AST3310: Astrophysical plasma and stellar interiors

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Chapter 1

Our Sun and the stars

1.1 Introduction

The stars that populate our universe are all based on the same basic physical setup. There are a few types which are called stars, but aren't stars according to the definition of stars as "a concentration of gas in space which generates a power output due to thermally ignited nuclear fusion in its core". Among these are white dwarf stars, which are the cooling stellar cores left over from a powerful blast that has ejected the outer layers of the progenitor star. Brown dwarf stars are another example, but brown dwarfs are failed stars that are not able to generate enough heat and pressure in their cores to ignite nuclear fusion, and so are doomed to radiate the energy from the compression of their progenitor gas cloud and the radioactive elements in their interiors. In these notes we will deal only with "real stars".

Stars have a heat generating engine in their central regions. The energy and heat is produced through nuclear fusion of light elements into heavier ones. That process is possible because the energy per nuclear particle needed to bind protons and neutrons together into an atomic nucleus, is lower and lower the heavier the nucleus becomes. That makes it energetically advantageous to fuse light elements, and the only reason this does not happen all the time, is that atomic nuclei have a strong repulsion force that makes it very hard to get them close enough to each other for the fusion to begin. The fusion of lighter into heavier elements has a limit where it no longer is energetically advantageous. That limit is iron consisting of 56 nuclear particles. For heavier elements, energy is needed to join more protons or neutron to the nuclei. The nuclear burning rates[1] of the fusion between different nuclei depends sensitively on temperature and pressure. The large dependence on temperature comes from the thermal motions needed to overcome the repulsion between two nuclei. The repulsion between two nuclei is the consequence of both nuclei having a positive electric charge which has to be overcome by the thermal motions. The larger the nuclei, the higher their charge, and consequently the Coulomb repulsion force becomes larger. To overcome that force, higher and higher temperatures are needed for fusion to happen. The burning of hydrogen into helium is most efficient at relatively low temperatures, because it only involves hydrogen with one proton in its nucleus, while heavier nuclei only fuse at higher temperatures because they contain more protons in their nuclei, and so have a larger charge. The consequence is that stars initially burn hydrogen into helium and when the energy production starts to wane because the amount of hydrogen available as fuel is depleting, the stellar core starts to contract, heats up and raises the core temperature to then ignite helium that fuses into heavier elements and so on. Stars need to have larger and larger masses to be able to burn heavier and heavier elements. Our Sun will only be able to burn helium in its last nuclear burning phase and will not be able to burn significant amounts of heavier elements. In order for a star to be able to burn the heaviest elements and produce iron (Fe), it needs to be much more massive than the Sun. Only stars of roughly more than 20 solar masses are capable of continuing cycles of heavier and heavier element fusion up to iron.

The energy produced by the fusion in the central parts of the star is transported to the stellar surface by either electromagnetic radiation or by the movement of hot gas from the interior of the star to the stellar surface. The transport of energy through radiation might seem strange, since the density of the gas in stellar interiors is very high, but it turns out to be effective enough. The transport of energy by moving parcels of gas is called convection. Convection is somewhat akin to what can be observed in a pot of hot water on the stove just before it boils. Patterns consisting of upwellings of hot water from below, and thin lanes of water that has cooled at the surface and moves back down to be reheated in the bottom of the pot.

Our Sun is an example of a rather quiescent medium size star, but which includes all these elements. It has a nuclear burning core where the fusion

[1] "Burning" is often used in this context, even though it is actually nuclear fusion, and not burning which involves the chemical process of oxidation of other materials, the process known in everyday life as fire.

of hydrogen into helium provides the energy that is initially transported by radiation the first two thirds of the solar radius, until the gas becomes convective and the gas motions transport the bulk of the energy to the surface where it is radiated into space.

There are stars that radiate more than a million times more energy than our Sun, as well as stars that radiate much less. In general, light stars radiate less energy, while heavy stars radiate more because heavy stars have higher pressure and temperatures in their centers, making the fusion more effective. It becomes so much more effective that the heavy stars are able to burn their available hydrogen much faster than light stars. Stars that are very light ($\sim 0.1 M_{\odot}$) have a lifetime that is much longer than the present age of the universe ($\sim$ 13.7 Gyr), while heavy stars can have lifetimes of only a few hundred million years. Dwarf stars like our Sun are estimated to have a lifetime of roughly 13 Gyr.

At some point, the fuel that is needed for the nuclear fusion at the core of an evolving star runs out, and the star is no longer able to resist the gravitational pull and it will contract. The contraction releases energy and for heavy stars a new fusion process is then able to start. For light stars, the contraction will not be able to raise the temperature sufficiently high enough to start a new fusion process. The last part of a star's life depends heavily on the initial mass. Stars with masses around one solar mass will end up as a white dwarf star, which has ejected its envelope as a consequence of a last desperate attempt to ignite fusion in a shell around the old core where there is still enough hydrogen to be consumed by fusion. A white dwarf has a nucleus which resists gravity through quantum effects. The electrons are forced into such a small volume that the quantum rules for fermions make it impossible to force more electrons into the same volume, because all available quantum states are already filled. Heavier stars of more than roughly 8 solar masses will after a number of contraction and expansion phases turn into a supernova type II where the stellar core compresses into a neutron star where most protons and electrons have fused into neutrons, and resist gravity by the quantum exclusion principle for neutrons which are fermions like electrons. The explosion of a supernova type II is very violent and is capable of raising the temperature of the gas to the extreme values that are needed for fusion into elements heavier than iron. Almost all the elements heavier than iron found on Earth are created through supernovae type II.

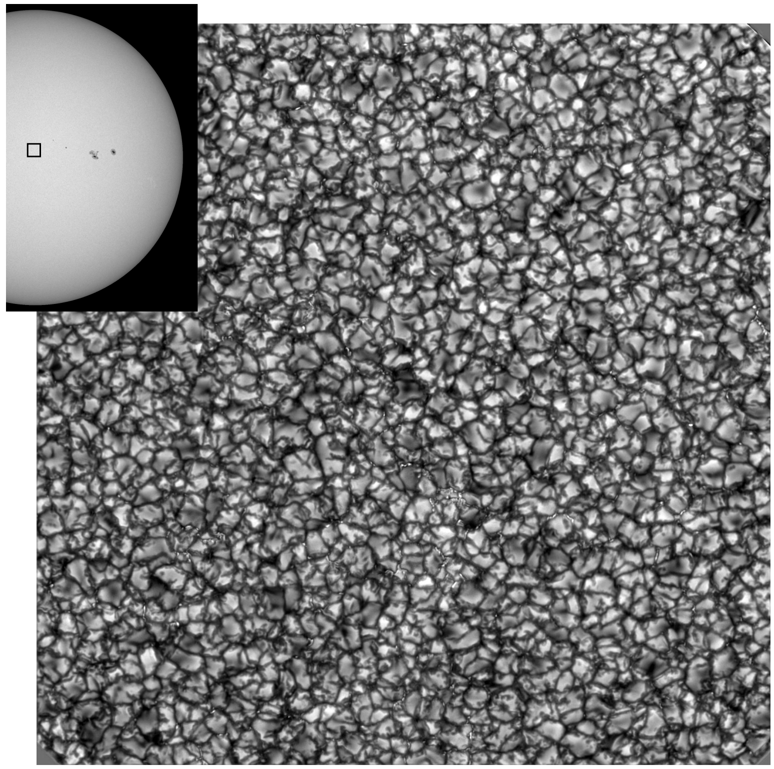

Figure 1.1: An image of the solar photosphere showing the imprint of the solar convection. Hot gas flows up from the solar interior creating bright bubbles. As these bubbles radiate their energy away, the gas contracts and flows back into the solar interior in the dark lanes surrounding the bright bubbles. This image is taken with the Swedish 1–m Solar Telescope (SST) on the island of La Palma, Spain. The inset in the upper left is a continuum image of part of the solar disk taken with NASA's Solar Dynamics Observatory (SDO). The small black square outlines the size of the SST image which measures about 58,0000 km on a side.

1.2 Our Sun

The Sun is a quiescent, middle-of-the-road star of spectral type G 2V. Even though it might seem as a rather boring star, its energy output of $3.8 \cdot 10^{26}$ W is still quite remarkable. For comparison, the total energy output by the nuclear arsenal of both the USA and the Soviet Union at the height of the cold war of ~77000 nuclear warheads, is equivalent to the energy produced by the Sun during $2 \cdot 10^{-5}$ s. The energy production in the core is dominated by hydrogen fusing into helium. The energy is transported from the core outwards by photons, which move only a very small distance before they are absorbed, to be reemitted and traveling further out. At $^2\!/_3$ the way to the surface, the gas becomes convectively unstable, and starts to overturn, creating large rising convective cells of hot gas, that travel to the surface where they cool by radiating their energy into space. When the gas cools, the density will increase and this will continue to the point where it is heavy enough to start sinking back into the sub-surface regions. This process creates a granular pattern on the surface of the Sun as shown in Fig. 1.1.

Other stars do not show the same structure. The heaviest and most luminous stars rely on radiative transport only, while certain classes of low mass stars have convection from the nuclear burning core, all the way to their surface. Some stars even have multiple convective layers sandwiched between radiative layers.

1.3 These notes

These notes are supposed to be a brief introduction to stellar interiors and the physics that control the structure and evolution of stars.

After having read these notes and completed the first two companion term projects, you will have created a model of a star, including all of the relevant physics. Finally in the the last term exercise you will have created a numerical simulation of gas convection as it takes place in the outer parts of the Sun and stars like it.

The notes are therefore ordered so that the physics important for the central parts of stars is treated first, followed by the physics relevant for layers further out. The details of the radiative transfer at the surface of a star, where photons can escape into space is not included in these notes.

The notes are very brief, and is build upon a number of classic text books, which will not be cited directly, but interested students might find more information and deeper explanations on some of the subjects touched upon here. The text books are: Stix: "The Sun", von Mieses: "Mathematical theory of compressible fluid flow" and Schroeder: "An introduction to Thermal Physics". See Appendix C for more details on these references.

Chapter 2

Stars

2.1 The main parameters of stars

In this chapter we will derive very simple relations between the main parameters of stars, their mass, their radius and their energy output or luminosity. This can only be done under some quite strict assumptions, but as it turns out, these assumptions are very close to being correct. Some of the equations used here will be explained in more detail later in these notes.

The first assumption we make is that stars are spherically symmetric, which for most stars is not exactly correct. That is mostly due to their rotation, so that their radius along their rotational axis is smaller than perpendicular to their rotational axis. Most stars on the main sequence are not rotating that fast, and so they are very close to being rotationally symmetric. Our sun is flattened by less than 0.001 %, even though its surface rotates with a velocity of roughly 2 km s^{-1}. An important effect of this assumption is that we cannot treat convection in detail, since convection transports gas up and down along the radius of the star, but more importantly it also transports gas perpendicular to the radial direction, and that is in conflict with the spherical symmetric assumption as that means that velocity only depends on radius. Convection is treated in more detail in Sect. 5.3.

The second assumption we have to make is that stars are in hydrostatic equilibrium. That means that the pressure gradient at all locations is in balance with the gravitational force. This assumption again excludes a few types of stars from this treatment, since a few types of stars have large changes in radius and luminosity due to pressure and gravitational instabilities. An example of variable stars that pulsate as a result of these instabilities are Cepheid stars.

A star in hydrostatic equilibrium has to be able to maintain a high pressure at its core, but since it looses energy from its surface in the form of radiation, it would cool down and the high pressure at its core cannot be maintained, unless the star produces energy at its core. That happens through fusion of lighter elements into heavier ones, and independent of which elements are fused, the process is extremely sensitive to temperature. Pretty much all stars have the same central temperature because the energy production is so sensitive to temperature. The energy production rate through the CNO cycle (see Chap. 3) is proportional to T^{16}, meaning that a very small change in temperature means a huge increase in energy production. A good approximation for the central temperature of stars is

$$T_{\mathrm{c}} \approx 1.5 \cdot 10^{7} \left(\frac{M}{M_{\odot}}\right)^{1/3} \mathrm{~K} \tag{2.1}$$

To get to relations between the main parameters controlling stellar structure we can use the virial theorem. It states in this case that the potential energy of a star is roughly equal to twice the total kinetic energy of the particles in the star. The potential energy of a star is roughly

$$E_{\mathrm{pot}} = \int\int \frac{G}{r} \,\mathrm{d}m_1 \,\mathrm{d}m_2 \sim \frac{GM^2}{R} \tag{2.2}$$

where G is the gravitational constant, r is the distance between mass 1 (m_1) and mass 2 (m_2). The kinetic energy of the particles in the star is roughly

$$E_{\mathrm{kin}} = \frac{3}{2} N k_{\mathrm{B}} T \sim \frac{M}{m_{\mathrm{p}}} k_{\mathrm{B}} T \tag{2.3}$$

where N is the number of particles in the star, k_{B} is Boltzmann's constant, m_{p} is the mass of a proton and T is a typical temperature. By using these expressions and the virial theorem, we get a relation for the radius of a star

$$R \sim 10^{11} \left(\frac{M}{M_{\odot}}\right)^{2/3} \mathrm{~cm} \tag{2.4}$$

when using cgs units. The radius of the sun is roughly $7 \cdot 10^{10}$ cm, so the relation is very precise considering we have neglected a number of constants along the way.

Exercise 2.1:
Show Eq. 2.4 using the expressions for the kinetic and potential energy, and calculate the numerical constant in Eq. 2.4 using SI units.

The other main characteristic of a star is the total energy it releases in the form of radiation. Energy from the core of a star moves slowly through the star until it reaches the surface, and then it is released into space. The released energy can be observed and the total energy released in the form of radiation is the star's luminosity (L). If the energy has difficulty moving through the star, then the energy is contained longer inside the star, and the amount of energy inside the star is therefore higher. Therefore the resistance to have energy being transported through the gas - called the opacity (κ) of the gas inside the star is important to know if we want to know the luminosity of a star. The opacity of a gas comes from atomic physics, and depends only on the ionisation degree and the atomic species present in the gas. If we assume it to be roughly constant for a star, then we can find the following relation

$$\frac{L}{4\pi R^2} = \left(\frac{c}{\kappa\rho}\right)\frac{\mathrm{d}U_{\mathrm{ph}}}{\mathrm{d}r} \tag{2.5}$$

where the left hand side is the flux of energy through the star (energy per area per second) and the left hand side is a diffusion constant involving the opacity and the gradient of the radiative energy. The energy of the photons is simply given as $U_{\mathrm{ph}} = aT^4/3$ where a is the radiative constant, so we can now write the luminosity as

$$\frac{L}{4\pi R^2} = \frac{ac}{3\kappa\rho}\frac{\mathrm{d}T^4}{\mathrm{d}r} \tag{2.6}$$

inserting values for ρ, T and R we find an expression that just says

$$L \propto M^3 \tag{2.7}$$

This expression also fits well with observations, and it shows that stars produce much more energy if they are heavy. Stars seem to be rather easy to understand, when looking at these expressions, and it should be surprising that using these simple expressions we actually find relations which hold. Remember that so far we have not looked at the atomic physics that control the energy production in the centre of stars and we haven't looked at how the energy is transported through the star either. The energy production does not need to be fusion at all, and the transport of energy only needs to be describable as a diffusion process in order for these relations to hold.

So stars are easy to understand - except the devil is of course hidden in the details, so the rest of this course will be to understand each of the processes involved in keeping a star alive and calculating how stars adjust their internal structure to be able to remain stable even though stars can have masses that span 4 orders of magnitude and luminosities that span more than 8 orders of magnitude.

Exercise 2.2:
Use Eq. 2.6, Eq. 2.4 and Eq. 2.1 to derive Eq. 2.7. Using values for the sun, find the constant of proportionality in Eq. 2.7 and then use that constant to find the luminosity of stars with masses of $0.1M_\odot$ and $10M_\odot$. You can find constants you need in App. A and values for solar parameters in App. B

Exercise 2.3:
Imagine a star that has mass M and radius R however its mass density has the following form:

$$\rho = \rho_c(1 - \left(\frac{r}{R}\right)^2) \tag{2.8}$$

where ρ_c is a constant and r is the distance from the centre of the star. (i) Find m(r). (ii) Find the relation between the total mass M and radius R. (iii) Calculate the average density of the star (Hint: $\rho_{\text{aver}} = 0.4\rho_c$)

The internal pressure of a star can be found by looking at hydrostatic equilibrium. Here it is assumed that the pressure and the gravitational force are balancing each other, so that the star is stable and does not change its structure. The radial component of the force balance then means that

$$\frac{\partial P}{\partial r} = -\frac{Gm\rho}{r^2} \tag{2.9}$$

If we now assume that we can use the whole stellar radius as ∂r, then we get

$$-\frac{P_c}{R} = -\frac{GM}{R^2}\frac{M}{{}^4\!/_3\,\pi R^3} \tag{2.10}$$

$$P_c = \frac{3G}{4\pi}\frac{M^2}{R^4} \tag{2.11}$$

Inserting solar values we can see that this value is a bit too low, so if we believe that the Sun is a good way to calibrate this equation, then we find

$$P_{\rm c} = 3.45 \cdot 10^{16} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \ \mathrm{Pa} \tag{2.12}$$

Chapter 3

Stellar engines

3.1 Introduction

In this chapter we shall investigate the energy source of stars. This is essential, because otherwise a star would have a very short lifetime if it could not produce more energy. If there was no energy source in the centre of a star, it would just cool and contract in a relatively short time. In order to create our model of a star, we need to understand how a star produces its energy and what decides the amount of energy that a star produces.

The power that keeps the star alive and which is finally emitted as electromagnetic radiation at its surface is provided by nuclear fusion. The fusion happens in steps depending mainly on temperature. The efficiency of a certain process depends steeply on temperature and pressure because temperature can be converted into thermal motions, and fast motions makes it more probable for two nuclei to overcome the electrostatic repulsion and get close together. The fusion of heavier and heavier elements requires higher and higher temperatures which again requires larger stellar masses as we have just seen in Chap. 2. A star will start by burning the element which will fuse at the lowest temperature. Coincidentally this is also the process which produces the most energy when fused into another element. Figure 3.1 shows how much energy is available when fusion happens. The curve shows the binding energy per nucleon for all the elements found naturally in the universe. The binding energy is the energy required to split the nuclei into free protons and neutrons. Naturally, hydrogen has zero binding energy because it consists of a single proton. Since it takes energy to split helium, that energy must then be available when fusing hydrogen into helium. That energy is converted into free energy and ends up as heat. Fusion of elements releases energy as long as the nuclei entering the process have lower binding energy than the nuclei that are produced. Iron (^{56}Fe) is then the last element that can be produced by fusion and release energy.

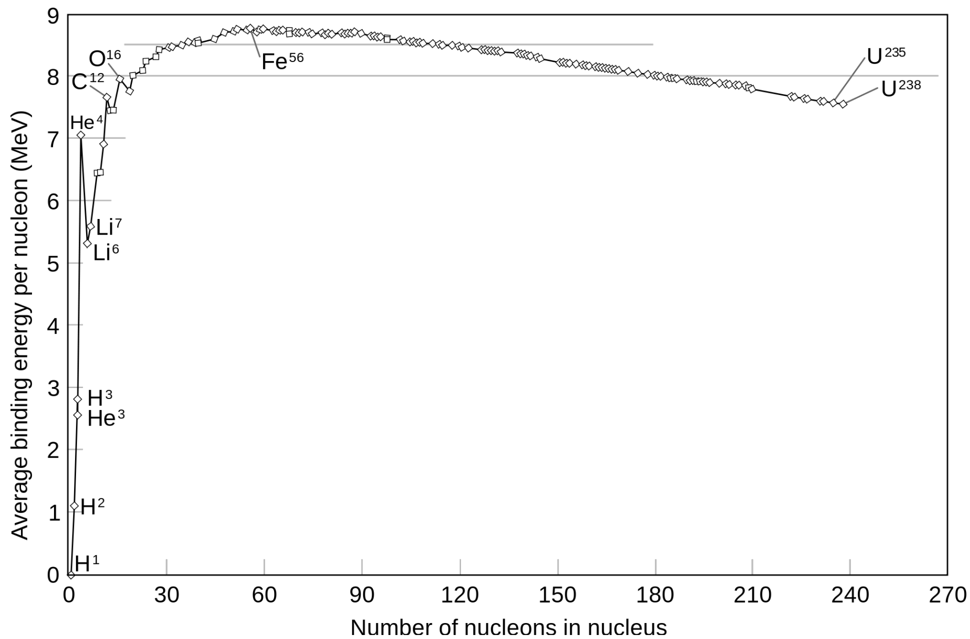

Figure 3.1: The binding energy per nucleon. Binding energy is the amount of energy needed to split the nucleus apart into isolated protons and neutrons. (From Fastfission and Wikipedia)

Exercise 3.1:
In the following we will assume that the ensemble of particles in the nuclear burning centre of the Sun is Maxwellian. That means that their velocity, and energy is distributed according to a Maxwell-Boltzmann distribution. In order for that to be true, it is necessary for the particles to have collided with each other a large number of times, before they are able to move to a region of the Sun which has a different temperature (which is a consequence of the particle momentum distribution). The distance a particle can travel before it collides with another particle is called the mean-free-path.

To calculate the mean free path, imagine we have a cylinder in which a particle moves from one end to the other. The cylinder has a cross section of 1 cm^2. The cylinder is filled with protons at rest each with a mass of m_p and they are placed uniformly so that they have a mass density equal to the average mass density of the Sun (get mass and radius of the Sun from App. B). The protons have a circular cross section with radius which is the Bohr radius a_0. Now imagine that you are trying to fire protons down through the cylinder without hitting the protons in the cylinder. You can assume that the cross sections of the protons in the cylinder do not overlap and that there is no space between the protons areas. How long can the cylinder be before the protons fill up the whole cross section of the cylinder?

Compare that length with the temperature gradient in the Sun, assuming that it is constant from the solar centre to the solar photosphere. Is it a good assumption that the particles are Maxwellian?

Exercise 3.2:
At the surface of the Sun (the photosphere), the scale height of the temperature is roughly 670 km, how does that compare with the mean free path of an atom using the same formula that you found in Exerc. 3.1? The density of the photosphere is about $\rho = 2.1 \cdot 10^{-4}\,\mathrm{kg\,m^{-3}}$, assuming pure hydrogen gas for convenience.

3.2 The proton-proton chain

The fusion of hydrogen into helium involves a chain of fusion reactions. One of the two known sets of fusion reactions is the proton-proton chain (or PP chain). The other set of fusion reactions is the CNO cycle which will be discussed in Sect. 3.3.

We can distinguish between three different branches in the PP chain of reactions, all resulting in ${}^4_2\mathrm{He}$ as the end product, but two branches involving the heaver elements lithium, beryllium and boron as intermediate products. The three branches have each their own temperature dependence and they will all be described here. The three branches have the same two reactions to begin with, and then branches off into different reactions that release energy. The first reaction in all three chains is the fusion of two hydrogen nuclei:

$$ {}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow \; {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu_\mathrm{e} + 0.15\,\mathrm{MeV} \tag{3.1} $$

The electron neutrino ν_e produced in this reaction is very weakly interacting and passes through the star without interacting, so the energy it possesses (0.25 MeV) is lost and cannot contribute to the pressure balance of the

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star. This first reaction in the PP chain is very slow, because it requires the decay of a proton to a neutron which on average takes a billion years for an individual proton. When the proton decays, a positron is necessarily created in order to satisfy charge conservation. The positron e^+ very quickly annihilates with an electron e^- and produces two photons γ which can be readily absorbed giving another 1.02 MeV. So the energy output from this reaction is

$$Q_{pp} = 0.15\,\text{MeV} + 1.02\,\text{MeV} \tag{3.2}$$

The energy output from each of the following reactions can be calculated in a similar way, taking into account that neutrino energy is lost and remembering that if a positron is created, then it is necessary to include the annihilation energy too.

The second step transforms deuterium into helium:

$$^2_1\text{D} + {}^1_1\text{H} \rightarrow {}^3_2\text{He} + 5.49\,\text{MeV} \tag{3.3}$$

These two steps are common to all the branches, but then there are three different paths to produce $^4_2$He.

Exercise 3.3:
Calculate how much mass disappears in the first two steps of the PP chains, by comparing mass input and mass output. How much energy does that correspond to? How large a percentage of the energy is lost due to the emission of the neutrino? How large is the difference in percentage between the numbers given in Eqs. 3.1 and 3.3 and the one you just calculated? Why?

PP I branch

The PP I branch is the most straight forward of the three PP branches. It directly converts $^3_2$He into $^4_2$He:

$$^3_2\text{He} + {}^3_2\text{He} \rightarrow {}^4_2\text{He} + 2\;{}^1_1\text{H} + 12.86\,\text{MeV} \tag{3.4}$$

This way three hydrogen nuclei are used for each $^3_2$He which then fuses into $^4_2$He and releases two $^1_1$H, all in all using four $^1_1$H to produce one $^4_2$He.

Exercise 3.4:
Show how we can calculate the energy of 12.86 MeV that is released by the reaction $^3_2\text{He} + {}^3_2\text{He} \rightarrow {}^4_2\text{He} + 2\;{}^1_1\text{H}$

PP II branch

The second and third proton-proton branches produce heavier elements in the intermediate steps, but they are all consumed during later steps. The PP II branch involves the following reactions:

$$
\begin{aligned}
{}^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} &\rightarrow {}^{7}_{4}\mathrm{Be} + 1.59\,\mathrm{MeV} && (3.5)\\
{}^{7}_{4}\mathrm{Be} + \mathrm{e}^{-} &\rightarrow {}^{7}_{3}\mathrm{Li} + \nu_{\mathrm{e}} + \delta E && (3.6)\\
{}^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} &\rightarrow 2\,{}^{4}_{2}\mathrm{He} + 17.35\,\mathrm{MeV} && (3.7)
\end{aligned}
$$

The step in the PP II branch involving an electron is an inverse β-decay, which produces free energy δE that depends on the initial energy quantum number of the produced ${}^{7}_{3}\mathrm{Li}$ nucleus. In the relevant temperature range where the PP II branch is effective, there are only two energy states the ${}^{7}_{3}\mathrm{Li}$ can be in: the ground state and an excited state. The produced neutrinos consequently carry either 0.86 MeV or 0.38 MeV, depending on if energy is used to excite the nucleus of ${}^{7}_{3}\mathrm{Li}$. About 90% of the time, ${}^{7}_{3}\mathrm{Li}$ is produced in the ground state, so that the neutrino energy is 0.86 MeV, while the other 10% of the time, ${}^{7}_{3}\mathrm{Li}$ is produced in the excited state and the neutrino energy is 0.38 MeV, giving a weighted average of 0.81 MeV. That energy is lost, because the neutrino does not interact with matter on its way through the Sun, so on average that reaction leaves only $\delta E = 0.05$ MeV to the energy budget of the Sun.

Exercise 3.5:
Why is an electron neutrino produced in the production of ${}^{7}_{3}\mathrm{Li}$?

Exercise 3.6:
The absorption coefficient per unit mass for a neutrino is $\kappa_\nu = 10^{-21}\,\mathrm{m}^2\,\mathrm{kg}^{-1}$. This absorption coefficient can be plugged into the equation for mean free path, but now using $l_{\mathrm{mfp}} = 1/(\kappa_\nu \rho)$. Find the mean free path at the centre of the Sun for the neutrinos.

PP III branch

The PP III branch is not very effective in the Sun. The PP III branch does not produce much energy unless the temperature is above 20 MK or the abundance of the heavy elements needed is much higher than in the Sun. For the Sun that has a core temperature of less than 15 MK, the PP III branch produces only a very small fraction of the total energy. The PP III branch involves the following reactions:

$$
\begin{aligned}
{}^3_2\mathrm{He} + {}^4_2\mathrm{He} &\rightarrow {}^7_4\mathrm{Be} + 1.59\,\mathrm{MeV} &(3.8)\\
{}^7_4\mathrm{Be} + {}^1_1\mathrm{H} &\rightarrow {}^8_5\mathrm{B} + 0.14\,\mathrm{MeV} &(3.9)\\
{}^8_5\mathrm{B} &\rightarrow {}^8_4\mathrm{Be} + \mathrm{e}^+ + \nu_\mathrm{e} + 6.88\,\mathrm{MeV} &(3.10)\\
{}^8_4\mathrm{Be} &\rightarrow 2\,{}^4_2\mathrm{He} + 3.00\,\mathrm{MeV} &(3.11)
\end{aligned}
$$

Note that the first step, (3.8), is identical to the first step in the PP II branch, (3.5). The positron created in (3.10) annihilates with an electron and provides the same energy as the positron created in the PP I chain. Step three requires ${}^8_5\mathrm{B}$ to go through a radioactive β-decay to ${}^8_4\mathrm{Be}$, and it produces very high energy neutrinos (7.2 MeV) that can be detected on Earth. These high energy neutrinos have been the source of the solar neutrino problem, where far fewer neutrinos were observed than the number predicted by solar models. It is now recognized that the neutrinos from the PP III chain must go through a conversion on their way from the Sun the to Earth. So the solar neutrino problem first believed to be a problem for astronomers became a triumph for particle physicists instead and resulted in the Noble Prize in Physics in 2015 for Kajita and McDonald.

Exercise 3.7:
For each of the PP branches, calculate how much energy is released in the whole branch (use the mass difference between what goes into the branch and what comes out) and calculate how much of that energy goes into neutrinos for each branch given as percentage of the energy produced in the branch.

Exercise 3.8:
The solar luminosity is $3.85 \cdot 10^{26}$ W, calculate how much mass the Sun looses every second if the Sun produced all its energy with the PPI chain. Assuming the solar luminosity is constant, calculate how long it takes the Sun to burn one earth mass. Assuming that the solar luminosity is constant (a very unrealistic assumption here), how long does it take for the Sun to convert all if its mass into energy.

Exercise 3.9:
Calculate the lifespan on the main sequence for a B - type star with $M_\star = 16M_\odot$ and $L_\star = 8000L_\odot$ and an M type star with $M_\star = 0.1M_\odot$

and $L_\star = 8 \cdot 10^{-4}$

Exercise 3.10:
Calculate the average energy released when 4 protons are fused into one ${}^4_2$He nucleus, taking into account all the PP branches and their using the fraction of the energy they produce in the Sun and how much they loose to neutrinoes.

3.3 The CNO cycle

The CNO cycle uses carbon, nitrogen and oxygen to produce helium. The process is only effective at temperatures above 10^7 K, so the CNO cycle contributes only a very small part ($\sim 1.7\%$) of the fusion energy production in the Sun. There are a number of sub branches of the CNO cycle, but only the dominant one is shown here:

$$\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{13}_{7}\mathrm{N} + 1.95\,\mathrm{MeV} && (3.12)\\
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + \mathrm{e}^+ + \nu_e + 0.49\,\mathrm{MeV} && (3.13)\\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + 7.55\,\mathrm{MeV} && (3.14)\\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + 7.30\,\mathrm{MeV} && (3.15)\\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + \mathrm{e}^+ + \nu_\mathrm{e} + 1.75\,\mathrm{MeV} && (3.16)\\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He} + 4.97\,\mathrm{MeV} && (3.17)
\end{aligned}$$

The neutrinos produced in steps 2 and 5 are both low energy neutrinos transporting away 0.71 and 1.00 MeV respectively. The other branches of the CNO cycle are only effective at much higher temperatures, so for instance the next most energy producing CNO cycle branch produces only 0.05% of the total energy produced by the CNO-cycle in the Sun.

Exercise 3.11:
Calculate the mass deficiency of the CNO cycle and the energy released by it. How much (in percentage) is the energy lost due to neutrinos?

Exercise 3.12:
Calculate the energy needed as input for the reaction ${}^4_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^8_4\mathrm{Be} + \gamma$

Exercise 3.13:
Why does a positron provide 1.02 MeV to the energy? Do the calculation that shows that it is in fact 1.02 MeV. Where does the electron come from?

Exercise 3.14:
How many PP reactions take place in the Sun every second (remember how much energy is produced by each branch and assume that all branches produce the same energy as PPI)? The luminosity of the Sun is $3.85 \cdot 10^{26}$ W. How many neutrinos are produced in these PP reactions in one second?

3.4 Energy production rate

Fusion processes have at least one strong barrier to overcome in order to be successful. The Coulomb repulsion between two positively charged atomic nuclei makes it very hard for them to interact. Most "collisions" are not actually collisions, but just the Coulomb repulsion at work. In order for the thermal motions to be strong enough to overcome the Coulomb repulsion, the kinetic energy of the incoming atomic nucleus must be larger than the electrostatic potential produced by the other atomic nucleus. For hydrogen the electrostatic potential energy is just:

$$U_{\mathrm{E}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} \tag{3.18}$$

where ϵ_0 is the permittivity of free space, q_1 and q_2 the charges of the two particles involved, and r is the distance between them. The potential energy is also the amount of work needed to overcome the potential, so by assuming that the particle comes from infinity and has to get within a distance of a proton radius, a rough approximation of how much kinetic energy a proton must posses in order for fusion to occur can be calculated by:

$$v^2 = \frac{1}{2\pi m_{\mathrm{p}}\epsilon_0}\frac{q_1 q_2}{r_{\mathrm{p}}} \tag{3.19}$$

where v is the initial velocity of the incoming proton, and $r_{\mathrm{p}} \simeq 10^{-15}\,\mathrm{m}$ is the size of a proton. The root-mean-square thermal velocity of a proton is given by:

$$\frac{1}{2}m_{\mathrm{p}}v_{\mathrm{rms}}^2 = \frac{3}{2}k_{\mathrm{B}}T \tag{3.20}$$

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where k_B is the Boltzmann constant. Since it is assumed that one of the protons is at rest, the velocity that goes into Eq. 3.19 is the relative velocity of the two protons. Assuming that the protons collide head-on, we can then assume that they only need half the velocity each for the collision to happen. The temperature for protons with rms velocity in a head-on collision can interact and fuse then becomes:

$$T = \frac{1}{24\pi\epsilon_0 k_B}\frac{q_1 q_2}{r_p} \tag{3.21}$$

which for two protons is $3 \cdot 10^{10}$ K. The velocities of protons are not identical, so it could be that only the fastest moving protons would fuse, and that would be ok, but the temperature in the centre of the Sun is only $15 \cdot 10^6$ K, which would mean that the protons able to overcome the Coulomb barrier would have to move at roughly 100 times the rms velocity. Only a very small fraction of the ensemble of protons have that velocity in a Maxwellian velocity distribution.

Exercise 3.15:
Assume that the protons are in thermal equilibrium so that they follow a Maxwellian velocity distribution

$$f_\nu(v) = \sqrt{\frac{2}{\pi}\left(\frac{m_p}{k_B T}\right)^3} v^2 \exp[\frac{-m_p v^2}{2k_B T}]$$

where $f_\nu(v)$ is the probability that a proton has the velocity v. The probability is normalised so that

$$\int_{-\infty}^{\infty} f_\nu(v)\partial v = 1$$

meaning that the probability is certain that a proton has *some* velocity v. Calculate how large a fraction of the protons have a velocity 10 times larger than the root-mean-square velocity

$$v_{rms} = \sqrt{\frac{3k_B T}{m_p}}$$

Hint: You should do a complicated integral here, but it turns out not to be necessary. Try to calculate the probability of finding a proton that has a velocity in the interval dv just around v_{rms} and compare it to what *the probability is of finding a proton in the velocity interval dv around $10v_{rms}$. Since v_{rms} is a typical velocity for the protons, what does this result mean?*

Exercise 3.16:
Determine the fraction of hydrogen atoms in a gas of T 10,000 K having speeds between $v_1 = 2 \cdot 10^4\,\mathrm{m\,s^{-1}}$ and $v_2 = 2.5 \cdot 10^4\,\mathrm{m\,s^{-1}}$.
Hint: We already know that the integral over all velocities of the Exerc. 3.15 is one, so you now only need to do the integral over the Maxwell distribution between v_1 and v_2, and since they are almost the same, you can make the same assumption as in Exerc. 3.15.

From these considerations it is obvious that some other process than just speed is needed to overcome the Coulomb barrier. It is the quantum tunneling effect that allows the incoming particle to the pass the Coulomb barrier. The tunneling effect relies among other things on the principle of non-locality, which means that there is a non-zero possibility of the particle actually being within the barrier at any time. In general, Heisenbergs uncertainty principle shows that $\Delta x \Delta p \gtrsim h$[1], so the location of the particles are never perfectly known, making it possible for a particle to 'appear' inside the Coulomb barrier. The smaller distance the particle has to tunnel, the larger the probability. When taking the tunneling into account, the reaction rate (the reactions per second) depends on the relative velocity between the reactants, and the cross section of the reactants. Cross sections should not be taken as a physical area, but more be thought of as a probability. The reaction rates pr unit mass can be expressed as:

$$r_{ik} = \frac{1}{\rho(1+\delta_{ik})} \int\int v\sigma(v)\,\mathrm{d}n_i\,\mathrm{d}n_k \tag{3.22}$$

where δ_{ik} is the Kronecker delta function, ρ is the mass density, $v = |\mathbf{v}_i - \mathbf{v}_k|$ is the relative velocity of the two particles, $\mathrm{d}n_i$ and $\mathrm{d}n_k$ are the distributions of the particles in velocity space and $\sigma(v)$ is the cross section of the reaction for relative velocity v.

The δ_{ik}–function is needed because the two integrals are combining particles of type i with velocity v_i and particle of type k with velocity v_k. Now if $i \neq k$ then the integrals make sure that the rate of particle type i and

[1]Actually the uncertainty principle is $\sigma_x \sigma_p \gtrsim \frac{1}{2}\hbar$ where σ_x and σ_p are the standard deviation of the distribution function of the position and momentum respectively and $\hbar$ is the reduced Planck's constant.

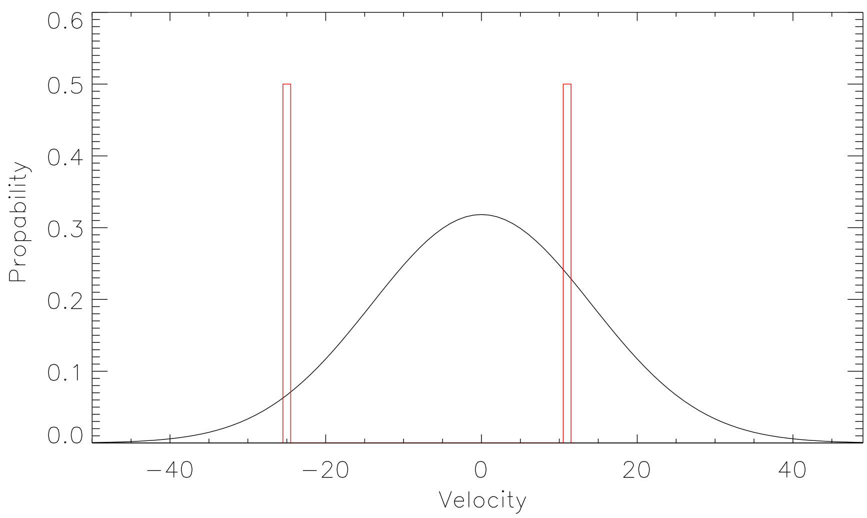

Figure 3.2: The normalized distribution of a large group of protons in velocity space (in black) and the velocity distribution of two identical protons (in red).

particle of type k fuse is calculated and summed. If the two particles types are the same, then we will be summing over the same distribution twice, and therefore the factor $(1 + \delta_{ik})$ appears in the denominator. To understand this better, think of the distribution of protons at a specific location in one dimensional space. Figure 3.2 shows this case for a distribution with many protons (in black) and for a simple case with just two protons (in red). If we call the particle with velocity -25 for particle one and the particle with velocity $+11$ for particle two, then mathematically we take one value for i and integrate over k and sum that over all values of i. In the case of the two protons, the factor n_i and n_k in the integrals make the reaction rate contribution zero unless i is -25 or 11. When we do the integral we therefore get two contributions:

$$\begin{array}{rcrcrcr} i &=& -25 & \text{and} & k &=& 11 \\ i &=& 11 & \text{and} & k &=& -25 \end{array}$$

This leads to summing the reaction from two different collisions, but since there are only two different particles, only one collision can take place – a collision between particle one and two, so we have to half the total reaction rate when the particles in the reaction are the same.

In stellar interiors, the distributions of the atoms are almost exactly Maxwellian (see Exerc. 3.1), so $\mathrm{d}n_i$ and $\mathrm{d}n_k$ are known. It is in general easier to calculate the reaction rates as a function of energy instead of relative velocity because for relative velocity one would have to integrate over both direction and speed for both particles.

Often the reaction rates (unit is $\mathrm{kg}^{-1}\ \mathrm{s}^{-1}$) are not given, but instead the function that allows the rate to be calculated. The proportionality function depends on energy or temperature, but is independent of the density of the particles involved in the fusion process. The proportionality function is often denoted by λ in the literature. It is connected to the rate per unit mass through:

$$r_{ik} = \frac{n_i n_k}{\rho(1+\delta_{ik})}\lambda_{ik} \tag{3.23}$$

The reaction rates (no longer per unit mass!) can now be formulated as:

$$\lambda_{ik} = \sqrt{\frac{8}{m\pi\,(kT)^3}} \int_0^\infty \exp\left(-\frac{E}{kT}\right) E\ \sigma(E)\,\mathrm{d}E \tag{3.24}$$

where $m = m_i m_k/(m_i + m_k)$ is the reduced mass and $E = {}^1\!/\!{}_2\, mv^2$ is the kinetic energy in the center of mass system. In the Sun, most reactions occur due to the tunneling effect, and for those reactions, the cross section is of the form:

$$\sigma(E) = E^{-1} S(E) \exp\left(-\sqrt{\frac{m}{2E}}\frac{Z_i Z_k e^2 \pi}{\epsilon_0 h}\right) \tag{3.25}$$

where $S(E)$ is a slowly varying function of E (which can be assumed to be of order one), Z_i and Z_k are the atomic numbers of particles i and k (number of protons in the nucleus), e is the elementary charge and h is Planck's constant.

Exercise 3.17:
What is the unit of λ in Eq. 3.24 (use your physical intuition)? What is the physical meaning?

Exercise 3.18:
For the proton-proton reaction, plot the two exponentials in Eq. 3.24 and their product in the same graph. Estimate the maximum and at what energy the maximum occurs.

The remaining integral is easy to evaluate if the reaction has a resonance - meaning that the reaction happens most efficiently just around a special energy, because then σ can be treated as a δ function and simply set $E = E_{\mathrm{R}}$, and finally

$$\lambda_{ik} \sim T^{-{}^{3}\!/_{2}} \exp\left(-E_R/kT\right) \tag{3.26}$$

Resonant energies are generally of little importance for the Sun, even though some of the reactions for the CNO chain do have strong resonant energy contributions to their reaction rates. The cross section for reactions that do not show resonant energies can be found through experiments and quantum mechanical calculations to be equal to Eq. 3.25. We can now see that there are two exponential functions in the integral of Eq. 3.24. The first (already apparent in Eq. 3.24) comes from the Maxwellian distribution the energy of the particles, while the second (shown in Eq. 3.25) comes from the electrostatic barrier between the two particles with charges Z_i and Z_k. The combination of the two exponentials creates a sharply peaked function - the Gamow peak[2].

After approximating the Gamow peak by a Gaussian and assuming $S(E)$ to be a constant the reaction rates can be found to be

$$\lambda_{ik} = 4S(E_{\max})\frac{\left({}^{2}\!/_{3}\, m\right)^{{}^{1}\!/_{2}} \left(b/2\right)^{{}^{1}\!/_{3}}}{\left(k_B T\right)^{{}^{2}\!/_{3}}} \exp\left(-\frac{3\left(b/2\right)^{{}^{2}\!/_{3}}}{\left(k_B T\right)^{{}^{1}\!/_{3}}}\right) \tag{3.27}$$

where $S(E_{\max})$ is the function S evaluated at the energy at which the assumed Gamow peak has its maximum and

$$b = \sqrt{\frac{m}{2}}\,\frac{e^2\pi}{\varepsilon_0 h}\, Z_i Z_k \tag{3.28}$$

Exercise 3.19:
Find the maximum of the function in Eq. 3.27 and an expression for the value of the temperature where the maximum is achieved.

The reaction rate's dependence on temperature is very general for non-resonant fusion reactions: A factor $T^{-{}^{2}\!/_{3}}$ multiplied with a exponential dependence on $T^{-{}^{1}\!/_{3}}$.

[2]The Gamow peak is named after the Russian physicist Georgiy Antonovich Gamow who considered the tunneling effect the first time in 1928, and was an early supporter of the Big Bang theory.

Table 3.1 gives a list of the equations for a number of reactions which are important in the Sun. The λ's are given in units of $\mathrm{cm}^3\,\mathrm{s}^{-1}$, so in order to get the reactions per second per cubic centimeter, they much be multiplied by the two relevant number densities n of particles as shown in Eq. 3.23.

Exercise 3.20:
What factor must you multiply the λ's in Table 3.1 with in order to convert them to SI units?

Exercise 3.21:
Calculate the maximum rate (λ) and at what temperature it is located for both the proton-proton and the ${}^{12}_{6}$C-proton fusion processes. Also calculate the full width of the Gaussian at $1/e$ of its maximum value for both of these processes.

Exercise 3.22:
Calculate the temperature of the different processes in the PP I branch for which the process is most effective. Which process needs the highest temperature?

Exercise 3.23:
Why do you think the expressions for the λs in Table 3.1 look so different from the expression in Eq. 3.27?

3.5 Evolution of element abundances

The rate of energy production from each of the production chains depends on which elements are available (for instance $^{12}_{6}$C for the CNO cycle and $^{4}_{2}$He for the PP II and PP III chains), and the rate of each of the different sub-steps in the cycle. The β-decay of the proton into a neutron in the first step of the PP chains (Eq. 3.1) is by far the slowest reaction in the PP chains, and controls the energy production of the Sun.

In order to be able to calculate the energy production, we need to consider change in number density of the different elements: the evolution of element abundances. But we first start by looking at the electron density, which is needed for the electron capture by $^{7}_{4}$Be in the second step of the PP II chain (Eq. 3.6). In the inner parts of the Sun, we can safely assume that both hydrogen and helium are ionized. We will forget about the heavier elements, because they do not supply that many electrons. The number densities can then be found by:

$$n_{\mathrm{p}} = \frac{\rho X}{1 m_u} \tag{3.29}$$

$$n_{\mathrm{He}} = \frac{\rho Y}{4 m_u} \tag{3.30}$$

$$\tag{3.31}$$

where X and Y are the fractional abundances by weight of hydrogen and helium respectively (see Sect. 4.3). The number of electrons coming from hydrogen and helium is one and two, so the total number of electrons $n_{\mathrm{e,tot}}$ is the sum of the electrons from hydrogen $n_{\mathrm{e,H}}$ and from helium $n_{\mathrm{e,He}}$:

$$\begin{aligned}
n_{\mathrm{e,tot}} &= n_{\mathrm{e,H}} + n_{\mathrm{e,He}} = n_{\mathrm{H}} + 2n_{\mathrm{He}} && (3.32)\\
&= \frac{\rho X}{m_u} + 2\frac{\rho Y}{4m_u} && (3.33)\\
&= \frac{\rho}{m_u}\left[X + \frac{(1-X)}{2}\right] && (3.34)\\
&= \frac{\rho}{2m_u}\left[2X + (1-X)\right] && (3.35)\\
&= \frac{\rho}{2m_u}(1+X) && (3.36)
\end{aligned}$$

When λ is the reaction rates (function of density of particles in the reaction, see Tables 3.2 and 3.3 to identify which rate symbol λ belongs to which reaction), the reactions in the center of the Sun can be written for all the elements that are consumed or produced in the three PP chains and the CNO cycle. To set up the evolution equation for ${}^1_1$H, we need to identify all the reactions in the PP chains and CNO cycle that involve ${}^1_1$H, so going through (Eq. 3.1) to (3.11), and take a factor 1/2 into account for fusion between identical particles, and a factor ± 2 for reactions that consume or produce two identical particles. The evolution equation for ${}^1_1$H then looks like:

$$
\begin{aligned}
\frac{\partial n_\mathrm{p}}{\partial t} \quad = \quad & -2\,(\frac{1}{2}\lambda_\mathrm{pp} n_\mathrm{p}^2) - \lambda_\mathrm{pd} n_\mathrm{p} n_{{}^2_1\mathrm{D}} + 2\,(\frac{1}{2}\lambda_{33} n^2_{{}^3_2\mathrm{He}}) \\
& -\lambda'_{17} n_\mathrm{p} n_{{}^7_3\mathrm{Li}} \\
& -\lambda_{17} n_\mathrm{p} n_{{}^7_4\mathrm{Be}} \\
& -\lambda_{\mathrm{p}12} n_\mathrm{p} n_{{}^{12}_{6}\mathrm{C}} - \lambda_{\mathrm{p}13} n_\mathrm{p} n_{{}^{13}_{6}\mathrm{C}} \\
& -\lambda_{\mathrm{p}14} n_\mathrm{p} n_{{}^{14}_{7}\mathrm{N}} - \lambda_{\mathrm{p}15} n_\mathrm{p} n_{{}^{15}_{7}\mathrm{N}}
\end{aligned}
\qquad (3.37)
$$

where the top line comes from the PP I branch (3.1), (3.3), and (3.4), the second line comes from the PP II branch (3.7), the third line comes from the PP III branch (3.9), while the fourth and fifth line come from the CNO cycle (3.12), (3.14), (3.15), and (3.17). The first term comes from (3.1) in which we include a factor 1/2 since we have fusion between two identical particles ${}^1_1$H, and a factor -2 since we consume two particles ${}^1_1$H. So effectively these two factors cancel. The same reasoning applies to the third term, which comes from (3.4), except that we have a factor $+2$ since the reaction produces two ${}^1_1$H.

Exercise 3.24:
Write up the evolution equations for ${}^2_1$D, ${}^3_2$He and ${}^4_2$He

The evolution equations for the remaining species are (see Tables 3.2 and

3.3 to identify the relevant reactions and their associated rate symbols λ):

$$\frac{\partial n_{^7_4\mathrm{Be}}}{\partial t} = \lambda_{34} n_{^3_2\mathrm{He}} n_{^4_2\mathrm{He}} - \lambda_{e7} n_{\mathrm{e}} n_{^7_4\mathrm{Be}} - \lambda_{17} n_{\mathrm{p}} n_{^7_4\mathrm{Be}} \tag{3.38}$$

$$\frac{\partial n_{^7_3\mathrm{Li}}}{\partial t} = \lambda_{\mathrm{e}7} n_{\mathrm{e}} n_{^7_4\mathrm{Be}} - \lambda_{17}^{'} n_{\mathrm{p}} n_{^7_3\mathrm{Li}} \tag{3.39}$$

$$\frac{\partial n_{^8_5\mathrm{B}}}{\partial t} = \lambda_{17} n_{\mathrm{p}} n_{^7_4\mathrm{Be}} - \lambda_8 n_{^8_5\mathrm{B}} \tag{3.40}$$

$$\frac{\partial n_{^8_4\mathrm{Be}}}{\partial t} = \lambda_8 n_{^8_5\mathrm{B}} - \lambda_8^{'} n_{^8_4\mathrm{Be}} \tag{3.41}$$

$$\frac{\partial n_{^{12}_6\mathrm{C}}}{\partial t} = \lambda_{p15} n_{\mathrm{p}} n_{^{15}_7\mathrm{N}} - \lambda_{p12} n_{\mathrm{p}} n_{^{12}_6\mathrm{C}} \tag{3.42}$$

$$\frac{\partial n_{^{13}_7\mathrm{N}}}{\partial t} = \lambda_{p12} n_{\mathrm{p}} n_{^{12}_6\mathrm{C}} - \lambda_{13} n_{^{13}_7\mathrm{N}} \tag{3.43}$$

$$\frac{\partial n_{^{13}_6\mathrm{C}}}{\partial t} = \lambda_{13} n_{^{13}_7\mathrm{N}} - \lambda_{p13} n_{\mathrm{p}} n_{^{13}_6\mathrm{C}} \tag{3.44}$$

$$\frac{\partial n_{^{14}_7\mathrm{N}}}{\partial t} = \lambda_{p13} n_{\mathrm{p}} n_{^{13}_6\mathrm{C}} - \lambda_{p14} n_{\mathrm{p}} n_{^{14}_7\mathrm{N}} \tag{3.45}$$

$$\frac{\partial n_{^{15}_8\mathrm{O}}}{\partial t} = \lambda_{p14} n_{\mathrm{p}} n_{^{14}_7\mathrm{N}} - \lambda_{15} n_{^{15}_8\mathrm{O}} \tag{3.46}$$

$$\frac{\partial n_{^{15}_7\mathrm{N}}}{\partial t} = \lambda_{15} n_{^{15}_8\mathrm{O}} - \lambda_{p15} n_{\mathrm{p}} n_{^{15}_7\mathrm{N}} \tag{3.47}$$

The large number of evolution equations for the different species can be significantly simplified, since some of the reactions happen very fast, while others take a long time. Examples of very fast reactions are the β-decay of $^8_5$B (λ_8), $^{13}_7$N (λ_{13}) and $^{15}_8$O (λ_{15}) which take only a matter of seconds. Another example is highly unstable $^8_4$Be which decays into two α particles (λ_8', Eq. 3.11) with a half life of $8.2 \cdot 10^{-17}$ s. On the other hand, in the center of the Sun, the lifetime of $^{14}_7$N against reactions with protons ($\lambda_{\mathrm{p}14}$, Eq. 3.15) is about 10^9 years, and the lifetime of protons against protons (λ_{pp}, Eq. 3.1) is of order 10^{10} years. We can therefore safely assume that the very fast reactions happen instantaneously.

We can make these lifetime considerations in order to simplify for example the evolution equation of $^1_1$H (Eq. 3.37). Deuterium comes into equilibriumn very fast (in Exerc. 3.24, you derived the evolution equation of $^2_1$D. The terms on the right hand side of this equation are very large and basically cancel), so that we can write:

$$\frac{\partial n_{^2_1\mathrm{D}}}{\partial t} = 0 \tag{3.48}$$

${}^7_4$Be transmutes into ${}^7_3$Li through electron capture (Eq. 3.6), a process that has a high probability, particularly compared to the production of ${}^7_4$Be and the destruction of ${}^7_3$Li, which is a stable isotope. This makes the change in abundance of these species tightly coupled so that we can write:

$$\frac{\partial n_{{}^7_3\mathrm{Li}}}{\partial t} + \frac{\partial n_{{}^7_4\mathrm{Be}}}{\partial t} = 0 \tag{3.49}$$

The isotopes of carbon, nitrogen and oxygen in the CNO cycle transmute into eachother and function as catalysts in the production of ${}^4_2$He. Their densities effectively do not change so that we can equate the sums in (Eq. 3.38) to (3.47) to zero. We can therefore write:

$$\frac{\partial n_{{}^{12}_6\mathrm{C}}}{\partial t} = \frac{\partial n_{{}^{13}_7\mathrm{N}}}{\partial t} = \frac{\partial n_{{}^{13}_6\mathrm{C}}}{\partial t} = \frac{\partial n_{{}^{14}_7\mathrm{N}}}{\partial t} = \frac{\partial n_{{}^{15}_8\mathrm{O}}}{\partial t} = \frac{\partial n_{{}^{15}_7\mathrm{N}}}{\partial t} = 0 \tag{3.50}$$

These considerations allow us to rewrite the evolution equation for ${}^1_1$H:

$$\frac{\partial n_\mathrm{p}}{\partial t} = -\frac{3}{2}\lambda_\mathrm{pp} n_\mathrm{p}^2 + \lambda_{33} n^2_{{}^3_2\mathrm{He}} - \lambda_{34} n_{{}^3_2\mathrm{He}} n_{{}^4_2\mathrm{He}} - 4\lambda_{p14} n_\mathrm{p} n_{{}^{14}_7\mathrm{N}} \tag{3.51}$$

which is a simplification since we need to consider fewer isotope densities.

In Exerc. 3.24, you derived the evolution equations of ${}^3_2$He and ${}^4_2$He. The evolution equation of ${}^3_2$He can be rewritten such that we drop the number density of ${}^2_1$D, so that we have:

$$\frac{\partial n_{{}^3_2\mathrm{He}}}{\partial t} = \frac{1}{2}\lambda_\mathrm{pp} n_\mathrm{p}^2 - \lambda_{33} n^2_{{}^3_2\mathrm{He}} - \lambda_{34} n_{{}^3_2\mathrm{He}} n_{{}^4_2\mathrm{He}} \tag{3.52}$$

For the evolution equation of ${}^4_2$He, we can consider the short lifetimes of ${}^8_5$B and ${}^8_4$Be mentioned earlier, and the short lifetime of ${}^7_4$Be against reactions with protons (Eq. 3.9), so that we have

$$\frac{\partial n_{{}^8_5\mathrm{B}}}{\partial t} = 0 \tag{3.53}$$

$$\frac{\partial n_{{}^8_4\mathrm{Be}}}{\partial t} = 0 \tag{3.54}$$

This allows us to rewrite the evolution equation of ${}^4_2$He to:

$$\frac{\partial n_{{}^4_2\mathrm{He}}}{\partial t} = \frac{1}{2}\lambda_{33} n^2_{{}^3_2\mathrm{He}} + \lambda_{34} n_{{}^3_2\mathrm{He}} n_{{}^4_2\mathrm{He}} + \lambda_{p14} n_\mathrm{p} n_{{}^{14}_7\mathrm{N}} \tag{3.55}$$

By assuming that the mass of the Sun does not change and therefore neglecting the small differences in total mass between the reactants and the reaction products, we may consider the evolution equations at constant mass density ρ. One can then write $n_\mathrm{p} = \rho X/m_\mathrm{p}$, $n_3 = \rho Y_3/m_3$, and $n_4 = \rho Y_4/m_4$ (with X and Y the mass fractions of hydrogen and helium, and the indices 3 and 4 refer to $^3_2$He and $^4_2$He). We can then write for the evolution of the mass fractions:

$$\frac{\partial X}{\partial t} = m_\mathrm{u} A_\mathrm{p} \left(-3r_\mathrm{pp} + 2r_{33} - r_{34} - 4r_{p14}\right) \tag{3.56}$$

$$\frac{\partial Y_3}{\partial t} = m_\mathrm{u} A_3 \left(r_\mathrm{pp} - 2r_{33} - r_{34}\right) \tag{3.57}$$

$$\frac{\partial Y_4}{\partial t} = m_\mathrm{u} A_4 \left(r_{33} + r_{34} + r_{p14}\right) \tag{3.58}$$

where the r's are the reaction rates per unit mass (Eq. 3.23), and A is the mass number ($m_\mathrm{p} = m_\mathrm{u} A_\mathrm{p}$).

Finally we write the full energy generation per unit mass as:

$$\varepsilon = \sum Q'_{ik} r_{ik} \tag{3.59}$$

where the sum is over all reactions. This is the expression for energy generation one needs to consider when making a stellar model. Figure 3.3 shows the result from a 1-dimensional model that illustrates that nuclear eneregy generation is concentrated in the core of the star.

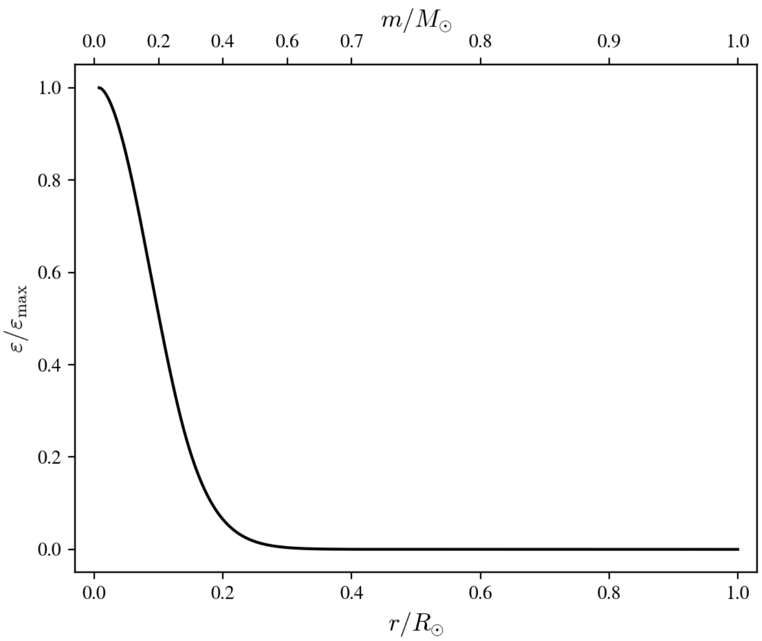

Figure 3.3: Nuclear energy generation in a model of the Sun including reactions from both the PP chains and the CNO cycle. The graph is normalized to $\varepsilon_{\rm max} = 0.9$ mW/kg. Most of the energy is generated within 30% of the solar radius $R_\odot$, or 3% of the volume. From the top axis which shows the equivalent mass fraction, we can see that this corresponds to about 60% of the solar mass. So energy generation is concentrated in a small core that contains most of the mass.

Reaction	Rate
${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{D} + e^+$	$N_A\lambda_{\rm pp} = 4.01 \cdot 10^{-15}\, T_9^{-2/3} \exp\left(-3.380 T_9^{-1/3}\right) \left(1 + 0.123\, T_9^{1/3} + 1.09\, T_9^{2/3} + 0.938\, T_9\right)$
${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \rightarrow {}^4_2\mathrm{He} + 2\,{}^1_1\mathrm{H}$	$N_A\lambda_{33} = 6.04 \cdot 10^{10}\, T_9^{-2/3} \exp\left(-12.276\, T_9^{-1/3}\right) \left(1 + 0.034\, T_9^{1/3} - 0.522\, T_9^{2/3} - 0.124\, T_9 + 0.353\, T_9^{4/3} + 0.213\, T_9^{5/3}\right)$
${}^3_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^7_4\mathrm{Be} + \gamma$	$N_A\lambda_{34} = 5.61 \cdot 10^{6}\, T_{9*}^{5/6}\, T_9^{-3/2} \exp\left(-12.826\, T_{9*}^{-1/3}\right)$ where $T_{9*} = T_9/(1 + 4.95\; 10^{-2}\, T_9)$
${}^7_4\mathrm{Be} + e^- \rightarrow {}^7_3\mathrm{Li} + \gamma$	$N_A\lambda_{\rm e7} = 1.34 \cdot 10^{-10}\, T_9^{-1/2} \left[1 - 0.537\, T_9^{1/3} + 3.86\, T_9^{2/3} + 0.0027\, T_9^{-1} \exp\left(2.515 \cdot 10^{-3}\, T_9^{-1}\right)\right]$
${}^7_3\mathrm{Li} + {}^1_1\mathrm{H} \rightarrow 2\,{}^4_2\mathrm{He}$	$N_A\lambda'_{17} = 1.096 \cdot 10^{9}\, T_9^{-2/3} \exp\left(-8.472\, T_9^{-1/3}\right) - 4.830 \cdot 10^{8}\, T_{9*}^{5/6}\, T_9^{-3/2} \exp\left(-8.472 T_{9*}^{-1/3}\right) + 1.06 \cdot 10^{10}\, T_9^{-3/2} \exp\left(-30.442\, T_9^{-1}\right)$ where $T_{9*} = T_9/(1 + 0.759\, T_9)$
${}^7_4\mathrm{Be} + {}^1_1\mathrm{H} \rightarrow {}^8_5\mathrm{B} + \gamma$	$N_A\lambda_{17} = 3.11 \cdot 10^{5}\, T_9^{-2/3} \exp\left(-10.262\, T_9^{-1/3}\right) + 2.53 \cdot 10^{3}\, T_9^{-3/2} \exp\left(-7.306\, T_9^{-1}\right)$
${}^{14}_7\mathrm{N} + {}^1_1\mathrm{H} \rightarrow {}^{15}_8\mathrm{O} + \gamma$	$N_A\lambda_{\rm p14} = 4.90 \cdot 10^{7}\, T_9^{-2/3} \exp\left(-15.228\, T_9^{-1/3} - 0.092\, T_9^2\right) \left(1 + 0.027\, T_9^{1/3} - 0.778\, T_9^{2/3} - 0.149\, T_9 + 0.261\, T_9^{4/3} + 0.127\, T_9^{5/3}\right) + 2.37 \cdot 10^{3}\, T_9^{-3/2} \exp\left(-3.011\, T_9^{-1}\right) + 2.19 \cdot 10^{4} \exp\left(-12.53\, T_9^{-1}\right)$

Table 3.1: The reaction rates in units cubic cm per second modified from Caughlan and Fowler (1988), also see Stix Table 3.1 . Temperature T_9 is in units of $10^9\,\mathrm{K}$. The electron capture by ${}^7_4\mathrm{Be}$ has an upper limit at temperatures below 10^6 K of $N_A\lambda_{e7} \leq 1.57 \cdot 10^{-7}/n_e$

Nuclear reactions of the PP chain.

Branch	Reaction	Q' [MeV]	Q_ν [MeV]	Rate symbol
all	${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \to {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu_\mathrm{e}$	1.177	0.265	λ_pp
	${}^2_1\mathrm{D} + {}^1_1\mathrm{H} \to {}^3_2\mathrm{He} + \gamma$	5.494		λ_pd
I	${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \to {}^4_2\mathrm{He} + 2\ {}^1_1\mathrm{H}$	12.860		λ_{33}
II & III	${}^3_2\mathrm{He} + {}^4_2\mathrm{He} \to {}^7_4\mathrm{Be} + \gamma$	1.586		λ_{34}
II	${}^7_4\mathrm{Be} + \mathrm{e}^- \to {}^7_3\mathrm{Li} + \nu_\mathrm{e} + \gamma$	0.049	0.815	λ_e7
	${}^7_3\mathrm{Li} + {}^1_1\mathrm{H} \to 2\ {}^4_2\mathrm{He}$	17.346		λ'_{17}
III	${}^7_4\mathrm{Be} + {}^1_1\mathrm{H} \to {}^8_5\mathrm{B} + \gamma$	0.137		λ_{17}
	${}^8_5\mathrm{B} \to {}^8_4\mathrm{Be} + \mathrm{e}^+ + \nu_\mathrm{e}$	8.367	6.711	λ_8
	${}^8_4\mathrm{Be} \to 2\ {}^4_2\mathrm{He}$	2.995		λ'_8

Table 3.2: The energy Q' is the part of the released energy that is delivered to the thermal bath in which the reactions occur, neutrinos escape from the star without interactions so their energy Q_ν is lost and does not contribute to the energy balance in the star. The subscript in the rate symbols λ reflect the reactants, where p is for proton ${}^1_1\mathrm{H}$, d for deuterium ${}^2_1\mathrm{D}$, and e for electron. The numbers reflect the atomic mass number A: 3 for ${}^3_2\mathrm{He}$, 4 for ${}^4_2\mathrm{He}$, etc. After Stix Table 2.1, who listed energy values according to Bahcall and Ulrich (1988) and Caulglan and Fowler (1988).

Nuclear reactions of the CNO cycle.

Reaction	Q' [MeV]	Q_ν [MeV]	Rate symbol
${}^{12}_6\mathrm{C} + {}^1_1\mathrm{H} \to {}^{13}_7\mathrm{N} + \gamma$	1.944		λ_p12
${}^{13}_7\mathrm{N} \to {}^{13}_6\mathrm{C} + \mathrm{e}^+ + \nu_e$	1.513	0.707	λ_{13}
${}^{13}_6\mathrm{C} + {}^1_1\mathrm{H} \to {}^{14}_7\mathrm{N} + \gamma$	7.551		λ_p13
${}^{14}_7\mathrm{N} + {}^1_1\mathrm{H} \to {}^{15}_8\mathrm{O} + \gamma$	7.297		λ_p14
${}^{15}_8\mathrm{O} \to {}^{15}_7\mathrm{N} + \mathrm{e}^+ + \nu_\mathrm{e}$	1.757	0.997	λ_{15}
${}^{15}_7\mathrm{N} + {}^1_1\mathrm{H} \to {}^{12}_6\mathrm{C} + {}^4_2\mathrm{He}$	4.966		λ_p15

Table 3.3: The isotopes of C, N, and O transmute into each other and function like catalysts in the production of ${}^1_1\mathrm{H}$ into ${}^4_2\mathrm{He}$, the total energy produced is 26.732 MeV. After Stix Table 2.2, who listed energy values according to Bahcall and Ulrich (1988) and Caulglan and Fowler (1988).

Chapter 4

Equation of state

4.1 Introduction

The gas, or plasma as it is often called, that a star consists of needs to be in equilibrium. That boils down to the sum of the forces that work on the gas is zero on average. If that was not the case, the star would either expand or contract. Some stars do actually expand and contract rhythmically, but for a large majority of stars, they are in a perfect equilibrium, and there is no large scale accelerations in their interiors. In order for us to build a model of a star that satisfies this requirement, we need to understand how the two major forces that controls the equilibrium works. These forces are the force derived from the gas pressure and the gravitational force. The gravitational force is well understood, but so far we have no way of estimating the pressure of the gas. In order for us to convert the thermal energy calculated in the previous chapter into a gas pressure, we need an equation of state. The equation of state gives the connection between density, temperature and pressure or any variables derived from these. In general, the equation of state is split into several terms. The simplest and the only relevant split is due to a number of different pressure forces that are present in stars.

4.2 Pressure forces

There are several possible forces that contribute to the pressure in stars. The most general ones are the gas pressure, radiative pressure and the pressure from degenerate electrons. For the Sun only the gas pressure and to a small degree the radiative pressure play a role.

The radiative pressure comes about when the radiative flux is so large that the otherwise small momentum carried by the photons becomes significant. Stars where the radiative pressure is important for the structure are generally stars with very large fluxes, so the heaviest stars. The radiative pressure in the Sun plays only a minor role in the central parts of the core, but is insignificant in the outer layers.

The pressure force from degenerate fermions is the main pressure force for two types of stars which are not stars in the traditional sense: White dwarfs and neutron stars. The force is a consequence of the Pauli exclusion principle which limits the number of fermions that can be forced into a given volume, because they cannot be in identical quantum states in the same location. When all the possible quantum states for the electrons in the compressed carbon and oxygen core of a white dwarf have been occupied it simply cannot compress anymore. Except of course when the electrons are no longer there, which is the case for neutron stars, where the electrons have been forced into the cores of the atoms, and fused with protons to produce neutrons. The neutrons themselves now provide the pressure force for the neutron star in the same way that the electrons do for the white dwarf star. None of these degenerate pressures are important for the Sun.

It is important to remember that pressure only provides a force, if there is a spatial gradient of the pressure. A location where the pressure is constant has no pressure force working on the gas.

4.2.1 Radiative pressure

The radiation pressure depends only on temperature and not on density:

$$P_{\mathrm{rad}} = \frac{a}{3} T^4 \tag{4.1}$$

where a is the radiation density constant, which is related to the Stefan-Boltzmann constant σ:

$$a = \frac{4\sigma}{c} \tag{4.2}$$

Exercise 4.1:
What is the radiation pressure in the centre of the Sun?

4.2.2 Gas pressure

The equation of state is rooted in atomic and molecular physics. That can easily be seen if we look at the simple equation of state for an ideal gas[1]

$$P_{\mathrm{gas}}V = Nk_{\mathrm{B}}T \tag{4.3}$$

where P is the pressure, V is a volume, N is the number of particles (atoms, molecules and electrons) in that volume, k_{B} is Boltzmann's constant and T is the temperature. As the number of particles N can be expressed as

$$N = \frac{m}{\mu m_{\mathrm{u}}} \tag{4.4}$$

where m is the mass inside the volume V, μ is the average atomic weight of the particles and m_{u} is the atomic mass unit. The density can be expressed as

$$\rho = \frac{m}{V} \tag{4.5}$$

and by combining Eq. 4.3 to 4.5

$$P_{\mathrm{gas}} = \frac{m}{\mu m_{\mathrm{u}}}\frac{k_{\mathrm{B}}T}{V} = \frac{\rho}{\mu m_{\mathrm{u}}}k_{\mathrm{B}}T \tag{4.6}$$

It is now easily seen that the pressure is intimately tied to the microphysics through the average weight of the particles, so in order to evaluate the pressure, the number and weight of each of the particles in the volume V must be known. One can think of scenarios where the electrons are not at the same temperature as the ions, or that the distribution of velocities of the ions are such that one can not assign a single temperature to the ions. In these notes we will not investigate such complications, because, as we have seen, the huge number of collisions between particles (see Exerc. 3.1) as well as the emission and reabsorption of photons in the Sun are able to smooth out any small-scale temperature differences and are able to average the energy between the different particle species very quickly (see Chap. 5).

It is important to notice that an ideal gas is a gas where the particles making up the gas are much smaller than the average distance between them, and the particles can have no inner structure that can absorb or emit energy (such as molecules). Conditions where we can use the ideal gas law are for example at low pressure when there are relatively few collisions (like

[1] An ideal gas is a gas consisting of particles that collide completely elastically, i.e. they do not have internal structure that can absorb or contain energy (as in realistic atomic nuclei, or ions with electrons orbiting a atomic nucleus).

in the Earth atmosphere that also has low temperature so little energy in the collisions). In stellar interiors we can also assume the ideal gas law to be valid since with at high temperature we can assume complete ionization of all elements so that the particles, electrons and nuclei, are “hard” bodies that have inelastic collisions.

Exercise 4.2:
What is the radial radiation pressure gradient, assuming the same temperature gradient as in Exerc. 3.1. What is the gas pressure gradient, assuming the same temperature gradient and that the density is constant? (Hint: assume $\mu = 0.6$ because all elements are assumed to be fully ionized.)

Exercise 4.3:
What is the relative importance of radiation pressure and gas pressure in the core of Sun? (Hint: assume $\mu = 0.6$ because all elements are assumed to be fully ionized, $T = T_{\rm core}$, $\alpha = 7.6 \cdot 10^{-16}$)

4.3 Abundances and molecular weight

Abundances of different elements are described usually by only three different parameters. They give the fractional abundances by weight for hydrogen (X), helium (Y) and "metals" (Z). Sometimes more are added, for instance the abundance of ^{3}He (Y_3) or carbon and nitrogen ($Z_{\rm CN}$).

For a case with only neutral hydrogen and neutral helium,

$$X = \frac{n_{\rm H} m_{\rm H}}{n_{\rm H} m_{\rm H} + n_{\rm He} m_{\rm He}} \tag{4.7}$$

$$Y = \frac{n_{\rm He} m_{\rm He}}{n_{\rm H} m_{\rm H} + n_{\rm He} m_{\rm He}} \tag{4.8}$$

$$Z = 0 \tag{4.9}$$

and then the number of ions are

$$\frac{X\rho}{1\,m_{\rm u}} \quad \text{hydrogen ions} \tag{4.10}$$

$$0 \quad \text{electrons from hydrogen} \tag{4.11}$$

$$\frac{Y\rho}{m_{\rm He}} = \frac{Y\rho}{4\,m_{\rm u}} \quad \text{helium ions} \tag{4.12}$$

$$0 \quad \text{electrons from helium} \tag{4.13}$$

so the total number of particles is:

$$n_{\rm tot} \quad = \quad \frac{X\rho}{m_{\rm u}} + \frac{Y\rho}{4m_{\rm u}} = \frac{\rho}{m_{\rm u}}\left(X + \frac{Y}{4}\right) \tag{4.14}$$

and now the mean molecular weight is just

$$\mu' \ = \frac{\rho}{n_{\rm tot}} \tag{4.15}$$

This number has the unit mass, but it is always given as a dimensionless number. This is done by dividing μ' by the atomic mass unit so that the mean molecular mass becomes a unit-less number and in this case is

$$\mu_0 = \frac{\rho/n_{\rm tot}}{m_{\rm u}} = \frac{\rho m_{\rm u}}{\rho\,(X + Y/4)\,m_{\rm u}} = \frac{1}{X + Y/4} \tag{4.16}$$

This is the typical definition of the mean molecular mass *per particle*, but one could in principle want it *per electron* instead and so it would be

$$\mu_0 = \frac{\rho/n_{\rm e}}{m_{\rm u}} \tag{4.17}$$

but this is not used very often, and if nothing is mentioned you can assume that it is per particle.

The mean molecular mass can be much more complicated for instance when electrons are involved or if many species of heavier elements are included. Assuming that all hydrogen and helium is neutral and that the heavier elements are fully ionized, then we may assume that the heavier elements have released half as many electrons as their atomic mass (assuming the nucleus is half neutrons and half protons), then the number of atoms and electrons for a specific atomic species i is

$$n_{Z_i} = \frac{Z_i\rho}{m_{Z_i}} = \frac{Z_i\rho}{A_i m_{\rm u}} \tag{4.18}$$

$$n_{{\rm e},Z_i} = \frac{1}{2}A_i\frac{Z_i\rho}{A_i m_{\rm u}} \tag{4.19}$$

where A_i is the atomic mass number of metal i and gives the number of nucleons (protons and neutrons combined). Z_i is the mass fraction of the metal we are looking at so that the total mass fraction of all the metals Z is

$$Z = \sum_i Z_i \tag{4.20}$$

The total number of particles is then

$$n_{\rm tot} = \sum_i n_{Z_i} + n_{{\rm e},Z_i} = \sum_i \frac{Z_i \rho}{A_i m_{\rm u}} + \frac{1}{2} A_i \frac{Z_i \rho}{A_i m_{\rm u}} \simeq \frac{Z\rho}{2m_u} \tag{4.21}$$

where the last equality is based on A_i being rather large so that the error we make by dropping the first term is small. The molecular average weight can now be written as

$$\mu_0 = \frac{1}{X + Y/4 + Z/2} \tag{4.22}$$

It is important to note here again that this expression assumes hydrogen and helium to be neutral but complete ionization of metals. That is not a very realistic situation and for stellar interior modelling we need to consider more involved expressions for μ_0.

Exercise 4.4:
Write down an expression for the mean molecular weight for a gas where the hydrogen has a mass fraction X and is fully ionized, and the mass fraction of helium is Y, where half of the helium is ionized once and the other half is ionized twice.

Exercise 4.5:
Write down an expression for the mean molecular weight of a gas with the hydrogen mass fraction X with all of the hydrogen being ionized, the helium mass fraction is Y with one third of the helium being neutral, one third ionized once and one third ionized twice. The gas also has a mass fraction Z_C of carbon atoms, which are ionized twice, and a mass fraction Z_N of nitrogen atoms which are ionized three times.

Exercise 4.6:
The mean molecular weight is usually defined per particle, but sometimes the mean molecular weight per electron is needed. Calculate the mean molecular weights per particle, per electron, and per nucleon for the Sun. Assuming X = 0.7346, Y = 0.2485 and Z = 0.017 and fully ionized hydrogen, helium and metals. Also assume that the mean number of protons of metals is equal with 8, thus the mean number of protons and neutrons is 16. You will have to go through the derivation of the

molecular mass in Sect. 4.3 to find the expression for each of these values, assuming that all atoms are ionized.

The mean molecular weight per particle can be calculated rather easily but one has to be careful. The formula is just the following

$$\mu_0 = \frac{1}{\sum(\text{particles per element} \times \text{massfraction/nucleons})} \tag{4.23}$$

where we count for the “particles per element” the ion plus the number of free electrons provided by the ion, and for the “nucleons” we count the number of protons and neutrons in the ion. This expression is completely general and can be used always for the mean molecular weight per particle.

Example 4.1:
As a simple example lets look at a gas of hydrogen which is not ionized and lets look at each of the elements in the terms in Eq. 4.23. Hydrogen provides no electrons, so the number of particles provided for each hydrogen atom is just one (the nucleus itself and no electrons), and the number of nuclear particles a hydrogen atom has is just one (a single proton). So now the mean molecular mass is just:

$$\mu_0 = \frac{1}{1X/1} = \frac{1}{X}$$

since there is only hydrogen, the mass fraction X has to be one (remember the sum of the mass fractions needs to be one), so for this gas the mean molecular mass $\mu_0 = 1$, which makes sense - all the particles are hydrogen atoms, and their mass is on average $1m_\mathrm{u}$ - remember that μ_0 is given in units of m_u.

Example 4.2:
Let us now mix in some non-ionized helium with a mass fraction of $Y = 0.3$, and so hydrogen needs to have a mass fraction $X = 0.7$. Helium also only provides one particle per nucleus (the nucleus itself and no electrons) but helium has 4 nuclear particles, so now

$$\mu_0 = \frac{1}{1X/1 + 1Y/4} = 1.29$$

which also makes sense. There are now not only the hydrogen that had a mean molecular weight of one, but now we have mixed in particles (helium) that are four times heavier, so the mean molecular mass per particle should go up.

Example 4.3:
Imagine a hydrogen gas (no other elements) but now half the hydrogen atoms are ionized. Now we need to split up the calculation in two parts because the two halves of the hydrogen gas are different. The mass fraction of hydrogen is again one since no other elements are present. The neutral part of the hydrogen atoms still provide only one particle per nucleus (the nucleus itself) and it has only one particle in its nucleus (a single proton). The other half of the hydrogen gas is ionized, so that part provides **two** particles per nucleus (the nucleus and one electron), and it still has only one particle in its nucleus (the single proton). So the mean molecular mass is now consisting of two hydrogen gasses with a mass fraction of $^1\!/_2$ X:

$$\mu_0 = \frac{1}{1\,(^1\!/_2\,X)\,/1 + 2\,(^1\!/_2\,X)\,/1} = \frac{1}{^3\!/_2\,X} = {}^2\!/_3$$

so now the mean molecular mass becomes $^2\!/_3$ because $X = 1$. The reason is that the electrons weigh almost nothing compared to a hydrogen atom, but they do count as a particle, so the mean molecular mass per particle becomes less than one.

Example 4.4:
Finally lets look at a gas consisting of hydrogen ($X = 0.5$), helium ($Y = 0.4$) and other elements ($Z = 0.1$). Half the hydrogen is ionized, one third of the helium is neutral, one third is ionized once and one third is ionized twice and the metals are also completely ionized (this is the standard approximation so the metals always enter the following way). We do the same as before, remembering all the different ionizations and

particles:

$$
\begin{aligned}
\mu_0 = &\\
&\frac{1}{1(\,^1\!/_2\,X)/1 + 2(\,^1\!/_2\,X)/1 + 1(\,^1\!/_3\,Y)/4 + 2(\,^1\!/_3\,Y)/4 + 3(\,^1\!/_3\,Y)/4 + Z/2}\\
&= \frac{1}{^3\!/_2\,X + \,^1\!/_2\,Y + \,^1\!/_2\,Z}\\
&= 1.0
\end{aligned}
$$

4.3.1 Corrections

There are two main corrections for the ionisation degree. Both of them are not very important for the Sun.

Electrostatic corrections take into account that the ionised particles also have an electric field around them, and therefore the collisions between particles are "softer" since they are surrounded by a "soft" electric field. The corrections are very small for the Sun, on the order of a few percent.

The partially degenerate electron correction, comes from the fact that deep in the Sun, the electrons are under such a large pressure, that they are affected by the Pauli exclusion principle that amongs other things says that if you force a large enough number of electrons into a very limited space, you cannot have electrons placed in the same location with the same momentum. There is therefore a correction to the electron pressure. For the Sun this only changes the electron pressure by a few percent at the very center of the core of the Sun, but since the total pressure is set by the hydrostatic equilibrium, the pressure provided by the ions just changes to acommodate the change in the electron pressure.

There are a number of other corrections which can be included, but the effects of them are rather small, and they make computations of stellar models much more cumbersome. In general a table for the equation of state is used instead, where one can lookup the values for the pressure P, when the density and temperature are known.

The average weight of the particles, or mean molecular weight as it is also known, can be calculated if one knows the temperature and density and one assumes that the gas is thermalized, ie., there are frequent collisions so that the velocity distribution is Maxwellian and the ionization degree follows Saha-Boltzmann statistics.

Chapter 5

Energy transport

5.1 Introduction

In this chapter we shall see how the vast amount of energy produced in the center of a star is transported throughout the star. The efficiency of the transportation of the energy controls the temperature of the stellar interior and consequently the efficiency of the energy production in the core. If the energy transport is extremely efficient, it would be able to remove more energy from the core than is produced and the core would cool down, producing less energy and therefore we need to understand how much energy is transported away from the stellar core in order for us to get a correct model of a star. The large amount of energy produced in the center of stars are transported to the surface in two different ways. They are both effective, but the local temperature, pressure and chemical composition decides which of the two dominates.

The first transport mechanism is by the emission, absorption and emission again of photons, a combined process that is referred to as scattering, and the very small difference between the photons traveling outwards and inwards is large enough to carry the energy outwards.

The second energy transport mechanism is through large scale gas motions, called convection. Convection happens when the radiative transport of energy is not able to transport the energy fast enough, making the gas unstable and setting it in motion. Hot gas wells up from the hot interior of the star, and when it reaches a location (the surface for the Sun) where another transport mechanism is efficient enough to move the energy, it cools and flows back down into the star.

In principle, heat conduction could also transport energy in stars, but it only has an effect in the cores of the most extreme stars with degenerate cores. The photon mean free path is very large in these cores because the electrons cannot absorb photons since any possible quantum states the electrons would have to jump to after the absorption of the photon are already filled with electrons, and that is not permitted according to the Pauli exclusion principle. The opposite is also true - no photons can be emitted from the core, since there are no free lower-level quantum states to fall into after the emission of the photon. We will not treat these extreme conditions in these notes.

5.2 Radiative transport

Radiative transport is the movement of energy by electromagnetic radiation. The gas inside a star is generally very opaque so the photons will travel a very short distance before being re-absorbed. The mean free path of a photon can be calculated by:

$$\lambda = \frac{1}{\kappa\rho} \tag{5.1}$$

where κ is the opacity (units of $\mathrm{m}^2\ \mathrm{kg}^{-1}$), and ρ is the mass density. Opacity is a measure of how effective a medium can block radiation. The opacity in the Sun is of the order one, and as the density in the Sun is somewhere between 1 and 10^5, the mean free path is very small. The photons perform a random walk through the star. By assuming that the photons are re-emitted instantly and that they are emitted in a random direction, they travel on average a certain distance L by being re-absorbed n times and on average traveling a distance λ between absorptions. This distance λ is the mean free path of the photons, and the three numbers are connected through:

$$L = \sqrt{n}\,\lambda \tag{5.2}$$

If the photons had been emitted along the same direction as they were traveling before they were absorbed, the equation would just be $L = n\,\lambda$, but because they are emitted in any direction, they do not travel in a straight line, but need to travel much further. The proof for the random walk equation can be easily done for one dimension, with a certain step size λ. Imagine a photon that is emitted at location zero, after which it travels either a distance $-\lambda$ or a distance $+\lambda$ before being re-absorbed. It is re-emitted and again travels a distance $-\lambda$ or $+\lambda$. The photon's displacement s_i after each step is then

$$s_i = \begin{cases} -\lambda & 50\% \text{ of the time} \\ +\lambda & 50\% \text{ of the time} \end{cases} \tag{5.3}$$

After n steps the position x of the photon is then

$$x(n) = \sum_{i=1}^{n} s_i \tag{5.4}$$

and the displacement squared is then

$$x^2(n) = \left(\sum_{i=1}^{n} s_i \right)^2 \tag{5.5}$$

The average position of the photon will always be at $x = 0$, but it only means that the possibility of finding the photon somewhere is a distribution centered at $x = 0$, not that the photon is stuck at $x = 0$ for all time, the probability of finding the photon at a certain distance from $x = 0$ increases with the number of times it has been reemitted. To figure out how large the broadening of the probability is, we can rewrite the square displacement:

$$x^2(n) = \left(\sum_{i=1}^{n} s_i \right)^2 = \sum_{i=1}^{n} s_i \sum_{j=1}^{n} s_j = \sum_{i=1}^{n} s_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} s_i s_j \tag{5.6}$$

Now consider the second sum with the product $s_i s_j$. There are four possibilities at each step, all four of them being equally probable. The product is positive if either both s_i and s_j are positive, or both are negative. If one of them is negative while the other is not, the product will be negative. This means we have:

$$s_i s_j = \begin{cases} +\lambda^2 & 50\% \text{ of the time} \\ -\lambda^2 & 50\% \text{ of the time} \end{cases} \tag{5.7}$$

which means that on average the second sum will be zero. But at the same time $s_i^2 = \lambda^2$ independently of whether λ is positive or negative. The average after n steps must therefore be

$$< x^2(n) >= \lambda^2 n \tag{5.8}$$

which is identical to Eq. 5.2.

5.2.1 Photon diffusion

As we have just seen, the photon mean free path is very small in the Sun. The transport of energy can therefore also be seen as a diffusion of photons from a more photon-dense volume (the core of the Sun) to a less photon-dense volume (the outer regions of the Sun). The diffusion equation in a 3-dimensional space in general form can be written as

$$\mathbf{j} = -D\nabla n \tag{5.9}$$

where n is density of the medium being diffused, $\mathbf{j}$ is the flux created by the diffusion and D is a diffusion constant. The diffusion constant is given as

$$D = \frac{1}{3}\, v\, \lambda \tag{5.10}$$

where v is the velocity of the particles. In this case the density of photons is given by the radiative energy density:

$$U = aT^4 \tag{5.11}$$

where a is the radiation density constant, and so the photon energy flux is:

$$F_{\rm rad} = -\frac{4}{3} ac\lambda T^3 \frac{\partial T}{\partial r} = -\frac{4}{3}\frac{acT^3}{\kappa\rho}\frac{\partial T}{\partial r} = -\frac{16}{3}\frac{\sigma T^3}{\kappa\rho}\frac{\partial T}{\partial r} \tag{5.12}$$

where we have used that $a = 4\sigma/c$ where σ is Stefan Boltzmann's constant.

We also know that the radiative flux has to carry all the energy produced by the star, so we also know that

$$F_{\rm rad} = \frac{L}{4\pi r^2} \tag{5.13}$$

so we can solve for the temperature gradient

$$\frac{L}{4\pi r^2} = -\frac{16}{3}\frac{\sigma T^3}{\kappa\rho}\frac{\partial T}{\partial r} \tag{5.14}$$

When considering models of stars, it is more convenient to take m as the independent variable instead of r. After multiplication by $\partial r/\partial m = (4\pi r^2\rho)^{-1}$, we can write

$$\begin{aligned}
\frac{\partial T}{\partial m} &= -\frac{3\kappa\rho}{64\pi r^2 \rho\sigma T^3}\frac{L}{4\pi r^2} \\
\frac{\partial T}{\partial m} &= -\frac{3\kappa L}{256\pi^2 r^4 \sigma T^3}
\end{aligned} \tag{5.15}$$

In principle, Eq. 5.15 is only true when all the energy is transported by radiation. The gradient in temperature is the gradient needed for the photons to be able to carry all the energy out through the Sun. If some other mechanism would set in that was more efficient than energy transport by radiation, the temperature gradient from Eq. 5.15 would not be correct.

5.3 Convective transport

Convective transport is the transport of energy by motions. It depends on the stability of the gas for small perturbations. Convection in stars is similar to what happens when you heat water in a pot on a stove. The stove heats the pot and the water at the bottom of the pot. Water is convectively unstable under normal circumstances, and the effect on the water is most obvious just before it boils. Large cells of water well up from below where it has been heated, when it reaches the surface, it cools when coming in contact with the colder air, and then flows back down to the bottom of the pot, usually along the cool sides of the pot. In stars the same effect happens, and in the Sun, the effect is almost identical. The gas is being heated from below, and the gas becomes convectively unstable, and the same large cells of upwelling gas can be seen on the solar surface, but since there are no "sides" on the Sun, the cells are surrounded by cooler downflow lanes as can be seen in Fig. 5.1.

To find out if the gas in a medium is convectively unstable, meaning that it will start to move even at the smallest perturbation, a number of thermodynamic principles must be understood. Thermodynamics is a very large subject, and only the absolute minimum needed to understand convective transport will be included in these notes.

5.3.1 Thermodynamics

The behavior of a gas can be explained with the help of a number of constants and principle laws. The first relation is the equation for added heat per unit mass:

$$\mathrm{d}q = \mathrm{d}u + P \, \mathrm{d}V \tag{5.16}$$

with the internal energy u and the specific volume $V = 1/\rho$. Be careful to note that we now use the symbol V here for specific volume which is volume per mass with units $\mathrm{m}^3 \ \mathrm{kg}^{-1}$, so different from traditional volume. It is further important to notice that $\mathrm{d}u$ is a total differential form, so that for

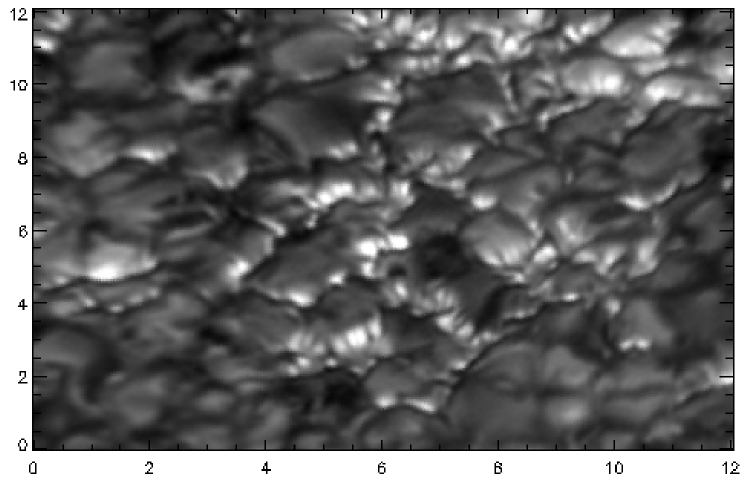

Figure 5.1: The convective pattern on the solar surface, showing the bubbles of hot gas sticking up above the cool and dark downflow valleys. The observation is made with the Swedish 1-m Solar Telescope on La Palma in the Canary Islands. Tick marks are arcseconds corresponding to roughly 725 km on the solar surface. We are looking here at the solar surface a bit from the side, the observation angle is 51°. Compare with Fig. 1.1 where we see the solar surface face-on, at the center of the solar disk, at an observing angle of 0°.

instance:

$$\mathrm{d}u = \left(\frac{\partial u}{\partial V}\right)_T \mathrm{d}V + \left(\frac{\partial u}{\partial T}\right)_V \mathrm{d}T \tag{5.17}$$

These expressions are rather useless for our purposes so we need to reform them into something more useful. We want to get an expression for what happens to the gas if we add just a small amount of energy $\mathrm{d}q$, and that involves both that we know something about the gas itself and how it changes its properties when it is being perturbed by outside disturbances. In order to understand these details, we need to be as general as possible. To that end we assume a very general equation of state $\rho(P,T)$ and $u(\rho,T)$, which usually is also dependent on the chemical composition, but we assume here that the chemical composition is constant or at least that the evolution of the chemical composition is so slow that it has no consequence. Be aware that the independent variables P and T for $\rho(P,T)$ and ρ and T for $u(\rho,T)$ are chosen specifically but could have been chosen differently – as long as two of the three P, T and ρ are kept as primary variables and the last as a derived quantity. In the following we will use the independent variables mentioned above. A number of derivatives will be needed later on and to simplify things they will be renamed by the following symbols:

$$\alpha = \left(\frac{\partial \ln \rho}{\partial \ln P}\right)_T = -\frac{P}{V}\left(\frac{\partial V}{\partial P}\right)_T \tag{5.18}$$

$$\delta = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P = \frac{T}{V}\left(\frac{\partial V}{\partial T}\right)_P \tag{5.19}$$

The subscripts T, P and later V mean that the derivatives are to be taken at constant temperature, pressure and specific volume respectively. We can now write the equation of state as

$$\frac{\mathrm{d}\rho}{\rho} = \alpha\frac{\mathrm{d}P}{P} - \delta\frac{\mathrm{d}T}{T} \tag{5.20}$$

Exercise 5.1:
Show Eq. 5.20 using Equations 5.16–5.19, alternatively show Eq. 5.20 using Eq. 5.18 and Eq. 5.19 repeating what is done in Eq. 5.17.

We also need the specific heats of which there are two. Specific heats are the amount of energy needed to raise the temperature of a unit mass of the gas by one Kelvin, at constant pressure (Eq. 5.21) or at constant specific volume (Eq. 5.22):

$$c_P = \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P \tag{5.21}$$

and

$$c_V = \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)_V = \left(\frac{\partial u}{\partial T}\right)_V \tag{5.22}$$

Eq. 5.16 can now be written as

$$\begin{aligned} \mathrm{d}q &= \left(\frac{\partial u}{\partial V}\right)_T \mathrm{d}V + \left(\frac{\partial u}{\partial T}\right)_V \mathrm{d}T + P\ \mathrm{d}V \\ &= c_V\,\mathrm{d}T + \left[\left(\frac{\partial u}{\partial V}\right)_T + P\right]\mathrm{d}V \end{aligned} \tag{5.23}$$

If we choose a case where the pressure is kept constant, then we can find a change in temperature that only changes the volume and we can then use the derivative of the volume with temperature to get

$$\mathrm{d}q = c_V\,\mathrm{d}T + \left[\left(\frac{\partial u}{\partial V}\right)_T + P\right]\left(\frac{\partial V}{\partial T}\right)_P \mathrm{d}T \tag{5.24}$$

At the same time a change in energy $\mathrm{d}q$ at constant pressure due to a change in temperature is the definition of heat capacity, so in this case

$$\mathrm{d}q = c_P\,\mathrm{d}T \tag{5.25}$$

We finally get

$$\begin{aligned}
c_P\,\mathrm{d}T &= c_V\,\mathrm{d}T + \left[\left(\frac{\partial u}{\partial V}\right)_T + P\right]\left(\frac{\partial V}{\partial T}\right)_P \mathrm{d}T \\
(c_P - c_V)\,\mathrm{d}T &= \left[\left(\frac{\partial u}{\partial V}\right)_T + P\right]\left(\frac{\partial V}{\partial T}\right)_P \mathrm{d}T \\
c_P - c_V &= \left[\left(\frac{\partial u}{\partial V}\right)_T + P\right]\left(\frac{\partial V}{\partial T}\right)_P
\end{aligned} \tag{5.26}$$

Even if we have an equation of state, u is still not a very useful variable to use, so in order to get rid of the derivative of u we can use the equation for the entropy

$$\mathrm{d}s = \frac{\mathrm{d}q}{T} = \frac{1}{T}\left(\,\mathrm{d}u + P\,\,\mathrm{d}V\right) \tag{5.27}$$

and if we insert the full derivative for u in Eq. 5.27, we can find

$$\begin{aligned}
\mathrm{d}s &= \frac{1}{T}\,\mathrm{d}u + \frac{1}{T}P\,\mathrm{d}V \\
&= \frac{1}{T}\left[\left(\frac{\partial u}{\partial T}\right)_V \mathrm{d}T + \left(\frac{\partial u}{\partial V}\right)_T \mathrm{d}V\right] + \frac{1}{T}P\,\mathrm{d}V \\
&= \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_V \mathrm{d}T + \frac{1}{T}\left[\left(\frac{\partial u}{\partial V}\right)_T + P\right]\mathrm{d}V
\end{aligned} \tag{5.28}$$

We compare that with the expression for the full derivative of s

$$\mathrm{d}s = \left(\frac{\partial s}{\partial T}\right)_V \mathrm{d}T + \left(\frac{\partial s}{\partial V}\right)_T \mathrm{d}V \tag{5.29}$$

and then we can see that

$$\left(\frac{\partial s}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_V \qquad \text{and} \qquad \left(\frac{\partial s}{\partial V}\right)_T = \frac{1}{T}\left[\left(\frac{\partial u}{\partial V}\right)_T + P\right] \tag{5.30}$$

Since

$$\frac{\partial}{\partial V}\left[\frac{\partial s}{\partial T}\right] = \frac{\partial}{\partial T}\left[\frac{\partial s}{\partial V}\right] \tag{5.31}$$

we can take the derivative with respect to V of the left expression in Eq. 5.30 then that should be the same as the derivative with respect to T of the right expression, so we get

$$\begin{aligned}
\frac{\partial}{\partial V}\left[\frac{1}{T}\left(\frac{\partial u}{\partial T}\right)_V\right] &= \frac{\partial}{\partial T}\left[\frac{1}{T}\left(\left(\frac{\partial u}{\partial V}\right)_T + P\right)\right] \\
\frac{1}{T}\frac{\partial^2 u}{\partial T \partial V} &= -\frac{1}{T^2}\left(\frac{\partial u}{\partial V}\right)_T + \frac{1}{T}\left(\frac{\partial^2 u}{\partial T \partial V}\right) - \frac{1}{T^2}P + \frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_V \\
0 &= -\left(\frac{\partial u}{\partial V}\right)_T - P + T\left(\frac{\partial P}{\partial T}\right)_V \\
\left(\frac{\partial u}{\partial V}\right)_T &= T\left(\frac{\partial P}{\partial T}\right)_V - P \qquad (5.32)
\end{aligned}$$

We now have an expression for the derivative of u we needed in Eq. 5.26, and by inserting we get

$$c_P - c_V = \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial P}{\partial T}\right)_V T \tag{5.33}$$

and since we can transform the second term

$$\left(\frac{\partial P}{\partial T}\right)_V = -\frac{\left(\frac{\partial V}{\partial T}\right)_P}{\left(\frac{\partial V}{\partial P}\right)_T} = \frac{P\delta}{T\alpha} \tag{5.34}$$

we finally arrive at

$$c_P - c_V = T\left(\frac{\partial V}{\partial T}\right)_P \frac{P\delta}{T\alpha} = \frac{P\delta^2}{\rho T\alpha} \tag{5.35}$$

which for an ideal gas transforms into

$$c_P - c_V = n k_{\mathrm{B}} \tag{5.36}$$

where k_{B} is Boltzmann's constant and n is the number density of atoms or molecules.

Exercise 5.2:
Using the first law of thermodynamics (Eq. 5.16) and assuming a process where the volume does not change, so that

$$\mathrm{d}q = c_V \,\mathrm{d}T = \mathrm{d}u$$

find an expression for the c_V for an ideal gas.

Exercise 5.3:
Deduce the expression for $c_P - c_V$ (Eq. 5.36) by using the ideal equation of state and Eq. 5.35.

Exercise 5.4:
Using the answers to Exerc. 5.2 and Exerc. 5.3, deduce an expression for c_P

Eq. 5.16 can now be transformed

$$\begin{aligned}
\mathrm{d}q &= \mathrm{d}u + P\,\mathrm{d}V \\
&= \left[\left(\frac{\partial u}{\partial V}\right)_T \mathrm{d}V + \left(\frac{\partial u}{\partial T}\right)_V \mathrm{d}T\right] + P\,\mathrm{d}V \\
&= \left(\frac{\partial u}{\partial V}\right)_T \mathrm{d}V + c_V\,\mathrm{d}T + P\,\mathrm{d}V \\
&= c_V\,\mathrm{d}T + \left[\left(\frac{\partial u}{\partial V}\right)_T + P\right]\mathrm{d}V \qquad (5.37)
\end{aligned}$$

using Eq. 5.32 and Eq. 5.34 we get

$$\begin{aligned}
\mathrm{d}q &= c_V\,\mathrm{d}T + \left[T\left(\frac{\partial P}{\partial V}\right)\right]\mathrm{d}V \\
&= c_V\,\mathrm{d}T + \left[T\frac{P\delta}{T\alpha}\right]\mathrm{d}V \\
&= c_V\,\mathrm{d}T + \frac{P\delta}{\alpha}\,\mathrm{d}V \qquad (5.38)
\end{aligned}$$

We now need to convert $\mathrm{d}V$ and that can be done using Eq. 5.20 for the general equation of state:

$$\begin{aligned}
\frac{d\rho}{\rho} &= \alpha\frac{\mathrm{d}P}{P} - \delta\frac{\mathrm{d}T}{T} \\
V\left(-\frac{1}{V^2}\right)\mathrm{d}V &= \alpha\frac{\mathrm{d}P}{P} - \delta\frac{\mathrm{d}T}{T} \\
\mathrm{d}V &= -\alpha V\frac{\mathrm{d}P}{P} + V\delta\frac{\mathrm{d}T}{T} \qquad (5.39)
\end{aligned}$$

Inserting that expression back into Eq. 5.38 gives

$$\begin{aligned}
\mathrm{d}q &= c_V\,\mathrm{d}T + \frac{P\delta}{\alpha}\left(-\frac{\alpha V}{P}\,\mathrm{d}P + \frac{V\delta}{T}\,\mathrm{d}T\right) \\
&= c_V\,\mathrm{d}T - \delta V\,\mathrm{d}P + \frac{P\delta^2 V}{\alpha T}\,\mathrm{d}T \\
&= \left(c_V + \frac{P\delta^2 V}{\alpha T}\right)\,\mathrm{d}T - \frac{\delta}{\rho}\,\mathrm{d}P
\end{aligned} \tag{5.40}$$

and finally using Eq. 5.35

$$\begin{aligned}
\mathrm{d}q &= \left(c_V + \frac{P\delta^2}{\rho T\alpha}\right)\,\mathrm{d}T - \frac{\delta}{\rho}\,\mathrm{d}P \\
&= c_P\,\mathrm{d}T - \frac{\delta}{\rho}\,\mathrm{d}P
\end{aligned} \tag{5.41}$$

We are now almost done. In our case we are interested in how the temperature, density and pressure would have to change inside the star in order for it to be stable to small perturbations. If a small perturbation moves a parcel of gas up or down, we need to know how its pressure, temperature and density changes. If the changes are such that it now gets the same temperature, density and pressure as its new surroundings, the star would be very stable, and nothing much would move. That same would be true if the parcel would start to move back towards its initial position after being moved up or down. We will look at how the temperature structure looks for a star where the gas can be perturbed up or down without becoming unstable. In such a case the temperature, density and pressure is exactly the same as the surroundings after it has moved, so there is no exchange of energy from the parcel to the surroundings. Such a process is adiabatic, and so we can use the definition of the adiabatic temperature gradient to be:

$$\nabla_{\mathrm{ad}} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_s \tag{5.42}$$

Using Eq. 5.41 and the assumption that there is no exchange of energy between the package of gas we are looking at and its surroundings, we now have

$$T\,\mathrm{d}s = 0 = \mathrm{d}q = c_P\;\mathrm{d}T - \frac{\delta}{\rho}\,\mathrm{d}P \tag{5.43}$$

or

$$\left(\frac{\mathrm{d}T}{\mathrm{d}P}\right)_s = \frac{\delta}{\rho c_P} \tag{5.44}$$

so that the adiabatic temperature gradient can now be rewritten as

$$\nabla_{\mathrm{ad}} = \left(\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}P}\right)_s = \frac{P\delta}{T\rho\, c_P} \tag{5.45}$$

Exercise 5.5:
Show that using the equation of state for an ideal gas (Eq. 4.3), Eq. 5.35 can be simplified to Eq. 5.36

Exercise 5.6:
By inserting the expression for the pressure of an ideal gas into Eq. 5.45 and find a value for ∇_{ad} for an ideal gas

5.3.2 Convective instability

Convection can easily be explained as a stratified gas or a fluid that is unstable to small changes in position. Imagine a small parcel or volume of gas is displaced slightly up towards the stellar surface. Assume that the movement is fast enough for the parcel not to exchange heat with its surroundings, while at the same time having time to get into pressure equilibrium. These assumptions might not seem very realistic, so we will see if they are realistic or not later in this chapter. In certain circumstances the small parcel of gas becomes buoyant (the density is lower than its surroundings) as an effect of that movement. If those circumstances are present, then the gas is convectively unstable, as the gas will continue to rise through the star until the circumstances change.

The assumptions we have made in this example are that the timescale for a pressure perturbation to travel across the volume of the gas parcel, is smaller than the timescale for a photon to cross the parcel taking into account the times it is absorbed and reemitted. That makes it obvious that we are talking about a parcel of gas that is many times larger than the mean free path of a photon, otherwise the photons would make the parcel the same temperature as the gas surrounding it in a time that is on the order of the photon flight time from one side of the parcel to the other. A pressure perturbation travels with sound speed. The sound speed for an ideal gas is

$$c_s^2 = \gamma\frac{P}{\rho} \tag{5.46}$$

where γ is the ratio between the specific heats

$$\gamma = \frac{c_P}{c_V} \tag{5.47}$$

and can also be written as

$$\gamma = 1 + \frac{2}{f} \tag{5.48}$$

where f is the number of degrees of freedom. A pressure perturbation will travel across its radius δr in a time δt

$$\delta t_s = \sqrt{\frac{\rho}{\gamma P}}\delta r \tag{5.49}$$

A photon will have a mean free path through the volume of

$$\lambda = \frac{1}{\rho\kappa} \tag{5.50}$$

where κ is the opacity (see Eq. 5.1). The number of steps it will have to take is (see Sect. 5.2)

$$N = \frac{\delta r^2}{\lambda^2} \tag{5.51}$$

and finally the time it would take would be

$$\delta t_r = \frac{N\lambda}{c} + N\delta t_{\mathrm{em}} \tag{5.52}$$

where c is the speed of light, and δt_{em} is the time it on average takes between an absorption and before the photon is re-emitted (in other words: the time the atom is in an excited state). In order for our assumption to be correct

$$\begin{aligned}
\delta t_s &< \delta t_r \\
\frac{\delta r}{c_s} &< \frac{N\lambda}{c} + N\delta t_{em} \\
\frac{\delta r}{c_s} &< N\left(\frac{\lambda}{c} + \delta t_{em}\right) \\
\delta r &< \frac{\delta r^2}{\lambda^2} c_s \left(\frac{\lambda}{c} + \delta t_{em}\right) \\
1 &< \frac{\delta r}{\lambda^2} c_s \left(\frac{\lambda}{c} + \delta t_{em}\right) \\
\delta r &> \frac{\lambda^2}{c_s}\left(\frac{c}{\lambda + c\delta t_{em}}\right) \\
\frac{\delta r}{\lambda} &> \frac{c}{c_s}\left(\frac{\lambda}{\lambda + c\delta t_{em}}\right)
\end{aligned} \tag{5.53}$$

Here we can see that the δr in units of the mean free path of the photons needs to be roughly c/c_s which is clearly large. The expression in the parentheses on the right hand side can then decrease the requirement for δr if the distance a photon can travel during the time it is absorbed by an atom is on the order of or larger than the mean free path λ. It is hard to estimate if this is realistic. It turns out that inside stars, the sound speed is very high so it makes this requirement less stringent.

Exercise 5.7:
Calculate the requirement for δr in the solar core, at the bottom of the convection zone and in the photosphere, using an ideal equation of state, so that the sound speed can be expressed as $c_s^2 = \gamma P/\rho$ where $\gamma = 5/3$. Use the values found in the appendices and assume that the opacity $\kappa = 1\,\mathrm{m}^2\,\mathrm{kg}^{-1}$

5.3.3 Onset of convection

Imagine now a parcel of gas that starts out at a certain height r, where it has a density ρ_0, pressure P_0 and temperature T_0. The gas of the star around the parcel has a density ρ_0^*, pressure P_0^* and temperature T_0^* which are equal to inside the parcel, i.e.

$$\rho_0 = \rho_0^* \quad , \quad P_0 = P_0^* \quad \text{and} \quad T_0 = T_0^* \tag{5.54}$$

The parcel is now displaced upward in the stably stratified stellar interior by

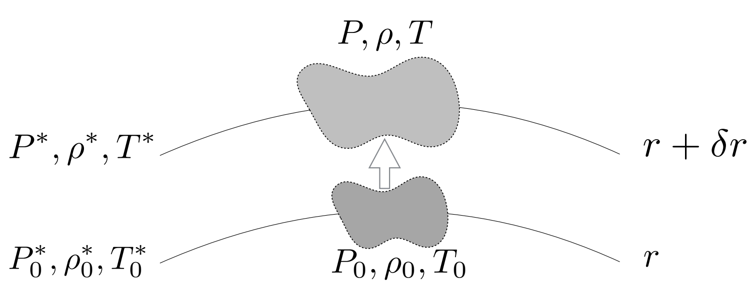

Figure 5.2: A convectively unstable parcel of gas inside a star

a distance δr from a position r to $r+\delta r$. During the adiabatic displacement, the density ρ_0 of the parcel changes to ρ while the density outside changes to ρ^*. The pressure inside the parcel P adjusts to the same level as the pressure outside the parcel P^*. If the density of the parcel after the displacement is lower than the surroundings, the parcel is buoyant and will move. So in order for the stratification to be convectively unstable, the density contrast can be written as

$$\begin{aligned} 0 > \rho - \rho^* &= \delta r \left[\frac{\rho - \rho_0}{\delta r} - \frac{\rho^* - \rho_0}{\delta r} \right] \\ &= \delta r \left[\frac{\rho - \rho_0}{\delta r} - \frac{\rho^* - \rho_0^*}{\delta r} \right] \\ &= \delta r \left[\left(\frac{\partial \rho}{\partial r} \right)_{\rm ad} - \frac{\partial \rho}{\partial r} \right] \end{aligned} \tag{5.55}$$

where the adiabatic gradient is part of the assumption of the parcel displacement being adiabatic. If we assume an ideal gas where the mean molecular weight μ can vary then

$$\left[\frac{\partial \rho}{\partial r} \right] = \frac{m_{\rm u}}{k_{\rm B}} \left[\frac{\partial \frac{P\mu}{T}}{\partial r} \right] = \frac{m_{\rm u}}{k_{\rm B}} \left[\frac{\mu}{T} \frac{\partial P}{\partial r} + \frac{P}{T} \frac{\partial \mu}{\partial r} + P\mu \frac{\partial T^{-1}}{\partial r} \right] \tag{5.56}$$

and then inserting into Eq. 5.55 gives

$$\begin{aligned} 0 &> \delta r \left[\frac{m_{\rm u}}{k_{\rm B}} \left(\left[\frac{\mu}{T} \frac{\partial P}{\partial r} + \frac{P}{T} \frac{\partial \mu}{\partial r} + P\mu \frac{\partial T^{-1}}{\partial r} \right]_{\rm ad} - \right. \right. \\ & \qquad \left. \left. \left[\frac{\mu}{T} \frac{\partial P}{\partial r} + \frac{P}{T} \frac{\partial \mu}{\partial r} + P\mu \frac{\partial T^{-1}}{\partial r} \right] \right) \right] \\ 0 &> \left[\frac{\mu}{T} \frac{\partial P}{\partial r} + \frac{P}{T} \frac{\partial \mu}{\partial r} + P\mu \frac{\partial T^{-1}}{\partial r} \right]_{\rm ad} - \\ & \qquad \frac{\mu}{T} \frac{\partial P}{\partial r} - \frac{P}{T} \frac{\partial \mu}{\partial r} - P\mu \frac{\partial T^{-1}}{\partial r} \\ 0 &> \left[\mu \frac{\partial P}{\partial r} + P \frac{\partial \mu}{\partial r} - \frac{P\mu}{T} \frac{\partial T}{\partial r} \right]_{\rm ad} - \\ & \qquad \mu \frac{\partial P}{\partial r} - P \frac{\partial \mu}{\partial r} + \frac{P\mu}{T} \frac{\partial T}{\partial r} \end{aligned} \tag{5.57}$$

Using that the pressure at the new height $r + \mathrm{d}r$ inside and outside the parcel is the same so that the gradient in pressure inside and outside the parcel is also the same,

$$0 > \left[P \frac{\partial \mu}{\partial r} - \frac{P\mu}{T} \frac{\partial T}{\partial r} \right]_{\rm ad} - P \frac{\partial \mu}{\partial r} + \frac{P\mu}{T} \frac{\partial T}{\partial r} \tag{5.58}$$

we finally get

$$-\mu\frac{\partial T}{\partial r} > -\mu\left(\frac{\partial T}{\partial r}\right)_{\rm ad} + T\left(\frac{\partial \mu}{\partial r}\right)_{\rm ad} - T\frac{\partial \mu}{\partial r} \tag{5.59}$$

$$\frac{\partial T}{\partial r} < \left(\frac{\partial T}{\partial r}\right)_{\rm ad} - \frac{T}{\mu}\left(\frac{\partial \mu}{\partial r}\right)_{\rm ad} + \frac{T}{\mu}\frac{\partial \mu}{\partial r} \tag{5.60}$$

The instability criterion says that if the temperature gradient is lower than the adiabatic temperature gradient, then we have a convective instability, and the gas will begin to move. There are two terms on the right, that can influence this criterion. If we assume that the mean molecular weight does not change under adiabatic expansion or contraction $(\partial\mu/\partial r)_{\rm ad} = 0$ and we are dealing with the Sun, then the mean molecular weight decreases with radius[1], so that the term $(\partial\mu/\partial r)$ is stabilizing – the temperature gradient can now be larger than the adiabatic temperature gradient. The instability criterion is often recast into

$$\nabla > \nabla_{\rm ad} \tag{5.61}$$

where $\nabla_{\rm ad}$ is defined by Eq. 5.42 and ∇ is then the same double logarithmic gradient, but for the stellar stratification being investigated. It is equivalent to Eq. 5.60 except for the terms depending on the mean molecular weight, since

$$\left(\frac{\partial \ln T}{\partial \ln P}\right) > \left(\frac{\partial \ln T}{\partial \ln P}\right)_{\rm ad}$$

$$\frac{P}{T}\frac{\partial T}{\partial r}\frac{\partial r}{\partial P} > \frac{P}{T}\left(\frac{\partial T}{\partial r}\right)_{\rm ad}\left(\frac{\partial r}{\partial P}\right)_{\rm ad} \tag{5.62}$$

$$\frac{\partial T}{\partial r} < \left(\frac{\partial T}{\partial r}\right)_{\rm ad} \tag{5.63}$$

because the parcel is supposed to be rising adiabatically and have the same pressure as outside, so the two gradients are identical.

Exercise 5.8:
Why does the inequality sign change from Eq. 5.62 to Eq. 5.63 ?

[1] For a gas consisting of only hydrogen and helium, both fully ionized, the mean molecular weight $\mu = 1/(2X + 3/4Y)$ which for the Sun decreases with radius because at the center of the Sun, the mass fraction of helium is much larger than at the surface. There are though regions in the outer parts of the Sun where helium goes from being twice ionized to only ionized once, which means that at that radius the mean molecular weight actually increases.

5.3.4 Convective energy flux

In general the energy flux at any point in a star, outside its energy producing core, has to obey

$$F_{\rm rad} + F_{\rm con} = \frac{L}{4\pi r^2} \tag{5.64}$$

where the $F_{\rm rad}$ is the radiative flux found in Eq. 5.12, $F_{\rm con}$ is the convective flux and L is the luminosity of the star. We have here implicitly assumed that heat conduction is not important. The convective flux can from first principles be written as

$$F_{\rm con} = \rho\, c_P\, v \Delta T \tag{5.65}$$

which just states that the convective flux is the excess energy $\rho\, c_P \Delta T$ a parcel of density ρ can move, with the velocity v. The trouble is of course to estimate what the velocity and temperature difference is. In the following we will try to estimate what the temperature difference between the parcel and the surroundings (ΔT) is and what the convective velocity (v) is.

The temperature difference

We imagine as in Sect. 5.3.3 that we have a rising parcel of gas, which is continuously in pressure equilibrium with its surroundings, but here we cannot assume that it rises adiabatically, because the time it takes for the parcel to rise might be long, and so it would be able to exchange energy with its surroundings.

To estimate the temperature difference, we write

$$\Delta T = \left[\left(\frac{\partial T}{\partial r}\right)_{\rm p} - \left(\frac{\partial T}{\partial r}\right)^{*} \right] \delta r = \left(\nabla^* - \nabla_p\right) \frac{T \delta r}{H_P} \tag{5.66}$$

where a subscript p is for the parcel, while a superscript $*$ is for the star. We have used the definition of the pressure scale height.

$$H_P = -P \frac{\partial r}{\partial P} \tag{5.67}$$

to change the gradient of r to a gradient in pressure:

$$\begin{aligned} \nabla &= \frac{\partial \ln T}{\partial \ln P} = P \frac{\partial r}{\partial P} \frac{\partial \ln T}{\partial r} \\ &= -H_P \frac{\partial \ln T}{\partial r} \\ &= -\frac{H_P}{T} \frac{\partial T}{\partial r} \end{aligned} \tag{5.68}$$

Exercise 5.9:
Assume that we are looking at a static atmosphere where gravity, and pressure are the only two forces present. Assume also that gravity can be represented by the gravitational acceleration and is constant and has the value g. Assume furthermore that gravity is aligned along $-r$. Write down the force balance, and use it to find an equation for P as a function of r. What is the pressure scale height, for an ideal gas, when the pressure scale height is defined as the height over which the pressure drops by a factor $1/e$?

We need to assume a value for the distance δr travelled by a parcel of gas in order to get further with Eq. 5.65. For historical reasons the distance usually assumed is half a mixing length

$$\delta r = \tfrac{1}{2}\, l_{\rm m} \tag{5.69}$$

Inserting this directly into Eq. 5.65 we get:

$$F_{\rm con} = \rho\, c_P\, v \frac{T l_{\rm m}}{2H_P} \left(\nabla^* - \nabla_{\rm p}\right) \tag{5.70}$$

If we split this into two terms by introducing $\nabla_{\rm ad}$ which is the temperature gradient for an adiabatically rising parcel we get:

$$F_{\rm con} = \rho\, c_P\, v \frac{T l_{\rm m}}{2H_P} \left(\nabla^* - \nabla_{\rm ad}\right) + \rho\, c_P\, v \frac{T l_{\rm m}}{2H_P} \left(\nabla_{\rm ad} - \nabla_{\rm p}\right) \tag{5.71}$$

These two terms can be understood quite easily. The first term is just the convective flux we would have if the parcels were rising adiabatically. The second term is the modification needed when taking into account the radiative losses of the rising parcel. The second term is just the radiative loss of the parcel so by using the radiative flux (Eq. 5.12) and assuming that the gradient in temperature from the outside to inside of the parcel is twice the temperature difference divided by the diameter d of the parcel:

$$f_{\rm rad} = -\frac{16}{3}\frac{\sigma T^3}{\kappa\rho}\frac{2\Delta T}{d} = -\frac{16}{3}\frac{\sigma T^4}{\kappa\rho H_P}\frac{l_{\rm m}}{d}\left(\nabla^* - \nabla_{\rm p}\right) \tag{5.72}$$

The flux is the radiative losses per unit area so we need to multiply it with the surface of the parcel S in order to get the total energy loss by the parcel. Since the second term of Eq. 5.71 is the difference in flux of the gas and the gas only can move up, then that term needs to be multiplied with the surface

Q normal to the velocity v. By multiplying with those geometric factors we get:

$$-\frac{16}{3}\frac{\sigma T^4}{\kappa\rho H_P}\frac{Sl_{\mathrm{m}}}{d}(\nabla^* - \nabla_{\mathrm{p}}) = Q\rho\, c_P\, v\frac{Tl_{\mathrm{m}}}{2H_P}(\nabla_{\mathrm{ad}} - \nabla_{\mathrm{p}}) \tag{5.73}$$

$$(\nabla_{\mathrm{p}} - \nabla_{\mathrm{ad}}) = \frac{32\sigma T^3}{3\kappa\rho^2 c_P v}\frac{S}{Qd}(\nabla^* - \nabla_{\mathrm{p}}) \tag{5.74}$$

We now finally need to figure out how fast the parcel rises so that we can find the convective flux.

Exercise 5.10:
Calculate the geometric factor $S/(Qd)$ for a spherical parcel with radius r_p

The convective velocity

After having found the temperature difference between the convective parcel and its surroundings, we need to find its rise-velocity in order to be able to use Eq. 5.65.

If we look at a buoyant parcel, then the buoyancy force working on it is just the gravitational force working on the difference in density:

$$f_{\mathrm{buo}} = -g\frac{\Delta\rho}{\rho} \tag{5.75}$$

and taking into account that the expression is per unit mass, so we need to divide by the density. The work the buoyancy force produces is just the buoyancy force times the distance over which it works. The parcel we are looking at cannot turn all the work it is subjected to into acceleration, since it also needs to push gas in front of it away while it rises. If we assume that half the work produced by the buoyancy force is used to accelerate the parcel and since we already have assumed that it moves a distance δr, then

$$^1\!/_2\, f_{\mathrm{buo}}\delta r = -\frac{g}{2}\frac{\Delta\rho}{\rho}\delta r = \frac{g\delta}{2}\frac{\Delta T}{T}\delta r = \frac{g\delta}{2T}\frac{T\,(\delta r)}{H_P}(\nabla^* - \nabla_{\mathrm{p}})\,(\delta r) \tag{5.76}$$

$$= \frac{g\delta}{2H_P}(\nabla^* - \nabla_{\mathrm{p}})\,(\delta r)^2 \tag{5.77}$$

where Eq. 5.20 and the fact that there is no change in pressure compared to the outside ($\mathrm{d}P = 0$) has been used. That amount of work is then equal to the kinetic energy of the parcel (remember again that all of these expressions are per unit mass):

$$\begin{aligned} {}^{1}\!/_{2}\, v^2 &= \frac{g\delta}{2H_P} \left(\nabla^* - \nabla_{\rm p}\right) (\delta r)^2 & (5.78) \\ v &= \sqrt{\frac{g\delta}{H_P}} \left(\nabla^* - \nabla_{\rm p}\right)^{1/2} (\delta r) \\ &= \sqrt{\frac{g\delta l_{\rm m}^2}{4H_P}} \left(\nabla^* - \nabla_{\rm p}\right)^{1/2} & (5.79) \end{aligned}$$

We can insert that into Eq. 5.74 to get

$$\begin{aligned} \left(\nabla_{\rm p} - \nabla_{\rm ad}\right) &= \frac{32\sigma T^3}{3\kappa\rho^2\, c_P} \sqrt{\frac{4H_P}{g\delta}} \left(\frac{S}{Ql_{\rm m}d}\right) \left(\nabla^* - \nabla_{\rm p}\right)^{1/2} \\ &= \frac{64\sigma T^3}{3\kappa\rho^2\, c_P} \sqrt{\frac{H_P}{g\delta}} \left(\frac{S}{Ql_{\rm m}d}\right) \left(\nabla^* - \nabla_{\rm p}\right)^{1/2} & (5.80) \end{aligned}$$

so finally inserting into Eq. 5.70, we get

$$\begin{aligned} F_{\rm con} &= \rho\, c_P \sqrt{\frac{g\delta}{H_P}} \frac{l_{\rm m}}{2} \left(\nabla^* - \nabla_{\rm p}\right) \frac{Tl_{\rm m}}{2H_P} \left(\nabla^* - \nabla_{\rm p}\right)^{1/2} \\ &= \rho\, c_P T \sqrt{g\delta} H_P^{-\frac{3}{2}} \left(\frac{l_{\rm m}}{2}\right)^2 \left(\nabla^* - \nabla_{\rm p}\right)^{\frac{3}{2}} & (5.81) \end{aligned}$$

So now we can find the energy flux and through that the temperature gradient our star has in the convection zone by using that the total flux is

$$F_{\rm rad} + F_{\rm con} = \frac{16\sigma T^4}{3\kappa\rho H_P} \nabla_{\rm stable} \tag{5.82}$$

where $\nabla_{\rm stable}$ is the temperature gradient needed for all the energy to be carried by radiation. In reality the radiative flux is

$$F_{\rm rad} = \frac{16\sigma T^4}{3\kappa\rho H_P} \nabla^* \tag{5.83}$$

where the ∇^* now is the actual temperature gradient. These two equations combined with Eq. 5.74 and Eq. 5.81 the temperature gradient of the star can be found.

So far we have encountered four different temperature gradients: $\nabla_{\rm p}$, $\nabla_{\rm stable}$, $\nabla_{\rm ad}$ and ∇^*. The relations between these gradients have been quite involved, but we can say something about their relative sizes. Pressure and temperature both decrease with radius so the gradients are all positive. If we furthermore look at a convectively unstable layer, then we already know from Eq. 5.61 that $\nabla^* > \nabla_{ad}$. Since for a convective gas, some of the energy is now moved through convection, the temperature gradient does not need to be as steep as in the case where all the energy has to be moved by radiation, so $\nabla^* < \nabla_{\rm stable}$. For a rising parcel of gas we know that it cools both by adiabatic expansion and by radiative losses, so the temperature must decrease faster than for a parcel which only cools by adiabatic expansion alone, i.e. $\nabla_{\rm p} > \nabla_{\rm ad}$. Finally we know that for a rising parcel, the temperature has to decrease slower than the surroundings, otherwise it would become unable to rise very quickly, so $\nabla_{\rm p} < \nabla^*$. Combining these we get

$$\nabla_{\rm stable} > \nabla^* > \nabla_p > \nabla_{ad} \tag{5.84}$$

Exercise 5.11:
Insert Eq. 5.81 and Eq. 5.83 into Eq. 5.82 to get an expression that contains an expression for $(\nabla^* - \nabla_p)$ as a function of ∇^* and $\nabla_{\rm stable}$.
Hint: If you can insert the expression for the pressure scale height(you found it in Exerc. 5.9) and end up with only known quantities for the gas, i.e. $\sigma, T, \kappa, \rho, c_P, g$ and δ in the expression, as well as the ∇'s.

Exercise 5.12:
Insert Eq. 5.74 into

$$(\nabla_p - \nabla_{ad}) = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p)$$

and use Eq. 5.79, to get a second order equation for $(\nabla^* - \nabla_p)^{1/2}$. Express the solution ξ to the resulting equation as a function of ∇^*, ∇_{ad}, U and the geometrical constant in front of U in the equation above, where

$$U = \frac{64\sigma T^3}{3\kappa\rho^2 c_P}\sqrt{\frac{H_P}{g\delta}}$$

Argue why this equation has only one viable solution.

Exercise 5.13:

Use your solution to Exerc. 5.12 into your solution of Exerc. 5.11 to eliminate ∇^* except in the form of ξ used in Exerc. 5.12.

There is still one free parameter left, which we have not determined: the mixing length l_m. One can give physical arguments for one value or another, but all in all, it is a free parameter that can be tuned to any value that suits the application. Generally it is assumed that

$$l_\mathrm{m} = \alpha H_P \tag{5.85}$$

where α is a parameter between $1/2$ and 2, again this is a free parameter, but at least slightly better constrained than l_m. That is not satisfying in the strictest sense, but a complete treatment of convection takes full compressible hydrodynamics simulations for some cases even magneto-hydrodynamic simulations, which are computationally very expensive to complete. The theory behind hydrodynamics and magneto-hydrodynamics will be treated later in these notes.

Example 5.1:

Inserting values for the middle of the convection zone, assuming that the gas is ideal and using the following values:

$$\begin{aligned}
\mu &= 0.6 \\
T &= 0.9\ 10^6\,\mathrm{K} \\
\rho &= 55.9\,\mathrm{kg\,m^{-3}} \\
R &= 0.84\,R_\odot \\
M(R) &= 0.99\,M_\odot \\
\kappa &= 3.98\,\mathrm{m^2\,kg^{-1}} \\
\alpha &= 1
\end{aligned}$$

then for purely radiative transport the following values can be found

$$\begin{aligned}
\nabla_\mathrm{stable} &= 3.26 \\
\nabla_\mathrm{ad} &= 2/5
\end{aligned}$$

so the gas is unstable, so the value for ∇_stable (the temperature gradient which would be necessary for all the energy to be carried by radiation) cannot be used. Calculating the convective and radiative energy flux ratios then gives:

$$
\begin{aligned}
H_P &= 32.5\,\mathrm{Mm} \\
U &= 5.94 \cdot 10^5 \\
\xi &= 1.175 \cdot 10^{-3} \\
\nabla^* &= 0.400 \\
v &= 65.62\,\mathrm{m\,s^{-1}} \\
\frac{F_{\mathrm{con}}}{F_{\mathrm{con}} + F_{\mathrm{rad}}} &= 0.88 \\
\frac{F_{\mathrm{rad}}}{F_{\mathrm{con}} + F_{\mathrm{rad}}} &= 0.12
\end{aligned}
$$

A further check you should make is that Eq. 5.84 hold. In this case I have:

$$
\begin{aligned}
\nabla_{\mathrm{ad}} < \nabla_{\mathrm{p}} &< \nabla^* < \nabla_{\mathrm{stable}} \\
\tfrac{2}{5} < 0.400000 &< 0.400001 < 3.26078
\end{aligned}
$$

In a physical sense we cannot write up this sort of expression, with such a precision, since most of the constants we use are not this precise, but in this case we use it only for internal consistency and as a mathematical check.

Chapter 6

Hydrodynamics

6.1 Introduction

In this chapter we will try to understand how convection actually works, making much fewer assumptions than in Chap. 5. That will also make it possible for us to see how good the approximation we have used actually is. To do that we will need to understand hydrodynamics. Hydrodynamics is the science of fluids. It describes the movements of fluids under the influence of external forces. Fluid is to be understood in the broadest terms, including gasses and as we shall see ensembles of particles that do not initially makes us think of fluids.

The hydrodynamic equations consist of only a few seemingly simple equations that hide a possibly very complex behavior. Especially one term in the equation of momentum gives rise to the complex behavior, which involves the transfer of momentum when a fluid package is stressed. The stress tensor term will be dealt with in detail to give a sense of the behavior to expect from that term under different conditions.

The fluid equations can be derived completely from statistical mechanics, but that is outside the scope of these notes. We will refer to more complete works on statistical mechanics instead, of which there are several. Sometimes results from statistical mechanics will be used, without giving the full derivation of the equations, but a short explanation of the equation will be given, that hopefully will make the equations look sensible.

6.2 Definition of a fluid

Hydrodynamics is a simplification of the full particle treatment of a large ensemble of particles. In general the main requirement of a medium to qualify as a fluid is that it can be split into volume elements which can be represented by a continuous function ρ defined as

$$\rho\, dV = \sum_{dV} m \tag{6.1}$$

with ρ being defined as the density of the fluid, dV is a small volume element and m is the mass of the individual particles that make up the fluid. If the mean free path of the particles are smaller than the typical diameter of the volume dV it is possible to create a meaningful mean velocity and energy of the particles inside dV, and also define ρ as done above. If this volume element is smaller than the relevant scales we are interested in the medium can be well approximated by hydrodynamics . If that is not true, then the medium we are looking at cannot be treated as a fluid. Another definition of a fluid involves the Knudsen[1] number:

$$Kn = \frac{\lambda}{L} \tag{6.2}$$

where λ is the mean free length between collisions for the particles in the fluid, while L is the typical scale of the system. The Knudsen number must be smaller than one (in general much smaller) for a fluid description to be valid. The Knudsen number is usually the number that is used since Eq. 6.1 is not as easily used in day to day work.

An example of a "fluid" is a large number of balloons, incased in a sports stadium with an average distance between them of one meter, then the Knudsen number would be $Kn = 1/100$ so as long one didn't look at scales much smaller than the scale of the stadium, they would behave as a fluid.

For neutral particles, the mean free path is given exactly as for photons

$$\lambda = \frac{1}{n\sigma} \tag{6.3}$$

where here σ is the scattering cross section, which typically is $10^{-19}\,\mathrm{m}^2$ for atomic particles and molecules and n is the number density of the particles,

[1] Martin Knudsen(1871–1949) was an engineer working primary on molecular gas flow at low pressure

which for air at ground level is roughly $10^{25}\,\mathrm{m}^{-3}$, giving $\lambda \sim 10^{-6}\,\mathrm{m}$ in the air surrounding us. That means that as long as we do not look at scales larger than roughly a tenth of a millimeter the Knudsen number will be less than 0.01, and se we can treat the air as a fluid.

If we use the same numbers for the interior of the Sun, then σ is still roughly the same, while the number density at the bottom of the convective zone is roughly $10^{31}\,\mathrm{m}^{-3}$, so $\lambda \sim 10^{-12}\,\mathrm{m}$ making the fluid approximation very applicable.

At the surface of the Sun, in the photosphere, the number density is roughly $10^{23}\,\mathrm{m}^{-3}$, so that $\lambda \sim 10^{-4}\,\mathrm{m}$ and again we can safely treat all relevant scales at the solar surface as a fluid (the typical scale at the solar surface is $1\,\mathrm{Mm} = 10^{6}\,\mathrm{m}$).

Exercise 6.1:
In this exercise consider the Sun as a particle. The collisional cross section of the Sun is not given by only the solar radius, but a a much larger radius, due to the gravitational pull exerted by the Sun. Estimate the radius where the gravitational potential of the Sun has decreased by a factor 1/e compared to the gravitational potential at the solar surface and use that for calculating the collisional cross section. If our galaxy consisted of stars identical to our Sun with the same collisional cross section, would the movements of stars then be considered to be governed by hydrodynamics? Would galaxies in a galaxy cluster?
Hint: Find a rough number for the number of stars in the milky way and assume that the milkyway is a disc with a radius of 20 kpc and a thickness of 0.5 kpc. For the galaxy cluster, you can for example use the local group of galaxies and assume that all the galaxies in the local group are identical to the milky way and distributed homogeneously in a sphere. For the galaxies you do not need to assume that the collisional cross section is larger than the physical radius, and you can assume that the galaxies are spherical with the same radius as the milky way.

6.3 From particles to fluid

When being certain that the gas or liquid under investigation actually behaves like a fluid, it is time to look at the equations that govern the dynamics of a fluid, regardless of wether it is a collection of molecules, the water in our oceans or the molecular gas in our galaxy.

6.3.1 Hydrostatics

Hydrostatics is a sub branch of hydrodynamics that only deals with fluids at rest. There are a number of simplifications that can be made and the only equation is that of momentum, including all the external body forces working on the fluid:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla P + \mathcal{F} \tag{6.4}$$

where $\mathbf{u}$ is the velocity vector. In hydrostatics there are no velocities so this equation is further simplified by the left hand side being zero. So here is only the balancing of body forces working on a fluid parcel. Most often the main body force is gravitation, which just gives:

$$\nabla P = -\frac{G\rho m}{r^2} \tag{6.5}$$

where m is the mass inside the radius r. Hydrostatics are not really a part of hydrodynamics, which is quite clear from the names, but often it is the transition from a dynamic region to a static region which gives problems, as there usually are large gradients, and always instabilities evolving which can only be treated by a hydrodynamic treatment. Such a place is the transition from the radiative zone to the convective zone in the Sun. If modeling that region, it is necessary to include a region that theoretically would be static, but because there is a dynamic layer just above, instabilities and down flows penetrate into the static region – making it non-static.

6.3.2 The continuity equation

The continuity equation, which in most cases is refereed to as the equation of conservation of mass. It describes any effects that might add or remove mass from the volume being modeled. Often there are no such sinks or sources present and the equation turns into an equation describing the conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho \mathbf{u}}{\partial \mathbf{x}} = \mathcal{S} \tag{6.6}$$

where $\mathcal{S}$ is source or sink terms. In vector notation which we will use to a large degree the equation reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \mathcal{S} \tag{6.7}$$

By setting $\mathcal{S}$ to zero, doing the divergence of the second term and dividing by ρ, one can rewrite the equation to read

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u} \tag{6.8}$$

where a new operator has been defined:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{6.9}$$

which is called the substantial derivative, and describes the evolution at a point following the fluid, in contrast to the $\partial/\partial t$ which describes changes at a set location in space.

6.3.3 The equation of motion

The momentum equation of equation of motion is an equation we have already used in Chap. 5, and also used in Sect. 6.3.1, and describes the change in velocity the fluid experience when being the subject of external and internal forces. The equation contains a term that looks simple, but is the principle complication when trying to model a fluid. If we look at the equation coming directly from statistical mechanics, then for an ensemble of particles we have

$$\frac{\partial\left(\rho u_i\right)}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho\left\langle v_i v_k\right\rangle\right) - \mathcal{F} = 0 \tag{6.10}$$

where the subscripts defines the direction, v is the velocity of the individual particles and $\langle ... \rangle$ is the number weighted average of the particles. The second term of this equation is the momentum that pass through the boundaries of the volume we are looking at while $\mathcal{F}$ is the body forces working on the whole of the volume, as for instance gravity.

We now decompose the particle velocity in two parts

$$v_i = u_i + w_i \tag{6.11}$$

where w_i is the difference between the bulk velocity u_i and the particle velocity v_i. That way we can write

$$\left\langle v_i v_k\right\rangle = u_i u_k + \left\langle w_i w_k\right\rangle \tag{6.12}$$

by using that the average of all the particles individual velocities v_i should give the bulk velocity u_i, then it follows that $\langle w_i\rangle = 0$. Eq. 6.10 can now be rewritten

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_k}\left(\overbrace{\rho u_i u_k}^{\text{advection}} + \overbrace{\rho \langle w_i w_k \rangle}^{\text{stress tensor}}\right) - \mathcal{F} = 0 \tag{6.13}$$

If we look at the first part of second term on the left side of Eq. 6.13 (the term called "advection") and see what physics that term stands for, then the isolated x component reads ($i = x$):

$$\begin{aligned}
\frac{\partial}{\partial x_k}(\rho u_x u_k) &= \frac{\partial}{\partial x}(\rho u_x^2) + \frac{\partial}{\partial y}(\rho u_x u_y) + \frac{\partial}{\partial z}(\rho u_x u_z) \\
&= +\frac{\partial}{\partial x}(\rho u_x^2) + \rho u_x \frac{\partial u_y}{\partial y} + u_y \frac{\partial \rho u_x}{\partial y} \\
&\quad + \rho u_x \frac{\partial u_z}{\partial z} + u_z \frac{\partial \rho u_x}{\partial z} \\
&= +\rho u_x \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\
&\quad + \left(u_x \frac{\partial \rho u_x}{\partial x} + u_y \frac{\partial \rho u_x}{\partial y} + u_z \frac{\partial \rho u_x}{\partial z}\right)
\end{aligned} \tag{6.14}$$

The terms are quite easy to understand. The first term describes the change in the x directed momentum due to the difference in the velocities advecting momentum into and out of the volume. The second term is then the change in x-directed momentum due to a difference in the momentum advected through the walls of the volume.

Exercise 6.2:
Insert Eq. 6.11 into the term $\langle v_i v_j \rangle$ and show that the result is Eq. 6.12, by using that $\langle w_i \rangle = \langle w_j \rangle = 0$

We can now split up the second part of the second term of Eq. 6.13 (called the "stress tensor") into a diagonal and a non-diagonal part

$$\rho \langle w_i w_k \rangle = P\delta_{ik} - \tau_{ik} \tag{6.15}$$

where τ_{ik} is the viscous stress tensor and P is the diagonal part of the stress tensor $\langle w_i w_k \rangle$ and is the well known thermal pressure and is given by

$$P = \tfrac{1}{3}\rho \left\langle |w^2| \right\rangle \tag{6.16}$$

where the expression $\langle |w| \rangle$ means the norm of the trace of the tensor $\langle w_i w_k \rangle$. While the terms in the viscous stress tensor are

$$\tau_{ik} = \rho \left\langle \tfrac{1}{3} |w^2| \delta_{ik} - w_i w_k \right\rangle \tag{6.17}$$

and these terms are the ones creating many of the interesting and complex behaviors seen in fluid dynamics. The whole of the momentum equation now becomes

$$\frac{\partial\left(\rho u_i\right)}{\partial t}=-\frac{\partial}{\partial x_k}\left(\rho u_i u_k+P\delta_{ik}-\tau_{ik}\right)+\mathcal{F} \tag{6.18}$$

This equation can look a bit intimidating at first, so lets look at the one of the symmetric components: where $i=x$ and $k=x$

$$\begin{aligned}\frac{\partial\rho u_x}{\partial t} &= -\frac{\partial}{\partial x}\left(\rho u_x^2+P-\tau_{xx}\right)+\mathcal{F}\\ &= -\frac{\partial}{\partial x}\left(\rho u_x^2+\tfrac{1}{3}\rho\left\langle|w^2|\right\rangle-\rho\left\langle\tfrac{1}{3}|w^2|-w_x^2\right\rangle\right)+\mathcal{F}\end{aligned} \tag{6.19}$$

If the average deviation from the main flow is the same in all directions (which it should be according to the definition of w_i) then

$$\left\langle w_x^2\right\rangle=\left\langle w_y^2\right\rangle=\left\langle w_z^2\right\rangle \tag{6.20}$$

and so Eq. 6.19 then just becomes

$$\begin{aligned}\frac{\partial\rho u_x}{\partial t} &= -\frac{\partial}{\partial x}\left(\rho u_x^2+P-\rho\left\langle\tfrac{1}{3}|w^2|-w_x^2\right\rangle\right)+\mathcal{F}\\ &= -\frac{\partial}{\partial x}\left(\rho u_x^2+P-\rho\left\langle\tfrac{1}{3}|w^2|-\tfrac{1}{3}|w^2|\right\rangle\right)+\mathcal{F}\\ &= -\frac{\partial}{\partial x}\left(\rho u_x^2+P\right)+\mathcal{F}\end{aligned} \tag{6.21}$$

in other words, there is only the advection of the momentum and the well known pressure to take care of. The asymmetric part is easier understood if we move away from the particle picture and try to understand the physical effect of the viscous stress tensor. The asymmetric stress tensor clearly transfers momentum from one direction to another. If we transfer coordinate system, the transfer of momentum should not change, so the stress tensor must be symmetric. Physically we know intuitively that a fluid moving at constant speed with no gradients in the velocity, should not transfer momentum to other directions, i.e. a flow which is along x should not suddenly start flowing along y or z just due to internal friction. So the viscous stress tensor clearly cannot have terms depending on the velocity, because that would lead to exactly such an effect. So we are left with terms containing derivatives of the velocity, which are symmetric. Another constraint comes from the imagning a fluid rotating like a solid body, there should again be no internal friction because a fluid element sees no gradients in velocity, even though there are gradients in a coordinate system that does not rotate with the fluid. We are finally left with terms that look like

$$\langle w_i w_k \rangle = \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \tag{6.22}$$

because these are symmetric and they provide no non-constant terms for rotations of the form

$$\mathbf{\Omega} \times \mathbf{r} \tag{6.23}$$

Exercise 6.3:
Show that the form of the tensor $\langle w_i w_k \rangle$ gives no non-constant terms for a solidly rotating cylinder. Begin by using cylindrical coordinates and remember that the calculating gradients in cylindrical coordinates introduces some extra factors in the expressions. Write down the expressions for u_ω and u_r for a solid rotating cylinder aligned with the z axis(i.e. no gradients in along z). Assume that the cylinder is infinitely wide in the r-ω plane. Then calculate the asymmetric terms in τ_{ik} and see if the contribution to the momentum equation is zero.

If we look at what physical effect terms like these provide in the equation for the momentum, then lets look at the case where $i = x$, and k takes all the directions possible, and there are no body-forces:

$$\begin{aligned}
\frac{\partial \rho u_x}{\partial t} \quad = \quad & -\rho u_x \frac{\partial u_x}{\partial x} - u_x \frac{\partial \rho u_x}{\partial x} - \frac{\partial P}{\partial x} \\
& -\rho u_x \frac{\partial u_y}{\partial y} - u_y \frac{\partial \rho u_x}{\partial y} - \frac{\partial}{\partial y}\left(-\eta\rho \frac{\partial u_x}{\partial y} - \eta\rho \frac{\partial u_y}{\partial x} \right) \\
& -\rho u_x \frac{\partial u_z}{\partial z} - u_z \frac{\partial \rho u_x}{\partial z} - \frac{\partial}{\partial z}\left(-\eta\rho \frac{\partial u_x}{\partial z} - \eta\rho \frac{\partial u_z}{\partial x} \right)
\end{aligned} \tag{6.24}$$

This equation seems again a bit intimidating, but lets look at the physics. We already know that the two first terms on each line represents x-directed momentum being advected into the volume we are looking at due to either a gradient in the x-momentum, or a gradient in the advecting velocity. The last term in the first line is also easy to understand, it's just the thermal pressure, so we are left with the last term on the second and third line. Lets look at the second line. The terms are double gradients, so the terms deal with a gradient of a gradient, which is not always easy to picture. In order

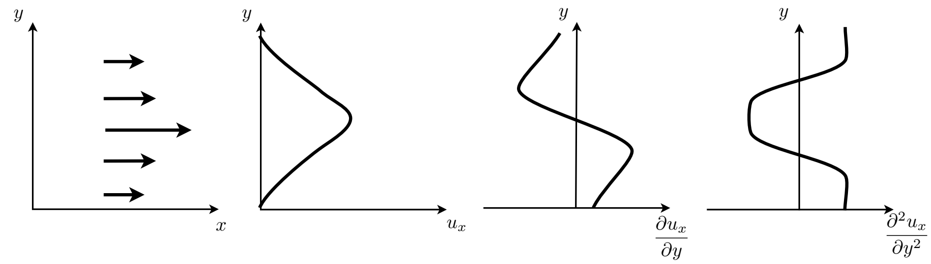

Figure 6.1: The effect of the first term of Eq. 6.25, which have the effect of smoothing out gradients in momentum

not to complicate the picture too much, assume that the density is constant (a very gross simplification for astrophysics, but is the standard assumption in meteorology) and assume further that there are only two dimensions x and y. There are now two separate terms we need to look at:

$$\frac{\partial^2 u_x}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u_y}{\partial x \partial y} \tag{6.25}$$

The terms can best be understood if we look at them separately. The first term is a double gradient in the y–direction, so lets imagine a flow which is directed along the x–axis and symmetric around $y = 0$ with a maximum at $y = 0$ (see Fig. 6.1). The internal friction in the fluid leads to the fast moving fluid at $y = 0$ is decelerated while the slower moving fluid next to it is being accelerated. In the end this effect by itself will lead to all the fluid moving at the same speed.

The second term of Eq. 6.25 is more complex. Here we are dealing with a double gradient in two different directions. Fig. 6.2 shows a flow with gradients in two directions. The effect of this term is now easier to understand, and partially comes because of mass conservation, but it is due to internal friction. The term makes the beam become narrower and narrower, and will continue to do so until other processes take over. The simple example of Fig. 6.2 is of course not realistic for very long if the mass density is to be constant, since the inner parts of the flow will be more and more tenuous because of the acceleration.

If we had not assumed ρ to be constant, then we would have another set of terms in the viscous stress tensor which deals with the gradients in both density and velocity. Those terms will not be treated here, but they are taking care of the number of collisions between particles in the flow directions. Imagine for instance that while looking at the first of the terms

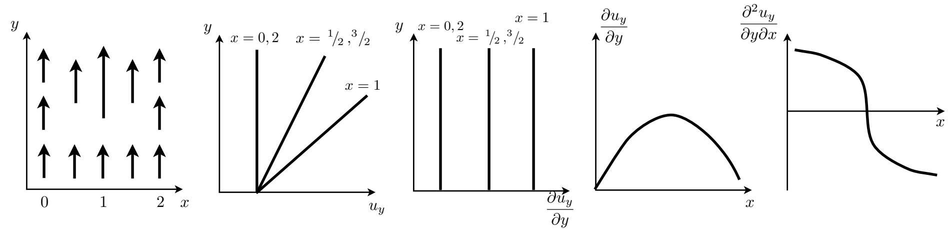

Figure 6.2: The effect of the second term of Eq. 6.25, which makes an accelerating narrow beam of fluid to become even narrower

here, but that the density also varied such that the density was low at low y and high at high y. There would be fewer collisions at low y than at high y, so the flows ability to feel the internal friction would be lower at low y than at high y, and consequently that would affect the internal friction in the fluid.

Exercise 6.4:
So far we have assumed the density to be constant. That removes some of the terms in the expression

$$\begin{aligned} \frac{\partial \rho u_x}{\partial t} &= \quad \ldots \quad -\frac{\partial}{\partial y}\left(-\eta\rho\frac{\partial u_x}{\partial y} - \eta\rho\frac{\partial u_y}{\partial x}\right) \\ &= \qquad -\frac{\partial}{\partial z}\left(-\eta\rho\frac{\partial u_x}{\partial z} - \eta\rho\frac{\partial u_z}{\partial x}\right) \end{aligned}$$

from Eq. 6.24. We have so far assumed that the density to be constant but now assume that the density is not constant. By using the chain rule this produces the the terms we have just looked at and but also terms that includes the gradient of the density. Give an example of when these terms are non zero and explain what the physical effect these terms have.

6.3.4 Energy

Internal energy is usually used in hydrodynamics. It behaves almost like mass, it can be advected with the gas motions and it is also a conserved quantity. That is the primary reason we use internal energy instead of temperature or pressure, even though these are connected through the equation of state. Internal energy is not exactly the same as mass, since internal energy has sources and sinks, which mass usually does not have. Examples of sinks could be the energy loss through radiation or conduction, and an example of an energy source could be the energy production of fusion. The internal energy equation therefore looks almost identical to the mass conservation equation:

$$\frac{\partial e}{\partial t} + \frac{\partial e\mathbf{u}}{\partial x} = -P\frac{\partial \mathbf{u}}{\partial x} + \mathcal{E} \tag{6.26}$$

where e is the internal energy per unit mass, P is the thermal pressure of the fluid, and $\mathcal{E}$ is any sources and sinks of energy. The first term on the right hand side represents the effect of work compressing the gas will increase the internal energy of the gas, while expanding the gas will decrease the internal energy, because the energy to do the work comes from the gas itself. This is the effect that is used in a fridge, where air is made to expand, cooling it down and then sent into the fridge. But since the air that are supposed to go into the fridge expands, something else has to compress, and that air is vented out into the surroundings, heating up the room the fridge is placed in. $\mathcal{E}$ represents the extra terms that provide or remove energy from the gas, including heat conduction, energy transport by radiation etc as already mentioned.

6.4 Waves in hydrodynamics

Waves are always grouped according to what their restoring force/forces is/are. There are in principle only one type of waves in pure hydrodynamics. If more body forces are introduced in Eq. 6.4, then more types of waves can be produced, but here only the pure wave-modes will be explained. In the case that the fluid is almost at rest and in equilibrium, then we make the assumption that any changes in the pressure and density are only small: $\rho = \rho_0 + \rho'$ and $P = P_0 + P'$ where the subscript 0 means an unchanging equilibrium value, while the superscript $'$ means a small perturbation on top of that. Since we are looking at small perturbations, it is also safe to assume that the velocity is small, so when looking at the continuity equation(Eq. 6.6) in one dimension it can be written:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\frac{\partial \rho u_x}{\partial x} \\ \frac{\partial \rho_0 + \rho'}{\partial t} &= -\frac{\partial\left(\rho_0 + \rho'\right) u_x}{\partial x} \qquad (6.27) \\ \frac{\partial \rho'}{\partial t} &= -\rho_0 \frac{\partial u_x}{\partial x} \qquad (6.28) \end{aligned}$$

where the step from Eq. 6.27 to Eq. 6.28 is true to first order. The argument is that terms that containing two small numbers multiplied with each other are so small compared with the others, that can be ignored. The numbers with subscript 0 are constants so can be moved outside the differentials. The same can be done for the equation of motion, giving

$$\begin{aligned} \frac{\partial \rho u_x}{\partial t} &= -\frac{\partial}{\partial x}\left[\rho u_x^2 + P\right] \\ \rho_0 \frac{\partial u_x}{\partial t} &= -\frac{\partial P'}{\partial x} \end{aligned} \tag{6.29}$$

Exercise 6.5:
Make the same assumptions made to reach Eq. 6.28 to prove Eq. 6.29

We are now left with three unknowns ρ',P' and u_x, and since we only have two equations we have to get rid of one of them. For small perturbations in an ideal fluid, we can assume that a small perturbation in the pressure is due to a small perturbation in the density at constant entropy. Writing that as

$$P' = \left(\frac{\partial P_0}{\partial \rho_0}\right)_s \rho' \tag{6.30}$$

we can insert Eq. 6.30 into Eq. 6.28 to get

$$\begin{aligned} \frac{\partial}{\partial t}\left[P'\left(\frac{\partial \rho_0}{\partial P_0}\right)_s\right] &= -\rho_0 \frac{\partial u_x}{\partial x} \\ \left(\frac{\partial \rho_0}{\partial P_0}\right)_s \frac{\partial P'}{\partial t} + P' \frac{\partial}{\partial t}\left(\frac{\partial \rho_0}{\partial P_0}\right)_s &= -\rho_0 \frac{\partial u_x}{\partial x} \end{aligned} \tag{6.31}$$

$$\frac{\partial P'}{\partial t} = -\rho_0 \left(\frac{\partial P_0}{\partial \rho_0}\right)_s \frac{\partial u_x}{\partial x} \tag{6.32}$$

where the step from Eq. 6.31 to Eq. 6.32 is due to the the term $\frac{\partial \rho_0}{\partial P_0}$ is set by the equation of state, which is not dependent on time, but only a function of the gas composition. We now have two equations with two unknowns: P' and ρ'. If we introduce the velocity potential $u_x = \frac{\partial \phi}{\partial x}$, then Eq. 6.29 becomes

$$P' = -\rho_0 \frac{\partial \phi}{\partial t} \tag{6.33}$$

Exercise 6.6:
Show that Eq. 6.29 becomes Eq. 6.33 when introducing a velocity potential.

Inserting Eq. 6.33 into Eq. 6.32 it gives

$$\frac{\partial^2 \phi}{\partial t^2} - \left(\frac{\partial P_0}{\partial \rho_0}\right)_s \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{6.34}$$

Equations of this form is called a wave equation, where

$$c_s^2 = \left(\frac{\partial P_0}{\partial \rho_0}\right)_s \tag{6.35}$$

where c_s is the wave speed. Looking back to Chap. 5.3.2 we find that for an ideal gas

$$c_s^2 = \left(\frac{\partial P_0}{\partial \rho_0}\right)_s = \gamma \frac{P}{\rho} \tag{6.36}$$

If we substitute $\eta = x + c_s t$ and $\xi = x - c_s t$, then Eq. 6.34 can be rewritten

$$\frac{\partial^2 \phi}{\partial t^2} - c_s^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{6.37}$$

$$\frac{\partial^2 \phi}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial t} \frac{\partial \xi}{\partial t} - c_s^2 \frac{\partial^2 \phi}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial x} = 0 \tag{6.38}$$

$$\frac{\partial^2 \phi}{\partial \eta \partial \xi} c_s \left(-c_s\right) - c_s^2 \frac{\partial^2 \phi}{\partial \eta \partial \xi} = 0 \tag{6.39}$$

$$\frac{\partial^2 \phi}{\partial \eta \partial \xi} = 0 \tag{6.40}$$

Integrating this with respect to ξ gives $\partial \phi / \partial \eta = F(\eta)$ where $F(\eta)$ is an arbitrary function of η. Integrating again , this time with respect to η we get $\phi = f_1(\eta) + f_2(\xi)$ where now f_1 and f_2 are arbitrary functions of their arguments. Writing this in full we simply have

$$\phi = f_1(x + c_s t) + f_2(x - c_s t) \tag{6.41}$$

and if we continue in the same way for density and pressure, we get expressions of the same form. For instance lets look at the density $\rho' = f_1(x + c_s t) + f_2(x - c_s t)$ and lets assume that $f_2 = 0$. Then $\rho' = f_1(x + c_s t)$, and so the density is the same as long as $x + c_s t$ is constant. So assume that the density has a certain value at a specific point in space x at $t = 0$, then at a later time, the density will have the same value at a point which is shifted along x by $c_s t$. As this is true for all positions, then it means that the whole profile of the density is just shifted in space by $c_s t$. As the only force that can drive this motion is pressure, the waves described by Eq. 6.34 is a pressure wave, which is also called a sound wave, and the pressure/density wave moves with the speed of sound c_s.

Exercise 6.7:
Calculate the speed of sound in the solar photosphere, and the solar corona, assuming the gas there to be an ideal gas. (Use the physical values you need from the appendix.)

Chapter 7

Magnetohydrodynamics

7.1 Introduction

Magneto hydrodynamics is an extension to hydrodynamics. Magneto hydrodynamics is the physics of electrically conducting fluids. As it turns out, the equations governing the electric and magnetic fields in such a fluid can be greatly simplified if the fluid is highly conductive, the fluid consists of particles which collide often and finally that there is on average as many negative as positive charges in the fluid. MHD is used almost everywhere in astrophysics, as magnetic fields are almost present everywhere. The magnetic field of the earth are known from everyday use when we look at a compass that point towards the magnetic north pole. The strength of the magnetic field at the earths surface is roughly 0.5 G $= 5 \cdot 10^{-5}$ T. The solar surface has concentrations of magnetic fields in sun spots with magnetic field strengths of roughly $2 \cdot 10^3$ G $= 0.2$ T or roughly 4000 times stronger. But the sun is a relatively quiescent star, while young stars generally have stronger magnetic fields. The most extreme magnetic fields are found around a subset of neutron stars with immensely strong magnetic fields of 10^{15} G called magnetars. As we shall see, all these astrophysical magnetic fields can completely dominate the dynamics of the gas and so MHD is necessary in order to understand the physics.

7.2 The assumption of MHD

MHD depends on a number of simplifications, just as HD does. The simplifications used in MHD are of electro-dynamic character. They come about because of the asymmetry between electric and magnetic fields. That asymmetry is easily understood because we have a large amount of electrically charged particles but we have so far not found any magnetic monopole particles. We also know that on large scales there seems to be no net electric charge, i.e. the universe seems to contain equal amounts of electrically positive and negatively charged particles. The consequence is that electric fields are on large scales rather small, while magnetic fields can easily be present. In almost all places in the universe, there will be a number of free electrons due to for instance UV radiation[1]. Even the almost neutral solar photosphere has enough free electrons from carbon and the alkali metals, to cancel the build up of any electric field.

A gas which contains a number of charge carriers large enough to affect the gas dynamics is called a plasma, and conditions for when a gas becomes a plasma is considered later. In this beginning phase it is enough to consider a gas where all ions have lost all of their electrons, so that all ions have a positive charge equal to their atomic number and that their free electrons are still in the neighbourhood. That implies that

$$\sum_i \mathcal{Z}_i n_i = n_e \tag{7.1}$$

where $\mathcal{Z}_i$ is the atomic charge number, n_i is the number density of that atom and n_e is the number density of electrons. If we now look at one specific ion then it will set up an electro static potential that is

$$\nabla^2\phi = -\frac{1}{\epsilon_0}\rho_e \tag{7.2}$$

where ρ_e is the *charge density* distribution which includes both the ion at the centre and the surrounding electrons and ions. The charge density will be given by

$$\rho_e = \mathcal{Z}_i e\delta(r) - n_e e \exp\left(-e\phi/kT\right) + n_i \mathcal{Z}_i e \exp\left(\mathcal{Z}_i e\phi/kT\right) \tag{7.3}$$

which consist of the ion (first term), the electrons (second term) and the surrounding ions (third term). The term describing the ion has a delta function $\delta(r)$ which is infinte at $r = 0$ and zero otherwise, and when it is integrated it has the value one at $r = 0$. The electron number density n_e and ion number density n_i is the number densities in the absence of the central

[1]Exceptions are the internal regions of very cold interstellar clouds, which can be completely screened from UV radiation, and cold planetary atmospheres where lightening is the effect of the build up of electric field.

ion we are looking at. By using the power expansion for the exponential function:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{7.4}$$

and keeping only the two first terms, then Eq. 7.3 becomes

$$\begin{aligned}\rho_e &= \mathcal{Z}_i e\delta(r) - n_e e + n_i \mathcal{Z}_i e - \frac{n_e e^2 \phi}{kT} - \frac{n_i \mathcal{Z}_i^2 e^2 \phi}{kT} \\ &= \mathcal{Z}_i e\delta(r) - \frac{e^2}{kT}\left(n_e + n_i \mathcal{Z}_i^2\right)\phi \end{aligned}\tag{7.5}$$

and use it to write Eq. 7.2 as:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{\mathcal{Z}_i e}{\epsilon_0}\delta(r) + \frac{e^2}{\epsilon_0 kT}\left(n_e + n_i \mathcal{Z}_i^2\right)\phi \tag{7.6}$$

where we use a spherical coordinate system, because the electric field is only dependent on the radius. The last term is a constant times the electric potential ϕ, and that constant has the dimension of inverse length squared and is called the Debye length:

$$L_D^{-2} = \frac{e^2}{\epsilon_0 kT}\left(n_e + \mathcal{Z}_i^2 n_i\right) \tag{7.7}$$

Inserting that into Eq. 7.6 gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{\mathcal{Z}_i e}{\epsilon_0}\delta(r) + L_D^{-2}\phi \tag{7.8}$$

Exercise 7.1:
Show that the unit of the second term on the right hand side of Eq. 7.5, disregarding the electric potential, is indeed inverse length squared.

Exercise 7.2:
Deduce Eq. 7.8

The solution to Eq. 7.8 has to satisfy $\phi \to 0$ as $r \to \infty$ and $\phi \to \mathcal{Z}_i e/\epsilon_0 r$ when $r \to 0$. Since the solution also has to be symmetric in space, then a good guess is

$$\phi = \frac{\mathcal{Z}_i e}{\epsilon_0 r}\exp\left(-r/L_D\right) \tag{7.9}$$

Exercise 7.3:
Show that Eq. 7.9 is a solution to Eq. 7.8 for $r > 0$ and that it satisfies $\phi \to 0$ as $r \to \infty$ and $\phi \to \mathcal{Z}_i e/\epsilon_0 r$ when $r \to 0$

Eq. 7.9 has a very clear physical interpretation. A potential set up by an ion (or more for that matter) is screened out exponentially by the surrounding electrons and ions with a characteristic length that is the Debye length. In order for us to be able to make the same simplification as we did when deducing the HD equations, we must now require that there are many charge carriers within the Debye sphere, so we must require that

$$n_e L_D^3 >> 1 \tag{7.10}$$

So if this requirement is met, there will be no large electric fields present, but at the same time, we cannot treat the gas as a collective in the same sense as we did in the equations for fluid mechanics, if we do not also limit our interest to scales which are much longer than the Debye length. Otherwise the electric and magnetic fields would be a patch work of small electric fields made up of a collection of small electric field contributions from individual ions and electrons.

As mentioned we can assume that on larger scale the fluid is neutral, but on small scales we cannot completely ignore electric fields, because it is possible for the electrons to move relative to the ions. Such movements would not violate the large scale neutrality, but would create electric currents. Such currents create magnetic fields as Ørsteds[2] famous experiment involving an electric current in a wire and compass, showing the direct connection between electric currents and magnetic fields. For particles moving in magnetic fields, they move in spirals – they gyrate around the magnetic field. The particles gyrate with a frequency called the Larmor frequency or cyclotron frequency

$$\omega_L = \frac{qB}{mc} \tag{7.11}$$

where q is the particle charge, B is the strength of the magnetic field, m is the particle mass and c is the speed of light. The exact Larmor frequency has an additional factor which of quantum mechanical nature, so Eq. 7.11 is for electrons which is the important one in this context. For a proton the gyro frequency is $\sim 0.01\,\mathrm{s}^{-1}$ around a magnetic field with the magnetic field strength of $10^{-9}\,\mathrm{T}$ (which is the typical value in interstellar clouds).

[2]Hans Christian Ørsted (1977-1851) was a famous danish scientist who investigated electricity, magnetism and acoustics, making the first experiment that showed a direct connection between electricity and magnetism

Exercise 7.4:
Calculate the Larmor frequency for both electrons and protons in a solar sunspot, where the magnetic field strength is $\sim 0.1\,\mathrm{T}$. The typical distance in the solar atmosphere is $\sim 1\,\mathrm{Mm}$ (the typical size of structures in the solar atmosphere). What velocity is needed in order for the gyroradius $a = v_\perp/\omega_L$ to be comparable to that size?

Imagine now that we have a plasma inside a container, and that we now displace all the electrons to one side by a distance dx. Then the electric field we set up as a result of that displacement is

$$E = \frac{n_e e}{\epsilon_0} dx \tag{7.12}$$

which is the same electric field as the one between two capacitor plates because at one side of the container, there will be many electrons while at the other end of the container, there will be only ions. The equation of motion for an electron inside the container will now be

$$\frac{\partial v}{\partial t} = -\frac{eE}{m_e} = -\frac{n_e e^2}{\epsilon_0 m_e} dx = -\omega_{pe}^2 dx \tag{7.13}$$

where ω_{pe} is the electron *plasma frequency*, because Eq. 7.13 is the equation for a harmonic oscillator. So by displacing some of the charge carriers(electrons or ions), will make the other charge carriers oscillate around their position with a frequency which is the plasma frequency.

So far we have not included thermal motions – we have in principle assumed to plasma to be very cold. Introducing thermal motions we can see that the plasma frequency might not be all that interesting in case the thermal motions completely cancel out the displacement dx assumed in Eq. 7.12, by fast movements. That effect is called Landau damping and simply means that if the charge-separation distance dx is generally much smaller than the distance the particles travel due to their thermal motions, then the charge separation cannot keep waves alive. The electric field that the charge separation produces effectively disappears if the movement that produces the charge separation is much smaller than the thermal velocities. That means that if we look at plasma with high thermal velocities it is not possible to have charge separation.

Exercise 7.5:
Calculate the typical thermal velocities of electrons in the solar corona. Within one plasma frequency the electrons move from $+dx$ to $-dx$ and back compared to the ions. Assume that $dx = L_D$ and compare the

distance the electrons have moved to the distance the electrons move due to their thermal motions within the same amount time. Is charge separation important in the solar corona? What about the solar photosphere?
Use the compendium to get the solar parameters you need to complete this exercise

7.3 The MHD equations

The MHD equations includes the HD equations, but now also includes a number of equations that describes the evolution of the electric and magnetic field. The foundation for those equation are Maxwells equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0} \tag{7.14}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.15}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{7.16}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_\mathrm{e} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{7.17}$$

where ρ_e and j_e is the electric charge and current densities, while $\mathbf{E}$ and $\mathbf{B}$ is the electric and magnetic vector fields. The two coefficients μ_0 and ϵ_0 are the permeability of vacuum and the permittivity of free space respectively. They represent the resistance to the inductance of a magnetic field and the resistance encountered when forming an electric field. As the magnetic and electric field are intimately connected, so are these two fundamental parameters:

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{7.18}$$

where c is the speed of light.

By taking the divergence of Amperes law (Eq. 7.17) and inserting the time derivative of Coulombs law (Eq. 7.14) and remembering that the divergence of the rotation of a vector field is always zero give the equation of charge conservation:

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{j}_\mathrm{e} = 0 \tag{7.19}$$

To get to the MHD equations we assume a number of things. First of all we assume that time-variations are slow compared to the light travel time (the factor in front of the time derivative of the electric field is actually

$1/c^2$ – much smaller than the other term), which allows us to ignore the *displacement current* in Eq. 7.17 so that it is just

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_\mathrm{e} \tag{7.20}$$

That assumption that time evolution is slow could also lead us to believe that Eq. 7.15 could be treated in the same way, but that is not the case because if that was the case then there is nothing left to compare $\nabla \times \mathbf{E}$ with. Another argument is that we know that the only way to create electric fields is to create charge separation, while magnetic fields are created by currents which can easily be very large. Eq. 7.20 states that magnetic fields are only created by electric currents. We can try to estimate how small drift velocities that can still produce magnetic fields as we observe them in our universe. As we have previously assumed, then the total positive and negative charges cancel out over large scales:

$$\rho_e = \sum_i \mathcal{Z}_i e n_i - e n_e = 0 \tag{7.21}$$

where $\mathcal{Z}_i$ is the atomic proton number for the ions with number density n_i, and we have now simplified everything to only have one species of atoms present and assuming that these atoms are completely ionized. As we have already seen the charge neutrality does not mean that there can be no currents, it just requires a small difference in velocity between the ions and the electrons:

$$\mathbf{j}_\mathrm{e} = \mathcal{Z}_i e n_i \mathbf{u}_i - e n_e \mathbf{u}_e = -e n_e \mathbf{u}' \tag{7.22}$$

where $\mathbf{u}_i$ and $\mathbf{u}_e$ is the velocity of the ions and electrons respectively and $\mathbf{u}' = \mathbf{u}_e - \mathbf{u}_i$ is the difference in velocity between the ions and electrons. This difference in velocity can generally be shown to be extremely small for most astrophysical instances.

Exercise 7.6:
In the solar photosphere, magnetic fields can reach field strengths of 10^3 G and generally there electron number density is roughly $10^{23}\,\mathrm{m}^{-3}$. By using Eq. 7.22 and Eq. 7.20 and assuming that the typical length is 1 Mm, give a rough estimate of what the speed $|\mathbf{u}'|$ needs to be in order to create the magnetic field strengths observed in the solar photosphere

The velocities are generally so small that we assume that collisions will mean that in spite of their being electric currents created by a difference in the speed of the ions and electrons, we still assume that all the particles (also neutral particles) can be assumed to have on single velocity. In order to get one of the final equations in MHD, we need to start by looking at the dynamics of the electrons in the rest frame of the ions. Remember that there are a small difference in velocity between the ions and electrons so for the electrons the equation of motion becomes:

$$m_e \frac{\partial \tilde{\mathbf{u}}'}{\partial t} = -e\left(\tilde{\mathbf{E}} + \tilde{\mathbf{u}}' \times \tilde{\mathbf{B}}\right) - m_e \nu_c \tilde{\mathbf{u}}' \tag{7.23}$$

where symbols with $\sim$ is given in the rest frame of the ions, and ν_c is the mean collision frequency between the electrons and the ions. There could easily be more forces included on the right hand side, such as gravity but even more importantly, the rest frame of the ions might not be inertial, so there could be inertial forces due to the non-linear movements of the ions. There could also be a collision term between the electrons and the neutrals, in case the plasma is not completely ionized, which would be almost identical to the collision term but now with a velocity that is the difference between the electrons and the neutrals. In this case we will concentrate on the ions and electrons. Because the electrons are so light compared to the ions we can assume that the electrons quickly reaches a state where there is a balance between the drag (the collisions with the ions) and the electromagnetic forces. That means that most of the time

$$m_e \nu_c \tilde{\mathbf{u}}' = -e\left(\tilde{\mathbf{E}} + \tilde{\mathbf{u}}' \times \tilde{\mathbf{B}}\right) \tag{7.24}$$

If we look at a reasonably uniform medium the gyromotion of the electrons cancel out, in the sense that they produce no large scale currents, so when we are only interested in the large scale motions – much larger than the gyroradius we can also exclude the term that produces the gyro motions of the electrons in Eq. 7.24. That now leaves only

$$m_e \nu_c \tilde{\mathbf{u}}' = -e\tilde{\mathbf{E}} \tag{7.25}$$

$$\tilde{\mathbf{u}}' = -\frac{e\tilde{\mathbf{E}}}{m_e \nu_c} \tag{7.26}$$

so we have an expression for the velocity of the electrons relative to the ions (not including the gyromotions of the electrons). The current produced by these electrons are then simply

$$\tilde{\mathbf{j}} = -en_e \tilde{\mathbf{u}}' = \sigma \tilde{\mathbf{E}} \tag{7.27}$$

which is Ohms law in the ions coordinate system and where

$$\sigma = \frac{n_e e^2}{m_e \nu_c} \tag{7.28}$$

is the *electric conductivity* of the plasma. Now we only need to transfer this equation into a rest frame before we can use it as one of the MHD equations. Since the electric current depends on the difference between two velocities, then the difference between them will still be the same, i.e. the electric current remains the same in our laboratory rest frame:

$$\mathbf{j} = \tilde{\mathbf{j}} \tag{7.29}$$

Transferring fields between two frames of reference which have a relative velocity $\mathbf{u}$ gives

$$\tilde{\mathbf{B}} = \mathbf{B} \tag{7.30}$$
$$\tilde{\mathbf{E}} = \mathbf{E} + \mathbf{u} \times \mathbf{B} \tag{7.31}$$

Inserting this back into the equation Eq. 7.27 we get

$$\mathbf{j} = \sigma\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) \tag{7.32}$$

Using this expression for $\mathbf{j}$ to eliminate the current in Eq. 7.20 and isolating the electric field $\mathbf{E}$ gives

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B} \tag{7.33}$$

and inserting that into the induction equation (Eq. 7.15) finally gives

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{u} \times \mathbf{B} - \nabla \times \left(\eta \nabla \times \mathbf{B}\right) \tag{7.34}$$

where we have introduced the resistive diffusion coefficient $\eta = 1/\mu_0 \sigma$. That equation can be rewritten by using a vector identity:

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2 \mathbf{A} \tag{7.35}$$

and using this and assuming that the resistive diffusion coefficient is constant Eq. 7.34 can be rewritten into

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} + \nabla \times \left(\mathbf{u} \times \mathbf{B}\right) \tag{7.36}$$

This is the final version of the equation controlling the magnetic field evolution in MHD.

Exercise 7.7:
Do the steps from Eq. 7.32 to Eq. 7.36 as explained in the text

7.3.1 Physical interpretation

The equation for the evolution of the magnetic field (Eq. 7.36), can be understood reasonably simply. The first term on the right hand side of Eq. 7.36 simply a diffusion term, similar to a heat diffusion term. The physical interpretation is just that the magnetic field will try to smooth itself out over time. The second term on the right hand side of Eq. 7.36 is an advection term, but where only velocities perpendicular to the magnetic field is take into account. It can be seen by trying to write the terms in the x–component:

$$\frac{\partial B_x}{\partial t} = \eta \frac{\partial^2 B_x}{\partial x^2} + \left[\frac{\partial}{\partial y} \left(u_x B_y - u_y B_x \right) - \frac{\partial}{\partial z} \left(u_z B_x - u_x B_z \right) \right] \tag{7.37}$$

If we imagine a magnetic field in the x direction and a velocity in the x direction, then it is clear from this equation that there is no change in any of the magnetic field components, since they all contain terms that are combinations of different combinations of u and B. Lets imagine a magnetic field in the z direction with a velocity field perpendicular to it along the x direction. Eq. 7.37 shows that only the last term gives a contribution

$$\frac{\partial B_x}{\partial t} = \frac{\partial}{\partial z} \left(u_x B_z \right) \tag{7.38}$$

which shows that if there is a velocity field which is not constant as a function of z, then there will be produced a field in the x direction. That effect can better be visualized by using the concept of field lines. A field line is defined as a connected set of line segments that is described by

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{Bz} \tag{7.39}$$

or put more simply, a field line is a line that at all points along its length it is tangential to the direction of the magnetic field. It is important to realize that this is true at each time t and that you can create a field line at each point in space (if the magnetic field strength is not identically zero). It means that it is not possible to "follow" a field line in time or space, but as we shall see later, it is possible to follow a specific amount of magnetic flux. The field line picture is only a tool we use to make it easier to visualize the field.

Frozen in condition and ideal MHD

Ideal MHD is MHD where there is no electrical resistivity, or similarly no magnetic diffusivity. That means that the fluid is a perfect conductor and this picture is an approximation in any astronomical setting. In the solar corona, the particle density is only 10^{15} m^{-3} (our atmosphere has roughly 10^{25} m^{-3}), but still the electrical conductivity is about the same as copper, so ideal MHD is a pretty good assumption in many cases. In order to get a more intuitive picture of the magnetic field, we use Gauss' theorem which states that for a surface S surrounding a volume V, then if $\mathbf{N}$ is a vector normal to the surface S and that $d\mathbf{S} = \mathbf{N}dS$ then for a vector field $\mathbf{A}$,

$$\int_V \nabla \cdot \mathbf{A} dV = \int_S \mathbf{A} \cdot d\mathbf{S} \tag{7.40}$$

so using that on Eq. 7.16 shows that the total magnetic flux crossing any surface is zero:

$$\int_S d\mathbf{S} \cdot \mathbf{B} = 0 \tag{7.41}$$

which is just showing that there are no magnetic charges in the universe. That also means that any field line going in through any closed surface S must also come back out of that closed surface. It also means that the amount of magnetic flux through *any* surface enclosed by C can be written:

$$\Phi = \int_C d\mathbf{S} \cdot \mathbf{B} \tag{7.42}$$

Now imagine a closed contour C moving with the fluid for a time dt. The enclosed surface consisting of the contour C, C' and the surface the contour has mapped out during the time dt makes up a closed volume. If we call a small step along C, $d\mathbf{s}$, then after dt, $d\mathbf{s}$ has swept out an area $d\mathbf{s} \times \mathbf{u}$. The total flux through the whole surface of the created volume, is the the sum of the flux passing through the original contour C, the final contour C' and the area swept out by the line segments $d\mathbf{s} \times \mathbf{u}$,

$$\int_C d\mathbf{S} \cdot \mathbf{B}(t) - \int_{C'} d\mathbf{S}'_C \cdot \mathbf{B}(t+dt) - \oint_C \mathbf{B}(t+dt) \cdot d\mathbf{s} \times \mathbf{u} dt = 0 \tag{7.43}$$

at time $t + dt$. So the difference in flux through the surface C and surface C' becomes

$$\begin{aligned} \delta\Phi &= \int_{C'} d\mathbf{S}'_C \cdot \mathbf{B}(t+dt) - \int_C d\mathbf{S}_C \cdot \mathbf{B}(t) \\ &= \int_C d\mathbf{S}_C \cdot [\mathbf{B}(t+dt) - \mathbf{B}(t)] - dt \oint \mathbf{B} \cdot d\mathbf{s} \times \mathbf{u} \\ &= dt \left[\int_C d\mathbf{S}_C \cdot \frac{\partial \mathbf{B}}{\partial t} - \oint_C \mathbf{B} \cdot d\mathbf{s} \times \mathbf{u} \right] \end{aligned} \tag{7.44}$$

then assuming that there is no magnetic diffusion then

$$\delta\Phi = dt \left[\int_C d\mathbf{S}_C \cdot \nabla \times (\mathbf{u} \times \mathbf{B}) - \oint_C \mathbf{B} \cdot d\mathbf{s} \times \mathbf{u} \right] \tag{7.45}$$

and using Stokes theorem which says

$$\int_C \nabla \times \mathbf{A} \cdot d\mathbf{S}_C = \oint_C \mathbf{A} \cdot d\mathbf{s} \tag{7.46}$$

then

$$\delta\Phi = dt \oint (d\mathbf{s} \cdot \mathbf{u} \times \mathbf{B} - \mathbf{B} \cdot d\mathbf{s} \times \mathbf{u}) \tag{7.47}$$

$$= dt \oint (\mathbf{B} \cdot d\mathbf{s} \times \mathbf{u} - \mathbf{B} \cdot d\mathbf{s} \times \mathbf{u}) \tag{7.48}$$

$$= 0 \tag{7.49}$$

and finally using a vector identity to get from Eq. 7.47 to Eq. 7.48. The consequence is that magnetic flux through any contour that moves with the fluid is constant in time, no matter how **B** and **U** looks.

Exercise 7.8:
Show that the vector identity used to get to Eq. 7.48 is true by writing out the three components of the first and second term in Eq. 7.47

Imagine now tubes of flux that move with the fluid, then if the tubes moved relative to the fluid, there would be tubes experiencing changes in flux, which cannot happen. So the flux tubes have to move with the fluid. That condition is called the “frozen in” condition, because the fluid and the flux move together.

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Resistive diffusion

Resistive diffusion is usually a small effect but it is a crucial effect in the solar atmosphere, where resistive diffusion is the effect that produces the giant solar flares and coronal mass ejections that heats the coronal plasma to tens of millions of degrees and can eject huge amounts of fast moving plasma that can disrupts satellite communication and induce large currents in high voltage power lines. The terms on the right hand side of Eq. 7.36 are in competition to control the evolution of the magnetic field. They have a magnitude of $\eta B/l^2$ and uB/l respectively, where B is the magnitude of the magnetic field, and l is the smallest characteristic scale of the magnetic field and velocity field. The ratio of these numbers is called the magnetic Reynolds number

$$R_m = \frac{uB}{l}\frac{l^2}{\eta B} = \frac{ul}{\eta} \tag{7.50}$$

and when R_m is very large compared to one, then the convective term is much more important than the resistive diffusion term, so the field is carried with the fluid. When R_m is small then the diffusion of the field is much stronger than the advection term, so the magnetic field will diffuse out of the fluid faster than it can be concentrated by the field. In almost all astrophysical circumstances, the resistive diffusion is very small.

Exercise 7.9:
Calculate the magnetic Reynolds number for the following astrophysical settings:

- The solar photosphere: The granular pattern has a typical size of 1Mm, velocity of 1 km s^{-1} and $\eta \sim 10^5$ m^2 s^{-1}
- The solar wind: The typical size of structures are roughly 1 earth radius, velocities are typically 400 km s^{-1} and the resistive diffusion coefficient is roughly 100 m^2 s^{-1}.
- The galactic disc: The typical size is 100 pc, the typical velocity is 10 km s^{-1} and the resistive diffusion coefficient is 10^5 m^2 s^{-1}

Is should be noted that as long as the collision frequency between electrons and ions is higher than the electron Larmor frequency, the electron drift is dominated by collisions, and then the resistive diffusion coefficient can be written as

$$\eta = 10^9\ T^{-\frac{2}{3}}\ \mathrm{m}^2\ \mathrm{s}^{-1} \tag{7.51}$$

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but it is important to notice that this is only true (and all of the previous discussion using the resistive diffusion coefficient), when the majority of the particles are electrically charged. If that is not the case, every electrically charged particle will have to "drag" the neutrals with them through collisions which will affect both the diffusion of the magnetic field, and the whole foundation of MHD. If there is a significant portion of the particles that are neutral, MHD can be changed by including a number of terms that treat the interaction between the neutrals and the electrically charged particles, but treating those terms here would be going too far. An amount of neutral particles large enough to affect the dynamics are possibly present in the upper solar photosphere, but that is a topic of cutting edge research at the moment.

Exercise 7.10:
The magnetic field in interstellar clouds have a typical size of 10^{-9} T, and the density of 10^3 m^{-3}. Assume that the magnetic field is passively advected by the gas as it coalesces and becomes the sun(use only average values for the present sun), and calculate the magnetic field strength that the sun should have by using the frozen-in condition where the amount of flux passing through a surface is constant, when the surface moves with the fluid.

7.4 The magnetic field and its forces

The magnetic field threads plasma over large distances. The forces it transmits to the plasma can therefore be a consequence of plasma movements at large distances. That is very different from hydrodynamics, where the forces experienced by a fluid parcel is only dependent on the neighboring fluid parcels. If we look at the force experienced by a charged particle in its own frame of reference, then it is simply

$$\mathbf{f}' = q\mathbf{E}' \tag{7.52}$$

where $\mathbf{f}$ and $\mathbf{E}$ is the force experienced by the particle with charge q and the electric field respectively. Both of these are here in the reference frame of the particle, so have been denoted with a mark. Transferring this to an inertial frame of reference, gives

$$\mathbf{f} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) \tag{7.53}$$

where the particle moves with the velocity u in the inertial frame of reference. A plasma by definition contains a large number of charge carriers in a unit

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volume, so if the overall charge of the volume is q_e, then the force per unit volume is

$$\mathbf{F} = q_e \mathbf{E} + \mathbf{j} \times \mathbf{B} \tag{7.54}$$

since the charge times the velocity is just the electric current. We have already seen that the electric field is much weaker than the magnetic field, so the force exerted by the electric field is therefore much smaller than the force exerted by the magnetic field, so that with the help of Eq. 7.20 we can write Eq. 7.54 to

$$\mathbf{F} = 0 + \frac{1}{\mu_0} \left(\nabla \times \mathbf{B}\right) \times \mathbf{B} \tag{7.55}$$

which finally can be rewritten using the vector identity for two vector fields $\mathbf{A}$ and $\mathbf{B}$

$$\nabla \left(\mathbf{A} \cdot \mathbf{B}\right) = \left(\mathbf{B} \cdot \nabla\right) \mathbf{A} + \left(\mathbf{A} \cdot \nabla\right) \mathbf{B} + \mathbf{B} \times \left(\nabla \times \mathbf{A}\right) + \mathbf{A} \times \left(\nabla \times \mathbf{B}\right) \tag{7.56}$$

into

$$\mathbf{F} = -\frac{1}{\mu_0} \left[\tfrac{1}{2} \nabla B^2 - \left(\mathbf{B} \cdot \nabla\right) \mathbf{B}\right] \tag{7.57}$$

Exercise 7.11:
Use Eq. 7.56 to show that Eq. 7.55 can be transformed into Eq. 7.57

The force is a regular body–force, so can be plugged directly into the momentum equation. The two terms have very different properties and both are similar to two reasonably well known forces. We can immediately see that magnetic fields that satisfy $\nabla \times \mathbf{B} = 0$ produces no force on the plasma. This type of field is called a potential field, because it can be created as the gradient of a scalar field but it is a special case of the more general force free fields. Force free fields is a large family of fields that do not produce any force on the plasma and they all satisfy $\nabla \times \mathbf{B} = \alpha(\mathbf{r})\mathbf{B}$, where $\alpha(\mathbf{r})$ is a scalar function. Fields where α is a constant are called linear force free, while fields where $\alpha(\mathbf{r})$ is non constant are called non linear force free. Such fields have a complete balance between the two terms in Eq. 7.57. For fields that are not force free the two terms do are not in balance, and the plasma will feel a force from the magnetic field.

Exercise 7.12:
Show that magnetic fields obeying $\nabla \times \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \alpha(\mathbf{x})\mathbf{B}$ produces no Lorentz force

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The magnetic pressure

The first term on the right hand side of Eq. 7.57 is just the gradient of scalar field so completely identical to the gas pressure, where now the pressure is no longer delivered by the thermal motions of the particles, but by the magnetic fields influence over the particles motions. The force works completely analogous to the thermal pressure, where the pressure for an ideal gas is $P_g = NkT$, then the magnetic pressure is just $P_m = B^2/2\mu_0$.

The magnetic tension

The second term is somewhat more complicated. Lets look at the x-component of the force

$$\left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)B_x \tag{7.58}$$

That is somewhat complicated to get a physical understanding from, but basically it says that you need a gradient of one of the components of the magnetic field, but at the same time you need the field to also be along that gradient. It basically means that if there is a curvature of the field, then there will be a force towards the centre of the curvature. To exemplify it in a slightly easier way, then imagine a magnetic field curving around the z-axis of a cylindrical coordinate system forming a magnetic cylinder. In the cylinrical coordinate system, the magnetic field would have the following components

$$B_\phi = B_0. \tag{7.59}$$
$$B_r = 0. \tag{7.60}$$
$$B_z = 0. \tag{7.61}$$

and the magnetic field would have a magnitude B_0 everywhere. This field clearly obeys $\nabla \cdot \mathbf{B} = 0$ and we can now calculate the two components of the Lorentz force: The magnetic pressure force gives no contribution as the field is constant everywhere. The second term has the following structure in cylindrical coordinates:

$$F_r = \frac{1}{\mu_0}\left[B_r\frac{\partial B_r}{\partial r} + \frac{B_\phi}{r}\frac{\partial B_r}{\partial \phi} + B_z\frac{\partial B_r}{\partial z} - \frac{B_\phi^2}{r}\right] = -\frac{B_\phi^2}{\mu_0 r} \tag{7.62}$$
$$F_\phi = \frac{1}{\mu_0}\left[B_r\frac{\partial B_\phi}{\partial r} + \frac{B_\phi}{r}\frac{\partial B_\phi}{\partial \phi} + B_z\frac{\partial B_\phi}{\partial z} + \frac{B_\phi B_r}{r}\right] = 0 \tag{7.63}$$
$$F_z = \frac{1}{\mu_0}\left[B_r\frac{\partial B_z}{\partial r} + \frac{B_\phi}{r}\frac{\partial B_z}{\partial \phi} + B_z\frac{\partial B_z}{\partial z}\right] = 0 \tag{7.64}$$

which gives only a contribution directed towards the centre of the coordinate system. The second term in the Lorentz force is therefore often called a tension force, because it works against any curvature of the field and will try to straighten the field out. The smaller the radius of the curvature, or the sharper the field has been bent, the greater the force. Here the force decreases with r because the field is closer and closer to be straight for increasing r.

Exercise 7.13:
Assume that a cylindrical magnetic flux tube is embedded in a plasma with $P = 10^8$ Pa. Assume that the field is along the z-axis in a cylindrical coordinate system, that the flux tube has a radius of 1 Mm and that there is no plasma inside the flux tube. What does the magnetic field need to be in order for the system to be in equilibrium ?

Lets look at two different twisted flux ropes. None of these are very realistic because it turns out that the balance of the magnetic pressure force and the magnetic tension force is an ill posed problem.

The force balance is simple that the two forces must balance each other, and as we have just seen, a curved magnetic field gives a simple expression for the tension force, so we imagine now a magnetic field that curves around the z-axis of a cylindrical coordinate system. Also assume that there is no gradients in the ϕ-direction, and that there are no other important forces present. In that case we just have the balance of the two forces:

$$\frac{B_\phi}{\mu_0 r} = \frac{1}{2\mu_0}\frac{\partial B^2}{\partial r} \tag{7.65}$$

The first flux rope has a simple functional form of the rotational part of the magnetic field: $B_\phi = B_0 r$, where B_0 is a constant. This field is of course not realistic as the field strength would grow to infinite values. If we insert that into Eq. 7.65, we get

$$\begin{aligned}
\frac{B_0^2 r^2}{\mu_0 r} &= \frac{1}{2\mu_0}\left[\frac{\partial B_\phi^2}{\partial r} + \frac{\partial B_z^2}{\partial r}\right] \\
2B_0^2 r &= B_0^2\frac{\partial r^2}{\partial r} + \frac{\partial B_z^2}{\partial r} \\
\frac{\partial B_z^2}{\partial r} &= 0 \\
B_z &= B_{z,0}
\end{aligned} \tag{7.66}$$

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where $B_{z,0}$ is an arbitrary constant. This is a very simple solution to the force balance equation and as mentioned not realistic, since the field strength would grow to infinity.

The second example of a flux rope is a bit more complicated. Here we again assume the same symmetry, but here

$$B_\phi = B_0 r \cos(r) \tag{7.67}$$

where B_0 is a constant. We also assume that this equation only holds for $0 \le r \le \pi/2$. For values of $r > \pi/2$ we assume that there is a gas pressure which has the same value as the magnetic pressure at $r = \pi/2$. We have already calculated the force from curved magnetic field in Eq. 7.62, so if we balance that against the other magnetic term in the force balance we just get

$$\frac{B_\phi^2}{\mu_0 r} - \frac{1}{2\mu_0}\frac{\partial(B_r^2 + B_\phi^2 + B_z^2)}{\partial r} = 0 \tag{7.68}$$

where we again assume that there are no other forces at play, or that they balance each other perfectly. Inserting Eq. 7.67 gives

$$\begin{aligned}
\frac{B_0^2 r^2 \cos(r)^2}{r} &= \frac{1}{2}\frac{\partial}{\partial r}\left(B_z^2 + B_0^2 r^2 \cos(r)^2\right) \\
\frac{\partial B_z^2}{\partial r} &= 2B_0^2 r \cos(r)^2 - B_0^2 r^2 \cos(r)^2 \\
B_z^2 &= \frac{-4(r-3)r^2 - 6(r-1)\cos(2r) + (3 - 6(r-2)r)\sin(2r)}{24}
\end{aligned}$$

7.5 Final MHD equations

So finally we have the equations of MHD, where we only need to include the Lorentz force in the equation of momentum, giving:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{u} && (7.69)\\
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla P + \nabla \cdot \tau + \mathbf{j} \times \mathbf{B} + G && (7.70)\\
\frac{\partial \mathcal{E}}{\partial t} &= F(\mathcal{E}, \mathbf{u}, \mathbf{B}, \eta, \tau) && (7.71)\\
\frac{\partial \mathbf{B}}{\partial t} &= \eta \nabla^2 \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}) && (7.72)
\end{aligned}$$

where $\mathcal{E}$ is the internal energy of the gas, G is the gravitational force and the Lorentz force has been added to the momentum equation. The internal energy depends on a number of things. It depends on the velocity through advection, the magnetic field and the resistive diffusion creates heat and finally the viscous stress also provides heat. The final equation which we have also used in the HD chapter is the equation of state which is the final equation needed to close this equation set.

7.6 MHD waves

As in HD, MHD has its own types of waves, driven by the restoring force of the magnetic field. As the magnetic field has both the tension force and the magnetic pressure that can act as a restoring force as well the the gas pressure, there is a number of different wave modes that can be exited in MHD, while HD only has one: The sound wave. To get to the equations for MHD waves, we need a few preparations. As we did in HD, we now assume that we have a medium in equilibrium, and that the waves are small perturbations on top of the equilibrium values, giving

$$\begin{aligned} \rho &= \rho_0 + \rho' \\ P &= P_0 + P' \\ \mathbf{u} &= \mathbf{u}_0 + \mathbf{u}' \\ \mathbf{B} &= \mathbf{B}_0 + \mathbf{B}' \end{aligned}$$

where the 0 subscripts denote the equilibrium values and the marked variables are the small perturbations. We further assume that the medium is homogeneous and that it does not move, which means that there are no spatial or time gradients of the equilibrium values and that $\mathbf{u}_0 = 0$. We make us of the equations for small perturbations in density at constant entropy (Eq. 6.30) again

$$\frac{P'}{\rho'} = \left(\frac{\partial P}{\partial \rho_0}\right)_s = c_s^2 \tag{7.73}$$

We now describe the small perturbations as wavelike, so that for instance

$$\mathbf{u}' = u^* \exp\left(-i\left(\omega t - \mathbf{k}\cdot\mathbf{x}\right)\right) \tag{7.74}$$

where u^* is just a scalar and $\mathbf{k}$ is the wave vector which is the direction of the wave propagation. Now we are ready to rewrite the mass conservation equation as

$$\frac{\partial \rho_0 + \rho'}{\partial t} = -\nabla\cdot\left[\left(\rho_0 + \rho'\right)\mathbf{u}'\right] \tag{7.75}$$

$$\frac{\partial \rho'}{\partial t} = -\nabla\cdot\left(\rho_0\mathbf{u}' + \rho'\mathbf{u}'\right) \tag{7.76}$$

and again as before we now ignore terms where two very small terms are multiplied, as in this case $\rho'\mathbf{u}'$. Then finally the equation of mass conservation becomes

$$\begin{aligned}
\frac{\partial \rho'}{\partial t} &= -\rho_0 \nabla \cdot \mathbf{u}' \\
-i\omega\rho' &= -\rho_0 \left(i\mathbf{k}\right) \cdot \mathbf{u}' \\
\rho' &= \frac{\rho_0}{\omega}\mathbf{k} \cdot \mathbf{u}' = \frac{P'}{c_s^2} \qquad (7.77) \\
P' &= \frac{c_s^2 \rho_0}{\omega}\mathbf{k} \cdot \mathbf{u}' \qquad (7.78)
\end{aligned}$$

where we have inserted the wavedescription of the perturbations and used Eq. 7.73.

We now continue to treat the equation for the magnetic field the same way and assuming that the magnetic diffusivity η is so small that we can ignore it here.

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \left(\mathbf{u} \times \mathbf{B}\right) \\
\frac{\partial \mathbf{B}'}{\partial t} &= -\nabla \times \left[\mathbf{u}' \times \left(\mathbf{B}_0 + \mathbf{B}'\right)\right] \\
\frac{\partial \mathbf{B}'}{\partial t} &= -\nabla \times \left(\mathbf{u}' \times \mathbf{B}_0\right) \\
-i\omega\mathbf{B}' &= -i\mathbf{k} \times \left(\mathbf{u}' \times \mathbf{B}_0\right) \\
\mathbf{B}' &= \frac{1}{\omega}\mathbf{k} \times \left(\mathbf{u}' \times \mathbf{B}_0\right) \qquad (7.79)
\end{aligned}$$

and finally we can now do the same with the momentum equation, where again we assume the dissipative effects are very small and can be ignored which removes the viscous stress tensor term:

$$
\begin{aligned}
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla P - \frac{1}{\mu_0}\left(\nabla \times \mathbf{B}\right) \times \mathbf{B} \\
\rho_0 \frac{\partial \mathbf{u}'}{\partial t} &= -\nabla P' - \frac{1}{\mu_0}\left(\nabla \times \mathbf{B}'\right) \times \mathbf{B}_0 \\
\rho_0 \left(-i\omega\right) \mathbf{u}' &= -i\mathbf{k}P' - \frac{1}{\mu_0}\left(i\mathbf{k} \times \mathbf{B}'\right) \times \mathbf{B}_0 \\
-i\omega\rho_0 \mathbf{u}' &= -i\mathbf{k}\frac{c_s^2 \rho_0}{\omega}\left(\mathbf{k}\cdot\mathbf{u}'\right) - \frac{i}{\mu_0}\left(\mathbf{k} \times \mathbf{B}'\right) \times \mathbf{B}_0 \\
-i\omega\rho_0 \mathbf{u}' &= -\frac{i c_s^2 \rho_0}{\omega}\left(\mathbf{k}\cdot\mathbf{u}'\right)\mathbf{k} - \frac{i}{\mu_0}\left[\mathbf{k} \times \left(\frac{1}{\omega}\mathbf{k} \times \left[\mathbf{u}' \times \mathbf{B}_0\right]\right)\right] \times \mathbf{B}_0 \\
\frac{\omega^2}{k^2}\mathbf{u}' &= \frac{c_s^2\left(\mathbf{k}\cdot\mathbf{u}'\right)}{k^2}\mathbf{k} + \frac{1}{\rho_0\mu_0 k^2}\left[\mathbf{k} \times \left(\mathbf{k} \times \left[\mathbf{u}' \times \mathbf{B_0}\right]\right)\right] \times \mathbf{B}_0 \\
\frac{\omega^2}{k^2}\mathbf{u}' &= c_s^2\left(\hat{\mathbf{k}}\cdot\mathbf{u}'\right)\hat{\mathbf{k}} + \frac{B_0^2}{\rho_0\mu_0}\left[\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \left[\mathbf{u}' \times \hat{\mathbf{B}}_0\right]\right)\right] \times \hat{\mathbf{B}}_0 \\
\frac{\omega^2}{k^2}\mathbf{u}' &= c_s^2\left(\hat{\mathbf{k}}\cdot\mathbf{u}'\right)\hat{\mathbf{k}} + c_A^2\left[\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \left[\mathbf{u}' \times \hat{\mathbf{B}}_0\right]\right)\right] \times \hat{\mathbf{B}}_0 \qquad (7.80)
\end{aligned}
$$

where we have used both Eq. 7.78, Eq. 7.79 and defined the Alfén speed $c_A = B_0/\sqrt{\rho_0\mu_0}$. Eq. 7.80 is clearly a wave equation with two components. The terms all have the same size, and if we ignore the directional information in the unit vectors on the right hand side the magnitude of the wave velocity can be found from:

$$\frac{\omega^2}{k^2}|u'| = x_1 c_s^2|u'| + x_2 c_A^2|u'| \qquad (7.81)$$

where x_1 and x_2 are functions of the directional vector expressions in Eq. 7.80 and have values between -1 and $+1$. That means that the wave speed is within the range of values given by

$$c_w^2 = x_1 c_s^2 + x_2 c_A^2 \qquad (7.82)$$

which has two solutions, so that the wave speed is between $c_A - c_s \leq c_w \leq c_A + c_s$ (here assuming that $c_A > c_s$).

There are several types of waves that comes out of this wave equation. If the velocities $\mathbf{u}'$ is along the magnetic field, then the second term of Eq. 7.80 is zero, and we only have a wave propagating if the wave vector is not perpendicular to $\mathbf{u}'$, and the speed will then be $c_s\sqrt{\hat{\mathbf{k}}\cdot\mathbf{u}'}$. This type of wave moves primarily in the direction of the equilibrium magnetic field $\mathbf{B_0}$ and it has a velocity c_s. From Eq. 7.79 we see that there is no effect on the magnetic field, and from Eq. 7.78 that the wave is compressing the gas. It it important to notice that this wave can move slightly across the field, but the velocity pertubation is always along the magnetic field. If it is not, the wave gets a contribution from the second term in Eq. 7.80. This wave is a very special solution of Eq. 7.80.

There are another special case, which is the Alfvén wave. The solution comes about if $\hat{\mathbf{k}} \cdot \mathbf{u}' = 0$, in other words the direction of the wave is perpendicular to the plasma movements. From Eq. 7.78 we see that the wave does not compress the gas, and from Eq. 7.79 we see that the magnetic field is pertubed, so this is a wave that is only a wave in the magnetic field. It is a wave motion of the magnetic field lines, which does makes the gas move back and forth perpendicular to the equilibrium magnetic field direction. It moves with the velocity c_A which in the sun is often much higher than the sound speed.

There are two types of waves that are combinations of the Alfvén wave and the sound wave. These are called the slow and the fast magnetosonic wave. The fast magnetosonic wave is a wave which is able to bend the magnetic field lines, as well as compress the gas so that both the gas pressure and the magnetic field is able to work as restoring forces in the wave propagation. The fast magnetosonic wave degenerates to an Alfvén wave when the wave moves along the magnetic field direction, while when it moves perpendicular to the magnetic field, it is able to move faster than both the sound speed and the Alfvén speed because the two restoring forces work in phase, giving it a wavespeed that is $c_s + c_A$. The slow magnetosonic wave is a wave where the two restoring forces are out of phase and the slow mode can only move along the magnetic field. For the slow mode, the magnetic field is also bent as in the Alfvén wave, but the pressure perturbations are out of phase, and the velocity can then at most become $c_A - c_s$ (assuming that $c_A > c_s$).

Appendices

Appendix A

List of constants

proton mass	$m_{\rm p}$	$=$	$1.6726 \cdot 10^{-27}$ kg
hydrogen mass	$m_{\rm H}$	$=$	$1.6738 \cdot 10^{-27}$ kg
$^3_2$He mass	$m_{^3_2{\rm He}}$	$=$	$5.0081 \cdot 10^{-27}$ kg
$^4_2$He mass	$m_{^4_2{\rm He}}$	$=$	$6.6464 \cdot 10^{-27}$ kg
electron mass	$m_{\rm e}$	$=$	$9.1094 \cdot 10^{-31}$ kg
atomic mass unit	$m_{\rm u}$	$=$	$1.6605 \cdot 10^{-27}$ kg
Boltzmann's constant	$k_{\rm B}$	$=$	$1.3806 \cdot 10^{-23}\ {\rm m^2\,kg\,s^{-2}\,K^{-1}}$
Stefan-Boltzmann constant	σ	$=$	$5.6704 \cdot 10^{-8}\ {\rm Wm^{-2}\,K^{-4}}$
radiation density constant	a	$=$	$\frac{4\sigma}{c} = 7.566 \cdot 10^{-16}\ {\rm J\,m^{-3}\,K^{-4}}$
speed of light	c	$=$	$2.9979 \cdot 10^{8}\ {\rm m\,s^{-1}}$
electric charge	e	$=$	$1.6022 \cdot 10^{-19}$ C
electron volt	$1\,{\rm eV}$	$=$	$1.6022 \cdot 10^{-19}$ J
permeability of free space	μ_0	$=$	$4\pi \cdot 10^{-7}\ {\rm V\,s\,A^{-1}\,m^{-1}}$
permittivity of free space	ϵ_0	$=$	$8.8542 \cdot 10^{-12}\ {\rm F\,m^{-1}}$
Planck's constant	h	$=$	$6.6261 \cdot 10^{-34}\ {\rm J\,s}$
gravitational constant	G	$=$	$6.6742 \cdot 10^{-11}\ {\rm N\,m^2\,kg^{-2}}$
Avogadro's number	$N_{\rm A}$	$=$	$6.0221 \cdot 10^{23}\ {\rm mol^{-1}}$
Bohr radius	a_0	$=$	$5.2918 \cdot 10^{-11}$ m
gas constant	$\mathcal{R}$	$=$	$8.3145\ {\rm J\,mol^{-1}\,K^{-1}}$

Appendix B

Solar parameters

The solar core

$$
\begin{aligned}
\rho &= 1.62 \cdot 10^5\,\mathrm{kg\,m^{-3}} \\
T &= 1.57 \cdot 10^7\,\mathrm{K} \\
P &= 3.45 \cdot 10^{16}\,\mathrm{Pa}
\end{aligned}
$$

Bottom of the solar convection zone

$$
\begin{aligned}
\rho &\approx 7.2 \cdot 10^3\,\mathrm{kg\,m^{-3}} \\
T &\approx 5.7 \cdot 10^6\,\mathrm{K} \\
P &\approx 5.2 \cdot 10^{14}\,\mathrm{Pa} \\
R &= 0.72 R_\odot
\end{aligned}
$$

Solar photosphere

$$
\begin{aligned}
\rho &= 2 \cdot 10^{-4}\,\mathrm{kg\,m^{-3}} \\
T &= 5778\,\mathrm{K} \\
P &= 1.8 \cdot 10^8\,\mathrm{Pa} \\
X &= 0.7346 \\
Y &= 0.2485 \\
Z &= 0.0169
\end{aligned}
$$

Solar corona

$$\begin{aligned} \rho &= 10^{-11}\,\mathrm{kg\,m^{-3}} \\ T &= 1.5 \cdot 10^{6}\,\mathrm{K} \\ P &= 0.1\,\mathrm{Pa} \end{aligned}$$

General parameters

$$\begin{aligned} L_\odot &= 3.846 \cdot 10^{26}\,\mathrm{W} \\ M_\odot &= 1.989 \cdot 10^{30}\,\mathrm{kg} \\ R_\odot &= 6.96 \cdot 10^{8}\,\mathrm{m} \\ t_{\mathrm{life}} &= 4.57 \cdot 10^{9}\,\mathrm{y} \\ \kappa &\approx 1\,\mathrm{m^2\,kg^{-1}} \end{aligned}$$

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Appendix C

References and background material

The following text books were used for these lecture notes and are recommended as background material:

Stix, M., 2002, “The Sun : an introduction”, 2nd edition, Springer

von Mieses, R., 1958, “Mathematical theory of compressible fluid flow”, available as paperback from Dover Publications (2004)

Schroeder, D.V., 1999, “An introduction to Thermal Physics”, Addison Wesley

Kippenhahn, R., Weigert, A. & Weiss, A., 2012, “Stellar structure and evolution”, 2nd edition, Springer