Cason Konzer

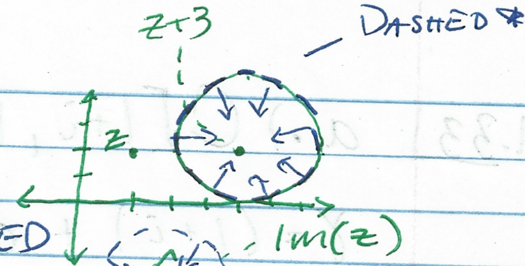
Complex Variables

HW #2

p.16 #1.27

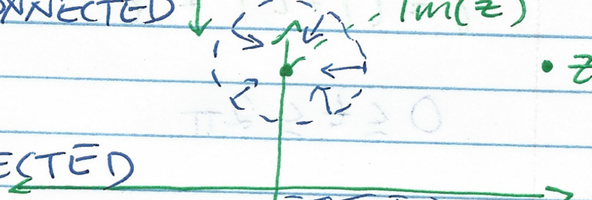
a) $|z+3| < 2$

OPEN, BOUNDED, CONNECTED



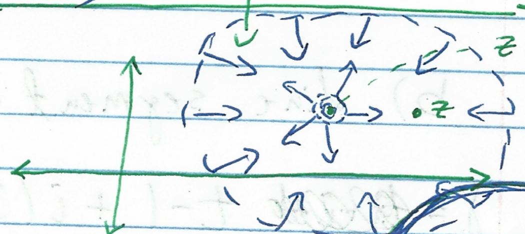
b) $|\mathrm{Im}(z)| < 1$

OPEN, BOUNDED, CONNECTED



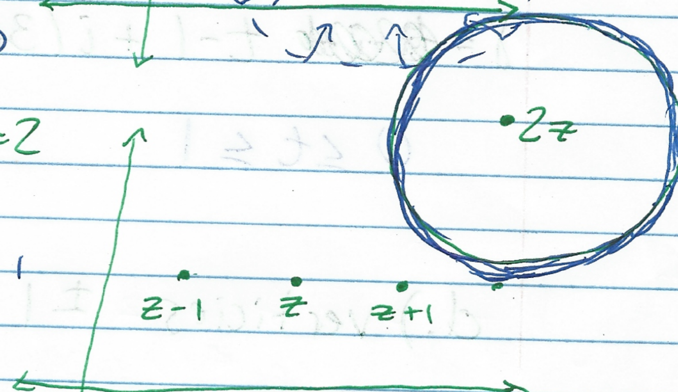
c) $0 < |z-1| < 2$

OPEN, BOUNDED, CONNECTED



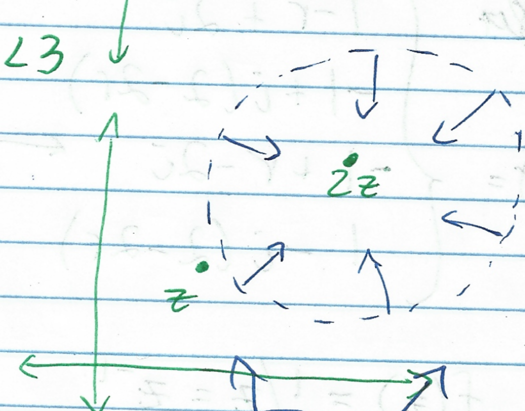
d) $|z-1| + |z+1| = 2$

NEITHER, BOUNDED, CONNECTED



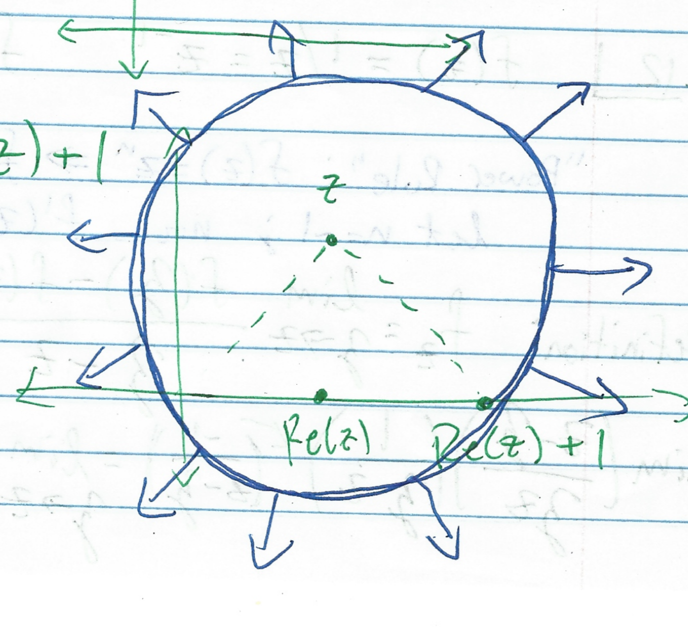
e) $|z-1| + |z+1| < 3$

OPEN, BOUNDED, CONNECTED

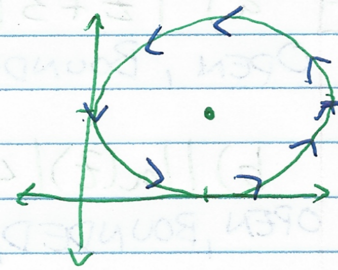


f) $|z| \geq \mathrm{Re}(z) + 1$

Closed, Unbounded, Connected.



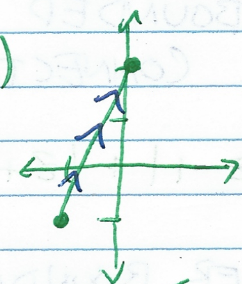
#1.33 a.) $C[1+i, 1]$; CC



$\gamma = (1+i) + e^{it}$

$0 \le t \le 2\pi$

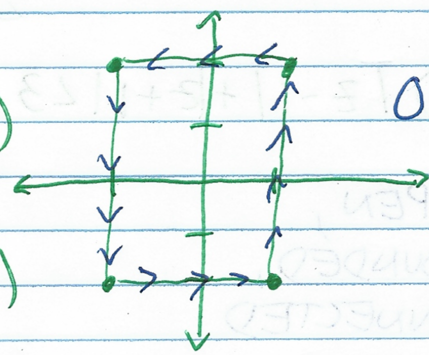
b.) line segment: $(-1-i) \to (2i)$



$\gamma = t-1 + i(3t-1)$

$0 \le t \le 1$

d.) verticies $\pm 1 \pm 2i$; CC



$$\gamma = \begin{cases} 1-r+2i \\ -1+i(2-2r) \\ -1+r-2i \\ 1-i(2-2r) \end{cases} \qquad 0 \le r \le 2$$

p. 31 **#2.12** $f(z) = 1/z = z^{-1}$ $\quad f'(z) = -1/z^2 = -z^{-2}$

"Power Rule": $f(z) = z^n \Rightarrow f'(z) = nz^{n-1}$

let $n=-1$; now $f'(z) = (-1)z^{(-1-1)} = -z^{-2}$

By Definition: $f_z = \lim_{\zeta \to z} \dfrac{f(\zeta)-f(z)}{\zeta - z} = \lim \left(\dfrac{1}{\zeta} - \dfrac{1}{z}\right)\left(\dfrac{1}{\zeta - z}\right)$

$= \lim \left(\dfrac{z-\zeta}{\zeta z}\right)\left(\dfrac{1}{\zeta - z}\right) \; \left(\dfrac{-1}{z-\zeta}\right) = \lim_{\zeta \to z} \dfrac{-1}{\zeta z} = \dfrac{-1}{z^2} = -z^{-2}$

C.R. $\{P_x = Q_y$ & $P_y = -Q_x\}$

Differentiable? Holomorphic? Derivatives?

#2.18

a) $f(z) = e^{-x}e^{-iy} = \underset{P}{e^{-x}\cos y}\ (\underset{Q}{-e^{-x}\sin y})i$

Derivatives
$P_x = -e^{-x}\cos y$
$P_y = -e^{-x}\sin y$
$Q_x = e^{x}\sin y$
$Q_y = -e^{x}\cos y$

$-Q_x = -e^{x}\sin y$

✓ $P_x = Q_y$ & $P_y = -Q_x$

This Eq. Satisfies the Cauchy Riemann Eqs. & Is thus differentiable Everywhere & Holomorphic.

c) $f(z) = x^2 + iy^2$

$\uparrow P \uparrow Q$

~~$D(x^n) = nx^{n-1}$~~

$P_x = 2x$
$P_y = 0$
$Q_x = 0$
$Q_y = 2y$

$P_x = Q_y$
$2x = 2y$

$P_y = -Q_x$
$0 = -0$

This E.Q. is both differentiable & Holomorphic where $x = y$ & $\forall (x,y) \in \mathbb{C}$

$2x = 2y$ given $x = y$

This implies either ($f(z) = x^2 + ix^2$ (1) or $y^2 + iy^2$ (2))

The derivatives are then

(1) $P_x = 2x$ $P_y = 0$
$Q_x = 2x$ $Q_y = 0$

(2) $P_x = 0$ $P_y = 2y$
$Q_x = 0$ $Q_y = 2y$

(1) $f'(z) = 2x + 2ix$

(2) $f'(z)\ 2y + 2iy$

$|z|^2 = (a+bi)(a-bi) = a^2+b^2 = z\bar{z}$

$|z| = \sqrt{a^2+b^2}$

(g) $f(z) = |z|^2 = \underbrace{x^2+y^2}_{P} = (z\bar{z})^2$

CR

$P_x = Q_y$

$P_y = -Q_x$

$P_x = 2x$

$P_y = 2y$

$Q_x = 0$

$Q_y = 0$

$P_x \neq Q_y$

$P_y \neq Q_x$

This E.Q. $f(z)$ is differentiable nowhere; thus it is neither holomorphic. There are no defined derivatives

$\bar{z} = Re(z) - Im(z)$

Derivatives

$P_x = 2y$

$Q_x = 2x$

$P_y = 2x$

$Q_y = 2y$

$P_x = Q_y$ ✓

$P_y = -Q_x$ ✓

(j) $f(z) = 4(Re\,z)(Im\,z) - i(\bar{z})^2$

$4(P)(Q)$

$f(z) = 4PQ - i(P - iQ) = 4PQ - Q + i(P)$

$-iP + i^2Q$

$z = Q(4P - 1) + iP$

$\bar{z} = a - bi \quad \bar{z}^2 = a^2 - 2abi - b^2$

$f(a+bi) = 4ab - i(a^2 - 2abi - b^2)$

$= 4ab - ia^2 + 2abi^2 + bi^2$

$= 4ab - 2ab - ia^2 + bi^2$

$= 2ab + i(a^2 + b^2)$

$x\ y \qquad x^2 \quad y^2$

(P) (Q)

This E.Q. is differentiable & Holomorphic everywhere $\forall (x,y) \in \mathbb{C}$

As $P_x = Q_x$ & $P_y = Q_y$ $P = \pm Q$ $\Rightarrow\Leftarrow$ thus the derivatives $\{P_x, Q_x, P_y, Q_y\} = 0$.

#2.20 Prove: If f is holomorphic in the region $G \subset \mathbb{C}$ & always real valued, then f is constant in G.

$f(z) = P(x,y) + iQ(x,y)$

$\overline{f(z)} = P(x,y) - iQ(x,y)$

As f is holomorphic, f satisfys

$P_x = Q_y$

$P_y = -Q_x$

$P = \int P_x \, dx$

$= \int Q_y \, dy$

$\int P_x \, dx = \int Q_y \, dy$

$Q = \int P_y \, dy = \int Q_x \, dx$

Because the E.Q. yields both $P = Q$ & $P = -Q$ when allowing $P_x = Q_x$ & $P_y = -Q_y$ there is a contradiction; thus P_x & P_y, Q_y & Q_x do not exist.

$P_x = Q_x$

$P_y = -Q_y$

$\Rightarrow$ $P = Q$, $P = -Q$

This yields part Q & P have no x or y.

If $Q = -Q$ $1 = -1$ ... $\Rightarrow\Leftarrow$

Thus $Q = 0$.

$P_{xy} = Q_{yy}$ $\quad Q_{yy} = -Q_{xx}$

$P_{yx} = -Q_{xx}$

$Q_{yx} = P_{xx}$ $\quad P_{xx} = -P_{yy}$

$+Q_{xy} = P_{yy}$

$f(z)_{xx} = P_{xx} \overset{i\,i\,Q_{x,y}}{-Q_{xx}} = P_{xx} + P_{yx}$ $\quad f_x = P_x + P_y$

$\cancel{= P_{yy} + P_{yx}}$

$f(z)_{yy} = P_{yy} - Q_{yy} = Q_{xy} - Q_{yy}$ $\quad f_y = Q_x - Q_y$

$f_{xy} = P_{xy} + P_{yy}$ $\quad \overset{-P_{yy}}{P_{xx}} + P_{yy} = Q_{xx} - Q_{yx}$

$f_{yx} = Q_{xx} - Q_{xx}$ $\quad 0 = Q_{xx} - Q_{yx}$

$Q = P \Leftarrow Q_{xx} - P_{xx} = 0$ $\quad Q_{yx} = Q_{xx} \Rightarrow Q_y = 1$