HW 4

Monday, February 15, 2021 8:00 AM

470hw4

2/15/21

MTH 470/570
Winter 2021
Assignment 4.

This assignment is due before class on Tuesday, February 16. Please submit your solutions via Blackboard. Solutions are required – answers must be justified.

p.52 #3.41. Convert the following expressions to the form $x + iy$. (Reason carefully.)
(a) $e^{i\pi}$. (c) i^i. (d) $e^{\sin(i)}$. (e) $\exp(\mathrm{Log}(3+4i))$ (f) $(1+i)^{1/2}$

3.51. Prove that $\exp(b \log a)$ is single valued if and only if b is an integer. (Note that this means that complex exponentials do not clash with monomials z^n, no matter which branch of the logarithm is used.) What can you say if b is rational?

p.68 #4.3. Integrate the function $f(z) = \overline{z}$ over these three paths from Example 4.1:
(a) The line segment from 0 to $1+i$.
(b) The arc of the parabola $y = x^2$ from 0 to $1+i$.
(c) The union of the line segment from 0 to 1 and the line segment from 1 to $1+i$.

4.8. Compute $\int_\gamma f(z)\, dz$ for the following functions f and paths γ:
Hint: All but one of these can be computed using the complex version of the Fundamental Theorem of Calculus.

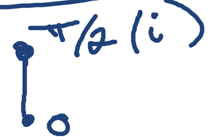
(a) $f(z) = z^2$ and $\gamma(t) = t + it^2$, $0 \le t \le 1$.
(b) $f(z) = z$ and γ is the semicircle from 1 through i to -1.
(c) $f(z) = \exp(z)$ and γ is the line segment from 0 to a point z_0. (The answer will be a function of z_0.)
(d) $f(z) = |z|^2$ and γ is the line segment from 2 to 3+i.

p:52, #3.41

a) $e^{i\pi} = \cos(\pi) + i\sin(\pi)$

$= 1 + i(0) = 1$

c) i^i $\quad i = e^{\frac{i\pi}{2}}$



$i^i = e^{i\,\mathrm{Log}(i)}$ $\quad \mathrm{Log}(i) = \ln|i| + i\,\mathrm{Arg}(i)$

$= 0 + \frac{i\pi}{2}$

$i^i = e^{i(i\pi/2)}$

$$= 0 + \frac{i\pi}{2}$$

$$i^i = e^{i(i\pi/2)}$$

$$= \boxed{e^{-\pi/2}} = \underline{.2079 + i(0)}$$

d) $e^{\sin(i)}$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\sin(i) = \frac{e^{i(i)} - e^{-i(i)}}{2i} = \frac{e^{-1} - e^{1}}{2i} = \frac{\left(\frac{1}{e} - \frac{e}{1}\right)}{2i}$$

$$\sin(i) = \left(\frac{1}{e} - \frac{e^2}{e}\right) \cdot \frac{1}{2i} = \frac{1 - e^2}{2ei} = \frac{i(1-e^2)}{-2e}$$

$$e^{\sin(i)} = e^{i\left(\frac{-1}{2e} + \frac{e^2}{2e}\right)} = e^{i\left(\frac{-1}{2e} + \frac{e}{2}\right)} = e^{-i/2e} e^{ei/2}$$

$$= \left(\cos\left(\frac{-1}{2e}\right) + i\sin\left(\frac{-1}{2e}\right)\right)\left(\cos(e/2) + i\sin(e/2)\right)$$

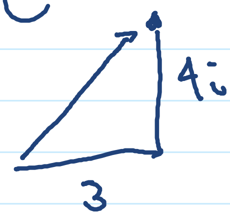
$$= \left(\cos(-1/2e)\cos(e/2)\right) - \left(\sin(-1/2e)\sin(e/2)\right)$$
$$+ i\left(\sin(-1/2e)\cos(e/2) + \sin(e/2)\cos(-1/2e)\right)$$

$$= \cos\left(\frac{-1}{2e} + \frac{e}{2}\right) + i\sin\left(\frac{-1}{2e} + \frac{e}{2}\right)$$

$$\boxed{= \cos\left(\frac{e^2 - 1}{2e}\right) + i\sin\left(\frac{e^2-1}{2e}\right)}$$

$$= \underline{.9885 + i(.1510)}$$

e) $e^{Log(3+4i)}$



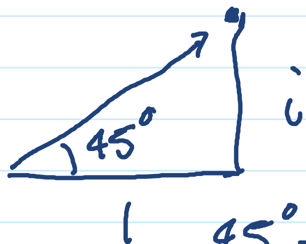
$$Log(3+4i) = \ln|3+4i| + i\,Arg(3+4i)$$
$$= \sqrt{3^2+4^2} + i\tan^{-1}(4/3)$$
$$= 25$$
$$= \ln(5) + i\tan^{-1}(4/3)$$

$$e^{Log(3+4i)} = e^{\ln(5) + i\tan^{-1}(4/3)} = e^{\ln(5)} e^{i\tan^{-1}(4/3)}$$

$$= 5\left(\cos(\tan^{-1}(4/3)) + i\sin(\tan^{-1}(4/3))\right)$$

$$= 5(.6 + i(.8)) = \boxed{3+4i}$$

f) $(1+i)^{1/2} = e^{1/2\,Log(1+i)}$



$45° = \pi/4$

$$Log(1+i) = \ln|1+i| + i\,Arg(1+i)$$
$$= \ln(\sqrt{2}) + i(\pi/4)$$

$$(1+i)^{.5} = e^{1/2(\ln\sqrt{2} + i\pi/4)} = e^{\frac{\ln\sqrt{2}}{2} + \frac{i\pi}{8}}$$

$$= e^{\frac{\ln\sqrt{2}}{2}} e^{i\pi/8} = \boxed{e^{\frac{\ln\sqrt{2}}{2}}\left(\cos(\pi/8) + i\sin(\pi/8)\right)}$$

$$= e^{\frac{\ln\sqrt{2}}{2}}(.9999 + i(.0069)) = 1.189 + .0082i$$

$$= e^{\frac{\ln 7 i}{2}}\left(.9999 + i(.0069)\right) = \underline{\underline{1.189 + .0082 i}}$$

3.51 Consider : $e^{b \log a} := e^{a^b}$

$\log a := \ln|a| + i\,\mathrm{Arg}(a)$

$$e^{b \log a} = e^{b\ln|a| + ib\,\mathrm{Arg}(a)} = e^{b\ln|a|}e^{ib\,\mathrm{Arg}(a)}$$

$$= e^{b\ln|a|}\left(\cos(b\,\mathrm{Arg}(a)) + i\sin(b\,\mathrm{Arg}(a))\right)$$

single valued $:= f := G \to \mathbb{C} \mid f$ is one-to-one

let $f(z) = e^{z} \mid a^b \in G \Rightarrow f(a^b) = e^{a^b}$

let some $\alpha, \beta \in G \mid f(a^b) = f(\alpha^\beta)$

$$\Rightarrow e^{a^b} = e^{\alpha^\beta} \Rightarrow e^{b\,\mathrm{Log}(a)} = e^{\beta\,\mathrm{Log}(\alpha)}$$

$$\therefore \ln(e^{a^b}) = \ln(e^{\alpha^\beta})$$

$$\Rightarrow a^b = \alpha^\beta \Rightarrow b\,\mathrm{Log}(a) = \beta\,\mathrm{Log}(\alpha)$$

$$\forall a^b, \alpha^\beta \in G\ \exists\ e^{a^b}, e^{\alpha^\beta} \in \mathbb{C}$$

$\therefore$ f is one-to-one

Consider $b = -\psi \mid \psi \in \mathbb{Z}$

$$e^{a^b} = e^{a^{-\psi}} = e^{-\psi \operatorname{Log}(a)}$$

in this instance we must now consider both positive & negative real roots of e^{a^b}

$\therefore$ f is not one-to-one.

Similarly consider $b = \sigma \mid \sigma \in \mathbb{Q}$

let $\sigma := \eta/\gamma \mid \eta, \gamma \in \mathbb{Z}$

Now $e^{a^b} = e^{a^{(\eta/\gamma)}} = e^{\eta/\gamma \operatorname{Log}(a)}$

written as $e^{\eta\gamma^{-1}\operatorname{Log}(a)}$, a similar issue arrises. We must now consider both positive & negative real roots of $b = \eta\gamma^{-1}$.

Again e^{a^b} now has multiple

Again e^{ab} now has multiple solutions in $\mathbb{C}$ & $f := G \to \mathbb{C}$ is one-to-many.

These Conclusions thus prove f is one-to-one and single valued only when $b \in \mathbb{Z}^+$. ∎

p. 68 / #4.31 $f(z) = \bar{z}$

$$\int_\gamma f(z)\,dz := \int_{t_i}^{t_f} f(\gamma(t))\,\dot{\gamma}(t)\,dt$$

a) $x = iy$, $0 \to 1+i$

$\gamma = t + it \;/\; 0 \le t \le 1$

$$\int_\gamma f(z)\,dz = \int_0^1 (t - it)(1+i)\,dt$$

$$= \int_0^1 (t - \cancel{it + it} + t)\,dt = \int_0^1 2t\,dt$$

$$= t^2 \Big|_0^1 = 1 - 0 = 1$$

$= t^2 \Big|_0^1 = 1 - 0 = 1$

b) $x^2 = iy$, $0 \to 1+i$

$\gamma = t + it^2 \;/\; 0 \le t \le 1$

$\int_\gamma f(z)\,dz = \int_\gamma^1 (t - it^2)(1 + 2it)\,dt$

$= \int_0^1 (t + 2it^2 - it^2 + 2t)\,dt = \int_0^1 (3t + it^2)\,dt$

$= 3\int_0^1 t\,dt + i\int t^2\,dt$

$= \frac{3t^2}{2}\Big|_0^1 + \frac{it^3}{3}\Big|_0^1 = 3/2 - 0 + i/3 - 0$

$= 3/2 + i/3 = 9/6 + 2i/6 = \boxed{1/6\,(9+2i)}$

c) $iy = 0$, $0 \to 1$ $\wedge$ $x = 0$, $1 \to 1+i$

$\gamma = \begin{cases} t \\ 1 + it \end{cases} \quad 0 \le t \le 1$

$\int f(z)\,dz = \int^1 t\,dt + \int^1 (1 - it)(i)\,dt$

$$\int_\gamma f(z)\,dz = \int_0^1 t\,dt + \int_0^1 (1-it)(i)\,dt$$

$$= \int_0^1 t\,dt + \int_0^1 i+t\,dt$$

$$= \int_0^1 t\,dt + \int_0^1 i\,dt + \int_0^1 t\,dt$$

$$= 2\int_0^1 t\,dt + i\int_0^1 dt = t^2\Big|_0^1 + it\Big|_0^1$$

$$= 1-0+i-0 = 1+i$$

4.81

a) $f(z) = z^2 \quad \gamma = t+it^2,\ 0 \le t \le 1$

$$\int_\gamma f(z)\,dz = \int_0^1 (t+it^2)^2(1+2i)\,dt$$

$$= \int_0^1 (t^2+2it^3-t^4)(1+2i)\,dt$$

$$= \int_0^1 (t^2+2it^3-t^4+2it^2-4t^3-2it^4)\,dt$$

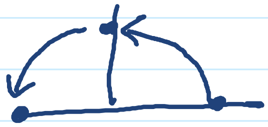
$$= \frac{t^3}{3} + \frac{\cancel{2}it^4}{2\cancel{4}} - \frac{t^5}{5} + \frac{2it^3}{3} - \frac{\cancel{4}t^4}{\cancel{4}} - \frac{2it^5}{5}\Bigg|_0^1$$

$$= \left(\frac{t^3}{3} - t^4 - \frac{t^5}{5}\right) + i\left(\frac{2t^3}{3} + \frac{t^4}{2} - \frac{2t^5}{5}\right)\Big|_0^1$$

$$= \left(\frac{1}{3} - 1 - \frac{1}{5}\right) + i\left(\frac{2}{3} + \frac{1}{2} - \frac{2}{5}\right)$$

$$= \left(\frac{5}{15} - \frac{15}{15} - \frac{3}{15}\right) + i\left(\frac{20}{30} + \frac{15}{30} - \frac{8}{30}\right)$$

$$= -\frac{13}{15} + \frac{27}{30}i = \boxed{\frac{1}{30}(-26 + 27i)}$$

b) $f(z) = z \qquad \gamma:$ 

$$\gamma = e^{it}, \quad 0 \le t \le \pi$$

$$\int_\gamma f(z)\,dz = \int_0^\pi e^{it}(ie^{it})\,dt$$

$$i\int_0^\pi e^{it}e^{it}\,dt = i\int_0^\pi e^{it+it}\,dt = i\int_0^\pi e^{2it}\,dt$$

$$\Downarrow$$

$$\frac{d}{dt}e^{2it} = 2ie^{2it} \qquad \frac{ie^{2it}}{2}\Big|_0^\pi$$

$$\int_\gamma f(z)\,dz = \frac{ie^{2i\pi}}{2} - \frac{ie^0}{2}$$

$$\frac{i}{2}\left(e^{2i\pi} - e^0\right) = \frac{i}{2}(1-1) = \boxed{0}$$

$$= \frac{i}{2}\left(e^{2i\pi} - e^{0}\right) = \frac{i}{2}(\cancel{1-1}) = \boxed{0}$$

c) $f(z) = e^{z}$, $x = iy$, $0 \to z_0$

$$\gamma = t z_0 \qquad 0 \le t \le z_0$$

$$\int_\gamma f(z)dz = \int_0^{z_0} e^{tz_0} z_0 \, dt = z_0 \int_0^{z_0} e^{tz_0} dt$$

$$\frac{d}{dt} e^{tz_0} = z_0 e^{tz_0} \qquad \Downarrow \quad e^{tz_0}\Big|_0^{z_0}$$

$$\int_\gamma f(z)dz = e^{z_0^2} - e^{0} = \boxed{e^{z_0^2} - 1}$$

d) $f(z) = |z|^2$, $x = iy$, $2 \to 3 + i$

$$\gamma = 2 + t + it , \quad 0 \le t \le 1$$

$$\int_\gamma f(z)dz = \int_0^1 \sqrt{(2+t)^2 + (it)^2}^{\,2} (1+i)\, dt$$

$$= \int_0^1 ((4 + 4t + t^2) + (-t^2))(1+i)\, dt$$

$$= \int_0^1 (4 + 4t)(1+i)\, dt$$

$$= \int_0^1 4 + 4t + 4i + 4it \, dt$$

$$= \int_0^1 4 + 4t + 4i + 4it \, dt$$

$$= 4t + \frac{4t^2}{2} + 4it + \frac{4it^2}{2} \Big|_0^1$$

$$= (4 - 0) + (4/2 - 0) + (4i - 0) + \left(\frac{4i}{2} - 0\right)$$

$$= 4 + 2 + i(4 + 2) = \boxed{6 + 6i}$$