MTH 470/570
Winter 2021
Assignment 5.

This assignment is due before class on Tuesday, March 2. Please submit your solutions via Blackboard. Solutions are required – answers must be justified.

p.72 #4.32. Suppose f and g are holomorphic in the region G and γ is a simple piecewise smooth G-contractible path. Prove that if $f(z) = g(z)$ for all $z \in \gamma$, then $f(z) = g(z)$ for all z inside γ.

4.34. Compute

$$I(r) := \int_{C[-2i,r]} \frac{dz}{z^2+1}$$

as a function of r, for $r > 0$, $r \neq 1, 3$.

4.35. Find

$$\int_{C[-2i,r]} \frac{dz}{z^2 - 2z - 8}$$

for r = 1, r = 3 and r = 5.

4.37. Compute the following integrals.

$$(a) \int_{C[-1,2]} \frac{z^2}{4-z^2}\,dz. \qquad (c) \int_{C[0,2]} \frac{\exp(z)}{z(z-3)}\,dz$$

$$(b) \int_{C[0,1]} \frac{\sin z}{z}\,dz \qquad (d) \int_{C[0,4]} \frac{\exp(z)}{z(z-3)}.$$

Problem C. Recall Green's Theorem from multivariate calculus:

Suppose $C : [a,b] \to \mathbf{R}^2$ is a closed curve which bounds the open set D, and the functions $P, Q : \mathbf{R}^2 \to \mathbf{R}^2$ have continuous derivatives on D. Then

$$\int_C P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)\ dt = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dx\ dy.$$

You will use Green's Theorem to prove an important case of Cauchy's Theorem.

Let $\gamma : [a,b] \to \mathbf{C}$ be a closed curve, and suppose that $f : \mathbf{C} \to \mathbf{C}$ is holomorphic on the open set Δ bounded by γ. Let $f(z) = U(z) + iV(z)$ and $\gamma(t) = x(t) + iy(t)$.

(a) Write $\int_\gamma f(z)\ dz$ in terms of U, V, x, y, and multiply out the integrand.

(b) Apply Green's Theorem to the real part and imaginary parts of your answer to (b) (separately) to obtain an equation of the form

$$\int_\gamma f(z)\ dz = \int\int_\Delta \cdots\ dx\ dy + i \int\int_\Delta \cdots\ dx\ dy$$

(c) Since f is, by hypothesis, holomorphic, use the Cauchy-Riemann equations to conclude that $\int_\gamma f(z)\ dz = 0$.