MATH 200: TEST 1

11.3 - VECTORS

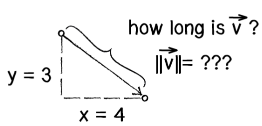
1. VECTOR OPERATIONS

- **Adding two vectors:**
  $$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$$

- **Scalar multiplication:**
  $$k\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}k\\2k\end{bmatrix}$$

- **Dot products:** this does NOT result in a vector, but a scalar
  $$\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}3\\4\end{bmatrix} = 1 * 3 + 3 * 4 = 15$$

- **Length of a vector,** or **magnitude**: || v || = $\begin{bmatrix}3\\4\end{bmatrix}$



Length of a vector

$$||v|| = \sqrt{3^2 + 4^2} = 25$$

  - **Length can also be expressed as a dot product**, if we $a \cdot a$ for $a = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$

- $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ = $(a_1)^2 + (a_2)^2 + (a_3)^2 \qquad (1)$
- Take the square root of (1), and we have $||a|| = \sqrt{a \cdot a}$

2. ANGLE BETWEEN VECTORS

- **Deriving from the law of cosines**, we have angle θ between two vectors u and v:

$cos(\theta)$ = $\frac{||u||*||v||}{u \cdot v}$ (length of two vectors, divide by dot product)

$||a \cdot b|| = ||a|| \cdot ||b|| sin(\theta)$

3. ANGLE OF RESULTANT FORCE

- Given a vector, the angle θ it makes with the x-axis can be found by:
  $v = ||v||(cos\,\theta, sin\,\theta)$
  So if given $v = (a, b)$ and $||v|| = x$, then the angle can be found by:
  $cos\,\theta = \frac{a}{||v||} = \frac{a}{x}$
- **For figure 11.2.17:**
  $F = ||F1|| + ||F2||$
  $= 200(cos\,30^o, sin\,30^o) + 300(cos\,70^o, sin\,70^o)$
  $= 275 + 381 = 469$

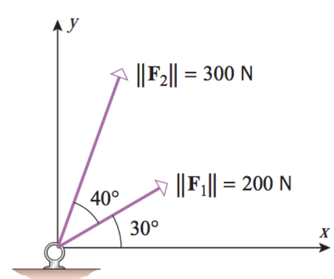
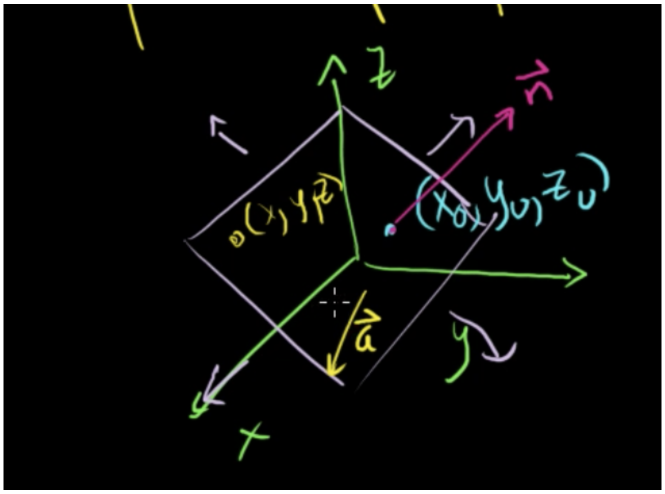


Figure 11.2.17

11.4 PLANES

1. NORMAL VECTOR TO A PLANE

- **Equation of a plane:** $Ax + By + Cx = D$ (in which for any three points x, y, z on the plane, it would satisfied the equation)
- **Normal vector**: a normal vector is a vector that is **PERPENDICULAR** to a plane



Line a (yellow) and normal plane (n)

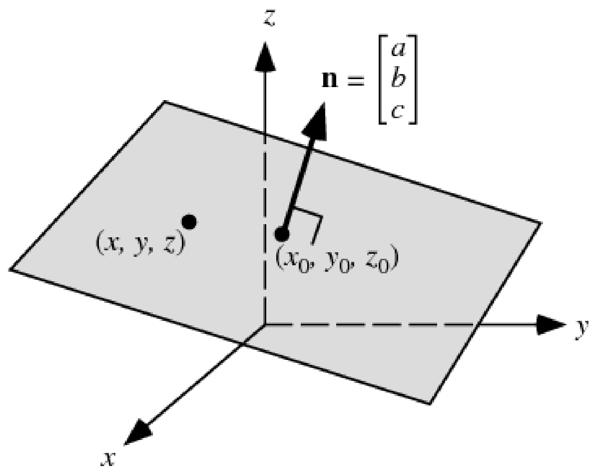


Image result for normal vector on a plane

- **Dot product property of a normal vector:** if we draw any line a on the plane (imagine a flat line on a cardboard), and we cross product it with n a normal vector, we would get 0

  $a \cdot n = 0$
  - **Given a normal vector to a plane:** $a_i + b_j + c_k$
  - **Given a vector that lies on the plane**: $(x - x_p)i + (y - y_p)j + (z - z_p)k$ (this is the result of two vectors (x, y, z) and (x_p, y_p, z_p)
  - **Then their dot product would be 0:**

    $ax - ax_p + by - by_p + cz - cz_p = 0$

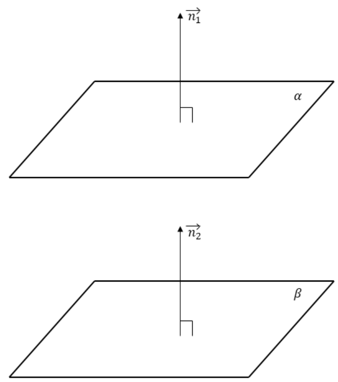
    $ax + by + cz = ax_p + by_p + cz_p \qquad (2)$
  - **Equation fo a plane is** $Ax + By + Cz = D$
  - Given equation (2), we can conclude that

    $a = A \qquad b = B \qquad c = C \qquad ax_p + by_p + cz_p = D$
- **Therefore an equation to a plane with the normal vector <A, B, C> with a point (D, E, F):**

  $A(x - D) + B(y - E) + C(z - F) = 0$
  - E.g: the normal vector to the plane $3x + 2y - 1z = \pi$ is $(3, 2, -1)$

2. CHECK IF TWO PLANES ARE PARALLEL

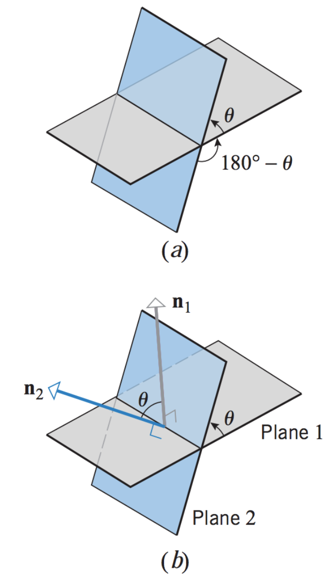
- **Theorem:** two planes are parallel if and only if their normals are parallel vectors
- **Two vectors are PARALLEL** if one is a scalar of the other (a, b, c) and $x(a, b, c)$



Parallel planes have parallel normals

3. INTERSECTING ANGLE BETWEEN TWO PLANES

- Two distinct intersecting planes determine two positive angles of intersection—an (acute) angle θ that satisfies the condition 0 ≤ θ ≤ π/2 and the supplement of that angle $180 - \theta$
- To find the angle θ between two planes:
  - acute angle: $|n_1 \cdot n_2| = ||n1|| \cdot ||n_2||cos(\theta)$



- Given $x + 2y - 2z = 5$ therefore the normal vector to that plane is $1, 2, -2$
- Given plane $6x = 3y + 2z = 8$ therefore $n_2 = (6, -3, 2)$
- Given the formula:
  $|n_1 \cdot n_2| = ||n1|| \cdot ||n_2|| cos\theta$
  $-4 = 3(7) \cdot cos\theta$
  $\theta = cos(\frac{4}{21}) = 79^o$

4. FIND EQUATION OF A PLANE

THE PLANE CONTAIN THREE POINTS P_1, P_2, P_3

- **Question:** Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$
- **Solution:** Create two vectors $\vec{P_1P_2} = (1, 1, 2)$ and $\vec{P_1P_3} = (2, -3, 3)$
  - Create a cross product of this vector, we will find an orthogonal vector to the plane

$$\vec{P_1P_2} \times \vec{P_1P_3} = \begin{bmatrix} i & j & k \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{bmatrix} = 9i + j - 5k$$

  - Equation to the plane using normal vector and point P_1 is $9(x - 1) + (y - 2) - 5(z + 1) = 0$

THE PLANE CONTAINS LINE (X,Y,Z) IS PERPENDICULAR TO A PLANE

- **Question:** Find an equation of the plane that contains the line

$x = -2 + 3t$
$y = 4 + 2t$
$z = 3 - t$
and is perpendicular to the plane $x - 2y + z = 5$

- **Solution:**
  - Line A exist on plane A
  - Line B exist on plane B
  - Find the cross product of A and B, which results in a vector perpendicular to both A and B, therefore

$$u * n = \begin{bmatrix} i & j & k \\ 3 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix} = (0, -4, 8)$$

  - A point on the line x = −2 + 3t, y = 4 + 2t, z = 3 - t is $(-2, 4, 3)$
    plane A = $-4(y-4) - 8(z-3) = 0$

The normal vector $(0, -4, 8)$ is perpendicular to plane A and also to plane B, therefore plane A is perpendicular to plane B?

THE PLANE CONTAINS LINE (X,Y,Z) IS PARALLEL TO INTERSECTION OF TWO PLANES

- **Parallel:** The plane X that contains the line x = 3t, y = 1 + t, z = 2t and is parallel to the intersection of the planes
  plane A: y + z = −1
  plane B: 2x − y + z = 0

- If x = 0, then $y + z = -1$ and $(-y + z) = 0$ → $y = -1/2$ and $z = 1/2$
- If x = 1, then $y + z = -1$ and $(-y + z) = -2$ → $y = -3/2, y = 1/2$

- Therefore we have two points $P_1(0, -1/2, 1/2)$ and $P_2(1, -3/2, 1/2)$
- The vector for the line of intersection = $P_2 - P_1 = (1, 1, -1)$
- Plane X is parallel to the vector $v = (3, 1, 2)$

$$u * v = \begin{bmatrix} i & j & k \\ 1 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix} = (3, -5, -2)$$

Equation for plane X = $3x - 5(y-1) - 2z = 0$

6. SUMMARY

- **Intersection line of plane A and B is NORMAL to n_A and n_B**
- **If plane C is normal to plane A and B,** then its normal n_C is normal to n_A to n_B

11.5 - PARAMETRIC EQUATIONS OF LINES

- **Theorem:** If a line L has two points P, P_0 in which $P\vec{P}_0$ is parallel to a vector $\vec{v}(a,b,c)$, then PP_0 is a scalar multiple of v or:
$$P\vec{P}_0 = t\vec{v} = t(a,b,c)$$

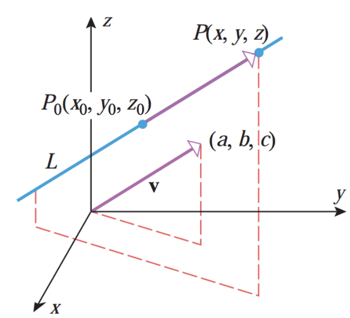
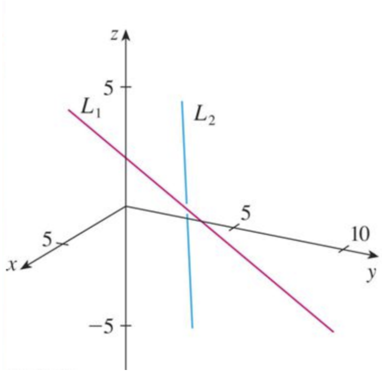


Figure 11.5.1

- **Parametric equation of a line (x,y,z)**
→ $(x - x_0) = ta, (y - y_0) = tb, (z - z0) = tc$
→ $x = ta + x_0, y = tb + y_0, z = tc + z_0$

- **Skew lines:** lines that are skew are not parallel and do not intersect, but instead they lie on different plane



Skew Lines

- **Distance between two parallel lines:** The shortest distance between two parallel lines is the length of the **perpendicular** segment between them.

HOMEWORK

Section 11.1: Problem 9, 11, 15, 23, 27, 37, 39, 41, 47, 49
Section 11.2: Problem 5, 7, 15 ,21 ,23, 29, 45, 47, 49, 53, 55
Section 11.3: Problem 3, 9, 13, 19, 25, 27, 33
Section 11.4: Problem 3, 11, 17, 19, 23, 29, 31, 33
Section 11.5: Problem 3, 7, 15, 17, 21, 29, 31, 47, 49
Section 11.6: Problem 3, 13, 15, 17, 19, 27, 29, 31, 33, 35

CHAPTER 11.1 — 3D

Problem 9: Find the center and radius of the sphere that has (1, −2, 4) and (3, 4, −12) as endpoints of a diameter

- Center = $(1+3)/2, (-2+4)/2, (4-12)/2 = (2, 1, -4)$
- $\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = r$
  So take midpoint and one point on the diameter:
  $\sqrt{(1)^2 + (3)^2 + (8)^2} = 8.6$

- **Problem 11:**
  (a) Show that (2, 1, 6), (4, 7, 9), and (8, 5, −6) are the vertices of a right triangle.
  (b) Which vertex is at the 90∘ angle?
  (c) Find the area of the triangle.

  - Find distance of 3 sides:
    Side A: $\sqrt{2^2+6^2+3^2}=7$
    Side B: $\sqrt{4^2+2^2+15^2}=\sqrt{245}$
    Side C: $\sqrt{36+16+144}=14$

    Since $a^2+b^2=c^2$ in a triangle, then $7^2+14^2=245$ which means it is a right triangle
  - The vertex that is 90^o at (2, 1, 6)
  - Area = $ab/2$ = $7*14/2=49$

- **Problem 15:**
  In each part, find an equation of the sphere with center (2, −1, −3) and satisfying the given condition.
  (a) Tangent to the xy-plane
  (b) Tangent to the xz-plane
  (c) Tangent to the yz-plane

  if it's tangent to the xy plane, that means its radius touch the xy-plane floor, so its radius is z
  a) $(x-2)^2+(y+1)^2+(z+3)^2=9$
  b) $(x-2)^2+(y+1)^2+(z+3)^2=1$
  c) $(x-2)^2+(y+1)^2+(z+3)^2=4$

Problem 23: Describe surface whose equation is given:
$x^2 + y^2 + z^2 + 10x + 4y + 2z - 19 = 0$

- Complete the square:
  $(x^2 + 10x + 25) + (y^2 + 4y + 4) + (z^2 + 2z + 1) = 49$
  $(x + 5)^2 + (y + 2)^2 + (z + 1)^2 = 11$
  So it's a sphere with center $(-5, -2, 1)$ and a radius of 7

Problem 25: Describe surface whose equation is given:
$2x^2 + 2y^2 + 2z^2 - 2x - 3y + 5z - 2 = 0$
$2(x^2 - x) + 2(y^2 - 1.5y) + 2(z^2 + 2.5z) = 2$

$2(x^2 - x + 0.5^2) + 2(y^2 - 1.5y + 0.75^2) + 2(z^2 + 2.5z + 1.25^2) = 6.75$
$2(x - 0.5)^2 + 2(y - 0.75)^2 + 2(z + 1.25)^2 = 6.75$
$(x - 0.5)^2 + (y - 0.75)^2 + (z + 1.25)^2 = 3.375$

So it's a sphere with center $(0.5, 0.75, -1.25)$ and a radius of $\sqrt{3.375}$

Problem 27: Describe surface whose equation is given:
$x^2 + y^2 + z^2 - 3x + 4y - 8z + 25 = 0$
$(x - 1.5)^2 + (y + 2)^2 + (z - 4)^2 = -25 + 22.5 = -2.75$
Since the radius is negative, this is invalid

CHAPTER 11.2 — VECTORS

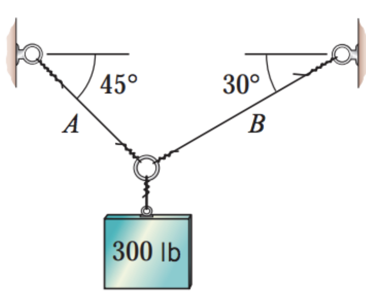
- **Problem 21:** Find unit vectors that satisfy the stated conditions.
  (a) Same direction as −i + 4j
  Component = (-1, 4)
  Unit vectors = $\frac{v}{||v||}$ = $\frac{(-1,4)}{\sqrt{1^2+4^2}} = \frac{-1}{\sqrt{17}}$ + $\frac{4}{\sqrt{17}}$

  (b) Oppositely directed to 6i − 4j + 2k.
  Component = (-6, 4, -2)
  Unit vectors = $\frac{v}{||v||}$ = $\frac{-6}{\sqrt{56}} + \frac{4}{\sqrt{56}} + \frac{-2}{\sqrt{56}}$

  (c) Same direction as the vector from the point A(−1, 0, 2) to the point B(3, 1, 1).
  $\vec{AB} = (3,1,1) - (-1,0,2) = (4,1,-1)$
  Unit vectors = $\frac{(4,1,-1)}{3\sqrt{56}}$

- **Problem 45:**
  - $F1 = 60(cos\, 90^o, sin\, 0^o) = (60, 0)$
  - $F2 = 30(cos\, 90^o, sin\, 90^o) = (0, 30)$
  - $F = F1 + F2 = (60, 30)$
  - Magnitude of $F = \sqrt{60^2 + 30^2} = 67.08\, lbs$

- **Problem 53:**
  - For A, the component is $A(cos\, 45^o, sin\, 45^o)$
  - For B, the component is $B(cos\, 30^o, sin\, 30^o)$
  - $A(cos\, 45^o) = B(cos\, 30^o)$
  - $A(sin\, 45^o) + B(sin\, 30^o) = 300$
  - $A = B(cos\, 30^o)/(cos\, 45^o) = 1.224B$
  - Therefore $0.86B + 0.5B = 300$
    $B = 219\, lbs$

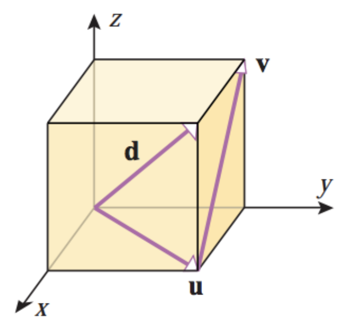


Problem 53

CHAPTER 11.3 - VECTOR PROJECTIONS

- **Problem 13:** Find r so that the vector from the point A(1, −1, 3) to the point B(3, 0, 5) is orthogonal to the vector from A to the point P (r, r, r).
  - $\vec{AB} = (3, 0, 5) - (1, -1, 3) = (2, 1, 2)$
  - $\vec{AP} = (r, r, r) - (1, -1, 3) = (r-1, r+1, r-3)$
  - So to be orthogonal, the dot product must be 0
    $(2r-2) + (r+1) + (2r-6) = 0$
    $5r - 7 = 0$
    $r = 5/7$

- **Problem 19:**



Problem 19

  - **a) Find the angle between d and u**
    $cos\theta = \frac{d \cdot u}{||d||||u||}$
    In the figure, $d = (a, a, a)$
    $u = (a, a, 0)$
    So

    $d \cdot u = (ai * ai) + (aj * aj) + (ak * 0) = a^2 + a^2$
    $||d|| = \sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = a\sqrt{3}$
    $||u|| = \sqrt{a^2 + a^2} = a\sqrt{2}$

    Therefore, $cos\theta = (2a^2)/a^2\sqrt{6}$
    $cos\theta = 2\sqrt{6}$
    $\theta = 35.26^o$

  - **b) Make a conjecture about the angle between the vectors d and v, and confirm your conjecture by computing the angle**
    $cos\theta = \frac{d \cdot v}{||d||||v||}$
    $v = (-a, 0, a)$
    Therefore $d \cdot v = -a^2 + 0 + a^2 = 0$

Since dot product of d and $v = 0$, this is a perpendicular angle
$\theta = 90^o$

- **Problem 25:** In each part, find the vector component of v along b and the vector component of v orthogonal to b
  - (a) v = 2i − j + 3k, b = i + 2j + 2k
    $v = (2, -1, 3)$ and $b = (1, 2, 2)$
    $v \cdot b = 6$
    - **The vector component of v along b :** $\frac{(v \cdot b)b}{||b||^2} = \frac{6(1,2,2)}{9} = (\frac{2}{3}, \frac{4}{3}, \frac{4}{3})$
    - **Vector component of v orthogonal to b:**
      $v - proj_b = (2, -1, 3) - (\frac{2}{3}, \frac{4}{3}, \frac{4}{3})$
      $= \frac{4}{3} - \frac{7}{3} + \frac{5}{3}$
  - b) $v = (4, -1, 7)$ and $b = (2, 3, -6)$
    - **The vector component of v along b:** $\frac{(v \cdot b)b}{||b||^2}$
      $= \frac{-37}{49}(2, -3, 6)$
    - **Vector component of v orthogonal to b:**
      $= (4, -1, 7) + \frac{37}{49}(2, 3, -6)$

- **Problem 27:** Express the vector v as the sum of a vector parallel to b and a vector orthogonal to b so $v = proj_b + ortho_b$
  - a) $v = (-3, 5)$ and $b = (1, 1)$
    $proj_b = \frac{2(1,1)}{2} = (1, 1)$
    $orthogonal = (-3, 5) - (1, 1) = (-4, 4)$
    $\vec{v} = (-3, 5) + (-4, 4)$

- **Problem 32:** If L is a line in 2-space or 3-space that passes through the points A and B, then the distance from a point P to the line L is equal to the length of the component of the vector $\vec{AP}$ that is orthogonal to the vector $\vec{AB}$. Use this result to find the distance from the point
  P (1, 0) to the line through A(2, −3) and B(5, 1).
  - $\vec{AP} = (1, 0) - (2, -3) = (-1, 3)$
  - $\vec{AB} = (5, 1) - (2, -3) = (3, 4)$

- $\vec{AP} \cdot \vec{AB} = 9$
- $||B||^2 = 25$
- $ortho_{AB} = \vec{AP} - proj_{AB} = (-1, 3) - \frac{9}{25}(3, 4) = (2.08, 1.56)$
- Distance from P to the line L = $\sqrt{2.08^2 + 1.56^2} = 2.6$

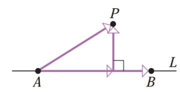


Figure Ex-32

- **Problem 33:** Find distance from point P (−3, 1, 2) to the line through A(1, 1, 0) and B(−2, 3, −4)

CHAPTER 11.4 - CROSS PRODUCTS

Problem 11: Find two unit vectors that are normal to the plane determined by the points A(0, −2, 1), B(1, −1, −2), and C(−1, 1, 0)

- Create two vectors that lie on the plane:
  $\vec{AB} = (1, -1, -2) - (0, -2, 1) = (1, 1, -3)$
  $\vec{BC} = (-1, 1, 0) - (1, -1, -2) = (-2, 2, 2)$

- If taken their cross product
  $$\begin{bmatrix} i & j & k \\ 1 & 1 & -3 \\ -2 & 2 & 2 \end{bmatrix} = (8, 4, 4)$$

- So the normal vector to the plane has component of $(8, 4, 4)$ and its opposite $-(8, 4, 4)$
- To find the unit vectors, we just divide that by the magnitude of the vector $\sqrt{8^2 + 4^2 + 4^2} = \sqrt{96}$
  $\frac{(8,4,4)}{\sqrt{96}}$

Problem 17: Find the area of the parallelogram that has u and v as adjacent sides.
u = i − j + 2k, v = 3j + k

$$\begin{bmatrix} i & j & k \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} = (-7, -1, 3)$$

- Area is the magnitude of this vector so $\sqrt{7^2 + 1^2 + 3^2} = \sqrt{59}$

Problem 19: Find area of triangle P (1, 5, −2), Q(0, 0, 0), R(3, 5, 1)

- Find the adjacent side of the triangle? Or can it be any 2 sides?
  $PQ = (-1, -5, 2)$
  $QR = (3, 5, 1)$

$$\begin{bmatrix} i & j & k \\ -1 & -5 & 2 \\ 3 & 5 & 1 \end{bmatrix} = (-15, 7, 10)$$

- That is magnitude of the vector: $\sqrt{15^2 + 7^2 + 10^2} = \frac{1}{2}\sqrt{374}$

Problem 29 : Consider the parallelepiped with adjacent edges

$u = 3i + 2j + k$
$v = i + j + 2k$
$w = i + 3j + 3k$

a) Find the volume:

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = abs(-9) = 9$$

b) Find the angle between u and the plane containing the face determined by v and w

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CHAPTER 11.5 - LINE

Problem 3: Find parametric equations for the line through P1 and P2 and also for the line segment joining those points.

(a) $P_1(3, -2)$ and $P_2(5, 1)$

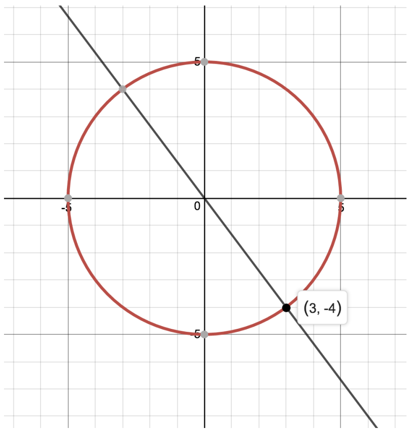
- Find the vector between $\vec{P_1P_2}$ = $(2, 3)$
- Use either P_1 or P_2
  $x = (3 + 2t)$
  $y = (-2 + 3t)$

b) $P_1(5, 2, -1)$, $P_2(2, 4, 2)$

- Find the vector between $\vec{P_1P_2}$ = $(3, 2, 3)$
- Use either P_1 or P_2
  $x = (5 + 3t)$
  $y = (2 + 2t)$
  $z = (-1 + 3t)$

Problem 17: Find an equation that passes through tangent line of the circle $x^2 + y^2 = 25$ that touches the point $(3, -4)$

- The vector of the black line is $(3, -4)$
- The dot product of the black line and the line L we need to find is $L(a, b)$:
  $3a - 4b = 0$
  - So the line is $3x - 4y$ therefore $y = (-4/3)x$
  - So the line is $y = 3/4x$
    Therefore the coordinates are $(4x, 3y)$
    $x = 4t + 3$
    $y = 3t - 4$



A line that is tangent to the circle at point (3,-4) is ***perpendicular*** *to the black line that is radius of the circle*

Problem 21: Find parametric equation for the line through (−2, 0, 5) that is parallel to the line given by x = 1 + 2t, y = 4 − t, z = 6 + 2t

Ans: So x = -2 + 2t, y = 0 - t, and z = 5 + 2t

Problem 29: Show that the lines L1 and L2 intersect, and find their point of intersection.

- L1 : x = 2 + t, y = 2 + 3t, z = 3 + t
- L2 : x = 2 + t, y = 3 + 4t, z = 4 + 2t

  Ans:

  - For the line to intersect, we need to find t_1 and t_2 such that

    $$2 + t_1 = 2 + t_2$$
    $$2 + 3t_1 = 3 + 4t_2$$
    $$3 + t_1 = 4 + 2t_2$$

    $$2 + t_1 - 2 - t_2 = 0$$
    $$2 + 3t_1 - 3 - 4t_2 = 0$$
    $$3 + t_1 - 4 - 2t_2 = 0$$

  - Matrix solver here

    $$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -4 & 1 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 3 & -4 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{(R3 - R1 → R3)}$$

Therefore $t_2(-1) = 1$ and $t_2 = -1$
$t_1 = -1$

- **Point of intersection:** Plug $(-1, -1)$ into the equation for (x, y, z) = $(1, -1, 2)$

Problem 47: Find the distance from the point P to the line L
P(-2,1,1)
L: x = 3 - t; y = t; z = 1 + 2t

- Line L is parallel to this vector $\vec{v} = (-1, 1, 2)$
- A point A on the line L is $(3, 0, 1)$
$\vec{AP} = (-2, 1, 1) - (3, 0, 1) = (-5, 1, 0)$
$\frac{||\vec{AP} \times \vec{AB}||}{||\vec{AB}||}$

$$||\vec{AP} \times \vec{AB}|| = \begin{bmatrix} i & j & k \\ -1 & 1 & 2 \\ -5 & 1 & 0 \end{bmatrix} = (-2, -10, 4) = \sqrt{4 + 100 + 16} = \sqrt{120}$$

$$||AB|| = \sqrt{1 + 1 + 4} = \sqrt{6}$$

Problem 49: Show that the lines L1 and L2 are parallel, and find the distance between them
L1 : $x = 2 - t, y = 2t$, z = 1 + t
L2 : x = 1 + 2t, y = 3 − 4t, z = 5 − 2t

- Prove they are parallel:
  $v =< -1, 2, 1 >$ and $v_2 =< 2, -4, -2 >$
  Since $v2 = -2v_1$ then the two lines are parallel to each other

- Let's pick a point P = $(2, 0, 1)$
  $L2$ is the line AB = $(2, -4, -2)$
  A point A on the line AB is $(1, 3, 5)$

- Distance from point P $(2, 0, 1)$ and point $A(1, 3, 5)$ can be found with
  $\vec{AP} = (2, 0, 1) - (1, 3, 5) = (1, -3, -4)$
- Use the cross product formula:
  $$||\vec{AB} \times \vec{AP}|| = \begin{bmatrix} i & j & k \\ 2 & -4 & -2 \\ 1 & -3 & -4 \end{bmatrix} = (10, 6, -2) = \sqrt{140}$$

  $$||\vec{AB}|| = \sqrt{4 + 16 + 4} = \sqrt{24}$$

  $$= \sqrt{140/24} = \sqrt{35/6}$$

CHAPTER 11.6 - PLANE

Problem 3: Find an equation of the plane that passes through the point P and has the vector n as a normal

- **Problem 15:**

1. Consider the plane $\mathcal{P} = 2x + y - 4z = 4$.

a) Find all points of intersection of $\mathcal{P}$ with the line

$$x = t, \quad y = 2 + 3t, \quad z = t.$$

b) Find all points of intersection of $\mathcal{P}$ with the line

$$x = 1 + t, \quad y = 4 + 2t, \quad z = t.$$

c) Find all points of intersection of $\mathcal{P}$ with the line

$$x = t, \quad y = 4 + 2t, \quad z = t.$$

Answer: a) To find the intersection we substitute the formulas for x, y and z into the equation for $\mathcal{P}$ and solve for t.

$$2(t) + (2 + 3t) - 4(t) = 4 \iff t = 2.$$

Now use $t = 2$ to find the point of intersection: $(x, y, z) = (2, 8, 2)$.

b) Substituting gives

$$2(1 + t) + (4 + 2t) - 4(t) = 4 \iff 6 = 4. \iff \text{no values of } t \text{ satisfy this equation.}$$

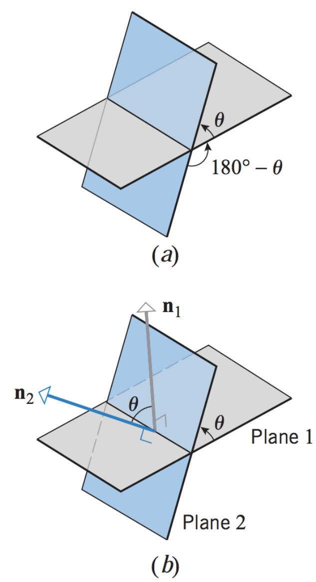
There are no points of intersection.

c) Substituting gives

$$2(t) + (4 + 2t) - 4(t) = 4 \iff 4 = 4. \iff \text{all values of } t \text{ satisfy this equation.}$$

- **Problem 27: Find equation for the plane that satisfies: The plane through the point (−1, 4, 2) that contains the line of intersection of the**
  - **plane A: 4x − y + z − 2 = 0**
  - **plane B: 2x + y − 2z − 3 = 0**

- **Step 1: Find the equation for the line of intersection between plane A and plane B:**
  - The line of intersection **(black line of blue plane and grey plane)** is NORMAL to plane A's normal (n_1) and plane B's normal n_2
  - $n_1 = (4, -1, 1)$ and $n_2 = (2, 1, -2)$
  - Do a cross product between n_1 and n_2



See bottom figure

$$\begin{bmatrix} i & j & k \\ 4 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$
$$= (2-1)i - (-8-2)j + (4+2)k = (1, 10, 6)$$

- **If x = 1,** then the line of intersection contains point $(1, 3, 1)$
- **So the equation for line of intersection:** $(1, 3, 1) + (1, 10, 6)t$

- **Step 2: find a vector that lies on plane C**
  - since the plane C contain the line of intersection, it must contain point $(1, 3, 1)$ and the question it said it also contain point $(1, -4, 2)$ so a vector lies on the plane is:

  $\vec{v_3}$ $=(1, 3, 1) - (-1, 4, 2) = (2, -1, -1)$

- **Step 3: find normal line of plane C for the equation**
  - Intersection line of plane (A, B, C) = $(1, 10, 6)$
  - Vector $(-2, -1, -1)$ lies on plane C
  - So normal equation
- A normal vector to both the vector $(1, 10, 6)$ and $(2, -1, -1)$:

$$\begin{bmatrix} i & j & k \\ 2 & -1 & -1 \\ 1 & 10 & 6 \end{bmatrix} = (4, 13, 21)$$

- So equation of line is $4x + 13y + 21z = 0$
  but since it contains point $(-1, 4, 2)$ we can make it
  $4(x+1) + 13(y-4) + 21(z-2) = 0$

Problem 29: Find equation of the plane through (1, 2, −1) that is perpendicular to the line of intersection of the planes 2x + y + z = 2 and x + 2y + z = 3.

- **Step 1:** Find the intersection line L_{ab} of Plane A and B
  - Find line of intersection's direction: $\begin{bmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ = $(-1, -1, 3)$

- **Step 2:** If plane C is perpendicular to L_{ab} then its normal $n_c$is also perpendicular to n_a and n_b

$$\begin{bmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$-1(x+1) - 1(y+2) + 3(z-1) = 0$$
$$-1 - x - 2 - y + 3z - 3 = 0$$
$$-6 - x - y + 3z = 0$$
$$-x - y + 3z = 6$$

Problem 31: The plane through (−1, 2, −5) that is perpendicular to the planes 2x − y + z = 1 and x + y − 2z = 3.

- So the normal line of this plan is also perpendicular to the plane (2, -1, 1) and (1, 1, -2)

$\begin{bmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ = (1, 5, 3)

- A point on the plane is (-1, 2, -5) therefore equation of plane

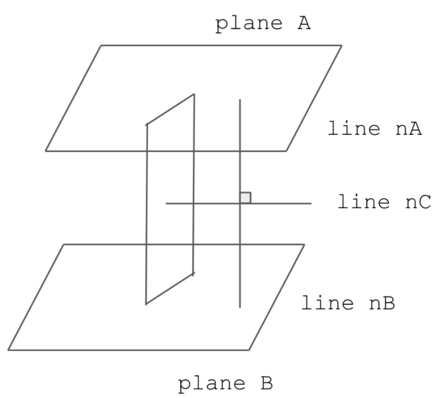
$$x + 5y + 3z = (-1 * 1 + 2 * 5 + 3 * -5) = -6$$

Problem 33: Find equation for the plane X whose points are equidistant from (2, −1, 1) and (3, 1, 5).

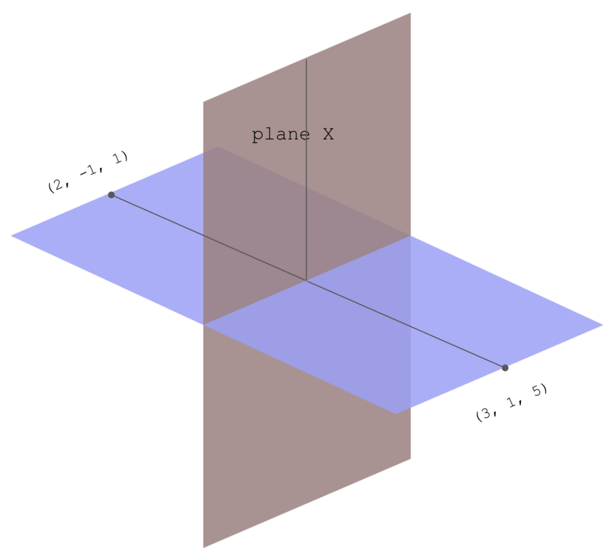
- So the 1/2 distance between these two points are:
  $[(2, -1, 1) + (3, 1, 5)]/2 = (2.5, 0, 3)$

  A line between these two points are normal to the plane X:
  $(2, -1, 1) - (3, 1, 5) = (-1, -2, -4)$

  $-x + -2y + -4z = -1 * 2.5 + 0 + 3 * -4 = -14.5$
- **So plane X =** $-x + -2y + -4z = -14.5$



Problem 31: *if plane C is perpendicular to plane A + B, then normal line to C* (n_C) *is perpendicular to norma line of A* (n_A) *and normal line of B* (n_B)



Problem 33

Problem 35: Find parametric equations of the line through the point (5, 0, −2)that is parallel to the planes x − 4y + 2z = 0 and 2x + 3y − z +1 = 0.

- If a line is parallel to plane A and plane B, then it is perpendicular to both the normal of the planes
$$\begin{bmatrix} i & j & k \\ 1 & -4 & 2 \\ 2 & 3 & -1 \end{bmatrix} = (-2, 5, 11)$$

So the parametric equation $x = 5 - 2t,\ y = 5t$ and $z = -2 + 11t$
