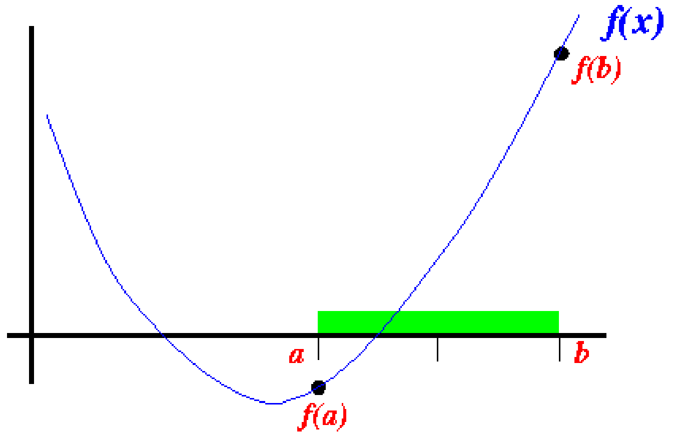
Bisection Method

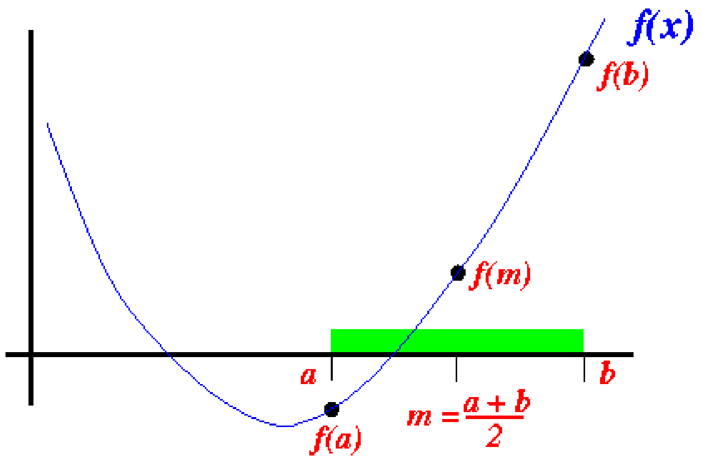
- One of the simplest and most reliable iterative method for the solution of non linear equations.
- Also known as binary chopping or half – interval method.
- Based on the repeated application of the intermediate value theorem.
- The Bisection Method is given and initial interval [a, b] that contains a root (We can choose a and b such that f(a) and f(b) are of opposite sign)
- The Bisection Method will cut the interval into 2 halves and check which half interval contains a root of the function.
- The Bisection Method will keep cut the interval in halves until the resulting interval is extremely small. The root is then approximately equal to any value in the final interval.

Graphical Representation

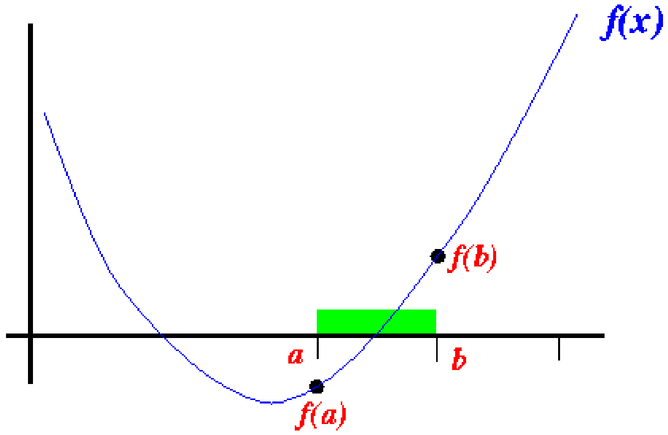
Suppose the interval [a, b] is as follows



We cut the interval [a, b] in middle: m = (a + b)/2



Because sign of f(a) ≠ sign of f(m) , we proceed with the search of new interval [a, b].



Bisection Method

- Let x_1 = a and x_2 = b.
- Let us also define an another point x0 to be the midpoint between a and b.
- i.e. $x_0 = \frac{x_1+x_2}{2}$
- Now there exists three conditions

i. If $f(x_0) = 0$ we have root at x_0`

ii. If $f(x_0)f(x_1)<0$, then there is a root between x_0 and x_1.

iii. If $f(x_0)f(x_2)<0$, then there is a root between x_0 and x_2.

iv.

Bisection method is thus computationally simple and the sequence of approximations always converges to the root for any f(x) which is continuous in the interval that contains the root. If the permissible error is ε then the approximate number of iterations required can be determined from the relation

$$\frac{b-a}{2^n} \leq \varepsilon$$

Or $$\log(b-a) - n\log 2 \leq \log\varepsilon$$

Or $$n \geq \frac{\log(b-a) - \log\varepsilon}{\log 2}$$

In particular, the minimum no. of iterations required for converging to a root in the interval (0,1) for a given ε are as under

ε	10^{-2}	10^{-3}	10^{-4}
n	7	10	14

Examples

Q1. Use the method of Bisection method to find a root of the equation

$f(x) = x^4 + 2x^3 - x - 1 = 0$ lying in the interval [0,1] at the end of sixth iteration. How many iterations are required if the permissible error is $\xi = 0.0005$

Solution: Here f(0) = -1

f(1) = 1 [Root lies between 0 and 1, $\xi_1 = \frac{0+1}{2} = 0.5$]

f(0.5) = -1.1875 [Root lies between 0.5 and 1, $\xi_2 = \frac{0.5+1}{2} = 0.75$]

f(0.75) = -0.5898 [Root lies between 0.75 and 1, $\xi_3 = \frac{0.75+1}{2} = 0.875$]

f(0.875) = 0.0510[Root lies between 0.875 and 0.75, $\xi_4 = \frac{0.75+0.875}{2} = 0.8125$]

f(0.8125) = - 0.3039 [Root lies between 0.875 and 0.8125, $\xi_5 = \frac{0.875+0.8125}{2} = 0.84375$]

f(0.84375) = -0.642[Root lies between 0.84375 and 0.875,
$\xi_6 = \frac{0.84375+0.875}{2} = 0.859375$]

Thus, approximate root at the end of sixth iteration is 0.859375

$|\xi_6 - \xi_5| = |0.859375 - 0.844375| = 0.015622 < 0.02$

Thus, if permissible error is $\varepsilon = 0.02$ then root at the end of sixth iteration gives required accuracy.

If permissible error is 0.0005 then by the equation

$$n \geq \frac{\log(1-0) - \log 0.0005}{\log 2} = 10.965$$

i.e. n = 11 iterations are required to achieve the degree of accuracy.

Q2. Using method of Bisection, find the cube root of 100, using six iterations. Determine whether number of iterations are enough for three significant digit accuracy.

Solution: Cube root of 100, is obtained by solving the equation $x^3 - 100 = 0$

$f(x) = x^3 - 100$

f(4) = 64-100= -36, f(5) = 125 - 100 = 25

Root lies between 4 and 5, $\xi_1 = \frac{4+5}{2} = 4.5$

f(4.5) = -8.875,root lies between 4.5 and 5

$$\xi_2 = \frac{4.5+5}{2} = 4.75$$

f(4.75) = 7.17,root lies between 4.5 and 4.75

$$\xi_3 = \frac{4.5+4.75}{2} = 4.625$$

f(4.625) = -1.068,root lies between 4.625 and 4.75

$$\xi_4 = \frac{4.625+4.75}{2} = 4.6875$$

f(4.6875) = 2.99,root lies between 4.625 and 4.6875

$$\xi_5 = \frac{4.625+4.6875}{2} = 4.65625$$

f(4.65625) = 0.95,root lies between 4.625 and 4.65625

$$\xi_6 = \frac{4.625+4.65625}{2} = 4.640625$$

Thus $\xi_6 = 4.640625$ is the approximate value of cube root at the end of sixth iteration.

$$|\xi_6 - \xi_5| = |4.640625 - 4.65625| = 0.015625$$

ξ_5, ξ_6 will agree to three significant digits if

$$|\xi_6 - \xi_5| < \frac{1}{2} * 10^{0-3+1} = \frac{1}{2} * 10^{-2} = 0.005$$

But 0.015625 is not less than 0.005 therefore two approximations do not agree to three significant digits. Hence number of iterations are not sufficient for three significant digit accuracy.

If $\varepsilon = 0.05$ then number of iterations required

$$n \geq \frac{\log(5-4) - \log 0.05}{\log 2} = 7.64$$

i.e 8 iterations will be required for three significant digit accuracy.

Q3. Find a root of the equation $\cos x = xe^x$ (measured in radian) using the bisection method at the end of sixth iteration.

Solution: Let $f(x) = \cos x = xe^x$

Since f(0) = 1 and f(1) = -2.18

Root lies between 0 and 1. The first approximation to the root is

$$\xi_1 = \frac{0+1}{2} = 0.5$$

$f(\xi_1) = 0.05$, Root lies between 0.5 and 1

$$\xi_2 = \frac{0.5+1}{2} = 0.75$$

$f(\xi_2) = -0.86$, Root lies between 0.5 and 0.75

$$\xi_3 = \frac{0.5+0.75}{2} = 0.625$$

$f(\xi_3) = -0.36$, Root lies between 0.5 and 0.625

$$\xi_4 = \frac{0.5+0.625}{2} = 0.5625$$

$f(\xi_4) = -0.14$, Root lies between 0.5 and 0.5625

$$\xi_5 = \frac{0.5+0.5625}{2} = 0.5312$$

$f(\xi_5) = -0.041$, Root lies between 0.5 and 0.5312

$$\xi_6 = \frac{0.5+0.5312}{2} = 0.5156$$

Hence the desired approximation to the root is 0.5156.

Secant Method

In numerical analysis, the **secant method** is a root-finding **algorithm** that uses a succession of roots of **secant** lines to better approximate a root of a function f. The **secant method** can be thought of as a finite-difference approximation of Newton's **method**.

The secant method is very similar to the bisection method except instead of dividing each interval by choosing the midpoint the secant method divides each interval by the secant line connecting the endpoints. The secant method always converges to a root of f(x)=0 provided that f(x) is continuous on [a,b] and f(a)f(b)<0.

Let f(x) be a continuous function on a closed interval [a,b] such that f(a)f(b)<0. A solution of the equation f(x)=0 for x∈[a,b] is guaranteed by the Intermediate Value Theorem.

Consider the line connecting the endpoint values (a,f(a)) and (b,f(b)). The line connecting these two points is called the secant line and is given by the formula

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

The point where the secant line crosses the x-axis is

$$0 - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

which we solve for x

$$x = a - \frac{b - a}{f(b) - f(a)} \cdot f(a)$$

If x_0, x_1 are two initial approximation to the root of f(x)=0 then the next approximation x_2 is given by $x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} \cdot f_1$

where $f_0 = f(x_0)$ and $f_1 = f(x_1)$

Generalised form can be written as $x_{i+1} = x_i - \frac{(x_i - x_{i-1})}{f_i - f_{i-1}} \cdot f_i$

Examples

1. Find the positive root of the equation $x^3 - 2x^2 + 3x - 4 = 0$ at the end of fifth iteration by using secant method

Solution: To obtain initial guess we use intermediate value theorem

$f(x) = x^3 - 2x^2 + 3x - 4$ Since $f(0) = -4$, $f(1) = -2$, $f(2) = 2$

Root lies between 1 and 2 .Taking initial approximation as $x_0 = 1$ and $x_1 = 2$ now we proceed to obtain successive approximation by secant method

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} \cdot f_1$$

Here $f_0 = -2$, $f_1 = 2$ $\qquad x_2 = 2 - \frac{(2-1)}{2-(-2)} \cdot 2 = 1.5$

$$f_2 = f(x_2) = f(1.5) = (1.5)^3 - 2(1.5)^2 + 3(1.5) - 4 = -0.625$$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f_2 - f_1} \cdot f_2 = 1.5 - \frac{(1.5-2)}{(-0.625-2)} \cdot (-0.625) = 1.619$$

$$f_3 = f(x_3) = f(1.619) = (1.619)^3 - 2(1.619)^2 + 3(1.619) - 4 = -0.1417$$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f_3 - f_2} \cdot f_3 = 1.619 - \frac{(1.619-1.5)}{(-0.1417-(-0.625))} \cdot (-0.1417) = 1.654$$

$$f_4 = f(x_4) = f(1.654) = (1.654)^3 - 2(1.654)^2 + 3(1.654) - 4 = 0.015$$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f_4 - f_3} \cdot f_4 = 1.654 - \frac{(1.654-1.619)}{(0.015-(-0.1417))} \cdot (0.015) = 1.6506$$

$$f_5 = f(x_5) = f(1.6506) = (1.6506)^3 - 2(1.6506)^2 + 3(1.6506) - 4 = -0.00013$$

$$x_6 = x_5 - \frac{(x_5 - x_4)}{f_5 - f_4} \cdot f_5 = 1.6506 - \frac{(1.6506-1.654)}{(-0.00013-(0.015))} \cdot (-0.00013) = 1.65063$$

which is the required root at the end of fifth iteration. It is correct to four decimal places.

2. Find the real root of the equation $x^3 - 2x - 5 = 0$ correct upto three decimal places by using secant method

Solution: To obtain initial guess we use intermediate value theorem

$f(x) = x^3 - 2x - 5 = 0$ Since $f(1) = -7$, $f(2) = -1$, $f(3) = 16$

Root lies between 2 and 3 .

Taking initial approximation as $x_0 = 2$ and $x_1 = 3$ now we proceed to obtain successive approximation by secant method

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} \cdot f_1$$

Here $f_0 = f(x_0) = -1$, $f_1 = f(x_1) = 16$ $\quad x_2 = 3 - \frac{(3-2)}{16-(-1)} \cdot 16 = 2.058823$

$$f_2 = f(x_2) = f(2.058823) = (2.058823)^3 - 2(2.058823) - 5 = -0.390799$$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f_2 - f_1} \cdot f_2 = 2.058823 - \frac{(2.058823 - 3)}{(-0.390799 - 16)} \cdot (-0.390799) = 2.081263$$

$$f_3 = f(x_3) = f(2.081263) = (2.081263)^3 - 2(2.081263) - 5 = -0.1417204$$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f_3 - f_2} \cdot f_3 = 2.081263 - \frac{(2.081263 - 2.058823)}{(-0.1417204 - (-0.390799))} \cdot (-0.1417204) = 2.094824$$

$$f_4 = f(x_4) = f(2.094824) = (2.094824)^3 - 2(2.094824) - 5 = 0.003042$$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f_4 - f_3} \cdot f_4 = 2.094824 - \frac{(2.094824 - 2.081263)}{(0.003042 - (-0.1417204))} \cdot (0.003042) = 2.094549$$

which is the required root. It is correct up to three decimal places.

3. Find the real root of the equation $xe^x = \cos x$ correct upto four decimal places by using secant method

Solution: To obtain initial guess we use intermediate value theorem

$f(x) = \cos x - xe^x = 0$ Since $f(0) = 1$, $f(1) = -2.17798$,

Root lies between 0 and 1 .

Taking initial approximation as $x_0 = 0$ and $x_1 = 1$ now we proceed to obtain successive approximation by secant method

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} \cdot f_1$$

Here $f_0 = f(x_0) = 1$, $f_1 = f(x_1) = -2.17798$

$$x_2 = 1 - \frac{(1-0)}{-2.17798 - (1)} \cdot \cdot (-2.17798) = 0.31467$$

$$f_2 = f(x_2) = f(0.31467) = \cos(0.31467) - 0.31467 \cdot \left(e^{0.31467}\right) = 0.51986$$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f_2 - f_1} \cdot f_2 = 0.31467 - \frac{(0.31467 - 1)}{(0.51986 - (-2.17798))} \cdot (0.519869) = 0.44673$$

$$f_3 = f(x_3) = f(c) = \cos(0.31467) - 0.31467 \cdot \left(e^{0.31467}\right) = 0.20354$$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f_3 - f_2} \cdot f_3 = 0.44673 - \frac{(0.44673 - 0.31467)}{(0.20354 - (0.51986))} \cdot (0.20354) = 0.53171$$

$$f_4 = f(x_4) = f(0.53171) = \cos(0.53171) - 0.53171(e^{0.53171}) = -0.04294$$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f_4 - f_3} \cdot f_4 = 0.53171 - \frac{(0.53171 - 0.44673)}{(-0.04294 - (0.20354))} \cdot (-0.04294) = 0.51690$$

$$f_5 = f(x_5) = f(0.51690) = \cos(0.51690) - 0.51690(e^{0.51690}) = 0.00260$$

$$x_6 = x_5 - \frac{(x_5 - x_4)}{f_5 - f_4} \cdot f_5 = 0.51690 - \frac{(0.51690 - 0.44673)}{(0.00260 - (-0.04294))} \cdot (0.00260) = 0.517746$$

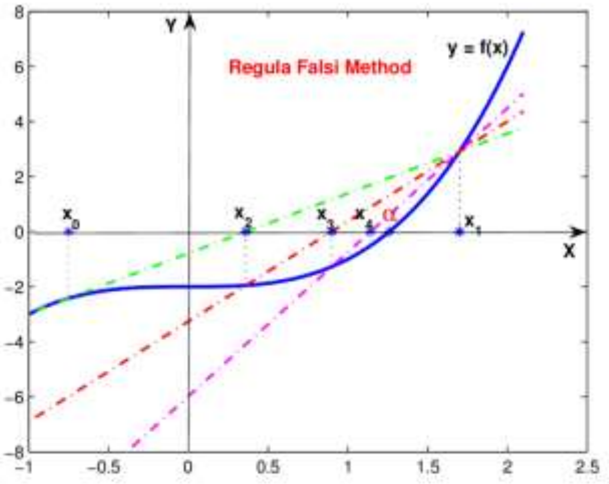
$$f_6 = f(x_6) = f(0.517746) = \cos(0.517746) - 0.517746(e^{0.517746}) = 0.000034$$

$$x_7 = x_6 - \frac{(x_6 - x_5)}{f_6 - f_5} \cdot f_6 = 0.517746 - \frac{(0.517746 - 0.51690)}{(0.000034 - (0.00260))} \cdot (0.000034) = 0.517757$$

which is the required root. It is correct up to four decimal places.

Regula-Falsi or Method of False Position

This method is based on the same principle as that of method of secant described in previous section. Only change is while choosing the initial approximations x_0, x_1 it is necessary that $f(x_0)f(x_1) < 0$. Similarly for obtaining further approximations, it must be ensured that $f_i \,.\, f_{i-1} < 0$ for i = 1,2,3,… (See in fig.) Formula used for secant method is also used for Regula-Falsi method.



Ex. 1 Use the Regula-Falsi method to find a real root of the equation $e^x - 4x = 0$, correct to three decimal places.

Sol.—

Here $f(x) = e^x - 4x$

Since f(0) = 1, f(1) = -1.28

Hence root lies between x = 0 and x = 1.

Let $x_0 = 0$, $x_1 = 1$, $f_0 = 1$ and $f_1 = -1.28$

We know, $x_{i+1} = x_i - \dfrac{(x_i - x_{i-1})}{f_i - f_{i-1}} f_i$

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} f_1 = 0.439$$

$$f_2 = e^{0.439} - 4(0.439) = -0.2048$$

Now we take the interval (0, 0.439) as $f_0 = 1$ and $f_2 = -0.2048$, $f_0\, f_2$ is –ve.

Let $x_1 = 0$ so that $f_1 = 1$

and $x_2 = 0.439$, $f_2 = -0.2048$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f_2 - f_1} f_2 = 0.3643$$

f_3= f(0.3643) = -0.0177

now the root lies in the interval (0 , 0.3643)

Let x_2 = 0 so that f_2 = 1

and x_3 = 0.3643 , f_3 = -0.0177

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f_3 - f_2} f_3 = 0.35796$$

f_4 = -0.00143

Root lies in the interval (0 , 0.35796)

Let x_3 = 0 so that f_3 = 1

and x_4 = 0.35796 , f_4 = -0.00143

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f_4 - f_3} f_4 = 0.35745$$

f_5 = -0.000121

Root lies in the interval (0 , 0.35745)

Let x_4 = 0 so that f_4 = 1

and x_5 = 0.35745 , f_5 = -0.000121

$$x_6 = x_5 - \frac{(x_5 - x_4)}{f_5 - f_4} f_5 = 0.35741$$

f_6 = -0.0000181

Thus, $|f_6| < 0.00002$ which exhibits degree of accuracy.

Root at end of fifth iteration is 0.35741 which is correct to four decimal places.

Unlike secant method there is guaranteed convergence in method of False-position

Ex. 2 Use the Regula-Falsi method to find a real root of the equation $x^3 - 2x - 5 = 0$, at the end of sixth iteration.

Sol.—

Here $f(x) = x^3 - 2x - 5 = 0$

Since f(0) = -5, f(1) = -6 , f(2) = -1 , f(3) = 16

Hence root lies between 2 and 3.

Let $x_0 = 2$, $x_1 = 3$, $f_0 = -1$ and $f_1 = 16$

We know, $x_{i+1} = x_i - \frac{(x_i - x_{i-1})}{f_i - f_{i-1}} f_i$

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} f_1 = 2.0588$$

$$f_2 = -0.3908$$

Now we take the interval (2.0588 , 3) as f_2 f_1 is –ve.

Let $x_1 = 3$, $f_1 = 16$

and $x_2 = 2.0588$, $f_2 = -0.3908$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f_2 - f_1} f_2 = 2.0812$$

$f_3 = -0.1479$

now we take the interval (2.0812 , 3) as f_3 f_1 is –ve.

Let $x_2 = 3$, $f_2 = 16$

and $x_3 = 2.0812$, $f_3 = -0.1479$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f_3 - f_2} f_3 = 2.0896$$

$f_4 = -0.0511$

Next, we take the interval (2.0896 , 3) as f_4 f_1 is –ve.

Let $x_3 = 3$ so that $f_3 = 16$

and $x_4 = 2.0896$, $f_4 = -0.0511$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f_4 - f_3} f_4 = 2.0925$$

$f_5 = -0.0229$

Next, we take the interval (2.0925 , 3) as f_5 f_1 is –ve

Let $x_4 = 1$ so that $f_4 = 16$

and $x_5 = 2.0925$, $f_5 = -0.0229$

$$x_6 = x_5 - \frac{(x_5 - x_4)}{f_5 - f_4} f_5 = 2.0938$$

Hence, the root at the end of sixth iteration is 2.0938.

Ex. 3 Solve the equation f(x) = $x - e^{-x} = 0$ by Regula-Falsi method with the initial approximations 0.5 and 1 correct upto three places of decimal.

Sol.—

Here $f(x) = x - e^{-x} = 0$

Since f(0) = -1, f(0.5) = -0.1065 , f(1) = 0.6321

Hence root lies between 0.5 and 1.

Let $x_0 = 0.5$, $x_1 = 1$, $f_0 = -0.1065$ and $f_1 = 0.6321$

We know, $x_{i+1} = x_i - \dfrac{(x_i - x_{i-1})}{f_i - f_{i-1}} f_i$

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} f_1 = 0.5721$$

$$f_2 = 0.007761$$

Now we take the interval (0.5 , 0.5721) as $f_0 f_2 < 0$

Let $x_1 = 0.5$, $f_1 = -0.1065$

and $x_2 = 0.5721$, $f_2 = 0.007761$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f_2 - f_1} f_2 = 0.5672$$

$f_3 = 0.000088$

Now we take the interval (0.5 , 0.5672) as $f_0 f_3 < 0$

Let $x_2 = 0.5$, $f_2 = -0.1065$

and $x_3 = 0.5672$, $f_3 = 0.000088$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f_3 - f_2} f_3 = 0.5671$$

Hence, the root of the equation correct upto three decimal places is 0.567

Ex. 4 Use the Regula-Falsi method to find a real root of the equation $\cos x - xe^x = 0$, taking initial approximately as $x_0 = 0$ and $x_1 = 1$, at the end of sixth iterations.

Sol.—

Here $f(x) = \cos x - xe^x$

Taking $x_0 = 0$, $x_1 = 1$f(0) = 1, f(1) = -2.17798

Hence root lies between x = 0 and x = 1.

We know, $x_{i+1} = x_i - \dfrac{(x_i - x_{i-1})}{f_i - f_{i-1}} f_i$

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} f_1 = 0.31467$$

$$f_2 = 0.51986$$

Now we take the interval (0.31467 , 1) as $f_1 f_2 < 0$

Let $x_1 = 1$ so that $f_1 = -2.17798$

and $x_2 = 0.31467$, $f_2 = 0.51986$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f_2 - f_1} f_2 = 0.44673$$

$f_3 = 0.20354$

Now the root lies in the interval (0.44673 , 1) as $f_1 f_3 < 0$

Let $x_2 = 1$ so that $f_2 = -2.17798$

and $x_3 = 0.44673$, $f_3 = 0.20354$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f_3 - f_2} f_3 = 0.49402$$

$f_4 = 0.07078$

Root lies in the interval (0.49402 , 1) as $f_1 f_4 < 0$

Let $x_3 = 1$ so that $f_3 = -2.17798$

and $x_4 = 0.49402$, $f_4 = 0.07078$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f_4 - f_3} f_4 = 0.50995$$

$f_5 = 0.02360$

Root lies in the interval (0.50995 , 1) as $f_1 f_5 < 0$

Let $x_4 = 1$ so that $f_4 = -2.17798$

and $x_5 = 0.50995$, $f_5 = 0.02360$

$$x_6 = x_5 - \frac{(x_5 - x_4)}{f_5 - f_4} f_5 = 0.51520$$

Hence the root at the end of sixth iteration is 0.51520. We note that on repeating above processes, the successive approximations are $x_7 = 0.51692$, $x_8 = 0.51748$, $x_9 = 0.51767$,

$x_{10} = 0.51775$

NEWTON RAPHSON METHOD

This method is one of the most powerful method and well known method, used for finding a root of f(x)=0,the formula many be derived in many ways the simplest way to derive this formula is by using the first two termsin Taylor series expansion of the form,

$f(x_{n+1}) = f(x_n)(x_{n+1}\text{-}\ x_n)\ f'(x_n)$

Setting $f(x_{n+1})$ =0 gives

Thus on simplification we get

$x_{n+1}= x_n - \frac{f(x_n)}{f\prime(x_n)}$

Let the curve f(x)=0 meet the x-axis at α,it means α is one of the original root of the f(x)=x =0.Let $x_0$be the point near the root α of the equation f(x)=0 then the equation of tangent $P[x_0, f(x_0)]\ is$

$$y - f(x_0) = f'(x_0)(\mathrm{x} - x_0)$$

This cut the x axis at $x_1= x_0 - \frac{f(x_0)}{f\prime(x_0)}$

This is the first approximation of the root α.If $P[x_1, f(x_1)]$ is the corresponding point to x_1 on the curve then the tangent at $P_1 is$

$$y - f(x_1) = f'(x_1)(\mathrm{x} - x_1)$$

This cut the x axis at $x_2= x_1 - \frac{f(x_1)}{f\prime(x_1)}$

This is the second approximation to the root α.Repeating this process we will get the root α with better approximations quite rapidly.

Remarkable Notes

1. When f'(x) is very large .i.e. Slop is very large then h will be small and hence the root can be calculated in even less time.
2. If we choose the initial approximation $x_0$close to the root then we get the root of the equation very quickly.
3. The process will evidently fail if f'(x) $= 0$ is in the neighbourhood of the root. In such cases the Regula falsie method should be used.
4. If the initial approximation to the root is not given, choose two say a and b such that f(a) and f(b) are of opposite signs if |f(a)|<|f(b)| then take a as the initial approximation.

Examples

1.Find the real root of the equation $x^3 + 2x - 5 = 0$ by applying Newton-Raphson method at the end of fifth iteration.

Answer: **Step:1**

Given $f(x) = x^3 + 2x - 5$

$$f'(x) = 3x^2 + 2$$

$$f''(x) = 6x$$

Since $f(0) = -5, f(1) = -2, f(2) = 7$

A root lies between 1 and 2.Since $f''(x) = 6x$ which is positive for interval (1,2) hence we choose

$x_0 = 2$ as $f(x_0)\, f''(x_0) > 0$

Step:2

BY Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{7}{14} = 1.5$$

Then f(x_1) $= 1.375$, f'(x_1) $= 8.75$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 1.5 - \frac{1.375}{8.75} = 1.343$$

Then f(x_2) $= 0.1083$, f'(x_2) $= 7.411$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.343 - \frac{0.1083}{7.411} = 1.329$$

Then f(x_3) $= 0.1083$, f'(x_3) $= 7.2987$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 1.329 - \frac{0.0053}{7.2987} = 1.3283$$

Then f(x_4) = 0.00227, f'(x_4) = 7.29314

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$x_5 = 1.3283 - \frac{0.00227}{7.29314} = 1.3283$$

Hence required root is 1.3283

2.Find by Newton-Raphson method $3x - cosx - 1 = 0$ correct to four decimal places.

Given $f(x) = 3x - cosx - 1$

$f'(x) = 3 + sinx$

Since $f(0) = -2, f(1) = 1.4597,$

A root lies between 0 and 1.Since $f''(x) = cosx$ which is positive for interval (0,1) hence we choose

$x_0 = 0.6(near\ to\ 1)$ as $f(x_0)\ f''(x_0) > 0$

Step:2

BY Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.6 - \frac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)} = 0.6071$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)} = 0.607$$

3.Find by Newton-Raphson method $sinx - xcosx = 0$.Assume initial guess value for $x = \frac{3\pi}{2}$,correct up to five decimal places.

Step:1

Given $f(x) = sinx - xcosx$

$f'(x) = cosx - cosx + xsinx = xsinx$

Given initial $x_0 = \frac{3\pi}{2}$

Step:2

BY Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = \frac{3\pi}{2} - \frac{\sin\left(\frac{3\pi}{2}\right) - \left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right)}{\frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right)} = 4.50018$$

$$x_2 = x_1 - \frac{f(x_1)}{f\prime(x_1)}$$

$$x_2 = 4.5001824 - \frac{\sin(4.50018) - (4.50018)\cos(4.50018)}{4.50018\sin(4.50018)} = 4.4934195$$

$$x_3 = x_2 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = 4.4934195 - \frac{\sin(4.4934195) - (4.5001824)\cos(4.4934195)}{4.4934195\sin(4.493195)} = 4.4934195$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 4.4934095 - \frac{\sin(4.4934095) - (4.4934095)\cos(4.4934095)}{(4.4934095)\sin(4.4934095)}$$

$= 4.4934095$

Modified Newton Raphson Method

If the derivative $f'(x)$varies but slightly on the interval [a,b] .

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_o)}$$

Convergence of Newton Raphson Scheme.

$Newton - Raphon\ formula\ is\ of\ the\ form$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$Consider = g(x) = x - \frac{f(x)}{f'(x)}$$

Differentiating= $g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$

and for convergence we $require\ that$, $\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$

Numerical Solution of Algebraic and Transcendental Equations

1. Geometrically roots of equation f(x) = 0 are the points where ---------
   a) Graph of y = f(x) cuts x – axis
   b) Graph of y = f(x) is parallel to x –axis
   c) Graph of y = f(x) is perpendicular to y axis
   d) Graph of y =f(x) is parallel to y – axis.
2. By intermediate value theorem, for equation f(x) = 0 to have at least one root say ξ in the interval (a,b) are --------
   a) f(x) is continuous on [a,b] and f(a)f(b) > 0
   b) f(x) is continuous on [a,b] and f(a) and f(b) < 0
   c) f(x) is continuous on (a,b) and f(a) = f(b)
   d) f(x) is continuous on (a,b) and f(a)f(b) = 0
3. If f(x) is continuous on [a,b] and f(a) f(b) < 0, then to find a root of f(x) = 0, initial approximation x_0 by bisection method is ---------
   a) $x_0 = \frac{a-b}{2}$ c) $x_0 = \frac{f(a)+f(b)}{2}$
   b) $x_0 = \frac{a+b}{2}$ d) $x_0 = \frac{a-b}{a+b}$
4. In bisection method, if permissible error is E for finding a root of f(x) = 0 then the approximate number of iterations required can be determined from the relation
   a) $\frac{a+b}{2} < \varepsilon$ b) $\frac{a+b}{a-b} < \varepsilon$ c) $\frac{a+b}{2^n} \le \varepsilon$ d) $\frac{b-a}{2^n} \le \varepsilon$
5. If x_0, x_1 are two initial approximations to the root of f(x) = 0, by secant method next approximations x_2 is given by
   a) $x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$ c) $x_2 = \frac{x_0 + x_1}{2}$
   b) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ d) $x_2 = x_1 + \frac{(x_1 + x_0)}{(f_1 + f_0)} \times f_1$
6. If x0 is initial approximation to the root of the equation f(x) =0, by Newton – Raphson method, first approximation x_1 is given by ----
   a) $x_2 = \frac{x_0 + x_1}{2}$ b) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ c) $x_1 = x_0 + \frac{f(x_0)}{f'(x_1)}$ d) $x_1 = x_0 + \frac{f'(x_0)}{f(x_0)}$

7. The condition of convergence of the computational scheme $x = \phi(x)$ in method of successive approximation (simple iteration method) for finding root of the equation f(x)=0 is ------

   a) $|\phi'(x)|>$ for all x in [a,b] c) $|\phi'(x)| \geq 1$ for all x

   b) $|\phi'(x)|<$ for all x in [a,b] d) $|\phi'(x)| \geq 0$ for all x

8. A root of the equation $x\log_{10}$ - 1.2 = 0 by using bisection method lies between___________

   a) 1 and 2 b) 0 and 1 c) 0.5 and 1 d) 2 and 3

9. A root of the equation $x^3 - 4x - 9 = 0$ using bisection methods lies between__________

   a) 2 and 3 b)1and 2 c)0 and 1 d) 3 and 4

10. A cube root of 100 using secant method, lies between_________

    a) 0 and 1 b) 4 and 5 c)1and 2 d) 5 and 6

11. A root of the equation $\cos x - xe^{x} = 0$ (measure in radian) by using secant method lies between ______________

    a) 2 and 3 b) 1and 2 c) 0 and 1 d) 2.5 and 3

12. Using bisection method, the first approximation to root ξ of the equation $x\sin x - 1 = 0$ (measure in radian), that lies between x = 1 and x = 1.5 is _____________

    a) 1.5 b) 0.25 c) 1 d) 1.25

13. Using Secant method, the first approximation to a root x_2 of the equiation $x^3 - 5x - 7 = 0$, if initial approximation are given as $x_0 = 2.5$ and $x_1 = 3$, is

    a) 2.7183 b) 3 c)2 d) 0

14. Using Newton –Raphson method, the first approximation to a root x1 of the equation $x^3 + 2x - 5 = 0$ in (1,2) if initial approximation x0 =2 is _______

    a) 0 b) 1.5 c) 3 d) 4

15. Using Successive approximation method (iterative method), the first approximation to a root x1 of the equation $x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$ in [3,4], taking initial approximation $x_0 = 3.6$ is ____

    a) 0 b) 1 c) 3.77815 d) 2

1.a	2.b	3.b	4.d	5.a
6.b	7.b	8.d	9.a	10.b
11.c	12.d	13.a	14.b	15.c