Computational Differentiation : Finance

Simulation Sciences Seminar

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Main reference paper

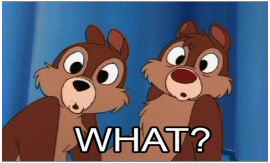
U. Naumann and J. du Toit.
Adjoint algorithmic differentiation tool support for typical numerical patterns in computational finance.
Journal of Computational Finance, 2016

Flow of the Presentation

- Use Case
  - Basic Terminologies
- Mathematical formulation of the problem
- Introduce dco/c++
- Checkpointing
- Performance Comparison

Use Case

Consider a simple European call option on an underlying driven by a local volatility process



Consider a simple European call option on an **underlying** driven by a local volatility process

Underlying

An underlying security is a stock, index, bond, interest rate, currency or commodity on which derivative instruments, such as futures and options, are based.

Consider a simple European **call option** on an underlying driven by a local volatility process

Call Option

Right to buy / sell

An **options contract** is an agreement between two parties to facilitate a potential transaction on the underlying security at a preset price, referred to as the strike price, prior to the expiration date **(maturity)**. The two types of contracts are **put** and **call** options.

Consider a simple **European call option** on an underlying driven by a local volatility process

Types of Option

Mainly two types of option : American and European
Name has nothing to do with geographic location.

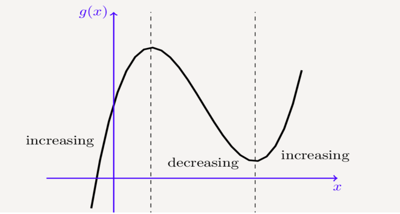
An **American option** is an option that can be exercised anytime during its life.
A **European option** is an option that can only be exercised at the end of its life, at its maturity.

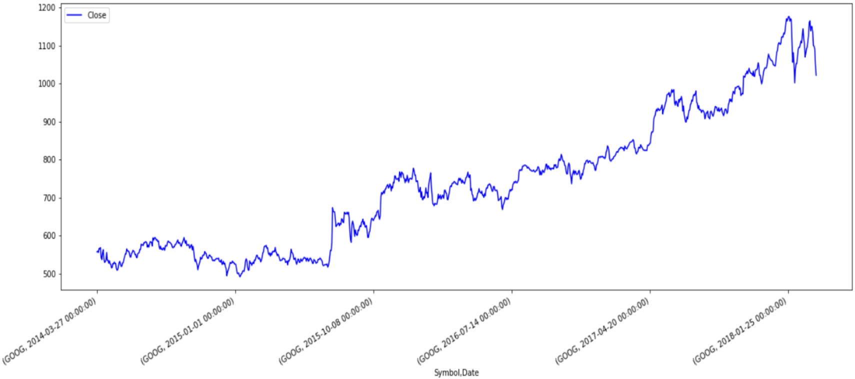
Consider a simple European call option on an underlying driven by a **local volatility process**

Volatility

Volatility refers to the **amount of uncertainty or risk** about the size of changes in a security's value.
Volatility can either be measured by using the **standard deviation or variance** between returns from that same security or market index.

Commonly, **the higher the volatility, the riskier the security** .





Google stock price close values : 1/4/2013 - 25/3/2018

Consider a simple European call option on an underlying driven by a local volatility process

Monte Carlo Pricing

Let $S = (S_t)_{t \geq 0}$ be the solution to the SDE :

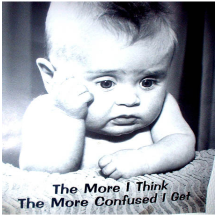
$dS_t = rS_t dt + \sigma(log(S_t), t)S_t dW_t$

where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion
r>0 is the risk free interest rate
σ is the local volatility function

The **price of the call option** is then given by :
$V = e^{-rT} E(S_T - K)^+$
T : Final time / maturity / expiration time

sigma σ

In practice σ will typically be computed from the market observed implied volatility surface and is often represented either as a bicubic spline or as a series of one-dimensional splines.



sigma σ

To keep things simple , we choose to represent σ as
$\sigma(x, t) = g(x).t$

$g : R- > R_+$ is given by :

$g(x) = \frac{p_m(x)}{q_n(x)} = \frac{a_0+a_1x+...+a_mx^m}{b_0+b_1x+...+b_nx^n}$

p_n and q_m are polynomials of order n and m respectively .

Sensitivities

Basically relevant derivatives

	Spot price (S)	**Volatility (σ)**	**Time to expiry (τ)**
Value (V)	Δ Delta	$\mathcal{V}$ Vega	Θ Theta
Delta (Δ)	Γ Gamma	Vanna	Charm
Vega ($\mathcal{V}$)	Vanna	Vomma	Veta
Theta (Θ)	Charm	Veta	
Gamma (Γ)	Speed	Zomma	Color
Vomma		Ultima	

Active and Passive variables

Active outputs : We are interested in their rate of change
Active inputs : We are interested in rate of change wrt them

Passive outputs : We are NOT interested in their rate of change
Passive inputs : We are NOT interested in rate of change wrt them

eg : Active input : S0
Passive input : a_i in calculation of sigma

How to differentiate

$z = f(a, b, c)$

Tangent mode / Forward mode : Derivative of output of every "layer" wrt input of that layer is calculated / stored . $\frac{\partial f}{\partial a}$, $\frac{\partial f}{\partial b}$, $\frac{\partial f}{\partial c}$

Reverse / Adjoint mode : Derivative of input of every "layer" wrt output of that layer is calculated / stored . $\frac{\partial a}{\partial f}$, $\frac{\partial b}{\partial f}$, $\frac{\partial c}{\partial f}$

How to differentiate : An example

Expression : $\mathbf{z = x * y + sin(x)}$

We are interested in the derivatives of output wrt input , i.e. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Steps to calculate z :

x = input x

y = input y

a = x * y

b = sin(x)

z = a + b

How to differentiate : Tangent / Forward mode

Chain rule : $\frac{\partial z}{\partial t} = \Sigma\left(\frac{\partial z}{\partial u_i}.\frac{\partial u_i}{\partial t}\right)$ where t is some input variable like x .

Our example : $\mathbf{z = x * y + sin(x)}$

$\frac{\partial x}{\partial t} = ?$

$\frac{\partial y}{\partial t} = ?$

$\frac{\partial a}{\partial t} = y * \frac{\partial x}{\partial t} + x * \frac{\partial y}{\partial t}$

$\frac{\partial b}{\partial t} = cos(x) * \frac{\partial x}{\partial t}$

$\frac{\partial z}{\partial t} = \frac{\partial a}{\partial t} + \frac{\partial b}{\partial t}$

Now , if we want $\frac{\partial z}{\partial x}$, $t = x$ and therefore $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$.

Once we "seed" these value , everything else is taken care of .

How to differentiate : Tangent / Forward mode

Chain rule : $\frac{\partial z}{\partial t} = \Sigma\left(\frac{\partial z}{\partial u_i}.\frac{\partial u_i}{\partial t}\right)$ where t is some input variable like x .

Our example : **z = x * y + sin(x)**

$\frac{\partial x}{\partial t} = 1$

$\frac{\partial y}{\partial t} = 0$

$\frac{\partial a}{\partial t} = y * 1 + x * 0$

$\frac{\partial b}{\partial t} = cos(x) * 1$

$\frac{\partial z}{\partial t} = \frac{\partial a}{\partial t} + \frac{\partial b}{\partial t}$

$\frac{\partial z}{\partial t} = y + cos(x)$

$\frac{\partial z}{\partial x} = y + cos(x)$

How to differentiate : Tangent / Forward mode

Advantage : The differential variables depend on the intermediate variables, calculate them together , no need to hold on to the the intermediate variables until later, **saving memory**.

Expression : z = x * y + sin(x)

x = input value

$\frac{\partial x}{\partial t} = ?$

y = input value

$\frac{\partial y}{\partial t} = ?$

a = x * y

$\frac{\partial a}{\partial t} = y * \frac{\partial x}{\partial t} + x * \frac{\partial y}{\partial t}$

b = sin(x)

$\frac{\partial b}{\partial t} = cos(x) * \frac{\partial x}{\partial t}$

z = a + b

$\frac{\partial z}{\partial t} = \frac{\partial a}{\partial t} + \frac{\partial b}{\partial t}$

How to differentiate : Tangent / Forward mode

Disadvantages : Since we set t = x to calculate $\frac{\partial z}{\partial x}$,
we will need to set t = y for $\frac{\partial z}{\partial y}$. This means number of passes = number of input variables .

Problem if number of input variables is large !

How to differentiate : Reverse / Adjoint mode

Turn the chain rule upside down : $\frac{\partial s}{\partial u} = \Sigma\left(\frac{\partial z_i}{\partial u} \cdot \frac{\partial s}{\partial z_i}\right)$

z_i : output variables of interest
u : input variables (x and y)
s : some output variable (either of the z_i)

For example problem : $\mathbf{z = x * y + \sin(x)}$

$\frac{\partial s}{\partial z} = ?$
$\frac{\partial s}{\partial b} = \frac{\partial s}{\partial z}$
$\frac{\partial s}{\partial a} = \frac{\partial s}{\partial z}$
$\frac{\partial s}{\partial y} = x * \frac{\partial s}{\partial a}$
$\frac{\partial s}{\partial x} = y * \frac{\partial s}{\partial a} + cos(x) * \frac{\partial s}{\partial b}$

Put s = z and in just 1 pass get $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$

How to differentiate : Reverse / Adjoint mode

s = z ; $\frac{\partial s}{\partial z} = 1$

For example problem : **z = x * y + sin(x)**

$\frac{\partial s}{\partial z} = 1$

$\frac{\partial s}{\partial b} = 1$

$\frac{\partial s}{\partial a} = 1$

$\frac{\partial s}{\partial y} = x * 1$

$\frac{\partial s}{\partial x} = y * 1 + cos(x) * 1$

$\frac{\partial z}{\partial y} = x$

$\frac{\partial z}{\partial x} = y + cos(x)$

How to differentiate : Reverse / Adjoint mode

Disadvantage : Now calculations and differential calculation cannot be interleaved . We need to save the intermediate variables also , and then calculate the differentials , thus leading to more memory use .

Disadvantage : If number of active output variables is large and number of active input variables is small , say 1 , tangent mode would be better since it requires only 1 pass .

Open source Automatic differentiation tools :

`www.autodiff.org`

`http://www.autodiff.org/?module=Tools`

Click on language preference -> List of tools

dco/c++

An Algorithmic Differentiation tool developed by Numerical Algorithms Group
Link for download :

https://www.nag.co.uk/content/downloads-dco-c-versions

dco/c++: Tangent Mode

```
#include "dco.hpp"
typedef dco::gt1s<double> DCO_MODE;
typedef DCO_MODE::type DCO_TYPE;

   ACTIVE_INPUTS<DCO_TYPE> X;
   PASSIVE_INPUTS XP;
   ACTIVE_OUTPUTS<DCO_TYPE> Y;
   PASSIVE_OUTPUTS YP;

   dco::derivative(X.S0)=1;
   price(X,XP,Y,YP);
   cout << "Y=" << dco::value(Y.V) << endl;
   cout << "dY/dX.S0=" << dco::derivative(Y.V) << endl;

   dco::derivative(X.S0)=0;
   dco::derivative(X.r)=1;
   price(X,XP,Y,YP);
   cout << "dY/dX.r=" << dco::derivative(Y.V) << endl;
```

[1]

Notice : price function called twice

dco/c++: Adjoint Mode

```
#include "dco.hpp"
typedef dco::ga1s<double> DCO_MODE;
typedef DCO_MODE::type DCO_TYPE;
typedef DCO_MODE::tape_t DCO_TAPE_TYPE;
DCO_TAPE_TYPE* & DCO_TAPE_POINTER=DCO_MODE::global_tape;

  ACTIVE_INPUTS<DCO_TYPE> X;
  PASSIVE_INPUTS XP;
  ACTIVE_OUTPUTS<DCO_TYPE> Y;
  PASSIVE_OUTPUTS YP;

  DCO_TAPE_POINTER = DCO_TAPE_TYPE::create();
  DCO_TAPE_POINTER->register_variable(X.S0);
  DCO_TAPE_POINTER->register_variable(X.r);
  ...

  price(X,XP,Y,YP);

  DCO_TAPE_POINTER->register_output_variable(Y.V);
  dco::derivative(Y.V)=1;
  DCO_TAPE_POINTER->interpret_adjoint();

  cout << "Y=" << dco::value(Y.V) << endl;
  cout << "dY/dX.S0=" << dco::derivative(X.S0) << endl;
  cout << "dY/dX.r=" << dco::derivative(X.r) << endl;
  ...

  DCO_TAPE_TYPE::remove(DCO_TAPE_POINTER);
```

[1]

Notice : price function called once

Memory

input................intermediate variables / values...........output

Problem : Since the memory requirements of the tape scale more or less linearly with the number of sample paths, this leads to infeasible peak memory requirements.
Solution : Store some values "elsewhere" , use them for calculations later.

Checkpointing

A checkpoint is a set of data which is stored (either to disk or to memory) at a point during a computation, and which allows the computation to be restarted (at some later time) from that point.

Advantage : Peak memory requirement reduces (A lot !) .

Disadvantage : Amount of computation increases (Not a lot).

$$F : (\mathbf{x}, \tilde{\mathbf{x}}) \xrightarrow{f_1} (\mathbf{u}, \tilde{\mathbf{u}}) \, , \begin{pmatrix} (\mathbf{u}, \tilde{\mathbf{u}}_1) \xrightarrow{g} (\mathbf{v}_1, \tilde{\mathbf{v}}_1) \\ (\mathbf{u}, \tilde{\mathbf{u}}_2) \xrightarrow{g} (\mathbf{v}_2, \tilde{\mathbf{v}}_2) \\ \vdots \\ (\mathbf{u}, \tilde{\mathbf{u}}_N) \xrightarrow{g} (\mathbf{v}_N, \tilde{\mathbf{v}}_N) \end{pmatrix} , (\mathbf{v}_1, \ldots, \mathbf{v}_N, \tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_N) \xrightarrow{f_2} (y, \tilde{\mathbf{y}})$$

[1]

N : sample paths
N evaluations of g : explosion in tape size
Mutual independence of evaluation of g : work in parallel

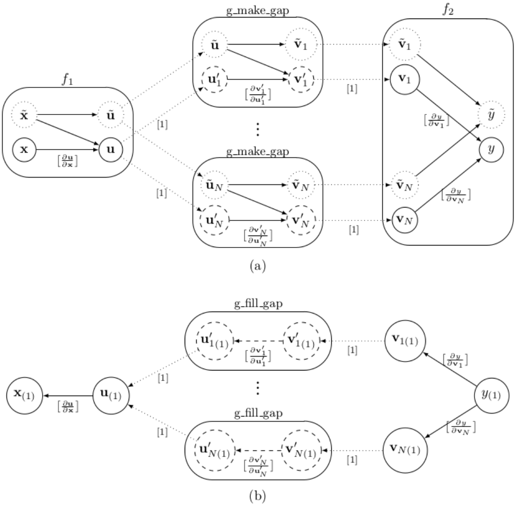
$X_t = log(S_t)$
$X_{t0} = log(S_0)$
Z_i : standard normal random number
$\Delta = T/M$ some integer M
$t_i = i\Delta, i = 1, 2, ...M =$ Monte Carlo time steps

$$dX_t = \left(r - \tfrac{1}{2}\sigma^2(X_t, t)\right)dt + \sigma(X_t, t)\, dW_t.$$

$$X_{t_{i+1}} = X_{t_i} + \left(r - \tfrac{1}{2}\sigma^2(X_{t_i}, t_i)\right)\Delta + \sigma(X_{t_i}, t_i)\sqrt{\Delta}Z_i$$

N sample paths generated to use in MC integrator to calculate V

Pathwise adjoint calculation :



[1]

Performance

n	`mc/primal`	`mc/cfd`	`mc/a1s`	`mc/a1s_ensemble`	$\mathcal{R}$
10	0.3s	6.1s	1.8s (2GB)	1.3s (1.9MB)	4.3
22	0.4s	15.7s	- ($>$ 3GB)	2.3s (2.2MB)	5.7
34	0.5s	29.0s	- ($>$ 3GB)	3.0s (2.5MB)	6.0
62	0.7s	80.9s	- ($>$ 3GB)	5.1s (3.2MB)	7.3
142	1.5s	423.5s	- ($>$ 3GB)	12.4s (5.1MB)	8.3
222	2.3s	1010.7s	- ($>$ 3GB)	24.4s (7.1MB)	10.6

[1]

N = 10000

R : Runtime of AD code / Runtime of primal code

R is sensitive to compiler flags , memory hierarchy , cache sizes , level of optimization of primal code

References

[1] U. Naumann and J. du Toit.
Adjoint algorithmic differentiation tool support for typical numerical patterns in computational finance.
Journal of Computational Finance, 2016

[2]https://rufflewind.com/2016-12-30/reverse-mode-automatic-differentiation
[3]https://www.probabilitycourse.com/chapter4/4_1_3_functions_continuous_var.php
[4]http://www.picturequotes.com/the-more-i-think-the-more-confused-i-get-quote-20923
[5]https://www.wikiwand.com/en/Greeks_(finance)

Questions ?