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Advanced Programming

Functional Design: Monoid, Foldable, Functor, Monad

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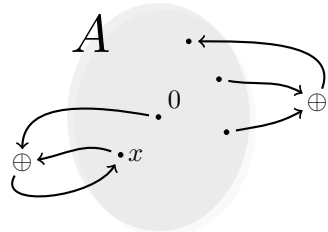
SOFTWARE QUALITY RESEARCH

Monoid

Definition

- Type A
- Binary operator $\oplus : A \times A \mapsto A$ (combine)
- Zero element $0 \in A$ (empty)
- Laws ...



Associativity $(x \oplus y) \oplus z \;=\; x \oplus (y \oplus z)$ for any $x, y, z \in A$

Identity $x \oplus 0 \;=\; x \;=\; 0 \oplus x$ for any $x \in A$

Associativity:
```
forAll { (x:A, y:A, z:A) =>
    combine(combine(x, y), z) ==combine(x, combine(y, z)) }
```

Identity:
```
forAll { (x: A) =>combine(x, empty) ==x }
forAll { (x: A) =>combine(empty, x) ==x }
```

Examples: (Integers, 0, +), (Integers, 1, *), (Strings, "", +)

Monoid

A type class

```
1  trait Monoid[A]:
2    self =>
3    def combine(a1: A, a2: A): A
4    def empty: A

5    object laws:
6      def associative (using Arbitrary[A], Equality[A]) =
7        forAll { (a1: A, a2: A, a3: A) =>
8          self.combine(self.combine(a1, a2), a3) ===
9            self.combine(a1, self.combine(a2, a3)) }

10     def unit (using Arbitrary[A], Equality[A]) =
11       forAll { (a: A) =>
12         (self.combine(a, self.empty) === a) &&
13         (self.combine(self.empty, a) === a) }

14     def monoid (using Arbitrary[A], using Equality[A]) =
15       associative && unit
```

- Type class: Monoid, type A, op, zero, abstract trait
- Type constructor: Monoid
- Laws: associativity, unit, both (imperative issues)
- Inner object's access to enclosing object
- Type constraints with implicits: Arbitrary, Equality

More on the Scalactic equality test `===` used above: https://www.scalactic.org/

Monoid

Instances

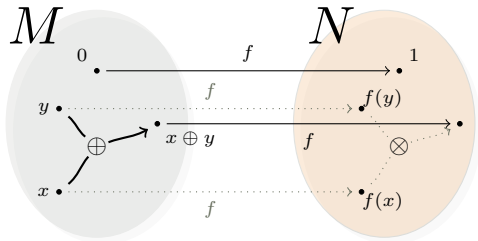
```
1 val stringMonoid: Monoid[String] = new:
2   def combine(a1: String, a2: String): String =
3     a1 + a2
4   val empty: String = ""

5 ...

7 object MonoidSpec extends ...
8   property("stringMonoid is a monoid") =
9     stringMonoid.laws.monoid
```

- A type class instance shows evidence that we can see a type as a Monoid
- Note that a type class instance is a value! How do we create one? We instantiate the trait.
- In place instantiation of abstract traits
- How do we "prove" that an instance is valid? We property test the Monoid Laws!
- Exercise: Take a piece of paper to write an instance for Int, 1, and multiplication
- Mentimeter on more instances

Monoid

Morphisms

- Monoid $(M, \oplus, 0)$, monoid $(N, \otimes, 1)$
- **Homomorphism** $\equiv f : M \to N$ + laws below
- Homomorphisms both ways $\equiv$ **isomorphism**



Distributive $f(x \oplus y) = f(x) \otimes f(y)$ for any $x, y \in M$
Preserves identity $f(0) = 1$

Distributive: `forAll { (x:A, y: A) =>f(M.combine(x, y)) ===N.combine(f(x), f(y)) }`
Preserves identity: `f(M.empty) ===N.empty`

Example homomorphism: `f(s: String): Int =s.size`
from Strings (with "" and concat) to integers (with 0 and +)
e.g. `"foo".size + "bar".size ==("foo"+ "bar").size`

Example isomorphisms: `String` and `List[Char]` through `toList` and `mkString`, `(Boolean, false, ||)` and `(Boolean, true, &&)` through negation

Foldable

Type Class

```
1 trait Foldable[F[_]]:
2   extension [A](as: F[A])
3     // map into a monoid and reduce using 'combine'
4     def foldMap[B: Mondoid](f: A => B): B

6     def foldRight[B](z: B)(f: (A, B) => B): B
7     def foldLeft[B] (z: B)(f: (B, A) => B): B

9     // If A is a monoid then
10    def concatenate[A](as: F[A])(using m: Monoid[A]): A =
11      as.foldLeft(m.empty)(m.combine)
```

Examples: `List`, `IndexedSeq`, `LazyList`, `Option` (almost all structures implemented in the course so far!) Also trees are foldable, any iterator can be cast as foldable

- Understand the type class and the interface
- Foldable is a **higher kind**, parameterized with a type constructor
- So type classes exist both for types and for higher kinds
- Folds work with Monoids operator, no diff btw left and right. **Why?**
- Reduce = a fold from monoid's empty with combine
- MapReduce (spark) can distribute calculations in monoids
- Behavioral interface, like Enumerable, Iteratable, etc. – (Monoid could be called "Addable")
- Mentimeter ($\times 2$)

Functor

Examples

Consider the following map functions from different parts of the course:

```
def map[A,B] (ga: Gen[A]) (f: A =>B): Gen[B]
def map[A,B] (la: List[A]) (f: A =>B): List[B]
def map[A,B] (oa: Option[A]) (f: A =>B): Option[B]
```

- All very similar, also `State`, `Par`, `Parser`, `Tree`
- Follow the same interface
- We could call it `Mappable`
- But for the sake of tradition we settle on `Functor`

Functor

Type Class

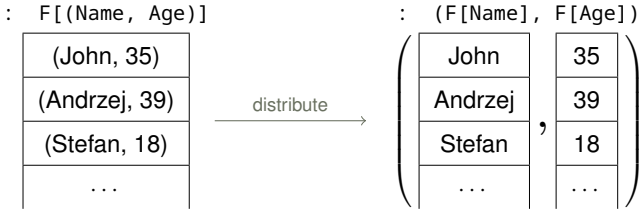
```scala
trait Functor[F[_]]:

  extension [A](fa: F[A])
    def map[B](f: A => B): F[B]

  extension [A, B](fab: F[(A, B)])
    def distribute: (F[A], F[B])=
      (fab.map { _._1 }, fab.map { _._2 })

  object functorLaws:
    def map[A](using Arbitrary[F[A]], Equality[F[A]]): Prop =
      forAll { (fa: F[A]) => fa.map[A](identity[A]) === fa }
```

- Understand the type class and the interface
- Functor is a **higher kind**, parameterized with a type constructor
- The law: structure preservation
- Why the `using` constraint?
- Even though `map` is a weak assumption we can derive useful functions (distribute)
- Distribute is a general unzipper for any functor, not just lists
- Distribute is explained in the next slide
- General zippers require `map2` (a Monad, or an Applicative Functor)

Examples: List, Gen, Option, LazyList are all functors (they have map, mappables)

Functor

Distribute




```
extension [A, B](fab: F[(A, B)])
  def distribute: (F[A], F[B])=
    (fab.map { _._1 }, fab.map { _._2 })
```

Note: This works for `List` and `LazyList` but also for `Gen` and `Par` with the same single implementation! This will continue working for any type constructor in the future for which we add a `Functor` instance.

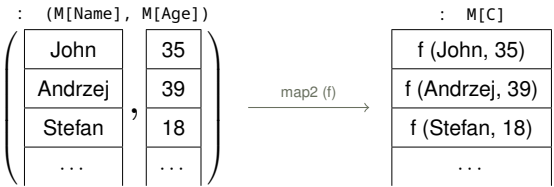
Exercise: Doodle an instance for `Option` (go back to the previous slide)

Map2

Some of the map2 functions from different parts of the course

```
def map2[A,B,C] (fa: Option[A], fb: Option[B]) (f: (A,B) => C): Option[C] =
  fa.flatMap { a => fb.map  { b => f (a,b) } }

def map2[A,B,C] (fa: List[A], fb: List[B]) (f: (A,B) => C): List[C] =
  fa.flatMap { a => fb.map  { b => f (a,b) } }

def map2[A,B,C] (fa: Gen[A], fb: Gen[B]) (f: (A,B) => C): Gen[C] =
  fa.flatMap { a => fb.map  { b => f (a,b) } }
```

- All very similar, also `State`, `Par`, `Parser`, the Map2-appable trait/interface (ApplicativeFunctor)
- `map2` is a 'kind-of' generalized binary zipWith (Beware! A different intuition for each instance!)



- Pic makes "sense" for `Gen`
- What happens for `List` ?
- What happens for `Option` ?
- `Parser` ?

Monad

Type Class

```
1 trait Monad[F[_]]
2   extends Functor[F]:

4   def unit[A](a: => A): F[A]

6   extension [A](fa: F[A])
7     def flatMap[B](f: A => F[B]): F[B]

9     def map[B](f: A => B): F[B] =
10      fa.flatMap[B] { a => unit(f(a)) }
```

- Understand the type class and the interface
- Monad is a higher kind, parameterized with a type constructor
- Monad is both a functor and an applicative functor
- It could be called "FlatMappable"
- It captures computations that can be sequenced, and transformed

Examples: All types in ADPRO!

Some monads in the Scala standard library [20]: `FilterMonadic`, `LazyList`, `TraversableMethods`, `Iterator`, `ParIterableLike`, `ParIterableLike`, `ParIterableViewLike`, `TraversableLike`, `WithFilter`, `MonadOps`, `TraversableProxyLike`, `TraversableViewLike`, `LeftProjection`, `RightProjection`, `Option`, `WithFilter`, `Responder`, `Zipped`, `ControlContext`, `Parser`

Monad Laws

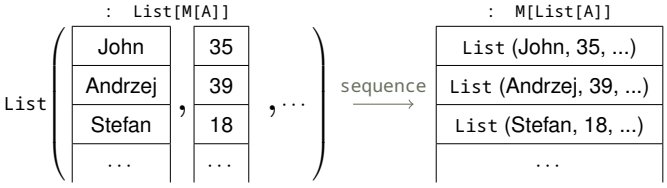
- Laws for flatMap and unit: associative, identity
- Discussion of type constraints, where is `arbAFA` used?

```
2     def associative[A, B, C](
3       using Arbitrary[F[A]], Arbitrary[A => F[B]], Arbitrary[B => F[C]], Equality[F[C]]) =
4         forAll { (x: F[A], f: A => F[B], g: B => F[C]) =>
5           (x.flatMap(f).flatMap(g)) === (x.flatMap { a => f(a).flatMap(g) })
6         }

8     def identityRight[A]
9       (using Arbitrary[F[A]], Equality[F[A]]) =
10        forAll { (x: F[A]) =>
11          x.flatMap(unit) === x
12        }

14    def identityLeft[A]
15      (using Arbitrary[A], Arbitrary[A => F[A]], Equality[F[A]]) =
16        forAll { (y: A, f: A => F[A]) =>
17          unit(y).flatMap(f)  === f(y)
18        }
```

Sequence is generalized n-ary zip

```
def sequence[A] (lfa: List[F[A]]): F[List[A]]
```

- We can derive sequence for any `Monad` instance.
- The function is implemented in terms of the above interface
- The function produces a computation that produces lists. The computed list is build by polling each of the computations in the parameter for their head, and simply recomposing to the list.



- Makes “sense” for `Gen`
- What happens for `Option` ?
- What happens for `List` ?
- Parser resembles `Option` and `Gen`