Assignment 1.1

Computational Fluid Dynamics

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NON-DIMENSIONAL NUMBERS

In a weakly compressible barotropic flow, the density is pressure dependent and is expressed by:

$$\rho(p) = \rho_0 + \frac{p}{c^2} \tag{1.1}$$

Here c is the speed of sound of the fluid assumed to be constant. The equations for conservation of momentum and mass are respectively given by:

$$\begin{cases} \text{momentum:} & \rho \frac{Du_\alpha}{Dt} = -p_{,\alpha} + \mu u_{\alpha,\beta\beta} + \rho g \\ \text{mass:} & \frac{\partial \rho}{\partial t} + (\rho u_\alpha)_{,\alpha} = 0 \end{cases} \tag{1.2}$$

By substitution of equation 1.1 into the conservation laws we can wright them in a pressure dependent fashion:

$$\begin{cases} \left(\rho_0 + \frac{p}{c^2}\right) \frac{Du_\alpha}{Dt} = -p_{,\alpha} + \mu u_{\alpha,\beta\beta} + \left(\rho_0 + \frac{p}{c^2}\right) g \\ \frac{\partial\left(\rho_0 + \frac{p}{c^2}\right)}{\partial t} + \left[\left(\rho_0 + \frac{p}{c^2}\right) u_\alpha\right]_{,\alpha} = \frac{1}{c^2}\frac{\partial p}{\partial t} + \left[\left(\rho_0 + \frac{p}{c^2}\right) u_\alpha\right]_{,\alpha} = 0 \end{cases} \tag{1.3}$$

To formulate a non-dimentional form, we will introduce necessary reference quantities: $L_r, U_r and \rho_r$. The nondimensional parameters are: $p' = \frac{p}{\rho_r u_r^2}; x' = \frac{x}{L_r}; u' = \frac{u}{u_r}; \rho_0' = \frac{\rho_0}{\rho_r}; t' = \frac{u_r t}{L_r}$. By substituting these dimensionless parameters in the pressure dependent conservation of momentum law we obtain,

$$\left(\rho_0'\rho_r + \frac{p'\rho_r u_r^2}{c^2}\right) \frac{D(u' u_r)_\alpha}{D(t' \frac{L_r}{u_r})} = -\frac{1}{L_r}(p'\rho_r u_r^2)_{,\alpha'} + \frac{\mu}{L_r^2}(u' u_r)_{\alpha,\beta'\beta'} + \left(\rho_0'\rho_r + \frac{p'\rho_r u_r^2}{c^2}\right) g \tag{1.4}$$

$$\Leftrightarrow \frac{\rho_r u_r^2}{L_r}\left(\rho_0' + p'\frac{u_r^2}{c^2}\right)\frac{Du_\alpha'}{Dt'} = -\frac{\rho_r u_r^2}{L_r} p'_{,\alpha'} + \frac{\mu u_r}{L_r^2} u'_{\alpha,\beta'\beta'} + \rho_r g \left(\rho_0' + p'\frac{u_r^2}{c^2}\right) \tag{1.5}$$

where one should notice that the spacial derivative is made non-dimensional as well. Dividing by $\frac{\rho_r u_r^2}{L_r}$, we find:

$$\left(\rho_0' + p'\frac{u_r^2}{c^2}\right)\frac{Du_\alpha'}{Dt'} = -p'_{,\alpha'} + \frac{\mu}{\rho_r u_r L_r} u'_{\alpha,\beta'\beta'} + \frac{L_r g}{u_r^2}\left(\rho_0' + p'\frac{u_r^2}{c^2}\right) \tag{1.6}$$

$$\Leftrightarrow \left(\rho_0' + \mathrm{Ma}^2 p'\right)\frac{Du_\alpha'}{Dt'} = -p'_{,\alpha'} + \mathrm{Re}^{-1} u'_{\alpha,\beta'\beta'} + \mathrm{Fr}^{-2}\left(\rho_0' + \mathrm{Ma}^2 p'\right) \tag{1.7}$$

where the following non-dimensial numbers are used:

$$\text{Reynolds number: } \mathrm{Re} = \frac{\rho_r u_r L_r}{\mu} \tag{1.8}$$

$$\text{Froude number: } \mathrm{Fr} = \frac{u_r}{\sqrt{g L_r}} \tag{1.9}$$

$$\text{Mach number: } \mathrm{Ma} = \frac{u_r}{c} \tag{1.10}$$

We use the approach to derive the non-dimensional form for the conservation of mass:

$$\frac{1}{c^2}\frac{\partial p}{\partial t}+\left[\left(\rho_0+\frac{p}{c^2}\right)u_\alpha\right]_{,\alpha}=\frac{1}{c^2}\frac{\partial(p^{'}\rho_r u_r^2)}{\partial(t^{'}\frac{L_r}{u_r})}+\frac{u_r}{L_r}\left[\left(\rho_0^{'}\rho_r+\frac{p^{'}\rho_r u_r^2}{c^2}\right)u_\alpha^{'}\right]_{,\alpha'}$$
$$=\frac{\rho_r u_r^3}{L_r c^2}\frac{\partial p^{'}}{\partial t^{'}}+\frac{u_r\rho_r}{L_r}\left[\left(\rho_0^{'}+p^{'}\frac{u_r^2}{c^2}\right)u_\alpha^{'}\right]_{,\alpha'}=0$$

Dividing by a factor $Ma^2\frac{u_r\rho_r}{L_r}$ we obtain:

$$\frac{\partial p^{'}}{\partial t^{'}}+\left[\left(Ma^{-2}\rho_0^{'}+p^{'}\right)u_\alpha^{'}\right]_{,\alpha'}=0 \tag{1.11}$$

Leaving out the primes we can rewrite the non-dimensional conservation equations for momentum and mass in a concise manner:

$$\begin{cases}\text{momentum:} & \left(\rho_0+Ma^2 p\right)\frac{Du_\alpha}{Dt}=-p_{,\alpha}+Re^{-1}u_{\alpha,\beta\beta}+Fr^{-2}\left(\rho_0+Ma^2 p\right)\\ \quad\text{mass:} & \frac{\partial p}{\partial t}+\left[\left(Ma^{-2}\rho_0+p\right)u_\alpha\right]_{,\alpha}=0\end{cases} \tag{1.12}$$

By carefully investigating these non-dimensional numbers within these non-dimensional conservation laws we can retrieve their physical interpretations.
Evaluating the Machs number, we observe that is scales the dependance of pressure and ρ_0 on the density. With an increasing Machs number the density will be relatively more pressure dependent than it will on ρ_0. More physically, this can be interpreted as a compressibility measure of the fluid: the higher the Mach number the more the pressure determines the actual value of the density.
Reynolds number only scales the $u_{\alpha,\beta\beta}$ term, which is the viscous term. Since the Reynolds term scales the viscous term reciprocal, the viscous or friction forces will dominate over the inertial forces at low Reynolds numbers. So the Reynolds number can be interpreted as a measure of the ratio between the inertial and the viscous forces.
Froudes number scales the last term in the non-dimensional equation of conservation of momentum reciprocally. The last term represents the gravitational force working on the fluid. At low Froude numbers, the gravitational term dominates over the inertial forces. So the Froudes number can be interpreted as the ratio between the inertial and gravitational forces.
The flow of the fluid is fully characterized by the nondimensional numbers. This means that if the scaled down problem is described by the exact same dimensionless form as the normal problem, the dimensionless numbers needs to be the same. When the density and the viscosity stays the same, we can derive the following conditions:

$$Re_1=Re_2$$
$$\rightarrow\frac{\rho u_1 L_1}{\mu}=\frac{\rho u_2 L_2}{\mu}=\frac{\rho u_2 s L_1}{\mu}$$
$$\rightarrow u_1=su_2$$

$$Fr_1=Fr_2$$
$$\rightarrow\frac{u_1}{\sqrt{gL_1}}=\frac{u_2}{\sqrt{gL_2}}=\frac{u_2}{\sqrt{gsL_1}}$$
$$\rightarrow u_1=\frac{u_2}{\sqrt{s}}$$

Here describes the subscript 1, the normal problem and subscript 2, the scaled version. For both conditions to be true, it needs to follow that $s=1$. This is not possible since s is defined to be smaller than 1. From this, we can conclude that it is not possible to have the same dimensionless form for the scaled problem as the normal version if the density and the viscosity stays exactly the same.

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THE STREAMFUNCTION

For a potential flow model we can describe the velocity field as a gradient of a vector field $\phi(\mathbf{x})$, referred to as the velocity potential. Mathematically this can be expressed as,

$$\phi_{,\alpha} = u_\alpha \tag{2.1}$$

where $u_\alpha = (u_1, u_2)^T$ is the velocity field. To complete the BVP the following boundary conditions have to be met,

$$u_\alpha \hat{n}_\alpha = \begin{cases} 0, & \mathbf{x} \in \Gamma_1 \\ -1, & \mathbf{x} \in \Gamma_2 \\ 0, & \mathbf{x} \in \Gamma_3 \\ 1, & \mathbf{x} \in \Gamma_4 \end{cases} \tag{2.2}$$

where the description of the domain can be found back in the assignment description. Furthermore, we can introduce the streamfunction, $\Psi(\mathbf{x})$, defined as,

$$\frac{\partial \Psi(\mathbf{x})}{\partial x_1} = -u_2 \tag{2.3a}$$

$$\frac{\partial \Psi(\mathbf{x})}{\partial x_2} = u_1 \tag{2.3b}$$

Contourlines of the streamfunction are the level lines of the function $\Psi(\mathbf{x})$ and are mathematically expressed as

$$\Psi(\mathbf{x}) = C \tag{2.4}$$

for C constant. Drawing these lines in the domain for various C will evolve in a level line plot or contour plot. On specific contourlines the difference in function value $\Psi(\mathbf{x})$ is logically 0. Herefore, we can introduce an infinitisimally small vectorfield $d\mathbf{r} = (dx_1, dx_2)^T$ representing the direction in which $\Psi(\mathbf{x})$ remains unchanged. Now, choosing two infinitisimally close points on a contourline the following must hold,

$$\begin{aligned} d\Psi(\mathbf{x}) &= \Psi(x_1 + dx_1, x_2 + dx_2) - \Psi(x_1, x_2) \\ &= \frac{\partial \Psi(\mathbf{x})}{\partial x_1} dx_1 + \frac{\partial \Psi(\mathbf{x})}{\partial x_2} dx_2 \\ &= \nabla \Psi(\mathbf{x}) \cdot d\mathbf{r} \\ &= 0 \end{aligned} \tag{2.5}$$

It follows that $\nabla\Psi(\mathbf{x}) \perp d\mathbf{r}$. We can also show that $\nabla\Psi(\mathbf{x}) \perp \mathbf{u}$ since,

$$\begin{aligned} \nabla \Psi(\mathbf{x}) \cdot \mathbf{u} &= \begin{bmatrix} \frac{\partial \Psi(\mathbf{x})}{\partial x_1} \\ \frac{\partial \Psi(\mathbf{x})}{\partial x_2} \end{bmatrix} \cdot \mathbf{u} \\ &= \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= -u_2 u_1 + u_1 u_2 \\ &= 0 \end{aligned} \tag{2.6}$$

Since we have a 2-dimensional flow model it follows that $\mathbf{u} \parallel d\mathbf{r}$. Therefore, the contourlines of $\Psi(\mathbf{x})$ are equivalent to the lines that define the direction of the velocity field, referred to as streamlines.
Now by imposing that $\Psi(\mathbf{x})$ is twicely continuously differentiable we can massage the divergence of the velocity into a logical statement:

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \frac{\partial}{\partial x_1}(u_1) + \frac{\partial}{\partial x_2}(u_2) && \text{i} \\
&= \frac{\partial}{\partial x_1}(u_1) - \frac{\partial}{\partial x_2}(-u_2) && \text{ii} \\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial \Psi(\mathbf{x})}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial \Psi(\mathbf{x})}{\partial x_1}\right) && \text{iii} \\
&= \frac{\partial^2 \Psi(\mathbf{x})}{\partial x_1 \partial x_2} - \frac{\partial^2 \Psi(\mathbf{x})}{\partial x_2 \partial x_1} && \text{iv} \\
&= \frac{\partial^2 \Psi(\mathbf{x})}{\partial x_1 \partial x_2} - \frac{\partial^2 \Psi(\mathbf{x})}{\partial x_1 \partial x_2} && \text{v} \\
&= 0 && \text{vi}
\end{aligned} \tag{2.7}$$

Where (i) by definition of the divergence operator, (iii) by definition of the streamfunction, (iv) imposing twicely differentiability on the streamfunciton, (v) interchanging order of differentiating has the same outcome for twicely continuously differentiable functions, (vi) logical statement $a - a = 0$. For $\Psi(\mathbf{x})$ twicely differentiable we have shown that the flow is solenoidal since $\nabla \cdot \mathbf{u} = 0$.
We can now describe the BVP for $\Psi(\mathbf{x})$ as follows:

$$\nabla \Psi(\mathbf{x}) = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}, \quad \mathbf{x} \in \Omega \tag{2.8}$$

$$u_\alpha \hat{n}_\alpha = \begin{cases} 0, & \mathbf{x} \in \Gamma_1 \\ -1, & \mathbf{x} \in \Gamma_2 \\ 0, & \mathbf{x} \in \Gamma_3 \\ 1, & \mathbf{x} \in \Gamma_4 \end{cases} \tag{2.9}$$

3
THE STATIONARY CONVECTION-DIFFUSION EQUATION

The BVP for the homogeneous 1D stationary convection-diffusion equation is defined as follows:

$$\begin{aligned} u\phi_{,1} - \epsilon\phi_{,11} = 0, \quad x_1 \in \Omega := \langle 0, 1\rangle, \\ \phi(0) = -\epsilon\phi_{,1}(0), \\ \phi(1) = b, \end{aligned} \tag{3.1}$$

The general solution solution of the differential equation can be formalized using the result of its characteristic polynomial,

$$ur - \epsilon r^2 = 0 \tag{3.2}$$

by solving with respect to r. The general solution explicitly reads as:

$$\phi(x_1) = C_1 e^{r_1} + C_2 e^{r_2} = C_1 + C_2 e^{\frac{u}{\epsilon}} \tag{3.3}$$

with C_1, C_2 arbitrary constants fully determined by both boundary conditions. Determining these constants yields expansion of the boundary condition defined in equation 3.1,

$$C_1 + C_2 = -\epsilon\left(C_2 \frac{u}{\epsilon}\right) \tag{3.4a}$$

$$C_1 + C_2 e^{\frac{u}{\epsilon}} = b \tag{3.4b}$$

Solving equation 3.4a w.r.t C_1 and inserting the result in equation 3.4b yields the following:

$$\begin{aligned} C_1 = b - C_2 e^{\frac{u}{\epsilon}} : \\ b - C_2 e^{\frac{u}{\epsilon}} + C_2 = -\epsilon\left(C_2 \frac{u}{\epsilon}\right) : \\ C_2 = \frac{b}{e^{\frac{u}{\epsilon}} - (u+1)} : \\ C_1 = b\left(1 - \frac{e^{\frac{u}{\epsilon}}}{e^{\frac{u}{\epsilon}} - (u+1)}\right) = b\frac{-(u+1)}{e^{\frac{u}{\epsilon}} - (u+1)} \end{aligned} \tag{3.5}$$

The exact solution of the BVP is:

$$\phi(x_1) = \frac{b}{e^{\frac{u}{\epsilon}} - (u+1)}\left(e^{\frac{u}{\epsilon}x_1} - (u+1)\right) \tag{3.6}$$

Perturbing the system on its Dirichlet BC by a small amount, db, yields,

$$d\phi(x_1) = \phi(x_1, b + db) - \phi(x_1, b) = \frac{db}{e^{\frac{u}{\epsilon}} - (u+1)}\left(e^{\frac{u}{\epsilon}x_1} - (u+1)\right) \tag{3.7}$$

Furthermore the change in fluid property compared to the perturbation is given by,

$$\left|\frac{d\phi(x_1)}{db}\right| = \left|\frac{e^{\frac{u}{\epsilon}x_1} - (u+1)}{e^{\frac{u}{\epsilon}} - (u+1)}\right| \tag{3.8}$$

For ϵ going to 0^+ we observe that,

$$\lim_{\epsilon \downarrow 0} \left| \frac{e^{\frac{u}{\epsilon} x_1} - (u+1)}{e^{\frac{u}{\epsilon}} - (u+1)} \right| = 0, \qquad \text{for all} x_1 \in \Omega \tag{3.9}$$

which logically follows since the exponent in the denumerator dominates over the exponent in the numerator in the given domain. Therefore, we can conclude that the BVP is well-conditioned since it has a mild dependance on its parameters in the whole domain.