MTH 470/570
Winter 2021
Assignment 3 key

p.62, #3.1. Show that if $f(z) = \dfrac{az+b}{cz+d}$ is a Mobius transformation, then
$$f^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Solve the equation $f(w) = z$ for w: $\dfrac{aw+b}{cw+d} = z \iff aw + b = cwz + dz \iff w(-cz+a) = dz - b \iff w = \dfrac{dz-b}{-cz+a}$. We know from class that f is one-to-one, so we are done.

#3.7. Show that the Mobius transformation $f(z) = \dfrac{1+z}{1-z}$ maps the unit circle (minus the point z = 1) onto the imaginary axis.

A typical point on the unit circle is $z = \cos\theta + i\sin\theta$, so $f(z) = \dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$
$\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} \cdot \dfrac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta} = \dfrac{1-\cos^2\theta-\sin^2\theta+2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} = \dfrac{2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta}$.
This is a pure imaginary number. Since the image of a circle is either a circle or a line, it must be the entire imaginary axis.

#3.14. Find Mobius transformations satisfying each of the following. Write your answers in standard form, as $\dfrac{az+b}{cz+d}$. (It's worth noting that the coefficients are not uniquely determined, since, for example, $\dfrac{az+b}{cz+d} = \dfrac{2az+2b}{2cz+2d}$.)

(a) $1 \to 0$ implies $a = -b$, $3 \to \infty$ implies $d = -3c$,
and $2 \to 1$ implies $2a+b = 2c+d$ implies $a = -c$. Setting $c = 1$, we have $\dfrac{az+b}{cz+d} = \dfrac{-z+1}{z-3}$.
You should check that this satisfies all of the given conditions (for our answers to (b) and (c), as well).

(b) $1 \to 0$ implies $b = -a$, $2 \to \infty$ implies $d = -2c$, and
$1+i \to 1$ implies $a(1+i) + b = c(1+i) + d$ implies $ai = c(-1+i)$. Setting $c = 1$, we have $a = \frac{-1+i}{i} = 1+i$, so $\dfrac{az+b}{cz+d} = \dfrac{(1+i)z+(-1-i)}{z-2}$.

(c) $0 \to i$ implies $b = di$, $\infty \to -i$ implies $a = -ci$, and
$1 \to 1$ implies $a+b = c+d$ implies $-ci+di = c+d$. Setting $c = 1$, we have $d(-1+i) = 1+i$, so $d = \dfrac{1+i}{-1+i} \cdot \dfrac{-1-i}{-1-i} = -i$. Therefore $\dfrac{az+b}{cz+d} = \dfrac{-iz+1}{z-i}$.

3.30. Prove that $\overline{\sin(z)} = \sin(\overline{z})$ and $\overline{\cos(z)} = \cos(\overline{z})$.

First, let's evaluate $\sin z$ for complex z,
Let $z = x + yi$. By definition of $\sin z$,

$$\begin{aligned}
\sin(z) &= \frac{1}{2i}\left(e^{i(x+yi)} - e^{-i(x+yi)}\right) = \frac{1}{2i}\left(e^{-y+xi} - e^{y-xi}\right) \\
&= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right) \\
&= \frac{1}{2}\left(e^{-y}(-i\cos x + \sin x) + e^{y}(i\cos x + \sin x)\right) \\
&= \frac{1}{2}\left((e^{y} + e^{-y})\sin x + i(e^{y} - e^{-y})\cos x\right),
\end{aligned}$$

so $\sin\overline{z} = \frac{1}{2}(e^{y} + e^{-y})\sin x + i(e^{-y} - e^{y})\cos x) = \overline{\sin z}$.
Similarly for $\cos z$: By definition of $\cos z$,

$$\begin{aligned}
\cos(z) &= \frac{1}{2}\left(e^{i(x+yi)} + e^{-i(x+yi)}\right) = \frac{1}{2}\left(e^{-y+xi} + e^{y-xi}\right) \\
&= \frac{1}{2}\left(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right) \\
&= \frac{1}{2}\left((e^{y} + e^{-y})\cos x + i(-e^{y} + e^{-y})\sin x\right),
\end{aligned}$$

so $\cos\overline{z} = \frac{1}{2}(e^{y} + e^{-y})\sin x + i(e^{y} - e^{-y})\sin x) = \overline{\cos z}$.
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\# 3.32. Prove that the zeros of $\sin z$ are all real numbers. Conclude that they are precisely the integer multiples of π.

Suppose that $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = 0$, where $z = x + yi$. Multply both sides by $2i(e^{iz})$, to obtain $e^{2iz} = 1$. Now $e^{2iz} = e^{-2y+2ix} = 1$. Therefore $|e^{-2y+2ix}| = e^{-2y} = 1$, which implies that $y = 0$, that is, z is a real number. We know from trigonometry that the real zeros of sin are the integer multiples of π.

Problem I. Let C be the circle with center a and radius r. Suppose that 0 does not lie on C. (In other words, 0 could be inside or outside of the circle.) Show that the image of C under the function $f(z) = 1/z$ is the circle with center $\dfrac{\overline{a}}{|a|^2 - |r|^2}$ and radius $\dfrac{r}{|\,|a|^2 - r^2\,|}$.

Circle C has equation $|z - a| = r$, that is, $(z - a)(\overline{z - a}) = r^2$, or

$$(*) \qquad z\overline{z} - (a\overline{z} + \overline{a}z) + |a|^2 = r^2.$$

The complex number $1/z$ lies on the second circle (call it C_2) if
$\left|\dfrac{1}{z} - \dfrac{\overline{a}}{|a|^2 - r^2}\right| = \dfrac{r}{|\,|a|^2 - r^2\,|}$, that is, $\left(\dfrac{1}{z} - \dfrac{\overline{a}}{|a|^2 - r^2}\right)\left(\dfrac{1}{\overline{z}} - \dfrac{a}{|a|^2 - r^2}\right) = \left(\dfrac{r}{|\,|a|^2 - r^2\,|}\right)^2$,
or

$$(**) \qquad \frac{1}{z\overline{z}} - \frac{(a/z) + (\overline{a}/\overline{z})}{|a|^2 - r^2} + \frac{|a|^2}{(|a|^2 - r^2)^2} = \frac{r^2}{(|a|^2 - r^2)^2}.$$

We complete the problem by showing that (*) and (**) are equivalent. Start by clearing fractions from (**) to obtain

$$(|a|^2 - r^2)^2 - (a\overline{z} + \overline{a}z))(|a|^2 - r^2)z\overline{z} + |a|^2 z\overline{z} = r^2 z\overline{z}$$

which factors as

$$(|a|^2 - r^2) \ \ (|a|^2 - r^2 - (a\overline{z} + \overline{a}z) + z\overline{z}) = 0$$

If $|a| = r$, then 0 lies on the circle $|z - a| = r$, which by hypothesis it does not – so the first factor above is not 0. Therefore the second factor is 0, which in fact is equivalent to equation (*). We are done.