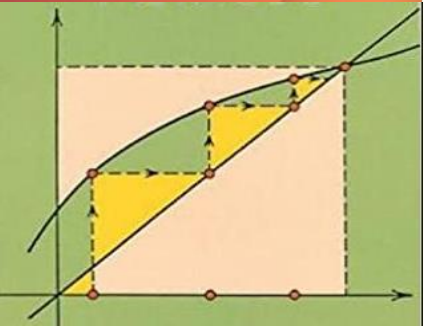
Unit-III

Numerical Methods

What is Numerical method?

- Numerical method is an approach for solving complex mathematical problems using simple arithmetic operations.
- Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations.
- It involves the formulation of model of physical situations that can be solve by arithmetic operations



Need of Numerical Methods

- Mathematical Models are central piece of Science and Engineering.
- Some models have closed form solutions, therefore they can be solved analytically. Many models cannot be solved analytically or the analytic solution is too costly to be practical.
- All models can be solved computationally and the result may not be the exact answer but it can be useful.

Why are numerical methods used in Engineering?

- Engineers use mathematical modeling which includes various equations and data to describe and predict the behavior of systems.
- Computers are widely used which give accurate results and are cheap and affordable to all.
- Many software packages are available that can be used to solve the problems.

Application of Numerical Methods

Interpolation

The most of the experimental or observed data is in the form of ordered pairs $(x_0, y_0), (x_1, y_1), (x_2, y_2)\ldots\ldots(x_n, y_n)$ which is the tabular form of an unknown function $y=f(x)$. The process of demining the value of y for any $x \in [x_0,x_n]$ is known as interpolation. The points $x_0, x_1, x_2,\ldots.x_n$ are known as interpolation or mesh points. The process of computing the value of y at any outside point of a given range is called as extrapolation.

Interpolation

The theme of interpolation is to construct a new function $\Phi(x)$ which coincides with the unknown function $f(x)$ at the set of tabulated points. The process of finding $\Phi(x)$ is the interpolation. If the $\Phi(x)$ is a polynomial then process is called the polynomial interpolation. The process of finding $\Phi(x)$ is done with the help of calculus of finite differences which deals with the changes in the dependent variable due to changes in the independent variable.

Finite Differences

Consider the function $y = f(x)$. The values of y corresponding to different values of x are obtained by substituting the various values of x. Let the consecutive values of x differing by h be x_0, x_0+h, $x_0+2h = x_2, \ldots x_0 + nh = x_n$ and corresponding values of $y = f(x)$ be $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$ ….. $y_n = f(x_n)$.

The values of the independent variables x are called arguments and corresponding values of dependent variable y are called entries.

Difference operators

Δ	**Forward Difference Operator**
∇	Backward Difference Operator
E	Shift Operator
δ	Central Difference Operator
μ	Average Operator

Forward Differences

The first forward difference is defined by $\Delta f(x) = f(x+h) - f(x)$ where Δ is called the forward or descending difference operator.(Δ= Delta)

In particular $\Delta f(x_0) = f(x_0+h) – f(x_0)$

$\Delta y_0 = y_1 - y_0$

$\Delta f(x_0+h) = f(x_0+2h) – f(x_0+h)$

$\Delta y_1 = y_2 – y_1$

……….

$\Delta y_{n-1} = y_n – y_{n-1}$

The differences of first forward differences are called the second forward differences.

In general, n^{th} forward difference is defined by

$$\Delta^n f(x_1) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_0$		
x_3	y_3		$\Delta^2 y_2$			
		Δy_3				
x_4	y_4					

1. Construct a forward difference table for the following data

x	0	10	20	30
y	0	0.174	0.347	0.518

Solution: The forward difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		0.174		
10	0.174		-0.001	
		0.173		-0.001
20	0.347		-0.002	
		0.171		
30	0.518			

2. Construct a forward difference table for $y = f(x) = x^3 + 2x + 1$ for $x = 1,2,3,4,5$.

Solution: $y = f(x) = x^3 + 2x + 1$ for $x = 1,2,3,4,5$

x	1	2	3	4	5
y	4	13	34	73	136

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4				
		9			
2	13		12		
		21		6	
3	34		18		0
		39		6	
4	73		24		
		63			
5	136				

Backward Differences

The first backward difference is defined by

$$\nabla f(x) = f(x) - f(x + h)$$

where is called the backward or ascending difference operator and h is interval difference.

$$\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$$

Backward Difference Table

X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
X_0	Y_0					
		∇y_1				
X_1	Y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
X_2	Y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		
X_3	Y_3		$\nabla^2 y_4$			
		∇y_4				
X_4	Y_4					

The Central Difference

The first order central difference is defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

Where δ is the central difference operator.

Central Difference Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_4$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

The Shift Operator E

The shift operator E is defined by

$Ef(x) = f(x+h)$

$E^2f(x) = E(Ef(x)) = E(f(x+h)) = f(x+2h)$

$E^3f(x) = f(x+3h)$

.

.

.

$E^nf(x) = f(x+nh)$

The inverse shift operator E^{-1} is defined as

$E^{-1} f(x) = f(x-h)$

$E^{-2} f(x) = f(x-2h)$

.

.

.

$E^{-n} f(x) = f(x-nh)$

The Average Operator μ

The average operator μ is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2} \right) + f\left(x - \frac{h}{2} \right) \right]$$

And

$$\mu^2 f(x) = \mu[\mu f(x)] = \mu\left[\frac{f\left(x + \frac{h}{2} \right) + f\left(x - \frac{h}{2} \right)}{2} \right] = \frac{1}{4}[f(x+h) - f(x-h)]$$

Relation between the operators

Relationship between $\Delta, \nabla, \delta, \mu, E$ **or** E^{-1}.

a) $\Delta\nabla = \nabla\Delta = \delta^2$

b) $\mu^2 = 1 + \dfrac{\delta^2}{4}$

c) $\Delta = E - 1$

d) $\nabla = 1 - E^{-1}$

e) $\delta = E^{1/2}\nabla$

f) $\delta = E^{-1/2}\Delta$

Exercise

1. If $f(x) = x^2 + 1$, h=2, E(x) is given by

a) $x^2 + 3$

b) $x^2 - 4x + 5$

c) $x^2 + 4x$

d) $x^2 + 4x + 5$

2. For $f(x) = x^2$, h = 1, first forward difference of f(x) is given by

a) 2x-1

b) 2x-1

c) 3x+1

d) 3x-1

3. If $f(x) = x^2 - 2$, h=1, first backward difference ∇ f(x) is given by

a) $2x - 1$

b) $x - 5$

c) $3x + 2$

d) $2x - 5$

4. For f(x) = x^2 , h=1, δ f(x) is given by

a) -2x   b) $2x^2$ c) 2x   d) 3x

5. For f(x) = x^2, h = 1, $\Delta\nabla f(x)$ is given by

a) - 2   b) 1   c) 2   d) -1

6. For f(x) = x^2, h = 1, $\delta^2 f(x)$ is given by

a) 2   b) 0   c) $2x^2$ d) 3x

7. For f(x) = x^2, h = 1, $\mu^2 f(x)$ is given by

a) x^2 +1   b) x^2 - 1/2

c) x^2 d) x^2+1/2

8. For f(x) = x^2 , h = 2, second forward difference $\Delta^2 f(x)$ is given by

a) 6 b) 12 c) 4 d) 8

9. The set of points (x_1,y_1), I = 0, 1, 2,3 are (0,1), (1,1), (2,7), (3,25) then the value of $\Delta^2 y_0$ is

a) 12 b) 18 c) 6 d) -12

10. The set of points (x_1,y_1), I = 0, 1, 2,3 are (0,1), (1,1), (2,7), (3,25) then the value of $\Delta^3 y_1$ is

a) 12 b) 18 c) 6 d) -12

11. The set of points (x_1,y_1), I = 0, 1, 2,3 are (0,1), (1,1), (2,7), (3,25) then the value of $\Delta^2 y_1$ is

a) 18 b)12 c) 6 d) -12

Interpolation

In engineering applications, data collected from the field are usually discrete and the physical meanings of the data are not always well known. To estimate the outcomes and, eventually, to have a better understanding of the physical phenomenon, a more analytically controllable function that fits the field data is desirable. The process of finding the coefficients for the fitting function is called **curve fitting**; the process of estimating the outcomes in between sampled data points is called **interpolation**; whereas the process of estimating the outcomes beyond the range covered by the existing data is called **extrapolation**.

Lagrange's Interpolating polynomials

Lagrange polynomials are used for Polynomial interpolation. For a given set of points (x_j , y_j) with no two x_j values equal, the Lagrange polynomial is the polynomial of lowest degree that assumes at each value x_j the corresponding value y_j, so that the functions coincide at each point.

Given a set of $k + 1$ data points (x_0 , y_0) , … , (x_j , y_j) , … , (x_k , y_k) where no two x_j are the same, then **the Lagrange's interpolating polynomial** is given by

$$L(x) = \sum_{j=0}^{k} y_j l_j(x)$$

where

$$l_j(x) = \prod_{\substack{0\le m\le k \\ m\ne j}} \frac{x - x_m}{x_j - x_m} = \frac{(x - x_0)}{(x_j - x_0)} \cdots\cdots \frac{(x - x_{j-1})}{(x_j - x_{j-1})} \frac{(x - x_{j+1})}{(x_j - x_{j+1})} \cdots \frac{(x - x_k)}{(x_j - x_k)}, 0 \le j \le k$$

1. Lagrange's interpolating polynomial

i) Passing through two points (x_0, y_0), (x_1, y_1)

$$y = \frac{(x-x_1)}{(x_0-x_1)}y_0 + \frac{(x-x_0)}{(x_1-x_0)}y_1$$

ii) Passing through three points (x_0, y_0), (x_1, y_1) (x_2, y_2)

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

iii) Passing through four points(x_0, y_0), (x_1, y_1) (x_2, y_2) (x_3, y_3)

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Q1. Find Lagrange's long interpolation polynomial passing through set of points

x	**0**	**1**	**2**
y	2	3	6

use it to find y at x = 1.5 and find $\int_0^2 y\,dx$. (May – 2014, Nov 19)

Solution: Here $y = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$ ……(1)

$$L_0(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x^2 - 3x + 2)$$

$$L_1(x) = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -1(x^2 - 2x)$$

$$L_2(x) = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}(x^2 - x)$$

Putting in (1)

$$y = 2.\ \frac{1}{2}(x^2 - 3x + 2) + 3(-1)(x^2 - 2x) + 6.\ \frac{1}{2}(x^2 - x)$$

$$= x^2 + 2$$

$\therefore$

$$y\,|_{x=1.5} = (1.5)^2 + 2$$

$$= 2.25 + 2 = 4.25$$

And $\int_0^2 y\,dx = \int_0^2 (x^2 + 2)\,dx = \left(\frac{x^3}{3} + 2x\right)_0^2$

$$= (8/3 + 4) - 0 = 20/3$$

Exercise

1. Lagrange's polynomial through the points

x	0	1
y	4	3

Is given by

a) $y = 2x - 3$ c) $y = x + 4$

b) $y = - x + 3$ d) $y = -x + 4$

2. If Lagrange's interpolation polynomial passing through the points

x	0	2	3
y	1	3	2

Then the value of y at $x = 1$ is given by

a) 7/3 b) 5/3 c) 8/3 d) 5/4

3. If Lagrange's polynomial passes through

x	0	1
y	-4	-4

Then dy/dx at $x = 1$ is given by

a) 0 b) 2 c) 1 d) ½

4. If Lagrange's polynomial passes through

x	0	1
y	1	2

Then $\int_0^2 y dx$ is equal to

a) -1 b) – 2 c) – 1 d) 2

5. Lagrange's polynomial through the points

x	0	1	2
y	4	0	6

Is given by

a) $y = 5x^2 - 3x + 4$

c) $y = 5x^2 + 3x + 4$

b) $y = 5x^2 - 9x + 4$

d) $y = 5x^2 - 9x + 4$

Exercise

Q1. Find Lagrange's interpolating polynomial passing through set of points

x	0	1	2
y	4	3	6

Use it to find y at x =2, $\frac{dy}{dx}$ at x =0.5 and find $\int_0^3 y\,dx.$ (Nov 14, May 16, May 19)

Q2. Find Lagrange's interpolating polynomial passing through the points (May 17)

x	0	1	3
y	3	4	12

Q3. Find Lagrange's interpolating polynomial passing through set of points

x	0	1	2
y	2	5	10

Use it to find y at x =0.5 and find $\frac{dy}{dx}$ at x = 0. (Nov 18)

Newton’s Forward Difference Interpolation

It is used for evenly spaced data.

We know

$$f(x_0+uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots.$$

$$Ef(x_0) = f(x_0+h)$$

$$OR$$

$$E^u f(x_0) = f(x_0+uh)$$

$$\therefore \quad f(x_0+uh) = E^u f(x_0) = E^u y_0$$

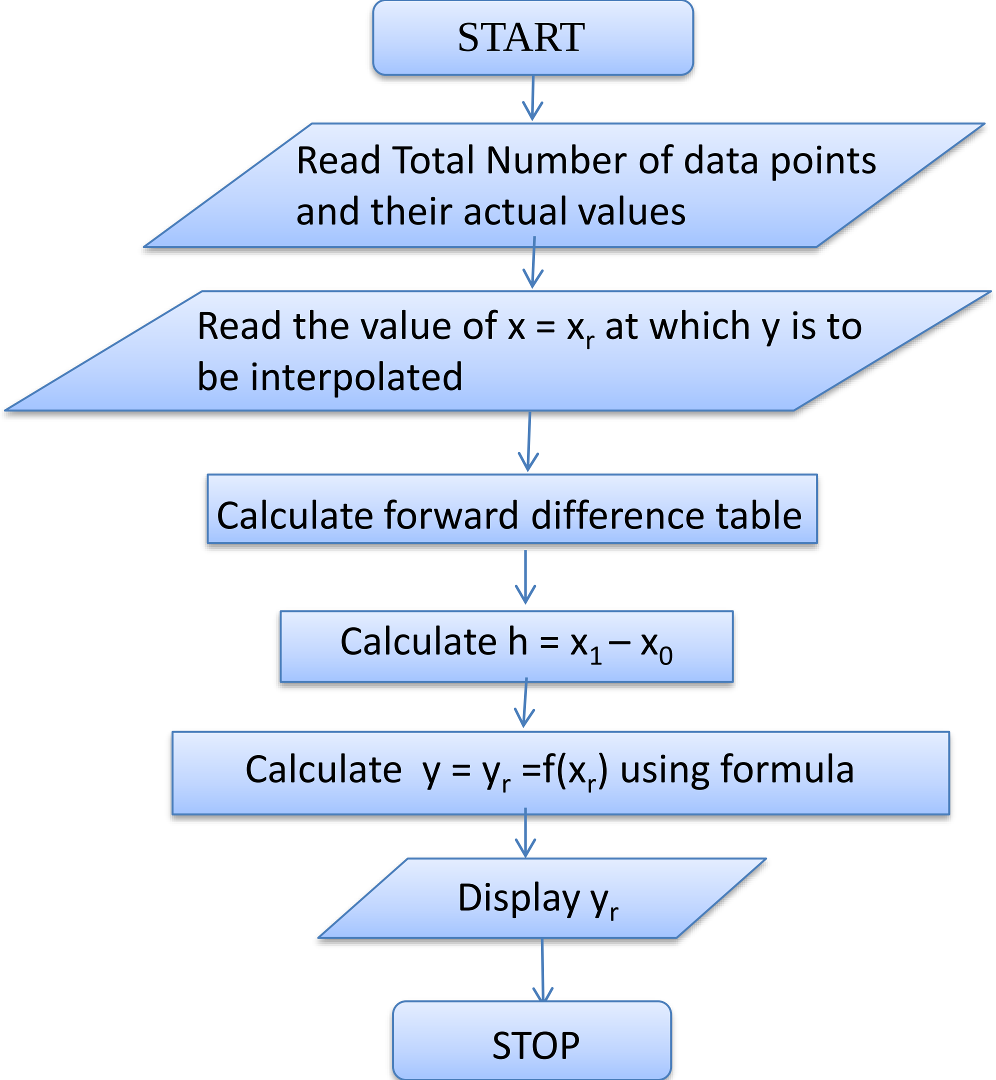
But $E = 1+\Delta$

$$\therefore \quad f(x_0+uh) = (1+\Delta)^u y_0$$

Expanding by Binomial theorem

$$\boxed{f(x_0+uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots.}$$

Putting for u = $(x-x_0)/h$ in (1), we get **Newton-Gregory forward difference interpolation polynomial** passing through a set of points (x_i, y_i), i = 0(1)n .


START
Read Total Number of data points and their actual values
Read the value of $x = x_r$ at which y is to be interpolated
Calculate forward difference table
Calculate $h = x_1 – x_0$
Calculate $y = y_r = f(x_r)$ using formula
Display y_r
STOP

Q1. Find value of y for x = 0.5 for the following table of x, y values using Newton's forward difference formula:

X	0	1	2	3	4
y	1	5	25	100	250

Ans: Preparing forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		4			
1	5		16		
		20		39	
2	25		55		-19
		75		20	
3	100		75		
		150			

Here h = 1, x_{given} = 0.5

$x_0 = 0$, $y_0 = 1$

u= $\frac{x_{given} - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$

At x = 0.5

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

$$y = 1 + 0.5(4) + \frac{0.5(0.5-1)}{2}16 + \frac{0.5(0.5-1)(0.5-2)}{6}39 +$$

$$\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(-19)$$

y = 4.1796

$\therefore$ x = 0.5 , y = 4.1796

Q2. An experiment carried out on a circuit give the following reading:

V_{in} (volts)	0	0.01	0.02	0.03	0.04	0.05
V_{out} (volts)	0	1.52	2.65	3.84	4.65	6.25

Using the above table evaluate V_{out} at V_{in} = 0.025 volts

Solution:

Newton's Backward Difference Interpolation

We know $E^{-1} = 1 - \nabla$

If we express x as $x = x_n + uh$

Then $f(x) = f(x_n + uh) = E^u f(x_n)$

$$\therefore \quad y = (E^{-1})^{-u} y_n$$

$$y = (1 - \nabla)^{-u} y_n$$

Expanding by Binomial theorem , we get

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots\ldots$$

where

$$x = x_n + uh$$

Q1. The distance travelled by a point p in XY – plane in a mechanism is given by y in the following table. Estimate distance travelled by p when x = 4.5.

x	1	2	3	4	5
y	14	30	62	116	198

Solution: Backward difference table:

x	**y**	∇ **y**	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	14				
		16			
2	30		16		
		32		6	
3	62		22		0
		54		6	
4	116		28		
		82			
5	198				

n = 5,

$$y = y_5 + u\nabla y_5 + \frac{u(u+1)}{2!}\nabla^2 y_5 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_5 + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_5$$

$y_5 = 198$, $x = x_n + uh$, $x = 4.5$, $h = 1$, $x_n = x_5 = 5$

$$y = 198 + (x-5)82 + \frac{(x-5)(x-4)}{2!}28 + \frac{(x-5)(x-4)(x-3)}{3!}6 + 0$$

$$y = x^3 + 2x^2 + 3x + 8$$

At $x = 4.5$, $y = 153.125$

Exercise

Q1. If **f(x)** is known at the following data points then

x_i	0	1	2	3	4
f_i	1	7	23	55	109

find **f(0.5)** and **f(1.5)** using Newton's forward difference formula.

Q2. Find the value of tan 0.12

x	0.10	0.15	0.20	0.25	0.30
y = tan x	0.1003	0.1511	0.2027	0.2553	0.3093

Q3. Find value of y(300) by Newton's Backward difference formula

x	50	100	150	200	250
y	618	724	805	906	1032

Q4. A test performed on NPN transistor gives the following result:

Base current I_b	0	0.01	0.02	0.03	0.04	0.05
Collector Current I_c	0	1.2	2.5	3.6	4.3	5.34

Calculate i) The value of the collector current for the base current of 0.005 mA.

ii) The value of base current required for a collector current of 4.0mA

Need of Numerical Integration

Many a times we encounters integrals that are very difficult or even impossible to solve analytically. For example the Fresnel integral given by

$$S(x) = \int_0^x \sin(x^2)dx \qquad C(x) = \int_0^x \cos(x^2)dx$$

cannot be evaluated by the usual methods of calculus. So in such cases we need methods form numerical analysis to evaluate such integrals.

Also we need numerics when the integrand of the integral to be evaluated consists of an empirical function, where we are given some recorded values of that functions.

So in all such cases method used in such kind of problems is called method of numerical integration.

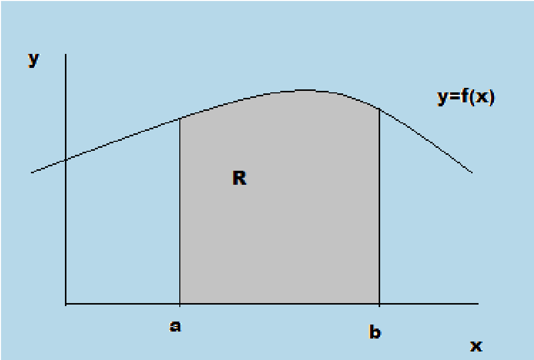
Numerical Integration

Numerical Integration means the numeric evaluation of integrals

$$J = \int_a^b f(x)dx$$

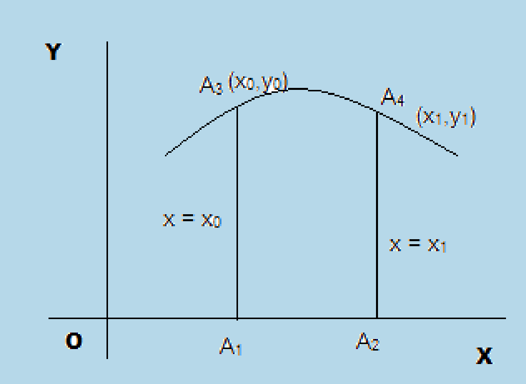
where a and b are given and f is a function given analytically by a formula or empirically by a table of values.

Geometrically, J is the area under the curve of f between a and b.



Trapezoidal rule

Consider the area A_1, bounded by the ordinates $x = x_0$, $x = x_1$, X – axis and the curve joining the points A3 and A4. Approximating curve joining the two points (x_0,y_0), (x_1,y_1) is given by Newton's interpolating polynomial



$$y = y_0 + u\Delta y_0$$

Where $u = \dfrac{x - x_0}{h}, \Delta y_0 = y_1 - y_0, x_1 - x_0 = h$

$$A_1 = \int_{x_0}^{x_1} y\,dx = \int_{x}^{x_1} (y + u\Delta y_0)\,dx$$

Trapezoidal rule

When $x = x_0, u = 0, x = x_1, u = \dfrac{x_1 - x_0}{h} = \dfrac{h}{h} = 1, dx = hdu$

$$\therefore \quad A_1 = \int_0^1 \{y_0 + u(y_1 - y_0)\} hdu$$

$$= h\left[y_0 u + \frac{u^2}{2}(y_1 - y_0) \right]_0^1 = h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$A_1 = \frac{h}{2}[y_0 + y_1]$$

Similarly we can find area under the curve joining the points $(x_1,y_1),(x_2,y_2)$ from above relation

$$A_2 = \frac{h}{2}[y_1 + y_2]$$

Trapezoidal rule

Proceeding in this manner, we get

$$A_3 = \frac{h}{2}[y_2 + y_3]$$

…………….

$$A_r = \frac{h}{2}[y_{r-1} + y_r]$$

$\therefore \quad \int_{x_0}^{x_n} y\,dx =$ Total area under the curve passing through the points $(x_0,y_0),(x_1,y_1),\ldots(x_n,y_n)$.

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[(y_0 + y_1) + (y_1 + y_2) + \ldots(y_{r-1} + y_r)\ldots(y_{n-1} + y_n)]$$

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots.y_{n-1})]}$$

Example

Q. Use Trapezoidal rule to estimate the value of :

$$\int_0^2 \frac{x}{\sqrt{2+x^2}}dx \quad \text{by taking } h = 0.5$$

Solution:

$$f(x) = \frac{x}{\sqrt{2+x^2}}$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots. + y_{n-1})\right]$$

$x_0 = 0$, $y_0 = 0$

$x_1 = 0.5$, $y_1 = \dfrac{0.5}{\sqrt{2+(0.5)^2}} = 0.333$

Example

$x_2 = 1.0$, $y_2 = 0.57735$

$x_3 = 1.5$, $y_3 = 0.72761$

$x_4 = 2.0$, $y_4 = 0.816497$

$$\int_0^2 \frac{x}{\sqrt{2+x^2}}\,dx = \frac{0.5}{2}\left[(0+0.816497)+2(0.3333+0.57735+0.72761)\right]$$

$$\int_0^2 \frac{x}{\sqrt{2+x^2}}\,dx = 1.02327$$

1. A curve passes through the set of points

X	0	1	2	3	4
y	1	2.72	7.39	20.09	54.6

Value of $\int_0^4 y dx$ by trapezoidal rule is given by

a) 58 b) 54.95 c) 53.87 d) 55.60

2. The value of $\int_0^{\pi/2} \sqrt{cos\theta} d\theta$ evaluated by trapezoidal rule taking h = π/4 is given by (Given: cos π/4 = 0.8409)

a) 1.8981 b) 1.0526 c) 2.55 d)2.9898

3. For the tabulated data, speeds of moving object at different times are recorded as

θ	0	π/6	π/3	π/2
sin θ	0	0.5	0.866	1

Evaluation of $\int_0^{\pi/2} \sin\theta d\theta$ using trapezoidal rule gives

a) 0.977 b) 0.988 c) 0.966 d) 0.954

4. Value of π obtained by evaluating the integral $\int_0^1 \frac{1}{1+x^2}\, dx$ by trapezoidal rule with h = ½ is given by (Given : $\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\pi}{4}$)

a) 3.15 b) 3.2 c) 3.1 d) 3.3

5. A curve passes through the set of points

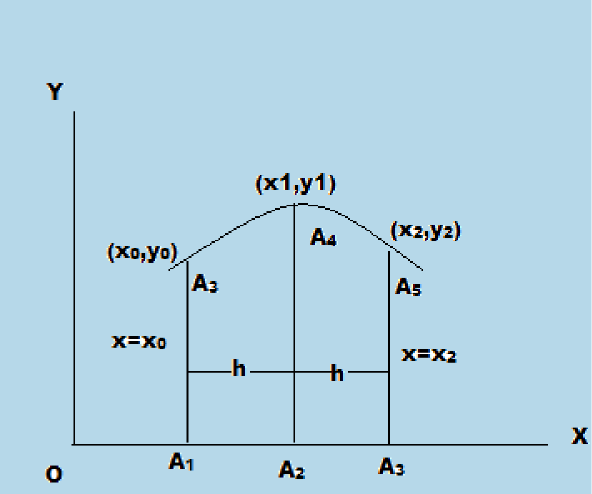
X	0	1	2	3
y	1	3	7	13

Value of $\int_0^3 y\, dx$ by trapezoidal rule is given by

a) 21 b) 15 c) 19 d) 17

Simpson's 1/3rd rule

In this method, we find the value of integral $\int_{x_0}^{y_0} y\,dx$ first by finding the area of the double strip under the curve passing through the points$(x_0,y_0),(x_1,y_1),(x_2,y_2)$.



Equation of curve is given by Newton's interpolating polynomial through (x_0,y_0), (x_1,y_1), (x_2,y_2) .

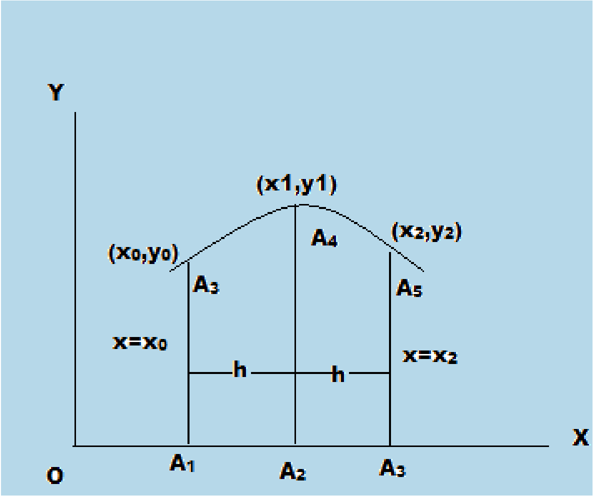
$$y = y_0 + u\Delta y_0 + \frac{u(u+1)}{2!}\Delta^2 y_0, u = \frac{x - x_0}{h}$$

Simpson's 1/3rd rule

Area A_1 of the double strip is

$$A_1 = \int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_2} y \frac{dx}{du}\,du$$

when x = x_0 ,u = 0, x = x_2 , u = 2



$$A_1 = \int_0^2 \left\{ y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 \right\} h\,du$$

$$= h\left[y_0 u + \frac{u^2}{2}(y_1 - y_0) + \left(\frac{u^3}{6} - \frac{u^2}{4}\right)(y_2 - 2y_1 + y_0) \right]_0^2$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \left(\frac{8}{6} - 1\right)(y_2 - 2y_1 + y_0) \right]$$

Simpson's 1/3rd rule

$$A_1 = h\left[2y_1 + \frac{1}{3}\left(y_2 - 2y_1 + y_0\right)\right]$$

$$= \frac{h}{3}\left[6y_1 + y_2 - 2y_1 + y_0\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$A_2 = \frac{h}{3}\left[y_2 + 4y_3 +\right]$$

Similarly, the area A_2 of the next double strip bounded by the curve passing through the points (x_2,y_2), (x_3,y_3),(x_4,y_4) is given by

$$A_2 = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

Simpson's 1/3rd rule

Proceeding in this manner we can find the area of consecutive strips A_3, A_4, A_5, …., A_{n-2} .

Thus, A_{n-2} = Area of last strip

$$A_{n-2} = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Total area under the cur +ve is given by

$$A = A1 + A2 + A3 + \ldots\ldots.$$

$$A = \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots. + (y_{n-2} + 4y_{n-1} + y_n)\right]$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right]$$

Example

Q. Evaluate $\int_{1}^{2} \frac{dx}{x}$ using Simpson's (1/3) rule taking h=0.25

(Dec - 2014)

Solution: h = 0.25

$$y_o = 1, y_1 = \frac{1}{1.25} = 0.8, y_2 = \frac{1}{1.5} = 0.667, y_3 = \frac{1}{1.75} = 0.57143, y_4 = 0.5$$

$$I_1 = \int_{1}^{2} \frac{1}{x} dx = \frac{0.25}{3}[(1+0.5)+4(0.8+0.57143)+2(0.667)]$$

$$= \frac{0.25}{3}(8.31972)$$

$$= 0.69331$$

1. The value of $\int_1^2 \frac{1}{x} dx$ evaluated by Simpson's 1/3rd rule taking h = 0.5 is given by

a) 0.6612 b) 0.6842 c) 0.6742 d) 0.6944

2. The value of $\int_1^2 e^{-x^2}$ dx evaluated by Simpson's 1/3rd rule taking h = 0.2 is given by

Given:

X	0	0.2	0.4	0.6	0.8
e^{-x^2}	1	0.960	0.8521	0.6977	0.5272

a) 0.5878 b) 0.6577 c) 0.4353 d) 0.5345

3. The value of $\int_0^{\pi/2} \frac{sinx}{x} dx$ evaluated by simpson's 1/3rd rule taking h = π/8 is given by

Given:

x	0	π/8	π/4	3π/8	π/2
$\frac{sinx}{x}$	0	0.9744	0.9003	0.7842	0.6366

a) 1.3700 b) 1.7300 c) 1.500 d) 1.4300

4. Value of $\log_e 2$ obtained by evaluating the integral $\int_0^1 \frac{1}{1+x}dx$, using Simpson's 1/3rd rule with h = ½ is given by

(Given: $\int_0^1 \frac{1}{1+x}dx = \log_e 2$)

a) 0.5934 b) 0.6560 c) 0.6944 d) 0.6140

Simpson's 3/8th rule

Here we first determine $\int_{x_0}^{x_3} y\,dx$ by finding the area under the curve passing through the points (x_0,y_0), (x_1,y_1), (x_2,y_2) , (x_2,y_2) bounded by the ordinates $x = x_0$, $x = x_3$ and X – axis.

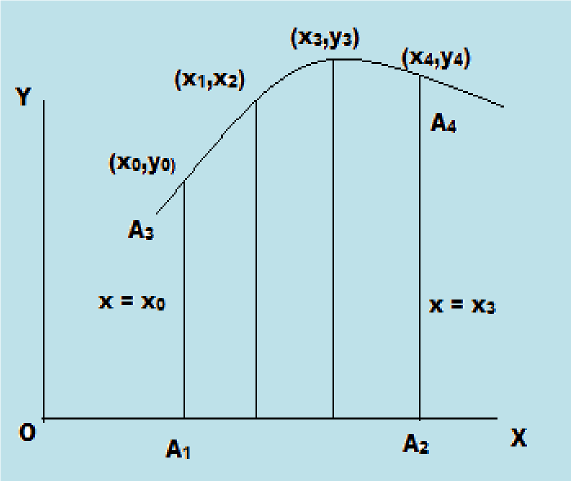
Newton's polynomial through these points is given by

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $$u = \frac{x - x_0}{h}$$

Simpson's 3/8th rule

Area A_1 of the triple strip



$$\int_{x_0}^{x_3} ydx = h\int_0^3 \left\{ y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 \right\} du$$

$$= h\left[y_0 u + \frac{u^2}{2}(y_1 - y_0) + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)(y_2 - 2y_1 + y_0)\right.$$

$$\left. + \frac{1}{6}\left(\frac{u^4}{4} - 3\frac{u^3}{3} + 2\frac{u^2}{2}\right)(y_3 - 3y_2 + 3y_1 - y_0)\right]_0^3$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{h}{8}\left[24y_0 + 36y_1 - 36y_0 + 18y_2 - 36y_1 + 18y_0 + 3y_3 - 9y_2 + 9y_1 - 3y_0\right]$$

$$A_1 = \int_{x_0}^{x_1} ydx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Simpson's 3/8th rule

Similarly, the next area bounded by the curve passing through the points (x_3,y_3), (x_4,y_4), (x_5,y_5) and (x_6,y_6) is given by

$$A_2 = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

Finding successive areas

$$A_{last} = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding all area's

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9) + 3(y_1 + y_2 + y_4 + y_5 + ...)\right]$$

Example

Q. Evaluate: $\int_0^1 \frac{1}{1+x}dx$ by Simpson's $\left(\frac{3}{8}\right)th$ rule.

Solution: Taking 6 subintervals $h = \frac{1-0}{6} = \frac{1}{6}$

X	0	1/6	2/6	3/6	4/6	5/6	1
Y	1	6/7=0.8571	6/8=0.75	6/9=0.666	6/10=0.6	6/12=0.5545	0.5

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{8}\left[(y_o + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3}{8}\left(\frac{1}{6}\right)\left[\left(1+\frac{6}{12}\right) + 3\left(\frac{6}{7}+\frac{6}{8}+\frac{6}{10}+\frac{6}{11}\right) + 2\left(\frac{6}{9}\right)\right]$$

$$= 0.6931$$

1. Value of $\log_e 7$ obtained by evaluating the integral 7 obtained by evaluating the integral $\int_0^6 \frac{1}{1+x} dx$ using Simpson's 3/8th rule with h = 2 is given by (Given: $\int_0^6 \frac{1}{1+x} dx = \log_e 7$)

a) 1.931 b) 2.124 c) 1.912 d) 2.057

2. The table below shows the temperature f(t) as function of time

Time, t	1	2	3	4
Temperature,f(t)	81	75	80	83

Using Simpson's 3/8th rule the value of $\int_1^4 f(t)\, dt$ is

a) 215.87 b) 240.87 c) 235.87 d) 225.87

3. Speeds of moving object at different times are recorded as

t (hrs)	0	1	2	3
v (km/hr)	20	40	45	30

Using Simpson's 3/8th rule, distance travelled in 3 hours is given by

a) 116.5 b) 114.375 c) 118.525 d)120.125

Exercise

Q1. Use Trapezoidal rule to estimate the value of : $\int_0^2 \frac{1}{1+x^4}\,dx$ by taking h = 0.5. (May 17)

Q2. Use Simpson's 1/3rd rule to find $\int_0^{0.6} dx$ by taking seven ordinates. (May 14)

Q3. Use Simpson's 1/3rd rule to find $\int_1^2 \frac{dx}{x}$ by taking h = 0.25. (Nov 15)

Q4. Stating the formula for Simpson's 1/3rd rule, evaluate $\int_{1}^{1.04} f(x)$ from the following data: (Dec 17)

X	1	1.01	1.02	1.03	1.04
y	3.953	4.066	4.182	4.182	4.421

Q5. Use Simpson's 1/3rd rule to find $\int_{1}^{2} \frac{dx}{x^2}$ by taking h = 0.25. (Nov 18)

Q6. Evaluate $\int_{0}^{1} \frac{dx}{1+x^2}$ using Simpson's 3/8 rule taking h = 1/6. (Nov 14)

Q7. Evaluate $\int_{0}^{3} \frac{dx}{1+x}$ by Simpson's 3/8 rule by taking 7 ordinates.

(Nov 16, May19)

THANK YOU