MTH 470/570
Winter 2021
Assignment 8

This assignment is due before class on Tuesday, April 13. Please submit your solutions via Blackboard. Solutions are required – answers must be justified.

The problems are on the next page. On this page I will outline the key technique to finding most residues (from class), the proof that this technique works (not done in class), and a hint for problem #9.12 that follows from the proof.

Definition: We say that $c$ is a simple zero (also known as a zero of degree 1) of the analytic function $g$ if the power series for $g$ at $z = c$ is $g(z) = b_1(z-c)+b_2(z-c)^2+\cdots$, and $a_1 \neq 0$.

Theorem: (Textbook Proposition 9.14) Suppose functions $f$ and $g$ are holomorphic at $z = c$, and $c$ is a simple zero of $g$. Then $\mathrm{Res}_{z=c}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{f(c)}{g'(c)}$.

Proof: Since $f$ is holomorphic at $c$, $f(z) = a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots$.
Since $g$ is holomorphic at $c$ and $c$ is a simple zero of $g$ , $g(z) = b_1(z-c)+b_2(z-c)^2+\cdots$.
Now let's find the Laurent series of $f(z)/g(z)$ by long division:

$$
\begin{array}{ccccc|ccccc}
 & & & & & \frac{a_0}{b_1}(z-c)^{-1} & + & \cdots & & \\
\hline
b_1(z-c) & + & b_2(z-c)^2 & + & \cdots & a_0 & + & a_1(z-c) & + & a_2(z-c)^2 + \cdots \\
 & & & & & a_0 & + & \frac{a_0 b_2}{b_1}(z-c) & + & \cdots \\
\hline
 & & & & & & & \cdots & &
\end{array}
$$

The ellipses are for terms that don't matter. The long division shows that the residue of the quotient $\dfrac{f(z)}{g(z)}$ at $z = c$ is $\dfrac{a_0}{b_1}$. By looking at the power series above for $f$ and $g$, you can see that $f(c) = a_0$ and $g'(c) = b_1$. In other words, $\mathrm{Res}_{z=c}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{f(c)}{g'(c)}$.

Problem 9.12 asks for the residue in the case where $c$ is a double zero of $g$, that is, $g(c) = b_2(z-c)^2 + b_3(z-c)^3 + \cdots$. You can prove the given formula by performing the corresponding long division.

p. 149 #9.2. Find the poles or removable singularities of the following functions and determine their orders:

(a) $(z^2+1)^{-3}(z-1)^{-4}$ (b) $z\cot(z)$ (c) $z^{-5}\sin(z)$

(d) $\dfrac{1}{1-\exp(z)}$ (e) $\dfrac{z}{1-\exp(z)}$

#9.8 Use residues to evaluate the following integrals:

(b) $\displaystyle\int_{C[i,2]} \frac{dz}{z(z^2+z-2)}$

(c) $\displaystyle\int_{C[0,2]} \frac{\exp(z)}{z^3+z}\,dz$

(d) $\displaystyle\int_{C[0,1]} \frac{dz}{z^2\sin z}$

#9.12. Extend Proposition 9.14 by proving, if $f$ and $g$ are holomorphic at $z_0$, which is a double zero of $g$, then

$$\mathrm{Res}_{z=z_0}\left(\frac{f(z)}{g(z)}\right) = \frac{6f'(z_0)g''(z_0) - 2f(z_0)g'''(z_0)}{3g''(z_0)^2}$$

#9.18. a) Suppose $f$ is an entire function and $a, b \in \mathbf{C}$ with $a \neq b$ and $|a|, |b| < R$. Evaluate

$$\int_{C[0,R]} \frac{f(z)}{(z-a)(z-b)}\,dz,$$

b) Use this result to sketch a proof of Liouville's Theorem 5.13.

Hint: Suppose that $f$ is a bounded entire function. Let $a, b$ be arbitrary complex numbers. Show that, for $z \in C[0,R]$, $R \cdot \frac{f(z)}{(z-a)(z-b)} \to 0$ as $R \to \infty$. Use the fact that,

Whenever $|g(z)| \leq M$ for $z \in \gamma$, it follows that $\left|\int_\gamma g(z)\,dz\right| \leq M\cdot$(arclength of $\gamma$)

to conclude that the above integral converges to zero as R increases. Then use the answer to part (a) to show that $f$ must be a constant function.