HW 8 - Cason Konzer

Friday, April 9, 2021 12:17 AM

Cason Konzer

Residues

MTH 470/570
Winter 2021
Assignment 8

This assignment is due before class on Tuesday, April 13. Please submit your solutions via Blackboard. Solutions are required – answers must be justified.

The problems are on the next page. On this page I will outline the key technique to finding most residues (from class), the proof that this technique works (not done in class), and a hint for problem #9.12 that follows from the proof.

Definition: We say that c is a simple zero (also known as a zero of degree 1) of the analytic function g if the power series for g at $z = c$ is $g(z) = b_1(z-c)+b_2(z-c)^2+\cdots$, and $a_1 \neq 0$.

Theorem: (Textbook Proposition 9.14) Suppose functions f and g are holomorphic at $z = c$, and c is a simple zero of g. Then $\operatorname{Res}_{z=c}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{f(c)}{g'(c)}$.

Proof: Since f is holomorphic at c, $f(z) = a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots$.
Since g is holomorphic at c and c is a simple zero of g , $g(z) = b_1(z-c) + b_2(z-c)^2 + \cdots$.
Now let's find the Laurent series of $f(z)/g(z)$ by long division:

$$
\begin{array}{lllll|ccccccc}
 & & & & & & & \frac{a_0}{b_1}(z-c)^{-1} & + & \cdots & & \\
\hline
b_1(z-c) & + & b_2(z-c)^2 & + & \cdots & a_0 & + & a_1(z-c) & + & a_2(z-c)^2 & + & \cdots \\
 & & & & & a_0 & + & \frac{a_0 b_2}{b_1}(z-c) & + & \cdots & & \\
\hline
 & & & & & & & \cdots & & & &
\end{array}
$$

The ellipses are for terms that don't matter. The long division shows that the residue of the quotient $\dfrac{f(z)}{g(z)}$ at $z = c$ is $\dfrac{a_0}{b_1}$. By looking at the power series above for f and g, you can see that $f(c) = a_0$ and $g'(c) = b_1$. In other words, $\operatorname{Res}_{z=c}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{f(c)}{g'(c)}$.

Problem 9.12 asks for the residue in the case where c is a double zero of g, that is, $g(c) = b_2(z-c)^2 + b_3(z-c)^3 + \cdots$. You can prove the given formula by performing the corresponding long division.

p. 149 #9.2. Find the poles or removable singularities of the following functions and determine their orders:

(a) $(z^2+1)^{-3}(z-1)^{-4}$ (b) $z\cot(z)$ (c) $z^{-5}\sin(z)$

(d) $\dfrac{1}{1-\exp(z)}$ (e) $\dfrac{z}{1-\exp(z)}$

#9.8 Use residues to evaluate the following integrals:

(b) $\displaystyle\int_{C[i,2]} \frac{dz}{z(z^2+z-2)}$

(c) $\displaystyle\int_{C[0,2]} \frac{\exp(z)}{z^3+z}\,dz$

(d) $\displaystyle\int_{C[0,1]} \frac{dz}{z^2\sin z}$

#9.12. Extend Proposition 9.14 by proving, if f and g are holomorphic at z_0, which is a double zero of g, then

$$\operatorname{Res}_{z=z_0}\left(\frac{f(z)}{g(z)}\right) = \frac{6f'(z_0)g''(z_0) - 2f(z_0)g'''(z_0)}{3g''(z_0)^2}$$

#9.18. a) Suppose f is an entire function and $a, b \in \mathbf{C}$ with $a \neq b$ and $|a|, |b| < R$. Evaluate

$$\int_{C[0,R]} \frac{f(z)}{(z-a)(z-b)}\,dz,$$

b) Use this result to sketch a proof of Liouville's Theorem 5.13.

Hint: Suppose that f is a bounded entire function. Let a, b be arbitrary complex numbers. Show that, for $z \in C[0,R]$, $R \cdot \frac{f(z)}{(z-a)(z-b)} \to 0$ as $R \to \infty$. Use the fact that,

Whenever $|g(z)| \leq M$ for $z \in \gamma$, it follows that $\left|\int_\gamma g(z)\,dz\right| \leq M\cdot$(arclength of γ)

to conclude that the above integral converges to zero as R increases. Then use the answer to part (a) to show that f must be a constant function.

9.12) $f(z) = a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots$ & $g(z) = b_2(z-c)^2 + b_3(z-c)^3 + \cdots$

So by long division ...

$$\begin{array}{r} \dfrac{a_0}{b_2(z-c)^2} + \dfrac{(a_1b_2 - a_0b_3)}{b_2^{\,2}(z-c)} + \\ \hline b_2(z-c)^2 + b_3(z-c)^3 + \cdots \;\big)\; a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots \\ -\left(a_0 + \dfrac{b_3 a_0 (z-c)}{b_2} + \dfrac{b_4 a_0 (z-c)^2}{b_2} + \cdots\right) \\ \hline \dfrac{(z-c)(a_1b_2 - a_0b_3)}{b_2} + \dfrac{(z-c)^2(a_2b_2 - a_0b_4)}{b_2} \\ -\left(\dfrac{(z-c)(a_1b_2 - a_0b_3)}{b_2} + \dfrac{(z-c)^2 b_3(a_1b_2 - a_0b_3)}{b_2^{\,2}}\right) \\ \hline \dfrac{(z-c)^2\big(b_2(a_2b_2 - a_0b_4) - b_3(a_1b_2 - a_0b_3)\big)}{b_2^{\,2}} \end{array}$$

$$\frac{f(z)}{g(z)} = \frac{a_0}{b_2}(z-c)^{-2} + \frac{(a_1b_2 - a_0b_3)}{b_2^{\,2}}(z-c)^{-1} + \frac{\big(b_2(a_2b_2 - a_0b_4) - b_3(a_1b_2 - a_0b_3)\big)}{b_2^{\,3}} + \cdots$$

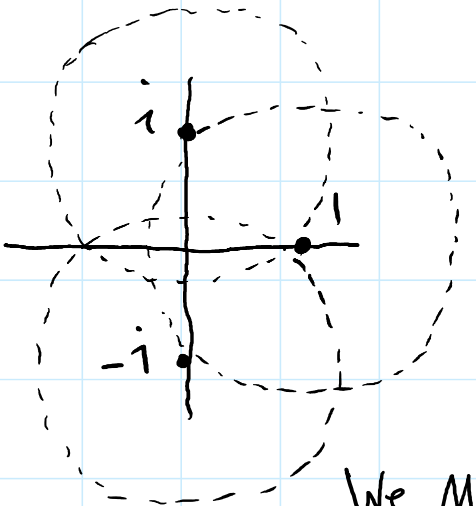
$$\operatorname*{Res}_{z\to c}\big(\eta(z)\big) = \eta_{-1} \quad \text{for } \eta(z) = \cdots + \eta_{-1}(z-c)^{-1} + \eta_0 + \eta_1(z-c) + \cdots$$

let $\dfrac{f(z)}{g(z)} = \eta(z)$ Thus $\eta(z) = \eta_{-2}(z-c)^{-2} + \left(\dfrac{(a_1b_2 - a_0b_3)}{b_2^{\,2}}\right)(z-c)^{-1} + \eta_0 + \cdots$

$$\text{Thus } \operatorname*{Res}_{z\to c}\left(\frac{f(z)}{g(z)}\right) = \operatorname*{Res}_{z\to c}\big(\eta(z)\big) = \eta_{-1} = \boxed{\left(\frac{a_1b_2 - a_0b_3}{b_2^{\,2}}\right)}$$

given $g(z)$ is a double zero @ $z = c$.

p.149 #17.2 a) $(z^2+1)^{-3}(z-1)^{-4} = g(z)$



$$(z^2+1) = (z+i)(z-i)$$

$$(z^2+1)^3 = (z+i)^3(z-i)^3$$

$$g(z) = \frac{1}{(z+i)^3(z-i)^3(z-1)^4}$$

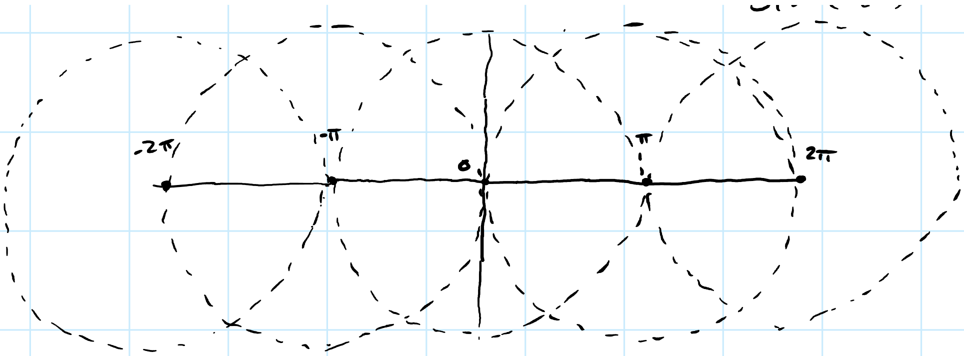
We Mus Consider these 3 singularities

1] $\dfrac{(z+i)^{-3}(z-i)^{-3}}{(z-1)^4}$ pole of order 4 @ $z=1$

i] $\dfrac{(z+i)^{-3}(z-1)^{-4}}{(z-i)^3}$ pole of order 3 @ $z=i$

-i] $\dfrac{(z-i)^{-3}(z-1)^{-4}}{(z+i)^3}$ pole of order 3 @ $z=-i$

b) $z\cot(z) = g(z) = \dfrac{z\cos(z)}{\sin(z)}$ We Must Consider Singularities at all $n\pi$ where



at all $n\pi$; where $\sin(n\pi) = 0 \mid n \in \mathbb{Z}$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + - \cdots$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + - \cdots$$

$$\frac{1}{z} + \frac{z(3!-2!)}{3!2!} + \frac{z^3\left(5!4!(3!-2!) - 3!3!2!(5!-4!)\right)}{5!4!3!3!2!}$$

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + - \cdots \;\Big)\; 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + - \cdots$$

$$-\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + - \cdots\right)$$

$$\frac{z^2(3!-2!)}{3!2!} - \frac{z^4(5!-4!)}{5!4!} + \frac{z^6(7!-6!)}{7!6!}$$

$$-\left(\frac{z^2(3!-2!)}{3!2!} - \frac{z^4(3!-2!)}{3!3!2!} + \frac{z^6(3!-2!)}{5!3!2!} + \cdots\right)$$

$$\frac{z^4\left(5!4!(3!-2!) - 3!3!2!(5!-4!)\right)}{5!4!3!3!2!} - \frac{z^6\left(7!6!(3!-2!) - 5!3!2!(7!-6!)\right)}{7!6!5!3!2!}$$

$$-\left(\frac{z^4\left(5!4!(3!-2!) - 3!3!2!(5!-4!)\right)}{5!4!3!3!2!} - \frac{z^6\left(5!4!(3!-2!) - 3!3!2!(5!-4!)\right)}{5!4!3!3!3!2!}\right)$$

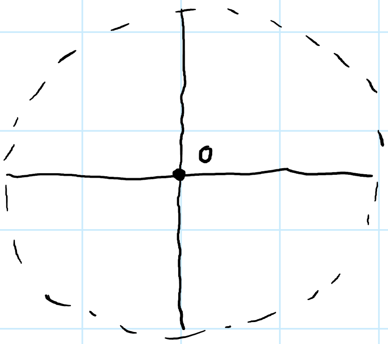
$$\frac{z^6\left(7!6!5!3!2!\left(5!4!(3!-2!) - 3!3!2!(5!-4!)\right) - 5!4!3!3!3!2!\left(7!6!(3!-2!) - 5!3!2!(7!-6!)\right)\right)}{7!6!5!5!4!3!3!3!3!2!2!}$$

So $\dfrac{\cos(z)}{\sin(z)} = \dfrac{1}{z} + \dfrac{z}{3} + \dfrac{z^3}{45} + \dfrac{2z^5}{945} + \cdots$

LAST $\dfrac{z\cos(z)}{\sin(z)} \quad 1 + \dfrac{z^2}{3} + \dfrac{z^4}{45} + \dfrac{z^6}{945} + \cdots$

Thus all the singularities seen @ $z = n\pi$ are removeable

c) $z^{-5}\sin(z) = g(z) = \dfrac{\sin(z)}{z^5}$



A singularity is noted here where $z=0$ | $g(z) = \frac{0}{0}$

We Know

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - + \cdots$$ So

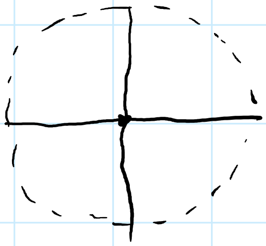
$$\frac{\sin(z)}{z^5} = \frac{1}{z^4} - \frac{1}{z^2 3!} + \frac{1}{5!} - \frac{z^2}{7!} + \frac{z^4}{9!} - + \cdots$$

sin(0) is analytic & we have a 4^{th} order Laurent series $\boxed{\text{thus } 0 \text{ is a pole of order } 4.}$

d) $\dfrac{1}{1-\exp(z)} = g(z)$ $\qquad e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!} + \cdots$

So $\qquad 1 - e^z = -z - \dfrac{z^2}{2!} - \dfrac{z^3}{3!} - \dfrac{z^4}{4!} - \dfrac{z^5}{5!} - \cdots$

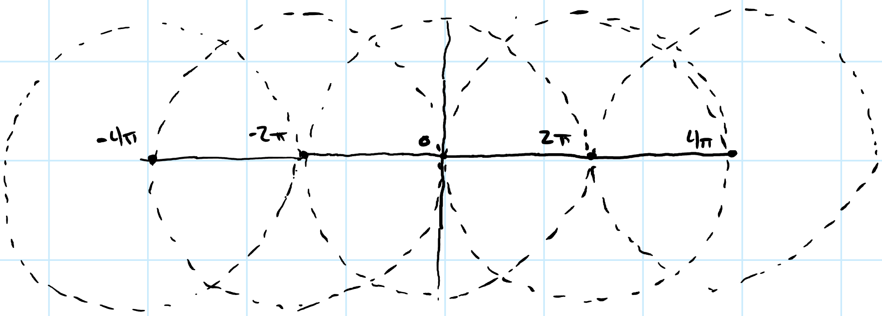
$$\frac{1}{1-e^z} = \sum_{k=0}^{\infty} (e^z)^k = 1 + e^z + e^z e^z + (e^z)^3 + (e^z)^4 + \cdots$$



we consider the singularity at

$z=0 \mid g(z) = \frac{1}{1-1} = \frac{1}{0}$

But : $e^{2\pi} = \cos(2\pi) + i\sin(2\pi) = 1 + 0 = 1$



$\boxed{\text{So any } e^{2k\pi} = 1 \text{ so there are many removable singularities @ } z = 2k\pi \mid k = \mathbb{Z}}$

X X X X X X X X X X

e) $\frac{z}{1-e^{z}} = g(z)$ from before ; $\frac{1}{1-e^{z}} = 1 + e^{z} + (e^{z})^{2} + (e^{z})^{3} + \ldots$

So $g(z) = z + ze^{z} + z(e^{z})^{2} + z(e^{z})^{3} + \ldots$

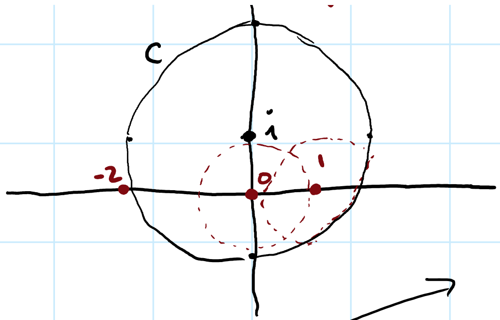
The singularitirs occure in the same locations $z = 2k\pi \mid k \in \mathbb{Z}$ and are again removeable

---

#9.8 | b) $\int_{C[i\ 2]} \frac{dz}{z(z^2+z-2)} = \int_C \frac{dz}{z(z+2)(z-1)} = \int_C f(z)\,dz$

$f(z)$

Consider $\frac{(z+2)^{-1}(z-1)^{-1}}{z}$ & $\frac{z^{-1}(z+2)^{-1}}{(z-1)}$



Consider $\frac{\quad}{z}$, $\frac{\quad}{(z-1)}$

$$\frac{f(c)}{g'(c)} = \frac{1}{(z+2)(z-1)} \qquad \frac{f(c)}{g'(c)} = \frac{1}{z(z+2)}$$

$$\lim_{z\to 0} \frac{1}{(z+2)(z-1)} = \frac{1}{2(-1)} = -\frac{1}{2} \ ; \ \lim_{z \to 1} \frac{1}{z(z+2)} = \frac{1}{1(1+2)} = \frac{1}{3}$$

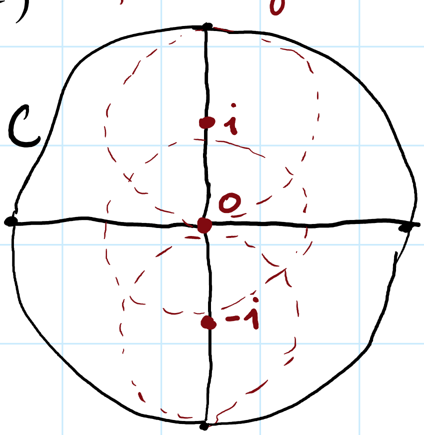
$$\text{So} \int_C f(z)\,dz = 2\pi i\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{\cancel{2}^{1}\pi i}{1}\cdot\cancel{\left(\frac{2}{6} - \frac{3}{6}\right)}\left(-\frac{1}{\cancel{6}_{3}}\right)$$

$$\boxed{\text{We can see} \int_C f(z)\,dz = \frac{-\pi i}{3}}$$

x x x x x x x x x x x x

c) $$\int_{C[0,2]} \frac{e^z\,dz}{z^3+z} = \int_C \frac{e^z\,dz}{z(z^2+1)} = \int_C \frac{e^z\,dz}{z(z+i)(z-i)} = \int_C f(z)\,dz$$

$f(z)$ The Singularities



① $$\frac{e^z (z+i)^{-1}(z-i)^{-1}}{z} = \frac{g(z)}{h(z)}$$

$$\frac{g(z)}{h'(z)} = \frac{e^z}{(z+i)(z-i)} = \zeta_1(z)$$

$$\lim_{z\to 0} \zeta_1(z) = \frac{e^0}{(i)(-i)} = \frac{1}{-(-1)} = 1 \ \ast_1$$

② $$\frac{g(z)}{h(z)} = \frac{e^z z^{-1}(z-i)^{-1}}{z+i}$$

③ $$\frac{g(z)}{h(z)} = \frac{e^z z^{-1}(z+i)^{-1}}{z-i}$$

$\tilde{h}(z) \quad z+i$ | $h(z) \quad z-1$

$$\frac{g(z)}{h'(z)} = \frac{e^z}{(z-i)z} = \xi_2(z) \qquad \frac{g(z)}{h'(z)} = \frac{e^z}{(z+i)z} = \xi_3(z)$$

$$\lim_{z\to -i} \xi_2(z) = \frac{e^{-i}}{(-2i)(-i)} = \frac{e^{-i}}{2i^2} = \frac{-e^{-i}}{2} \; ☆_2 \qquad \lim_{z\to i} \xi_3(z) = \frac{e^{i}}{(2i)i} = \frac{e^{i}}{2i^2} = \frac{e^{i}}{-2} = \frac{-e^{i}}{2} \; ☆_3$$

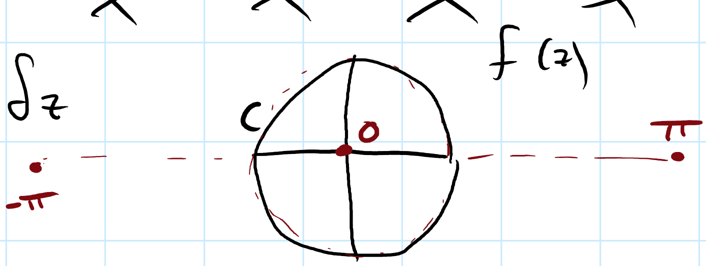
$$\Sigma = 1 - \frac{e^{-i}}{2} - \frac{e^{i}}{2} = \frac{2}{2} - \frac{e^{-i}}{2} - \frac{e^{i}}{2} = \frac{2-e^{i}-e^{-i}}{2}$$

$$\int_C f(z)\,dz = 2\pi i\,\Sigma = \frac{\cancel{2}\pi i}{1} \cdot \left(\frac{2-e^{i}-e^{-i}}{\cancel{2}}\right)$$

☆ $e^{i} = \cos(1) + i\sin(1)$ ☆

$$\boxed{\text{We Can Now See } \int_C f(z)\,dz = \pi i\left(2 - e^{i} - e^{-i}\right)}$$

d) $$\int_{C[0,1]} \frac{dz}{z^2 \sin z} = \int_C f(z)\,dz$$



$\sin(n\pi) = 0$

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15120} + \dots \qquad \underline{\text{From HW 7 ☆}}$$

$$\left(\frac{1}{z^2}\right)\left(\frac{1}{\sin(z)}\right) = \frac{1}{z^3} + \frac{1}{6z} + \frac{7z}{360} + \frac{31z^3}{15120} + \dots = f(z)$$

Now We Can See that $\text{Res}(f(z)) = \frac{1}{6}$

So $\displaystyle\int_C f(z)\,dz = \frac{2\pi i}{6} = \frac{\pi i}{3}$

9.12 [EXTENDED]

$f(z) = a_0 + a_1(z-c) + a_2(z-c)^2 + \dots$ & $g(z) = b_2(z-c)^2 + b_3(z-c)^3 + \dots$

$f'(z) = a_1 + 2a_2(z-c) + \dots$ & $g'(z) = 2b_2(z-c) + 3b_3(z-c)^2 + \dots$ &

$g''(z) = 2b_2 + 6b_3(z-c) + \dots$ & $g''' = 6b_3 + \dots$

So @ $z = z_0$; $a_0 = f(z_0)$, $a_1 = f'(z_0)$, Subbing into
$b_2 = \frac{g''(z_0)}{2}$ & $b_3 = \frac{g'''(z_0)}{6}$ $\quad \text{Res}\left(\frac{f(z)}{g(z)}\right) = \frac{a_1 b_2 - a_0 b_3}{b_2^2}$

$$\text{Res}\left(\frac{f(z)}{g(z)}\right) = \frac{\frac{f'(z_0)g''(z_0)}{2} - \frac{f(z_0)g'''(z_0)}{6}}{\left(\frac{g''(z_0)}{2}\right)^2} = \frac{\frac{3f'(z_0)g''(z_0) - f(z_0)g'''(z_0)}{\cancel{6}\,3}}{\frac{g''(z_0)^2}{\cancel{4}\,2}}$$

$$\boxed{\text{Thus } \text{Res}\left(\frac{f(z)}{g(z)}\right) = \frac{6f'(z_0)g''(z_0) - 2f(z_0)g'''(z_0)}{3g''(z_0)^2}}$$

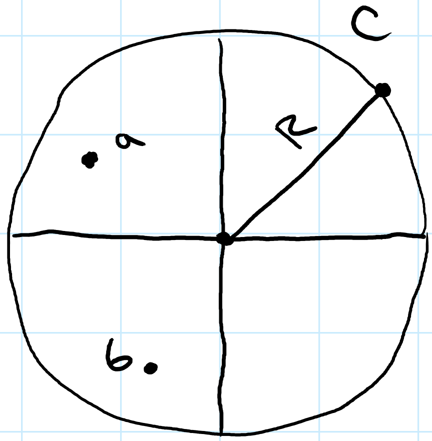
#9.18 | f is entire (has a power series)

$a, b \in \mathbb{C}$; $a \neq b$; $|a|$ & $|b| < R$

D f(z) f

C

$$\int_{C[0,R]} \frac{f(z)\,dz}{(z-a)(z-b)}$$



Residue of a; let $\dfrac{g(z)}{h(z)} = \dfrac{(z-b)^{-1} f(z)}{(z-a)}$

So $\dfrac{g(z)}{h'(z)} = \dfrac{f(z)}{(z-b)} = \eta(z)$

Now $\lim\limits_{z \to a} \eta(z) = \dfrac{f(a)}{(a-b)} = \text{Res}_a$.

Similarly, $\text{Res}_b = \dfrac{f(b)}{(b-a)}$

$$\Sigma = \frac{f(a)}{(a-b)} - \frac{f(b)}{(a-b)} = \frac{f(a) - f(b)}{(a-b)}$$

$$\boxed{\int_{C[0,R]} \frac{f(z)\,dz}{(z-a)(z-b)} = 2\pi i \left(\frac{f(a)-f(b)}{a-b} \right)}$$

$f(z)$

let $g(z) = \dfrac{f(z)}{(z-a)(z-b)}$, Assume $f(z)$ is bounded; thus $g(z)$ is bounded as $z \in C[0,R]$, as $\lim_{R\to\infty} \dfrac{Rf(z)}{(R-a)(R-b)} = \dfrac{\infty f(z)}{(\infty-a)(\infty-b)}$

we have $\dfrac{f(z)}{\infty} = 0$. $\therefore |g(z)| < M$, where M is some bound. It now follows

$$\left|\int_\gamma g(z)\,dz\right| \leq M \cdot 2\pi R \quad \cancel{\not}$$

$$\left|\frac{f(z)}{(z-a)(z-b)}\right| \leq M \quad \text{As } R \text{ is } + ;$$

letting $\check{g}(z) = R\,g(z) \leq M$

We have $\check{g}(z) \leq \dfrac{M}{R}$

Thus as $R \to \infty$ we have $\check{g}(z) \leq 0$

It Follows $\mathbb{R}\check{g}(z) = g(z) \leq 0$

Last $\int_\gamma 0 \, dz = 0$, Thus the given Function is Constant.