# HW 3 - Cason Konzer

Saturday, February 6, 2021 10:53 PM

MTH 470/570
Winter 2021
Assignment 3.

This assignment is due before class on Tuesday, February 9. Please submit your solutions via Blackboard. Solutions are required – answers must be justified.

p.62, #3.1. Show that if $f(z) = \dfrac{az+b}{cz+d}$ is a Mobius transformation, then

$$f^{-1}(z) = \frac{dz-b}{-cz+a}.$$

#3.7. Show that the Mobius transformation $f(z) = \dfrac{1+z}{1-z}$ maps the unit circle (minus the point z = 1) onto the imaginary axis.

#3.14. Find Mobius transformations satisfying each of the following. Write your answers in standard form, as $\dfrac{az+b}{cz+d}$.

(a) $1 \to 0$, $2 \to 1$, $3 \to \infty$.

(b) $1 \to 0$, $1+i \to 1$, $2 \to \infty$

(c) $0 \to i$ , $1 \to 1$, $\infty \to -i$ .

# 3.30. Prove that $\overline{\sin(z)} = \sin(\overline{z})$ and $\overline{\cos(z)} = \cos(\overline{z})$.

# 3.32. Prove that the zeros of $\sin z$ are all real numbers. Conclude that they are precisely the integer multiples of $\pi$.

Problem I. Let $C$ be the circle with center $a$ and radius $r$. Suppose that 0 does not lie on $C$. (In other words, 0 could be inside or outside of the circle.) Show that the image of $C$ under the function $f(z) = 1/z$ is the circle with center $\dfrac{\overline{a}}{|a|^2 - |r|^2}$ and radius $\dfrac{r}{|\,|a|^2 - r^2\,|}$.

Hints: This is an exercise in complex algebra. Start by writing down the equations for both circles. Use the fact that, for any complex number $w$, $w\overline{w} = |w|^2$, to eliminate all of the complex norms in both equations. Starting from the equation of the second circle, clear all of the fractions and factor out a common term to work toward the equation of the first circle.

p.62 #3.1 Let $f(z) = \dfrac{az+b}{cz+d}$ suffice $ad-bc \neq 0$

Thus, $f$ is a möbius transform.

Let $f(z) = w$ Now $w = \dfrac{az+b}{cz+d}$ ← $z \neq \frac{-d}{c}$

$wcz + wd = az + b \qquad z(wc - a) = b - wd$

$z = \frac{b - wd}{wc - a} \leftarrow w \neq \frac{a}{c}$ Swap w w/ z

Now $'w = \frac{-zd + b}{zc - a}$ ; $'w^{-1} = \frac{az + b}{cz + d}$

$z \neq \frac{a}{c}$

$$\boxed{\therefore \; f^{-1}(z) = {'w} = \frac{-zd + b}{zc - a} \quad \Big| \; z \neq \frac{a}{c}}$$

#3.7 $f(z) = \frac{1+z}{1-z}$ | $ad - bc \neq 0$ as $-z - z = -2z$

but $f(z) = w = \frac{1+z}{1-z}$ unit circle $= e^{i\theta} = \cos\theta + i\sin\theta$

$\ldots ; = \underbrace{|z| = 1}$

$w - wz = 1 + z \qquad w - 1 = z + wz$

$w - 1 = z(w+1) \qquad z = \frac{w-1}{w+1} \Rightarrow |z| = \left|\frac{w-1}{w+1}\right|$

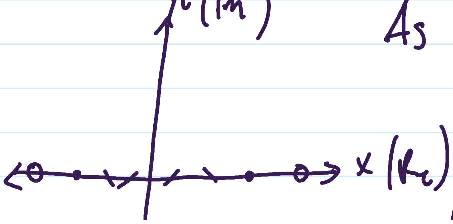
$\frac{|w-1|}{|w+1|} = |1| \Rightarrow |w-1| = |w+1|$ w = some $x + yi$

Re Im

$|x + yi - 1| = |x + yi - 1| \Rightarrow |(x+1) + yi| = |(x-1) + yi|$

$(x+1)^2 + \cancel{y^2} = (x-1)^2 + \cancel{y^2} \qquad \cancel{x^2} + 2x \cancel{+1} = \cancel{x^2} - 2x \cancel{+1}$

$2x = -2x \Rightarrow x = -x$ ; $2x = 0$




As x cannot be -x simultaneously;

$x = 0 \therefore f(z) = Re(0) + Im(y)$

Thus $f(e^{i\theta})$ is strictly imaginary

$\leftarrow \circ \cdot \times \cdot \times \cdot \circ \rightarrow x\ (\mathrm{Re})$

Thus $f(e^{i\theta})$ is strictly imaginary.

#3.14 Recall $\dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

a) $z_1 = 1 \quad w_1 = 0$
$z_2 = 2 \quad w_2 = 1$
$z_3 = 3 \quad w_3 = \infty$

$$\frac{(w-0)\cancel{(1-\infty)}}{\cancel{(w-\infty)}(1-0)} = w = \frac{(z-1)(2-3)}{(z-3)(2-1)}$$

$f(z) = w \qquad w = \dfrac{-(z-1)}{(z-3)} \qquad \therefore \boxed{f(z) = \dfrac{-z+1}{z-3}}$

b) $z_1 = 1 \quad w_1 = 0$
$z_2 = 1+i \quad w_2 = 1$
$z_3 = 2 \quad w_3 = \infty$

$$\frac{(w-0)\cancel{(1-\infty)}}{\cancel{(w-\infty)}(1-0)} = w = \frac{(z-1)(1+i-2)}{(z-2)(1+i-1)}$$

$$w = \frac{(z-1)(i-1)}{(z-2)(i)} = w \cdot 1 = w \cdot \frac{i}{i} = \frac{(z-1)(-i-1)}{-(z-2)} = w \cdot \frac{-1}{-1} = \frac{(z-1)(i+1)}{z-2}$$

$$\boxed{f(z) = \frac{z(i+1) - (i+1)}{z-2}}$$

c) $z_1 = 0 \quad w_1 = i$
$z_2 = 1 \quad w_2 = 1$
$z_3 = \infty \quad w_3 = -i$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{(z-0)\cancel{(1-\infty)}}{\cancel{(z-\infty)}(1-0)} = z$$

$$z = \left(\frac{w-i}{w+i}\right)\left(\frac{(1+i)^2}{1-i^2}\right) \Rightarrow z(w+i) = (w-i)\left(\frac{\cancel{1}+2i-\cancel{1}}{\cancel{2}}\right)$$

$$zw + iz = (w-i)\,i$$

$$zw + iz = wi + 1 \qquad w(zw - wi) = 1 - iz \qquad w = \frac{1-iz}{z-i}$$

$$\boxed{\therefore f(z) = \frac{-iz+1}{z-i}}$$

**#3.30** Prove $\sin(\bar{z}) = \overline{\sin(z)}$ & $\overline{\cos(z)} = \cos(\bar{z})$

$$e^{z} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y) \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

$$e^{iz} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x) \qquad \cos z := \frac{e^{iz} + e^{-iz}}{2i}$$

$$e^{-iz} = e^{y}e^{-ix} = e^{y}(\cos x + i\sin x)^{-1}$$

$$\overline{\sin z} = \overline{\left(\frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}\right)} = \overline{\left(\frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x + i\sin x)^{-1}}{2i}\right)}$$

$$\overline{\sin z} = \left(\frac{e^{-y}(\cos x - i\sin x) - e^{y}(\cos x - i\sin x)^{-1}}{\overline{2i}}\right)$$

$$\overline{\sin z} = \left(\frac{e^{-y}e^{i\bar{x}} - e^{y}e^{-i\bar{x}}}{2i}\right) = \frac{e^{i\bar{x}-y} - e^{-i\bar{x}-y}}{2i} = \frac{e^{i(\bar{x}-iy)} - e^{-i(\bar{x}-iy)}}{2i}$$

$$\boxed{\overline{\sin z} = \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} = \sin\bar{z}}$$

$$\overline{\cos z} = \overline{\left(\frac{e^{iz} + e^{-iz}}{2}\right)}$$

$$\overline{\cos z} = \overline{\left(\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}\right)} = \overline{\left(\frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}\right)}$$

$$= \overline{\left(\frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x + i\sin x)^{-1}}{2}\right)}$$

$$\left(e^{-y}(\cos x - i\sin x) + e^{y}(\cos x - i\sin x)^{-1}\right) \quad \left(e^{-y}e^{i\bar{x}} + e^{y}e^{i\bar{x}}\right)$$

$$= \left( \frac{e^{-y}(\cos x - i\sin x) + e^{y}(\cos x - i\sin x)^{-1}}{2} \right) = \left( \frac{e^{-y}e^{i\bar{x}} + e^{y}e^{i\bar{x}}}{2} \right)$$

$$= \left( \frac{e^{i\bar{x}-y} + e^{i\bar{x}+y}}{2} \right) = \left( \frac{e^{i(+\bar{x}-yi)} + e^{i(-\bar{x}+yi)}}{2} \right)$$

$$= \left( \frac{e^{i(\bar{x}-yi)} + e^{-i(\bar{x}-yi)}}{2} \right) = \frac{e^{i\bar{z}} + e^{-i\bar{z}}}{2}$$

$$\therefore \quad \overline{\cos z} = \frac{e^{i\bar{z}} + e^{-i\bar{z}}}{2} = \cos \bar{z}$$

#3.32 | Consider $\sin z = 0$ $\qquad \dfrac{e^{iz} - e^{-iz}}{2i} = 0$

To begin $2i(0) = 0 = e^{iz} - e^{-iz} \Rightarrow e^{iz} = e^{-iz}$

$$\frac{e^{iz}}{-e^{iz}} = 1 \Rightarrow e^{2iz} = 1 \qquad e^{2i(x+yi)} = 1$$

$\cos(\pi n) = 1$ ; $\sin(\pi n) = 0$ As $1 \in \mathbb{R}$

Thus $z$ is purely real. as $\pi \in \mathbb{R}$ & $n \in \mathbb{R}$

Zeros of $\sin(z) = \pi n \mid n \in \mathbb{Z}$

$\therefore$ zeros are purely real & integer multiples of $\pi$

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"Problem I" $C$ = Circle w/ center @ $a$ & radius $r$. $0$ does not lie on $C$. $f(z) = \frac{1}{z}$ $\quad w\bar{w} = |w|^2$

$f^{-1}(z) = w$ ; the image of $C$ is centered @ $\frac{\bar{a}}{|a|^2 - |r^2|}$ w/ radius $\frac{r}{||a|^2 - r^2|}$

$$\therefore \frac{\bar{a}}{a\bar{a} - r\bar{r}}$$

$$= \frac{x - yi}{x^2 + y^2 - r^2}$$

$$a = (x - yi)(x^2 + y^2 - r^2)$$

$$a = x^3 + y^2 x - r^2 x - yix^2 - y^3 i + ryi$$

$$\therefore \frac{r}{|a\bar{a} - r^2|} = \frac{r}{(a\bar{a} - r^2)(a\bar{a} + r^2)}$$

$$r = \frac{r}{a^2\bar{a}^2 - r^4} = r a^2 \bar{a}^{-2} - r5$$

$a$ = some $x + yi$

$$r = r(a^2\bar{a}^2 - r^4)$$

$$1 = a^2\bar{a}^2 - r^4$$

$$= (x+yi)^2(x - yi^2) - r^4$$

$$r = x^4 + y^4 - r^4$$