Time Series Cheat Sheet

By Marcelo Moreno - King Juan Carlos University
As part of the Econometrics Cheat Sheet Project

THIS IS A WORK IN PROGRESS

NOT INTENDED FOR GENERAL PURPOSE YET

Basic concepts

Definitions

Time series - is a succession of quantitative observations of a phenomena ordered in time.

There are some variations of time series:

- **Panel data** - consist of a time series for each observation of a cross section.
- **Pooled cross sections** - combines cross sections from different time periods.

Stochastic process - sequence of random variables that are indexed in time.

Components of a time series

- **Trend** - is the long-term general movement of a series.
- **Seasonal variations** - are periodic oscillations that are produced in a period equal or inferior to a year, and can be easily identified on different years (usually are the result of climatology reasons).
- **Cyclical variations** - are periodic oscillations that are produced in a period greater than a year (are the result of the economic cycle).
- **Residual variations** - are movements that do not follow a recognizable periodic oscillation (are the result of eventual non-permanent phenomena that can affect the studied variable in a given moment).

Type of time series models

- **Static models** - the relation between y and x's is contemporary. Conceptually:
$$y_t = \beta_0 + \beta_1 x_t + u_t$$
- **Distributed-lag models** - the relation between y and x's is not contemporary. Conceptually:
$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + ... + \beta_{q+1} x_{t-q} + u_t$$
The long term cumulative effect in y when Δx is:
$$\beta_1 + \beta_2 + ... + \beta_{q+1}$$

Assumptions and properties

OLS model assumptions under time series

Under this assumptions, the estimators of the OLS parameters will present good properties. **Gauss-Markov assumptions extended applied to time series**:

ts1. **Parameters linearity and weak dependence**.
   a. y_t must be a linear function of the β's.
   b. The stochastic $\{(x_t, y_t) : t = 1, 2, ..., T\}$ is stationary and weakly dependent.

ts2. **No perfect collinearity**.
   - There are no independent variables that are constant: $Var(x_j) \neq 0$
   - There is not an exact linear relation between independent variables.

ts3. **Conditional mean zero and correlation zero**.
   a. There are no systematic errors: $E(u_t|x_{1t}, ..., x_{kt}) = E(u_t) = 0 \rightarrow$ **strong exogeneity** (a implies b).
   b. There are no relevant variables left out of the model: $Cov(x_{jt}, u_t) = 0$ for any $j = 1, ..., k \rightarrow$ **weak exogeneity**.

ts4. **Homoscedasticity**. The variability of the residuals is the same for any t: $Var(u_t|x_{1t}, ..., x_{kt}) = \sigma^2$

ts5. **No auto-correlation**. The residuals do not contain information about other residuals: $Corr(u_t, u_s|x) = 0$ for any given $t \neq s$.

ts6. **Normality**. The residuals are independent and identically distributed (**i.i.d.** so on): $u \sim N(0, \sigma^2)$

ts7. **Data size**. The number of observations available must be greater than $(k+1)$ parameters to estimate. (IS already satisfied under asymptotic situations)

Asymptotic properties of OLS

Under the econometric model assumptions and the Central Limit Theorem:

- Hold (1) to (3a): OLS is **unbiased**. $E(\hat{\beta}_j) = \beta_j$
- Hold (1) to (3): OLS is **consistent**. $plim(\hat{\beta}_j) = \beta_j$ (to (3b) left out (3a), weak exogeneity, biased but consistent)
- Hold (1) to (5): **asymptotic normality** of OLS (then, (6) is necessarily satisfied): $u \sim_a N(0, \sigma^2)$.
- Hold (1) to (5): **unbiased estimate of** σ^2. $E(\hat{\sigma}^2) = \sigma^2$
- Hold (1) to (5): OLS is BLUE (Best Linear Unbiased Estimator) or **efficient**.
- Hold (1) to (6): hypothesis testing and confidence intervals can be done reliably.

Trends and seasonality

Spurious regression - is when the relation between y and x is due to factors that affect y and have correlation with x, $Corr(x, u) \neq 0$.

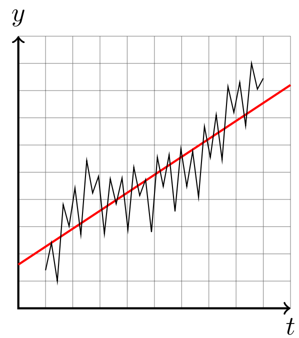
Trends

Two time series can have the same (or contrary) trend, that should lend to a high level of correlation. This, can provoke a false appearance of causality, the problem is in what is known as **spurious regression** (the non-fulfillment of ts3). For example, given the model:
$$y_t = \beta_0 + \beta_1 x_t + u_t$$
where:
$$y_t = \alpha_0 + \alpha_1 Trend + v_t$$
$$x_t = \gamma_0 + \gamma_1 Trend + v_t$$
Adding a trend to the model can solve the problem:
$$y_t = \beta_0 + \beta_1 x_t + \beta_2 Trend + u_t$$
The trend can be linear or non-linear (quadratic, cubic, exponential, etc.)

Seasonality

A time series with can manifest seasonality. That is, the series is subject to a seasonal variations or pattern, usually related to climatology conditions.

For example, the GDP is usually higher in summer and lower in winter.



This problem is part of what is known as **spurious regression** (the non-fulfillment of ts3). A seasonal adjustment can solve the problem.

A simple seasonal adjustment could be creating stational binary variables and adding them to the model. For example, for quarterly series:
$$y_t = \beta_0 + \beta_1 Q2_t + \beta_2 Q3_t + \beta_3 Q4_t + \beta_4 x_{1t} + ... + \beta_k x_{kt} + u_t$$
Another way is to seasonaly adjust (sa) the variables, and then, do the regression with the final regression:
$$z_t = \beta_0 + \beta_1 Q2_t + \beta_2 Q3_t + \beta_3 Q4_t + v_t \rightarrow (sa)\tilde{v}_t = (sa)\tilde{z}_t$$
$$(sa)\tilde{y}_t = \beta_0 + \beta_1 (sa)\tilde{x}_{1t} + ... + \beta_k (sa)\tilde{x}_{kt} + u_t$$
There are much better and complex methods to seasonaly adjust a time series.

Stationarity and weak dependence

Stationarity means stability of the joints distributions of a process as time progresses. It allows to correctly identify the relations –that stay unchange with time– between variables.

Stationary and non-stationary process

- **Stationary process** (strong stationarity) - is the one in that the probability distribution are stable in time: if any collection of random variables is taken, and then are shifted h periods, the joint probability distribution should stay unchanged.
- **Non-stationary process** - is, for example, a series with trend, where at least the mean changes with the time.
- **Covariance stationary process** - is a weaker form of stationarity:
  - $E(x_t)$ is constant.
  - $Var(x_t)$ is constant.
  - For any t, $h \geq 1$, the $Cov(x_t, x_{t+h})$ depends only of h, not of t.

Weak dependence time series

It is important because it replaces the random sampling assumption, giving for granted the validity of the central limit theorem (requires stationality and a form of weak dependence). Weakly dependent process are **also known as integrated of order zero, I(0)**.

- **Weak dependence** - restricts how close the relationship between x_t and x_{t+h} can be as the time distance between the series increases (h).

A stationary time process $\{x_t : t = 1, 2, ..., T\}$ is weakly dependent when x_t and x_{t+h} are almost independent as h increases without a limit.
A covariance stationary time process is weakly dependent if the correlation between x_t and x_{t+h} tends to 0 fast enough when $h \to \infty$ (they are not asymptotically correlated).
Some examples of stationary and weakly dependent time series are:

- **Moving average** - $\{x_t\}$ is a moving average of order one MA(1).
$$x_t = e_t + \alpha_1 e_{t-1}$$
where $\{e_t : t = 0, 1, ..., T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
- **Auto-regressive process** - $\{y_t\}$ is a auto-regressive process of order one AR(1).
$$y_t = \rho_1 y_{t-1} + e_t$$
where $\{e_t : t = 0, 1, ..., T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
If $|\rho_1| < 1$, then $\{y_t\}$ is an AR(1) stable process that is weakly dependent. It is stationary in covariance, $Corr(y_t, y_{t+1}) = \rho_1$.

A series with a trend cannot be stationary, but can be weakly dependent (and stationary if the series is de-trended).

Strong dependence time series

Most of the time, economics series are strong dependent (or high persistent in time). Some special cases of unit root processes:

- **Random walk** - an AR(1) process with $\rho_1 = 1$.
$$y_t = y_{t-1} + e_t$$
where $\{e_t : t = 1, 2, ..., T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
This is a random walk ($\{e_t\}$ *i.i.d.* is the reason). It is not stationary, is persistent.
- **Random walk with a drift** - an AR(1) process with $\rho_1 = 1$ and a constant.
$$y_t = \alpha_0 + y_{t-1} + e_t$$
where $\{e_t : t = 1, 2, ..., T\}$ is an *i.i.d.* sequence with zero mean and σ_e^2 variance.
This is a random walk ($\{e_t\}$ *i.i.d.* is the reason). It is not stationary, is persistent.

Transforming unit root to weak dependent

Unit root processes are integrated of order one, I(1). This means that the first difference of the process is weakly dependent (and usually, stationary). For example, a random walk:
$$\Delta y_t = y_t - y_{t-1} = e_t$$
where $\{e_t\} = \{\Delta y_t\}$ is *i.i.d.*
Usually, I(1) series are converted to logarithms in order to, getting the first difference to achieve weak dependence (and possibly stationarity), and obtain the (approx.) percentage change:
$$\Delta log(y_t) = log(y_t) - log(y_{t-1}) \approx (y_t - y_{t-1})/y_{t-1}$$
Getting the first difference of a series also deletes its trend.
[GRÁFICO]

I(1) detection

- The approx. way: in a AR(1) model (if the series have a trend, is better to de-trend before measuring the AC).

In a AR(1) model:
- If $|\rho_1| < 1 \to I(0)$
- If $|\rho_1| = 1 \to I(1)$
- If $|\rho_1| > 1 \to tryAR(2)andtest$

Predictions

Two types of prediction:

- Of the mean value of y for a specific value of x.
- Of an individual value of y for a specific value of x.

If the values of the variables (x) approximate to the mean values ($\overline{x}$), the confidence interval amplitude of the prediction will be shorter.

Heteroscedasticity in time series

The residuals u_i of the population regression function do not have the same variance σ^2:

$$Var(u|x) = Var(y|x) \neq \sigma^2$$

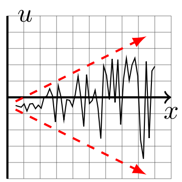
Is the **breaking of the fifth (5) econometric model assumption**.

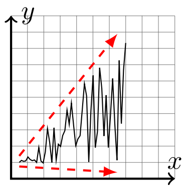
Consequences

- OLS estimators still are unbiased.
- OLS estimators still are consistent.
- OLS is **not efficient** anymore, but still a LUE (Linear Unbiased Estimator).
- **Variance estimations of the estimators are biased**: the construction of confidence intervals and the hypothesis contrast are not reliable.

Detection

- **Graphical analysis** - look for scatter patterns on x vs. u or x vs. y plots.





- **Formal tests** - White, Bartlett, Breusch-Pagan, etc. Commonly, the null hypothesis: H_0 = Homoscedasticity

Correction

- Use OLS with a variance-covariance matrix estimator robust to heteroscedasticity, for example, the one proposed by White.
- If the variance structure is known, make use of Weighted Least Squares (WLS) or Generalized Least Squares (GLS).
- If the variance structure is not known, make use of Feasible Weighted Least Squared (FWLS), that estimates a possible variance, divides the model variables by it and then apply OLS.
- Make assumptions about the possible variance:
  - Supposing that σ_i^2 is proportional to x_i, divide the model variables by the square root of x_i and apply OLS.
  - Supposing that σ_i^2 is proportional to x_i^2, divide the model variables by x_i and apply OLS.
- Make a new model specification, for example, logarithmic transformation.

Auto-correlation

The residual of any observation, u_t, is correlated with the residual of any other observation. The observations are not independent.

$$Corr(u_t, u_s|x) \neq 0 \text{ for any } t \neq s$$

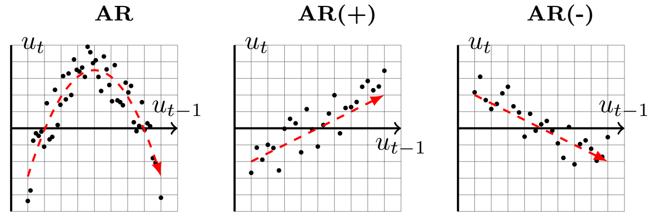
The "natural" context of this phenomena is time series. Is the **breaking of the sixth (6) econometric model assumption**.

Consequences

- OLS estimators still are unbiased.
- OLS estimators still are consistent.
- OLS is **not efficient** anymore, but still a LUE (Linear Unbiased Estimator).
- **Variance estimations of the estimators are biased**: the construction of confidence intervals and the hypothesis contrast are not reliable.

Detection

- **Graphical analysis** - look for scatter patterns on u_{t-1} vs. u_t or make use of a correlogram.



- **Formal tests** - Durbin-Watson, Breusch-Godfrey, etc. Commonly, the null hypothesis: H_0 : No auto-correlation

DURBIN WATSON EXPLANATION AND GRAPHICAL ANALYSIS (AND MOST COMMON VALUES?)

Correction

- Use OLS with a variance-covariance matrix estimator robust to auto-correlation, for example, the one proposed by Newey-West.
- Use Generalized Least Squares. Supposing $y_t = \beta_0 + \beta_1 x_t + u_t$, with $u_t = \rho u_{t-1} + \varepsilon_t$, where $|\rho| < 1$ and ε_t is white noise.
  - If ρ is known, create a quasi-differentiated model where u_t is white noise and estimate it by OLS.
  - If ρ is not known, estimate it by -for example- the Cochrane-Orcutt method, create a quasi-differentiated model where u_t is white noise and estimate it by OLS.