Boundary Conditions

Idea for vertical boundary conditions: use forward and backward difference approximation to get expressions for $\left[\frac{\partial\phi}{\partial y}\right]_{i,0}^{n}$ (top) and $\left[\frac{\partial\phi}{\partial y}\right]_{i,n_y-1}^{n}$ (bottom)

Horizontal (periodic)

$$\phi_{-1,j} = \phi_{x-1,j}^{n}$$

$$\phi_{x,j}^{n} = \phi_{0,j}$$

Vertical

Vertical velocity

$$w_{i,-1}^{n} = w_{i,n_y} = 0$$

Top:

$$\left[\frac{\partial w}{\partial y}\right]_{i,0}^{n} = \frac{4w_{i,1}^{n} - w_{i,2}^{n}}{2\Delta y}$$

Bottom:

$$\left[\frac{\partial w}{\partial y}\right]_{i,n_y-1}^{n} = \frac{w_{i,n_y-3}^{n} - 4w_{i,n_y-2}^{n}}{2\Delta y}$$

Horizontal velocity

$$\left[\frac{\partial u}{\partial y}\right]_{i,0}^{n} = \left[\frac{\partial u}{\partial y}\right]_{i,ny-1}^{n} = 0$$

Energy

$$e = \frac{P}{\gamma - 1}$$

$$\left[\frac{\partial e}{\partial y}\right]_{i,j}^{n} = \frac{1}{\gamma - 1}\left[\frac{\partial P}{\partial y}\right]_{i,j}^{n} = -\frac{g}{\gamma - 1}\rho_{i,j}^{n} = -\frac{g\mu m_u}{\gamma - 1}\frac{P_{i,j}^{n}}{T_{i,j}^{n}}$$

Top:

$$\left[\frac{\partial e}{\partial y}\right]_{i,0}^{n} = -\frac{g\mu m_u}{\gamma - 1}\frac{P_{i,0}^{n}}{T_{i,0}^{n}}$$

Bottom:

$$\left[\frac{\partial e}{\partial y}\right]_{i,n_y-1}^{n} = -\frac{g\mu m_u}{\gamma - 1}\frac{P_{i,n_y-1}^{n}}{T_{i,n_y-1}^{n}}$$

Density

$$\rho = C_2 \frac{e}{T}, \qquad C_2 \equiv \frac{(\gamma - 1)\mu m_u}{k_B}$$

$$\left[\frac{\partial \rho}{\partial y}\right]_{i,j}^{n} = \frac{C_2}{[T^2]_{i,j}^{n}} \left(T_{i,j}^{n} \left[\frac{\partial e}{\partial y}\right]_{i,j}^{n} - e_{i,j}^{n} \left[\frac{\partial T}{\partial y}\right]_{i,j}^{n} \right)$$

Found from initial conditions:

$$\frac{\partial T}{\partial y} = -\nabla \mathrm{g} \frac{\mathrm{\mu m_u}}{k_B}$$

Top:

$$\left[\frac{\partial \rho}{\partial y}\right]_{i,0}^{n} = \frac{C_2}{[T^2]_{i,0}^{n}} \left(T_{i,0}^{n} \left[\frac{\partial e}{\partial y}\right]_{i,0}^{n} + e_{i,0}^{n} \nabla \mathrm{g} \frac{\mathrm{\mu m_u}}{k_B} \right)$$

Bottom:

$$\left[\frac{\partial \rho}{\partial y}\right]_{i,n_y-1}^{n} = \frac{C_2}{[T^2]_{i,n_y-1}^{n}} \left(T_{i,n_y-1}^{n} \left[\frac{\partial e}{\partial y}\right]_{i,n_y-1}^{n} + e_{i,n_y-1}^{n} \nabla \mathrm{g} \frac{\mathrm{\mu m_u}}{k_B} \right)$$

Horizontal momentum

Top:

$$\left[\frac{\partial \rho u}{\partial y}\right]_{i,0}^{n} = u_{i,0}^{n} \left[\frac{\partial \rho}{\partial y}\right]_{i,0}^{n} + \rho_{i,0}^{n} \left[\frac{\partial u}{\partial y}\right]_{i,0}^{n} = u_{i,0}^{n} \left[\frac{\partial \rho}{\partial y}\right]_{i,0}^{n}$$

Bottom:

$$\left[\frac{\partial \rho u}{\partial y}\right]_{i,n_y-1}^{n} = u_{i,n_y-1}^{n} \left[\frac{\partial \rho}{\partial y}\right]_{i,n_y-1}^{n} + \rho_{i,n_y-1}^{n} \left[\frac{\partial u}{\partial y}\right]_{i,n_y-1}^{n} = u_{i,n_y-1}^{n} \left[\frac{\partial \rho}{\partial y}\right]_{i,n_y-1}^{n}$$

Vertical momentum

Top:

$$\left[\frac{\partial \rho w}{\partial y}\right]_{i,0}^{n} = w_{i,0}^{n} \left[\frac{\partial \rho}{\partial y}\right]_{i,0}^{n} + \rho_{i,0}^{n} \left[\frac{\partial w}{\partial y}\right]_{i,0}^{n} = \rho_{i,0}^{n} \left[\frac{\partial w}{\partial y}\right]_{i,0}^{n}$$

Bottom:

$$\left[\frac{\partial \rho u}{\partial y}\right]_{i,n_y-1}^{n} = w_{i,n_y-1}^{n} \left[\frac{\partial \rho}{\partial y}\right]_{i,n_y-1}^{n} + \rho_{i,n_y-1}^{n} \left[\frac{\partial w}{\partial y}\right]_{i,n_y-1}^{n} = \rho_{i,n_y-1}^{n} \left[\frac{\partial w}{\partial y}\right]_{i,n_y-1}^{n}$$