MTH 470/570
Winter 2021
Assignment 4 key

This assignment is due before class on Tuesday, February 16. Please submit your solutions via Blackboard. Solutions are required – answers must be justified.

3.41 (a) $e^{i\pi} = \cos\pi + i\sin\pi = -1$.
(c) $i^i = e^{i\text{Log}(i)} = e^{i\cdot i\pi/2} = e^{-\pi/2}$.
(d) $e^{\sin(i)}$. First compute $\sin(i) = \frac{1}{2i}(e^{i\cdot i} - e^{-i\cdot i}) = \frac{1}{2i}(e^{-1} - e^{1}) = \frac{1}{2}(e^1 - e^{-1})i$.
Now $e^{\sin(i)} = \cos(\frac{1}{2}(e^1 - e^{-1})) + i\sin(\frac{1}{2}(e^1 - e^{-1})) \approx .385 + .923i$
(e) $\exp(\text{Log}(3+4i)) = \exp(\ \ln(5) + i\arctan(4/3)) = 5(3/5 + 4i/5) = 3 + 4i$.
(f) $(1+i)^{1/2}$. First compute $\text{Log}(1+i) = \ln(\sqrt{2}) + i\pi/4$, so
$(1+i)^{1/2} = \exp(\ \frac{1}{2}(\ln(\sqrt{2}) + i\pi/8)\ ) = 2^{1/4}\ (\cos(\pi/8) + i\sin(\pi/8)) \approx 1.099 + .455i$.

3.51. Prove that $\exp(b\log a)$ is single valued if and only if b is an integer. (Note that this means that complex exponentials do not clash with monomials z^n, no matter which branch of the logarithm is used.) What can you say if b is rational?

We know that $\log a$ takes on values of the form $x + yi + 2k\pi i = x + (y + 2k\pi)i$ where k ranges over all integers. Therefore
$\exp(b\log a) = \exp(bx + (y + 2kb\pi)i)) = e^{xb}(\ \cos(y + 2kb\pi) + i\sin(y + 2kb\pi)\ )$.

If b is an integer, then $\cos(y + 2kb\pi) = \cos y$ and $\sin(y + 2kb\pi) = \sin y$ for all integers k, so $\exp(b\log z)$ takes only one value, namely $e^{xb}(\cos y + i\sin y)$.

If b is not an integer, then $2b\pi$ is not an integer multiple of 2π, so at least one of the equations $\cos(y + 2b\pi) = \cos y$ and $\sin(y + 2b\pi) = \sin y$ is false, and so $e^{xb}(\ \cos y + i\sin y\ ) \neq e^{xb}(\ \cos(y + 2b\pi) + i\sin(y + 2b\pi)\ )$, that is, $\exp(b\log a)$ takes multiple values.

If b is rational and in lowest terms has denominator j, then $\exp(b\log a)$ takes on exactly j different values.

p.68 #4.3. Integrate the function $f(z) = \overline{z}$ over these three paths from Example 4.1:
(a) The line segment from 0 to $1+i$.
Let $\gamma(t) = t + it$ for $0 \le t \le 1$. Then $\displaystyle\int_\gamma \overline{z}\ dz = \int_0^1 (t - it)(1+i)\ dt = \int_0^1 2t\ dt = 1$.
(b) The arc of the parabola $y = x^2$ from 0 to $1+i$.
Let $\gamma_1(t) = t + it^2$ for $0 \le t \le 1$. Then
$$\int_{\gamma_1} \overline{z}\ dz = \int_0^1 (t - it^2)(1+2ti)\ dt = \int_0^1 t + 2t^3 + it^2\ dt = \frac{1}{2}t^2 + \frac{1}{2}t^4 + \frac{1}{3}it^3\Big|_0^1 = 1 + \frac{1}{3}i.$$
(c) The union of the line segment from 0 to 1 and the line segment from 1 to $1+i$.
Let $\gamma_2(t) = t$ and $\gamma_3(t) = 1 + it$, both for $0 \le t \le 1$.
Then $\displaystyle\int_{\gamma_2+\gamma_3} \overline{z}\ dz = \int_0^1 t\cdot 1dt + \int_0^1 (1 - it)i\ dt = \frac{1}{2}t^2 + it + \frac{1}{2}t^2\Big|_0^1 = 1 + i$.

4.8. Compute $\int_\gamma f(z)\,dz$ for the following functions f and paths γ:

Hint: All but one of these can be evaluated using the complex version of the Fundamental Theorem of Calculus.

(a) $f(z) = z^2$ and $\gamma(t) = t + it^2$, $0 \le t \le 1$.

$$\int_\gamma z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i.$$

(b) $f(z) = z$ and γ is the semicircle from 1 through i to -1.

$$\int_\gamma z\,dz = \frac{1}{2}z^2\Big|_1^{-1} = 0.$$

(c) $f(z) = \exp(z)$ and γ is the line segment from 0 to a point z_0. (The answer will be a function of z_0.)

$$\int_\gamma e^z\,dz = e^z\Big|_0^{z_0} = e^{z_0} - 1.$$

(d) $f(z) = |z|^2$ and γ is the line segment from 2 to 3+i.

$|z|^2 = x^2+y^2$ has no complex antiderivative, so we must resort to more primitive methods.

$\gamma(t) = (2+t) + it$, $\gamma'(t) = 1+i$, $0 \le t \le 1$, so

$$\int_\gamma |z|^2\,dz = \int_0^1 ((2+t)^2 + t^2)(1+i)\,dt = \int_0^1 2t^2 + 4t + 4 + (2t^2 + 4t + 4)i\,dt$$

$$\frac{2}{3}t^3 + 2t^2 + 4t\Big|_0^1 + (\frac{2}{3}t^3 + 2t^2 + 4t)i\Big|_0^1 = \frac{20}{3} + \frac{20}{3}i.$$