AST3310 Project 2 - Modelling Stellar Energy Transport

Caspar William Bruenech

April 11, 2020

1 Introduction

As a continuation of the AST3310 numerical stellar models, this project aims to produce a numerical model of the interior of a star by looking at the transport of energy through the stellar medium, via both convection and radiation. We will be performing a numerical integration of various physical properties of the star, starting with the initial values on the surface and moving in towards the core. Additionally, we will be implementing the energy production model created in project 1 in order to estimate the luminosity at the different mass shells as we integrate inwards. The goal of the project is to produce a star with a large outer convection zone and with a realistically sized core. This will be done by first producing multiple candidates with varying initial parameter values, and by seeing how these changes affect the physical properties of the star. We will then fine tune the parameters to produce a model that satisfies our critera. Finally, we will perform an analysis on the model by plotting various data produced by the model, including a cross section of the star, showing where we have convective and radiative energy transport. This will then be compared to a model produced with solar values.

2 Method

2.1 Parameter Evolution Equations

Our model of the star will be a one-dimensional system, where the idea is to compute various physical quantities at given mass shells. To do this, we will start with the values of these quantities at the stellar surface, and then move our way into the star towards the core. There are five physical quantities of the star that we will compute at each given mass shell; the mass m of the volume under the current radius, the radius r, the pressure P, the temperature T and the luminosity L. In order to maintain higher numerical stability, we will use the mass m as our independent variable, which gives us the gradients of these quantities with respect to m

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}, \tag{1}$$

$$\frac{\partial P}{\partial m} = -\frac{GM}{4\pi r^4}, \tag{2}$$

$$\frac{\partial L}{\partial m} = \epsilon, \tag{3}$$

where ϵ is the stellar energy production at the current mass shell, computed using the numerical model from Project 1. Additionally we have the temperature gradient $\frac{\partial \ln T}{\partial \ln P} = \nabla$, which can also be expressed in terms of the radius r as (CITE LEC NOTES)

$$\nabla = -\frac{H_p}{T}\frac{\partial T}{\partial r},$$

where H_p is the pressure scale height, defined as

$$H_p = -P\frac{\partial r}{\partial P}. \tag{4}$$

By inserting this, multiplying with $\partial m/\partial m$, and rearranging we get

$$\frac{\partial T}{\partial m} = \nabla \frac{T}{P}\frac{\partial P}{\partial m}. \tag{5}$$

We so far then have four coupled differential equations describing the evolution of four parameters. However, we still need expressions for the density ρ and the total pressure P. We will assume that all the pressure comes from the sum of radiative pressure and gas pressure. The radiative pressure P_{rad} can be expressed as

$$P_{rad} = \frac{4\sigma}{3c}T^4,$$

where σ is the Stefan-Boltzmann constant. Since we are assuming the gas in the star to be an ideal gas, we can find the pressure of the gas P_{gas} using the ideal gas law, combined with the approximation of the total number of particles inside the mass m as

$$N = \frac{m}{\mu m_u},$$

where μ is the mean atomic weight of the particles and m_u is the atomic mass unit. This gives us the expression for the gas pressure as

$$P_{gas} = \frac{m}{\mu m_u}\frac{k_B T}{V} = \frac{\rho k_B}{\mu m_u}T,$$

where we have used that the density ρ can be expressed as $\rho = \frac{m}{V}$.

Thus, we get an expression for the total pressure

$$P = P_{gas} + P_{rad} = \frac{4\sigma}{3c}T^4 + \frac{\rho k_B}{\mu m_u}T. \tag{6}$$

This we can rewrite to an expression for the density ρ as

$$\rho = \left(P - \frac{4\sigma T^4}{3c}\right)\frac{\mu m_u}{k_B T}. \tag{7}$$

There are several quantities that need to be defined and computed here. Primarily we have the mean atomic weight μ which is a unitless number defined as the density over the total number of particles (including electrons) over the atomic mass unit. In our star, we are assuming complete ionization of all elements, which means that hydrogen and helium atoms will contribute respectively with 2 and 3 particles such that the total number of particles from these two elements can be expressed as

$$n_{tot}^{H,He} = \frac{\rho}{m_u}\left(2X + \frac{3}{4}Y\right),$$

where X and Y denote, respectively, the mass fractions of hydrogen and helium.

For the metallic elements, we will assume their nuclear consists of same amount of neutrons and protons, which means that they must have half as many electrons as their atomic mass. Thus the total number of particles is the sum over all the metals present in the star

$$n_{tot}^{metals} = \sum_i \frac{Z_i\rho}{A_i m_u} + \frac{1}{2}A_i\frac{Z_i\rho}{A_i m_u} \simeq \frac{Z\rho}{2m_u},$$

where we have assumed that A_i is large enough such that the first term is approximately zero. We can then compute the mean atomic weight as

$$\mu = \frac{\rho}{n_{tot}m_u} = \frac{\rho}{mu}\left(\frac{\rho}{m_u}\left(2X + \frac{3}{4}Y\right) + \frac{Z\rho}{2m_u}\right) = \frac{1}{2X + 3Y/4 + Z/2}.$$

Secondly we have the heat capacity at constant pressure, C_p. We are modeling the gas in the star as an ideal gas, which means that we have

$$C_p - C_v = \frac{k_B}{\mu m_u}.$$

Since we are only assuming the presence of monoatomic gases in the star, it can be shown from the equipartition theorem that the heat capacity at constant volume is given by

$$C_v = \frac{3}{2}\frac{k_B}{\mu m_u},$$

which gives us the expression for the heat capacity at constant pressure

$$C_p = \frac{k_B}{\mu m_u} + \frac{3}{2}\frac{k_B}{\mu m_u} = \frac{5k_B}{2\mu m_u}. \tag{8}$$

2.2 The Temperature Gradient

The main objective for the analytical part of this project is to find an expression for the temperature gradient ∇. In our stellar model we will assume that there is only energy transportation in the form of radiation and convection. This means that the temperature gradient depends on whether the mass shell is convectively stable or not. Additionally, we must have that the total energy flux F, defined as the total luminosity divided by the spherical area of radius r, must be equal to the energy flux of radiation and convection. I.e.

$$F_{tot} = \frac{L}{4\pi r^2} = F_{rad} + F_{con}.$$

From the lecture notes[1], we also have that

$$F_{rad} + F_{con} = \frac{16\sigma T^4}{3\kappa\rho H_p}\nabla_{stable}, \tag{9}$$

where ∇_{stable} is the temperature gradient for convective stability (radiative transport), and κ is the opacity. When the energy is being transported by radiation, there is no convection, which means that $F_{con} = 0$. Combining then the two above equations we get

$$F_{rad} = \frac{L}{4\pi r^2} = \frac{16\sigma T^4}{3\kappa\rho H_p}\nabla_{stable}. \tag{10}$$

Solving for the temperature gradient then get

$$\nabla_{stable} = \frac{3\kappa\rho H_p L}{64\pi r^2\sigma T^4}. \tag{11}$$

The next task is then to find an expression for the temperature gradient in the convective zones of the star, denoted as ∇^*. To do this, we will use expressions for the convective energy flux,

$$F_{con} = \rho C_p T\sqrt{g}H_p^{-3/2}\frac{l_m^2}{4}(\nabla^* - \nabla_p)^{3/2}, \tag{12}$$

the radiative flux,

$$F_{rad} = \frac{16\sigma T^4}{3\kappa\rho H_p}\nabla^*. \tag{13}$$

and Eq. (9) to produce an expression for $(\nabla^* - \nabla_p)$. Combining these we get

$$F_{con} = \frac{16\sigma T^4}{3\kappa\rho H_p}\left(\nabla_{stable} - \nabla^*\right)$$

$$\rho C_p T\sqrt{g}H_p^{-3/2}\frac{l_m^2}{4}(\nabla^* - \nabla_p)^{3/2} = \frac{16\sigma T^4}{3\kappa\rho H_p}\left(\nabla_{stable} - \nabla^*\right)$$

$$(\nabla^* - \nabla_p)^{3/2} = \frac{64\sigma T^3}{3\kappa\rho^2 C_p l_m^2}\sqrt{\frac{H_p}{g}}\left(\nabla_{stable} - \nabla^*\right). \tag{14}$$

Using then that

$$(\nabla_p - \nabla_{ad}) = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p), \tag{15}$$

where ∇_{ad} is the adiabatic temperature gradient of a rising gas parcel which has the same temperature, pressure, and density as the surrounding of its final position after moving. It is defined as

$$\nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{surroundings}.$$

Using thermodynamics and the assumption of no energy exchange between the gas parcel and the stellar surroundings, this can be shown to equal (for an ideal gas) CITE LEC NOTES

$$\nabla_{ad} = \frac{P}{T\rho C_p}. \tag{16}$$

Additionally we have

$$(\nabla_p - \nabla_{ad}) = \frac{32\sigma T^3}{3\kappa\rho^2 C_p v}\frac{S}{Qd}(\nabla^* - \nabla_p), \tag{17}$$

which, when combined with the expression for the velocity of the rising gas parcel

$$v = \sqrt{\frac{gl_m^2}{4H_p}}(\nabla^* - \nabla_p)^{1/2},$$

we get can get an expression for $(\nabla^* - \nabla_p)^{1/2}$. Inserting the expression for the velocity we get

$$\begin{aligned}(\nabla_p - \nabla_{ad}) &= \frac{32\sigma T^3}{3\kappa\rho^2 C_p}\sqrt{\frac{4H_p}{gl_m^2}}(\nabla^* - \nabla_p)^{-1/2}\frac{S}{Qd}(\nabla^* - \nabla_p) \\ &= \frac{32\sigma T^3}{3\kappa\rho^2 C_p}\sqrt{\frac{4H_p}{gl_m^2}}\frac{S}{Qd}(\nabla^* - \nabla_p)^{1/2}.\end{aligned}$$

We can then insert this into Eq. (15), giving us

$$\frac{32\sigma T^3}{3\kappa\rho^2 C_p}\sqrt{\frac{4H_p}{gl_m^2}}\frac{S}{Qd}(\nabla^* - \nabla_p)^{1/2} = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p).$$

By now denoting

$$U \equiv \frac{64\sigma T^3}{3\kappa\rho^2 C_p}\sqrt{\frac{H_p}{g}},$$

$$\Omega \equiv \frac{S}{Qd},$$

and

$$\xi \equiv (\nabla^* - \nabla_p)^{1/2},$$

the equation becomes

$$\frac{U\Omega}{l_m}\xi = (\nabla^* - \nabla_{ad}) - \xi^2.$$

Rewriting this, we get

$$\xi^2 + \frac{U\Omega}{l_m}\xi + \nabla_{ad} = \nabla^*. \tag{18}$$

We can then insert this into Eq. (14) in order to eliminate ∇^*. We note that the left side of this equation is simply ξ^3, and the factor on the right side is U/l_m^2. Thus the equation becomes

$$\frac{l_m^2}{U}\xi^3 - \nabla_{stable} = -\xi^2 - \frac{U\Omega}{l_m}\xi - \nabla_{ad}.$$

We collect all the ξ terms and rewrite, giving us

$$\xi^3 + \frac{U}{l_m^2}\xi^2 + \frac{U^2\Omega}{l_m^3} + \frac{U}{l_m^2}\left(\nabla_{ad} - \nabla_{stable}\right) = 0. \tag{19}$$

This is a odd-degree polynomial, which means it must have at least one real root. Thus we can, numerically, find a value for ξ and input this into equation (18) in order to get a value for the temperature gradient of the star ∇^*.

We now have ways of computing the temperature gradients required to model the temperature change. However, we still need to know when to use which gradient, or in other words, when do we have convective instability (convection and radiation energy transport), and when do we have convective stability (only radiation)? This is given by the Schwarzschild criterion[2], which can be written as

$$\nabla_{stable} > \nabla_{ad} \rightarrow \text{Unstable}. \tag{20}$$

So when the temperature gradient of the star in case of no convection (∇_{stable}) is larger than the adiabatic temperature gradient, we will have convection, and the temperature gradient will be given by ∇^*.

2.3 Numerical Model

The idea of this project is to produce a model which takes the initial radius, density, temperature and pressure as input and outputs the values of the parameters at given mass shells in the range $R \in [R_0, 0)$, $L \in [L_0, 0)$, and $m \in [M_0, 0)$. To do this, we will perform a numerical integration of the coupled differential equations defined above with the mass m as our independent variable. The integration will be performed using the Runge-Kutta 4 method. At each mass shell, the convective stability criterion will be checked, and the corresponding temperature gradient will be used in Eq. (5) when computing the gradients. Additionally, we will be implementing dynamic step size in order to ensure that no quantity changes more than a given percentage each step. This can be done by precalculating the gradients and setting the step size such that the parameter with the largest gradient does not change more than a given fraction p. The overall goal is tune the input parameters such that we produce a star that has

- A continous convection zone near the surface of the star, and has a width of at least 15% of the radius R_0,
- Has L, m, and R going to at least 5% of their respective initial values,
- Has a core with a width of at least 10% of the radius, with the core being defined as where the luminosity is less than 95% of L_0.

3 Implementation & Results

Using the solar initial parameter values

$$L_\odot = 3.846 \times 10^{26}\text{W}, \quad R_\odot = 6.96 \times 10^{8}\text{m}, \quad M_\odot = 1.989 \times 10^{30}\text{kg},$$

$$\rho_\odot = 19.9936 \times 10^{-5}\text{kg}\,\text{m}^{-3}, \quad T_\odot = 5770\text{K},$$

and the mass fraction values from project 1, a mean molecular weight value was computed to be $\mu = 0.62$.

In order to determine what parameters to tweak in order to produce a model which satisfied our criteria, a total of eight stars were modeled, one model for a high and low value for each of the four initial parameters. The resulting cross-section plot of each model is shown in figure 1.

Using the data from the models in 1, the parameters were tweaked to produce a star that satisfied the given criteria. The resulting star had the initial parameter values

$$R_0 = 1.5R_\odot, \quad \rho_0 = 1.2\rho_\odot, \quad T_0 = 0.9T_\odot = 5193K, \quad P_0 = 0.8P_\odot.$$

The final values of the mass, radius and luminosity following the integration loop were, respectively, 2.1%, 0.1%, and 0.0% of their initial values. Additionally, the star has an outer convection zone of size 20.72% and a core of size 16.80% of R_0. A plot of the cross-section, the temperature gradients, the relative energy flux, and the energy production, all as a function of the radius are shown in figure 2, while figure 3 show all the computed parameters as a function of radius.

4 Discussion

By changing the initial parameters, it was observed that increasing the radius, increasing the initial density, reducing the surface temperature, and decreasing the initial pressure all contributed to a larger outer convection zone, with the increase in radius providing the biggest boost, as seen in Fig. (1). To understand why this happens, we need to study the convective stability criterion and the temperature gradients. If the star is to have a larger outer convection zone, the adiabatic temperature gradient needs to be smaller than the stable temperature gradient for a larger part of the radius. We remind that these two gradients are given by

$$\nabla_{stable} = \frac{3\kappa\rho H_p L}{64\pi r^2 \sigma T^4}, \qquad \nabla_{ad} = \frac{P}{T\rho C_p}.$$

Stellar Cross Sections with Varying Initial Parameter Values

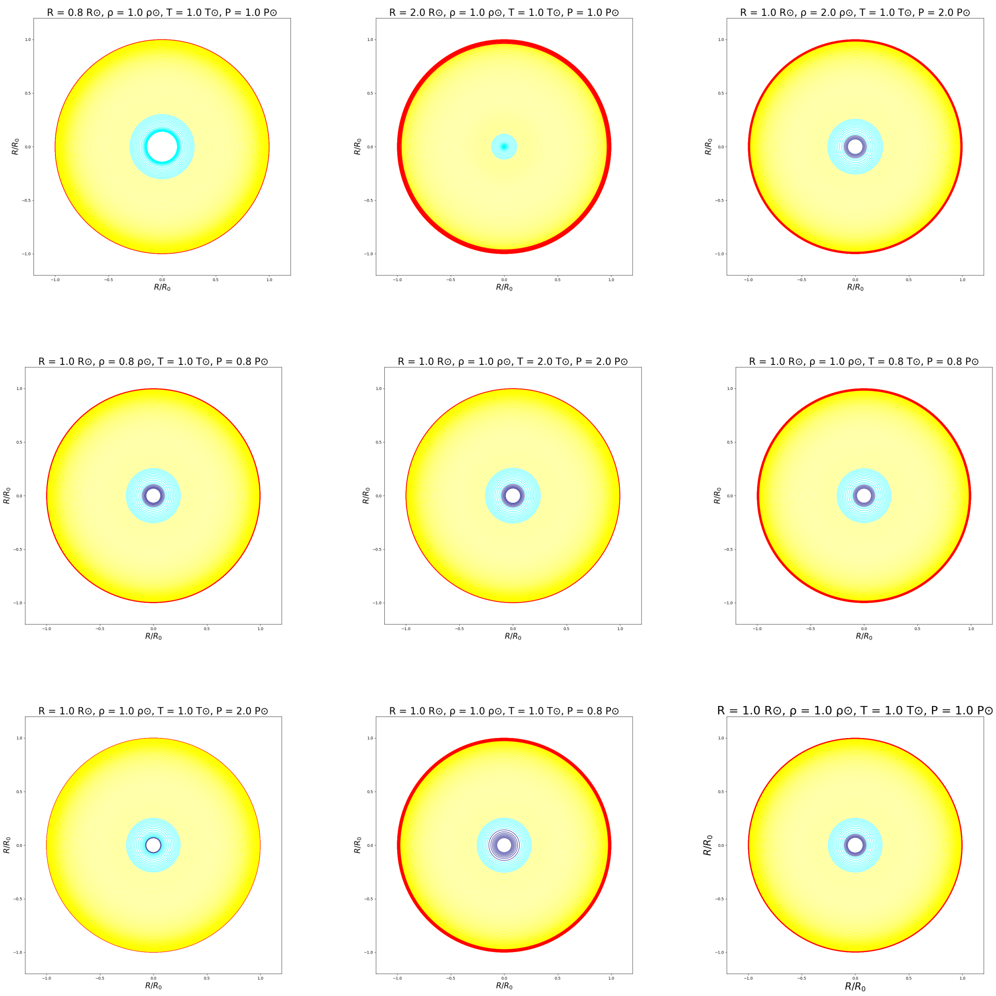

Figure 1: Cross sections of models with varying initial parameters.

Best Model Visualization

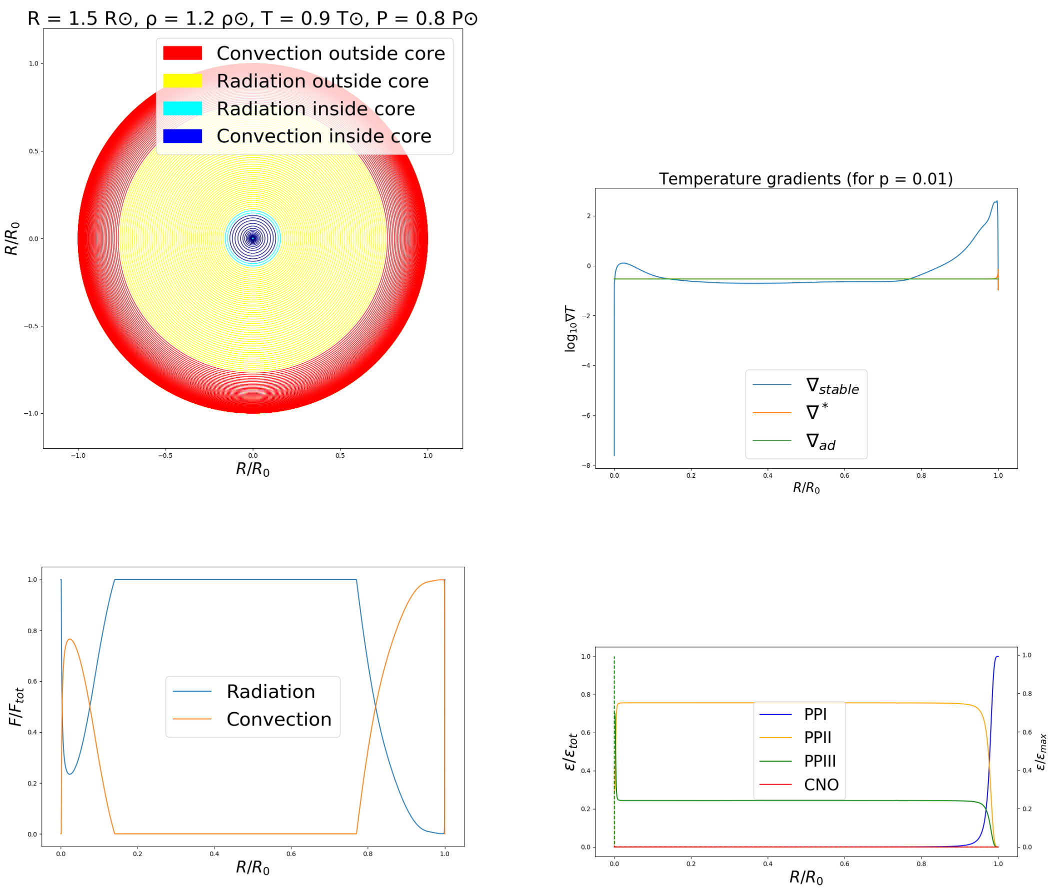

Figure 2: Various data of the best model. Top row shows the cross section and temperature gradients as function of radius, while the bottom show shows the relative radiative and convective flux, and the energy production per branch, both as function of the relative radius.

Best Model Parameters

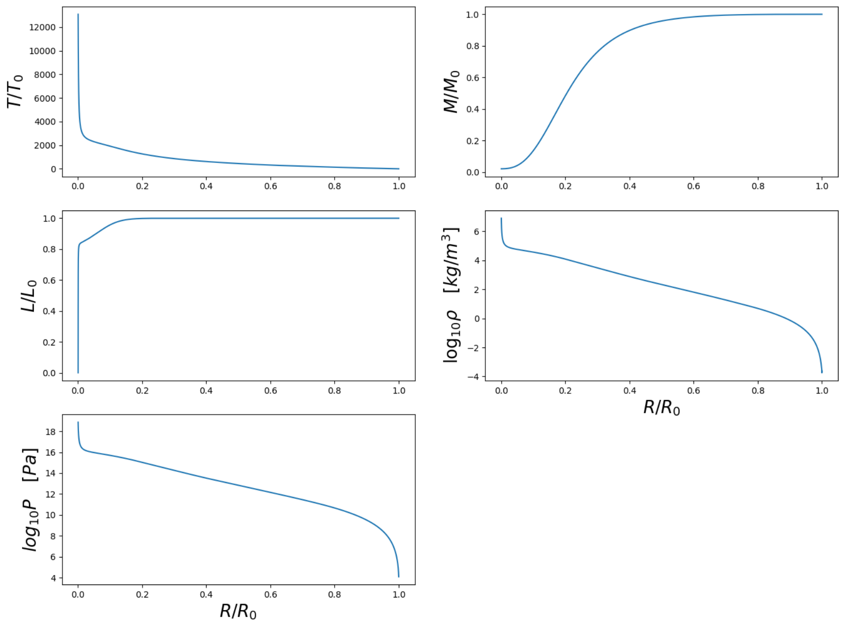

Figure 3: Computed parameters of the best model as a function of the relative radius.

We see that decreasing the initial temperature T_0 increases both gradients, but ∇_{stable} is increased by a factor of 3 higher than the adiabatic gradient, thus explaining the observations. Additionally, the pressure P is only present in the adiabatic gradient, and is linearly proportional to it, thus decreasing this decreases the adiabatic gradient, and further increases the size of the convection zone, again as observed. The same argument can be made for the density ρ, which is also confirmed by the data. Finally, there is the case for the radius r. As already mentioned, it was observed that increasing the initial value of this parameter not only contributed to the increased outer convection zone size, but it also had the biggest impact. However, ∇_{stable} is inversely proportionally to the square of the radius, thus we would expect a higher radius to drastically reduce the outer convection zone. This is true, but we must note it is only true for the initial value it has. And in fact, the best model star does have a small radiation zone on the very edge of the surface, as seen on the temperature gradient plot in Fig. (2), but the outer convection zone is still very large because of how these parameters change as we go further into the star. As we integrate towards the core, the radius and luminosity get smaller, while the density, pressure, and temperature all increase, as we see in the parameter plots in Fig. (3). While the temperature is the biggest contributor to the change in the stable gradient value, it is most likely the density that causes the gradient to jump very early in the surface of the star. The temperature changes very slowly before reaching the core, but the density (and pressure) jumps by several orders of magnitude very rapidly within the first few % of R. The gradient then gradually decreases as the density and pressure keep increasing, while the temperature and radius are becoming smaller. As we see in the plots, the temperature changes so slowly in the first 50% of R_0, that the increase in density outweighs the decrease in temperature. We then get a radiation zone once the stable gradient has decreased to a value below the adiabatic temperature gradient, before again increasing as we enter the core where the temperature has a steeper increase.

An interesting observation of the model is what happens in the very centre of the core. Here we observe that there is a radiation zone, with all the parameters making a significant jump in their respective values. This is most likely caused by the fact that the pressure is inversely proportional to the fourth power of the radius, thus it will go to ∞ as $R \to 0$. Indeed we see that it jumps several orders of magnitude as the radius approchaes zero. As a result, the stable temperature gradient plummets, and we get a new zone of radiation. Additionally, in the plot of the energy production, Fig. (3), we observe that essentially all the energy is produced in the limit $R \to 0$. This is again a result of the dramatic increase in the density; the denser the gas, the higher the pressure, the higher the temperature, which leads to a higher reaction rate, as seen in Proj. 1, thus more energy is produced. The energy production plot also shows the energy production from each branch relative to the maximum produced energy. This shows that the $pp-II$ chain dominates, with the contribution from the other branches only becoming visible when zooming substantially. This is in line with what we observed in Proj. 1, where we saw that the $pp-II$-branch contributed to around

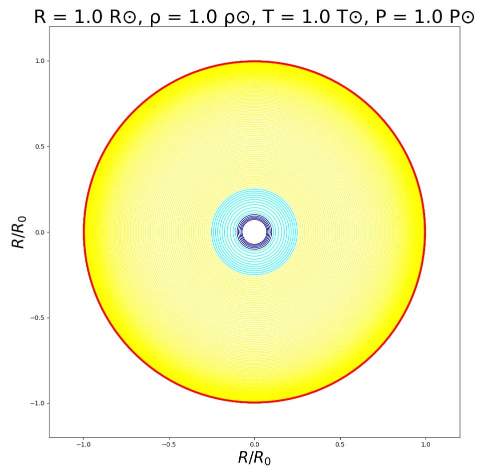

Figure 4: A cross section produced with solar parameters.

99.7% of the total energy produced.

In the bottom left plot in Fig. (2) we see how the flux of the energy produced by radiation and convection varies as a function of the radius. As expected, we see that the areas which have convection also have radiation, while the majority of the radius is dominated by pure radiative flux. We notice the abovementioned radiation zone where $R \approx 0$. By further zooming in on the plot, one can also observe the very small radiative zone on the very surface of the star. All of these observations are supported by the other plots.

Finally, we can compare the cross-section of the best model with that of the sun, where all the parameters are $1 \times X_{\odot}$. A plot of this is shown in Fig. (4). The most striking difference is the size of the convection zone, with the model having a zone over 20x larger than the one in the sun. Additionally, we see that the sun has a significantly large radiation zone inside the core, with a small convection zone, while the opposite is observed in the best model. It is hard to say whether the radiation zone in the very centre of the model is something we would have observed in the sun or not, as in this model the integration loop has run longer such that the radius ended on a smaller value. Thus we can not realistically compare the two models for radiae smaller than the smallest value in the solar model. With this in mind, one can conclude that while the size of the various zones are different in the two models are different, the structure is much the same, with an outer convection zone, a large outer radiation zone, and a radiation and convection zone within the core.

5 Conclusion

In this project we have, using thermodynamics and various assumptions about stellar interiors, we have produced a numerical model for simulating the energy transport within the star. By finding expressions for the temperature gradients for convective and radiative energy transport, combined with the Schwarschild criteria for convective stability, we have successfully implemented both convection and radiation at realistic positions within the star. By tweaking the initial parameters, we observed how these contributed to increasing the outer convection zone, with an increasing the radius and decrease in the pressure on the surface having the largest impact. With these observations, a star was produced with a radius of 1.5 times the solar radii, 1.2 times the surface density, and 0.8 times the pressure, with a surface temperature of 5193K, which satisfied the initially set criteria of a 20% outer convection zone and a core with a minimum size of 10% of the radius. The model has a significantly larger convection zone inside the core when compared to the solar model, with a radiation zone at the very center, which was most likely the result of running the integration loop longer and letting the parameters go further to zero.

References

[1] B V Gudiksen. "AST3310: Astrophysical plasma and stellar interiors". In: (2015).

[2] Karl Schwarzschild. *Gesammelte Werke Collected Works*. 1992. DOI: 10.1007/978-3-642-58086-4.