Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

The Traveling Salesman Problem Under Squared Euclidean Distances

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Vrije Universiteit Amsterdam

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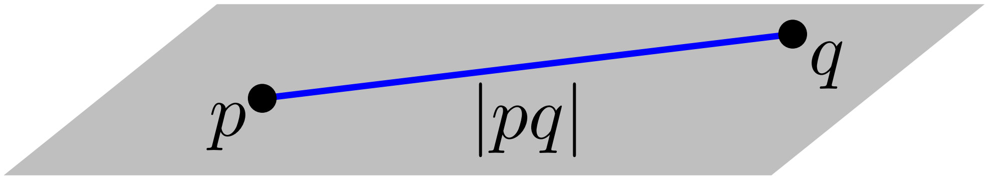
Universität Würzburg

What's the Problem?

Notation. For points $p = (p_1, \dots, p_d), q = (q_1, \dots, q_d) \in \mathbb{R}^d$, denote by $|pq| = \sqrt{\sum_{i=1}^{d}(p_i - q_i)^2}$ their *Euclidean distance*.



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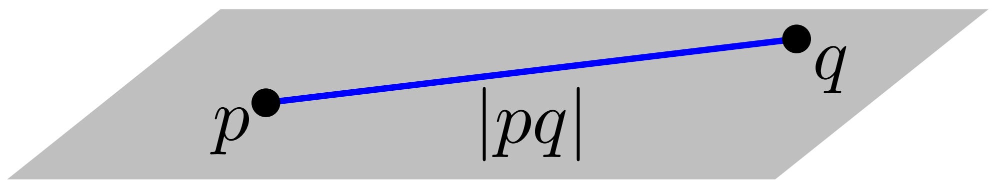

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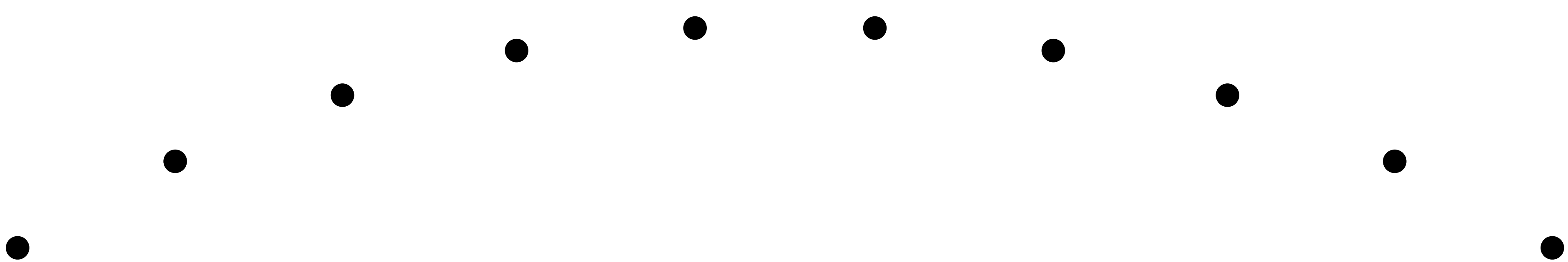

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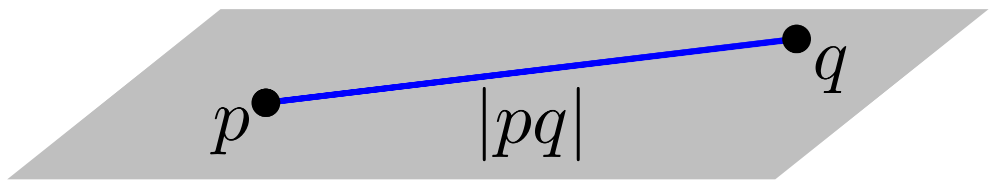

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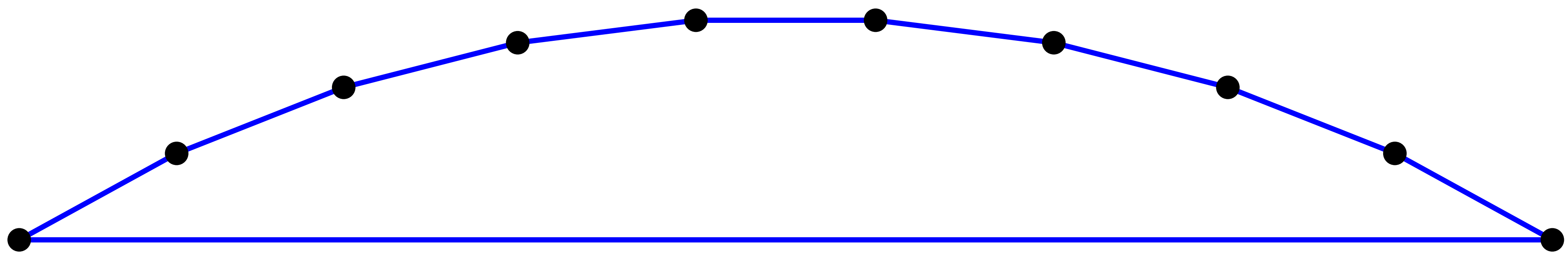

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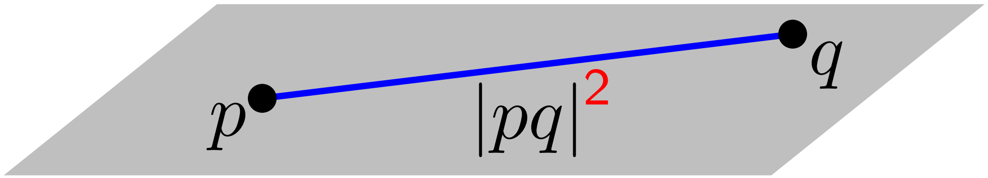

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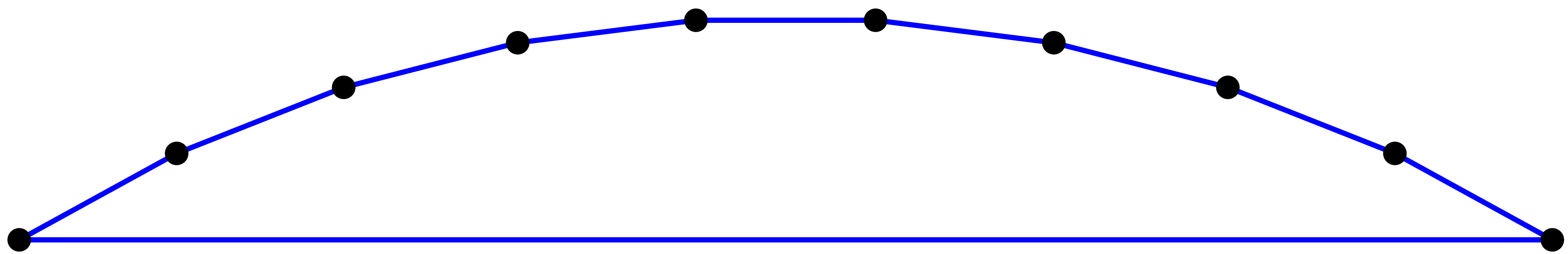

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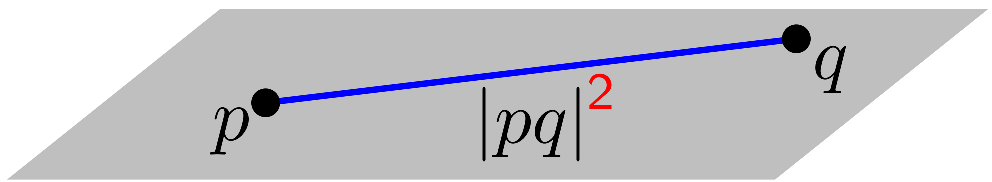

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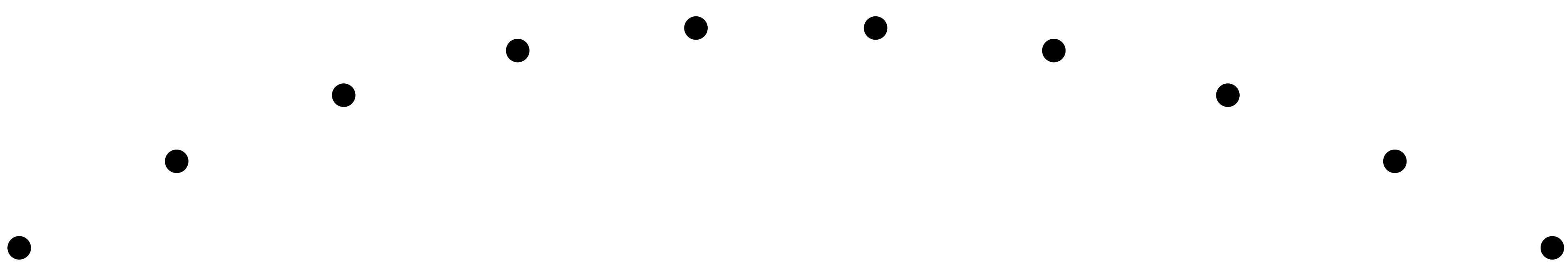

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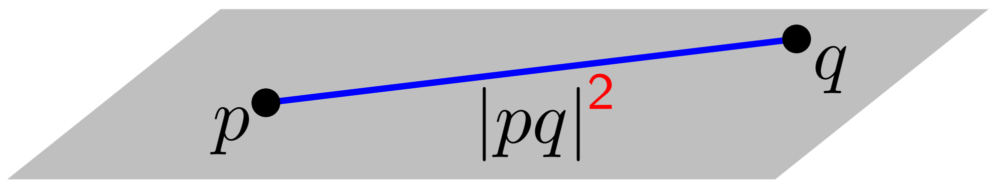

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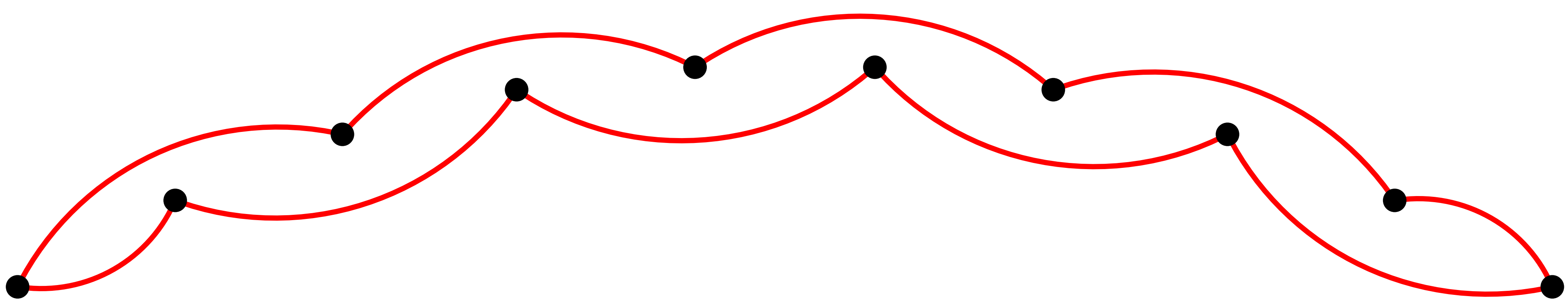

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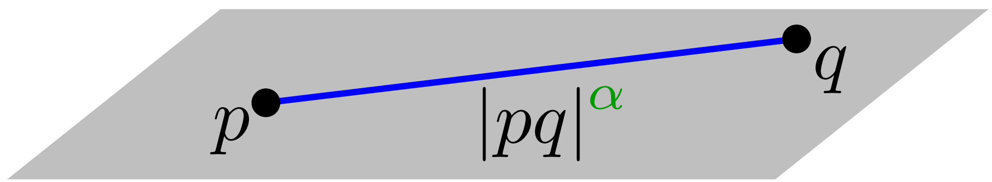

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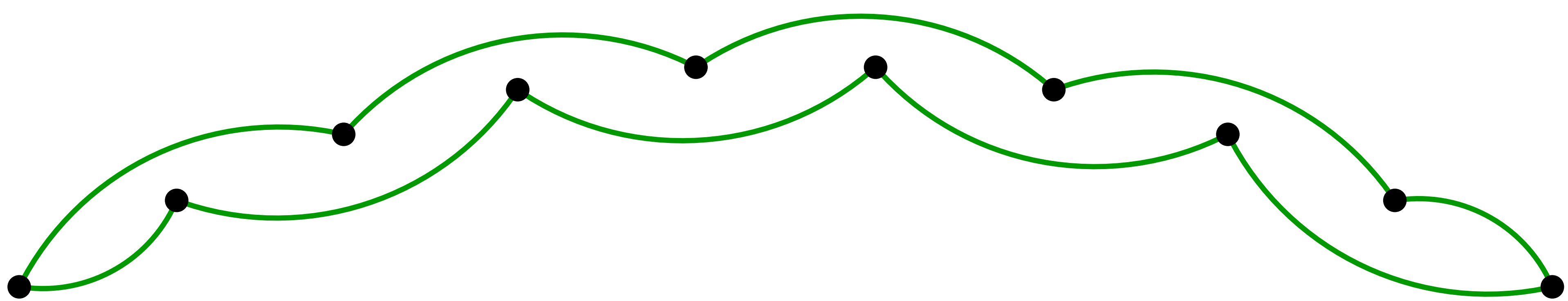

The Metric/Euclidean Case ($\alpha = 1$)

Theorem. [folklore]
The MST yields a 2-approximation for metric TSP.

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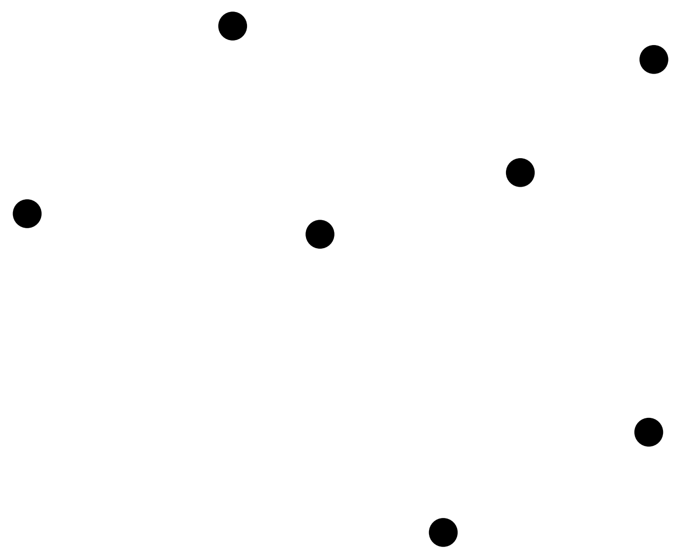

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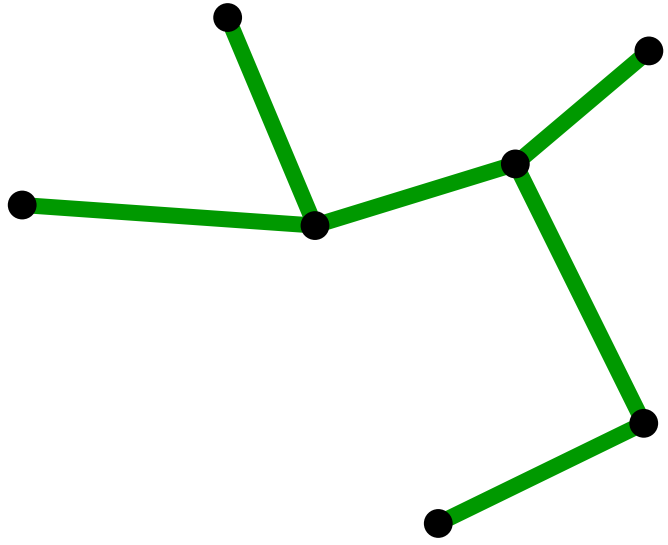

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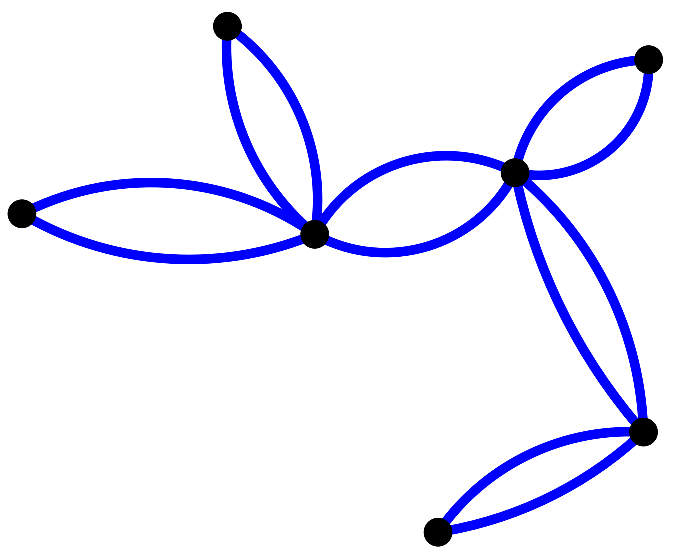

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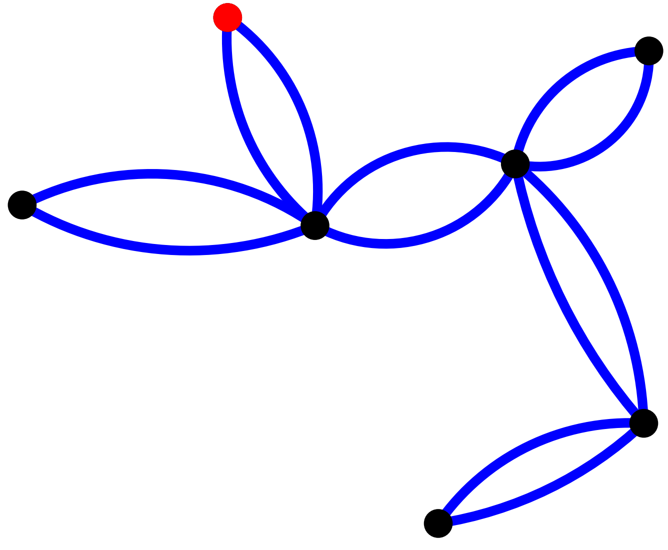

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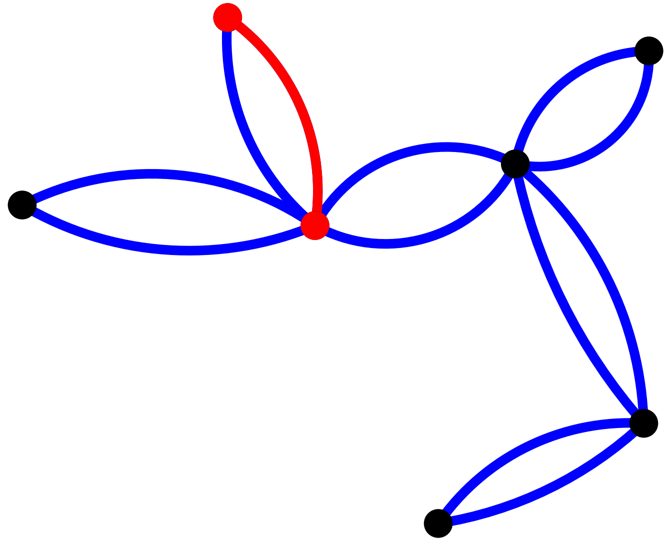

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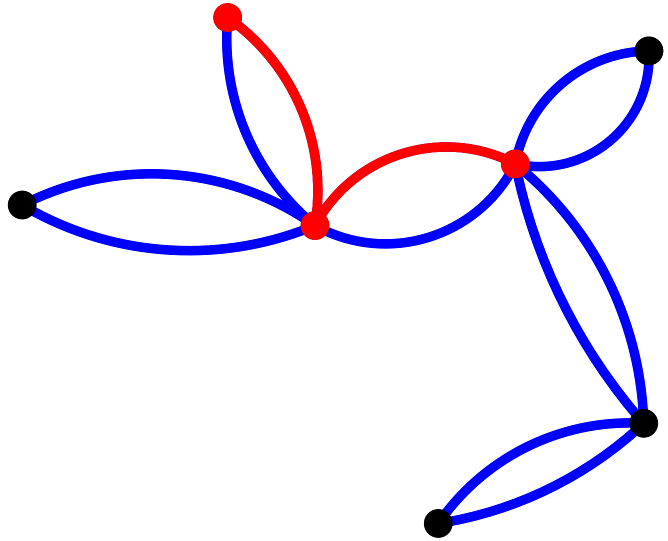

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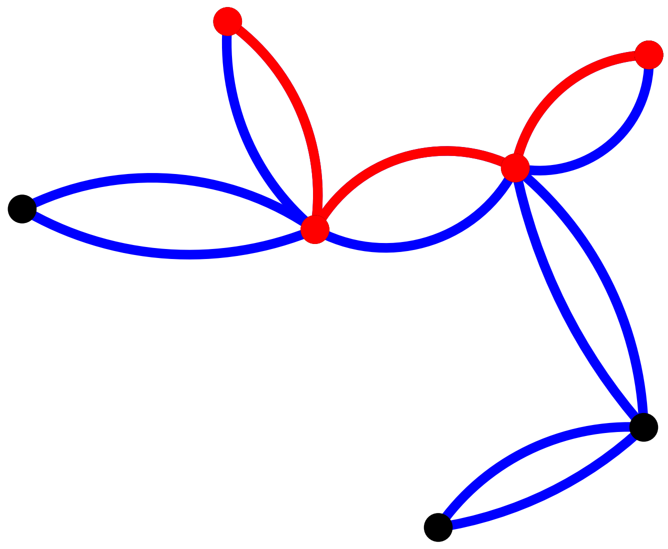

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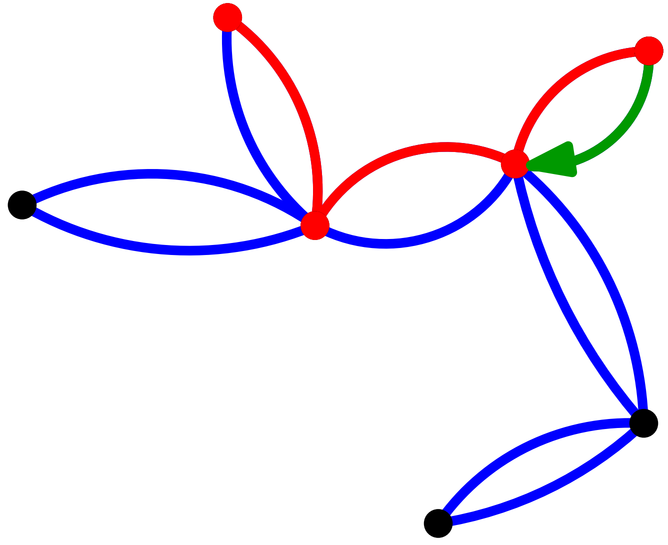

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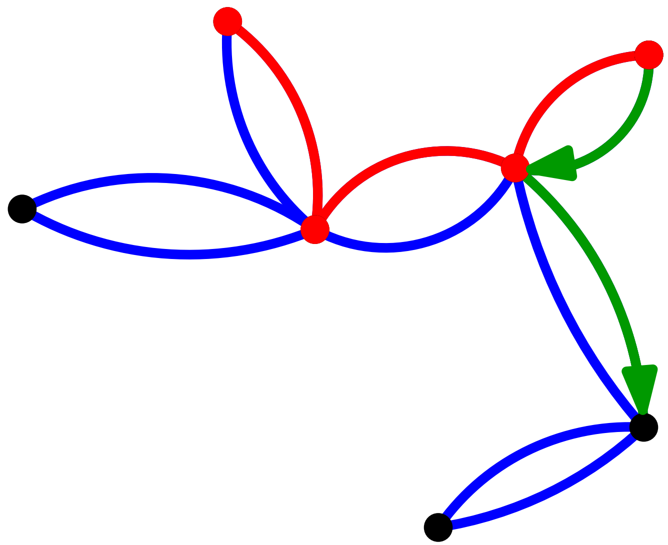

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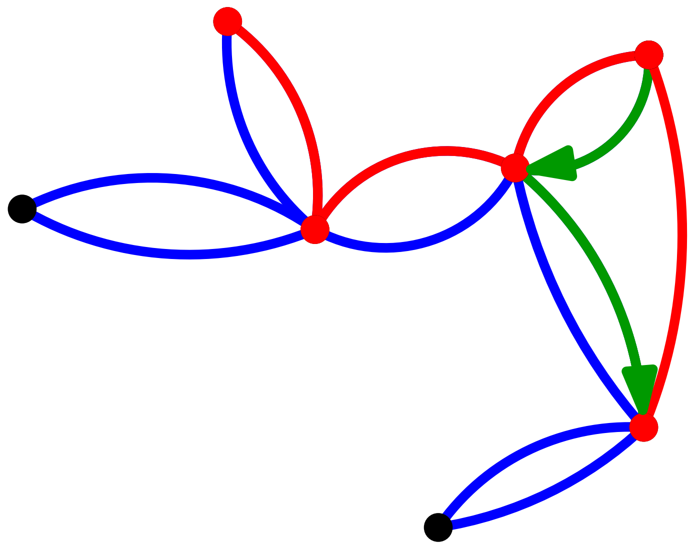

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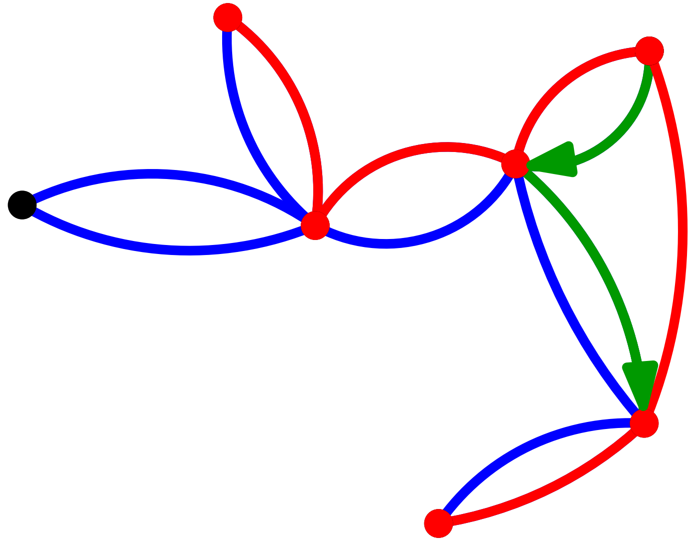

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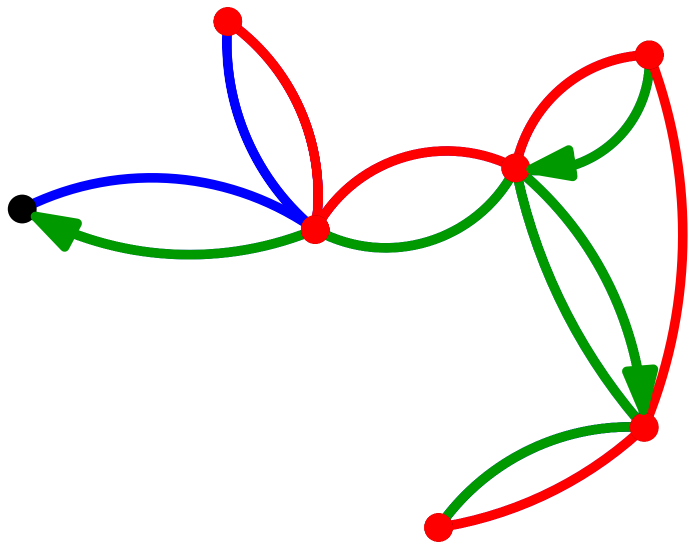

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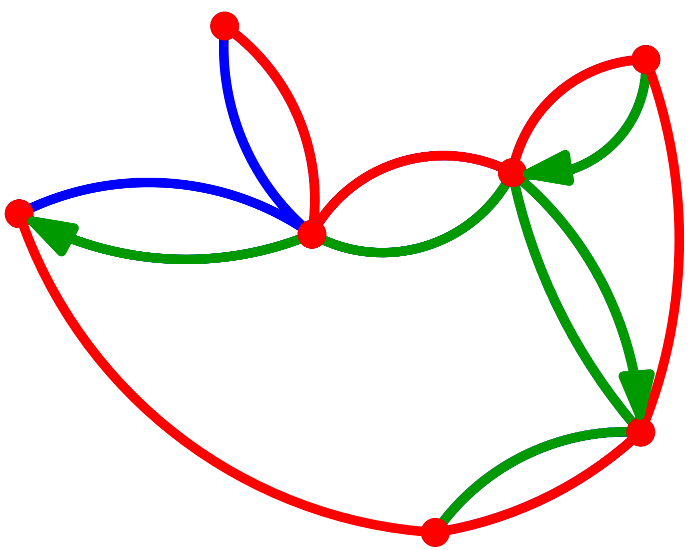

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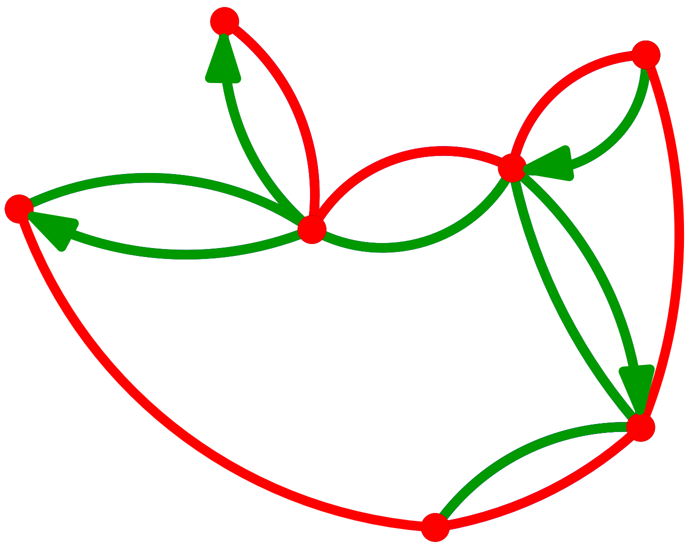

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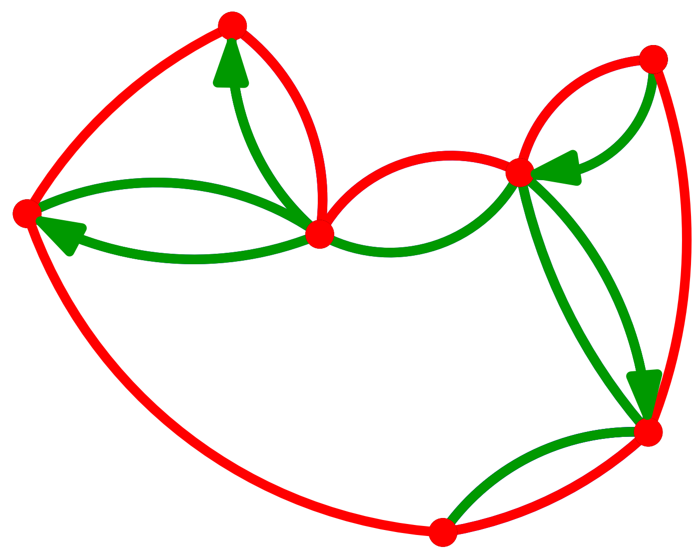

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But what about TSP(d, α)
for $\alpha \neq 1$?

Motivation

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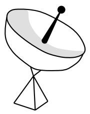

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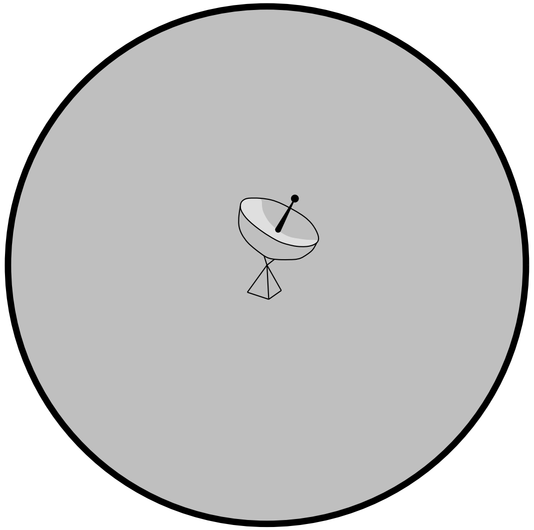

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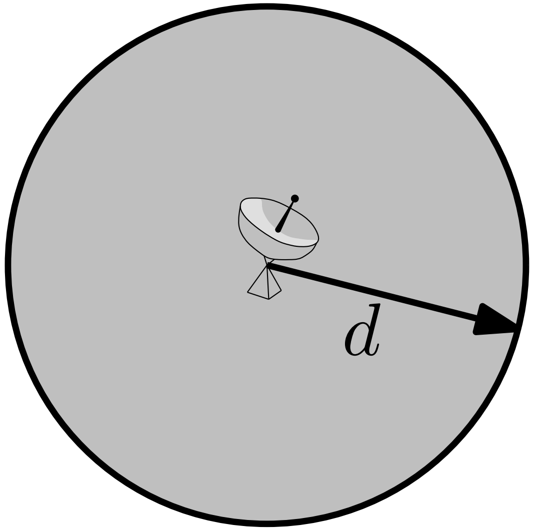

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  (*"distance-power gradient"*)

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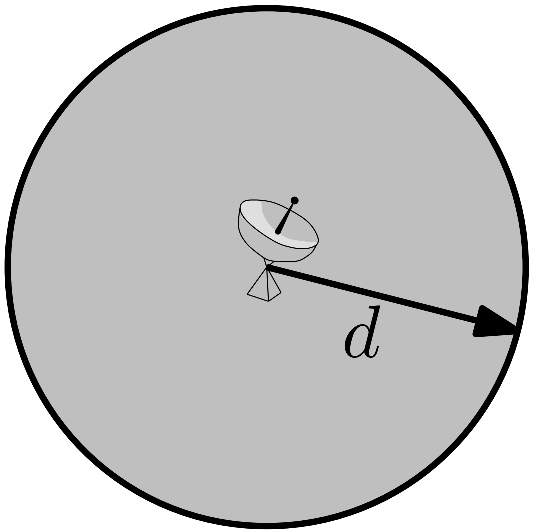

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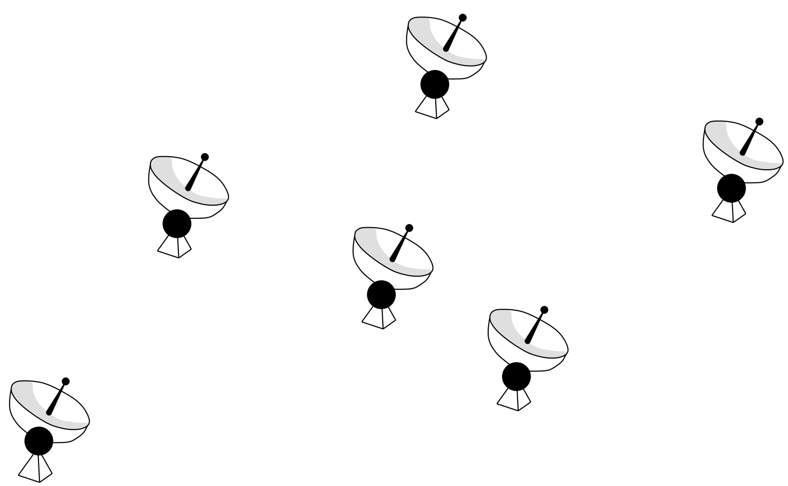

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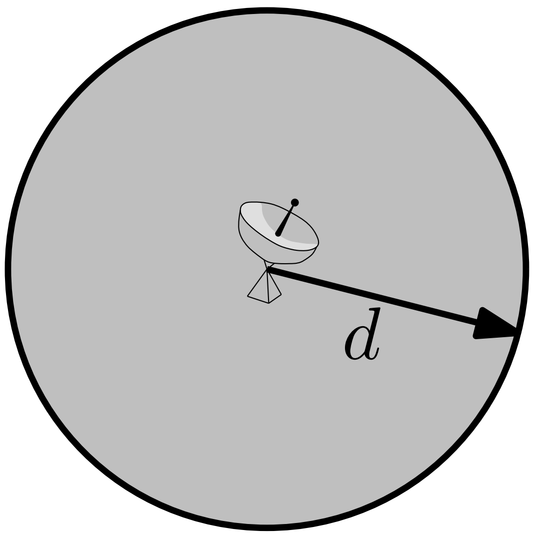

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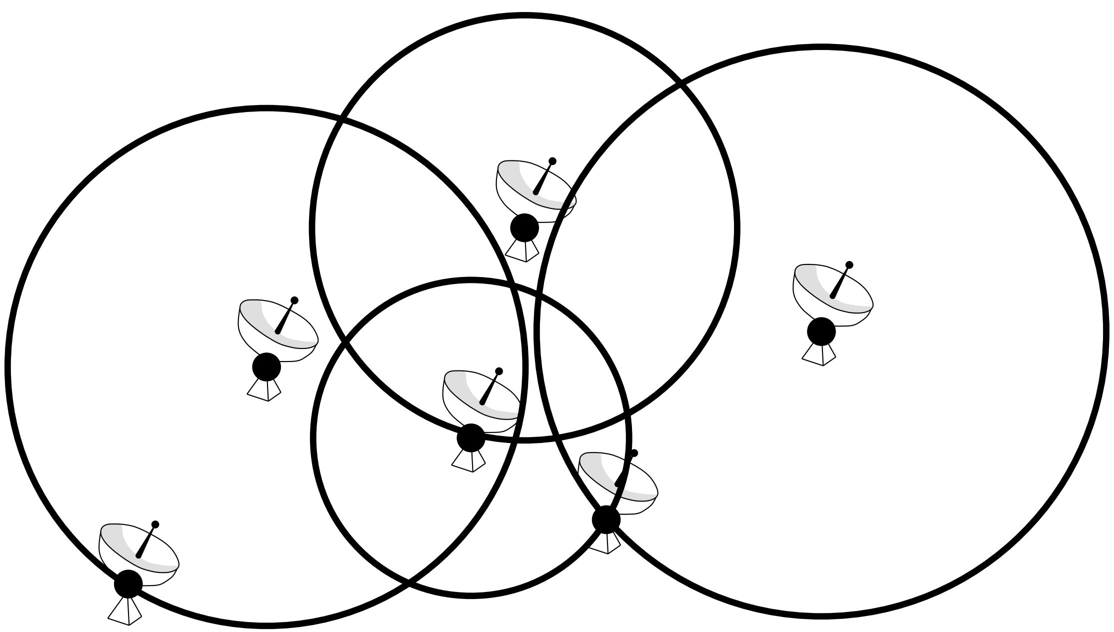

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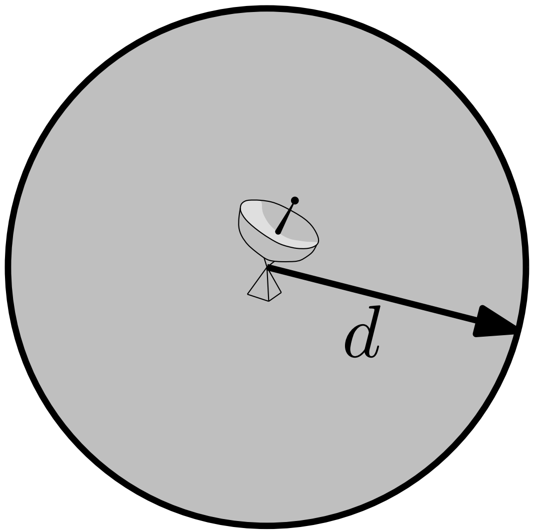

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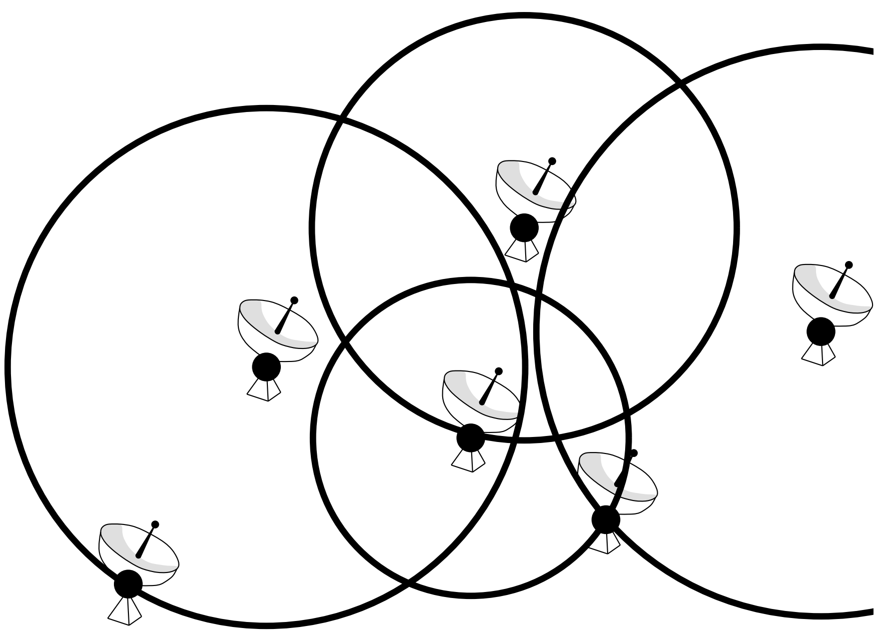

- range assignment ρ

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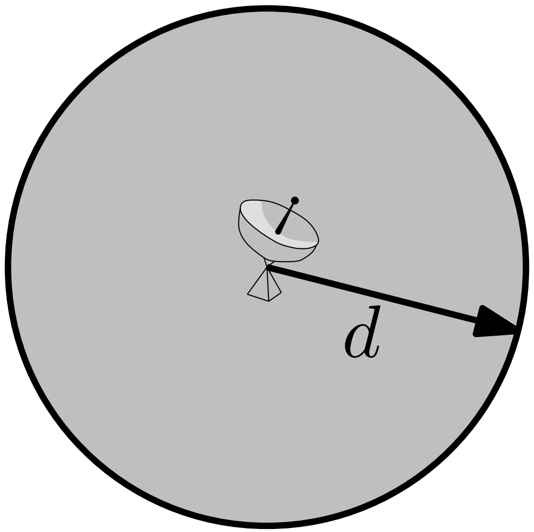

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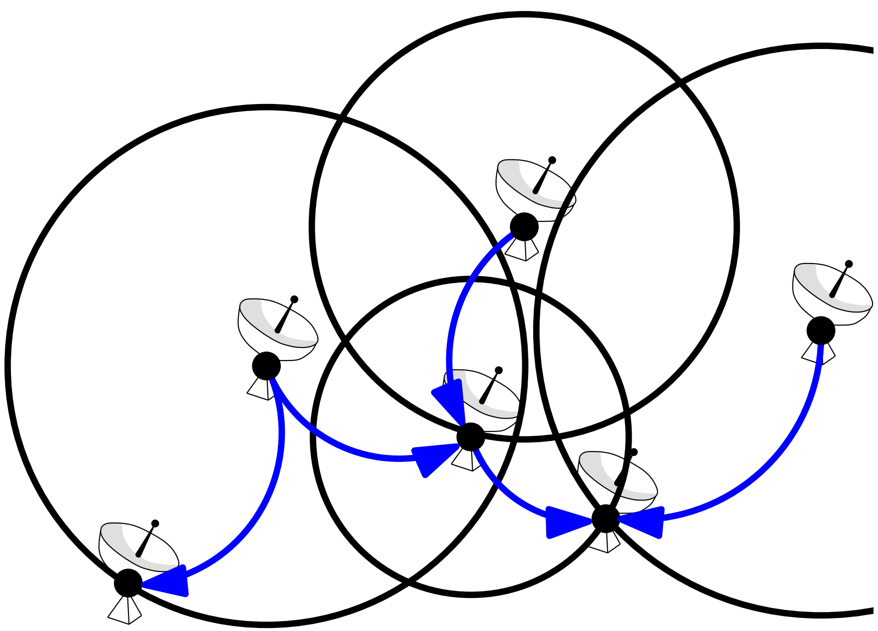

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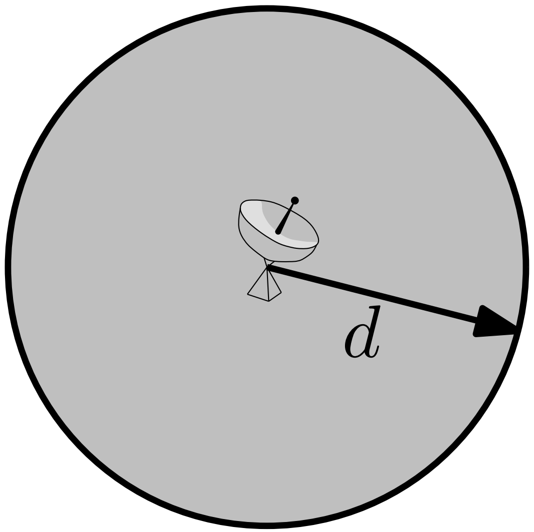

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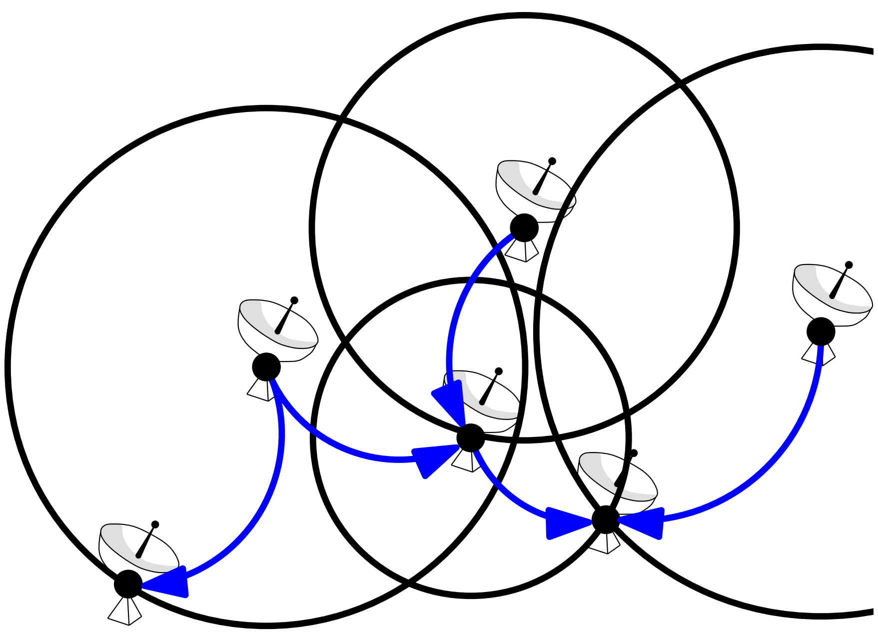

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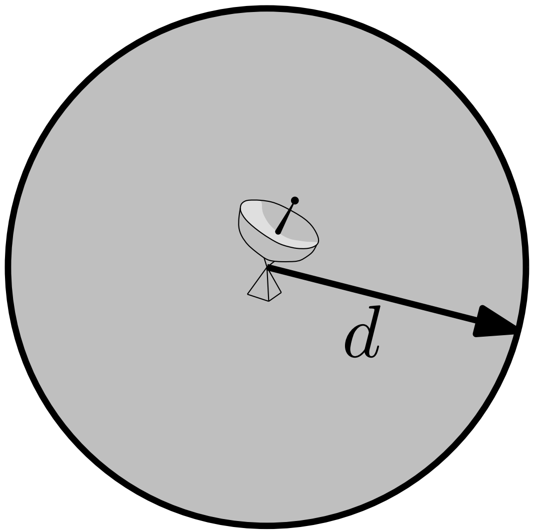

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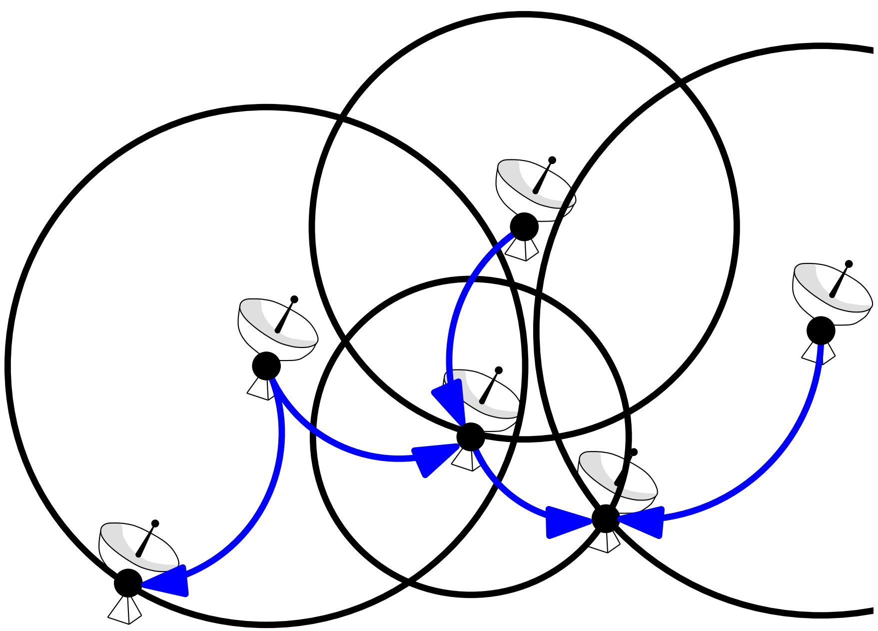
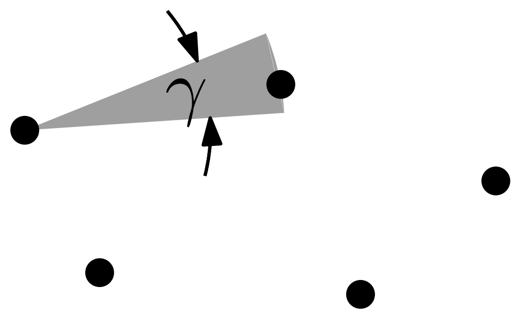


- range assignment ρ induces dir. communication graph G_ρ
- optimization problem: compute *min-energy* range assignment ρ s.t. G_ρ ...
  - is strongly connected
  - contains broadcast tree
  - *contains tour* [Funke...'08]

Motivation

2. Directional antennas with circular sectors [Caragiannis...'08]



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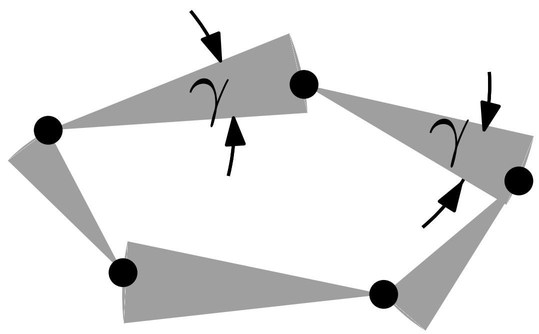

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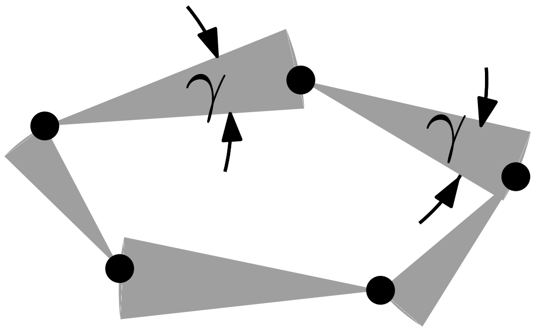

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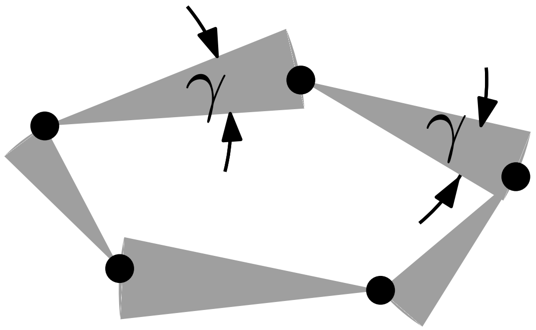

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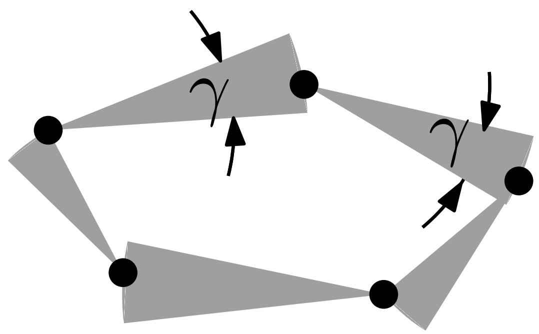

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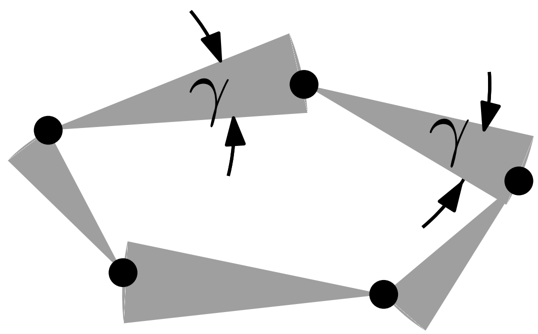

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3. Complexity

Are things becoming simpler or harder?

Motivation

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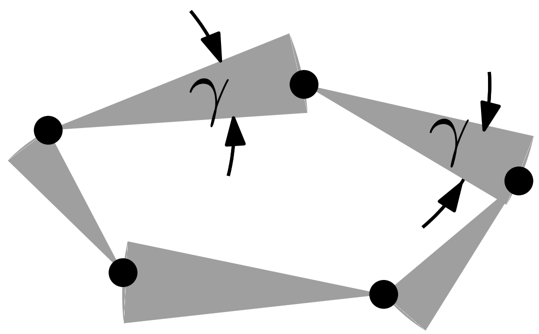

G_ρ strongly connected $\xrightarrow{\gamma \to 0}$ G_ρ tour

3. Complexity

Are things becoming simpler or harder?

Is Arora's PTAS for Euclidean TSP a "lucky coincidence"?

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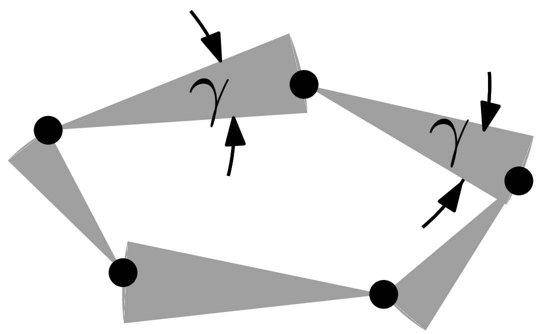

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3. Complexity

Are things becoming simpler or harder?

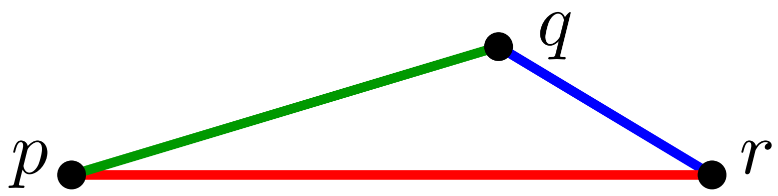
Is Arora's PTAS for Euclidean TSP a "lucky coincidence"?

If it is, how well can we approximate, say, TSP$(2,2)$?

Previous Work

Definition. $\mathrm{dist}(\cdot,\cdot)$ fulfills the τ-*relaxed triangle inequality* if any three points p, q, r satisfy

$$\mathrm{dist}(p,r) \leq \tau \cdot (\mathrm{dist}(p,q) + \mathrm{dist}(q,r)).$$



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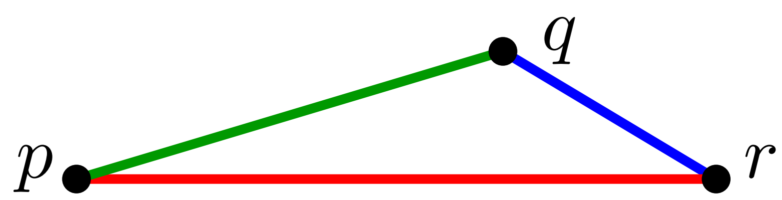

Lemma. [Funke...'08]

$|\cdot|^2$ fulfills the 2-relaxed triangle inequality.

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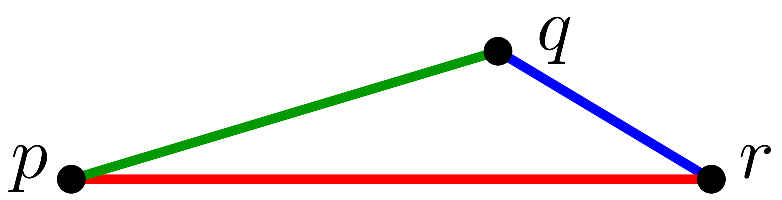

Lemma. [Funke...'08] For $\alpha \geq 1$, $|\cdot|^\alpha$ fulfills the $2^{\alpha-1}$-relaxed triangle inequality.

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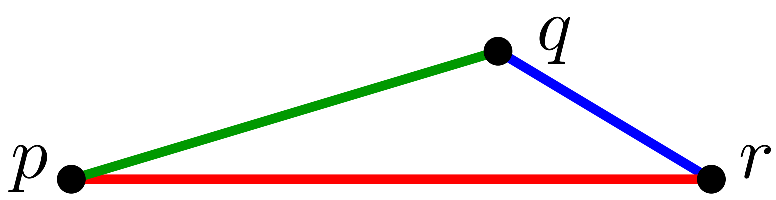

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Good news. Can apply algorithms for Δ_τ-TSP to $\text{TSP}(\,\cdot\,,\alpha)$!

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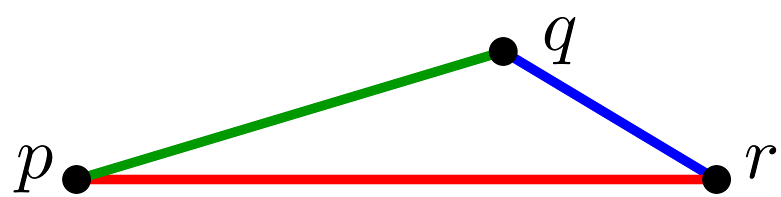

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Δ_τ	TSP$(\cdot, \alpha)$	
$\tau^2 + \tau$	$4^{\alpha-1} + 2^{\alpha-1}$	[Andreae'01]
4τ	$2^{\alpha+1}$	[BenderChekuri'00]

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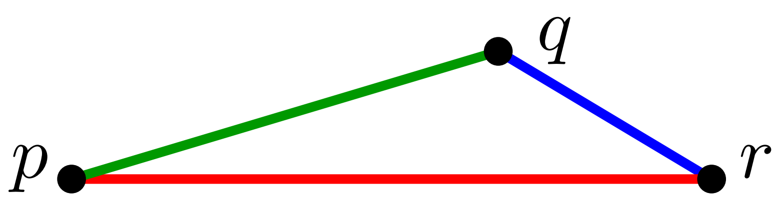

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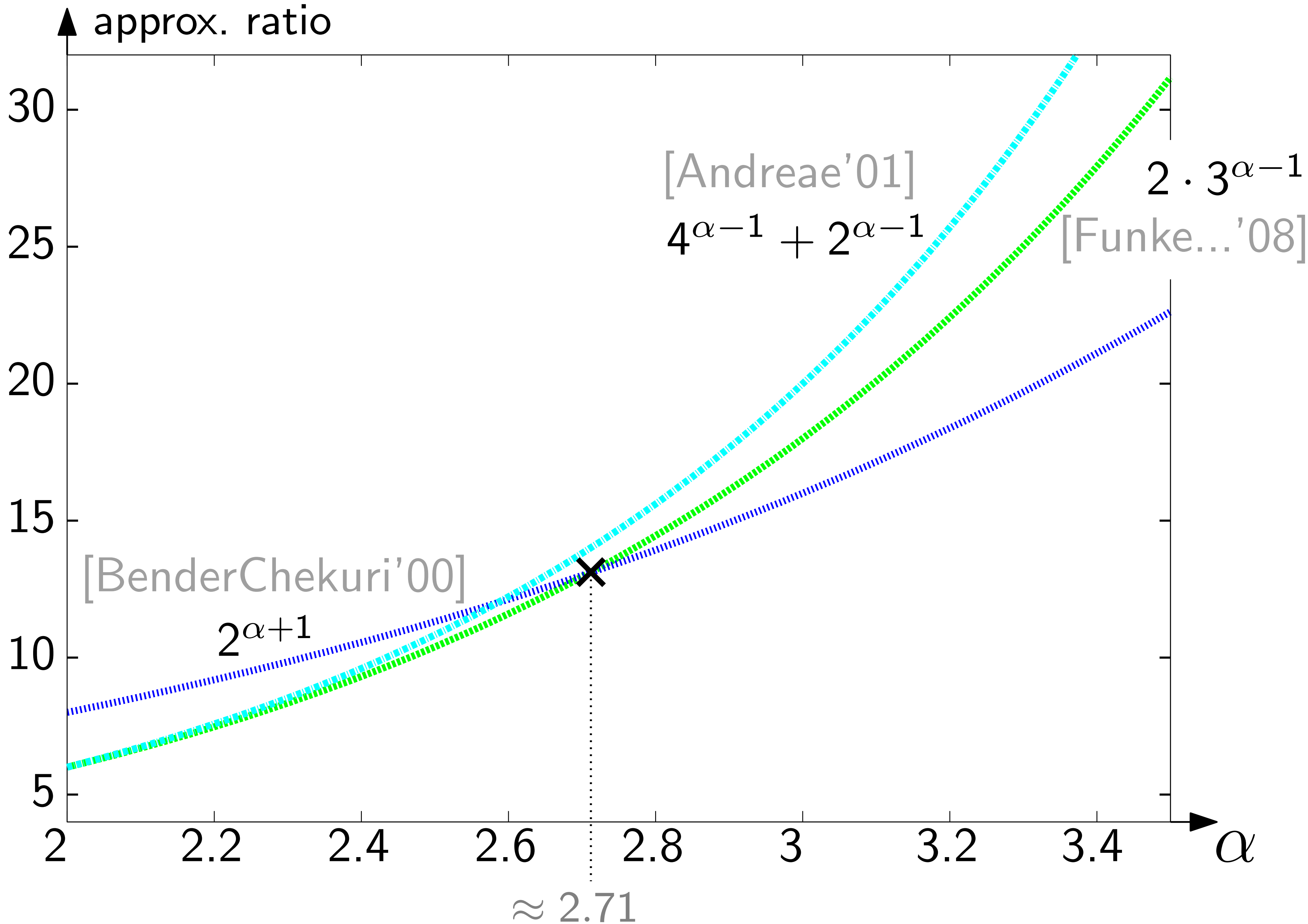

approx. ratio
30
25
20
15
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2.2
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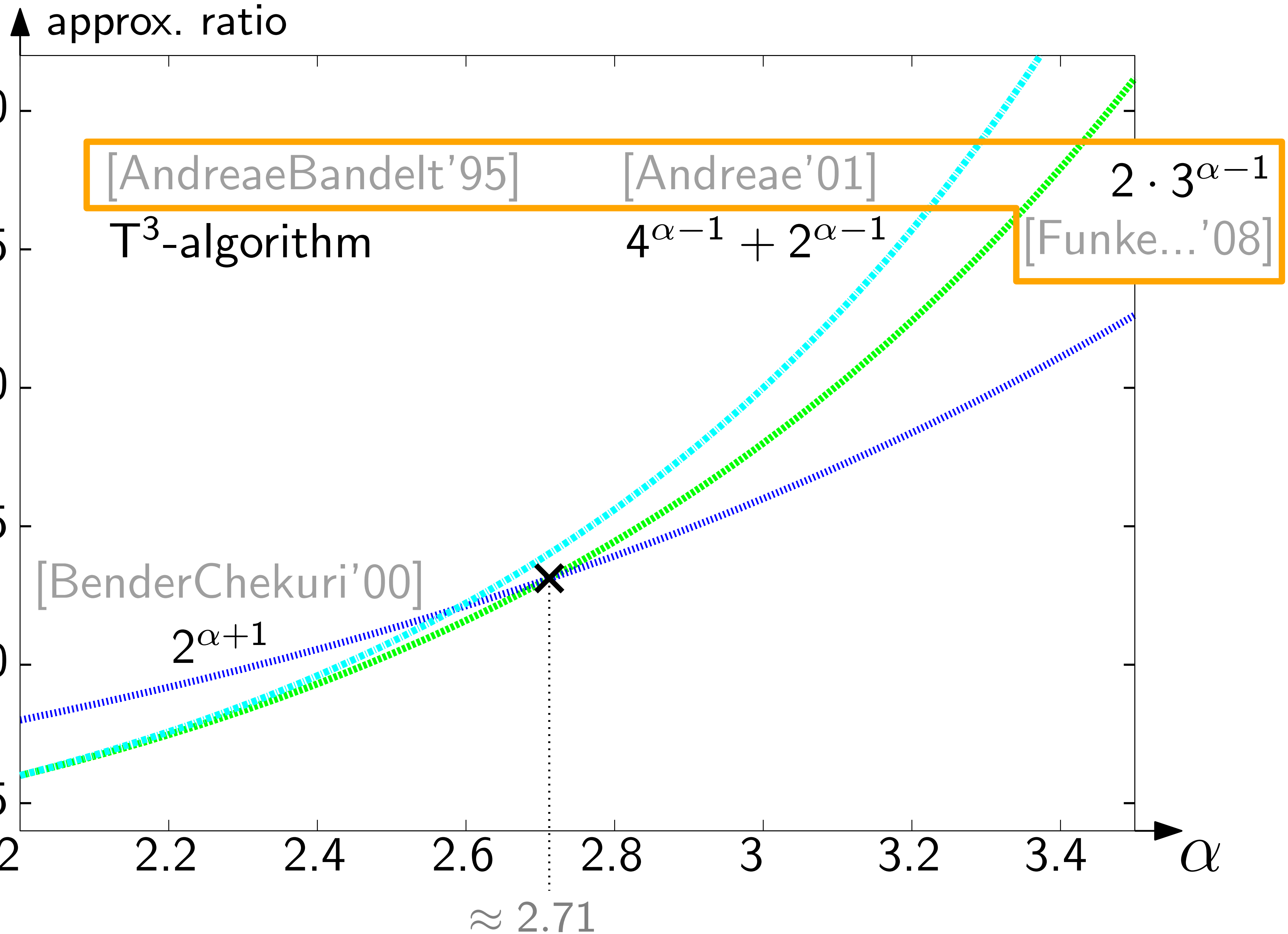
Our Results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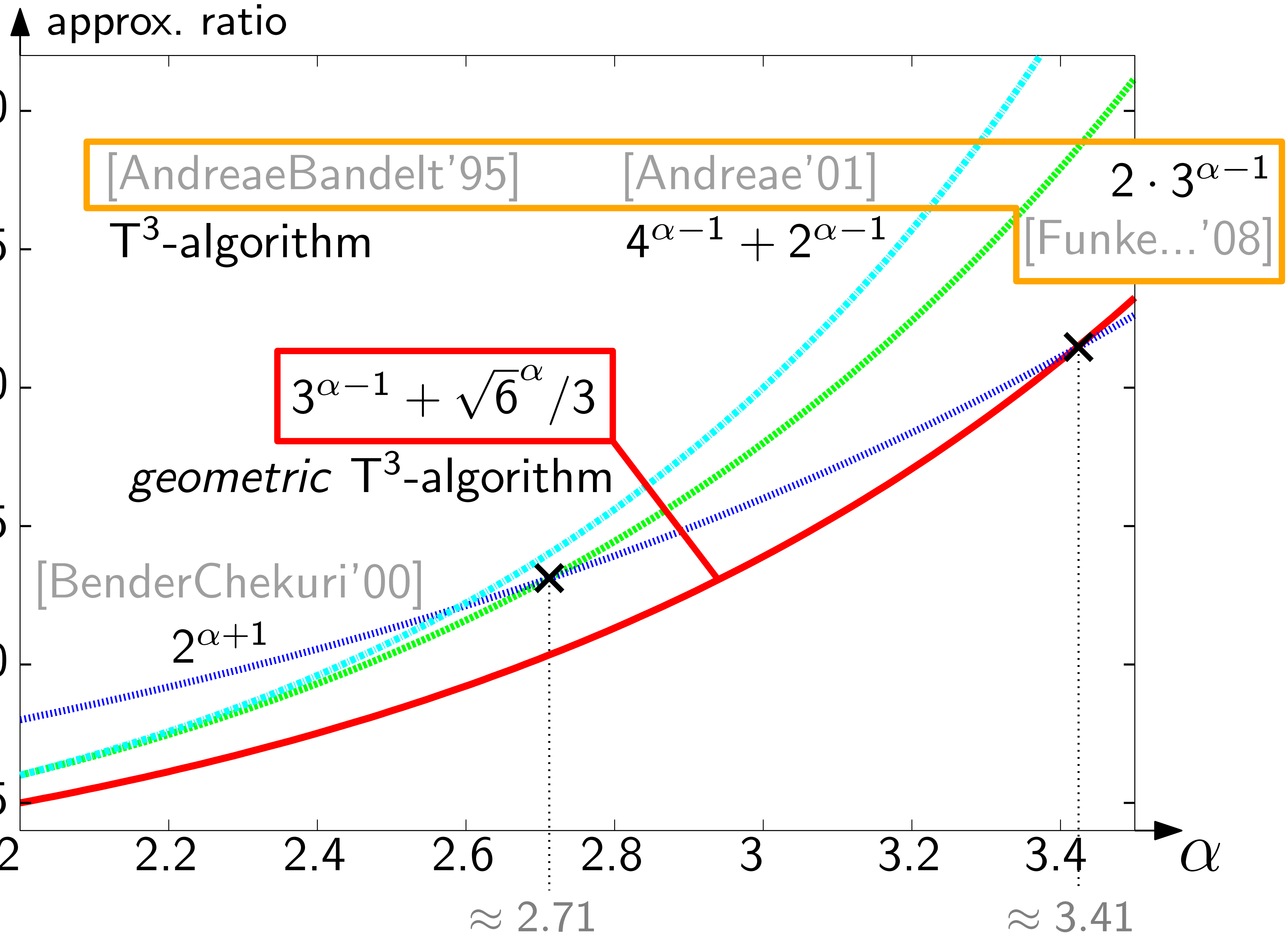
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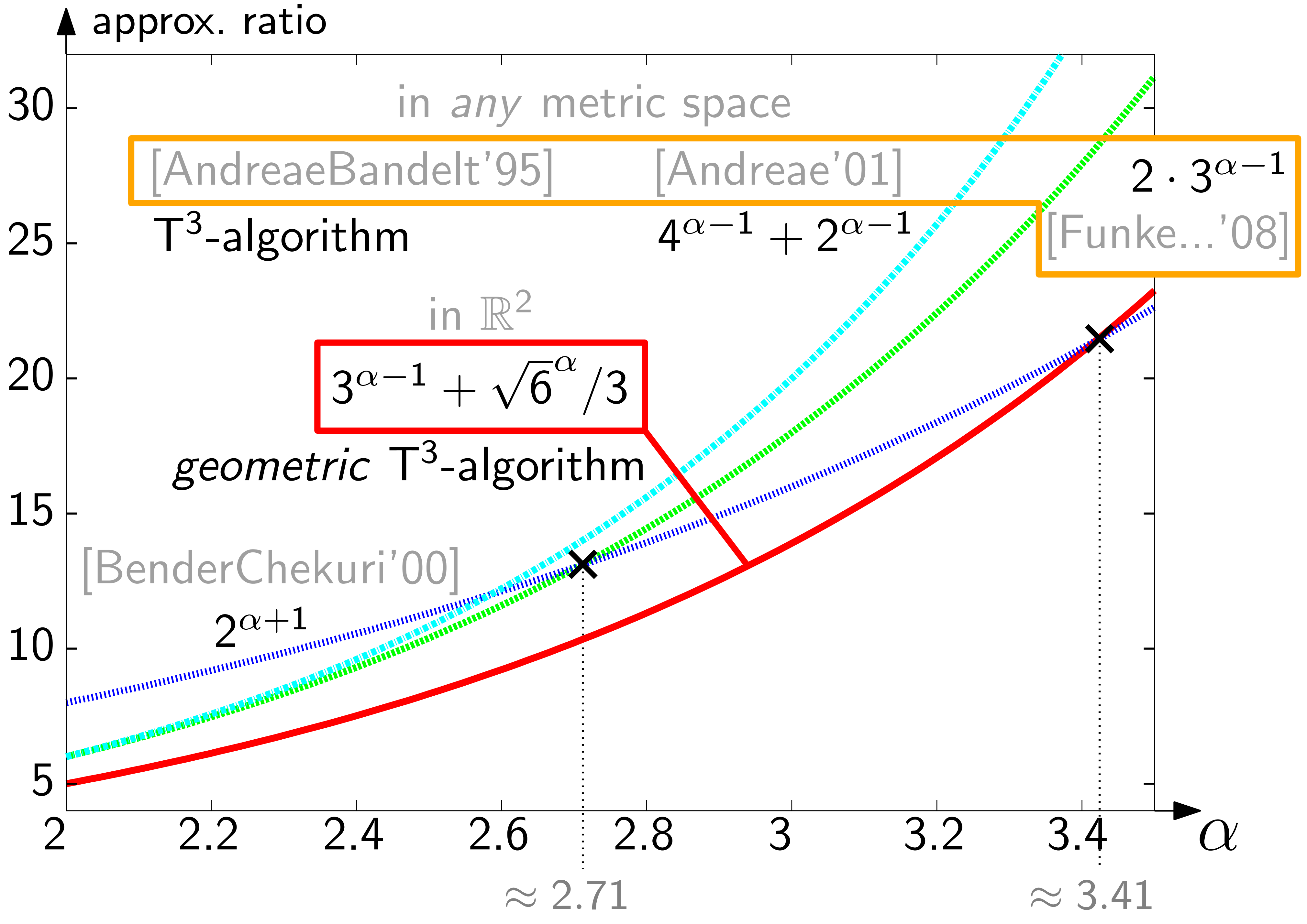

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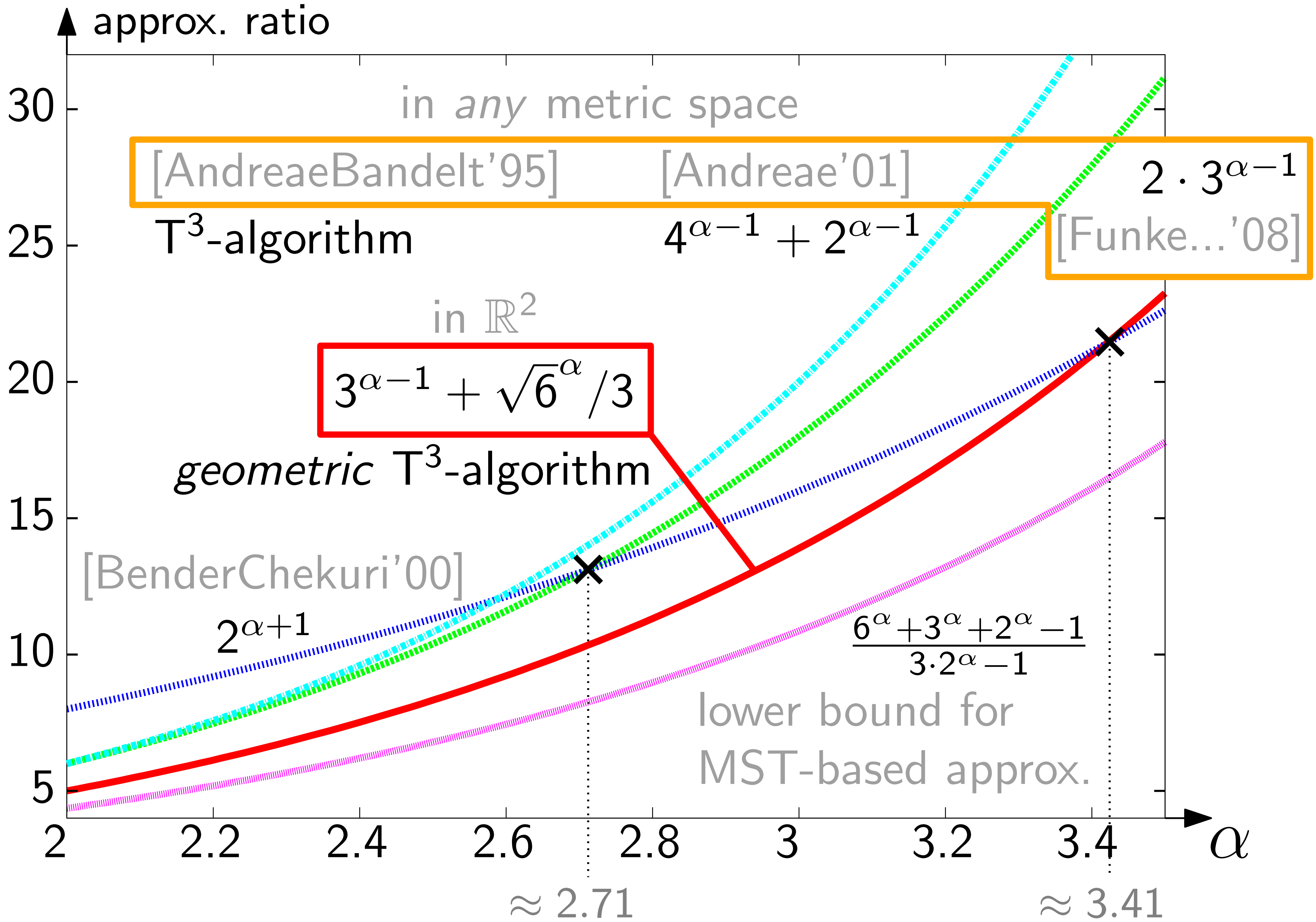

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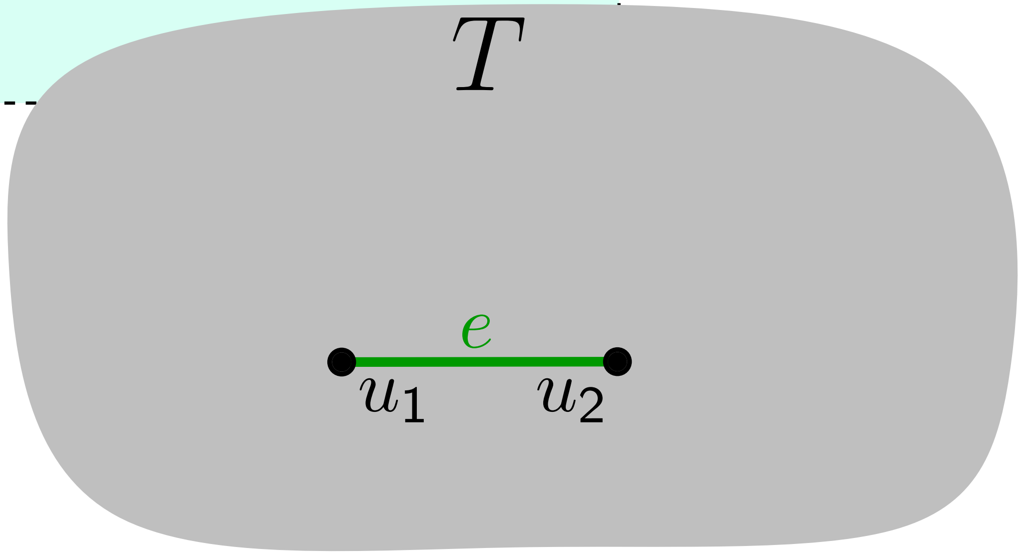
The T^3-Algorithm

[Sekanina'60, AndreaeBandelt'95]

$\textsc{CycleInCube}(T, e = u_1u_2)$



The T^3-Algorithm

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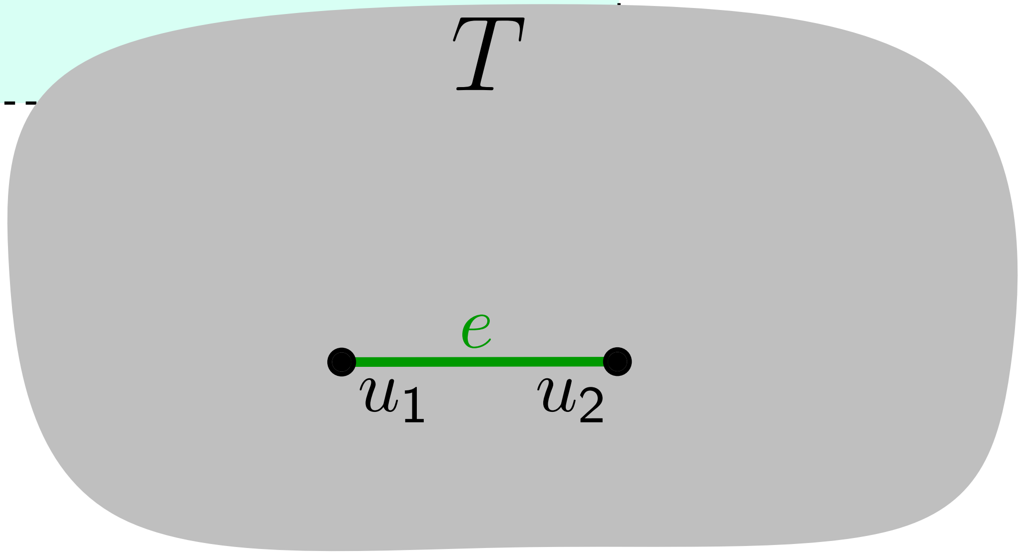
CycleInCube(T, $e = u_1u_2$)

Take MST of given point set!



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for $i \leftarrow 1$ **to** 2 **do**

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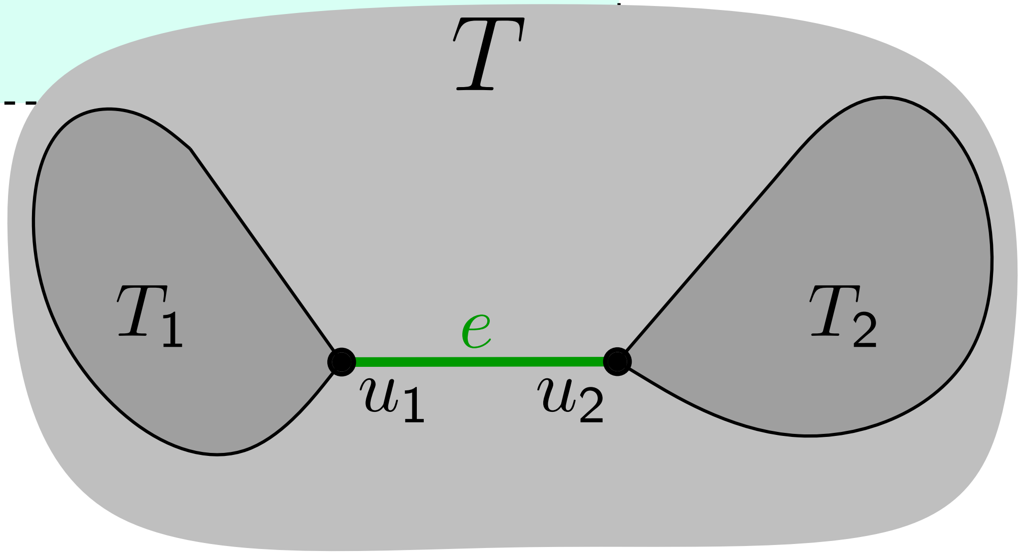

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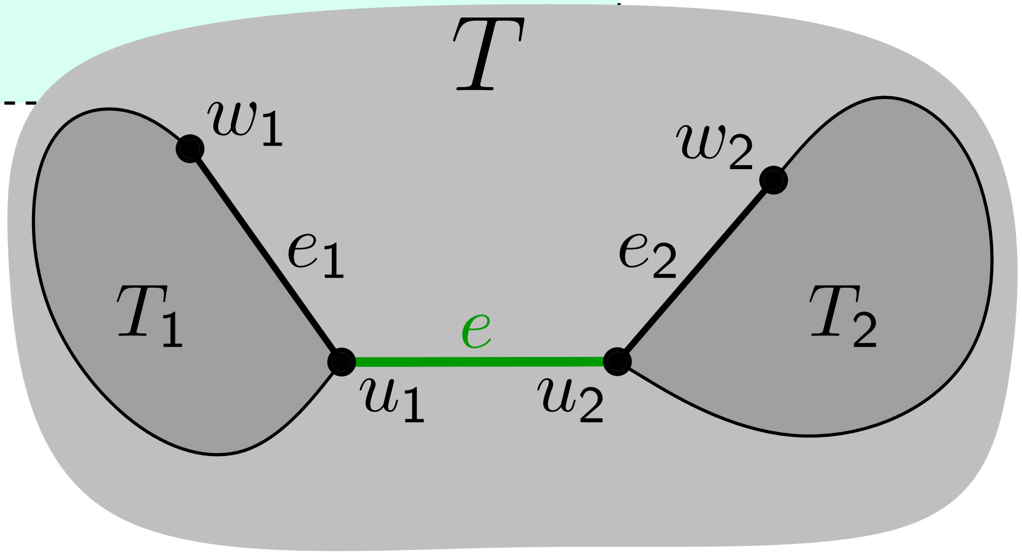

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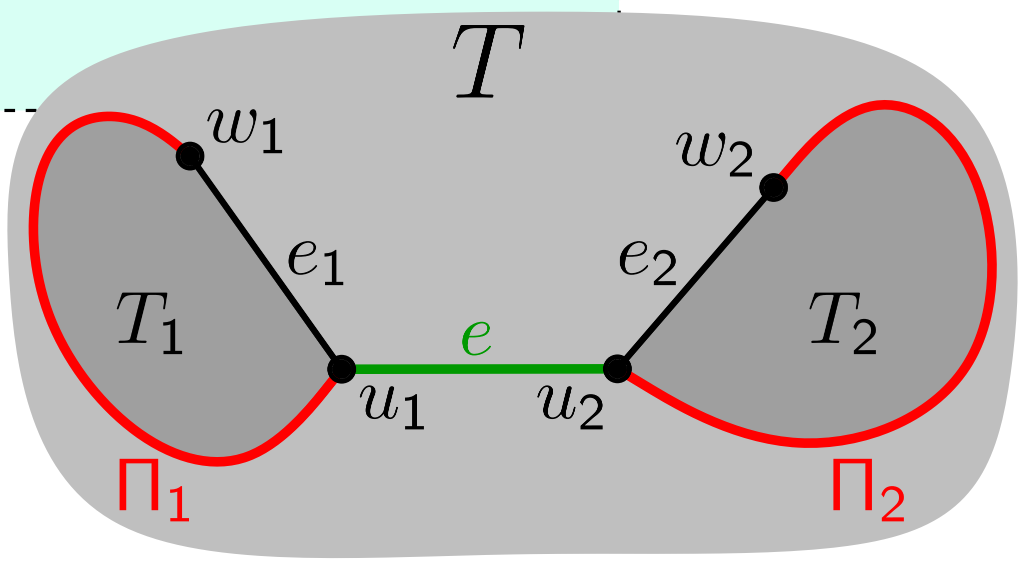

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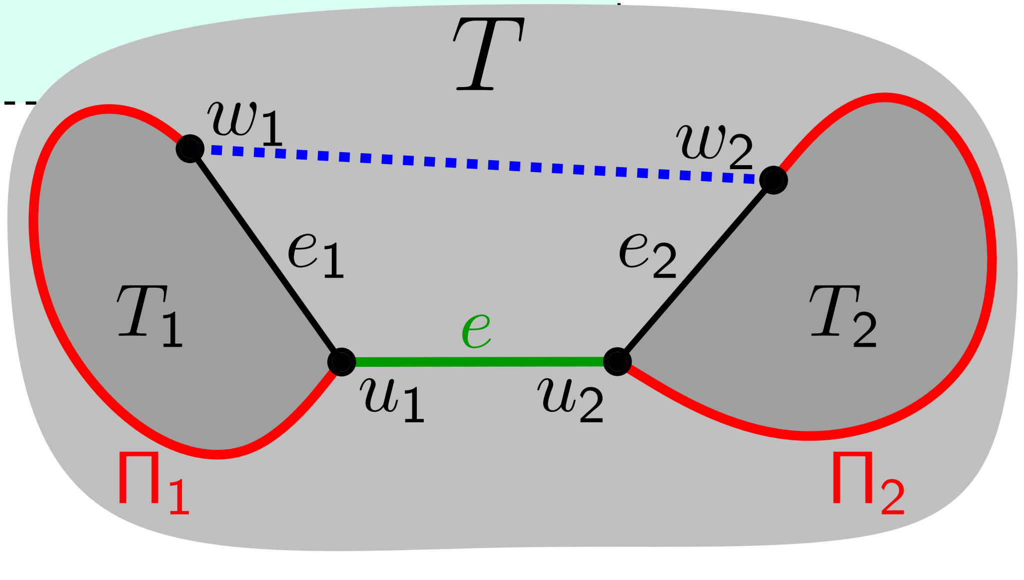
return $\Pi_1 + e + \Pi_2 + w_1w_2$



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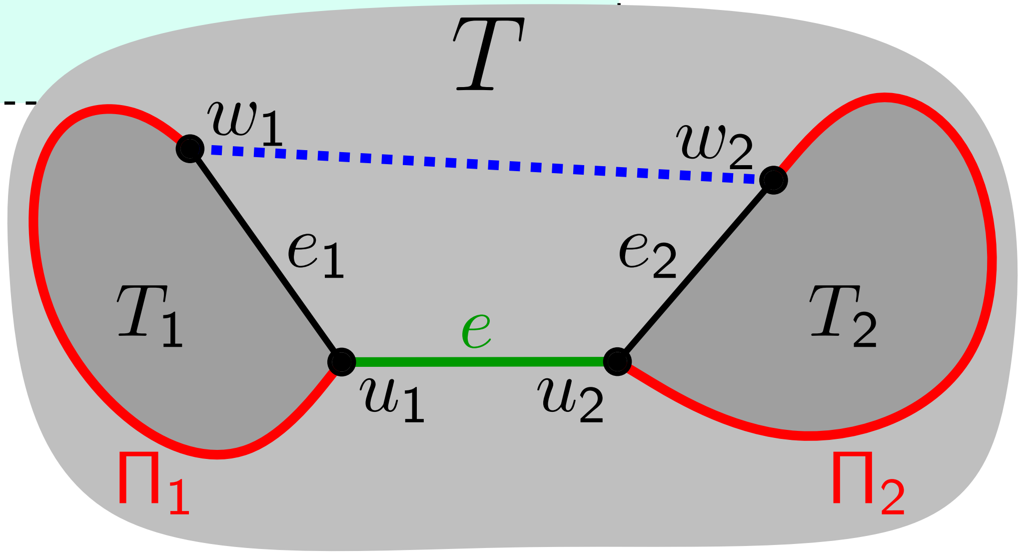

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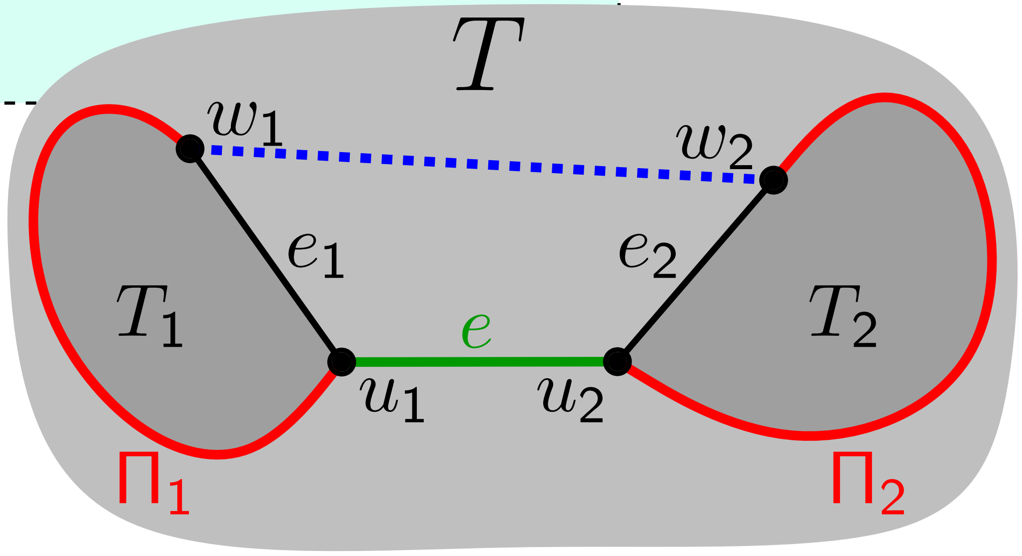

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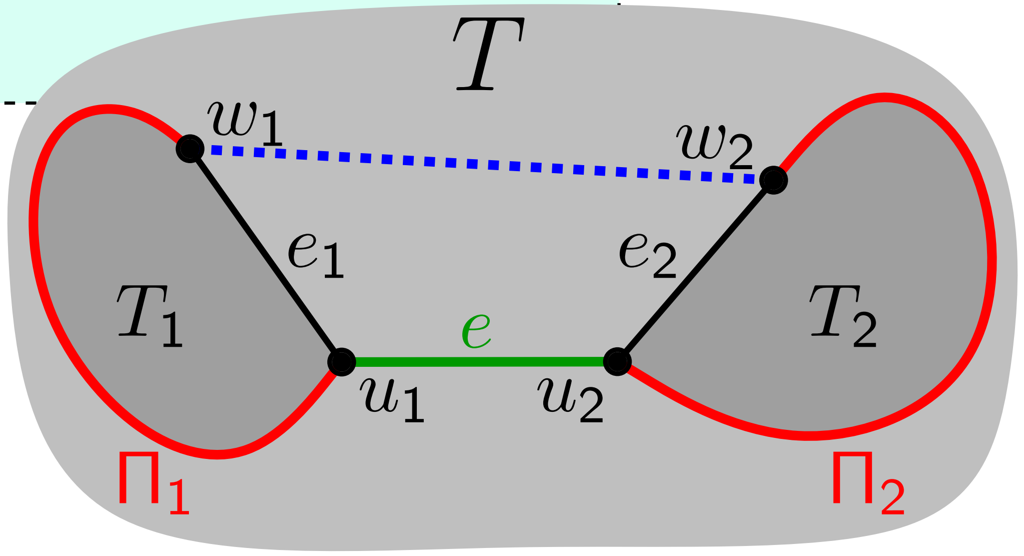

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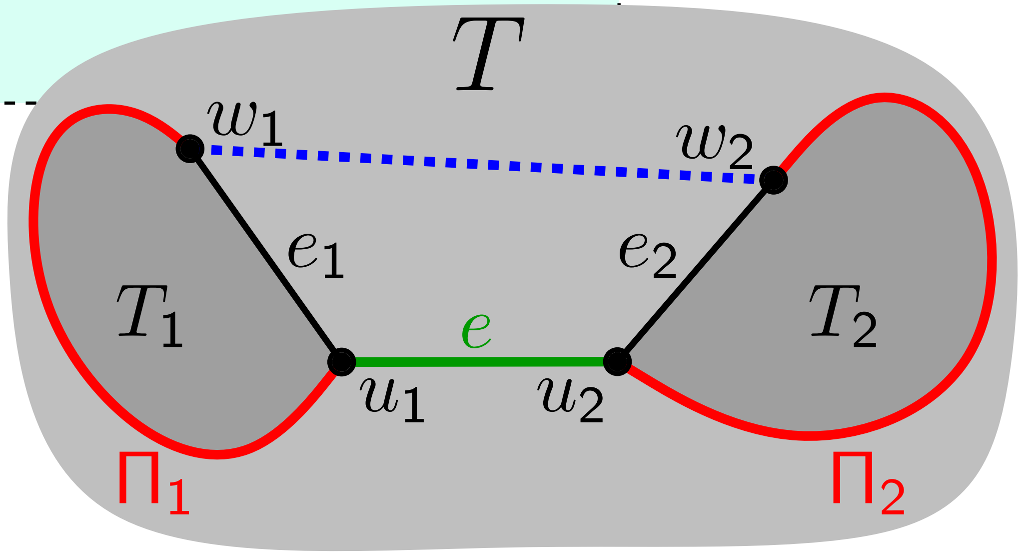

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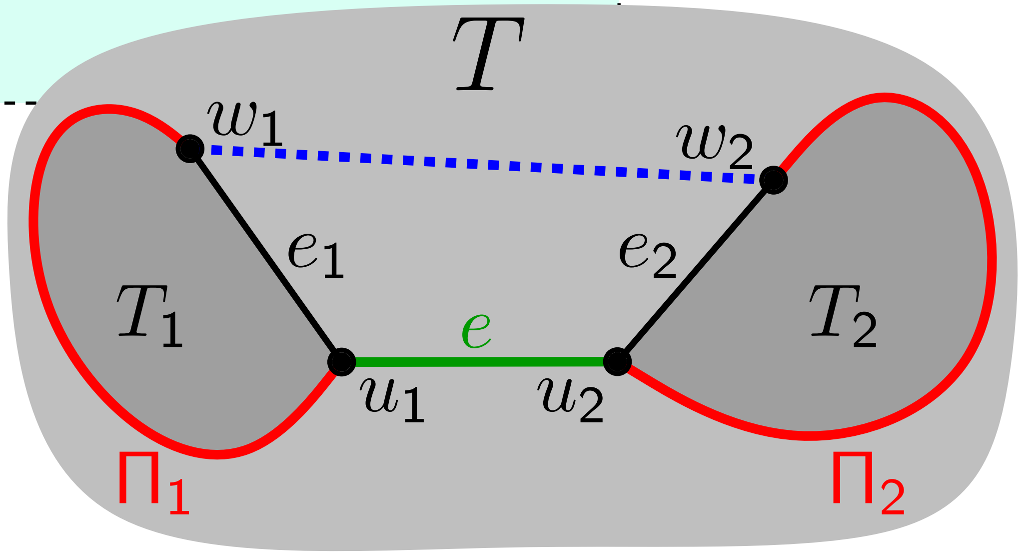
(2- or) *3-shortcut*
uses edges e, e_1, and e_2



The T^3-Algorithm

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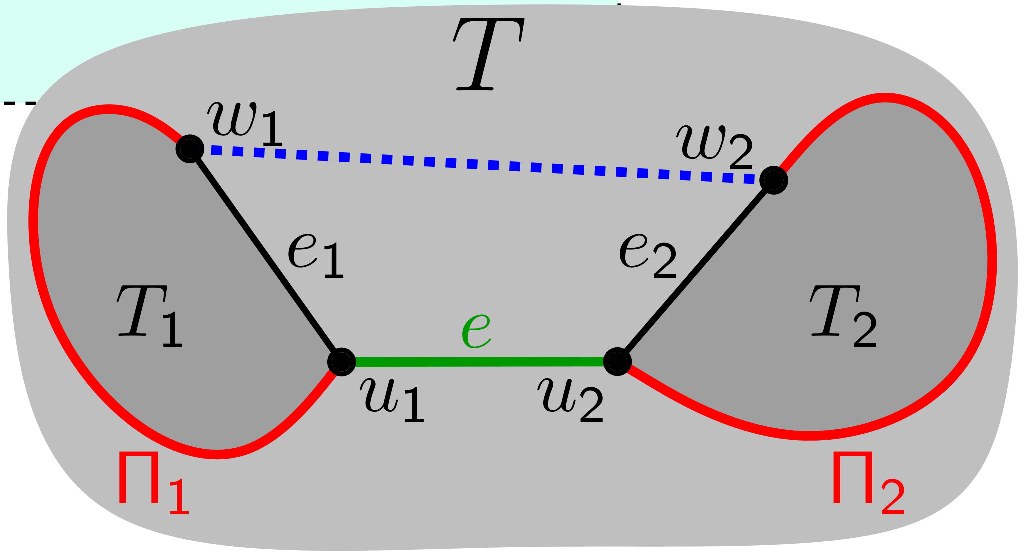

Observation.
Every edge is used at most ...?

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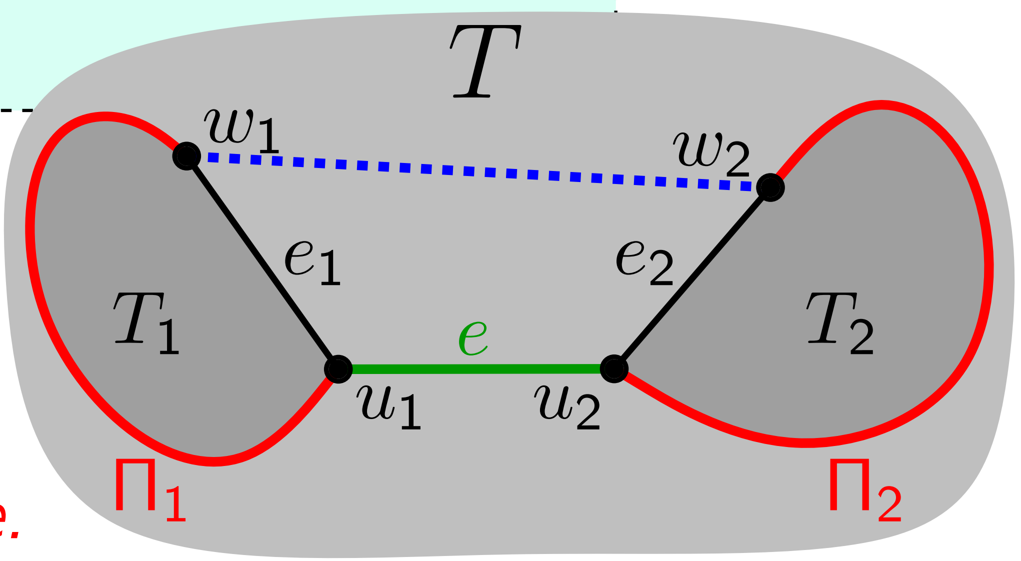

Observation.
Every edge is used at most *twice.*

Result #1

Observation. Every edge is used at most twice.

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Lemma. Let e be a 3-shortcut using a, b, c. Let $\alpha \geq 1$. Then $|e|^\alpha \leq 3^{\alpha-1}\big(|a|^\alpha + |b|^\alpha + |c|^\alpha\big)$.

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Corollary. For $\alpha \geq 2$, the T^3-algorithm yields a $(2 \cdot 3^{\alpha-1})$-approximation for TSP$(\,\cdot\,, \alpha)$.

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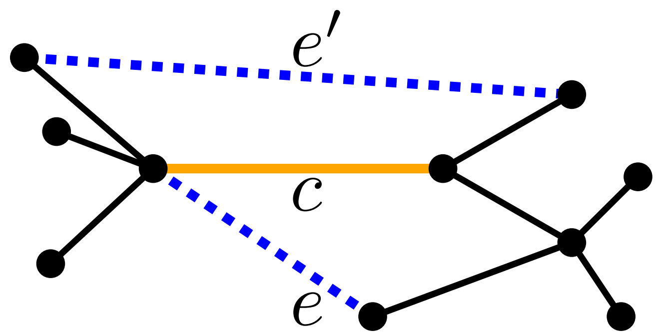
Proof. Every edge c of the MST (w.r.t. $|\cdot|^\alpha$) contributes at most twice at most $3^{\alpha-1}|c|^\alpha$ to the T^3-tour.



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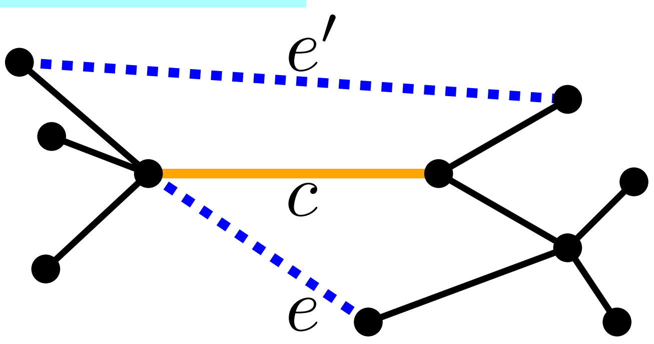

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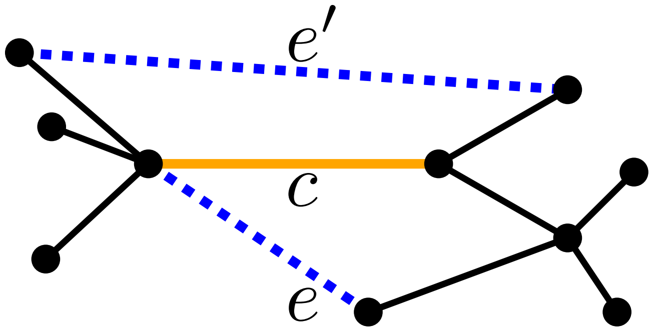

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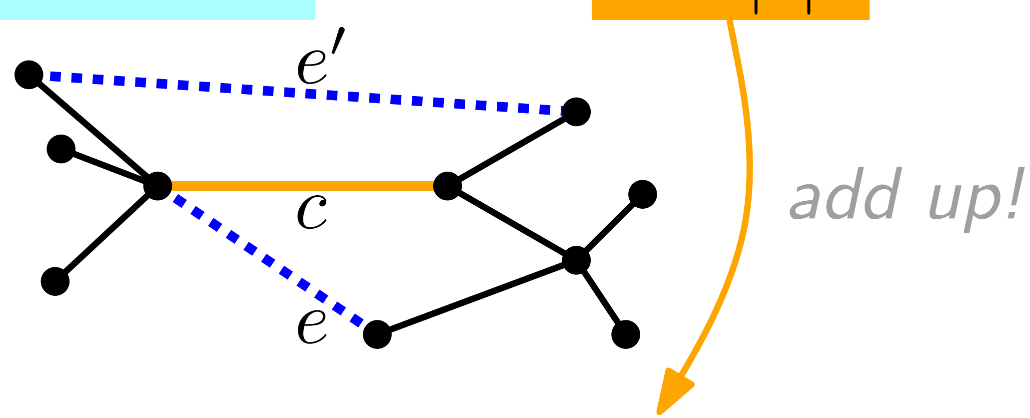

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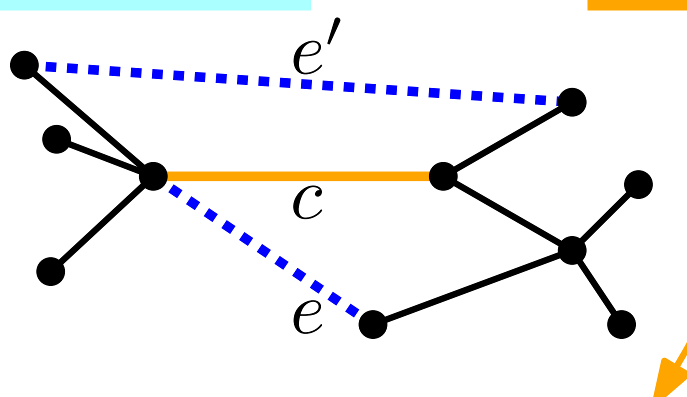

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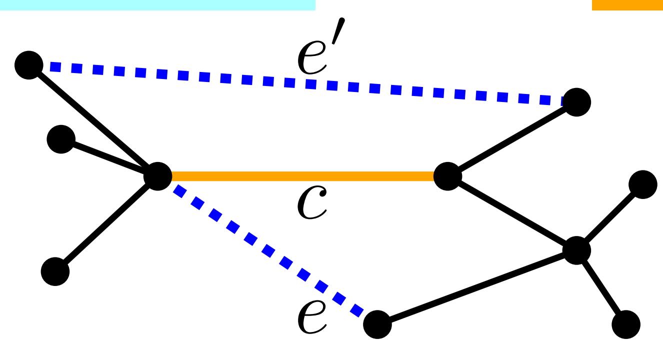

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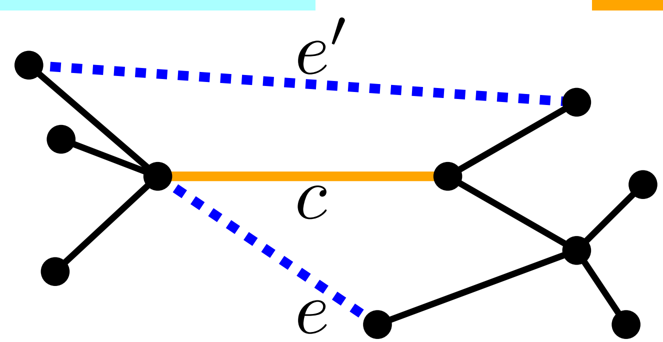

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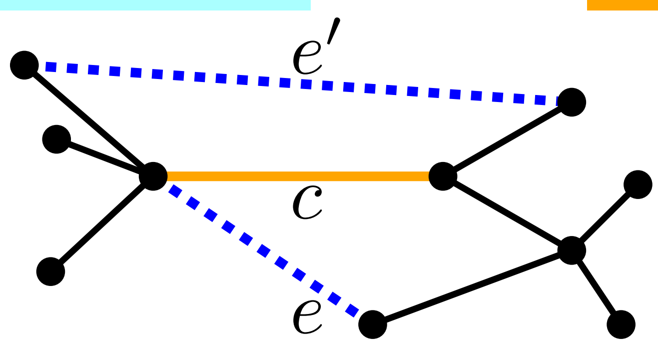

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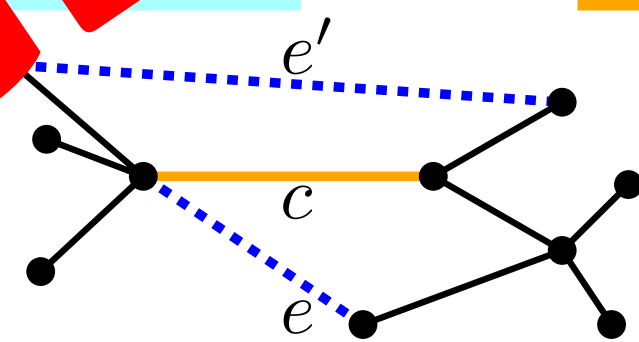

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NO GEOMETRY!

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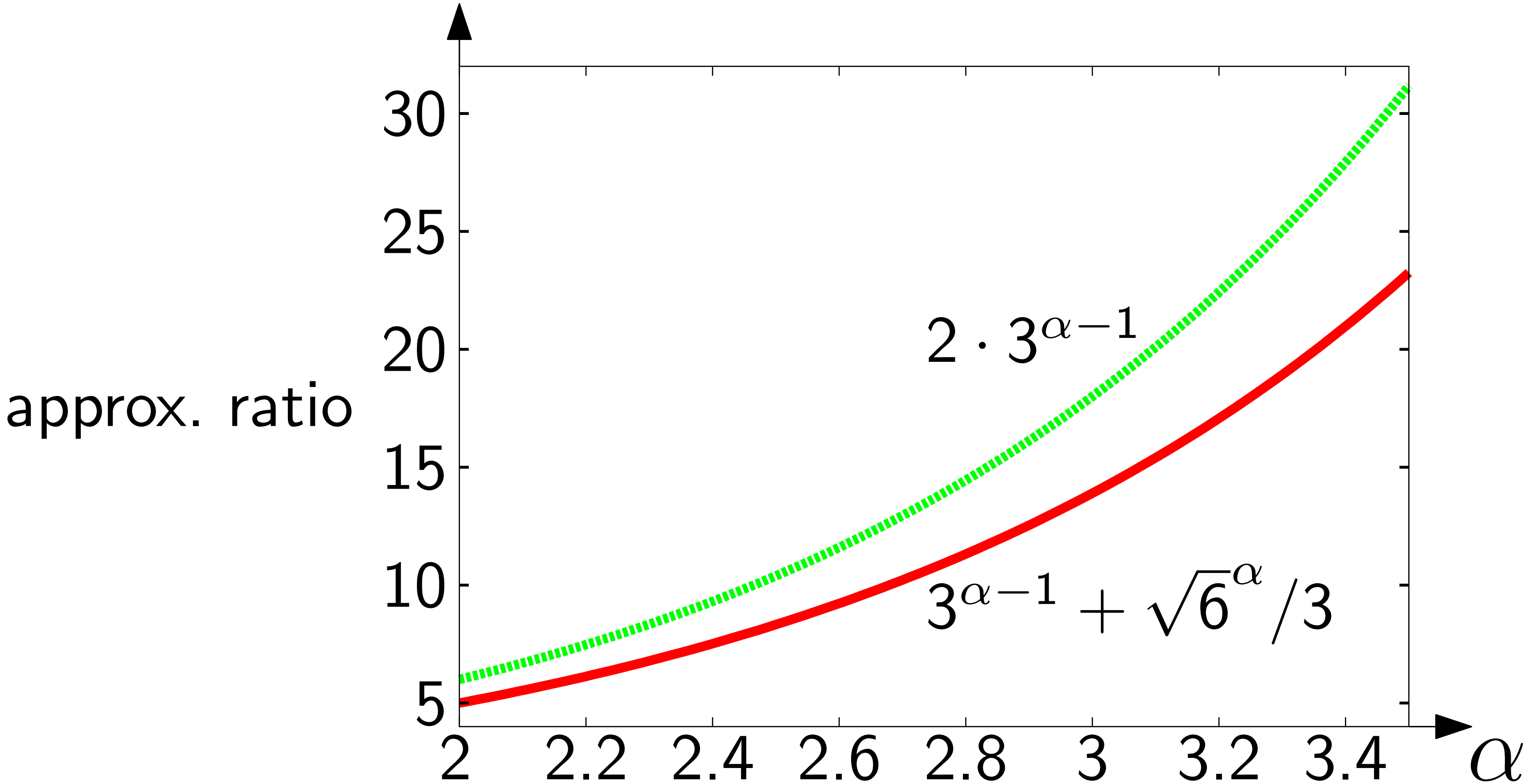

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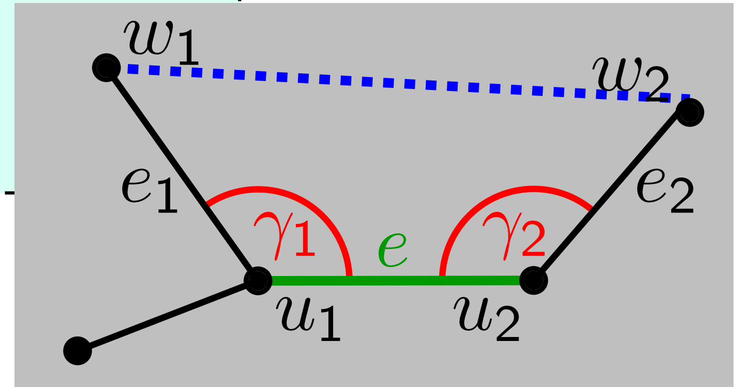

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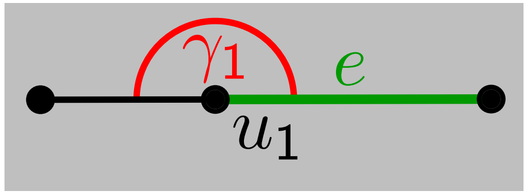

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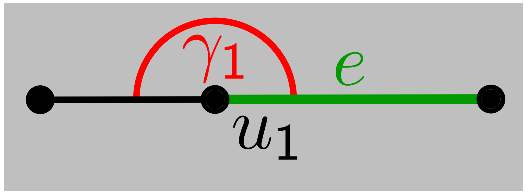

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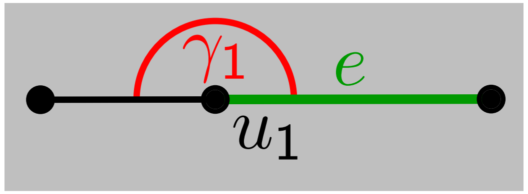

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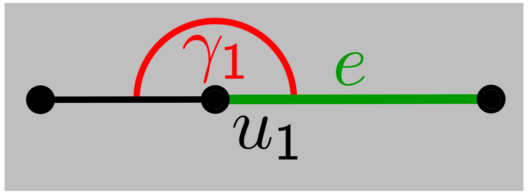

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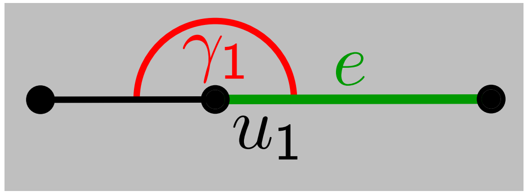

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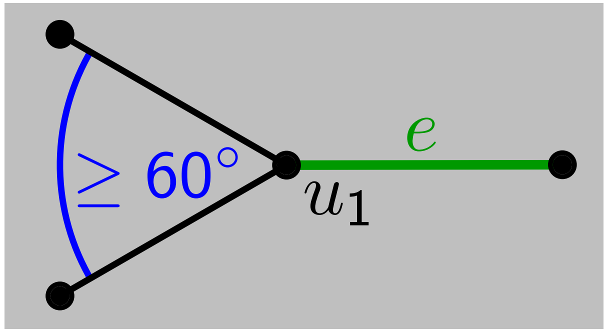

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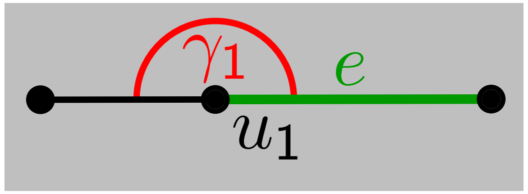

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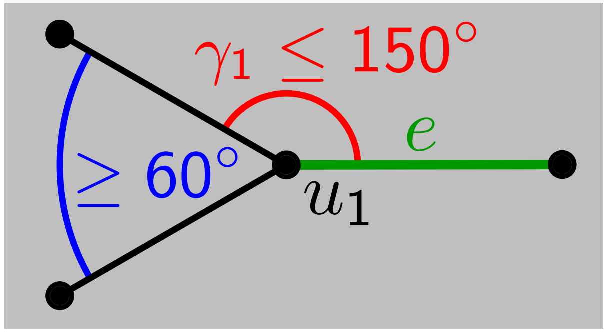

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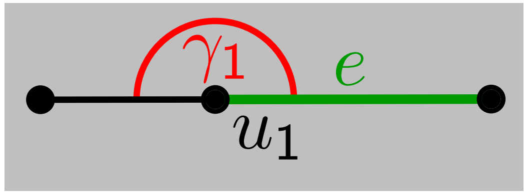

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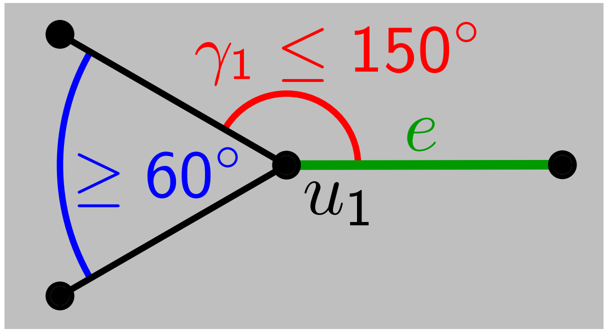

Thus, there is an edge e_1 incident to u_1 with $\angle ee_1 \leq 150°$.

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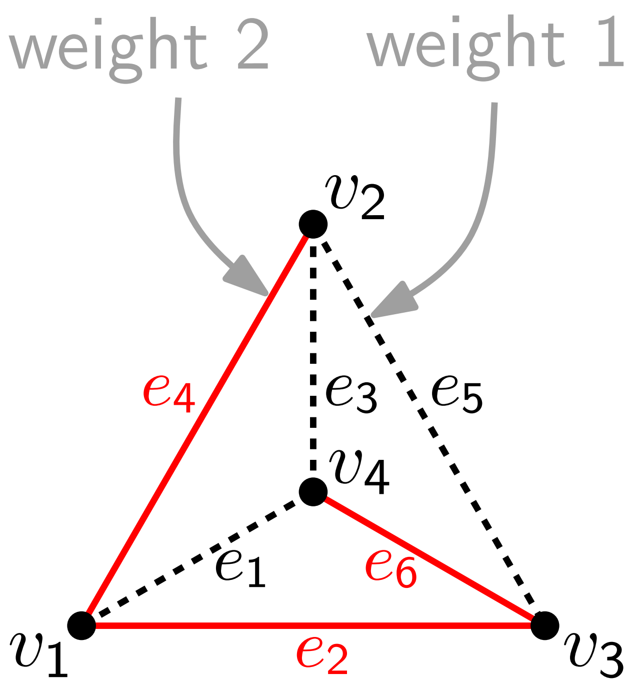

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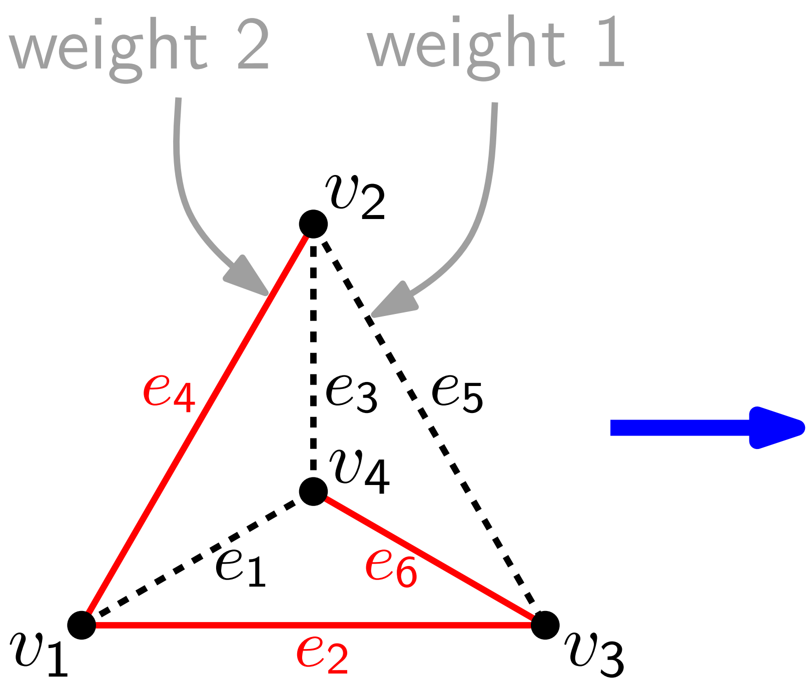

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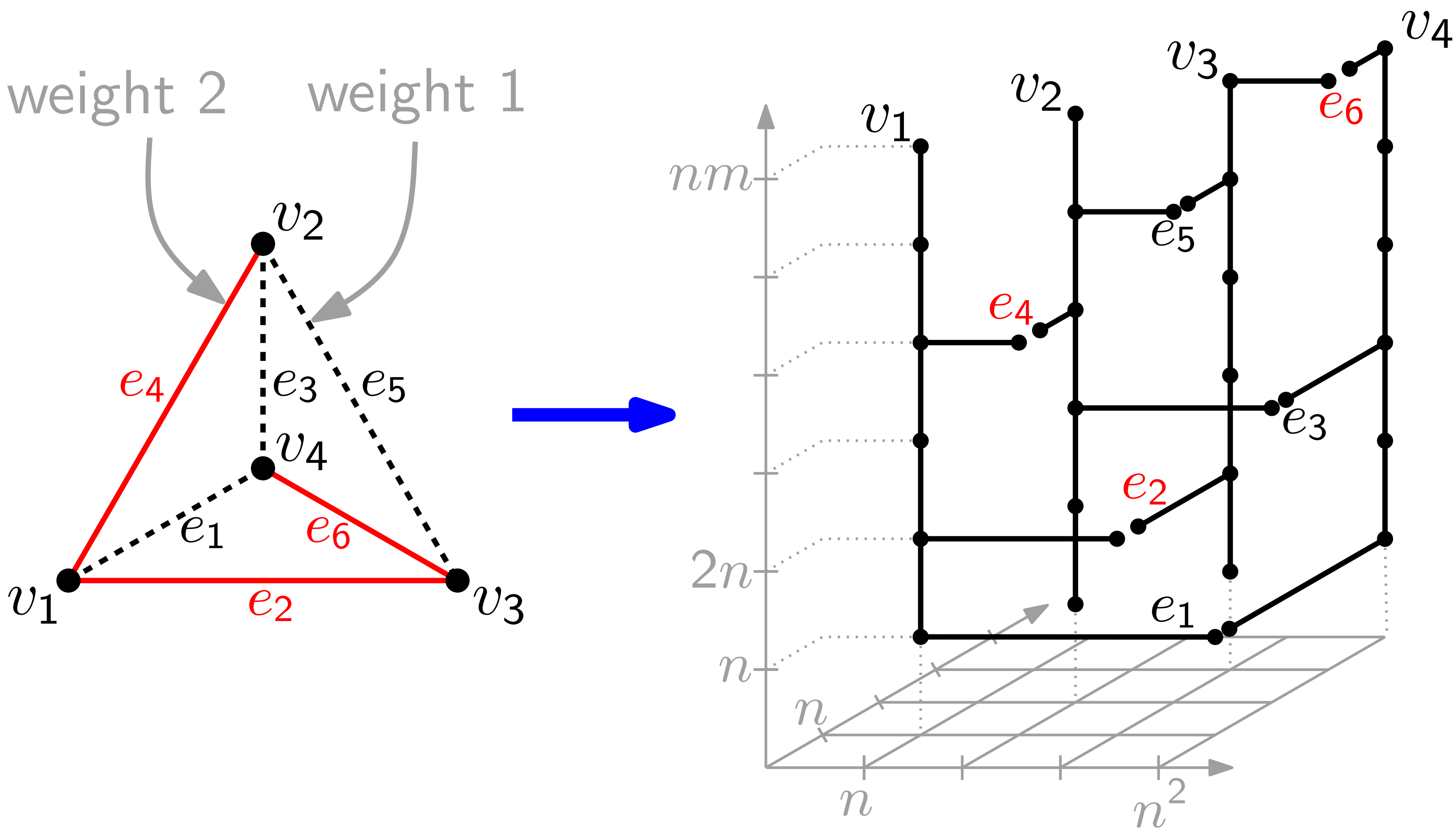

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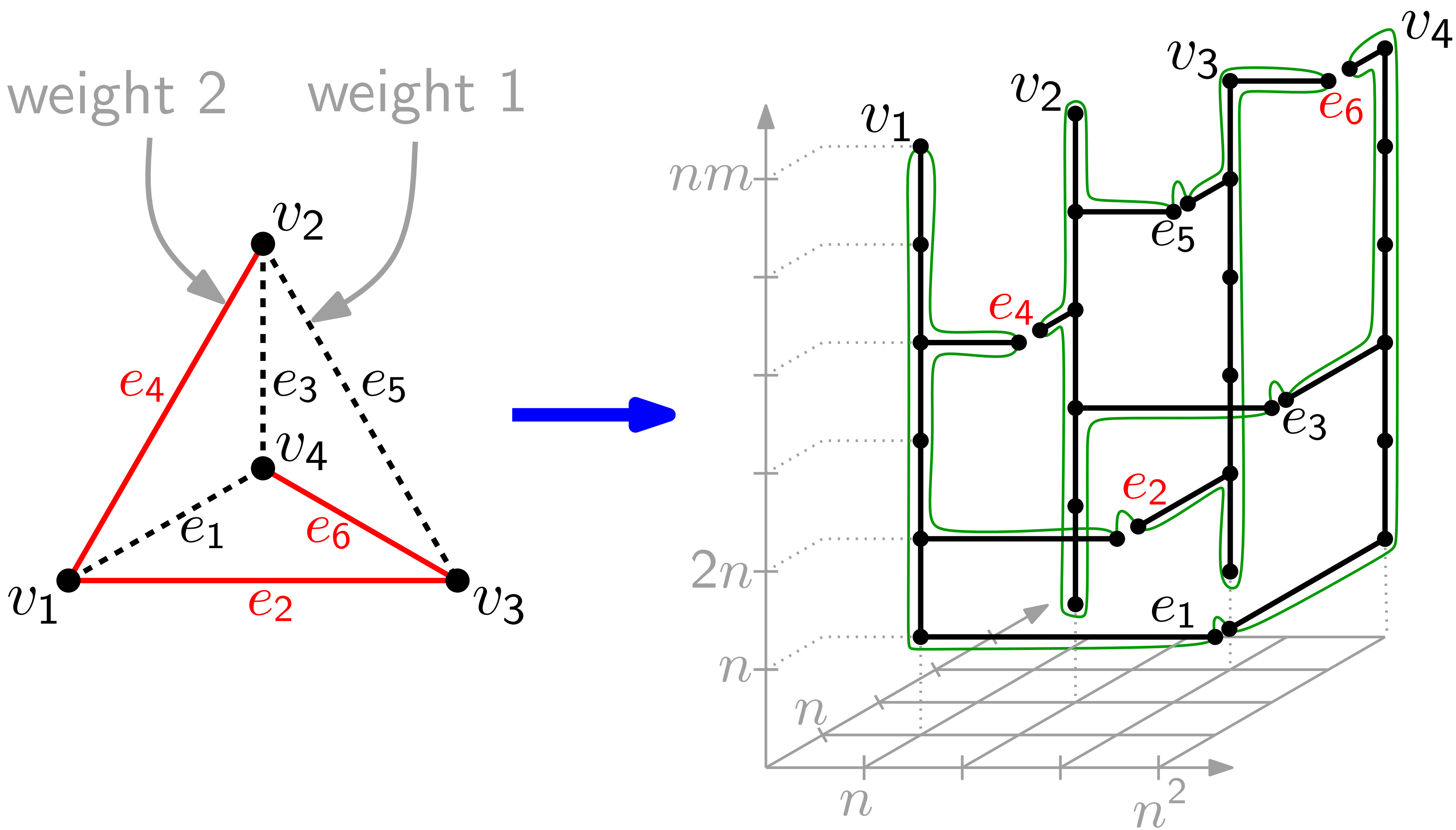

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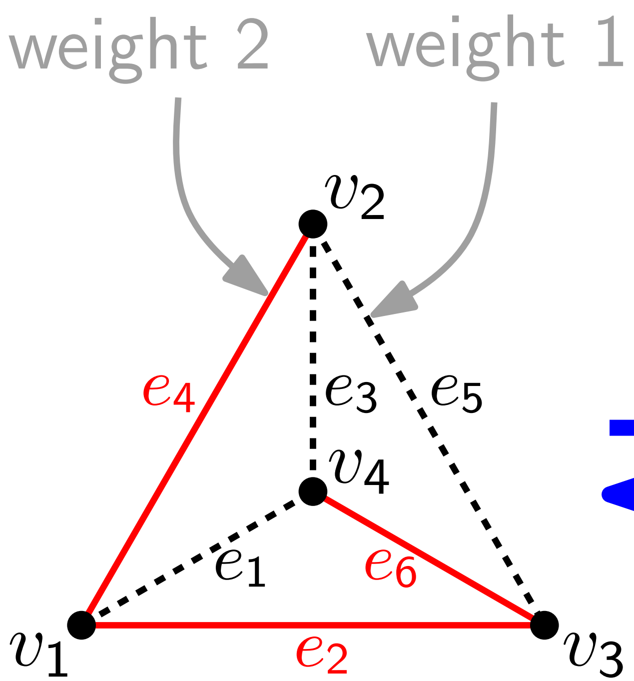

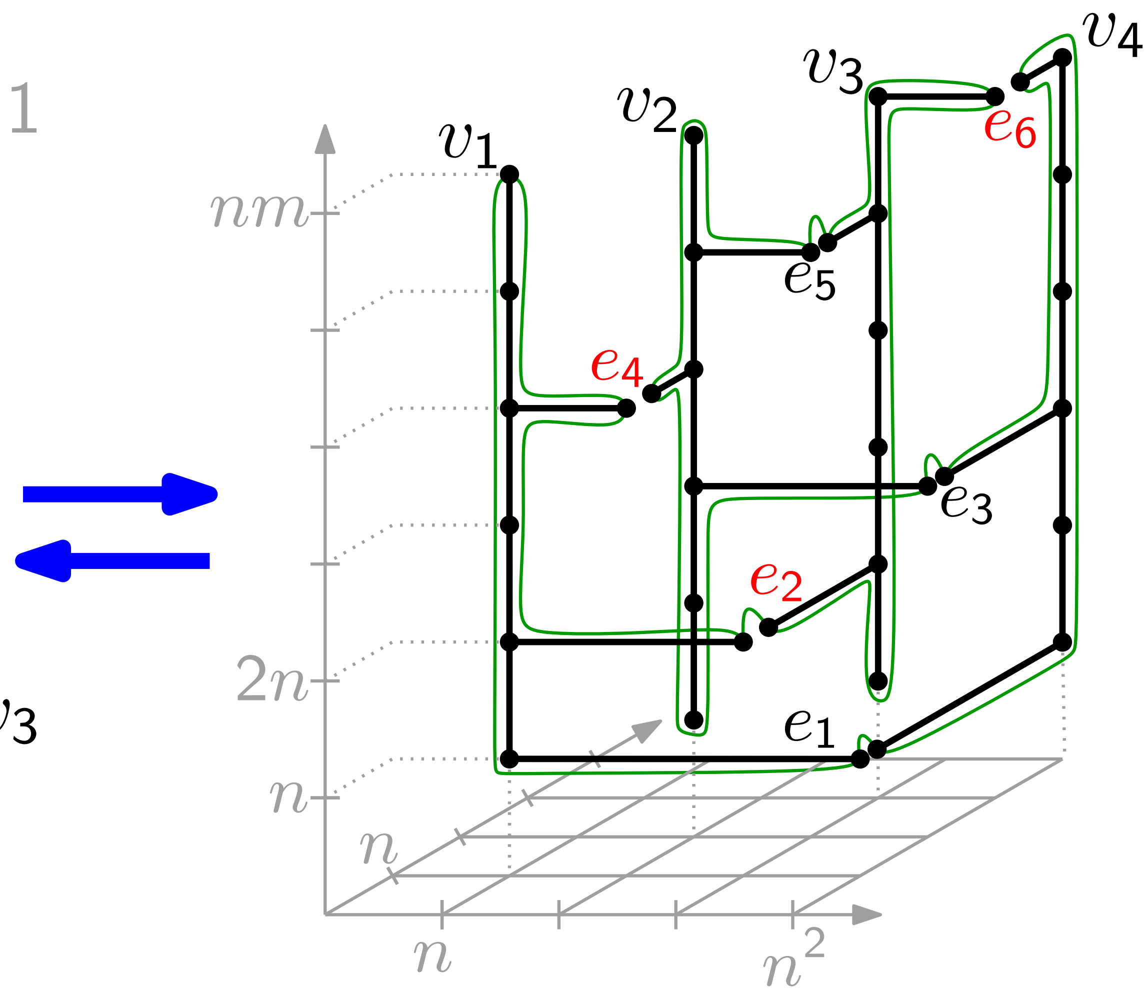

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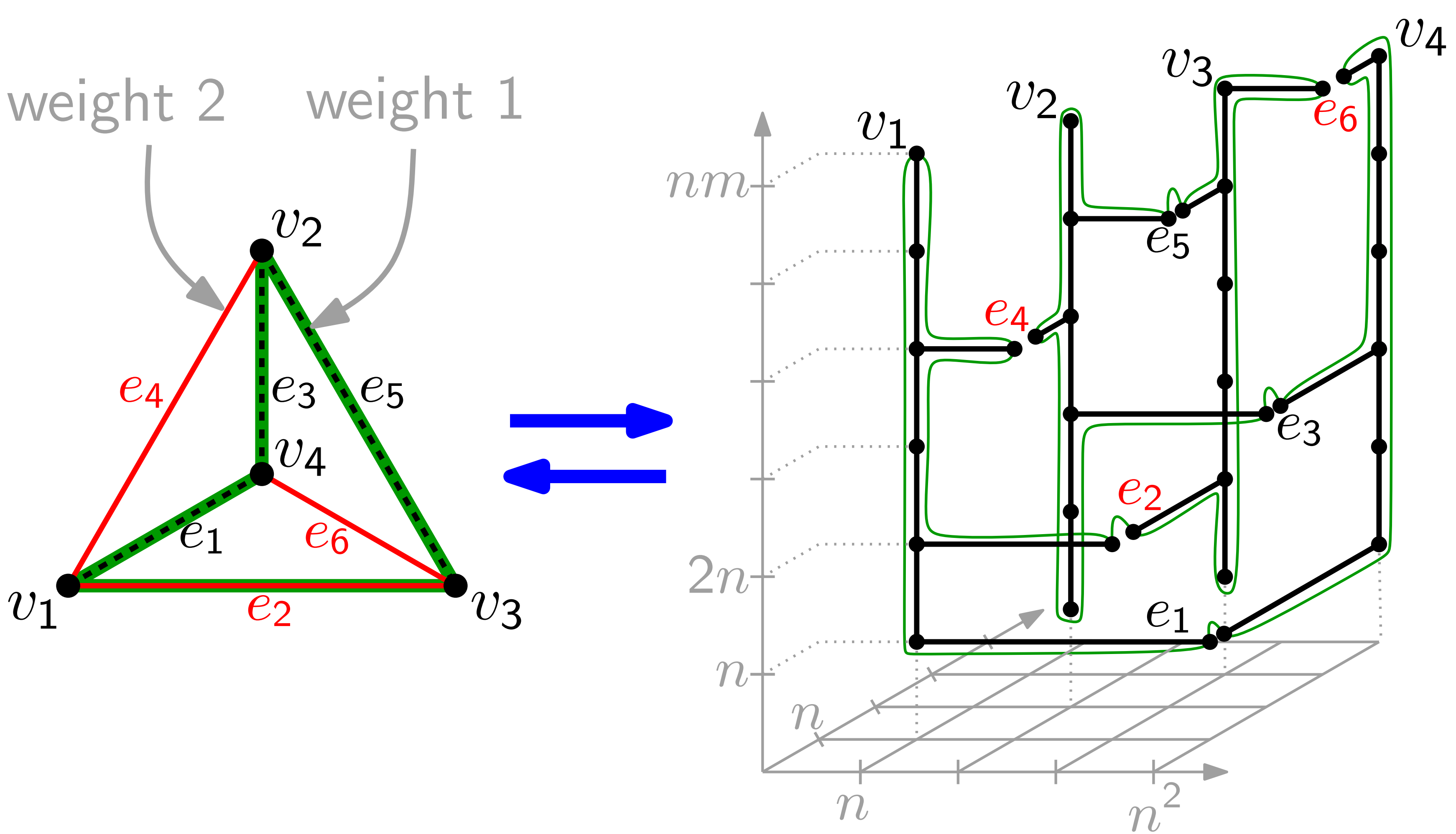

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  - Rev-TSP(d, α) is APX-hard for any $d \geq 3$ and $\alpha > 1$.
  - Rev-TSP($2, \alpha$) has a PTAS for any $\alpha \geq 2$.
  - Rev-TSP($\cdot\,, \alpha$) has a quasi-PTAS for any $0 < \alpha < 1$.
  - What about Rev-TSP($2, \alpha$) with $1 < \alpha < 2$?

    (At least as hard as TSP in weighted planar graphs!)