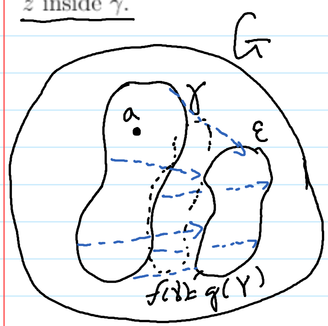
HW 5 - Cason Konzer

Monday, March 1, 2021 8:10 PM

p.72 #4.32. Suppose f and g are holomorphic in the region G and γ is a simple piecewise smooth G-contractible path. Prove that if $f(z) = g(z)$ for all $z \in \gamma$, then $f(z) = g(z)$ for all z inside γ.



Known

* No singularities are inside γ *

f & g are differentiable everywhere

f & g map γ to ε w/in G

Thus $\gamma \sim_G \varepsilon$. By the Corollary to Cauchy's Theorem: $\int_\gamma f = \int_\gamma g = 0$

Now Using CIF: consider $z \in \gamma$

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz \quad \& \quad g(a) = \frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z-a}\,dz$$

We know $f(z) = g(z)$ sub ε_0

$$\text{Now } f(a) = \frac{1}{2\pi i}\int_\gamma \frac{\varepsilon_0\,dz}{z-a} = g(a) \quad \& \quad f(a) = \varepsilon = g(a)$$

let $a = z$ inside γ & $f(z) = g(z)$ $\forall$ z inside γ ∎

4.34. Compute

$$I(r) := \int_{C[-2i,r]} \frac{dz}{z^2+1}$$

as a function of r, for $r > 0$, $r \neq 1, 3$.

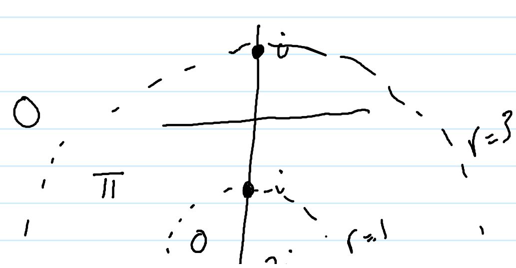
$$\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i} \Rightarrow 1 = A(z-i) + B(z+i)$$

$$@\ z = i, \quad 1 = B(2i) \quad B = 1/2i = \frac{-i}{2} \quad \therefore \frac{1}{z^2+1} = \frac{\frac{i}{2}}{z+i} + \frac{-\frac{i}{2}}{z-i}$$

$$@\ z = -i \quad 1 = A(-2i) \quad A = -1/2i = \frac{i}{2}$$

Now,

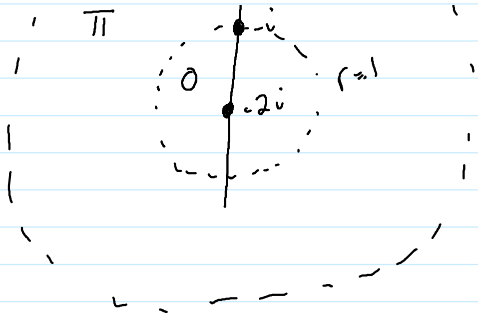
$$I(r) = \int_{C[\ldots]} \overset{①}{\frac{i/2}{z+i}} + \overset{③}{\frac{-i/2}{z-i}}\,dz$$



$$I(r) = \int_{C[-2i,r]} \frac{\cdots}{z+i} + \frac{\cdots}{z-i}\,dz \qquad \pi$$

$$\frac{2\pi i \cdot -i}{2} = \pi$$

$$f(z) = i/2 = f(-2i) \qquad \cancel{\frac{2\pi i \cdot i}{2}} = -\pi$$



$$I(r) = \begin{cases} 0 & ; \ r < 1 \\ -\pi & ; \ 1 < r < 3 \\ 0 & ; \ r > 3 \qquad (-\pi + \pi) \end{cases}$$

4.35. Find

$$I(r) = \int_{C[-2i,r]} \frac{dz}{z^2 - 2z - 8}$$

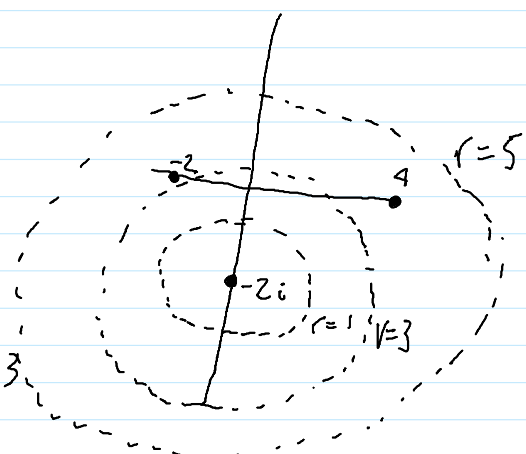
for r = 1, r = 3 and r = 5.

$$\frac{1}{z^2 - 2z - 8} = \frac{A}{(z-4)} + \frac{B}{(z+2)} \ ; \ 1 = A(z+2) + B(z-4)$$

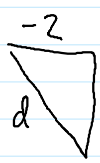
@ $z = -2$, $1 = B(-6)$, $B = -1/6$ $\therefore \frac{1}{z^2 - 2z - 8} = \frac{1/6}{z-4} + \frac{-1/6}{z+2}$

@ $z = 4$, $1 = A(6)$, $A = 1/6$

$$I(r) = \int_{C[-2i,r]} \frac{1/6}{z-4} + \frac{-1/6}{z+2}\,dz$$

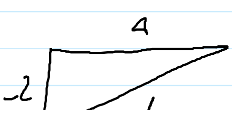


$(0,-2) \to (-2,0)$



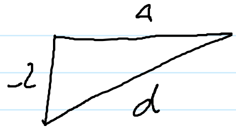
$d = \sqrt{8} < \sqrt{9} = 3$

$(0,-2) \to (0,4)$



$d = \sqrt{20} < \sqrt{25} > \sqrt{9}$

$(0,-2) \to (0,4)$



$d = \sqrt{20} < \sqrt{25} > \sqrt{9}$

$I(1) = 0 = I(3)$, $I(5) = 2\pi i\left(1/6 - 1/6\right) = 0$

Thus $I(r)\big|_{r=1,3,5} = 0$

4.37. Compute the following integrals.

$$(a) \int_{C[-1,2]} \frac{z^2}{4-z^2}\, dz. \qquad (c) \int_{C[0,2]} \frac{\exp(z)}{z(z-3)}\, dz$$

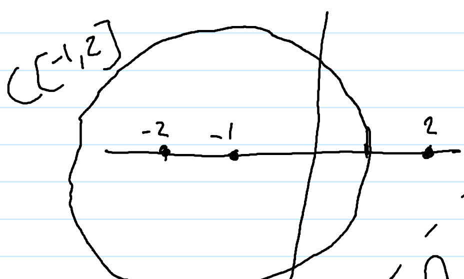
$$(b) \int_{C[0,1]} \frac{\sin z}{z}\, dz \qquad (d) \int_{C[0,4]} \frac{\exp(z)}{z(z-3)}.$$

a) $\dfrac{z^2}{(4-z^2)} = \dfrac{A}{(2+z)} + \dfrac{B}{(2-z)}$; $z^2 = (2-z)A + (2+z)B$

@ $z=2$, $4 = 4B$, $B=1$ $\qquad \dfrac{z^2}{4-z^2} = \dfrac{1}{2+z} + \dfrac{1}{2-z} = \dfrac{1}{z+2} + \dfrac{-1}{z-2}$

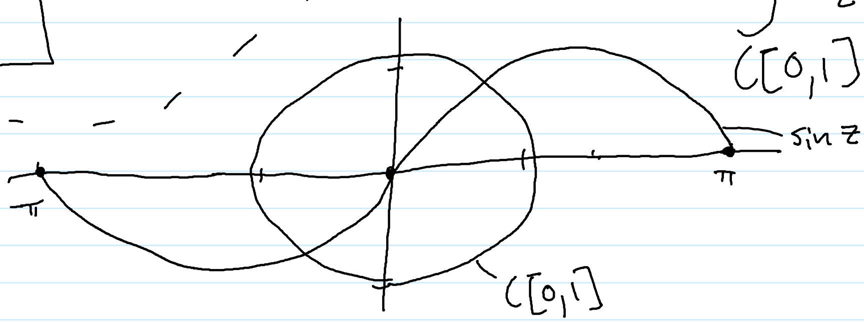
@ $z=-2$, $4 = 4A$, $A=1$

$$\int_{C[-1,2]} \frac{1}{z+2} + \cancel{\frac{-1}{z-2}}^{\,0}\, dz$$



b) $\displaystyle\int_{C[0,1]} \frac{\sin z}{z-0}\, dz$

$\boxed{I = 2\pi i}$



$\sin(z)$ is holomorphic in $\mathbb{C}$

sin(z) is holomorphic in $\mathbb{C}$ $\qquad T \qquad C[0,1]$

$$I = 2\pi i f(a) = 2\pi i \sin(0) \quad \therefore \boxed{I = 0}$$

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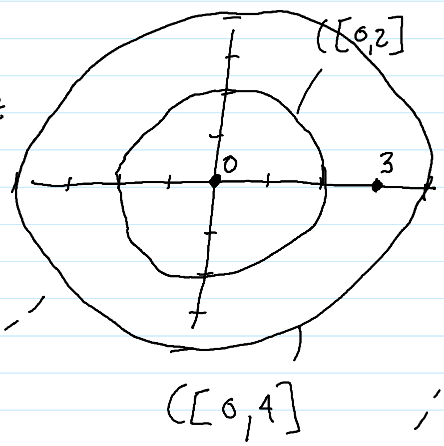
c) $\dfrac{e^z}{z(z-3)} = \dfrac{A}{z} + \dfrac{B}{z-3}$; $e^z = A(z-3) + Bz$

@ $z=0$, $1 = -3A$, $A = -1/3$ $\therefore \dfrac{e^z}{z(z-3)} = \dfrac{-1/3}{z-0} + \dfrac{e^3/3}{z-3}$

@ $z=3$, $e^3 = 3B$, $B = e^3/3$ $\qquad \gamma = C[0,2]$

$$I = \int_\gamma \frac{f\,dz}{z-a} = \int_{C[0,2]} \frac{-1/3}{z-0} + \cancel{\frac{e^3/3}{z-3}}\,dz$$



$$I = 2\pi i\left(-1/3\right) \qquad I = \frac{-2\pi i}{3}$$

d)

$$I = \int_{C[0,4]} \frac{-1/3}{z-0} + \frac{e^3/3}{z-3}\,dz = 2\pi i\left(\frac{-1}{3} + \frac{e^3}{3}\right) \therefore \boxed{I = \frac{2\pi i}{3}\left(e^3 - 1\right)}$$

Problem C. Recall Green's Theorem from multivariate calculus:

Suppose $C : [a, b] \to \mathbf{R}^2$ is a closed curve which bounds the open set D, and the functions $P, Q : \mathbf{R}^2 \to \mathbf{R}^2$ have continuous derivatives on D. Then

$$\int_C P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)\ dt = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dx\ dy.$$

You will use Green's Theorem to prove an important case of Cauchy's Theorem.

Let $\gamma : [a, b] \to \mathbf{C}$ be a closed curve, and suppose that $f : \mathbf{C} \to \mathbf{C}$ is holomorphic on the open set Δ bounded by γ. Let $f(z) = U(z) + iV(z)$ and $\gamma(t) = x(t) + iy(t)$. $\quad z = x + iy$

$dz = dx + i\,dy$

(a) Write $\int f(z)\ dz$ in terms of U, V, x, y, and multiply out the integrand.

$dz = dx + i\,dy$

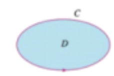
(a) Write $\int_\gamma f(z)\,dz$ in terms of U, V, x, y, and multiply out the integrand.

$$I = \int_{x(t)+iy(t)} (U(z) + iV(z))(dx + i\,dy) = \int_\gamma U(z)\,dx + iV(z)\,dx + iU(z)\,dy - V(z)\,dy$$

$$= \int_\gamma U\,dx - V\,dy + i(V\,dx + U\,dy) = \boxed{\int_{(x(t)+iy(t))} U(x+iy)\,dx - V(x+iy)\,dy + i\int_{(x(t)+iy(t))} V(x+iy)\,dx + U(x+iy)\,dy}$$

(b) Apply Green's Theorem to the real part and imaginary parts of your answer to (b) (separately) to obtain an equation of the form

Green's Theorem $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$



$$\int_\gamma f(z)\,dz = \int\int_\Delta \cdots\,dx\,dy + i\int\int_\Delta \cdots\,dx\,dy$$

$$\int_\gamma f\,dz = \iint_\Delta \left(\frac{-\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) dx\,dy + i\iint_\Delta \left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right) dx\,dy$$

$$\boxed{\underline{I} = \iint_\Delta -(V_x + U_y)\,dx\,dy + i\iint_\Delta (U_x - V_y)\,dx\,dy}$$

(c) Since f is, by hypothesis, holomorphic, use the Cauchy-Riemann equations to conclude that $\int_\gamma f(z)\,dz = 0$.

$$\text{C.R. Eqs.} \equiv \left(U_x = V_y \ \wedge\ U_y = -V_x\right)$$

Thus $$\underline{I} = \iint_\Delta -(V_x - V_x)\,dx\,dy + i\iint_\Delta (U_x - U_x)\,dx\,dy$$

$$= \iint_\Delta (-0 + i0)\,dx\,dy = \iint \cancel{0\,dx\,dy} = \iint 0 = 0$$

$$\therefore\ \underline{I} = 0$$

Diff possible approach...

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$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b f(x(t)+iy(t))(x'(t)+iy'(t))\,dt$$

$$= \int_a^b \Big(U(x(t)+iy(t)) + iV(x(t)+iy(t))\Big)(x'(t)+iy'(t))\,dt$$

$$= \int_a^b (U+iV)(\dot{x}+i\dot{y})\,dt = \int_a^b U\dot{x}-V\dot{y}\,dt + i\int_a^b V\dot{x}+U\dot{y}\,dt \qquad dt=dx\,dy$$

$$= \iint_\Delta -V_x-U_y\,dt + i\iint_\Delta U_x-V_y\,dt$$

time depends on position, $t(x,y) \Rightarrow dt=dx\,dy$