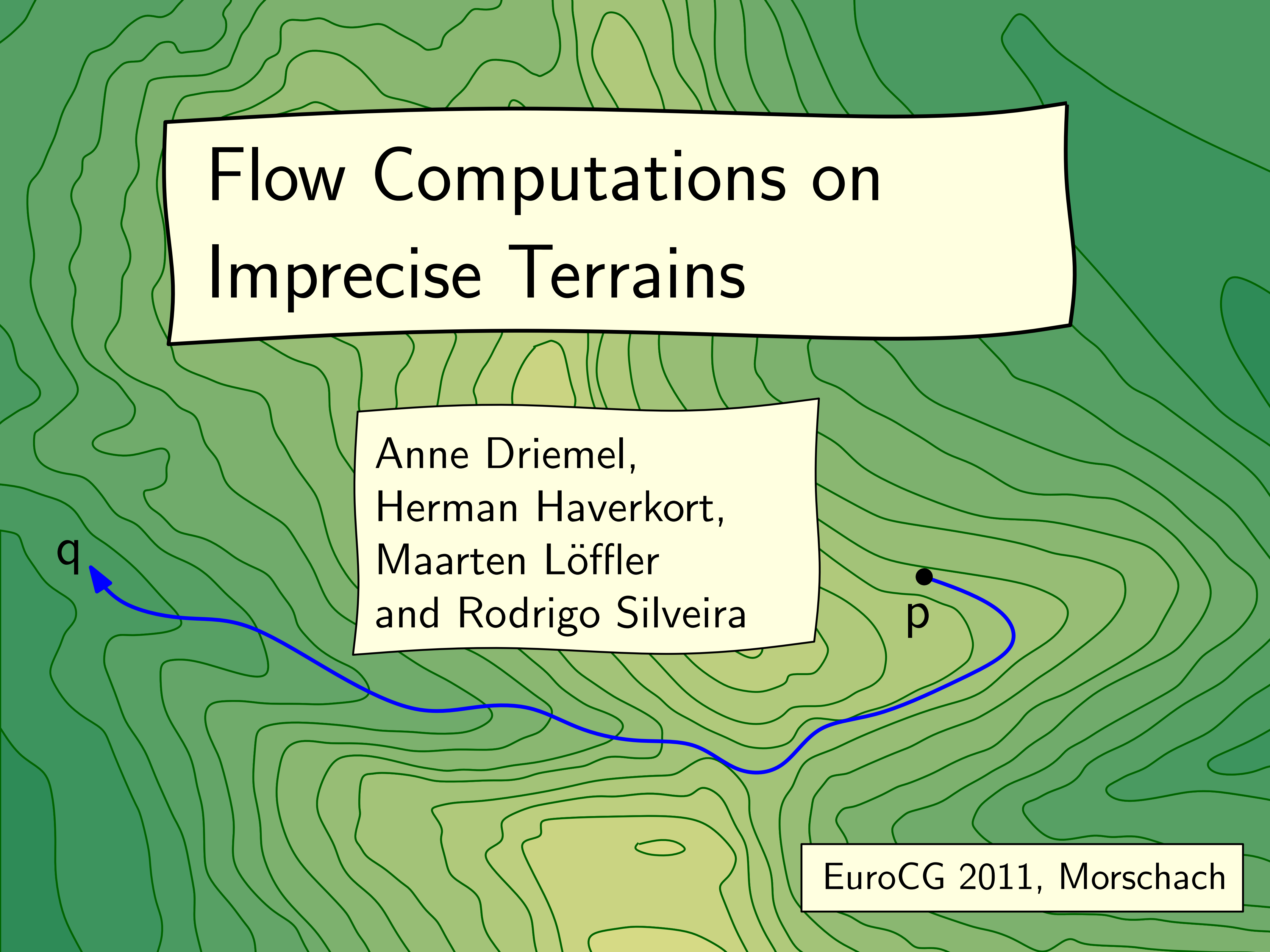
Flow Computations on Imprecise Terrains

Anne Driemel,
Herman Haverkort,
Maarten Löffler
and Rodrigo Silveira

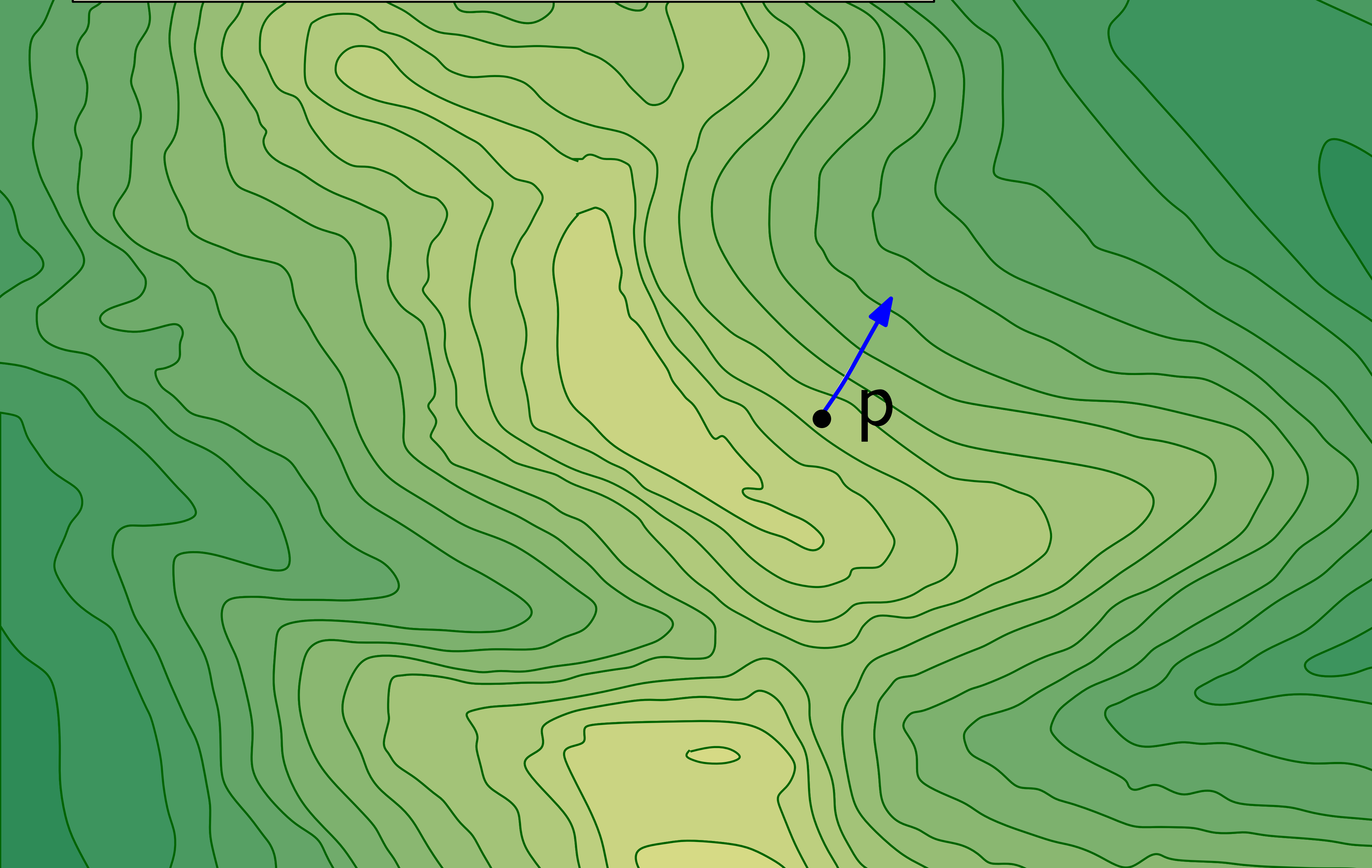


EuroCG 2011, Morschach

Why study water flow on terrains?

– Analysis of flash floods
– Streamflow forecasting
– Erosion prediction for river beds
..
– Interesting geometric problems

Assumption: Water flows downwards in the direction of steepest descent.



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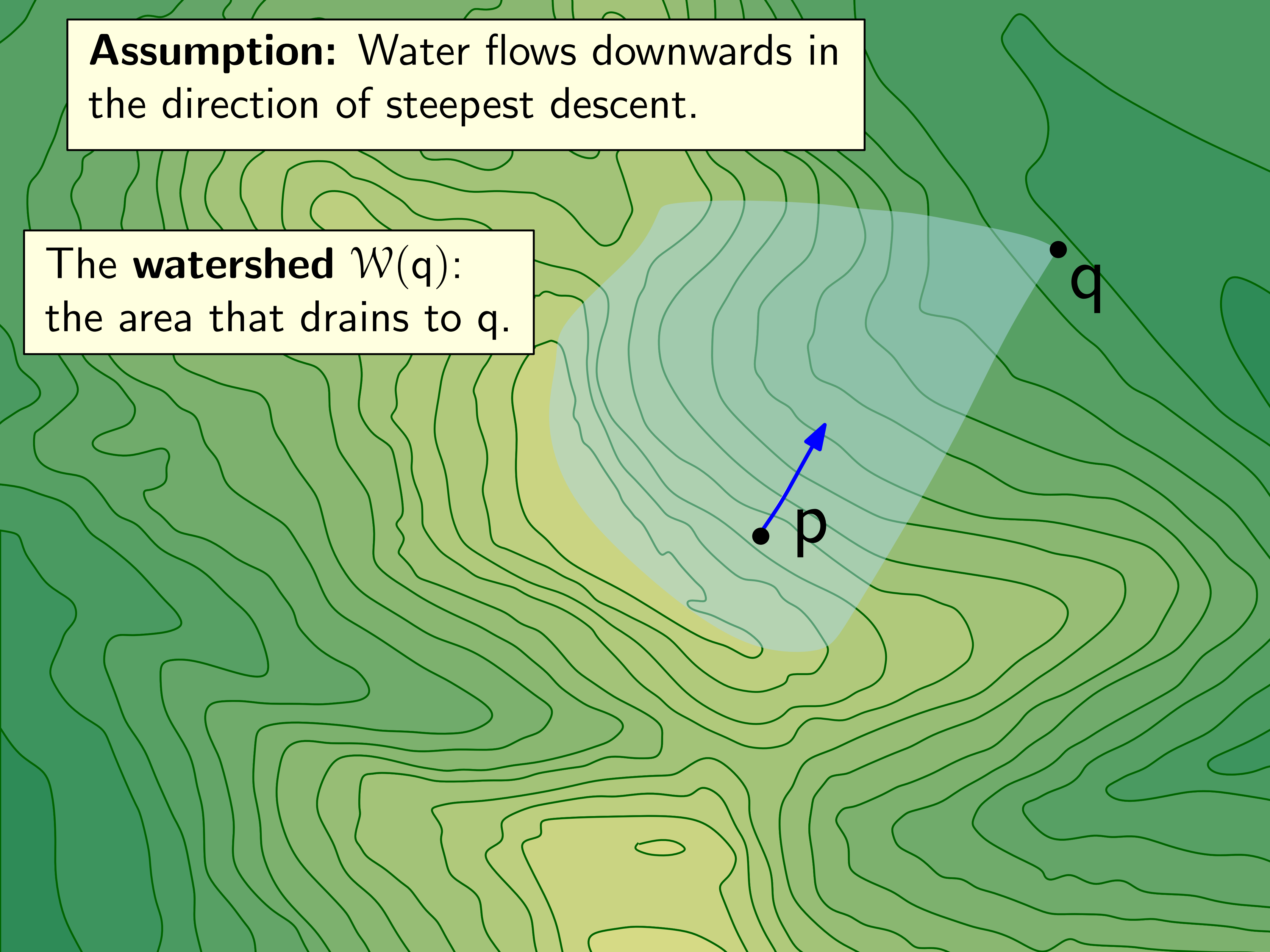
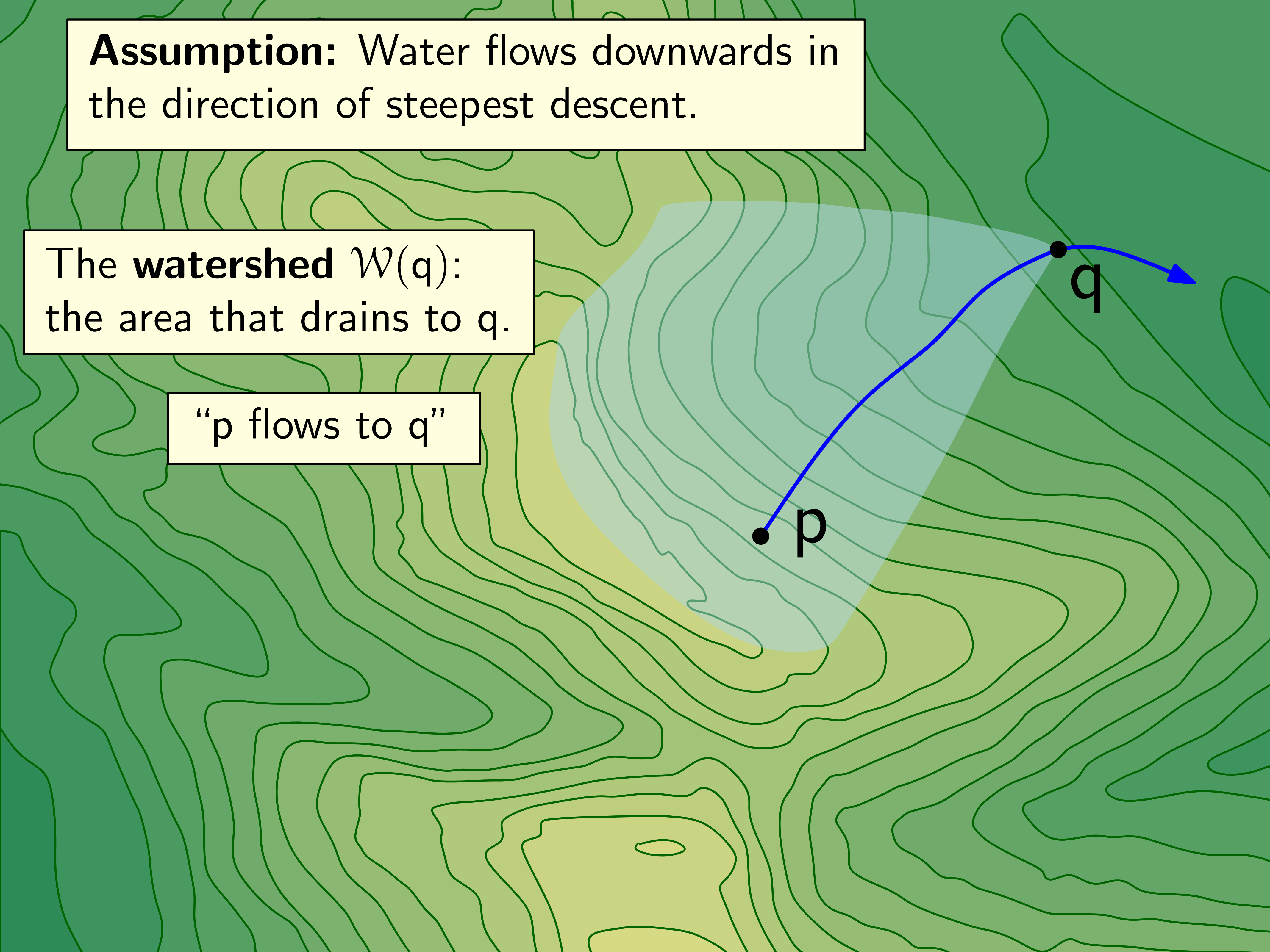
The **watershed** $\mathcal{W}(q)$: the area that drains to q.

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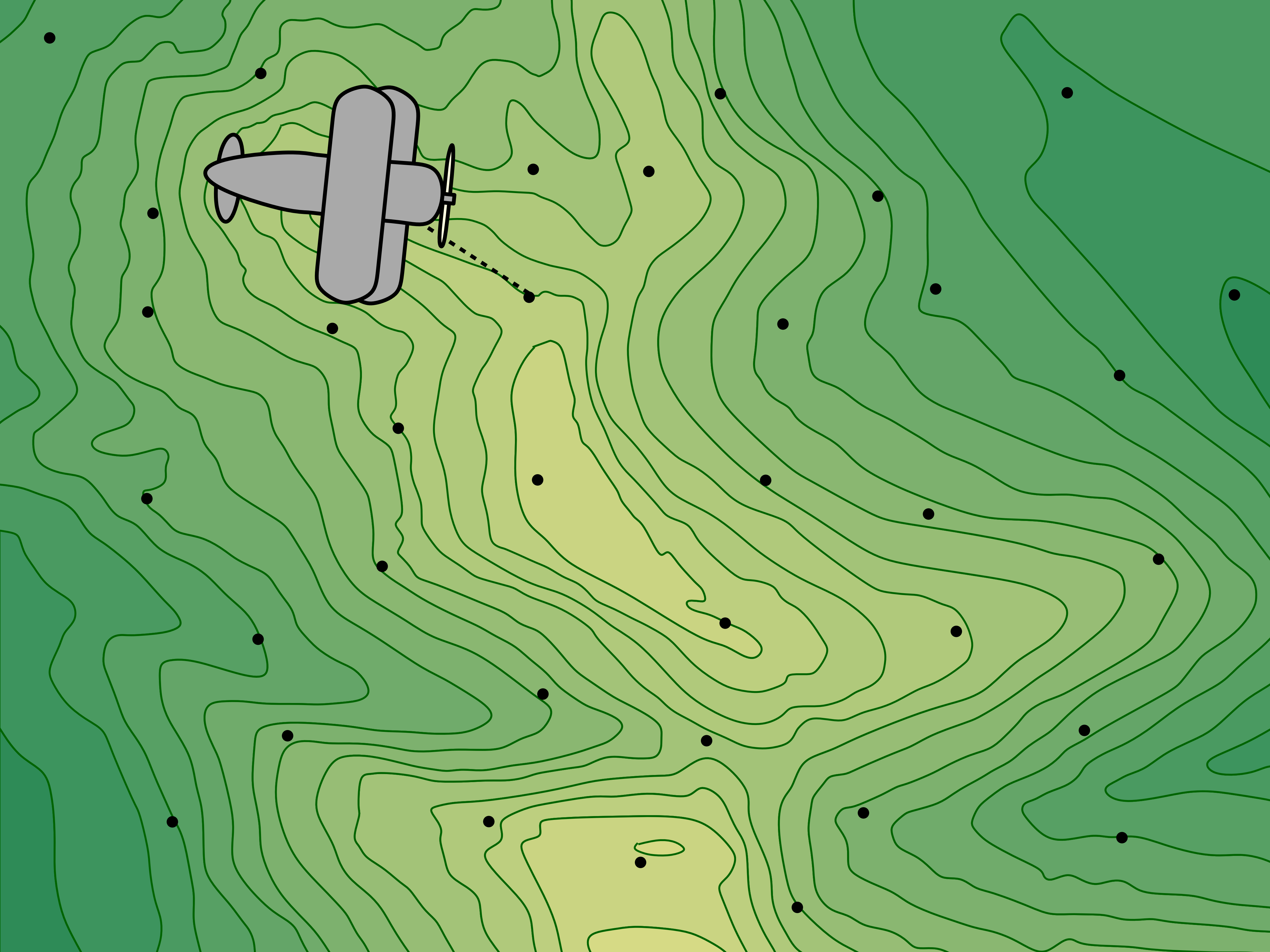
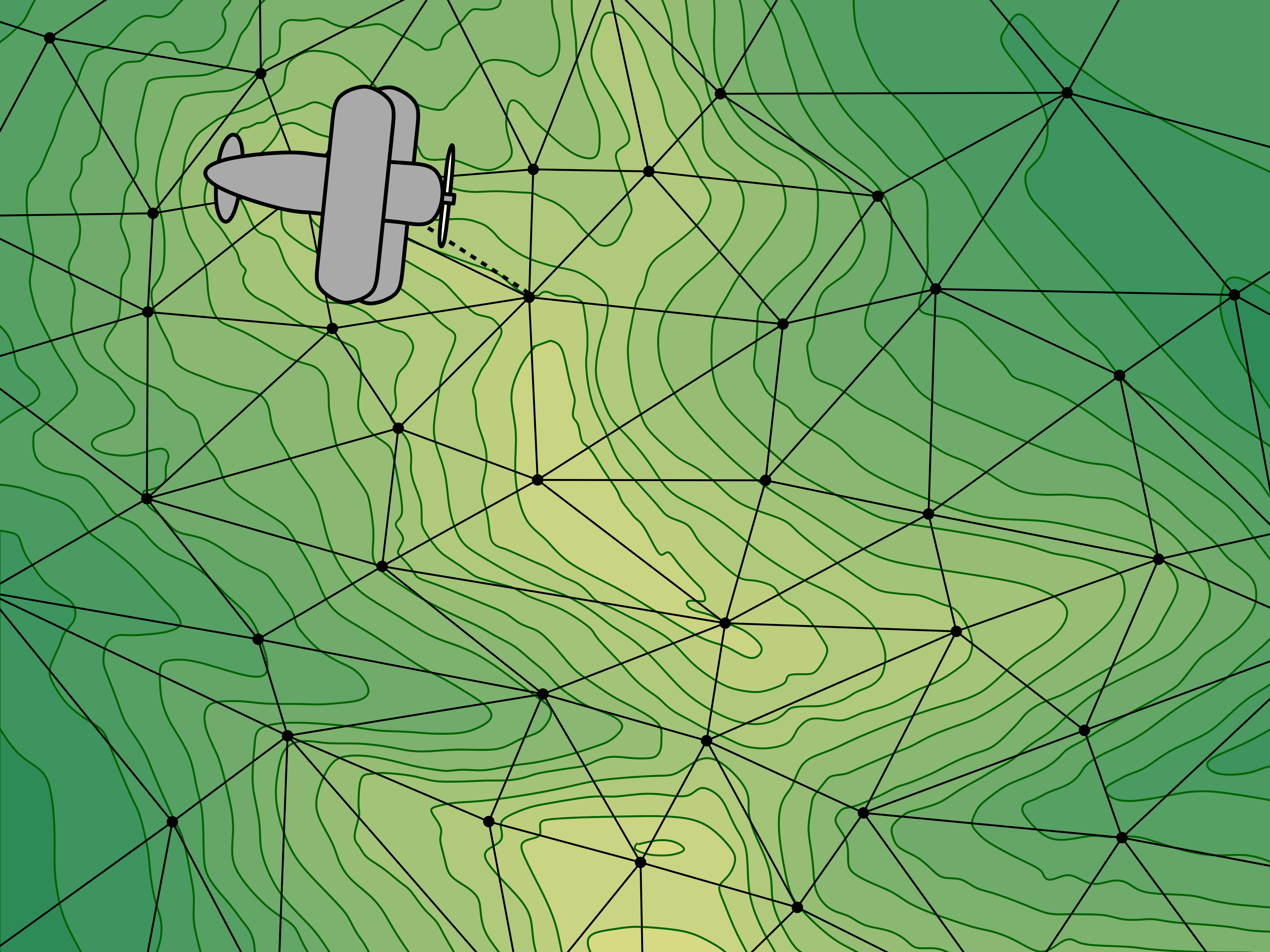
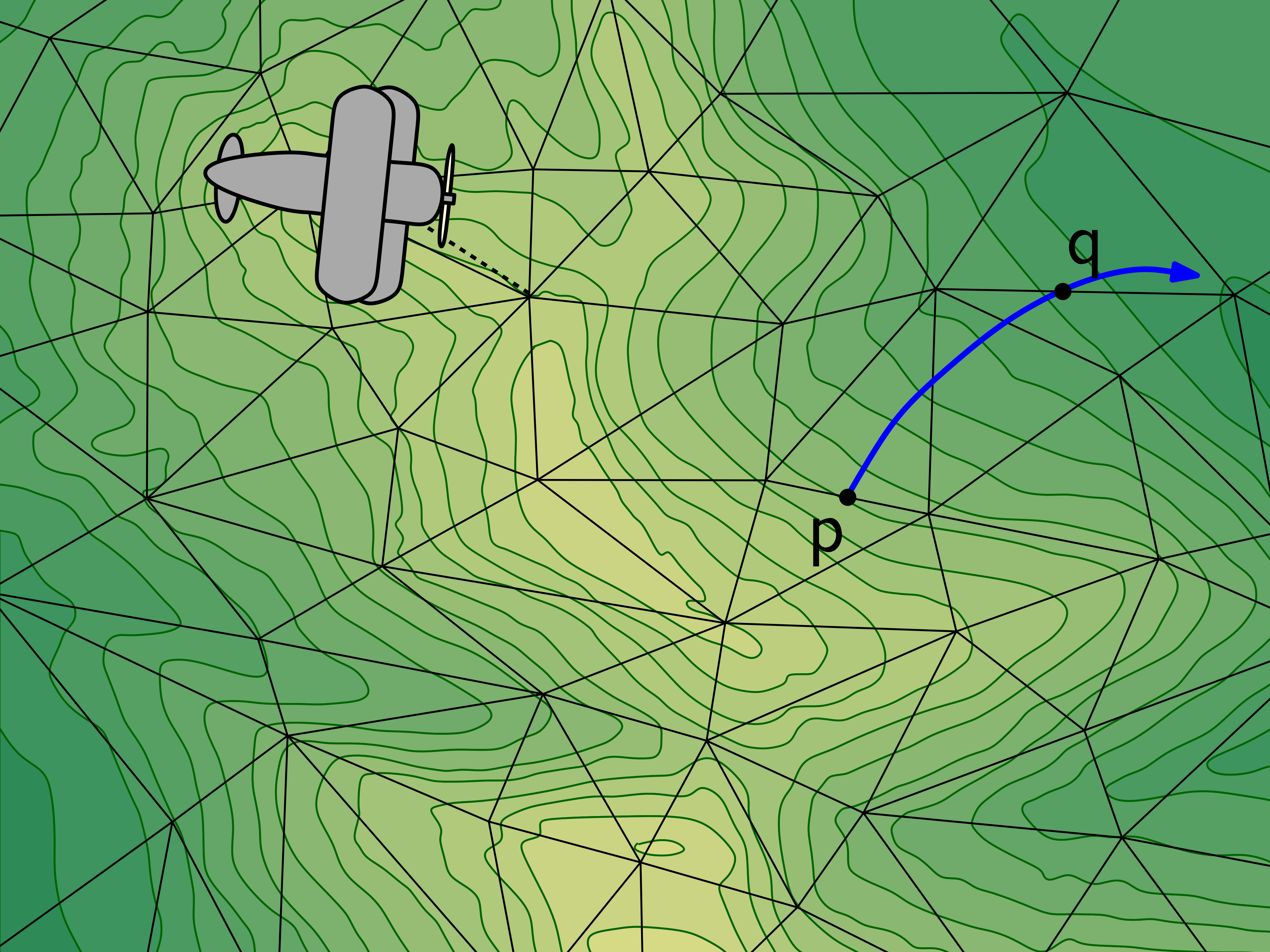
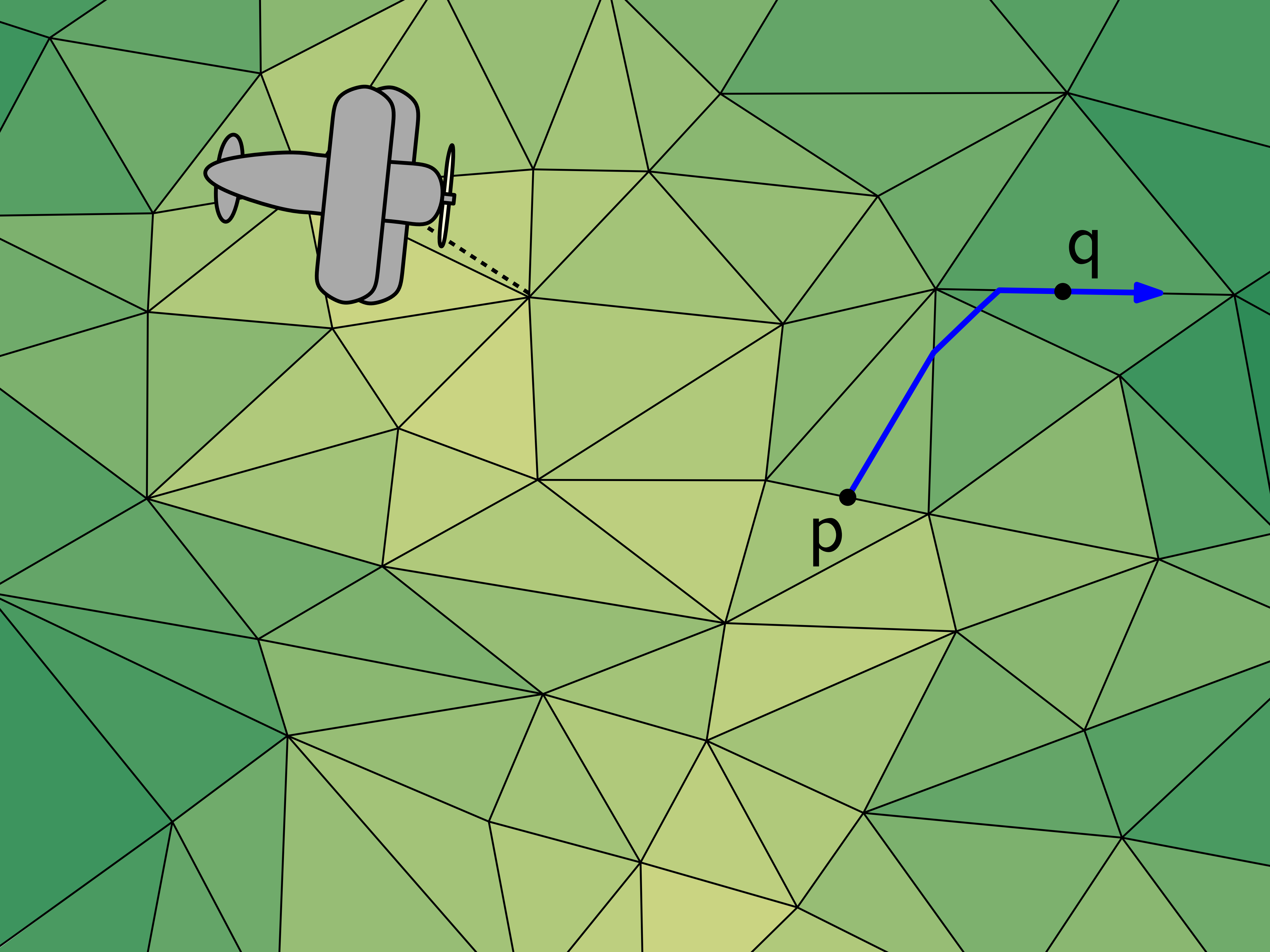
"p flows to q"
p
q






q
p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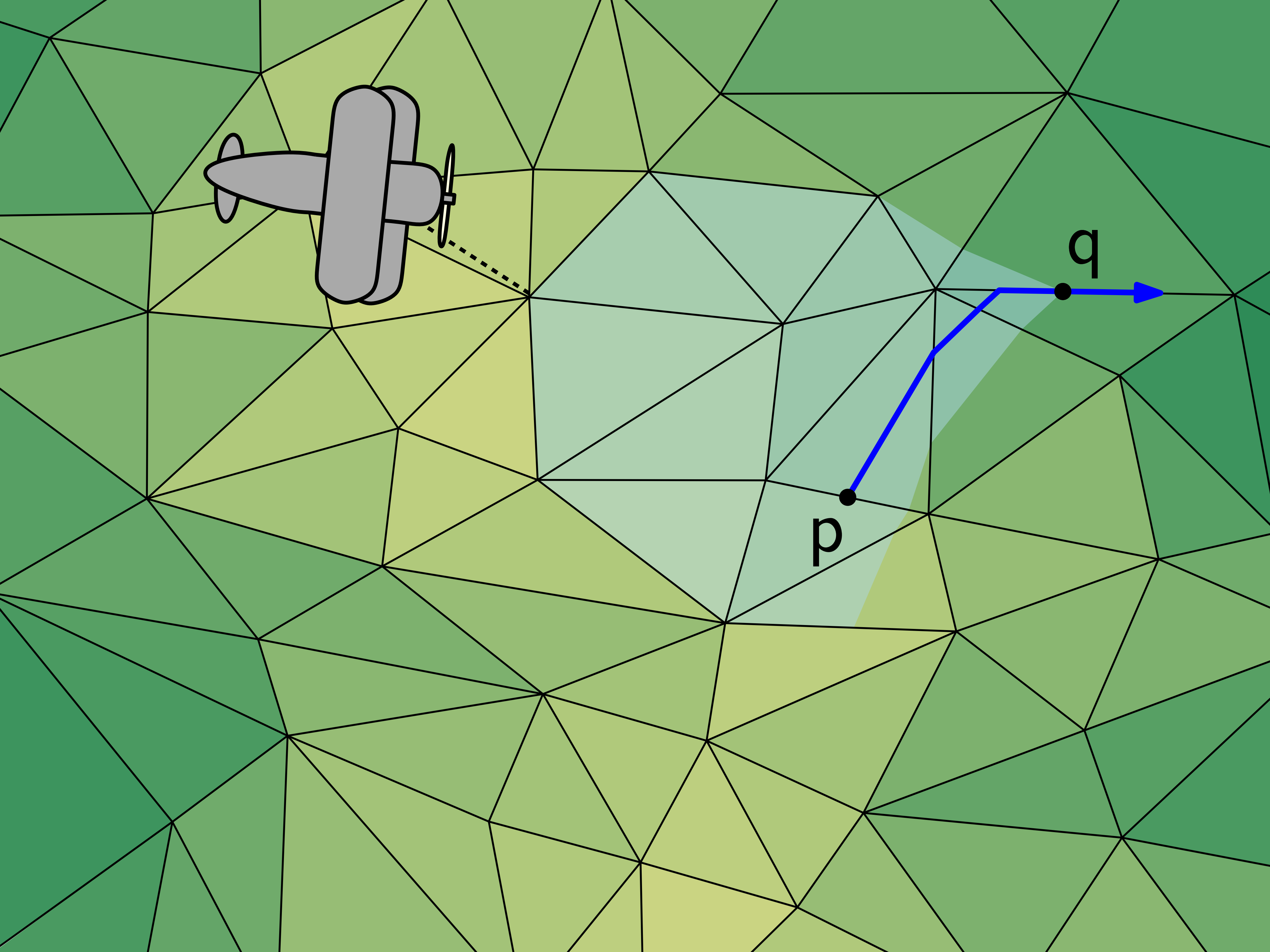
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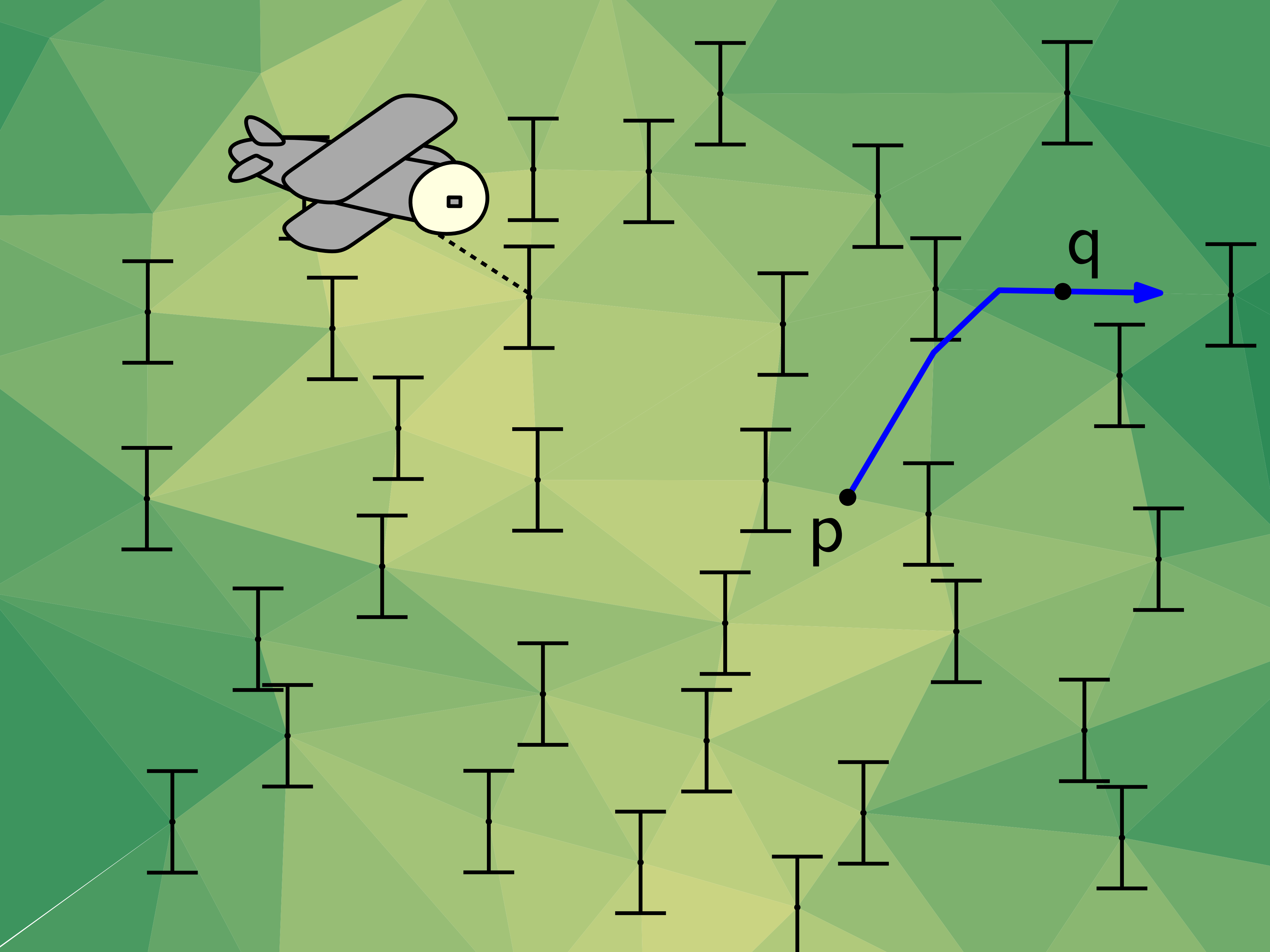
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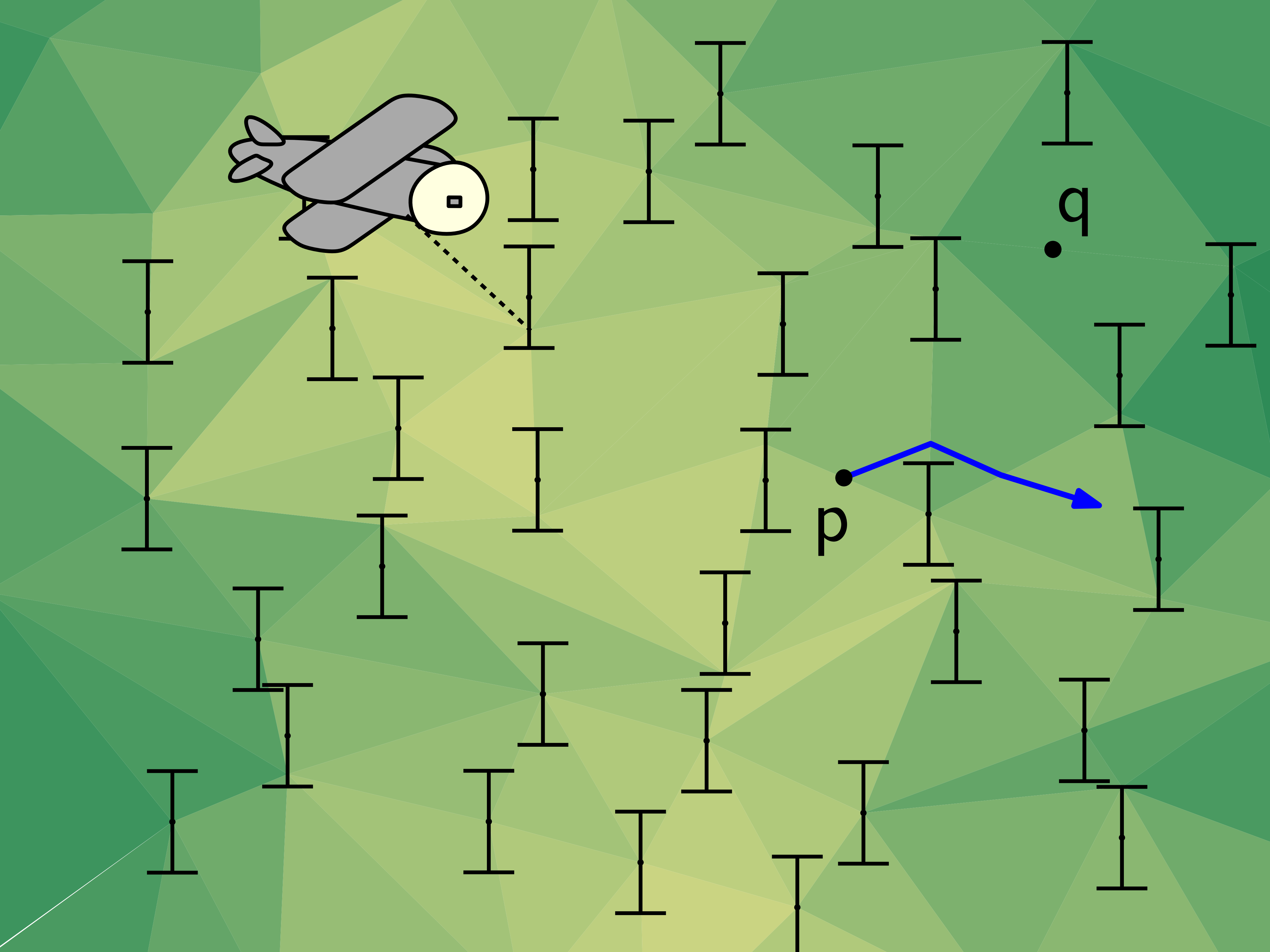
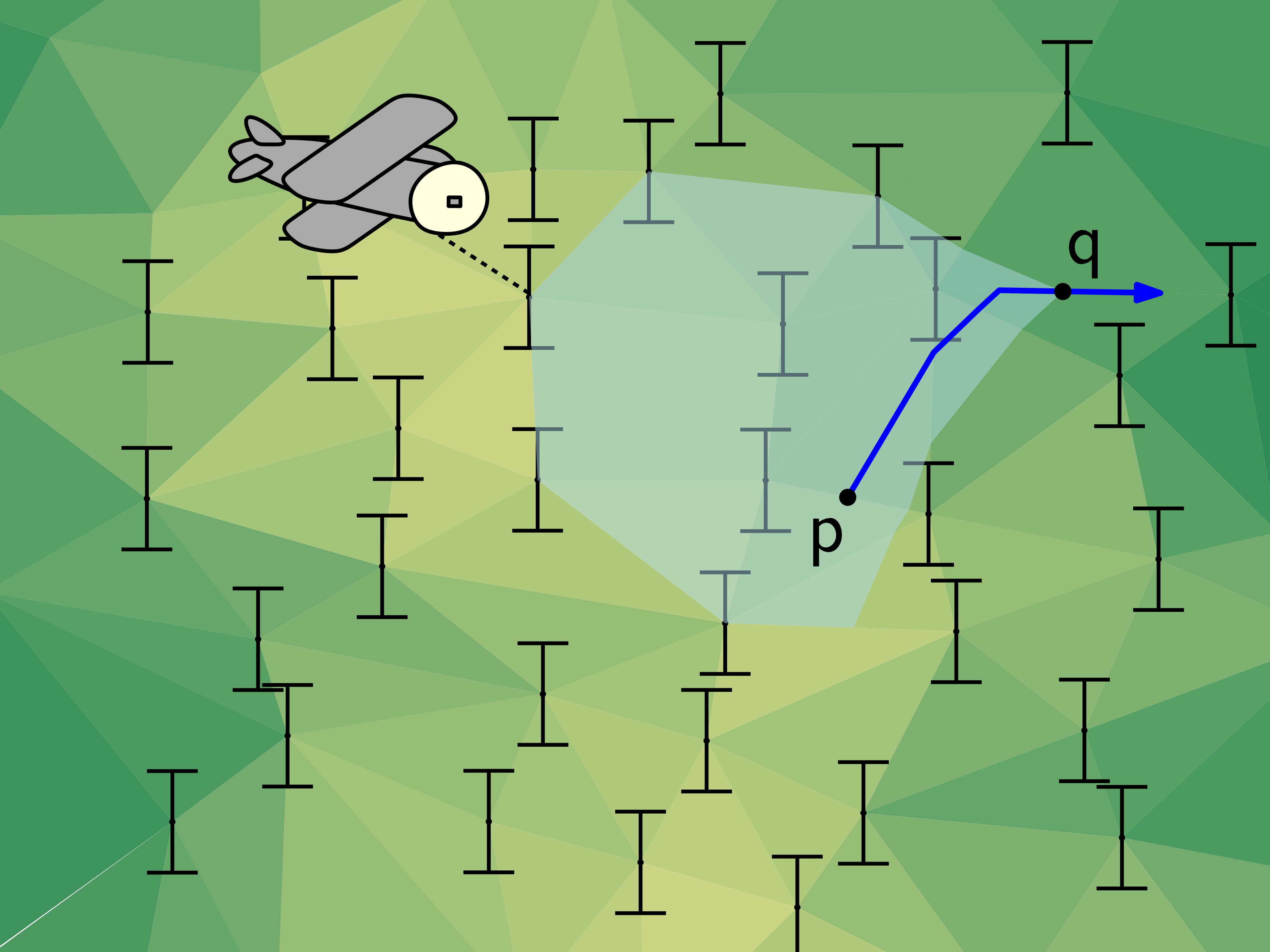

q
p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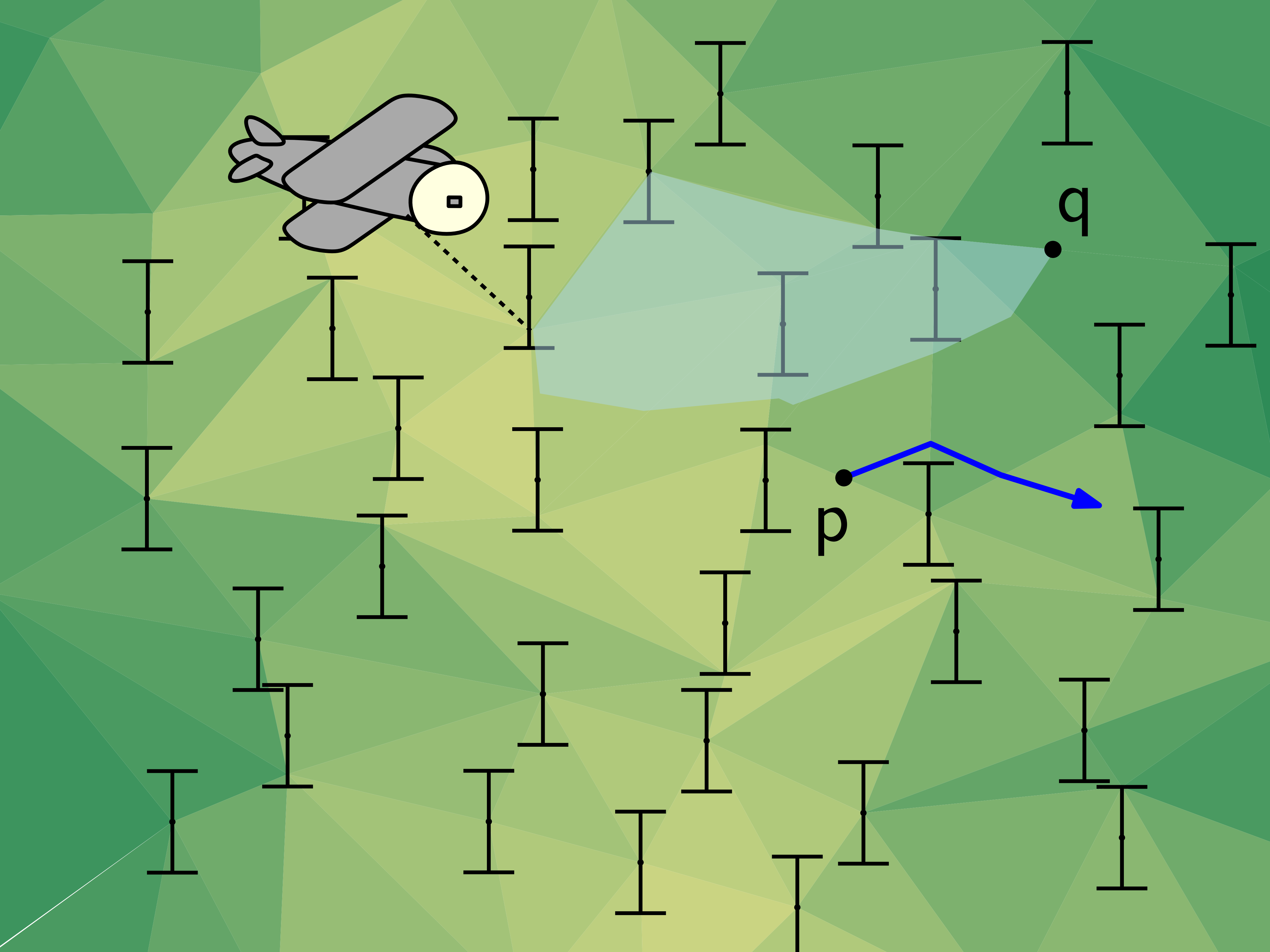
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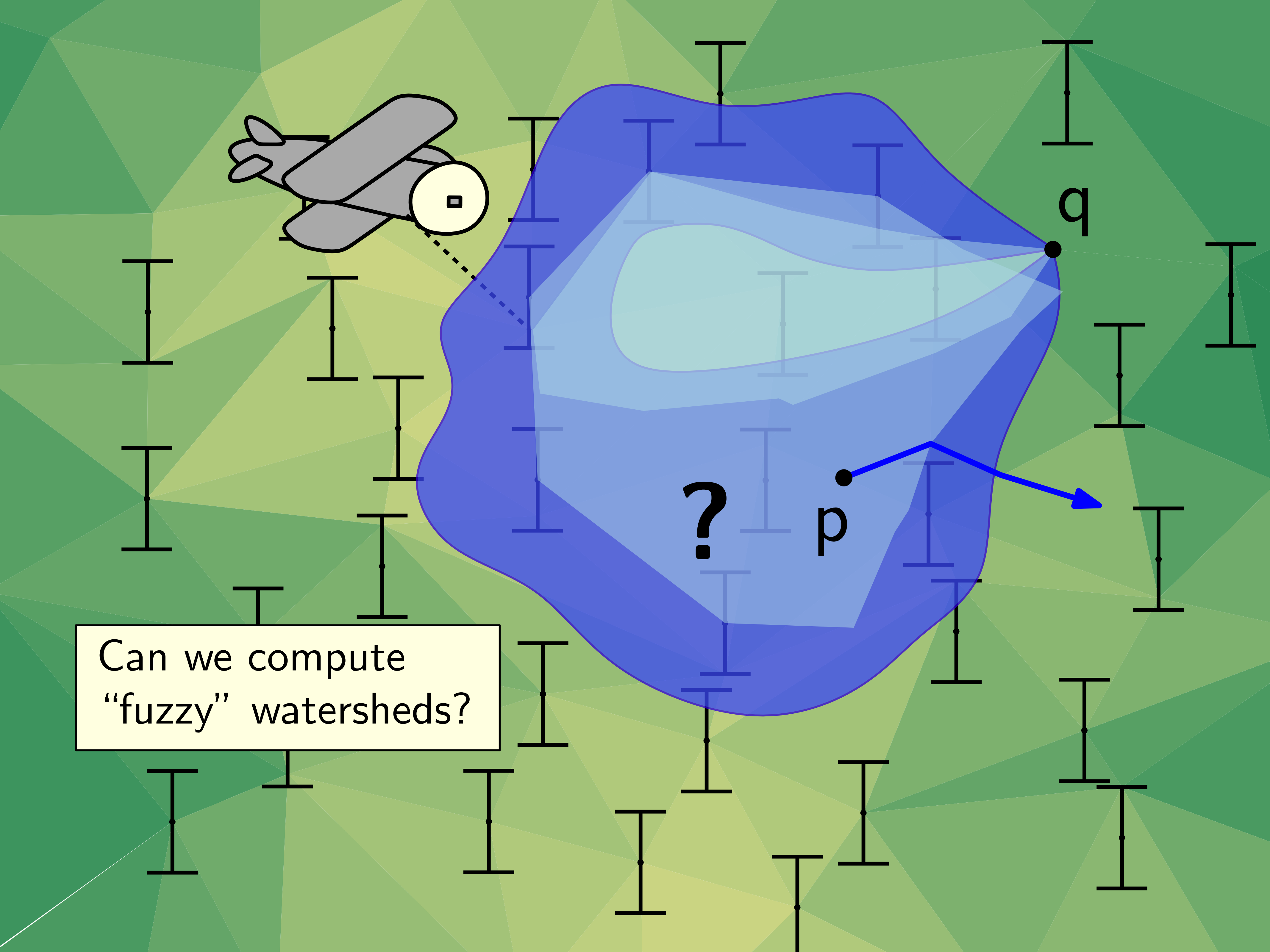
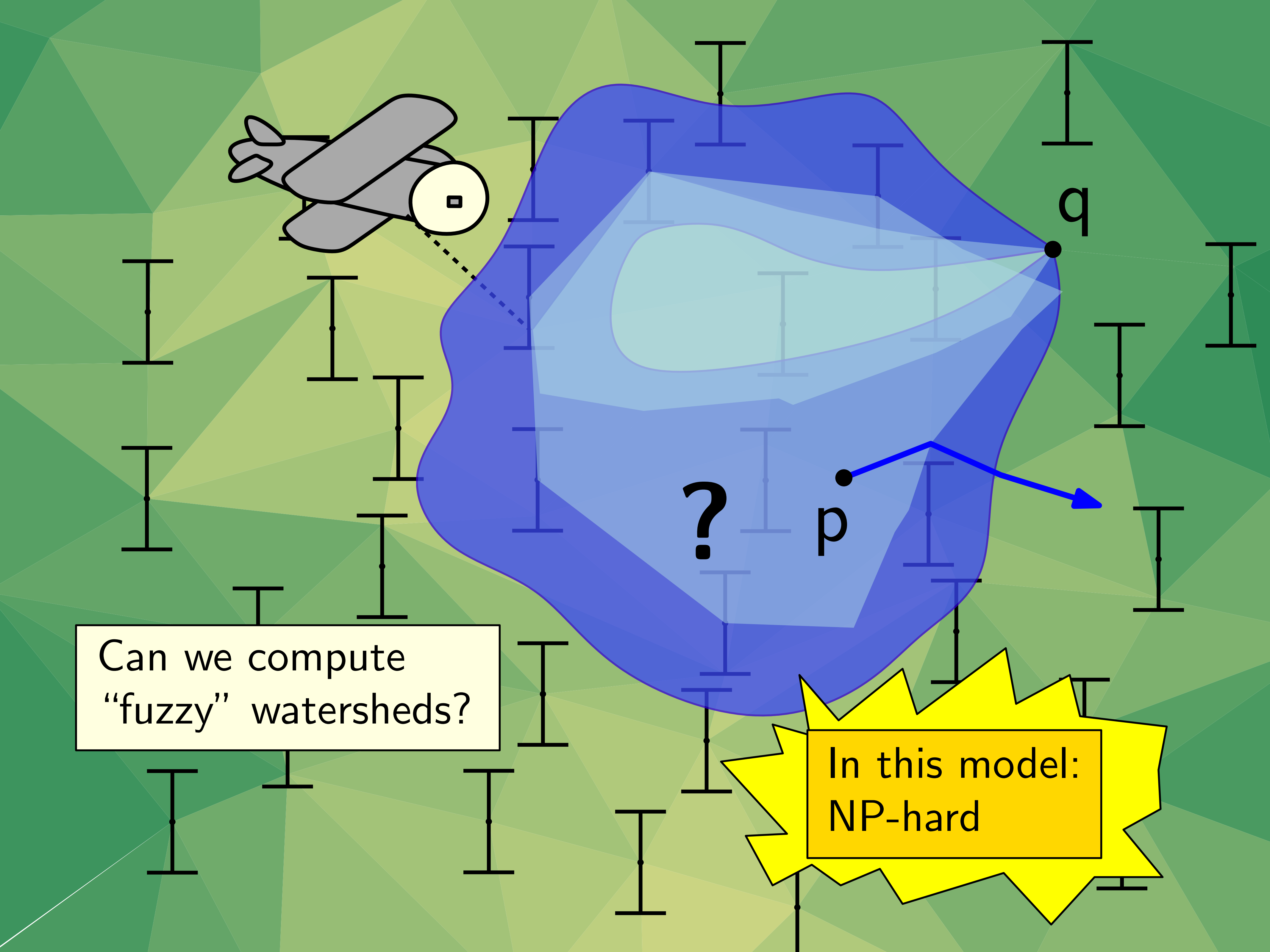

q
?
p
Can we compute
“fuzzy” watersheds?


q
?
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Can we compute
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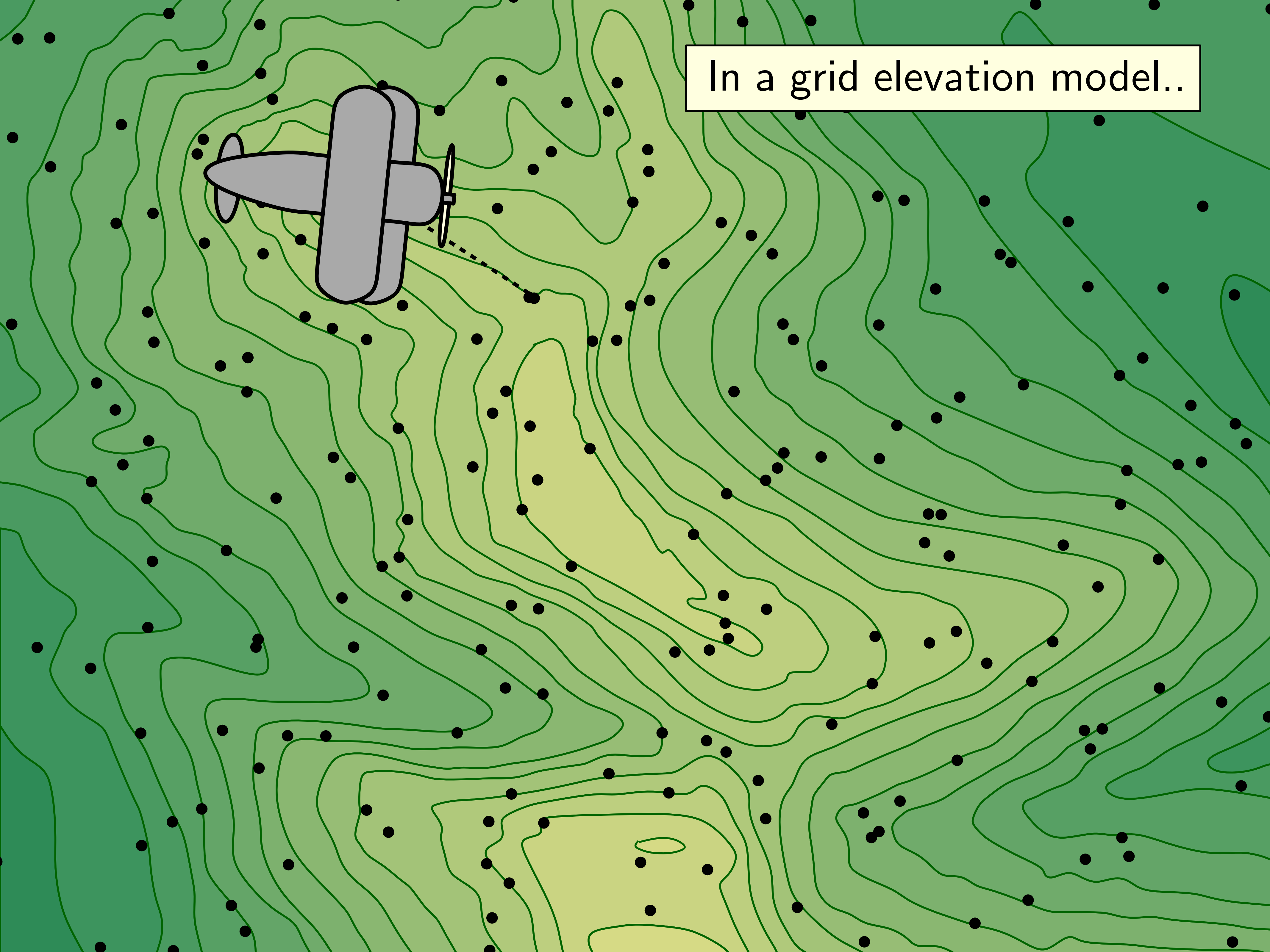
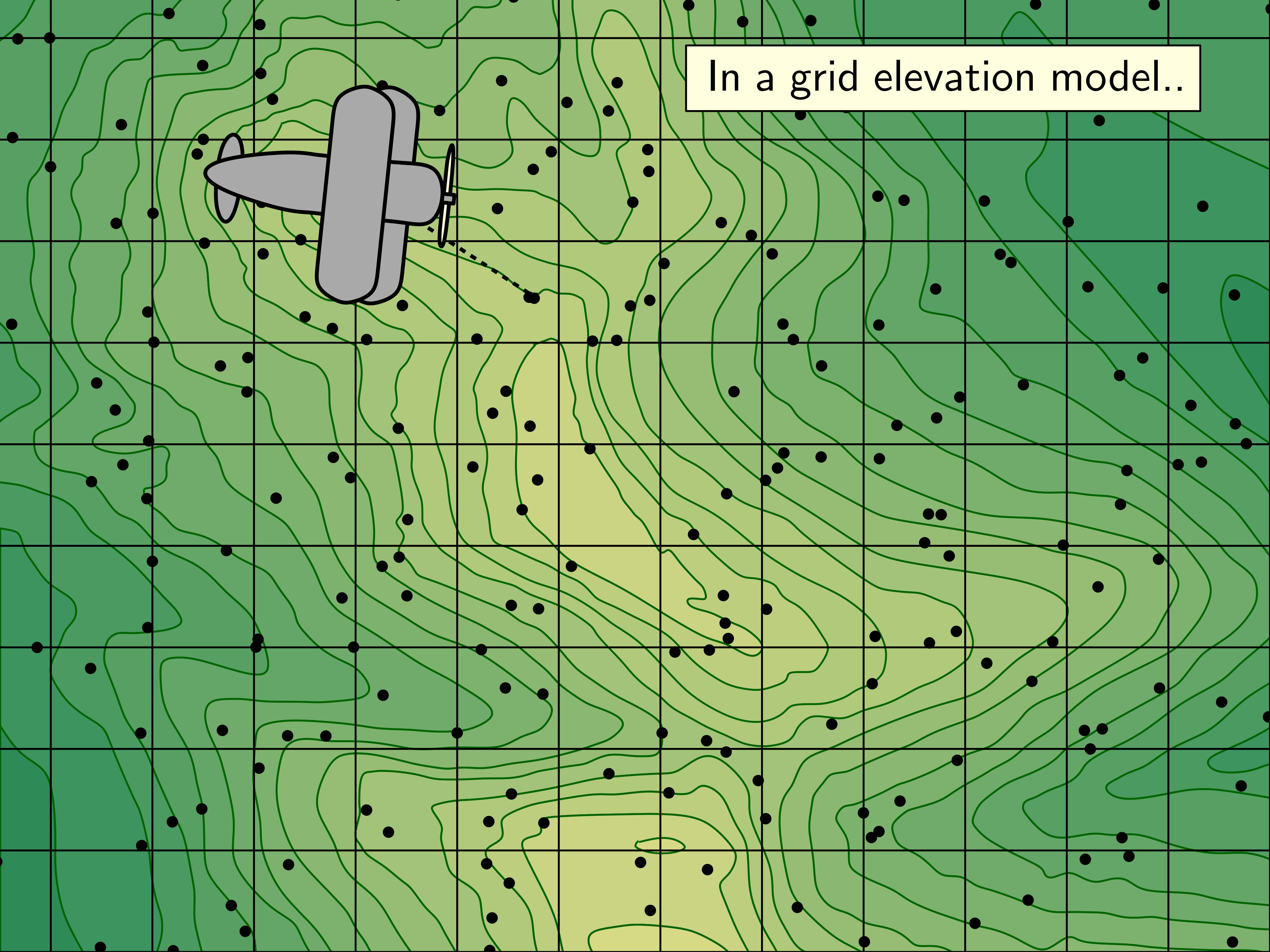
In this model:
NP-hard


In a grid elevation model..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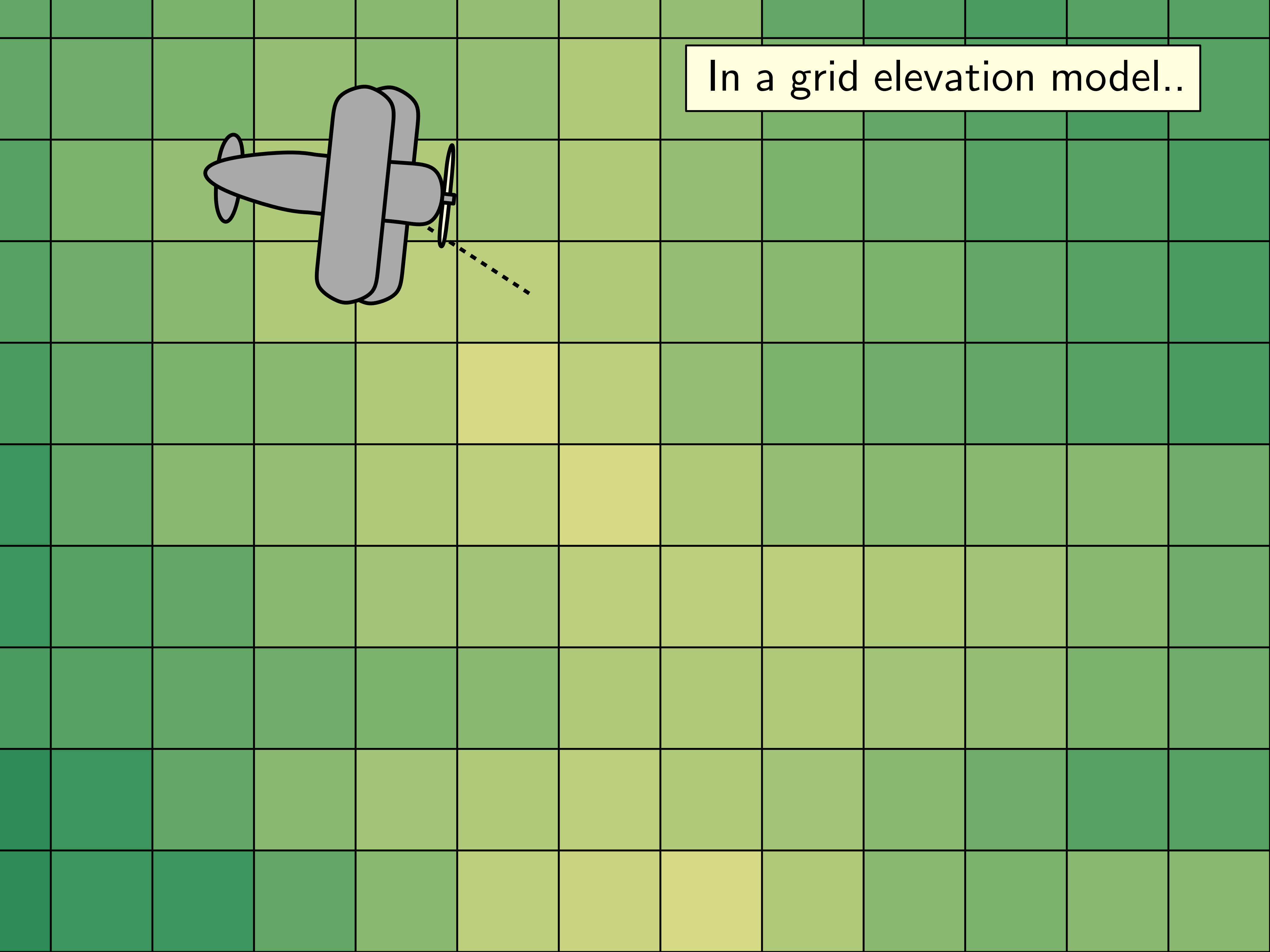
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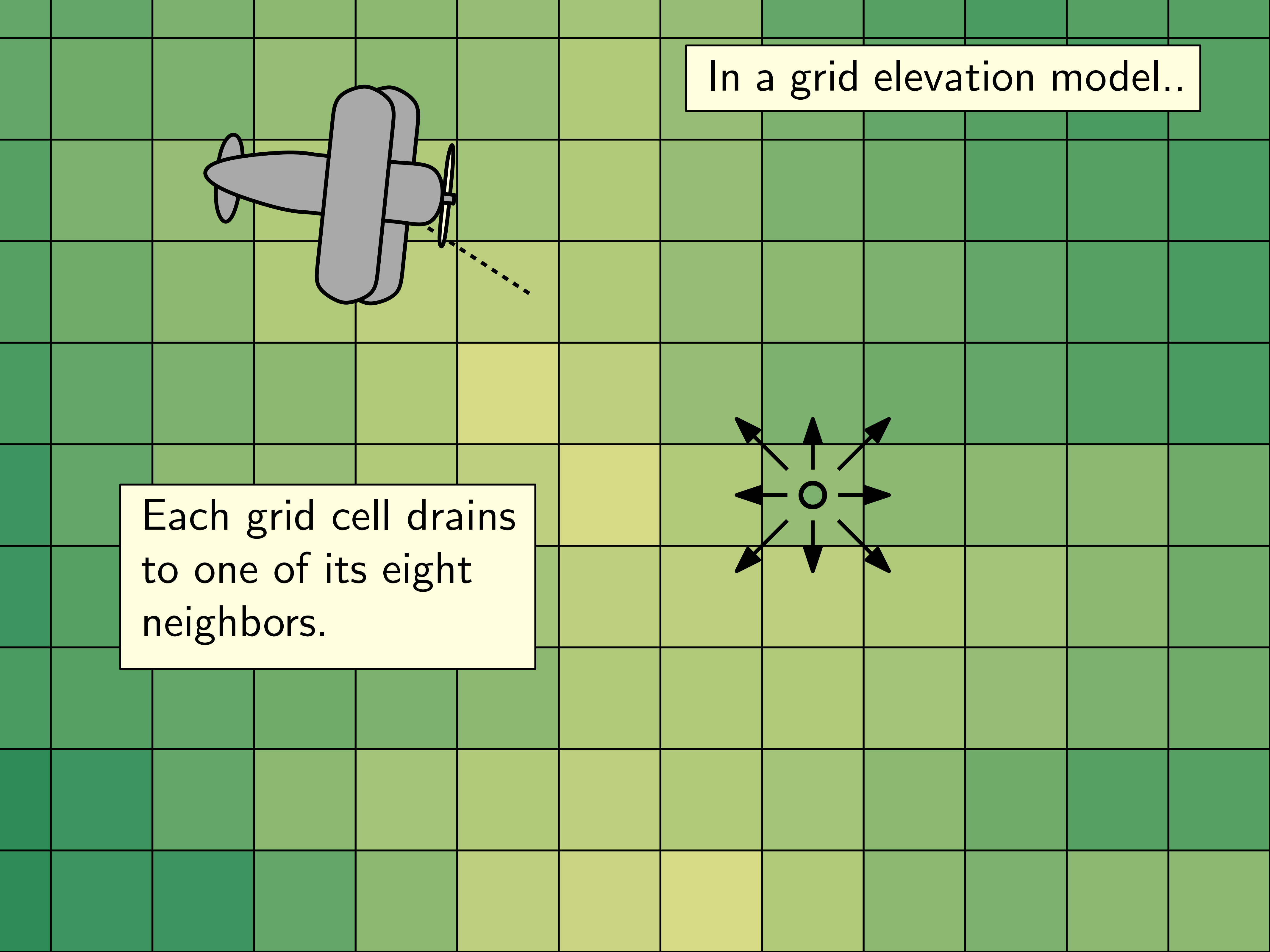
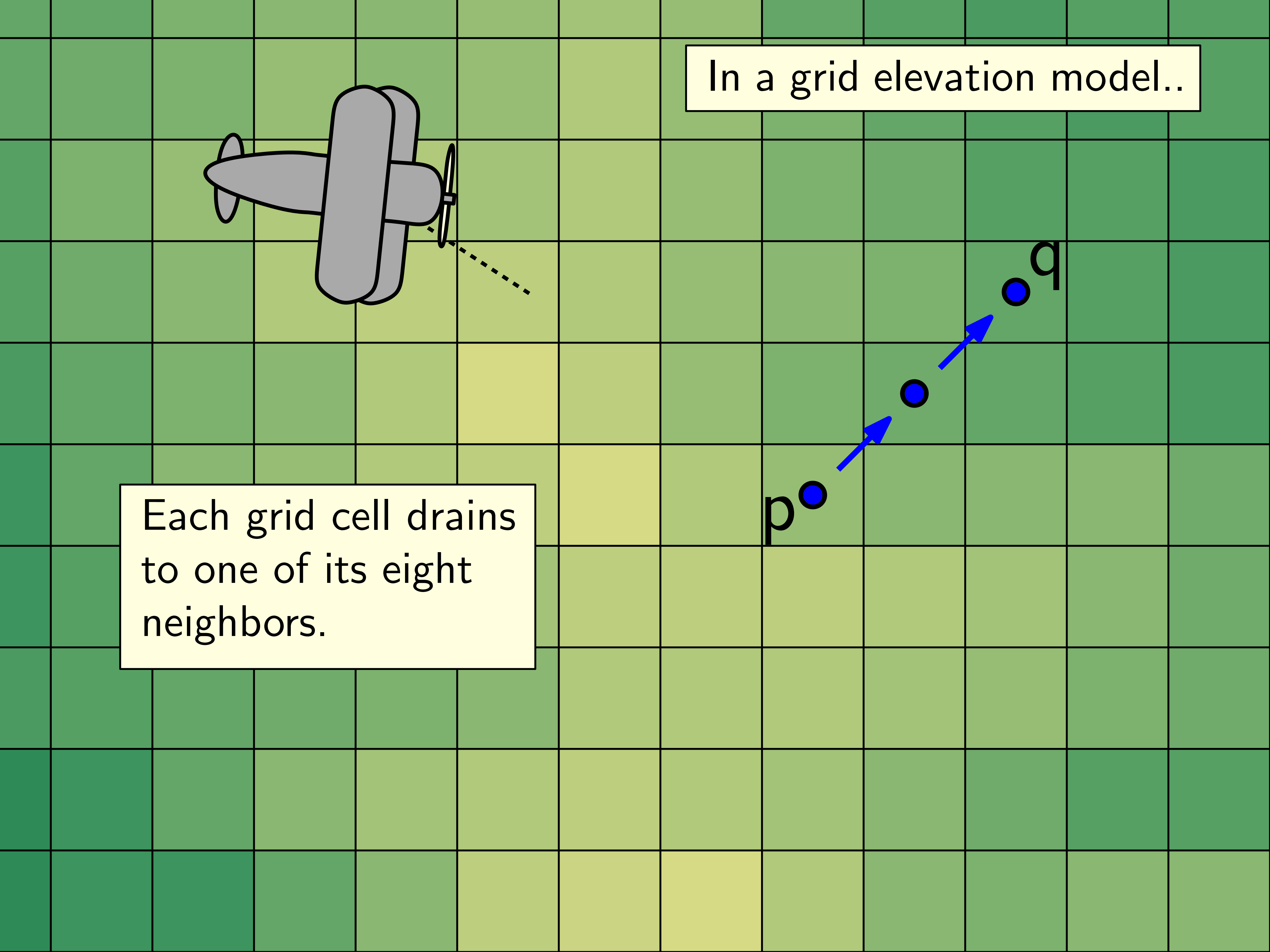

In a grid elevation model..
Each grid cell drains to one of its eight neighbors.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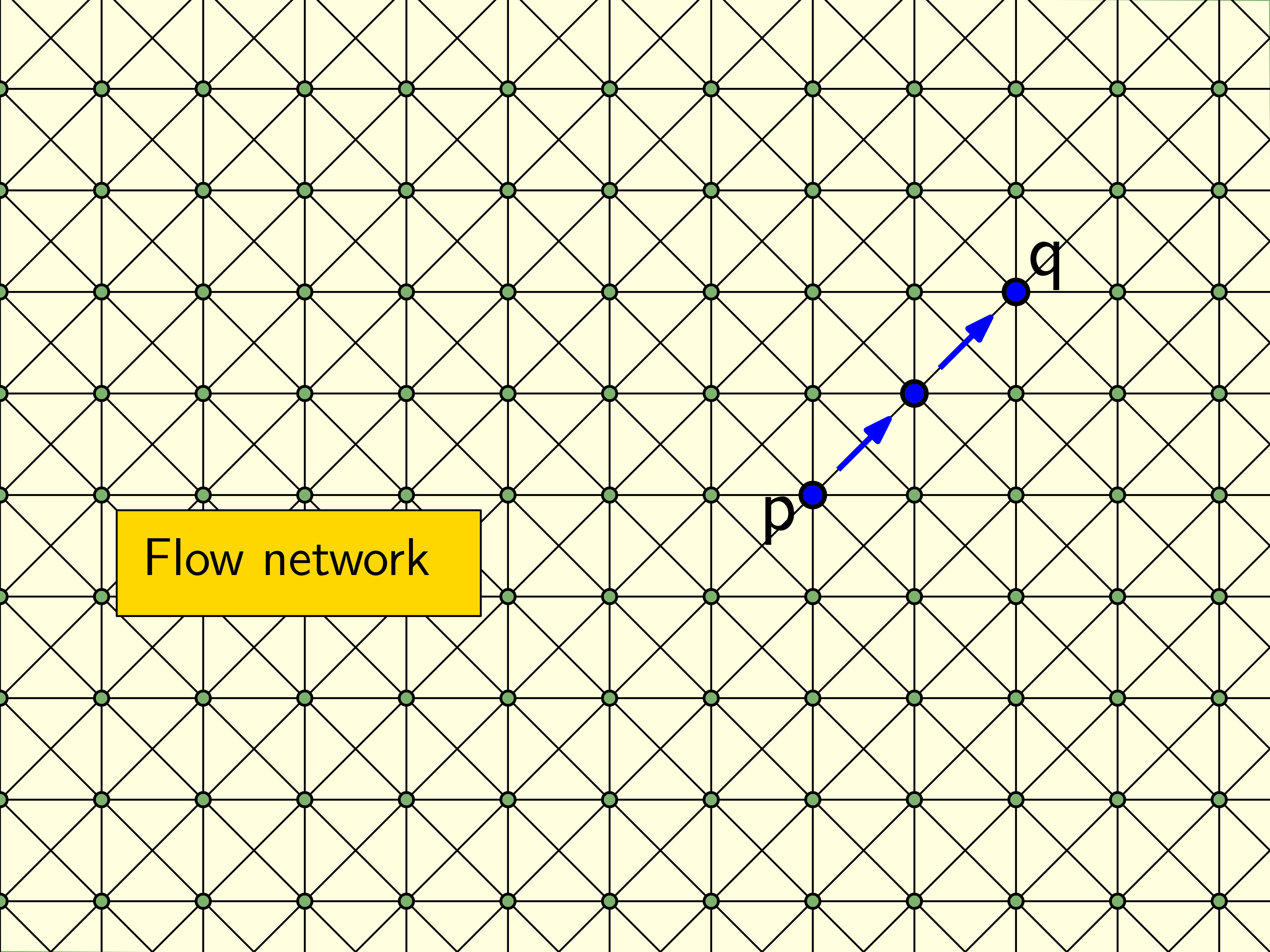
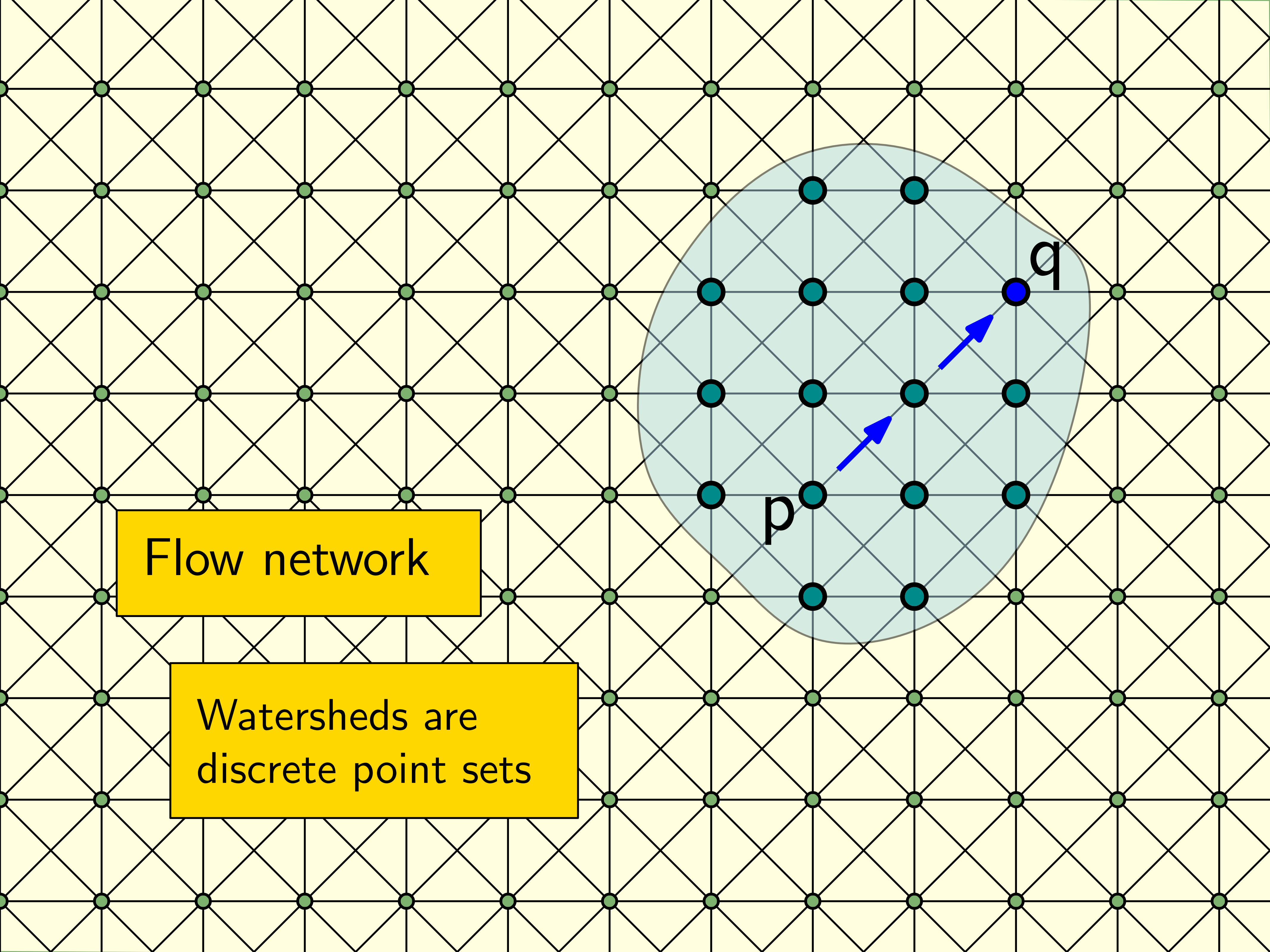
p
q


Flow network
p
q

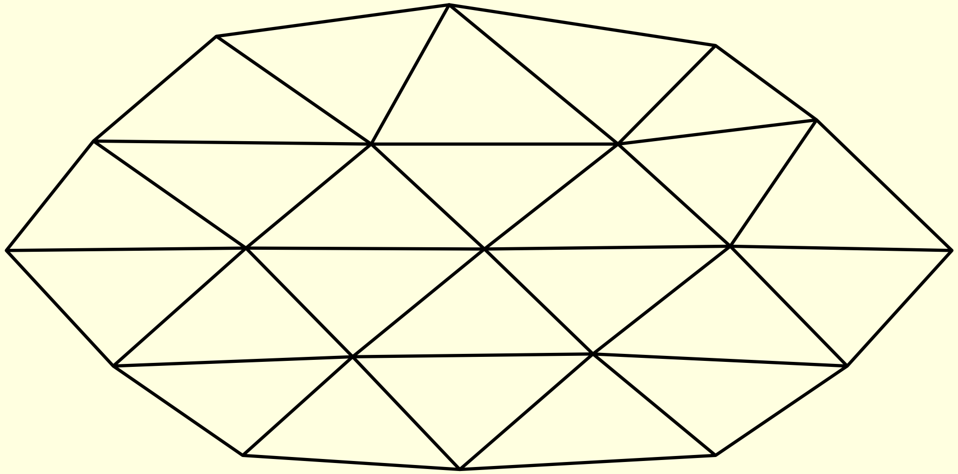
Flow network

Watersheds are discrete point sets



Definitions

An *imprecise terrain* is a graph in which each node v has the form $v = (x, y, [z_-, z_+])$.



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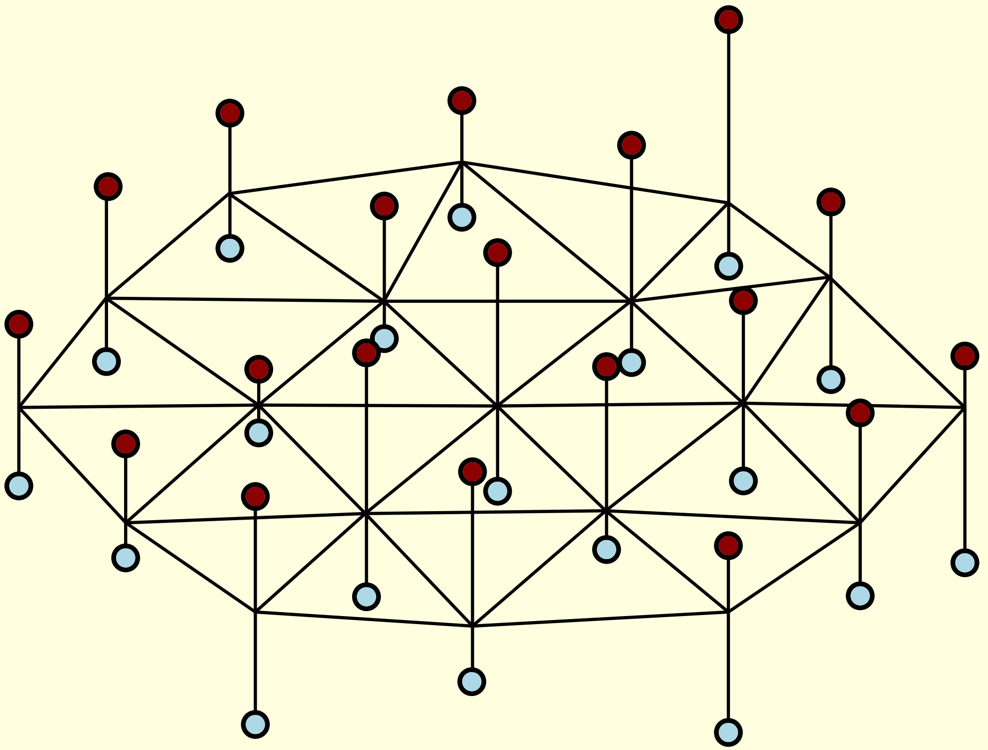

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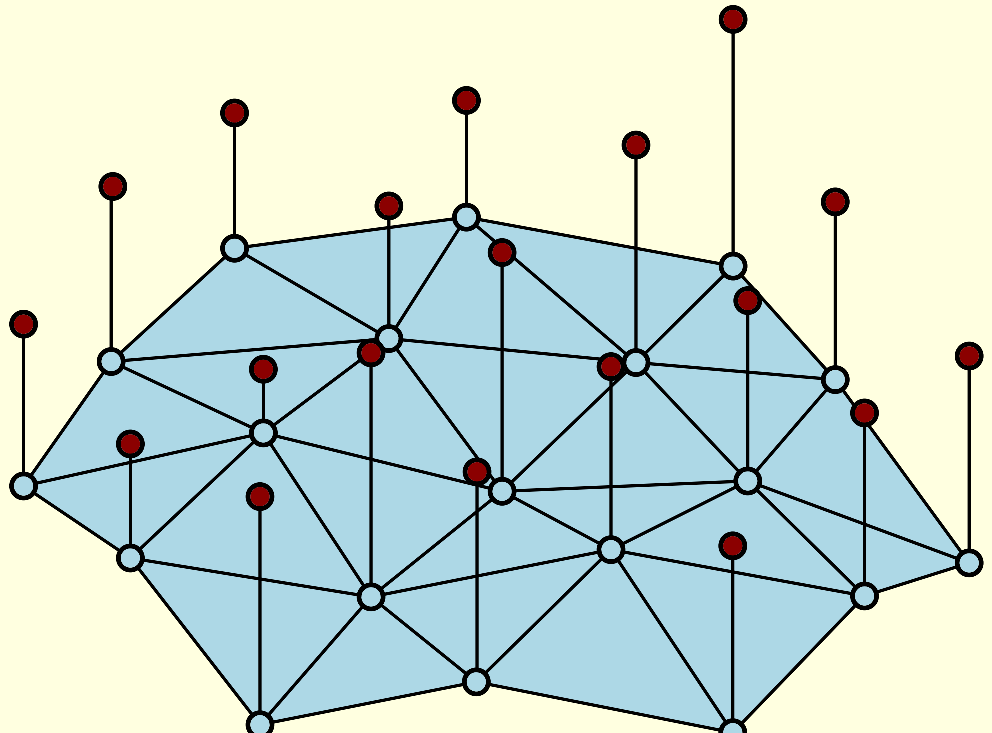

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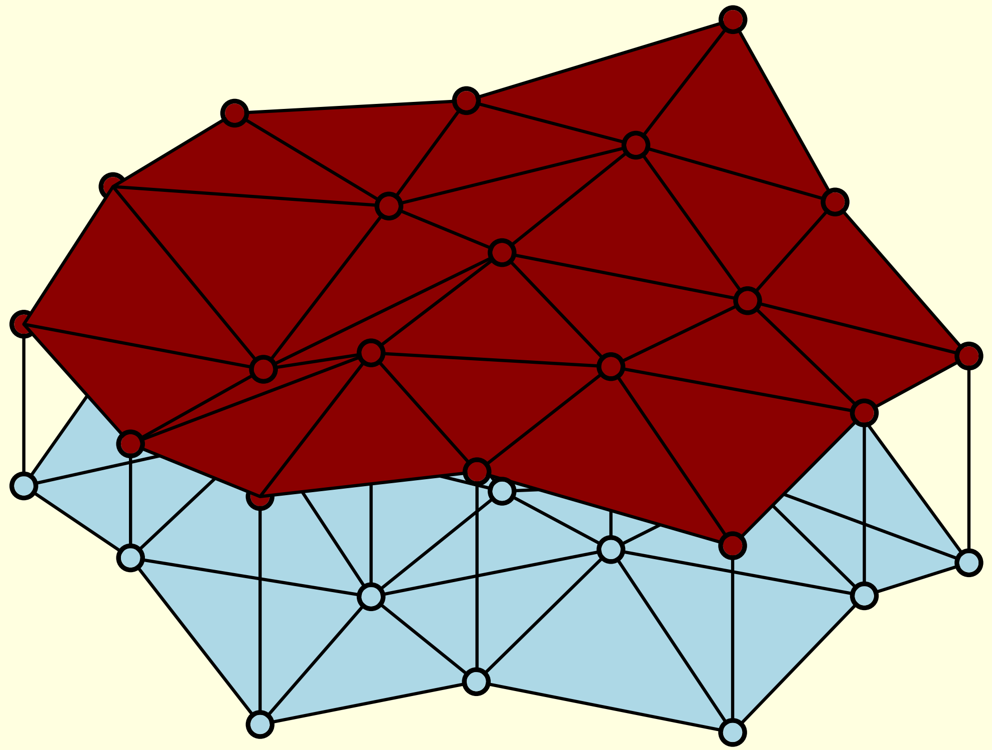
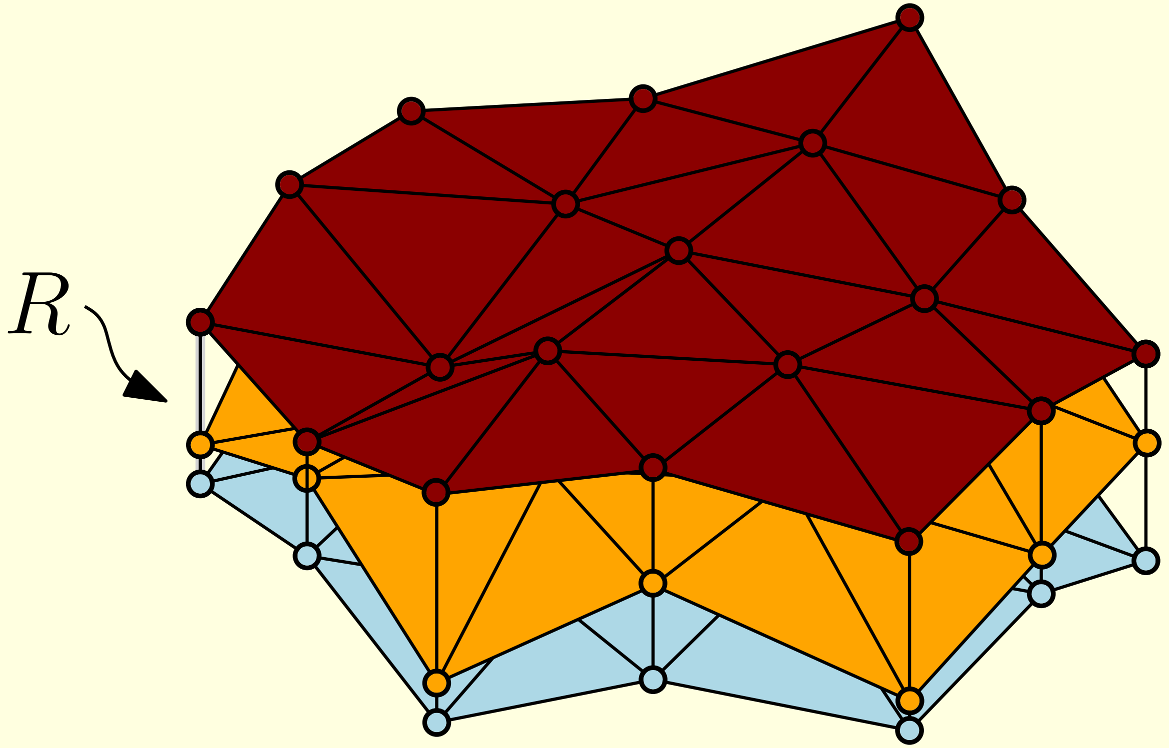


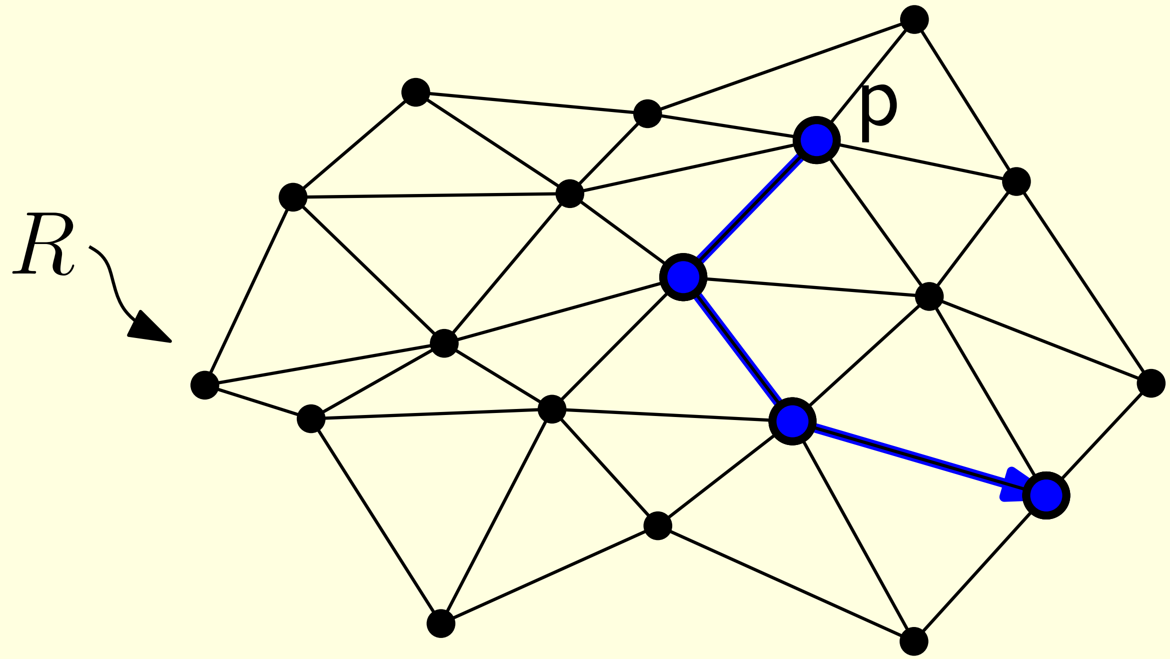
Definitions

A *realization* R is the graph with $v = (x, y, z)$, such that $z \in [z_-, z_+]$.



Definitions

In a fixed realization water flows from a node to its steepest descent neighbor.



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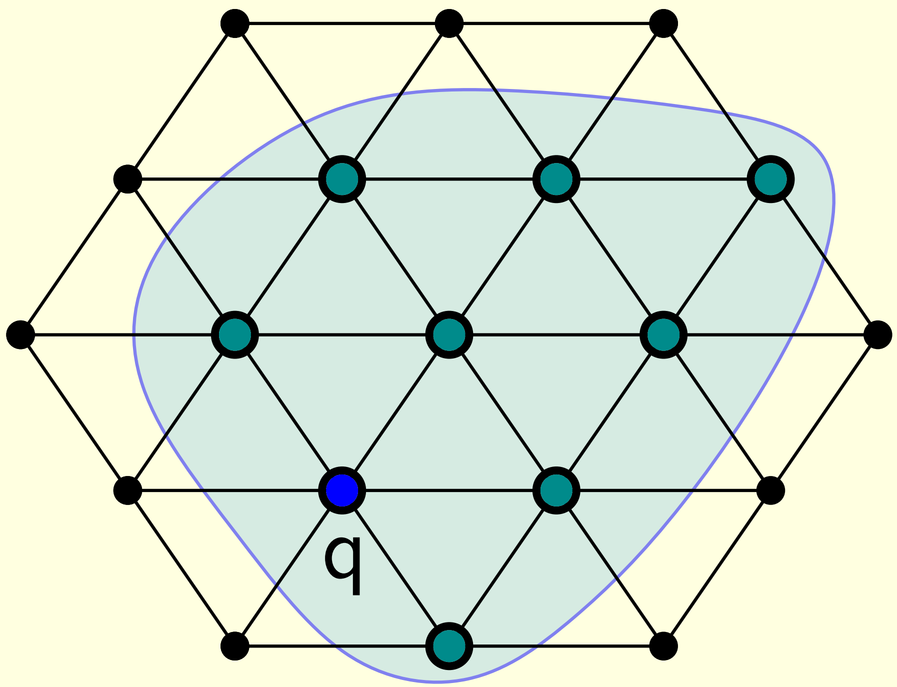
$$\mathcal{W}_\cup(\mathsf{q}) := \bigcup_R \{\mathsf{p} : \mathsf{p} \text{ flows to } \mathsf{q} \text{ in } R\}$$



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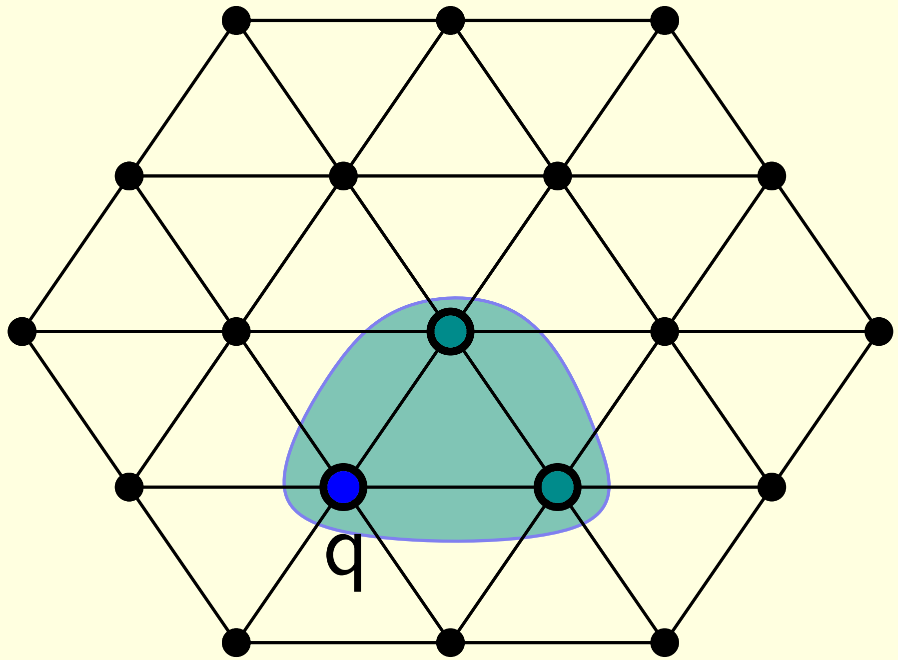

Results

The *potential watershed* :

$$\mathcal{W}_{\cup}(Q) := \bigcup_{R} \{\mathsf{p} : \mathsf{p} \text{ flows to } \mathsf{q} \text{ in } R, \mathsf{q} \in Q\}$$

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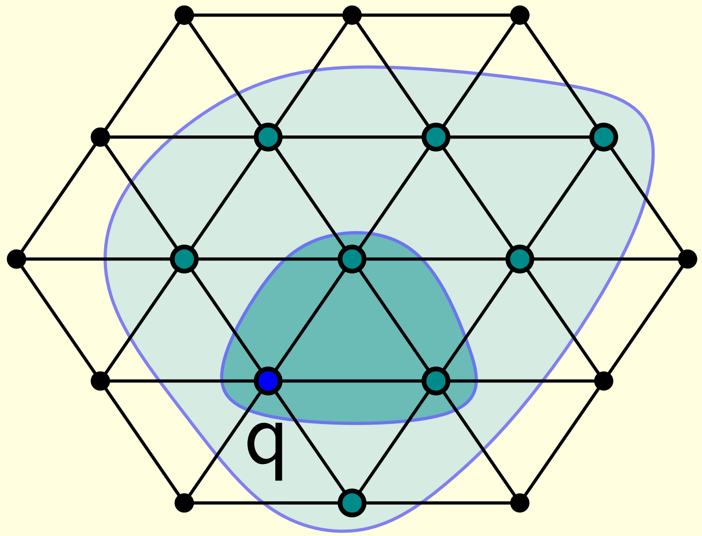

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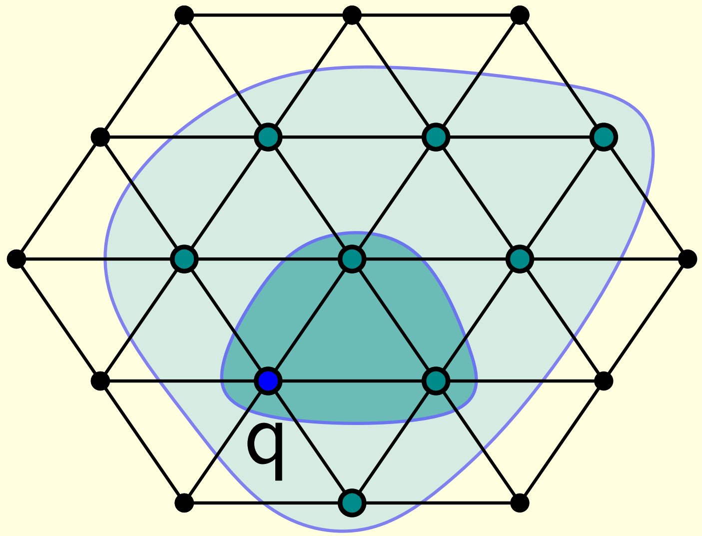

We can compute both in $O(n \log n)$ time; on grid terrains: $O(n)$

Understanding Core Watersheds

Core watersheds do not give a good definition of persistent water flow..

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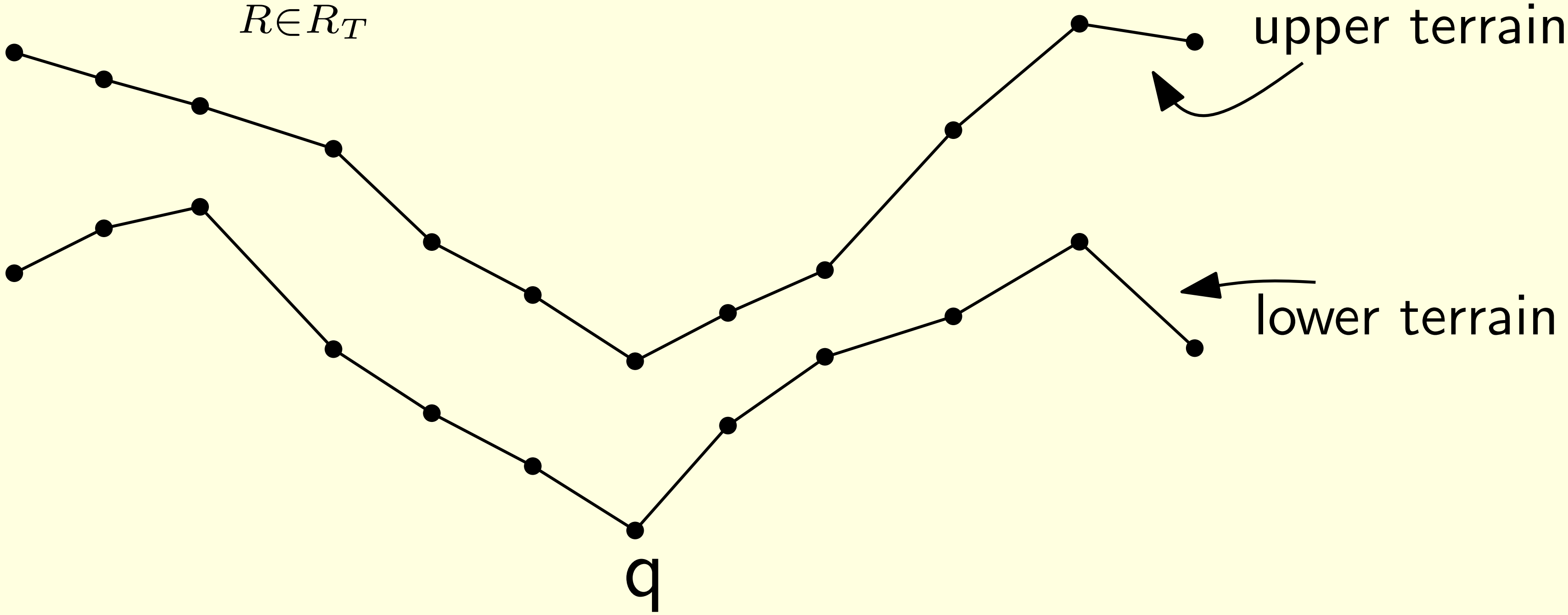

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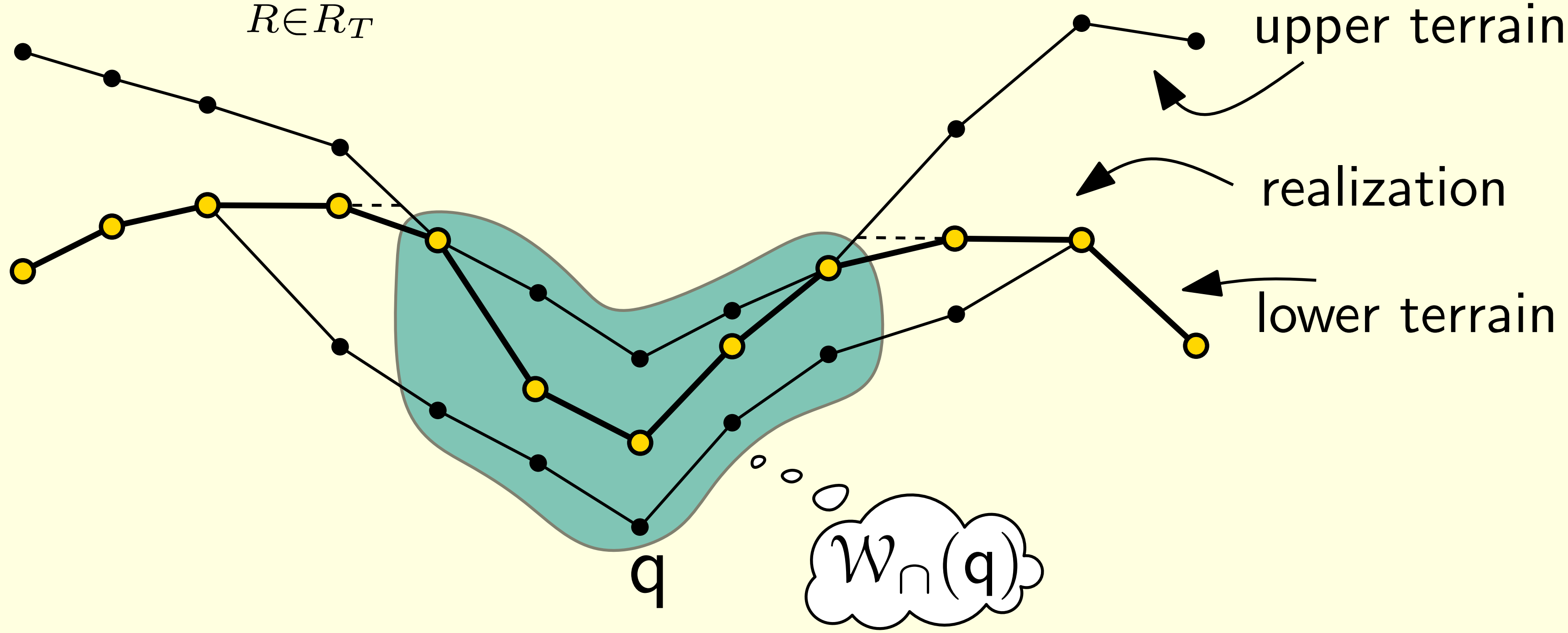

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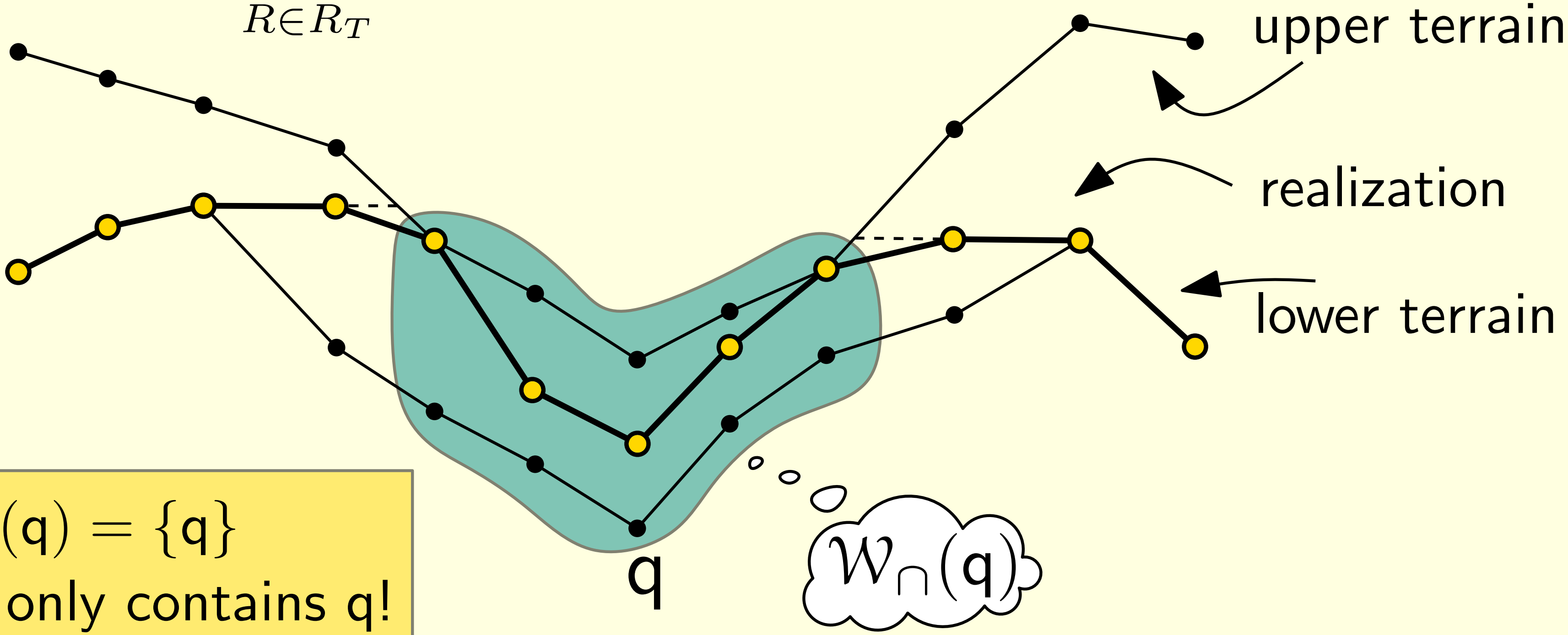

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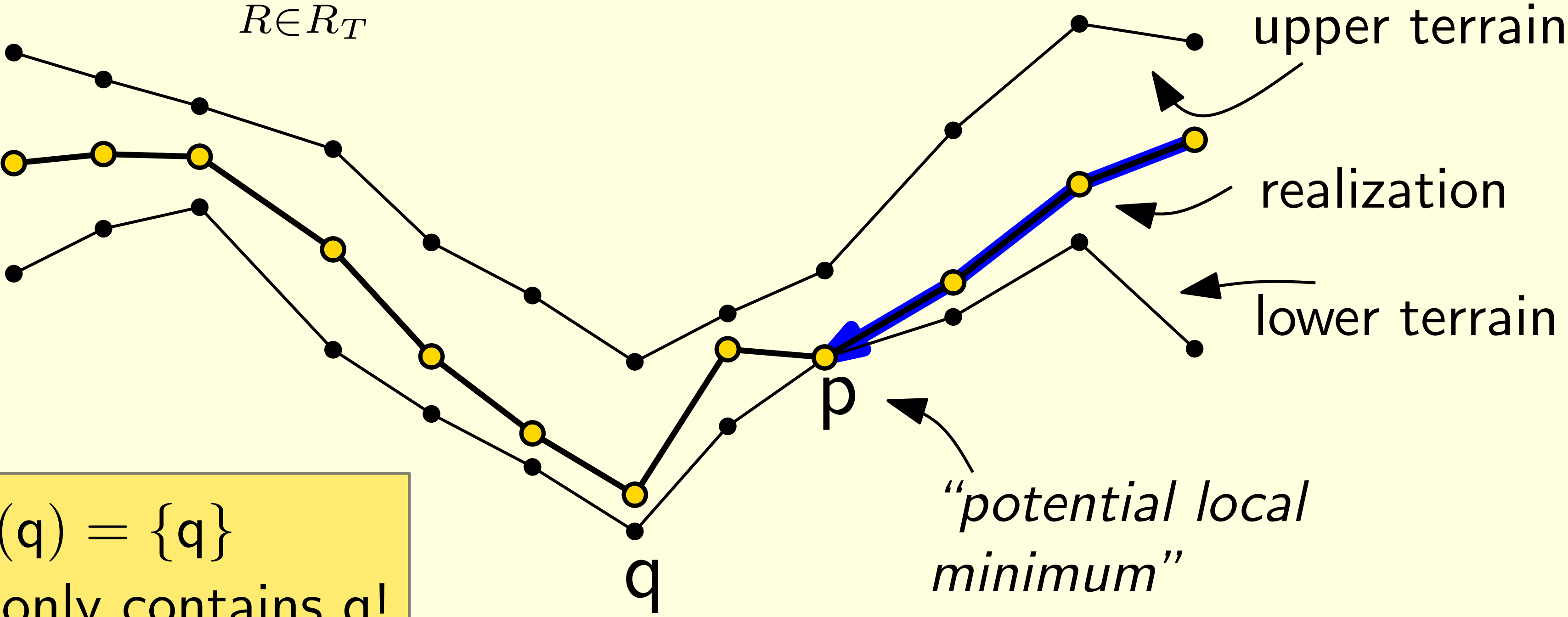

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Core Watersheds are the complement of the set of nodes that have alternative destinations

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Contained in this set:

(i) potential local minima

(ii) nodes outside $\mathcal{W}_\cup(\mathsf{q})$

(iii) nodes with flow paths to (i) or (ii)

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$V_{\min}$

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Caution! Avoid the flow paths through q.

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Understanding Core Watersheds

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ha lesti

Alternative Definition:

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"Persistent Watersheds"

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Persistent Watersheds – Properties

In general, persistent watersheds and potential watersheds are not nested!

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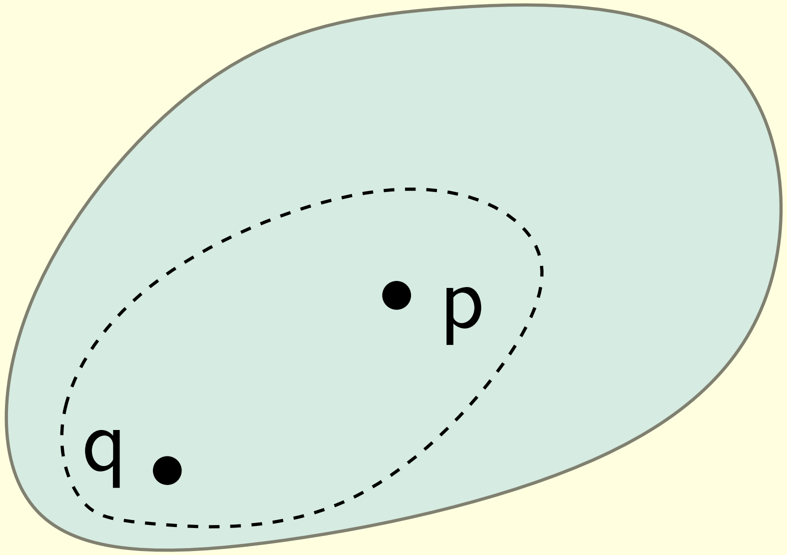
On regular* terrains, we can prove:

Let $\mathsf{p} \in \mathcal{W}_{\cap}(\mathsf{q})$



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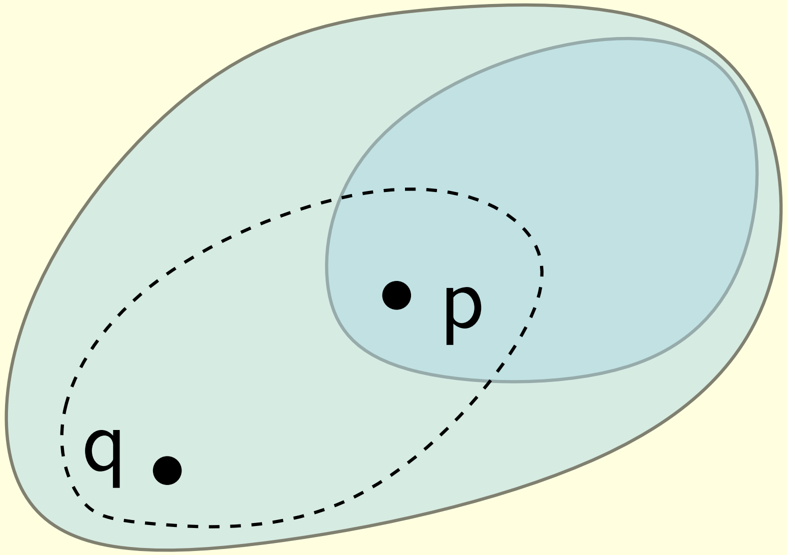

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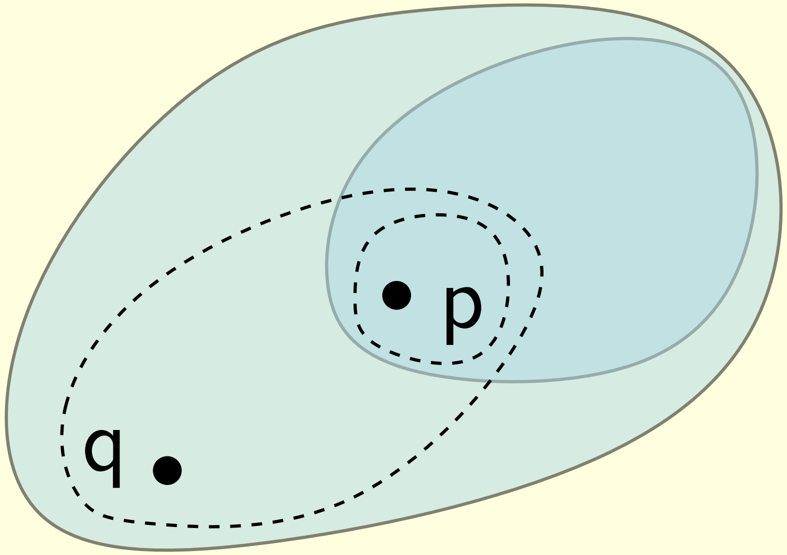
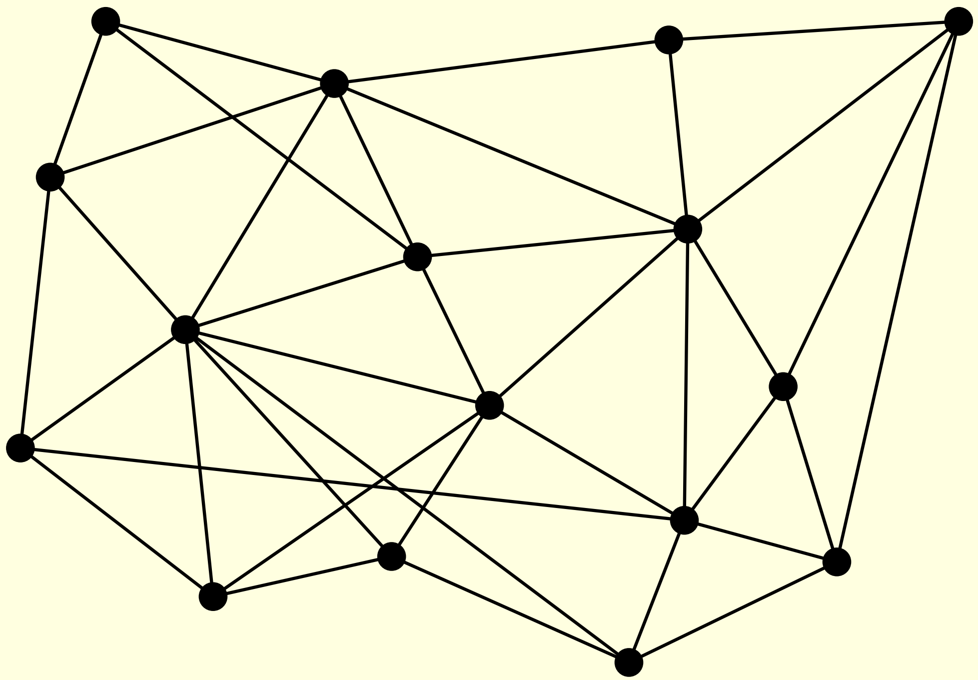
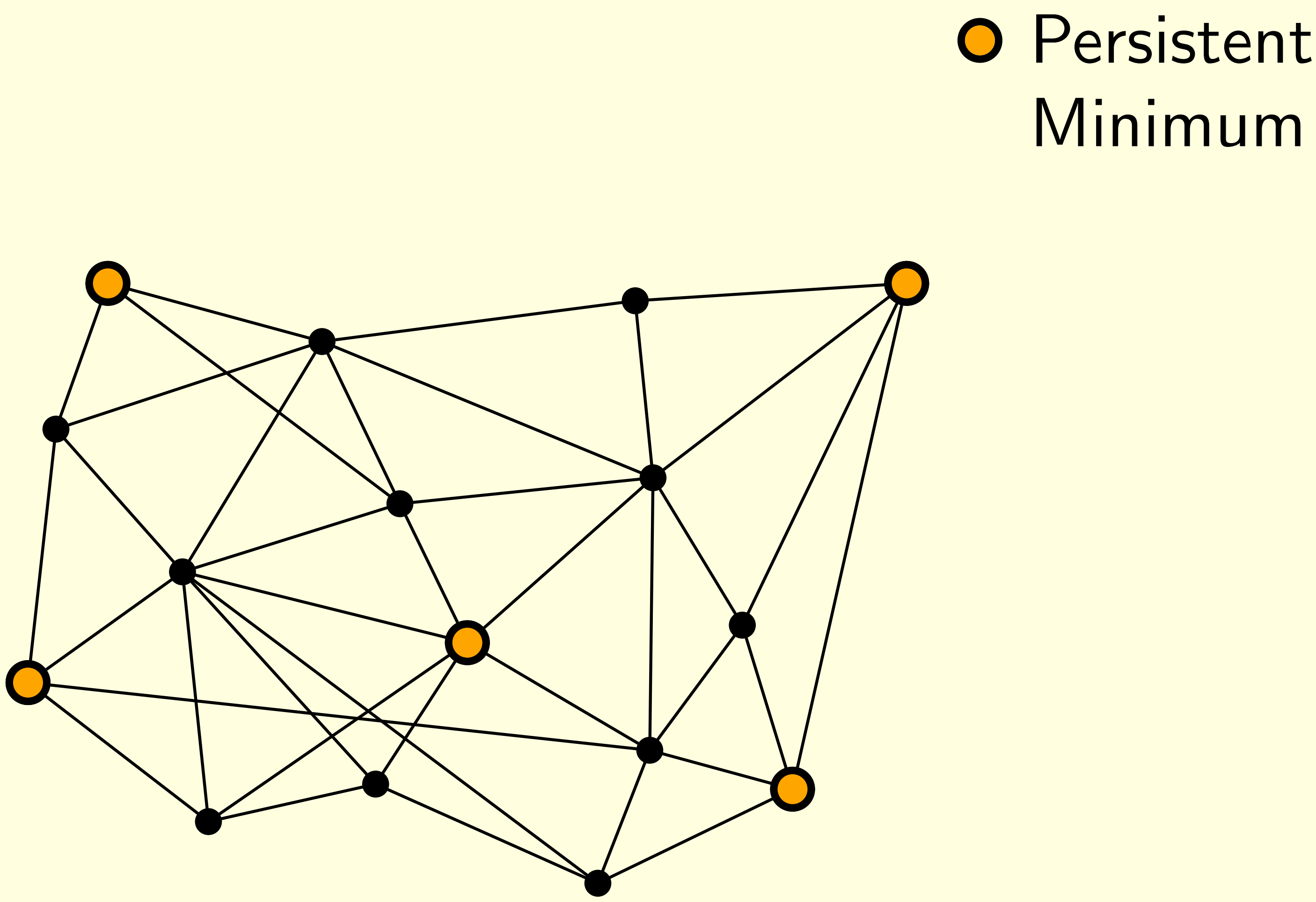


* after removing avoidable local minima

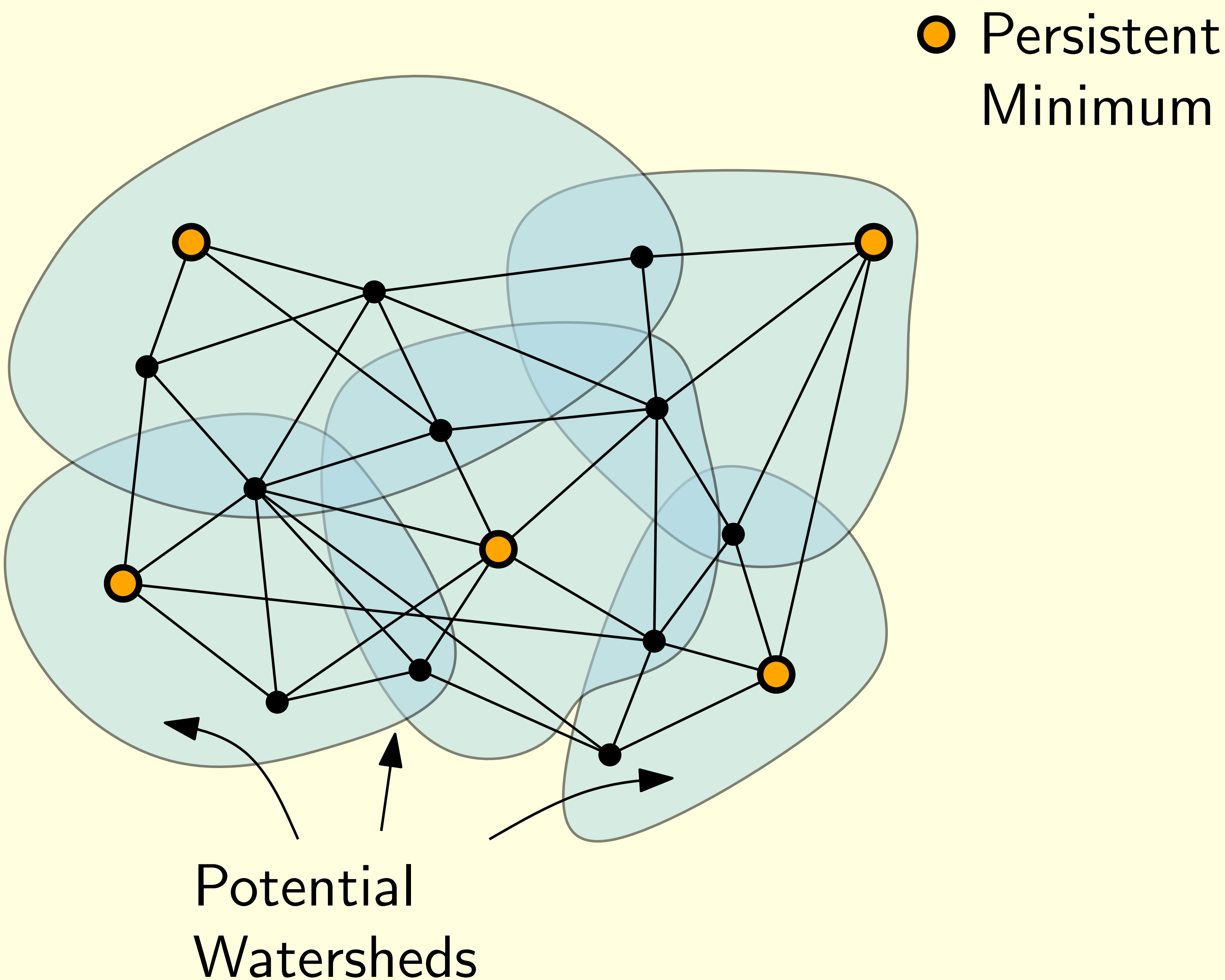
Fuzzy watersheds



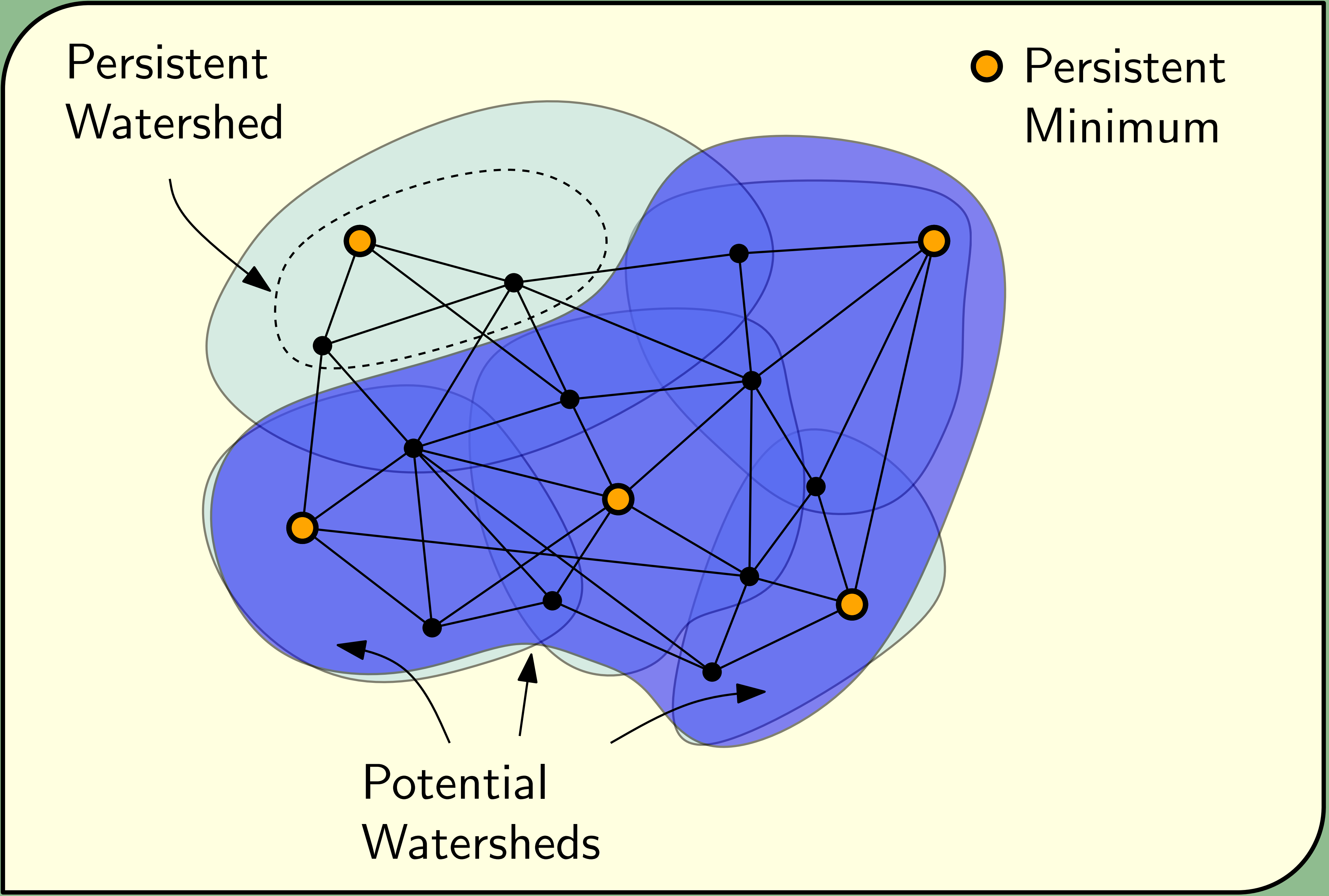
Fuzzy watersheds



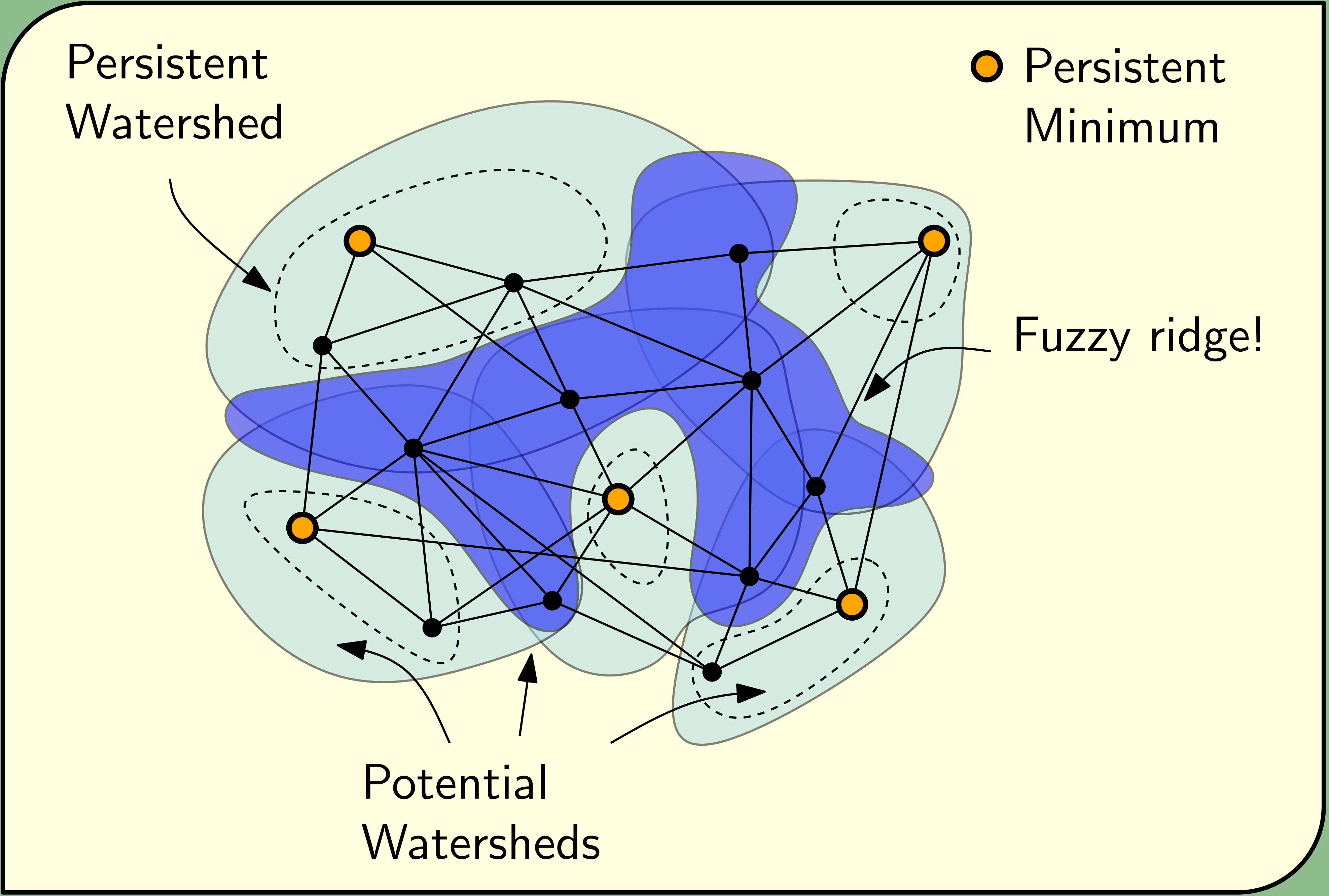
Fuzzy watersheds



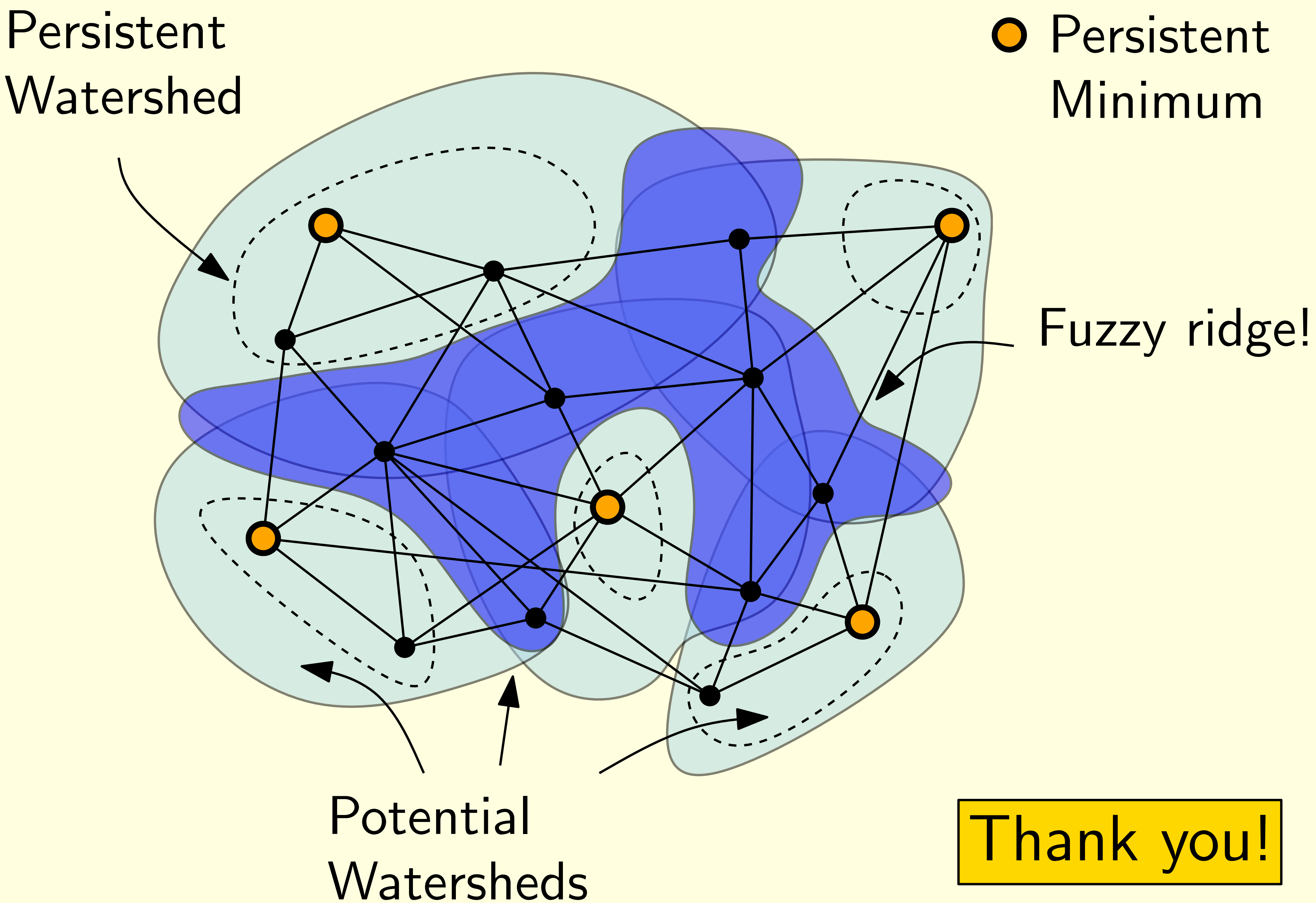
Fuzzy watersheds



Fuzzy watersheds



Fuzzy watersheds



Thank you!