HW 6 - Cason Konzer

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Problem Z. Find the value of the constant C that makes the theorem below true, and prove the theorem.

Theorem 5.1. (continued) Suppose f is holomorphic in the region G and γ is a positively oriented, simple, closed, piecewise smooth, G-contractible path. If w is inside γ then $f'''(w)$ exists, and

$$f'''(w) = C \int_\gamma \frac{f(z)}{(z-w)^4} dz.$$

Hint: Follow the argument from class for the cases of f' and f''. (The proof from class is simpler than the proof in the textbook.) The algebraic identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ is likely to be helpful.

we know $f''(w) = \frac{1}{\pi i} \int_\gamma \frac{f(z)}{(z-w)^3} dz$

we also know $f'(w) = \lim_{h \to 0} \frac{f(w+h) - f(w)}{h}$

thus $f'''(w) = (f''(w))' = \frac{1}{h\pi i} \int_\gamma \frac{f(z)}{(z-w-h)^3} - \frac{f(z)}{(z-w)^3}$

$\Rightarrow f'''(w) = \frac{1}{h\pi i} \int_\gamma \frac{f(z)\left[(z-w)^3 - (z-w-h)^3\right]}{(z-w-h)^3(z-w)^3} dz \quad \left(\lim_{h \to 0}\right)$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$ factoring,

$(z-w)^3 - (z-w-h)^3 = (z-w-z+w+h)\left((z-w)^2 + (z-w)(z-w-h) + (z-w-h)^2\right)$

$= h(z^2 - 2zw + w^2 + z^2 - 2zw - zh + w^2 + wh + z^2 - 2zw - 2zh + w^2 + 2hw + h^2)$

$= h(3z^2 - 6zw + 3w^2 - 3zh + 3wh + h^2)$

$\Rightarrow f'''(w) = \frac{h}{h\pi i} \int_\gamma f(z) \frac{(3z^2 - 6zw + 3w^2 - 3zh + 3wh + h^2)}{(z-w-h)^3(z-w)^3} dz \quad \left(\lim_{h \to 0}\right)$

$\Rightarrow f'''(w) = \frac{1}{\pi i} \int_\gamma f(z) \frac{(3z^2 - 6zw + 3w^2)}{(z-w)^3(z-w)^3} = \frac{1}{\pi i} \int_\gamma f(z) \frac{3(z^2 - 2zw + w^2)}{(z-w)^6}$

$\Rightarrow f'''(w) = \frac{1}{\pi i} \int_\gamma f(z) \frac{3(z-w)^2}{(z-w)^6} dz = \frac{3}{\pi i} \int_\gamma \frac{f(z)}{(z-w)^4} dz$ ∎

$$\Rightarrow f'''(w) = \frac{1}{\pi i}\int_\gamma f(z)\,\frac{3\,(z-w)}{(z-w)^6}\,dz = \frac{\ }{\pi i}\int_\gamma \frac{\ }{(z-w)^4}\,dz$$

$$\boxed{\therefore\ C = \frac{3}{\pi i}}$$

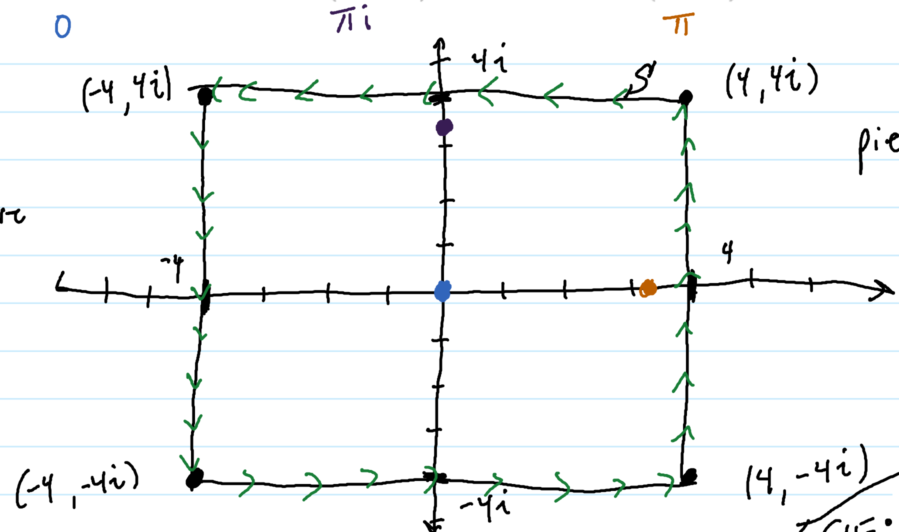
5.1. Compute the following integrals, where S is the boundary of the square with vertices $-4-4i$, $4-4i$, $4+4i$, $-4+4i$, (oriented counterclockwise).

(a) $\int_S \frac{\exp(z^2)}{z^3}\,dz$. (b) $\int_S \frac{\exp(3z)}{(z-\pi i)^2}\,dz$. (e) $\int_S \frac{\sin(z/3)}{(z-\pi)^4}\,dz$.

0   πi π

all $f(z)$ are holomorphic

All values for "a" are inside S



S is a piecewise smooth curve & simple

all integrals are bounded

CIF: $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz.$

a) $\int_S \frac{e^{z^2}}{z^3}\,dz = \int_S \frac{e^{z^2}}{(z-0)^3}\,dz$, $f(0)'' = \frac{1}{\pi i}\int_S \frac{f(z)}{(z-0)^3}\,dz$

$\frac{d}{dz} e^{z^2} = 2z\,e^{z^2}$, $\frac{d^2}{dz^2} e^{z^2} = \frac{d}{dz} 2z\,e^{z^2} = 2e^{z^2} + 4z^4 e^{z^2} = 2e^{z^2}(1+2z^2)$

$f''(0) = 2e^0(1+2(0)) = 2(1) = 2$, $2 = \frac{1}{\pi i}\int_S \frac{e^{z^2}}{z^3}\,dz$

$$\boxed{\therefore \int_S \frac{e^{z^2}}{z^3}\,dz = 2\pi i}$$

b) $\int_S \frac{e^{3z}}{(z-\pi i)^2}\,dz$

$f'(\pi i) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-\pi i)^2}\,dz$ $\frac{d}{dz} e^{3z} = 3e^{3z}$

$$f'(\pi i) = \frac{1}{2\pi i}\int_{S} \frac{f(z)}{(z-\pi i)^2}\,dz \quad , \quad \frac{d}{dz} e^{3z} = 3e^{3z}$$

$$f'(\pi i) = 3e^{3\pi i} = 3(\cos(3\pi) + i\sin(3\pi)) = \frac{1}{2\pi i}\int_{S} \frac{e^{3z}}{(z-\pi i)^2}\,dz$$

$$\boxed{\therefore \int_{S} \frac{e^{3z}}{(z-\pi i)^2}\,dz = 6\pi i(\cos(3\pi) + i\sin(3\pi)) = 6e^{3\pi i}}$$

c) $$\int_{S} \frac{\sin(\frac{z}{3})}{(z-\pi)^4}\,dz \quad , \quad f'''(\pi) = \frac{3}{\pi i}\int_{S} \frac{f(z)}{(z-\pi)^4}\,dz$$

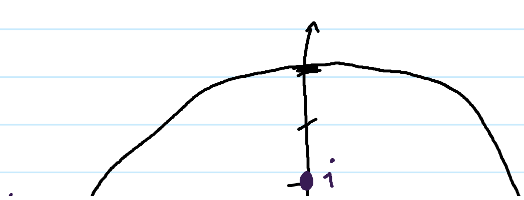
$$\frac{d}{dz}\sin\left(\frac{z}{3}\right) = \frac{\cos(z/3)}{3}, \quad \frac{d^2}{dz^2} = \frac{-\sin(z/3)}{9}, \quad \frac{d^3}{dz^3} = \frac{-\cos(z/3)}{27}$$

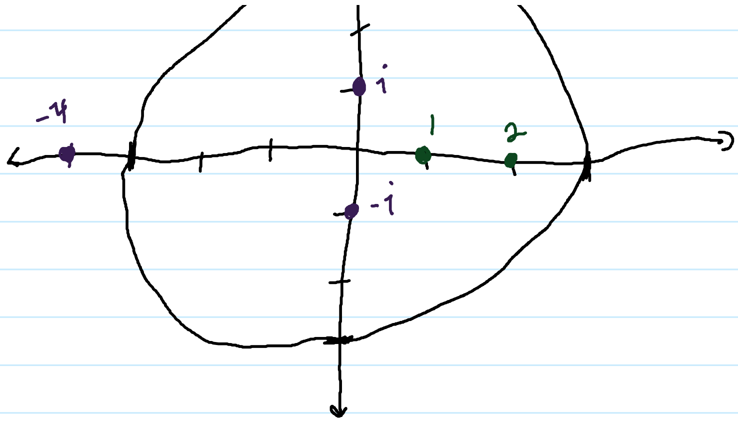
$$f'''(\pi) = \frac{-\cos(\frac{\pi}{3})}{27} = \frac{3}{\pi i}\int_{S} \frac{\sin(\frac{z}{3})}{(z-\pi)^4}\,dz$$

$$\boxed{\therefore \int_{S} \frac{\sin(z/3)}{(z-\pi)^4}\,dz = \frac{-\pi i\cos(\pi/3)}{81} = \frac{-\pi i(\frac{1}{2})}{81} = \frac{-\pi i}{162}}$$

5.3. Integrate the following functions over the circle C[0, 3]:

(h) $\dfrac{1}{(z+4)(z^2+1)}$. (i) $\dfrac{\exp(2z)}{(z-1)^2(z-2)}$





h) $\dfrac{1}{(z+4)(z^2+1)} = \dfrac{1}{(z+4)(z+i)(z-i)} = \dfrac{A}{\underset{-4}{(z+4)}} + \dfrac{B}{\underset{-i}{(z+i)}} + \dfrac{C}{\underset{i}{(z-i)}}$

$1 = A(z+i)(z-i) + B(z+4)(z-i) + C(z+4)(z+i)$

@ $z=i$, $1 = 0 + 0 + C(i+4)(i+i)$, $C = \dfrac{1}{(i+4)2i} = \dfrac{1}{-2+8i}\dfrac{(-2-8i)}{(-2-8i)} = \dfrac{-2(1+4i)}{4+64}$

$2+3z$

$C = \dfrac{-1-2i}{34}$ @ $z=i$, $1 = 0 + B(-i+4)(-i-i) + 0$ $\quad B = \dfrac{1}{(-i+4)(-2i)}$

$B = \dfrac{1}{-2-8i}\dfrac{(-2+8i)}{(-2+8i)} = \dfrac{-2(1-4i)}{4+64} = \dfrac{-(1-2i)}{32}$ $\quad B = \dfrac{-1+2i}{34}$

@ $z=-4$, $1 = A(-4+i)(-4-i) + 0 + 0$, $A = \dfrac{1}{(-4+i)(-4-i)} = \dfrac{1}{16+1}$ $\quad A = 1/17$

$$\frac{1}{(z+4)(z^2+1)} = \frac{(1/17)}{(z+4)} + \frac{\left(\frac{-1+2i}{34}\right)}{(z+i)} + \frac{\left(\frac{-1-2i}{34}\right)}{(z-i)}$$

$$\int_{C[0,3]} \frac{dz}{(z+4)(z^2+1)} = \int_{C[0,3]} \frac{(1/17)}{(z-(-4))}dz + \int_{C[0,3]} \frac{\left(\frac{-1+2i}{34}\right)}{(z-(-i))}dz + \int_{C[0,3]} \frac{\left(\frac{-1-2i}{34}\right)}{(z-i)}dz$$

$$\int_{C[0,3]} \frac{(1/17)}{(z-(-4))}dz = \boxed{0}, \quad \int_{C[0,3]} \frac{\left(\frac{-1+2i}{34}\right)}{(z-(-i))}dz = 2\pi i\left(\frac{-1+2i}{34}\right) = \frac{-\pi i + 2\pi i^2}{17} = \boxed{\frac{\pi(-2-i)}{17}}$$

$$\int_{C[0,3]} \frac{\left(\frac{-1-2i}{34}\right)}{(z-i)}dz = 2\pi i\left(\frac{-1-2i}{34}\right) = \frac{-\pi i - 2\pi i^2}{17} = \boxed{\frac{\pi(2-i)}{17}}$$

$\therefore$

$$\int_{C[0,3]} \frac{}{(z-i)}$$

$\therefore$

$$\boxed{\int_{C[0,3]} \frac{dz}{(z+4)(z^2+1)} = 0 + \frac{\pi(-2-i)}{17} + \frac{\pi(2-i)}{17} = \frac{\pi(-2-i+2-i)}{17} = \frac{\pi(-2i)}{17} = \frac{-2\pi i}{17}}$$

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i) $\dfrac{e^{2z}}{(z-1)^2(z-2)} = \dfrac{A}{(z-1)^2} + \dfrac{B}{(z-2)}$ $\quad e^{2z} = A(z-2) + B(z-1)^2$

1 $\quad$ 2

@ $z=2$, $e^4 = 0 + B(1)^2$, $B = e^4$ @ $z=1$, $e^2 = A(-1) + 0$, $A = -e^2$

$$\frac{e^{2z}}{(z-1)^2(z-2)} = \frac{-e^2}{(z-1)^2} + \frac{e^4}{(z-2)}, \quad \int_{C[0,3]} \frac{e^{2z}dz}{(z-1)^2(z-2)} = \int_{C[0,3]} \frac{-e^2}{(z-1)^2} dz + \int_{C[0,3]} \frac{e^4}{(z-2)} dz$$

$$\int_{C[0,3]} \frac{-e^2}{(z-1)^2} dz = 2\pi i(-e^2) \qquad \int_{C[0,3]} \frac{e^4}{(z-2)} dz = 2\pi i e^4$$

$\therefore$

$$\boxed{\int_{C[0,3]} \frac{e^{2z}dz}{(z-1)^2(z-2)} = 2\pi i(-e^2) + 2\pi i e^4 = 2\pi i(e^4 - e^2)}$$

5.15. Suppose f is entire with bounded real part, i.e., writing $f(z) = u(z) + iv(z)$, there exists $M > 0$ such that $|u(z)| \leq M$ for all $z \in \mathbf{C}$. Prove that f is constant. (Hint: Consider the function $\exp(f(z))$.)

Consider the function $g(z) = e^{f(z)}$ $\quad|\quad$ $f(z) = u(z) + iv(z)$

So $g(z) = e^{u(z)+iv(z)} = e^{u(z)} e^{iv(z)} = e^{u(z)}(\cos(v(z)) + i\sin(v(z)))$

now $|g(z)| = |e^{u(z)}||e^{iv(z)}| = |e^{u(z)}| \cdot 1 = e^{u(z)} = e^{|u(z)|}$

now $|\xi(z)| = |e^{u(z)}||e^{iv(z)}| = |e^{u(z)}| \cdot 1 = e^{u(z)} = e^{|u(z)|}$

We know $0 \leq |u(z)| \leq M \quad \forall z \in \mathbb{C}$ so $e^{|u(z)|} \leq e^{M}$

since $|\xi(z)| = e^{|u(z)|}$, $|\xi(z)| \leq e^{M}$, as $M > 0$, $e^{M} > 1$

since $e^{|f(z)|} = e^{|u(z)|}$, so $|f(z)| = |u(z)|$

as $|u(z)| \leq M$, $|f(z)| \leq M$ $\quad$ $f(z)$ is bounded & entire

Thus by Liouville's Theorem $|f'(z)| \leq \frac{M}{R}$ $\quad \lim_{R \to \infty} |f'(z)| \leq 0$

Thus as only $|0| \leq 0$, $f'(z) = 0$.

Such as $\xi'(z) = \frac{d}{dz} e^{f(z)} = f'(z) e^{f(z)} = 0 \cdot e^{f(z)} = 0$

$\boxed{\therefore \xi'(z) = 0, \ \xi(z) \text{ is constant } \blacksquare}$