Additional Cheat Sheet

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The Econometrics Cheat Sheet Project

OLS matrix notation

The general econometric model:

$$y_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_k x_{ki} + u_i$$

Can be written in matrix notation as:

$$y = X\beta + u$$

Let's call $\hat{u}$ the vector of estimated residuals ($\hat{u} \neq u$):

$$\hat{u} = y - X\hat{\beta}$$

The **objective** of OLS is to **minimize** the SSR:

$$\min \mathrm{SSR} = \min \sum_{i=1}^n \hat{u}_i^2 = \min \hat{u}^\top \hat{u}$$

- Defining $\hat{u}^\top \hat{u}$:
$$\hat{u}^\top \hat{u} = (y - X\hat{\beta})^\top (y - X\hat{\beta}) =$$
$$= y^\top y - 2\hat{\beta}^\top X^\top y + \hat{\beta}^\top X^\top X \hat{\beta}$$
- Minimizing $\hat{u}^\top \hat{u}$:
$$\frac{\partial \hat{u}^\top \hat{u}}{\partial \hat{\beta}} = -2X^\top y + 2X^\top X\hat{\beta} = 0$$
$$\hat{\beta} = (X^\top X)^{-1}(X^\top y)$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} n & \sum x_1 & \dots & \sum x_k \\ \sum x_1 & \sum x_1^2 & \dots & \sum x_1 x_k \\ \vdots & \vdots & \ddots & \vdots \\ \sum x_k & \sum x_k x_1 & \dots & \sum x_k^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sum y \\ \sum y x_1 \\ \vdots \\ \sum y x_k \end{bmatrix}$$

The second derivative $\frac{\partial^2 \hat{u}^\top \hat{u}}{\partial \hat{\beta}^2} = X^\top X > 0$ (is a min.)

Variance-covariance matrix of $\hat{\beta}$

Has the following form:

$$\mathrm{Var}(\hat{\beta}) = \hat{\sigma}_u^2 \cdot (X^\top X)^{-1} =$$
$$= \begin{bmatrix} \mathrm{Var}(\hat{\beta}_0) & \mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \dots & \mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_k) \\ \mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_0) & \mathrm{Var}(\hat{\beta}_1) & \dots & \mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(\hat{\beta}_k, \hat{\beta}_0) & \mathrm{Cov}(\hat{\beta}_k, \hat{\beta}_1) & \dots & \mathrm{Var}(\hat{\beta}_k) \end{bmatrix}$$

where: $\hat{\sigma}_u^2 = \frac{\hat{u}^\top \hat{u}}{n-k-1}$

The standard errors are in the diagonal of:

$$\mathrm{se}(\hat{\beta}) = \sqrt{\mathrm{Var}(\hat{\beta})}$$

Error measurements

- $\mathrm{SSR} = \hat{u}^\top \hat{u} = y^\top y - \hat{\beta}^\top X^\top y = \sum (y_i - \hat{y}_i)^2$
- $\mathrm{SSE} = \hat{\beta}^\top X^\top y - n\overline{y}^2 = \sum (\hat{y}_i - \overline{y})^2$
- $\mathrm{SST} = \mathrm{SSR} + \mathrm{SSE} = y^\top y - n\overline{y}^2 = \sum (y_i - \overline{y})^2$

Variance-covariance matrix of u

Has the following shape:

$$\mathrm{Var}(u) = \begin{bmatrix} \mathrm{Var}(u_1) & \mathrm{Cov}(u_1, u_2) & \dots & \mathrm{Cov}(u_1, u_n) \\ \mathrm{Cov}(u_2, u_1) & \mathrm{Var}(u_2) & \dots & \mathrm{Cov}(u_2, u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(u_n, u_1) & \mathrm{Cov}(u_n, u_2) & \dots & \mathrm{Var}(u_n) \end{bmatrix}$$

When there is no heterocedasticity and no auto-correlation, the variance-covariance matrix of u has the form:

$$\mathrm{Var}(u) = \sigma_u^2 \cdot I_n = \begin{bmatrix} \sigma_u^2 & 0 & \dots & 0 \\ 0 & \sigma_u^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_u^2 \end{bmatrix}$$

where I_n is an identity matrix of $n \times n$ elements.

When there is **heterocedasticity** and **auto-correlation**, the variance-covariance matrix of u has the shape:

$$\mathrm{Var}(u) = \sigma_u^2 \cdot \Omega = \begin{bmatrix} \sigma_{u_1}^2 & \sigma_{u_{12}} & \dots & \sigma_{u_{1n}} \\ \sigma_{u_{21}} & \sigma_{u_2}^2 & \dots & \sigma_{u_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{u_{n1}} & \sigma_{u_{n2}} & \dots & \sigma_{u_n}^2 \end{bmatrix}$$

where $\Omega \neq I_n$.

- Heterocedasticity: $\mathrm{Var}(u) = \sigma_{u_i}^2 \neq \sigma_u^2$
- Auto-correlation: $\mathrm{Cov}(u_i, u_j) = \sigma_{u_{ij}} \neq 0, \ \forall i \neq j$

Variable omission

Most of the time, is hard to get all relevant variables for an analysis. For example, a true model with all variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + v$$

where $\beta_2 \neq 0$, v is the error term and $\mathrm{Cov}(v|x_1, x_2) = 0$.

The model with the available variables:

$$y = \alpha_0 + \alpha_1 x_1 + u$$

where $u = v + \beta_2 x_2$.

Relevant variable omission causes OLS estimators to be **biased** and **inconsistent**, because there is no weak exogeneity, $\mathrm{Cov}(x_1, u) \neq 0$. Depending on the $\mathrm{Corr}(x_1, x_2)$ and the sign of β_2, the bias on $\hat{\alpha}_1$ could be:

	$\mathrm{Corr}(x_1, x_2) > 0$	$\mathrm{Corr}(x_1, x_2) < 0$
$\beta_2 > 0$	(+) bias	(−) bias
$\beta_2 < 0$	(−) bias	(+) bias

- (+) bias: $\hat{\alpha}_1$ will be higher than it should be (it includes the effect of x_2) $\rightarrow \hat{\alpha}_1 > \beta_1$
- (−) bias: $\hat{\alpha}_1$ will be lower than it should be (it includes the effect of x_2) $\rightarrow \hat{\alpha}_1 < \beta_1$

If $\mathrm{Corr}(x_1, x_2) = 0$, there is no bias on $\hat{\alpha}_1$, because the effect of x_2 will be fully picked up by the error term, u.

Variable omission correction

Proxy variables

Is the approach when a relevant variable is not available because it is non-observable, and there is no data available.

- A **proxy variable** is something **related** with the non-observable variable that has data available.

For example, the GDP per capita is a proxy variable for the life quality (non-observable).

Instrumental variables

When the variable of interest (x) is observable but **endogenous**, the proxy variables approach is no longer valid.

- An **instrumental variable** (IV) **is an observable variable** (z) that is **related** with the variable of interest that is endogenous (x), and meet the **requirements**:
$$\mathrm{Cov}(z, u) = 0 \rightarrow \text{instrument exogeneity}$$
$$\mathrm{Cov}(z, x) \neq 0 \rightarrow \text{instrument relevance}$$

Instrumental variables let the omitted variable in the error term, but instead of estimate the model by OLS, it utilizes a method that recognizes the presence of an omitted variable. It can also solve error measurement problems.

- **Two-Stage Least Squares** (TSLS) is a method to estimate a model with multiple instrumental variables. The $\mathrm{Cov}(z, u) = 0$ requirement can be relaxed, but there has to be a minimum of variables that satisfies it.

  The TSLS **estimation procedure** is as follows:
  1. Estimate a model regressing x by z using OLS, obtaining $\hat{x}$:
  $$\hat{x} = \hat{\pi}_0 + \hat{\pi}_1 z$$
  2. Replace x by $\hat{x}$ in the final model and estimate it by OLS:
  $$y = \beta_0 + \beta_1 \hat{x} + u$$

  There are some <u>important</u> things to know about TSLS:
  - TSLS estimators are less efficient than OLS when the explanatory variables are exogenous. The **Hausman test** can be used to check it:
  $$H_0\text{: OLS estimators are consistent.}$$
  If H_0 is accepted, the OLS estimators are better than TSLS and vice versa.
  - There could be some (or all) instrument that are not valid. This is known as over-identification, **Sargan test** can be used to check it:
  $$H_0\text{: all instruments are valid.}$$

Information criterion

It is used to compare models with different number of parameters (p). The general formula:

$$\text{Cr}(p) = \log(\tfrac{\text{SSR}}{n}) + c_n \varphi(p)$$

where:

- SSR is the Sum of Squared Residuals from a model of order p.
- c_n is a sequence indexed by the sample size.
- $\varphi(p)$ is a function that penalizes large p orders.

Is interpreted as the relative amount of information lost by the model. The p order that min. the criterion is chosen.

There are different $c_n \varphi(p)$ functions:

- Akaike: $\text{AIC}(p) = \log(\frac{\text{SSR}}{n}) + \frac{2}{n} p$
- Hannan-Quinn: $\text{HQ}(p) = \log(\frac{\text{SSR}}{n}) + \frac{2\log(\log(n))}{n} p$
- Schwarz: $\text{Sc}(p) = \log(\frac{\text{SSR}}{n}) + \frac{\log(n)}{n} p$

$\text{Sc}(p) \leq \text{HQ}(p) \leq \text{AIC}(p)$

The non-restricted hypothesis test

Is an alternative to the F test when there are few hypothesis to test on the parameters. Let β_i, β_j be parameters, $a, b, c \in \mathbb{R}$ are constants.

- $H_0 : a\beta_i + b\beta_j = c$
- $H_1 : a\beta_i + b\beta_j \neq c$

$$\text{Under } H_0: \quad t = \frac{a\hat{\beta}_i + b\hat{\beta}_j - c}{\sqrt{\text{Var}(a\hat{\beta}_i + b\hat{\beta}_j)}}$$

$$= \frac{a\hat{\beta}_i + b\hat{\beta}_j - c}{\sqrt{a^2 \text{Var}(\hat{\beta}_i) + b^2 \cdot \text{Var}(\hat{\beta}_j) \pm 2ab\text{Cov}(\hat{\beta}_i, \hat{\beta}_j)}}$$

If $|t| > |t_{n-k-1,\alpha/2}|$, there is evidence to reject H_0.

ANOVA

Decompose the total sum of squared in sum of squared residuals and sum of squared explained: $\text{SST} = \text{SSR} + \text{SSE}$

Variation origin	Sum Sq.	df	Sum Sq. Avg.
Regression	SSE	k	SSE$/k$
Residuals	SSR	$n-k-1$	SSR$/(n-k-1)$
Total	SST	$n-1$	

The F statistic:

$$F = \frac{\text{SSA of SSE}}{\text{SSA of SSR}} = \frac{\text{SSE}}{\text{SSR}} \cdot \frac{n-k-1}{k} \sim F_{k,n-k-1}$$

If $F_{k,n-k-1} < F$, there is evidence to reject H_0.

Incorrect functional form

To check if the model **functional form** is correct, we can use **Ramsey's RESET** (Regression Specification Error Test). It test the original model vs. a model with variables in powers.

H_0: the model is correctly specified.

Test procedure:

1. Estimate the original model and obtain $\hat{y}$ and R^2:
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_k x_k$$
2. Estimate a new model adding powers of $\hat{y}$ and obtain the new R^2_{new}:
$$\tilde{y} = \hat{y} + \tilde{\gamma}_2 \hat{y}^2 + \cdots + \tilde{\gamma}_l \hat{y}^l$$
3. Define the test statistic, under $\gamma_2 = \cdots = \gamma_l = 0$ as null hypothesis:
$$F = \frac{R^2_{\text{new}} - R^2}{1 - R^2_{\text{new}}} \cdot \frac{n-(k+1)-l}{l} \sim F_{l,n-(k+1)-l}$$

If $F_{l,n-(k+1)-l} < F$, there is evidence to reject H_0.

Logistic regression

When there is a binary (0, 1) dependent variable, the linear regression model is no longer valid, we can use logistic regression instead. For example, a **logit model**:

$$P_i = \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_i + u_i)}} = \frac{e^{\beta_0 + \beta_1 x_i + u_i}}{1 + e^{\beta_0 + \beta_1 x_i + u_i}}$$

where $P_i = \text{E}(y_i = 1 \mid x_i)$ and $(1 - P_i) = \text{E}(y_i = 0 \mid x_i)$

The **odds ratio** (in favor of $y_i = 1$):

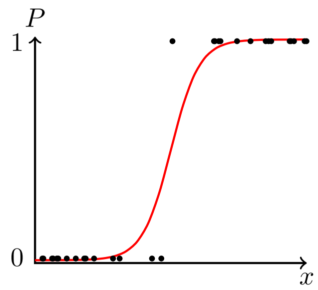
$$\frac{P_i}{1 - P_i} = \frac{1 + e^{\beta_0 + \beta_1 x_i + u_i}}{1 + e^{-(\beta_0 + \beta_1 x_i + u_i)}} = e^{\beta_0 + \beta_1 x_i + u_i}$$

Taking the natural logarithm of the odds ratio, we obtain the **logit**:

$$L_i = \ln\left(\frac{P_i}{1 - P_i}\right) = \beta_0 + \beta_1 x_i + u_i$$

P_i is between 0 and 1, but L_i goes from $-\infty$ to $+\infty$.

If L_i is positive, it means that when x_i increments, the probability of $y_i = 1$ increases, and vice versa.



Statistical definitions

Let ξ, η be random variables, $a, b \in \mathbb{R}$ constants, and P denotes probability.

Mean

Definition: $\quad E(\xi) = \sum_{i=1}^{n} \xi_i \cdot P[\xi = \xi_i]$

Population mean: $\text{E}(\xi) = \dfrac{1}{N} \sum_{i=1}^{N} \xi_i$

Sample mean: $\text{E}(\xi) = \dfrac{1}{n} \sum_{i=1}^{n} \xi_i$

Some properties:

- $\text{E}(a) = a$
- $\text{E}(\xi + a) = \text{E}(\xi) + a$
- $\text{E}(a \cdot \xi) = a \cdot \text{E}(\xi)$
- $\text{E}(\xi \pm \eta) = \text{E}(\xi) + \text{E}(\eta)$
- $\text{E}(\xi \cdot \eta) = \text{E}(\xi) \cdot \text{E}(\eta)$ only if ξ and η are independent.
- $\text{E}(\xi - \text{E}(\xi)) = 0$
- $\text{E}(a \cdot \xi + b \cdot \eta) = a \cdot \text{E}(\xi) + b \cdot \text{E}(\eta)$

Variance

Definition: $\quad \text{Var}(\xi) = \text{E}(\xi - \text{E}(\xi))^2$

Population variance: $\text{Var}(\xi) = \dfrac{\sum_{i=1}^{N} (\xi_i - \text{E}(\xi))^2}{N}$

Sample variance: $\text{Var}(\xi) = \dfrac{\sum_{i=1}^{n} (\xi_i - \text{E}(\xi))^2}{n-1}$

Some properties:

- $\text{Var}(a) = 0$
- $\text{Var}(\xi + a) = \text{Var}(\xi)$
- $\text{Var}(a \cdot \xi) = a^2 \cdot \text{Var}(\xi)$
- $\text{Var}(\xi \pm \eta) = \text{Var}(\xi) + \text{Var}(\eta) \pm 2 \cdot \text{Cov}(\xi, \eta)$
- $\text{Var}(a \cdot \xi \pm b \cdot \eta) = a^2 \cdot \text{Var}(\xi) + b^2 \cdot \text{Var}(\eta) \pm 2ab \cdot \text{Cov}(\xi, \eta)$

Covariance

Definition: $\quad \text{Cov}(\xi, \eta) = \text{E}[(\xi - E(\xi)) \cdot (\eta - E(\eta))]$

Population covariance: $\dfrac{\sum_{i=1}^{N} (\xi_i - \text{E}(\xi)) \cdot (\eta_i - \text{E}(\eta))}{N}$

Sample covariance: $\dfrac{\sum_{i=1}^{n} (\xi_i - \text{E}(\xi)) \cdot (\eta_i - \text{E}(\eta))}{n-1}$

Some properties:

- $\text{Cov}(\xi, a) = 0$
- $\text{Cov}(\xi + a, \eta + b) = \text{Cov}(\xi, \eta)$
- $\text{Cov}(a \cdot \xi, b \cdot \eta) = ab \cdot \text{Cov}(\xi, \eta)$
- $\text{Cov}(\xi, \xi) = \text{Var}(\xi)$
- $\text{Cov}(\xi, \eta) = \text{Cov}(\eta, \xi)$

VAR (Vector Autoregressive)

A VAR model captures **dynamic interactions** between time series variables. The VAR(p):

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + B_0 x_t + \cdots + B_q x_{t-q} + C D_t + u_t$$

where:

- $y_t = (y_{1t}, \ldots, y_{Kt})^\intercal$ is a vector of K observable endogenous time series variables.
- A_i's are $K \times K$ coefficient matrices.
- $x_t = (x_{1t}, \ldots, x_{Mt})^\intercal$ is a vector of M observable exogenous time series variables.
- B_j's are $K \times M$ coefficient matrices.
- D_t is a vector that contains all deterministic terms, that may be a: constant, linear trend, seasonal dummy, and/or any other user specified dummy variables.
- C is a coefficient matrix of suitable dimension.
- $u_t = (u_{1t}, \ldots, u_{Kt})^\intercal$ is a vector of K white noise series.

The process is **stable** if:

$$\det(I_K - A_1 z - \cdots - A_p z^p) \neq 0 \quad \text{for} \quad |z| \leq 1$$

this is, there are **no roots** in and on the complex unit circle.

For example, a VAR model with two endogenous variables ($K = 2$), two lags ($p = 2$), an exogenous contemporaneous variable ($M = 1$), a constant (const) and a trend (Trend_t):

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} a_{11,1} & a_{12,1} \\ a_{21,1} & a_{22,1} \end{bmatrix} \cdot \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{11,2} & a_{12,2} \\ a_{21,2} & a_{22,2} \end{bmatrix} \cdot \begin{bmatrix} y_{1,t-2} \\ y_{2,t-2} \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \cdot \begin{bmatrix} x_t \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \cdot \begin{bmatrix} \text{const} \\ \text{Trend}_t \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

Visualizing the separate equations:

$$y_{1t} = a_{11,1} y_{1,t-1} + a_{12,1} y_{2,t-1} + a_{11,2} y_{1,t-2} + a_{12,2} y_{2,t-2} + b_{11} x_t + c_{11} + c_{12} \text{Trend}_t + u_{1t}$$
$$y_{2t} = a_{21,1} y_{2,t-1} + a_{22,1} y_{1,t-1} + a_{21,2} y_{2,t-2} + a_{22,2} y_{1,t-2} + b_{21} x_t + c_{21} + c_{22} \text{Trend}_t + u_{2t}$$

If there is an unit root, the determinant is zero for $z = 1$, then some or all variables are integrated and a VAR model is no longer appropiate (is unstable).

VECM (Vector Error Correction Model)

If **cointegrating relations** are present in a system of variables, the VAR form is not the most convenient. It is better to use a VECM, that is, the levels VAR substracting y_{t-1} from both sides. The VECM($p-1$):

$$\Delta y_t = \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_{p-1} \Delta y_{t-p+1} + B_0 x_t + \cdots + B_q x_{t-q} + C D_t + u_t$$

where:

- y_t, x_t, D_t and u_t are as specified in VAR.
- $\Pi = -(I_K - A_1 - \cdots - A_p)$ for $i = 1, \ldots, p-1$; Πy_{t-1} is referred as the **long-term** part.
- $\Gamma_i = -(A_{i+1} + \cdots + A_p)$ for $i = 1, \ldots, p-1$ is referred as the **short-term** parameters.
- A_i, B_j and C are coefficient matrices of suitable dimensions.

If the VAR(p) process is unstable (there are roots), Π can be written as a product of $(K \times r)$ matrices α (**loading matrix**) and β (**cointegration matrix**) with $\text{rk}(\Pi) = \text{rk}(\alpha) = \text{rk}(\beta) = r$ (**cointegrating rank**) as follows $\Pi = \alpha\beta^\intercal$.

- $\beta^\intercal y_{t-1}$ contains the cointegrating relations.

For example, if there are three endogenous variables ($K = 3$) with two cointegratig relations ($r = 2$), the long term part of the VECM:

$$\Pi y_{t-1} = \alpha\beta^\intercal y_{t-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{bmatrix} = \begin{bmatrix} \alpha_{11} ec_{1,t-1} + \alpha_{12} ec_{2,t-1} \\ \alpha_{21} ec_{1,t-1} + \alpha_{22} ec_{2,t-1} \\ \alpha_{31} ec_{1,t-1} + \alpha_{32} ec_{2,t-1} \end{bmatrix}$$

where:

$$ec_{1,t-1} = \beta_{11} y_{1,t-1} + \beta_{21} y_{2,t-1} + \beta_{31} y_{3,t-1}$$
$$ec_{2,t-1} = \beta_{12} y_{1,t-1} + \beta_{22} y_{2,t-1} + \beta_{32} y_{3,t-1}$$