Boundary Conditions

Idea for vertical boundary conditions: use forward and backward difference approximation to get expressions for $\phi_{i,-1}^n$ (top) and ϕ_{i,n_y}^n (bottom). So

$$\phi_{i,-1}^n = 2\Delta y \left[\frac{\partial \phi}{\partial y}\right]_{i,1}^n - 3\phi_{i,1}^n + 4\phi_{i,0}^n$$

$$\phi_{i,n_y} = 4\phi_{i,n_y-1}^n - 3\phi_{i,n_y-2}^n + 2\Delta y \left[\frac{\partial \phi}{\partial y}\right]_{i,n_y-2}^n$$

Horizontal (periodic)

$$\phi_{-1,j} = \phi_{x-1,j}^n$$

$$\phi_{x,j}^n = \phi_{0,j}$$

Vertical

Vertical velocity

$$w_{i,-1}^n = w_{i,y-1} = 0$$

$$w_{-1,j} = w_{x-1,j} = 0$$

Horizontal velocity

$$\left[\frac{\partial u}{\partial y}\right]_{i,0}^n = \left[\frac{\partial u}{\partial y}\right]_{i,ny-1}^n = 0$$

Top: $j - 1 = -1 \Rightarrow j = 1$:

$$0 = 3u_{i,1}^n - 4u_{i,0}^n + u_{i,-1}^n \Rightarrow u_{i,-1}^n = 4u_{i,0}^n - 3u_{i,1}^n$$

Bottom: $j + 2 = ny \Rightarrow j = n_y - 2$

$$0 = -u_{i,n_y}^n + 4u_{i,n_y-1}^n - 3u_{i,n_y-3}^n \Rightarrow u_{i,n_y}^n = 4u_{i,n_y-1}^n - 3u_{i,n_y-2}^n$$

Energy

$$e = \frac{P}{\gamma - 1}$$

$$\left[\frac{\partial e}{\partial y}\right]_{i,j}^n = \frac{1}{\gamma - 1}\left[\frac{\partial P}{\partial y}\right]_{i,j}^n = -\frac{g}{\gamma - 1}\rho_{i,j}^n$$

Top: $j - 1 = -1 \Rightarrow j = 1$:

$$e_{j,-1}^n = -2\Delta y \frac{g}{\gamma - 1}\rho_{i,1}^n - 3e_{i,1}^n + 4e_{i,0}^n$$

Bottom: $j + 2 = ny \Rightarrow j = n_y - 2$

$$e_{i,n_y}^n = 4e_{i,n_y-1}^n - 3e_{i,n_y-2}^n + 2\Delta y \frac{g}{\gamma - 1}\rho_{i,n_y-2}^n$$

Density

$$\rho = H\frac{e}{T}, \qquad H \equiv \frac{(\gamma - 1)\mu m_u}{k_B}$$

$$\frac{\partial \rho}{\partial y} = \frac{H\partial\left(\frac{e}{T}\right)}{\partial y} = \frac{H}{T^2}\left(T\frac{\partial e}{\partial y} - e\frac{\partial T}{\partial y}\right)$$

$$\left[\frac{\partial \rho}{\partial y}\right]_{i,j}^{n} = \frac{H}{[T^2]_{i,j}^{n}}\left(T_{i,j}^{n}\left[\frac{\partial e}{\partial y}\right]_{i,j}^{n} - e_{i,j}^{n}\left[\frac{\partial T}{\partial y}\right]_{i,j}^{n}\right)$$

Found from initial conditions:

$$\frac{\partial T}{\partial y} = -\nabla \mathrm{g}\frac{\mathrm{\mu m_u}}{k_B}$$

Top: $j - 1 = -1 \Rightarrow j = 1$:

$$\rho_{j,-1}^{n} = 2\Delta y\frac{H}{[T^2]_{i,1}^{n}}\left(T_{i,j}^{n}\left[\frac{\partial e}{\partial y}\right]_{i,1}^{n} - e_{i,j}^{n}\left[\frac{\partial T}{\partial y}\right]_{i,1}^{n}\right) - 3\rho_{i,1}^{n} + 4\rho_{i,0}^{n}$$

Bottom: $j + 2 = ny \Rightarrow j = n_y - 2$

$$\rho_{i,n_y}^{n} = 4\rho_{i,n_y-1}^{n} - 3\rho_{i,n_y-2}^{n} + 2\Delta y\frac{H}{[T^2]_{i,n_y-2}^{n}}\left(T_{i,n_y-2}^{n}\left[\frac{\partial e}{\partial y}\right]_{i,n_y-2}^{n} - e_{i,n_y-2}^{n}\left[\frac{\partial T}{\partial y}\right]_{i,n_y-2}^{n}\right)$$