) = \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right)\left((1+i)^j + (1-i)^j\right) z^j}$$

Also $\cos(z) = \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda} z^{2\lambda}}{(2\lambda)!}$ & $e^{z} = \sum_{k=0}^{\infty} \frac{z^k}{k!}$

So $\boxed{e^{z}\cos(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda} z^{2\lambda}}{(2\lambda)!}}$

8.32 $f(z) = \dfrac{3}{(1-z)(z+2)}$, $z_0 = 0$, $\underbrace{|z| < 1}_{①} < \overbrace{|z| < 2}^{②} < \underbrace{|z|}_{③}$



$$\left\{ f(0) = \frac{3}{(1)(2)} = 3/2 \right\}$$

$$f(z) = \frac{A}{(1-z)} + \frac{B}{(2+z)} \; ; \; \frac{A(2+z) + B(1-z)}{(1-z)(2+z)}$$

$$= \frac{2A + B + Az - Bz}{(1-z)(2+z)} = \frac{3}{(1-z)(2+z)}$$

@ $z = 1$, $B = 0 \therefore A(3) = 3$ & $A = 1$.

@ $z = -2$, $A = 0 \therefore B(3) = 3$ & $B = 1$ $\therefore f(z) = \frac{1}{(1-z)} + \frac{1}{(2+z)} = f(z) + f(z)$

$$f(z) = \frac{1}{(1-z)} = 1 + z + z^2 + z^3 + \dots = \sum_{k=0}^{\infty} z^k$$

$$f' = -(z+2)^{-2}$$
$$f'' = 2(z+2)^{-3}$$
$$f''' = -6(z+2)^{-4}$$

$$f(z) = \frac{1}{(z+2)} = f(0) + \frac{f'(0)(z-0)}{1!} + \frac{f''(0)(z)^2}{2!} + \frac{f'''(0)z^3}{3!}$$

$$f(z) = \frac{1}{2} + \frac{\left(\frac{-1}{4}\right)z}{1} + \frac{2\left(\frac{1}{8}\right)z^2}{2} + \frac{-6\left(\frac{1}{16}\right)z^3}{6} + \frac{4!}{4!}(-1)^4\left(\frac{1}{2^5}\right)z^4 + \dots \frac{k!}{k!}\frac{(-1)^k z^k}{2^{k+1}}$$

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{2^{k+1}}$$

$$\boxed{f(z) = \sum_{k=0}^{\infty} z^k + \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{2^{(j+1)}} \quad \Big| \quad |z| < 1}$$

$$\boxed{f(z) = \sum_{i=0}^{\infty} \left(1 + \left(\frac{-1}{2^{(i+1)}}\right)^i\right) z^i} \quad |z| < 1$$

$$f(z) = \frac{1}{1-z} = \frac{\frac{1}{z}}{\frac{1}{z} - 1} = \frac{-\left(\frac{1}{z}\right)}{1 - \left(\frac{1}{z}\right)} \quad \text{thus} \quad f^L(z) = \sum_{k=0}^{\infty} \left(-\frac{1}{z}\right)\left(\frac{1}{z}\right)^k$$

$$\boxed{f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{2^{(k+1)}} - \sum_{j=0}^{\infty} (z^{-1})^{k+1} \quad \Big| \quad 1 < |z| < 2}$$

$$f^L(z) = -\sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^{k+1}$$

$$\boxed{f(z) = \sum_{i=0}^{\infty} \left(\frac{(-1)^i z^i}{2^{(i+1)}} - (z^{-1})^{(i+1)}\right)} \quad 1 < |z| < 2$$

$$f(z) = \sum_{i=0} \left(\frac{(\cdot)}{2^{(i+1)}} - (z^{-1}) \right) \quad 1<|z|<2$$

$$f(z) = \frac{1}{2+z} = \frac{\frac{1}{z}}{1+\frac{2}{z}} = \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{-2}{z}\right)}$$ thus $$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)\left(\frac{-2}{z}\right)^k = \sum_{k=0}^{\infty} (-2)^k \left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^k$$

$$\boxed{f(z) = \sum_{k=0}^{\infty} (-2)^k (z^{-1})^{k+1} - \sum_{j=0}^{\infty} (z^{-1})^{j+1} \quad 2<|z|}$$

$$\boxed{f(z) = \sum_{i=0}^{\infty} \left((-2)^i - 1\right)(z^{-1})^{i+1} \quad 2<|z|}$$

H)

$$g(z) = \frac{1}{1-\cos(z)}, \quad z_0 = 0; \quad \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

$$1-\cos(z) = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \dots \quad \left\{ g(0) = \frac{1}{1-\cos(0)} = \frac{1}{0} \right\}$$

$$g(z) = \frac{1}{1-\cos(z)} = \frac{1}{\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \dots}$$

$$\frac{2!}{z^2} + \frac{1}{6} + \frac{z^2}{120} - \frac{z^4}{2520} - \frac{89 z^6}{1814400}$$

$$1$$
$$-\left(1 - \frac{z^2}{4\cdot 3} + \frac{z^4}{6\cdot5\cdot4\cdot3} - \frac{z^6}{8\cdot7\cdot6\cdot5\cdot4\cdot3} + \frac{z^{10}2!}{z^2 10!} \dots\right)$$

$$\frac{z^2}{12} - \frac{z^4}{360} + \frac{z^6}{20160} - \frac{z^8}{1814400} + \dots$$

$$-\left(\frac{z^2}{12} - \frac{z^4}{6\cdot 4!} + \frac{z^6}{6\cdot 6!} - \frac{z^8}{6\cdot 8!} + \dots\right)$$

$$\frac{z^4}{240} - \frac{11 z^6}{20160} + \frac{13 z^8}{3628800}$$

$$-\left(\frac{z^4}{240} - \frac{z^6 (7)}{2880 (7)} + \frac{z^8 (42)}{6!(120)(42)} \dots\right)$$

$$-\frac{z^6}{5040} - \frac{29 z^8}{3628800}$$

$$-\left(-\frac{z^6}{} + \frac{z^8}{}(60) \dots\right)$$

$$\frac{(5)}{(5)} \frac{z^4}{6\cdot4\cdot3\cdot2\cdot1} - \frac{z^4}{6\cdot5\cdot4\cdot3}\frac{(2)}{(2)} = \frac{3z^4}{6!} = \frac{z^4}{240}$$

$$\frac{z^6}{8\cdot7\cdot6\cdot5\cdot4\cdot3}\left(\frac{6\cdot2}{6\cdot2}\right) - \frac{z^6}{6\cdot6\cdot5\cdot4\cdot3\cdot2}\left(\frac{8\cdot7}{8\cdot7}\right) = \frac{-44 z^6}{80640} = \frac{-11 z^6}{20160}$$

$$\frac{z^8}{6\cdot 8!}\left(\frac{10\cdot9}{10\cdot9}\right) - \frac{z^8(2!)}{10\cdot9\cdot8!}\left(\frac{6}{6}\right) = \frac{78 z^8}{21772800}$$

$$= \frac{39 z^8}{10886400} = \frac{13 z^8}{3628800}$$

$$\frac{7z^6}{20160} - \frac{11 z^6}{20160} = \frac{-4 z^6}{20160}$$

$$\frac{13 z^8}{} - \frac{42 z^8}{} = \frac{-29 z^8}{}$$

$$\frac{13z^8}{3628800} - \frac{42z^8}{3628800} = \frac{-29z^8}{3628800}$$

$$-\left(\frac{-z^6}{5040}\overset{5040}{} + \frac{z^8}{2520(4!)}\overset{3628800}{}\frac{(60)}{(60)} - \dots\right)$$

$$\frac{-29z^8 - 60z^8}{3628800} = \frac{-89z^8}{3628800}$$

$$\frac{-89z^8}{3628800} \pm \dots\dots$$

$$\frac{-89z^8}{3628800} \div \left(\frac{\cancel{z^2}}{2!}\right) \cdot \left(\frac{2!}{z^2}\right) = \frac{-89z^6}{1814400}$$

$\cos(n\pi) = 1$ $\quad \boxed{0 < |z| < \pi}$

$g(n\pi) = \frac{1}{0}$ $\quad$ ~~$n\pi < |z| < (n+1)\pi$~~

$$\boxed{g(z) = \frac{2}{z^2} + \frac{1}{6} + \frac{z^2}{120} - \frac{z^4}{2520} - \frac{89z^6}{1814400} \pm \dots = \frac{1}{1-\cos(z)}}$$

\# 8.20 $\quad g(z) = \frac{1}{\sin^2(z)}$, $z_0 = 0$ $\left[\text{for } c_{-4}z^{-4} + \dots + c_0z^0 + \dots + c_4z^4\right]$

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$\left\{g(0) = \frac{1}{\sin^2(0)} = \frac{1}{0^2}\right\}$$

$$\sin^2(z) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right)^2$$

$$\left\{g(n\pi) = \frac{1}{\sin^2(n\pi)} = \frac{1}{0^2}\right\}$$

$$z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \frac{z^8}{7!}$$

$$\boxed{0 < |z| < \pi}$$

$$-\frac{z^4}{3!} + \frac{z^6}{3!3!} - \frac{z^8}{1-1}$$

~~$n\pi < |z| < (n+1)\pi$~~

$\dots\ 3.5,$

$$\frac{1}{\sin(z)} = \left(\frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15120} + \dots\right); \left(\frac{1}{\sin(z)}\right)^2 = \left(\frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15120} + \dots\right)^2$$

$$\frac{1}{z^2} + \frac{1}{6} + \frac{7z^2}{360} + \frac{31z^4}{15120}$$

$$0 + \frac{1}{6} + \frac{z^2}{36}\left(\frac{10}{10}\right) + \frac{7}{6}\cdot\frac{z^4}{360}\left(\frac{7}{7}\right)$$

$$0 + 0 + \frac{7z^2}{360} + \frac{7z^4}{6\cdot 360}\left(7/7\right)$$

$$0 + 0 + 0 + \frac{31z^4}{15120}$$

$$\frac{1}{z^2} + 2\left(\frac{1}{6}\right) + (2\cdot 7 + 10)\left(\frac{z^2}{360}\right) + (2\cdot 31 + 2\cdot 7^2)\left(\frac{z^4}{15120}\right)$$

$$\frac{1}{z^2} + \frac{2}{6} + \frac{24z^2}{360} + \frac{160z^4}{15120}$$

$$\frac{16}{1512}$$

$$2/189$$

$$\boxed{\vartheta(z) = z^{-2} + \frac{1}{3} + \frac{z^2}{15} + \frac{2z^4}{189} + \ldots}$$