WI4011-17

Computational Fluid Dynamics

Assignment 1.1 2018-2019

October 2, 2018

The completion of Assignment 1.2 is required to pass the exam for WI4011-17. Provide clear and motivated answers to the questions. Try to use the information available in the book of Prof. Wesseling and the lecture notes. Assignment 1.2 should uploaded to Brightspace before October 21, 23:59. You are advised but not obliged to work in pairs of two students. If you work in a pair, please state both your names clearly on your work. Preferably use LaTeX. Only well-organised hand-written reports are accepted.

Note: the notation x_1 is used both to refer to the spatial coordinate in general and the specific coordinate of the center of the first control volume. However, what is referred to is (or should be) clear from the context.

The Method of Manufactured Solutions

We study discretization schemes for the following boundary value problem for the one-dimensional constant coefficient stationary convection diffusion equation defined as:

$$\begin{gathered} \phi_{,1}(x_1) - \epsilon\phi_{,11}(x_1) = q(x_1), \quad x_1 \in \Omega = \langle 0, 1\rangle, \\ \phi(0) = a,\ \phi(1) = b, \end{gathered} \tag{1}$$

where $0 < \epsilon \ll 1$ and $0 < a,\ b \in \mathbb{R}$.

- Assume $q(x_1) = 0,\ x_1 \in \Omega$. Write a small program for the discretization of (1). Your discretization should be based on a cell-centered finite

volume discretization on a uniform mesh with a central approximation of the convective and diffusive fluxes. Use the exact solution of $\phi(x_1)$ of (1) (with values of the parameters of your choice) to confirm the numerical solution $\phi_j,\ j = 1 \ldots J$ converges with error $\|e\|_{l_2} = \mathcal{O}(h^2)$ to $\phi(x_j),\ j = 1 \ldots J$, where $h = J^{-1}$ and J is the number of control volumes. The error $\|e\|_{l_2}$ is defined as:

$$\|e\|_{l_2} = J^{-\frac{1}{2}} \sqrt{\sum_{j=1}^{J} (\phi(x_j) - \phi_j)^2}. \tag{2}$$

You can use the sample MATLAB program available on Brightspace as a starting point or for inspiration.

- For many mathematical models, analytical solutions are only available for strongly simplified cases (specific values of the parameters, certain spatial symmetries in domain or boundary conditions). Comparing the numerical approximations of solutions for these simplied cases is not a thorough verification procedure for the implementation of your model. Luckily, there is the Method of Manufactured Solutions[1] for verification of your program. In this approach you will *define* a solution and approximate this self-defined solution numerically.

  Suppose we want to verify the implementation of a numerical approximation L_h of the (second order) differential operator L (with appropriate boundary conditions). Furthermore, we have theoretically determined the order p of the global error of numerical approximations to solutions to the boundary value problem for the homogeneous equation $L\phi = 0$ obtained with L_h.

  We can now proceed as follows: First choose a function $\phi_{\text{man}}(x_1) \in C^\infty$ that complies to the original boundary conditions and define $q(x_1) := L\phi_{\text{man}}(x_1)$. Naturally, $u(x_1) := \phi_{\text{man}}(x_1)$ will be a solution of the boundary value problem for the differential equation $Lu(x_1) = q(x_1)$. The discrete approximation of the latter differential equation is given by $L_h u_j = q_j$ and $u_j = \phi_{\text{man}}(x_j) + Ch^p$ if we make sure that the approximation $q_j = q(x_j)$ is sufficiently accurate to not be decisive for the order of the global error.

  If we can compute a sequence of approximations to the solution for decreasing values of h and observe the theoretically expected order

[1] It is good practice to apply this technique for verification of every numerical scheme you code, e.g. in your MSc thesis project.

of convergence p, this is a strong indication that the implementation is correct. The method provides a way to determine the global error needed for this verification without having an exact solution to compare our numerical solution with. We only need to be able to analytically find $q(x_1)$, maybe with some help of some computer algebra system like Maple or Wolfram Alpha.

Choose the following definition of $\phi_{\text{man}}(x_1)$:

$$\phi_{\text{man}}(x_1) = \alpha \cos\left(\tfrac{1}{2}\pi x_1\right) + \beta\left(1 - e^{x_1}\right). \tag{3}$$

- Why is the proposed manufactured solution preferable over one of polynomial form?
- How should α and β be chosen to make $\phi_{\text{man}}(x_1)$ comply with the boundary conditions of (1)?
- How should $q(x_1)$ be defined to make $\phi_{\text{man}}(x_1)$ a solution of (1)?

- Incorporate the sourceterm $q(x_1)$ you derived in the previous question in your program and establish experimentally that the numerical solution converges to (3) with $\|e\|_{l_2} = \mathcal{O}(h^2)$. Make sure you choose $q(x_1)$ in relation to the value of a and b in a way it significantly differs from the homogeneous solution!

Overlapping grids

In the lecture we have discussed a Péclèt-uniform discretization based on local grid refinement. Another commonly used approach is to use *overlapping grids*: a boundary conforming grid with uniform mesh width h and a background grid with uniform mesh width H, with $h \ll H$. We will refer to the mesh with mesh width h as the *fine* grid and the mesh with mesh width H as the *coarse* grid. Both grids do not share a common face, but overlap each other. The solution on one grid is used to approximate function values on the other grid and vice versa. To make the solution available at the necessary location (on the other grid), it is interpolated from the cell center values on the fine grid to the cell centers on the coarse grid and vice-versa (alternatively information could be retrieved to approximate flux values on either grid, but this approach is not persued here).

The lay-out of the problem is shown in Figures 1 and 2. We want to solve the following boundary value problem using this *overlapping-grids* or

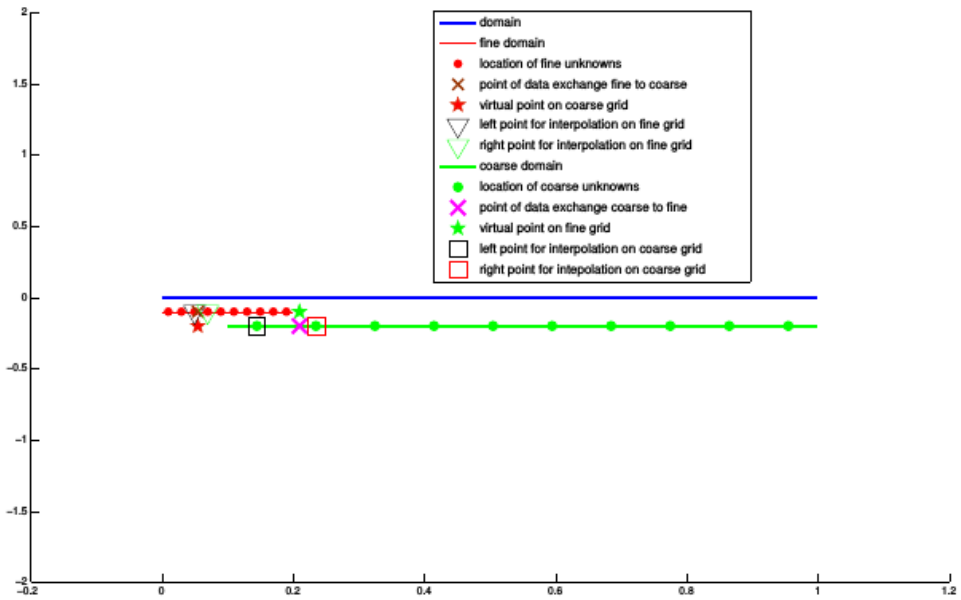

Figure 1: General layout of the overlapping-grid problem, indicating pivotal data points in the interpolation process.

Chimera method:

$$-\phi_{,11}(x_1) = \sin(x_1\pi). \quad x_1 \in \Omega = \langle 0,\ 1\rangle, \qquad (4)$$
$$\phi(0) = a,\ \phi(1) = b,$$

where $0 < a, b \in \mathbb{R}$.

- Use a cell centered finite volume discretisation with a central approximation to the (diffusive) fluxes. Assume the number of control volumes in the fine grid is J and in the coarse grid is K, and the grids show sufficient overlap to make the exchange of the solution between the two grids possible. The numerical approximations on the fine grid and coarse grid are given by $\phi_j = \phi(x_j) + \mathcal{O}(h^p),\ j = 1 \dots J$ and $\Phi_k = \Phi(X_k) + \mathcal{O}(H^p),\ k = 1 \dots K$, respectively. The values ϕ_{J+1} and Φ_0 are exchanged between the two grids, using linear interpolation. Assume interpolation from the fine to the coarse grid and vice versa can be symbolically prescribed as:

$$\phi(x_j) = \beta_0^C \Phi(X_k) + \beta_1^C \Phi(X_{k+1}) + \mathcal{O}(H^2) = \Phi(x_j), \qquad (5)$$
$$x_j \in [X_k,\ X_{k+1}]\,,$$
$$\Phi(X_k) = \beta_0^F \phi(x_j) + \beta_1^F \phi(x_{j+1}) + \mathcal{O}(h^2) = \phi(X_j),$$
$$X_j \in [x_j,\ x_{j+1}]\,,$$

  where $\beta_i^{C,F} \in [0, 1]$. Determine the local truncation error in the numerical scheme for $\phi_j,\ j = 1 \dots J$ and $\Phi_k,\ k = 1...K$ (6 different cases in total). Refer to the results presented in the lecture notes for the cases that are not influenced by the exchange between the two grids.

- After discretisation the problem can be described by the following linear system:

$$A\boldsymbol{\Phi} = \boldsymbol{r},\ \boldsymbol{\Phi} = [\phi_1,\ \phi_2, \dots,\ \phi_J,\ \Phi_1,\ \Phi_2, \dots,\ \Phi_K]^T\,. \qquad (6)$$

  The matrix $A \in \mathbb{R}^{(J+K)\times(J+K)}$ has the following block structure:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad (7)$$
$$A_{11} \in \mathbb{R}^{J\times J}, \quad A_{12}, A_{21}^T \in \mathbb{R}^{J\times K}, \quad A_{22} \in \mathbb{R}^{K\times K}.$$

  Give a complete description of the different blockmatrices that comprise the matrix A, and the vector $\boldsymbol{r}$.

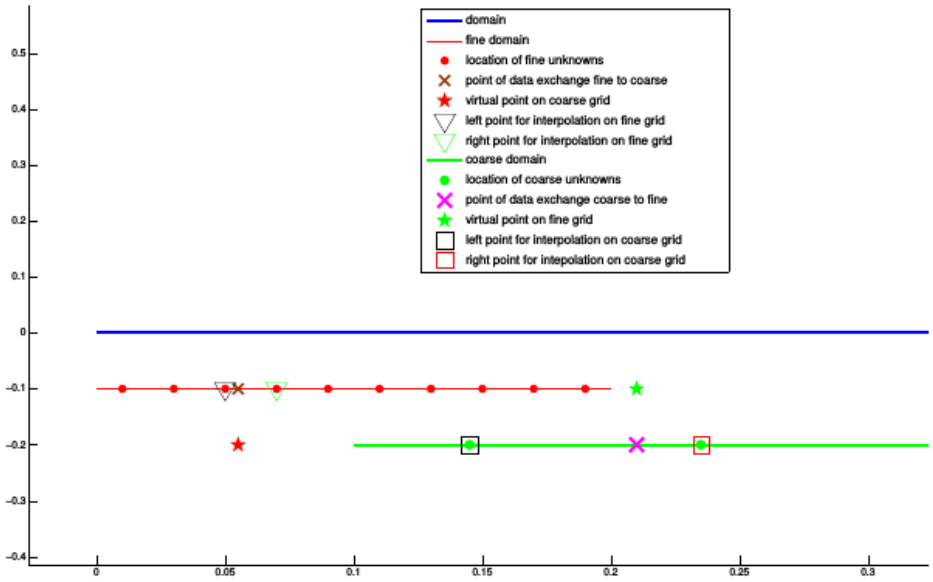

Figure 2: Close-up of the points where the interpolation is performed from coarse to fine grid and vice versa.

- Write a small program to compute an approximation to the solution of (4) using the overlapping grid method, where the fine and the coarse solution are obtained simultaneously. Investigate different configurations of the overlapping meshes and show through numerical experiments how the overlap of the coarse and fine grid influence the order of accuracy of the fine solution and the combined fine and coarse solution.

- For those cases where the order of the global error is less than for a single grid ($\mathcal{O}(h^p),\ p < 2$), two possible options are proposed:

  - Raise the order of accuracy of the interpolation to exchange information.
  - Interpolate the approximation to the flux $\phi_{,1}$ instead of the approximation to the solution.

  Discuss what you can expect of either approach: will this approach work at all and can the accuracy be improved? Bonus points for experimental evidence that supports your claims!