WI4011-17

Computational Fluid Dynamics

Assignment 1.1 2018-2019

September 12, 2018

The completion of Assignment 1.1 is required to pass the exam for WI4011-17. Provide clear and motivated answers to the questions. Try to use the information available in the book of Prof. Wesseling and the lecture notes. Assignment 1.1 should be uploaded to Brightspace before September 30, 23:59. You are advised, but not obliged, to work in pairs of two students. If you work in a pair, please state both your names clearly on your work. Preferably use LaTeX. Only well-organised hand-written reports are accepted.

Nondimensional numbers

Barotropic flow is a type of compressible flow, where the density ρ is a function of the pressure p. In *weakly compressible* barotropic flow the density can be expressed as:

$$\rho(p) = \rho_0 + \frac{p}{c^2}, \quad \frac{1}{c^2} = \frac{d\rho}{dp}, \quad \frac{|\rho - \rho_0|}{\rho_0} \ll 1, \tag{1}$$

where c is the speed of sound, assumed to be constant. Note that the fact that $\rho(t, \boldsymbol{x}) \approx \rho_0$ does *not* imply that $\rho_{,t}$ and $\rho_{,\alpha}$ are negligible. The equations for conservation of momentum and mass are given by:

$$\begin{aligned} \rho \frac{Du_\alpha}{Dt} &= -p_{,\alpha} + \mu u_{\alpha,\beta\beta} + \rho g, \\ \frac{\partial \rho}{\partial t} &+ (\rho u_\alpha)_{,\alpha} = 0, \end{aligned} \tag{2}$$

where u_α is the velocity component in the $\boldsymbol{e}_\alpha$ direction, μ is the viscosity and g is the gravitational acceleration. We are comparing the solution of this system of equations for two cases, where the second case is a geometrically scaled-down version of the first case ***of the same fluid***. We refer to the second problem as the *scaled* problem. The appropriate reference length scales for the first case and second case are L_1 and L_2, respectively. Here, $L_2 = sL_1$, where $0 < s < 1$.

- Reformulate (2) in the dependent variables $\{p, u_\alpha\}$.
- Show that when (2) is formulated in $\{p, u_\alpha\}$ and in dimensionless form, by introducing reference quantities L_{ref}, U_{ref} and ρ_{ref}, the resulting dimensionless formulation will contain three dimensionless numbers:

$$\text{the Reynolds number:} \quad \text{Re} = \frac{\rho_{\text{ref}} U_{\text{ref}} L_{\text{ref}}}{\mu}, \tag{3}$$

$$\text{the Froude number:} \quad \text{Fr} = \frac{U_{\text{ref}}}{\sqrt{g L_{\text{ref}}}}, \tag{4}$$

$$\text{and the Mach number:} \quad \text{M} = \frac{U}{c}. \tag{5}$$

- Explain the physical interpretation of these nondimensional numbers, by considering the scaling properties of the appropriate terms in (2). Discuss the limit of the model when either of the nondimensional numbers vanishes.
- Assume the fluid in the full-scale problem and the scaled-down problem is the same *(same density, same viscosity)*. Derive the condition under which both problems are described by the exact same dimensionless form of (2) derived when answering the previous question.

The stream function

We will study the streamfunction as a means of plotting streamlines for a *given* velocity field where the velocity field originates from a potential flow model.

- Assume the flow in the domain depicted in Fig. 1 is described by a potential flow model. Formulate the complete boundary value problem for the velocity potential $\phi(\boldsymbol{x})$, considering the following boundary conditions have been imposed for the velocity field u_α on the boundary

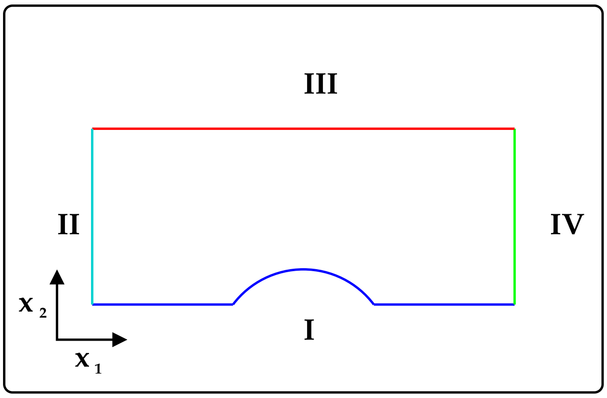

Figure 1: Computational domain for computation of the streamfunction from a given velocity field.

of the domain depicted in Fig. 1:

$$\begin{aligned} u_\alpha \hat{n}_\alpha &= 0, \ \boldsymbol{x} \in \Gamma_I, \\ u_\alpha \hat{n}_\alpha &= -1, \ \boldsymbol{x} \in \Gamma_{II}, \\ u_\alpha \hat{n}_\alpha &= 0, \ \boldsymbol{x} \in \Gamma_{III}, \\ u_\alpha \hat{n}_\alpha &= 1, \ \boldsymbol{x} \in \Gamma_{IV}, \end{aligned}$$

where $\hat{n}_\alpha$ is the outward pointing unit normal vector.

Streamlines are curves that define the streamfunction $\psi(\boldsymbol{x})$ in the following way:

$$\begin{aligned} \psi_{,1}(\boldsymbol{x}) &= -u_2(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega, \\ \psi_{,2}(\boldsymbol{x}) &= u_1(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega. \end{aligned} \tag{6}$$

- Show that contourlines of the field $\psi(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega$ are streamlines.
- Show that under the assumptions that $\psi(\boldsymbol{x})$ is twice continuously differentiable and that (6) holds, the velocity field $\boldsymbol{u}(\boldsymbol{x})$ is solenoidal.
- Formulate the complete boundary value problem for the scalar streamfunction $\psi(\boldsymbol{x})$, whose contourlines are streamlines of the velocity field in Fig. 1.

The stationary convection-diffusion equation

Consider the boundary value problem for the homogeneous one-dimensional stationary convection-diffusion equation defined on the domain Ω:

$$\begin{aligned} u\phi_{,1} - \epsilon\phi_{,11} &= 0,\ x_1 \in \Omega := \langle 0, 1\rangle, \\ \phi(0) &= -\epsilon\phi_{,1}(0), \\ \phi(1) &= b, \end{aligned} \tag{7}$$

where $u,\ \epsilon,\ b > 0$. Analyse the sensitivity of the solution at $x_1 = 1$ in the limit of $\epsilon \downarrow 0$. Is this behavior such that it would make you conclude the boudary value problem is ill-conditioned for very small values of ϵ? Discuss.