Muon Lifetime

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Abstract

We present results from an experiment which measures the lifetime of the muon. This measurement was done by detecting positively charged muons (μ^+) approaching the apparatus from the atmosphere, stopping them when they are incident on a thick aluminum slab, and measuring the time it takes to detect a resultant positron (e^+) from the muon's decay process. The distribution of delay times between muon (μ^+) and positron (e^+) detection provide an understanding of the muon lifetime. We found that the average time difference between μ^+ detection and e^+ detection was $\tau_{avg} = 2.216679 \pm 0.013 \times 10^{-6}$ s. Therefore the muon lifetime predicted by this experiment was within 0.01s of the expected lifetime $\tau_\mu = 2.196981(22) \times 10^{-6}$ s [1][2].

Introduction

In this experiment, we measured the lifetime of cosmic muons by measuring the interval of time it took to decay. We did this by operating an apparatus that stopped the muons from the atmosphere using a slab of aluminum. When the muons are stopped, the apparatus measures the interval of time between the initial detection of the muon, and the detection of the resulting positron after the decay. We used the distribution of intervals of time in order to fit a decaying exponential and recover the missing parameter τ, for the μ^+ lifetime.

The objective of this experiment was to statistically determine the average interval of time it takes for muons from the upper atmosphere to decay via the following decay process

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu. \tag{1}$$

Through the use of a photomultiplier and time-to-amplitude converter, we were able to convert the signal from the scintillation detector into an electronic voltage, representing the presence of a time delay due to muon decay. We then fitted a decaying exponential for the parameter τ, the muon lifetime:

$$N(t) = \lambda e^{-\lambda t}, \tag{2}$$

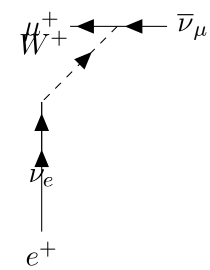
where $\lambda = 1/\tau$.

Theory

The decay process

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu \tag{3}$$

has a theoretical decay rate Γ.



Methods

After a charged particle passes through the scintillation detector, the flash of light that is produced is covnerted to a pulse of electrons via the photomulitplier tube. This pulse is then filtered for noise from the detector itself, and amplified by a discriminator/amplifier. After being amplified, the signal is passed into a coincidence circuit that allows us to measure coincidences through a combination of logic modules and delay lines. The condition for the timer to start is that a muon is detected by both of the detectors that are placed on the incoming side of the aluminum slab, and that the detector on the outgoing side of the slab does **not** detect this muon.

The condition to stop the timer is that a positron is detected by either of the detectors on the incoming and outgoing sides, but not in both detectors. In order to implement this, a circuit implementing an AND gate, as well as an XOR gate needs to be constructed. It is important to note that we have an exlusive OR gate here, and not a regular OR gate. The XOR gate is eliminating events that are necessarily detected by both the detector on the incoming side, and the detector on the outgoing side. The set of events for which this is true is characterized by having both a cosmic muon coming from the atmosphere, and also a background muon being detected at the same time. A regular OR gate would have included these events. This is important because we want to isolate detections of muons coming from a single source, namely, the cosmic muons. Using an OR gate would have contaminated our sample with muons that are from the background signal. The rate at which thes events occur is going to be the probability of detecting a positron on BOTH the incoming and outgoing side (B AND C), conditioned on having detected a true cosmic muon (both detectors on the incoming side - given A AND B) . One way to correct for the background noise if we didn't use an XOR gate would be to measure an estimate of the background noise, and subtract off this estimate before performing data analysis.

Results and Discussion

0.1 Statistics

The model we used in this experiment was that the distribution of the measurements of the lifetime of the muon, is distributed as an exponentially distributed random variable. This means that our model assumes the distribution function is given by:

$$f_X(x) = \lambda e^{-\lambda x}, \quad \lambda > 0 \tag{4}$$

And likewise, the cumulative distribution function is given by:

$$F_X(x) = 1 - e^{-\lambda x}, \quad \lambda > 0 \tag{5}$$

In this parametrization of the exponential distribution, we make the connection that

$$\lambda = \frac{1}{\tau} \tag{6}$$

One estimate for the value of the parameter λ could be the maximum likelihood estimator, which is given by:

$$\hat{\lambda}_{MLE} = \frac{N}{\sum_{i=1}^{N} x_i} \tag{7}$$

For our data, the value we calculated to be the maximum likelihood estimate on the value of the parameter λ was:

$$\hat{\lambda}_{MLE} = 0.00045851(\mathrm{ns}^{-1}) \tag{8}$$

This leads to a value for the average lifetime of

$$\tau = 2180.955923(\mathrm{ns}) \tag{9}$$

img/data.png

Figure 1: Plot of Distribution of Decay Times

According to **REFERENCE NEEDED**wikipedia, the accepted value for the mean lifetime of a muon is $(2.1969811 \pm 0.0000022) \times 10^{-6} s$. We have plotted the distribution of the data we collected, along with a vertical line showing this accepted value for the lifetime of a muon in figure 1.

img/mle_estimate.png

Figure 2: Plot of Maximum Likelihood Estimate for λ

In figure 2, we have shown a plot of the data, with the exponential distribution function using the maximum likelihood estimate for the parameter λ superimposed. We see a reasonably good fit to the data using this single exponential fit. We also performed this fit using a double exponential, and we were able to get similar results.

0.1.1 Confidence Interval

Next, we set out to compute the 99% confidence interval for our parameter λ. In order to do this, I need to define a pivot function that is a function of the observations $\{X_i\}$, as well as the parameter λ. In this case, I used the pivot function:

$$h(X_1, \ldots, X_n, \lambda) = 2\lambda \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} Y_i \tag{10}$$

where I have defined $Y_i \equiv 2\lambda X_i$. Under this definition for Y_i, the distribution of the Y_i is given by the χ_2^2 distribution. The proof of this is deferred to the appendix.

$$X \approx \text{Exp}(\lambda) \tag{11}$$

$$Y \equiv 2\lambda X \tag{12}$$

Then,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(2\lambda X \leq y) = \mathbb{P}\left(X \leq \frac{y}{2\lambda}\right) = F_X\left(\frac{y}{2\lambda}\right) \tag{13}$$

We know that

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Then,

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \frac{1}{2} e^{-y/2} \tag{15}$$

The χ_k^2 density function (for k degrees of freedom) is given by:

$$f_{\chi_k^2}(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2} \tag{16}$$

For $k = 2$, this coincides with our density function for Y:

$$f_{\chi_2^2}(x) = \frac{1}{2} e^{-x/2} \tag{17}$$

Now, because we assume our Y_i to be i.i.d as the chi-square distribution with 2 degrees of freedom, we have that the distribution function for our pivot function is given by:

$$f_h(x) = f_{\chi_{2N}^2} \tag{18}$$

Now we need to specify the α we want for our confidence interval. To get the 99% confidence interval, we use $\alpha = 0.01$. We can now compute the $1 - \alpha$ confidence interval for our parameter λ using the $\alpha/2$ and $1 - \alpha/2$ quantiles of the χ_{2n}^2 distribution. If I define the quantile function for the chi square distribution with $2n$ degrees of freedom to be $\chi_{2n}^2(x)$, then we can compute the confidence interval as:

$$\mathbb{P}\left(\chi_{2n}^2\left(\frac{\alpha}{2}\right) \leq h \leq \chi_{2n}^2\left(1 - \frac{\alpha}{2}\right)\right) = 1 - \alpha \tag{19}$$

$$\mathbb{P}\left(\frac{\chi_{2n}^2\left(\frac{\alpha}{2}\right)}{2\sum_{i=1}^{n} X_i} \leq \lambda \leq \frac{\chi_{2n}^2\left(1 - \frac{\alpha}{2}\right)}{2\sum_{i=1}^{n} X_i}\right) = 1 - \alpha \tag{20}$$

Therefore, we have that our $1 - \alpha$ confidence interval is given by:

$$\lambda \in \left[\frac{\chi_{2n}^2\left(\frac{\alpha}{2}\right)}{2\sum_{i=1}^{n} X_i}, \frac{\chi_{2n}^2\left(1 - \frac{\alpha}{2}\right)}{2\sum_{i=1}^{n} X_i}\right] \tag{21}$$

In the frequentist interpretation, we interpret this by saying the random confidence interval bounds the true [fixed] value of the population parameter λ with probability $1 - \alpha$. For our case, we had $\alpha = 0.01$, $N = 335508$, and then we took the sum of our datapoints. When we did this, we got our bounds for the confidence interval to be:

$$\hat{CI} = [0.000456478, 0.000460556] \tag{22}$$

As expected, this interval bounds the value for our maximum likelihood estimator.

Figure 3: Plot of 99% Confidence Interval on Estimate of λ

In figure 3, we plotted the exponential distribution function using the maximum likelihood estimator, as well as the exponential distribution using the upper and lower bounds on the 99% confidence interval for λ. As we can see, the confidence interval is sufficiently tight that one cannot distinguish between the maximum likelihood estimator and the lower and upper confidence interval bounds.

`img/log_distributions_confidence_interval.png`

Figure 4: Plot of Logarithm of 99% Confidence Interval Distributions

In order to see the difference between the distribution functions at the lower and upper bounds on the confidence interval, in figure 4 we've plotted the logarithm of the distribution functions to try to see some differentiation. At the tail of the distrbution, one can make out the difference between the lower and upper bounds.

0.1.2 Bayesian

In order to compare different methodologies, we also computed an estimate for the parameter λ using a bayesian approach. I used a standard gaussian as the prior, and computed the posterior distribution approximately in order to use the maximum a posteriori estimator as an estimate for the parameter λ. Note that in the bayesian framework, we do not consider a single "true" value of λ, instead we search for a distribution over all possible values of λ, and we guess a prior using the so-called nuissance parameters in order to encode our prior beliefs as to the distribution of the values of λ. We then take a point estimate of the posterior distribution to quote an estimate for the parameter λ.
We use a standard gaussian prior:

$$p(\lambda|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(\lambda-\mu)^2}{2\sigma^2}\right)} \tag{23}$$

We use the exponential distribution as our sampling distribution in order to compute the likelihood of the data:

$$L(\mathcal{D}|\lambda) = \prod_{i=1}^{N} \lambda e^{-x\lambda} \tag{24}$$

In order to simplify the computations, we used the log posterior distribution instead, in order to get rid of those pesty exponentials:

$$\ell(\lambda|\mathcal{D}) = N\log(\lambda) - \lambda\sum_{i=1}^{N} x_i - \frac{1}{2}\left[\left(\frac{\lambda-\mu}{\sigma}\right)^2 + \log(2\pi\sigma^2)\right] \tag{25}$$

Now, in order to get a point estimate for the parameter λ, we use the maximum a posteriori estimator of our posterior distribution: The MAP estimator is given by:

$$\hat{\lambda}_{MAP} = \text{argmax}_{\lambda}\ell(\lambda\mathcal{D}) \tag{26}$$

So we find the mode of our posterior distribution with respect to λ:

$$\frac{\partial\ell(\lambda|\mathcal{D})}{\partial\lambda} = \frac{N}{\lambda} - \sum_{i=1}^{N} x_i - \left(\frac{\lambda-\mu}{\sigma}\right)\frac{1}{\sigma} = 0 \tag{27}$$

Now we solve this for λ:

$$\lambda^2 + \lambda\left(\sum_{i=1}^{N} x_i - \frac{\mu}{\sigma^2}\right) - N = 0 \tag{28}$$

We can use the quadratic formula**REFERENCE NEEDED FROM EUCLID**:

$$\lambda = \frac{\left(\frac{\mu}{\sigma^2} - \sum_{i=1}^{N} x_i\right) \pm \sqrt{\left(\sum_{i=1}^{N} x_i - \frac{\mu}{\sigma^2}\right)^2 - 4(1)(-N)}}{2(1)} \tag{29}$$

$$\lambda = \frac{\frac{\mu}{\sigma^2} - \sum_{i=1}^{N} x_i \pm \sqrt{\left(\sum_{i=1}^{N} x_i - \frac{\mu}{\sigma^2}\right)^2 + 4N}}{2} \tag{30}$$

img/prior_versus_posterior.png

Figure 5: Plot of Normal Prior on λ versus Posterior Distribution

0.2 Pre-Lab Questions

- Consider an experiment that counts the number of events in a given amount of time. Suppose that, after many repetitions of the experiment, you find that the average number of events in a given amount of time is ν. What is the standard deviation of the results in your experiment?
  Consider the sequence of random variables: $\{X_1, \ldots, X_n\}$. This random process represents our experiment. If we assume the X_i are i.i.d and poisson distributed, then we know we are looking for an estimate for the parameter λ. In this case, we are told that empirical mean of the data is ν. For a poisson distributed random variable, the variance is λ, so the standard deviation is $\sqrt{\lambda}$.

- Suppose the count rate (events/sec) from the two detectors in the muon experiment are R_1 and R_2. What is the rate of accidental coincidences if the width of the coincidence window (measured in seconds) is W?

- How can the result of question 2 allow you to improve your count rate without significantly increasing the noise in your experiment?

- Assume events occur at a particular rate R (events/s). If you start looking for events at some time ($t = 0$), what is the probability that the first event you detect occurs between times T and $T + \Delta T$? Assume that Δt is a small interval of time so that $R\Delta T << 1$. Hint: First calculate the probability that zero events occur in the time interval T.

Error Analysis

Given that the time interval between muon detection events is much longer than the tolerance for a decay in our electronics (order microseconds), it's

Conclusion

References

[1] J. Beringer et al. (Particle Data Group) (2012). *PDGLive Particle Summary 'Leptons (e, mu, tau, ... neutrinos ...)'* (PDF). Particle Data Group. Retrieved 2013-01-12.

[2] C. Patrignani and Particle Data Group (2016). *Review of Particle Physics.* Chinese Physics C. 40 (10): 100001.

[3] A K De and P Ghosh and S Mitra and P C Bhattacharya and A K Das (1972) *The muon flux of cosmic rays at sea level* Journal of Physics A: General Physics 10.1088/0305-4470/5/8/016

[4] Euclid (300BC) *Euclid's Elements* Vol.2