MTH 470/570
Winter 2021
Assignment 2 key

#1.27. Sketch the sets defined by the following constraints and determine whether they are open, closed, or neither; bounded; connected. (Proofs are not required.)

(a) $|z+3| < 2$. This set is the interior of the circle of radius 2 about 3. It is open, bounded and connected.

(b) $|Im(z)| < 1$. This is the strip between the horizontal lines $y = -1$ and $y = 1$. It is open, unbounded and connected.

(c) $0 < |z-1| < 2$. This is the circle of radius 2 about 1, with the center point 1 removed. It is open, bounded and connected.

(d) $|z-1| + |z+1| = 2$. This is the line seqment from -1 to 1. It is closed, bounded, connected.

(e) $|z-1| + |z+1| < 3$. This is the interior of the ellipse with foci $-1, 1$ and focal diameter 3. It is open, bounded and connected.

(f) $|z| \geq Re(z) + 1$. In other symbols, $\sqrt{x^2+y^2} \leq x+1 \quad \to \quad x^2+y^2 \leq x^2+2x+1 \quad \to$ $y^2 \leq 2x+1 \quad \to \quad x \geq \frac{1}{2}(y^2-1)$. This set is a rightward-facing parabola, along with all points to its right. It is closed, unbounded and connected.

#1.33. Find a parametrization for each of the following paths:

(a) the circle $C[1+i, 1]$, oriented counter-clockwise.

$\gamma(t) = (1+\cos t) + (1+\sin t)\mathbf{i}$, $0 \leq t \leq 2\pi$.

(b) the line segment from $-1-i$ to $2i$.

$\gamma(t) = (-1+t) + (-1+3t)\mathbf{i}$, $0 \leq t \leq 1$.

(d) the rectangle with vertices $\pm 1 \pm 2i$, oriented counter-clockwise.

$$\gamma(t) = \begin{cases} (-1+2t) - 2\mathbf{i}, & 0 \leq t \leq 1 \\ 1 + (-2+4(t-1))\mathbf{i}, & 1 < t \leq 2 \\ (1-2(t-2)) + 2\mathbf{i}, & 2 < t \leq 3 \\ -1 + (2-4(t-3))\mathbf{i}, & 3 \leq t \leq 4. \end{cases}$$

p.31, #2.12. Consider the function $f : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ given by $f(z) = 1/z$. Apply the definition of the derivative to give a direct proof that $f(z) = -1/z^2$.

$$f'(z) = \lim_{h\to 0} \frac{\frac{1}{(z+h)} - \frac{1}{z}}{h} = \lim_{h\to 0} \frac{z-(z+h)}{z(z+h)h} = \lim_{h\to 0} \frac{-h}{z(z+h)h} = \lim_{h\to 0} \frac{-1}{z(z+h)} = \frac{-1}{z^2}.$$

#2.18. For which values of z are the following functions differentiable? Holomorphic? Determine their derivatives at points where they are differentiable.

(a) $f(z) = e^{-x}e^{-iy} = e^{-x}\cos y - \mathbf{i}e^{-x}\sin y$, where $z = x + y\mathbf{i}$.

Method 1: Apply the Cauchy-Riemann equations:

$\frac{\partial}{\partial x}(e^{-x}\cos y) = \frac{\partial}{\partial y}(-e^{-x}\sin y) \iff -e^{x}\cos y = -e^{x}\cos y$, true for all x, y, and

$\frac{\partial}{\partial y}(e^{-x}\cos y) = -\frac{\partial}{\partial x}(-e^{-x}\sin y) \iff -e^{x}\sin y = -e^{x}\sin y$, true for all x, y.

Therefore f is differentiable and holomorphic at all z.

Method 2: Notice that $f(z) = e^{-z}$. We know from class that the exponential function e^z and the function $-z$ are both differentiable everywhere, so their composition f is differentiable and holomorphic everywhere. In other words, f is an entire function.

Just as for the real exponential function, $f(z) = -e^{-z}$.

(c) $f(z) = x^2 + iy^2$. Apply C-R:

$$\frac{\partial}{\partial x}(x^2) = \frac{\partial}{\partial y}(y^2) \iff 2x = 2y \iff x = y, \text{ and}$$

$$\frac{\partial}{\partial y}(x^2) = -\frac{\partial}{\partial x}(y^2) \iff 0 = 0.$$

So: f is differentiable at all z where $Re(z) = Im(z)$, and holomorphic nowhere.

(g) $f(z) = |z|^2 = x^2 + y^2$

Apply C-R:

$$\frac{\partial}{\partial x}(x^2 + y^2) = \frac{\partial}{\partial y}(0) \iff 2x = 0 \iff Re(z) = 0, \text{ and}$$

$$\frac{\partial}{\partial y}(x^2 + y^2) = -\frac{\partial}{\partial x}(0) \iff 2y = 0 \iff y = 0.$$

So: f is differentiable only at $z = 0$, and holomorphic nowhere.

(j) $f(z) = 4(Re\, z)(Im\, z) - i(\overline{z})^2 = 4xy - \mathbf{i}(x - yi)^2 = 4xy - \mathbf{i}x^2 - 2xy + \mathbf{i}y^2$
$= 2xy + (y^2 - x^2)\mathbf{i}$.

Method 1: Use the C-R equations.

$$\frac{\partial}{\partial x}(2xy) = \frac{\partial}{\partial y}(y^2 - x^2) \iff 2y = 2y, \text{ and}$$

$$\frac{\partial}{\partial y}(2xy) = -\frac{\partial}{\partial x}(y^2 - x^2) \iff 2x = 2x.$$

Method 2: Note that $f(z) = -iz^2$.

Both methods lead to the conclusion that f is differentiable and holomorphic at all z.

$f'(z) = -2iz$.

2.20. Prove: If f is holomorphic in the region $G \subset C$ and always real valued, then f is constant in G.

Suppose that $f(x + yi) = P(x + yi)$ (and $Q(x + yi) = 0$). Then the Cauchy-Riemann equations tell us that $\frac{\partial P}{\partial x} = 0$ and $\frac{\partial P}{\partial y} = 0$. This means that P is constant in x and also in y – P is a costant function. Therefore f is a constant function, too.