MTH 470/570
Winter 2021
Assignment 5 key

p.72 #4.32. Suppose f and g are holomorphic in the region G and γ is a simple piecewise smooth G-contractible path. Prove that if $f(z) = g(z)$ for all $z \in \gamma$, then $f(z) = g(z)$ for all z inside γ.

Let z lie inside γ. We have the string of equations

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dw = \frac{1}{2\pi i}\int_\gamma \frac{g(w)}{z-w}\,dw = g(z)$$

The first and last equations are instances of Cauchy's Integral Formula. The middle equation is true because these integrals depend on the values of $f(w)$ and $g(w)$ only for $w \in \gamma$, where $f(w) = g(w)$.

4.34. Compute

$$I(r) := \int_{C[-2i,r]} \frac{dz}{z^2+1}$$

as a function of r, for $r > 0$, $r \neq 1, 3$.

$\displaystyle\int_{C[-2i,r]} \frac{dz}{z^2+1} = \int_{C[-2i,r]} \frac{i/2}{z+i} - \frac{i/2}{z-i}\,dz.$

When $0 < r < 1$, $C[-2i,r]$ contains neither i nor $-i$, so the integral is 0.

When $1 < r < 3$, $C[-2i,r]$ contains $-i$ but not i, so the integral is $2\pi i \cdot i/2 = -\pi$.

When $r > 3$, $C[-2i,r]$ contains both $-i$ and i, so the integral is $2\pi i \cdot i/2 + 2\pi i \cdot -i/2 = 0$.

4.35. Find

$$\int_{C[-2i,r]} \frac{dz}{z^2-2z-8}$$

for r $= 1$, r $= 3$ and r $= 5$.

$\displaystyle\int_{C[0,r]} \frac{dz}{z^2-2z-8} = \int_{C[0,r]} \frac{1/6}{z-4} - \frac{1/6}{z+2}\,dz.$

$C[0,1]$ contains neither -2 nor -4, so the integral is 0.

$C[0,3]$ contains -2 but not 4, so the integral is $2\pi i \cdot -1/6 = -\pi i/3$.

$C[0,5]$ contains both -2 and 4, so the integral is $2\pi i(1/6 - 1/6) = 0$.

4.37. Compute the following integrals.

(a) $\displaystyle\int_{C[-1,2]} \frac{z^2}{4-z^2}\,dz$. $\quad \displaystyle\frac{1}{4-z^2} = \frac{1/4}{z+2} - \frac{1/4}{z-2}$, so

$$\int_{C[-1,2]} \frac{z^2}{4-z^2}\,dz = \frac{1}{4}\int_{C[-1,2]} \frac{z^2}{z+2}\,dz - \frac{1}{4}\int_{C[-1,2]} \frac{z^2}{z-2}\,dz.$$

$C[-1,2]$ contains -2, so by CIF, the first integral is $2\pi i \cdot (-2)^2 = 8\pi i$.

$C[-1,2]$ does not contain 2, so by Cauchy's Theorem, the second integral is 0.

Therefore $\displaystyle\int_{C[-1,2]} \frac{z^2}{4-z^2}\,dz = \frac{1}{4} \cdot 8\pi i = 2\pi i.$

(b) By CIF, $\displaystyle\int_{C[0,1]} \frac{\sin z}{z}\, dz = 2\pi i \sin 0 = 0.$

(In fact, $\dfrac{\sin z}{z}$ is an entire function, so this also follows from Cauchy's Theorem.) .

(c) $\displaystyle\int_{C[0,2]} \frac{\exp(z)}{z(z-3)}\, dz. \qquad \frac{1}{z(z-3)} = \frac{1/3}{z-3} - \frac{1/3}{z},$ so

$$\int_{C0,2]} \frac{\exp(z)}{z(z-3)}\, dz = \frac{1}{3}\int_{C[0,2]} \frac{\exp(z)}{z-3}\, dz - \frac{1}{3}\int_{C[0,2]} \frac{\exp(z)}{z}\, dz$$

$C[0,2]$ does not contain 3, so by Cauchy's Theorem, the first integral is 0.
$C[0,2]$ contains 0, so by CIF, the second integral is $2\pi i \exp[0] = 2\pi i$.
Therefore $\displaystyle\int_{C[0,2]} \frac{\exp(z)}{z(z-3)}\, dz = -\frac{2}{3}\pi i.$

(d) $\displaystyle\int_{C[0,4]} \frac{\exp(z)}{z(z-3)}$. This is just like (c), expect that C[0,4] contains both 0 and 3, so

$$\int_{C0,4]} \frac{\exp(z)}{z(z-3)}\, dz = \frac{1}{3}\int_{C[0,4]} \frac{\exp(z)}{z-3}\, dz - \frac{1}{3}\int_{C[0,4]} \frac{\exp(z)}{z}\, dz = \frac{2}{3}e^3\pi i - \frac{2}{3}\pi i = \frac{2e^3-2}{3}\pi i$$

Problem C. Recall Green's Theorem from multivariate calculus:

Suppose $C : [a,b] \to \mathbf{R}^2$ is a closed curve which bounds the open set D, and the functions $P, Q : \mathbf{R}^2 \to \mathbf{R}^2$ have continous derivatives on D. Then

$$\int_C P(x(t),y(t)) \cdot x'(t) + Q(x(t),y(t)) \cdot y'(t)\, dt = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy.$$

You will use Green's Theorem to prove an important case of Cauchy's Theorem.

Let $\gamma : [a,b] \to \mathbf{C}$ be a closed curve, and suppose that $f : \mathbf{C} \to \mathbf{C}$ is holomorphic on the open set $Delta$ bounded by γ. Let $f(z) = U(z) + iV(z)$ and $\gamma(t) = x(t) + iy(t)$.

(a) Write $\displaystyle\int_\gamma f(z)\, dz$ in terms of U, V, x, y, and multiply out the integrand.

$\displaystyle\int_\gamma f(z)\, dz = \int_\gamma (U((x+iy)) + iV(x(t)+iy(t)))(x'(t)+iy'(t))\, dt =$

$\displaystyle\int_\gamma (\, U(x(t)+iy(t))\cdot x'(t) - V(x(t)+iy(t))\cdot y'(t)\,) + i(\, V(x(t)+iy(t))x'(t) + U(x(t)+iy(t))\cdot y'(t)\,)\, dt$
.

(b) Apply Green's Theorem to the real part and imaginary parts of your answer to (b) (separately) to obtain an equation of the form

$$\int_\gamma f(z)\, dz = \int\int_\Delta \cdots\, dx\, dy + i\int\int_\Delta \cdots\, dx\, dy$$

Apply Green's Theorem to the real part of (a) (with $P = U$ and $Q = -V$) to obtain

$$\int_\gamma U(x(t)+iy(t)) \cdot x'(t) - V(x(t)+iy(t)) \cdot y'(t)\, dt = \int\int_\Delta -\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\, dx\, dy$$

and to the imaginary part of (a) (with $P = V$ and $Q = U$) to obtain

$$\int_\gamma V(x(t)+iy(t))x'(t) + U(x(t)+iy(t)) \cdot y'(t) = \int\int_\Delta \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\, dx\, dy,$$

so, substituting, we have

$$\int_\gamma f(z)\ dz = \int\int_\Delta -\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\ dx\ dy + i\int\int_\Delta \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\ dx\ dy$$

(c) Since f is, by hypothesis, holomorphic, use the Cauchy-Riemann equations to conclude that $\int_\gamma f(z)\ dz = 0$.

The Cauchy-Riemann equations tell us that $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$, so the righthand side of (b) is 0. Therefore $\int_\gamma f(z)\ dz = 0$, and we are done.