Part IB — Complex Analysis
Theorems

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Analytic functions
Complex differentiation and the Cauchy–Riemann equations. Examples. Conformal mappings. Informal discussion of branch points, examples of $\log z$ and z^c. [3]

Contour integration and Cauchy's theorem
Contour integration (for piecewise continuously differentiable curves). Statement and proof of Cauchy's theorem for star domains. Cauchy's integral formula, maximum modulus theorem, Liouville's theorem, fundamental theorem of algebra. Morera's theorem. [5]

Expansions and singularities
Uniform convergence of analytic functions; local uniform convergence. Differentiability of a power series. Taylor and Laurent expansions. Principle of isolated zeros. Residue at an isolated singularity. Classification of isolated singularities. [4]

The residue theorem
Winding numbers. Residue theorem. Jordan's lemma. Evaluation of definite integrals by contour integration. Rouché's theorem, principle of the argument. Open mapping theorem. [4]

Contents

0 Introduction

1 Complex differentiation

1.1 Differentiation

Proposition. Let f be defined on an open set $U \subseteq \mathbb{C}$. Let $w = c + id \in U$ and write $f = u + iv$. Then f is complex differentiable at w if and only if u and v, viewed as a real function of two real variables, are differentiable at (c, d), *and*

$$\begin{aligned} u_x &= v_y, \\ u_y &= -v_x. \end{aligned}$$

These equations are the *Cauchy–Riemann equations*. In this case, we have

$$f'(w) = u_x(c, d) + iv_x(c, d) = v_y(c, d) - iu_y(c, d).$$

1.2 Conformal mappings

Theorem (Riemann mapping theorem)**.** Let $\mathcal{U} \subseteq \mathbb{C}$ be the bounded domain enclosed by a simple closed curve, or more generally any simply connected domain not equal to all of $\mathbb{C}$. Then $\mathcal{U}$ is conformally equivalent to $D = \{z : |z| < 1\} \subseteq \mathbb{C}$.

1.3 Power series

Proposition. The uniform limit of continuous functions is continuous.

Proposition (Weierstrass M-test)**.** For a sequence of functions f_n, if we can find $(M_n) \subseteq \mathbb{R}_{>0}$ such that $|f_n(x)| < M_n$ for all x in the domain, then $\sum M_n$ converges implies $\sum f_n(x)$ converges uniformly on the domain.

Proposition. Given any constants $\{c_n\}_{n\geq 0} \subseteq \mathbb{C}$, there is a unique $R \in [0, \infty]$ such that the series $z \mapsto \sum_{n=0}^\infty c_n(z-a)^n$ converges absolutely if $|z - a| < R$ and diverges if $|z - a| > R$. Moreover, if $0 < r < R$, then the series converges uniformly on $\{z : |z - a| < r\}$. This R is known as the *radius of convergence*.

Theorem. Let

$$f(z) = \sum_{n=0}^\infty c_n(z-a)^n$$

be a power series with radius of convergence $R > 0$. Then

(i) f is holomorphic on $B(a; R) = \{z : |z - a| < R\}$.

(ii) $f'(z) = \sum nc_n(z-1)^{n-1}$, which also has radius of convergence R.

(iii) Therefore f is infinitely complex differentiable on $B(a; R)$. Furthermore,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Corollary. Given a power series

$$f(z) = \sum_{n\geq 0} c_n(z-a)^n$$

with radius of convergence $R > 0$, and given $0 < \varepsilon < R$, if f vanishes on $B(a, \varepsilon)$, then f vanishes identically.

1.4 Logarithm and branch cuts

Proposition. On $\{z \in \mathbb{C} : z \notin \mathbb{R}_{\leq 0}\}$, the principal branch $\log : U \to \mathbb{C}$ is holomorphic function. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}z} \log z = \frac{1}{z}.$$

If $|z| < 1$, then

$$\log(1 + z) = \sum_{n \geq 1} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots .$$

2 Contour integration

2.1 Basic properties of complex integration

Lemma. Suppose $f : [a, b] \to \mathbb{C}$ is continuous (and hence integrable). Then

$$\left|\int_a^b f(t)\ \mathrm{d}t\right| \leq (b-a)\sup_t |f(t)|$$

with equality if and only if f is constant.

Theorem (Fundamental theorem of calculus)**.** Let $f : U \to \mathbb{C}$ be continuous with antiderivative F. If $\gamma : [a, b] \to U$ is piecewise C^1-smooth, then

$$\int_\gamma f(z)\ \mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

2.2 Cauchy's theorem

Proposition. Let $U \subseteq \mathbb{C}$ be a domain (i.e. path-connected non-empty open set), and $f : U \to \mathbb{C}$ be continuous. Moreover, suppose

$$\int_\gamma f(z)\ \mathrm{d}z = 0$$

for any closed piecewise C^1-smooth path γ in U. Then f has an antiderivative.

Proposition. If U is a star domain, and $f : U \to \mathbb{C}$ is continuous, and if

$$\int_{\partial T} f(z)\ \mathrm{d}z = 0$$

for all triangles $T \subseteq U$, then f has an antiderivative on U.

Theorem (Cauchy's theorem for a triangle)**.** Let U be a domain, and let $f : U \to \mathbb{C}$ be holomorphic. If $T \subseteq U$ is a triangle, then $\int_{\partial T} f(z)\ \mathrm{d}z = 0$.

Corollary (Convex Cauchy)**.** If U is a convex or star-shaped domain, and $f : U \to \mathbb{C}$ is holomorphic, then for *any* closed piecewise C^1 paths $\gamma \in U$, we must have

$$\int_\gamma f(z)\ \mathrm{d}z = 0.$$

2.3 The Cauchy integral formula

Theorem (Cauchy integral formula)**.** Let U be a domain, and $f : U \to \mathbb{C}$ be holomorphic. Suppose there is some $\overline{B(z_0; r)} \subseteq U$ for some z_0 and $r > 0$. Then for all $z \in B(z_0; r)$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial \overline{B(z_0;r)}} \frac{f(w)}{w - z}\ \mathrm{d}w.$$

Corollary (Local maximum principle)**.** Let $f : B(z, r) \to \mathbb{C}$ be holomorphic. Suppose $|f(w)| \leq |f(z)|$ for all $w \in B(z; r)$. Then f is constant. In other words, a non-constant function cannot achieve an interior local maximum.

Theorem (Liouville's theorem)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (i.e. holomorphic everywhere). If f is bounded, then f is constant.

Corollary (Fundamental theorem of algebra)**.** A non-constant complex polynomial has a root in $\mathbb{C}$.

2.4 Taylor's theorem

Theorem (Taylor's theorem)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic. Then f has a convergent power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

on $B(a, r)$. Moreover,

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\partial B(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\, \mathrm{d}z$$

for any $0 < \rho < r$.

Corollary. If $f : B(a, r) \to \mathbb{C}$ is holomorphic on a disc, then f is infinitely differentiable on the disc.

Corollary. If $f : U \to \mathbb{C}$ is a complex-valued function, then $f = u + iv$ is holomorphic at $p \in U$ if and only if u, v satisfy the Cauchy–Riemann equations, and that u_x, u_y, v_x, v_y are continuous in a neighbourhood of p.

Corollary (Morera's theorem)**.** Let $U \subseteq \mathbb{C}$ be a domain. Let $f : U \to \mathbb{C}$ be continuous such that

$$\int_\gamma f(z)\, \mathrm{d}z = 0$$

for all piecewise-C^1 closed curves $\gamma \in U$. Then f is holomorphic on U.

Corollary. Let $U \subseteq \mathbb{C}$ be a domain, $f_n ; U \to \mathbb{C}$ be a holomorphic function. If $f_n \to f$ uniformly, then f is in fact holomorphic, and

$$f'(z) = \lim_n f_n'(z).$$

2.5 Zeroes

Lemma (Principle of isolated zeroes)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic and not identically zero. Then there exists some $0 < \rho < r$ such that $f(z) \neq 0$ in the punctured neighbourhood $B(a, \rho) \setminus \{a\}$.

Corollary (Identity theorem)**.** Let $U \subseteq \mathbb{C}$ be a domain, and $f, g : U \to \mathbb{C}$ be holomorphic. Let $S = \{z \in U : f(z) = g(z)\}$. Suppose S contains a non-isolated point, i.e. there exists some $w \in S$ such that for all $\varepsilon > 0$, $S \cap B(w, \varepsilon) \neq \{w\}$. Then $f = g$ on U.

2.6 Singularities

Proposition (Removal of singularities)**.** Let U be a domain and $z_0 \in U$. If $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, and f is bounded near z_0, then there exists an a such that $f(z) \to a$ as $z \to z_0$.

Furthermore, if we define

$$g(z) = \begin{cases} f(z) & z \in U \setminus \{z_0\} \\ a & z = z_0 \end{cases},$$

then g is holomorphic on U.

Proposition. Let U be a domain, $z_0 \in U$ and $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Suppose $|f(z)| \to \infty$ as $z \to z_0$. Then there is a unique $k \in \mathbb{Z}_{\geq 1}$ and a unique holomorphic function $g : U \to \mathbb{C}$ such that $g(z_0) \neq 0$, and

$$f(z) = \frac{g(z)}{(z - z_0)^k}.$$

Theorem (Casorati-Weierstrass theorem)**.** Let U be a domain, $z_0 \in U$, and suppose $f : U \setminus \{z_0\} \to \mathbb{C}$ has an essential singularity at z_0. Then for all $w \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w$.

In other words, on any punctured neighbourhood $B(z_0; \varepsilon) \setminus \{z_0\}$, the image of f is dense in $\mathbb{C}$.

Theorem (Picard's theorem)**.** If f has an isolated essential singularity at z_0, then there is some $b \in \mathbb{C}$ such that on each punctured neighbourhood $B(z_0; \varepsilon) \setminus \{z_0\}$, the image of f contains $\mathbb{C} \setminus \{b\}$.

2.7 Laurent series

Theorem (Laurent series)**.** Let $0 \leq r < R < \infty$, and let

$$A = \{z \in \mathbb{C} : r < |z - a| < R\}$$

denote an annulus on $\mathbb{C}$.

Suppose $f : A \to \mathbb{C}$ is holomorphic. Then f has a (unique) convergent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial \overline{B(a,\rho)}} \frac{f(z)}{(z - a)^{n+1}}\, \mathrm{d}z$$

for $r < \rho < R$. Moreover, the series converges uniformly on compact subsets of the annulus.

Lemma. Let $f : A \to \mathbb{C}$ be holomorphic, $A = \{r < |z - a| < R\}$, with

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

Then the coefficients c_n are uniquely determined by f.

3 Residue calculus

3.1 Winding numbers

Lemma. Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous closed curve, and pick a point $w \in \mathbb{C} \setminus \text{image}(\gamma)$. Then there are continuous functions $r : [a, b] \to \mathbb{R} > 0$ and $\theta : [a, b] \to \mathbb{R}$ such that

$$\gamma(t) = w + r(t)e^{i\theta(t)}.$$

Lemma. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1-smooth closed path, and $w \notin \text{image}(\gamma)$. Then

$$I(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w}\,\mathrm{d}z.$$

3.2 Homotopy of closed curves

Proposition. Let $\phi, \psi : [a, b] \to U$ be homotopic (piecewise C^1) closed paths in a domain U. Then there exists some $\phi = \phi_0, \phi_1, \cdots, \phi_N = \psi$ such that each ϕ_j is piecewise C^1 closed and ϕ_{i+1} is obtained from ϕ_i by elementary deformation.

Corollary. Let U be a domain, $f : U \to \mathbb{C}$ be holomorphic, and γ_1, γ_2 be homotopic piecewise C^1-smooth closed curves in U. Then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Corollary (Cauchy's theorem for simply connected domains)**.** Let U be a simply connected domain, and let $f : U \to \mathbb{C}$ be holomorphic. If γ is any piecewise C^1-smooth closed curve in U, then

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

3.3 Cauchy's residue theorem

Theorem (Cauchy's residue theorem)**.** Let U be a simply connected domain, and $\{z_1, \cdots, z_k\} \subseteq U$. Let $f : U \setminus \{z_1, \cdots, z_k\} \to \mathbb{C}$ be holomorphic. Let $\gamma : [a, b] \to U$ be a piecewise C^1-smooth closed curve such that $z_i \neq \text{image}(\gamma)$ for all i. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^k I(\gamma, z_i)\operatorname{Res}(f; z_i).$$

3.4 Overview

3.5 Applications of the residue theorem

Lemma. Let $f : U \setminus \{a\} \to \mathbb{C}$ be holomorphic with a pole at a, i.e f is meromorphic on U.

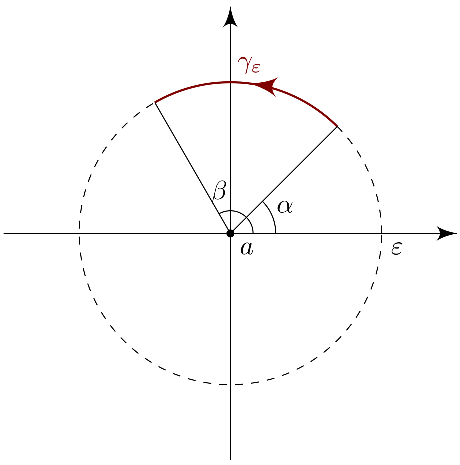
(i) If the pole is simple, then

$$\operatorname{Res}(f, a) = \lim_{z \to a} (z - a)f(z).$$

(ii) If near a, we can write
$$f(z) = \frac{g(z)}{h(z)},$$
where $g(a) \neq 0$ and h has a simple zero at a, and g, h are holomorphic on $B(a, \varepsilon) \setminus \{a\}$, then
$$\mathrm{Res}(f, a) = \frac{g(a)}{h'(a)}.$$

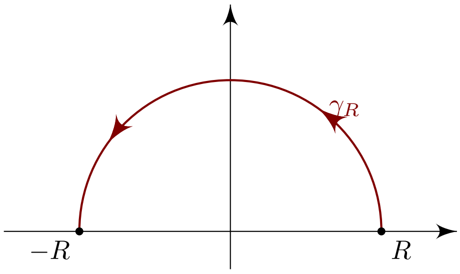
(iii) If
$$f(z) = \frac{g(z)}{(z-a)^k}$$
near a, with $g(a) \neq 0$ and g is holomorphic, then
$$\mathrm{Res}(f, a) = \frac{g^{(k-1)}(a)}{(k-1)!}.$$

Lemma. Let $f : B(a, r) \setminus \{a\} \to \mathbb{C}$ be holomorphic, and suppose f has a simple pole at a. We let $\gamma_\varepsilon : [\alpha, \beta] \to \mathbb{C}$ be given by
$$t \mapsto a + \varepsilon e^{it}.$$



Then
$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\, \mathrm{d}z = (\beta - \alpha) \cdot i \cdot \mathrm{Res}(f, a).$$

Lemma (Jordan's lemma)**.** Let f be holomorphic on a neighbourhood of infinity in $\mathbb{C}$, i.e. on $\{|z| > r\}$ for some $r > 0$. Assume that $zf(z)$ is bounded in this region. Then for $\alpha > 0$, we have
$$\int_{\gamma_R} f(z) e^{i\alpha z}\, \mathrm{d}z \to 0$$
as $R \to \infty$, where $\gamma_R(t) = Re^{it}$ for $t \in [0, \pi]$ is the semicircle (which is *not* closed).



3.6 Rouchés theorem

Theorem (Argument principle)**.** Let U be a simply connected domain, and let f be meromorphic on U. Suppose in fact f has finitely many zeroes $z_1, \cdots, z_k$ and finitely many poles $w_1, \cdots, w_\ell$. Let γ be a piecewise-C^1 closed curve such that $z_i, w_j \notin \text{image}(\gamma)$ for all i, j. Then

$$I(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = \sum_{i=1}^{k} \text{ord}(f; z_i) I_\gamma(z_i) - \sum_{j=1}^{\ell} \text{ord}(f, w_j) I(\gamma, w_j).$$

Corollary (Rouchés theorem)**.** Let U be a domain and γ a closed curve which bounds a domain in U (the key case is when U is simply connected and γ is a simple closed curve). Let f, g be holomorphic on U, and suppose $|f| > |g|$ for all $z \in \text{image}(\gamma)$. Then f and $f + g$ have the same number of zeroes in the domain bound by γ, when counted with multiplicity.

Lemma. The local degree is given by

$$\deg(f, a) = I(f \circ \gamma, f(a)),$$

where

$$\gamma(t) = a + re^{it},$$

with $0 \leq t \leq 2\pi$, for $r > 0$ sufficiently small.

Proposition (Local degree theorem)**.** Let $f : B(a, r) \to \mathbb{C}$ be holomorphic and non-constant. Then for $r > 0$ sufficiently small, there is $\varepsilon > 0$ such that for any $w \in B(f(a), \varepsilon) \setminus \{f(a)\}$, the equation $f(z) = w$ has exactly $\deg(f, a)$ distinct solutions in $B(a, r)$.

Corollary (Open mapping theorem)**.** Let U be a domain and $f : U \to \mathbb{C}$ is holomorphic and non-constant, then f is an open map, i.e. for all open $V \subseteq U$, we get that $f(V)$ is open.

Corollary. Let $U \subseteq \mathbb{C}$ be a simply connected domain, and $U \neq \mathbb{C}$. Then there is a non-constant holomorphic function $U \to B(0, 1)$.