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Invited Review

Irregular packing problems: A review of mathematical models

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ABSTRACT

Irregular packing problems (also known as nesting problems) belong to the more general class of cutting and packing problems and consist of allocating a set of irregular and regular pieces to larger rectangular or irregular containers, while minimizing the waste of material or space. These problems combine the combinatorial hardness of cutting and packing problems with the computational difficulty of enforcing the geometric non-overlap and containment constraints. Unsurprisingly, nesting problems have been addressed, both in the scientific literature and in real-world applications, by means of heuristic and metaheuristic techniques. However, more recently a variety of mathematical models has been proposed for nesting problems. These models can be used either to provide optimal solutions for nesting problems or as the basis of heuristic approaches based on them (e.g. matheuristics). In both cases, better solutions are sought, with the natural economic and environmental positive impact. Different modeling options are proposed in the literature. We review these mathematical models under a common notation framework, allowing differences and similarities among them to be highlighted. Some insights on weaknesses and strengths are also provided. By building this structured review of mathematical models for nesting problems, research opportunities in the field are proposed.

1. Introduction

Cutting and packing problems arise in many industrial and logistics applications, as problems to be solved or as part of more complex optimization problems. They share a nice underlying geometric structure and, on top of it, many different settings and specific characteristics define a wide variety of particular problems. Therefore, it is not surprising they have recently attracted a great deal of attention in the scientific literature, and a typology for cutting and packing problems was introduced in Wäscher, Haußner, and Schumann (2007). Apart from the obvious classification criterion based on dimensionality, a second element that divides the cutting and packing field into two clearly different parts is the type of pieces being considered, whether they are regular or irregular. Regular, and more specifically rectangular, pieces appear in many cutting problems and in the vast majority of packing problems, in which products are usually packaged into rectangular boxes. Other regular, non-rectangular pieces, such as circles, have also been studied in specific applications. Nevertheless, there are also many applications in which the pieces to be cut or packed are irregularly shaped. Typical examples appear in the clothing industry, furniture, leather, glass, or sheet metal cutting. Fig. 1 shows an example of nesting problem found in garment industries. It illustrates a set of 64 pieces with regular and irregular shapes feasibly packed into a rectangular board.

Irregular cutting and packing problems, aka nesting problems, consist of packing a set of irregular-shaped pieces into a board or a set of boards. The pieces must be placed completely inside the boards in such a way that they do not overlap each other, while the waste material is minimized. Minimizing waste has not only economic, but also environmental impact if the use of raw materials can be kept to a minimum. The geometric structure of nesting problems is much more complex than the case of rectangular pieces. Checking if pieces are included in the board and do not overlap is a hard problem when irregular pieces are involved. As a consequence, nesting problems are much harder than their counterparts that deal with rectangular pieces and the results obtained so far are much more limited regarding the size (number of pieces) of the problems tackled. Their difficulty and their many applications make nesting problems the most challenging problems in the field of cutting and packing.

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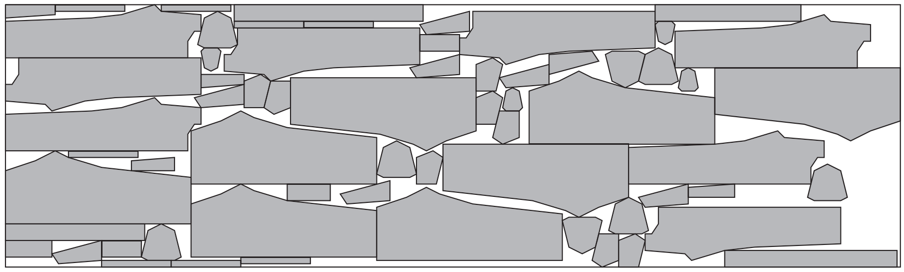


Fig. 1. An example of nesting problem.

The objectives of this paper are to review the mathematical models proposed in the literature for nesting problems and to serve as a starting point for a new wave of research on this field. Several types of models have been proposed in the last decades: mixed-integer linear programming, non-linear programming, and constraint programming models. The characteristics of the models are strongly related to the geometric methodologies employed. Bennell and Oliveira (2008) provide a tutorial that covers the geometric tools used in irregular cutting and packing problems. Raster point, direct trigonometry, no-fit polygon and phi-functions have been applied to the modeling. In this study, we carefully review all these modeling approaches and their corresponding geometric tools. We focus on two-dimensional nesting problems, since this is the version addressed by most of the models proposed so far, although some references to the three-dimensional case are added when an extension to the 3D case is mentioned. References and surveys on heuristic and metaheuristic algorithms developed for nesting problems can be found in Bennell and Oliveira (2009), Hopper and Turton (2001), Dowsland and Dowsland (1995). We also focus on problems with no restrictive assumption on the shape of the pieces and the cutting process. Nevertheless, a few references of models developed for general convex pieces are given. For solution methods addressing problems with convex pieces and guillotine cuts, we refer to Han, Bennell, Zhao, and Song (2013) and Bennell, Cabo, and Mart Anez Sykora (2018).

The remainder of this paper is organized as follows. In Section 2, nesting problems are defined. Section 3 revises the geometric tools according to their usage in the mathematical models. Mixed-integer linear programming models, non-linear programming models and constraint programming models are described in Section 4, Section 5 and Section 6, respectively. In Section 7 an outlook on future research is presented and in Section 8 we conclude with a categorization of the mathematical models.

2. Problem definition

In nesting problems, we have a board geometrically represented by $\mathbf{P}^0$ or a set of boards $\mathbf{P}^{0b}$, $b = 1, \dots, n$, and a set of m pieces, where the geometrical representation of piece i is denoted by $\mathbf{P}^i$, $i = 1, \dots, m$. The boards can have any shape (irregular, regular or strip), holes, and defects. In the set of pieces, there is at least one piece with irregular shape. Pieces and boards can be described by their vertices, arcs, or the union of primitive geometric figures. The problem consists of packing all pieces or a subset of them into a board or into a set of boards such that they do not overlap each other and the waste of material is minimized. If a piece is packed into the board, it must be entirely inside the board.

The placement of a piece on the board is given by a vector, which can be defined by the Minkowski sum operator $\oplus$ as follows.

Definition 1. Given a piece represented by $\mathbf{P}$ and a vector $\mathbf{v}$, set $\mathbf{P} \oplus \mathbf{v} = \{\mathbf{p} + \mathbf{v} : \mathbf{p} \in \mathbf{P}\}$ defines piece $\mathbf{P}$ translated by $\mathbf{v}$, where $\mathbf{p}$ is a point in $\mathbf{P}$ with coordinates $(\mathbf{x}^p, \mathbf{y}^p)$.

Some applications allow the free rotation or a set of rotations to the pieces. By denoting θ the angle of rotation of a piece and Θ the set of rotations allowed to the piece, the rotation function can be given by:

$$\mathbf{P}(\theta) = \{(\mathbf{x}^p \cos(\theta) - \mathbf{y}^p \sin(\theta), \mathbf{x}^p \sin(\theta) + \mathbf{y}^p \cos(\theta)) : \mathbf{p} = (\mathbf{x}^p, \mathbf{y}^p) \in \mathbf{P}\}.$$

We can distinguish two types of constraints in the problem:

1. If a piece is packed it must be inside a board: $\mathbf{P}^i(\theta_i) \oplus v^i \subseteq \mathbf{P}^{0b}$, for some b.
2. If pieces i and j are packed into the same board, they cannot overlap: $int(\mathbf{P}^i(\theta_i) \oplus \mathbf{v}^i) \cap int(\mathbf{P}^j(\theta_j) \oplus \mathbf{v}^j) = \emptyset$, where $int(\mathbf{P})$ is the interior of a piece.

The expression of the objective function (minimize the waste of material) depends on the application. In the irregular strip packing problem, if the board is a rectangle with width W and unlimited length L, the objective is to minimize the length used to pack all pieces. The objective function in knapsack problems is to maximize the value c_i of the pieces packed into the board, which can be defined, for example, by their area. In cutting stock and bin packing problems, the objective function is usually to minimize the number of boards necessary to pack all pieces. However, the objective function can be easily adapted from an application to another.

3. Geometric tools in mathematical formulations

In the development of mathematical models and solution methods for nesting problems, the first challenge researchers face is the development of geometric tools to represent the problem. The geometric tool chosen to deal with the problem affects model and method types, solution precision, time dedicated to implementing and computational results. Then, it is crucial to use a geometric tool that meets the application needs, such as solution precision and convergence speed. Geometric tools are used during the method search and in a pre-processing phase, allowing us to deal with the computational and mathematical representation of pieces, boards and solution layouts. Along this paper, the word "piece" will be consistently applied, even when polygons are used to represent (or approximate) the actual piece. Bennell and Oliveira (2008) provide a review of the geometric tools developed in the literature to deal with nesting problems. Among them, raster point method, direct trigonometry, phi-functions and no-fit polygon have been used in the mathematical models. In the following, we briefly describe these tools and their usage in the modeling approach.

3.1. Raster point

The method consists of representing pieces and boards by a grid, where the board is discretized in small areas and each one is associated with a value to identify if the area is empty, or occupied by a piece, or if there is overlap among pieces. Different strategies of codification have been used in the literature. Despite being simple to implement, the disadvantage of raster point methods is the

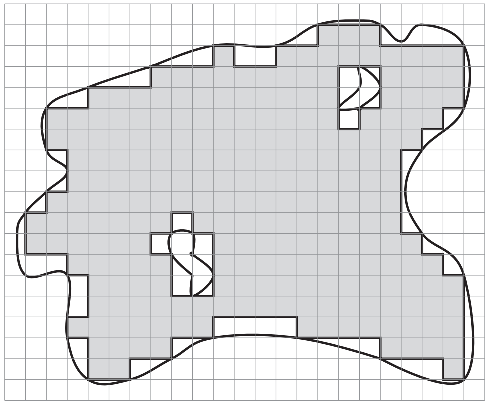


Fig. 2. Raster representation of an irregular board with holes.

lack of precision in representing pieces and boards, which can result in gaps among pieces. To mitigate this effect, a refined grid can be used. However, it affects the algorithm efficiency in terms of memory and computational running time. This tool has been mainly used in heuristics. In the models reviewed in this paper, only Baldacci, Boschetti, Ganovelli, and Maniezzo (2014) used this type of codification to solve the irregular knapsack problem, where the board is irregular with defects and different levels of quality. Fig. 2 illustrates a raster presentation of an irregular board with defects.

3.2. Direct trigonometry

Methods that use direct trigonometry are more precise than raster point methods. There are many direct trigonometry strategies applied to nesting problems in the literature. The most common strategies applied to mathematical models are the D-function and the usage of circles to describe pieces. The D-function was proposed by Konopasek (1981) and determines the relative position between a vertex of a piece and an edge of another piece. It was derived from the distance equation between a point and a line. In the modeling approach, D-functions were used by Scheithauer and Terno (1993), Cherri et al. (2016b) and Cherri, Cherri, and Soler (2018). The circle covering representation can be used to represent pieces. Rocha, Rodrigues, Gomes, and F. M. B. Toledo (2014) present three types of circle covering representation. In the complete circle covering representation, a set of circles is defined such that the circles cover the piece. The inner circle covering representation consists of defining a piece by inscribed circles. In the partial circle covering representation, neither the piece is entirely inside of a set of circles, nor the set of circles is entirely inside of the piece. The inner circle covering and the complete circle covering representations were used in Jones (2014) and Rocha, Gomes, Rodrigues, Toledo, and Andretta (2016), respectively. In addition, Rocha et al. (2014) compare all circle covering representations using their non-linear programming model. Fig. 3 illustrates the direct trigonometry tools that have been used in the modeling approach. Fig. 3(a) represents the D-function analysis of the relative position of the reference point of piece j and an edge of piece i. Fig. 3(b) and (c) show the inner circle covering representation and the complete circle covering representation, respectively.

Another direct trigonometry tool used in the mathematical modeling approach is the separation lines. It can also be illustrated according to Fig. 3(a). This tool was used in Peralta, Andretta, and Oliveira (2018) and consists of defining a line using two consecutive vertices of a piece. The non-overlapping between two pieces is ensured if their vertices are on different sides of the line or on the line. The main difference between D-function and separation lines is in the derivation of the non-overlapping constraints that is shown in Section 5.2.

3.3. No-fit polygon and inner-fit polygon

The no-fit polygon was proposed by Art (1966) and is used to define the overlapping region between two pieces. The inner-fit polygon is derived from the no-fit polygon and is the region on the board in which a piece can be placed. In this representation, each piece has a reference point, which can be any point inside or outside the piece. Considering that the position of piece i is fixed, the no-fit polygon between pieces i and j is the polygon described by the locus of points where if the reference point of piece j is placed, then the pieces are in contact. We denote the no-fit polygon between two pieces i and j by NFP_{ij}. The frontier of NFP_{ij} and its exterior are the feasible region where piece j can be placed so as not to overlap piece i. Fig. 4 shows an example of a no-fit polygon. The inner-fit polygon can be seen as the no-fit polygon between a piece and the outer region of the board. Fig. 5 shows the inner-fit polygon of piece j considering a rectangular board. There are several methods in the literature to obtain no-fit polygons such as the sliding method (Burke, Hellier, Kendall, & Whitwell, 2007; 2010), the Minkowski sum (Bennell & Song, 2008) and the decomposition of polygons combined with the Minkowski sum (Agarwal, Flato, & Halperin, 2002).

Mathematical models that use no-fit polygon can be found in Scheithauer and Terno (1993), Daniels, Li, and Milenkovic (1994), Dean (2002), Fischetti and Luzzi (2009), Alvarez-Valdes, Martinez, and Tamarit (2013), Toledo, Carravilla, Ribeiro, Oliveira, and Gomes (2013), Leao, Toledo, Oliveira, and Carravilla (2016), Cherri et al. (2016b), Rodrigues and Toledo (2017), Carravilla, Ribeiro, and Oliveira (2003), Ribeiro and Carravilla (2009) and Cherri, Carravilla, Ribeiro, and Toledo (2017).

3.4. Phi-function and quasi phi-function

Phi-function is a tool that has been used in the literature to describe the relations between two objects (phi-objects). In nesting problems, phi-objects represent pieces and boards, and phi-functions are used to define their relation. Phi-objects are

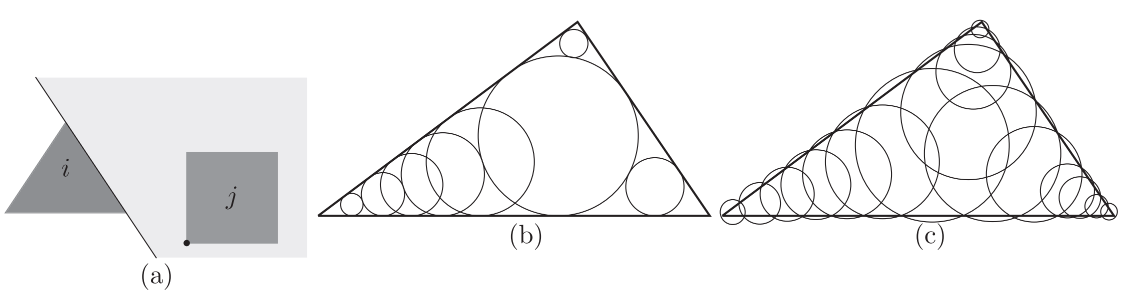


Fig. 3. (a) D-function representation. (b) Inner circle covering representation. (c) Complete circle covering representation.

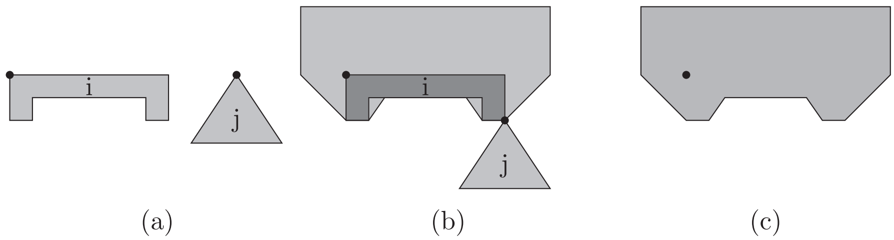


Fig. 4. (a) Pieces. (b) Construction of NFP_{ij}. (c) NFP_{ij}.

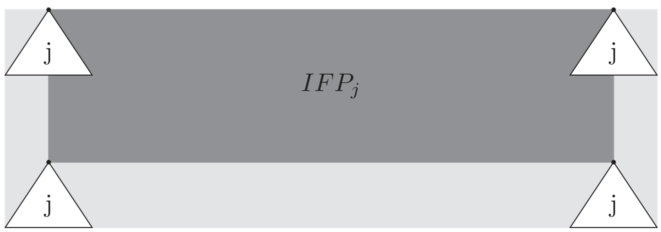


Fig. 5. Construction of the inner-fit polygon of piece j (IFP_j).

canonically closed sets (in topology) and do not have frontier self-intersections, in particular, a phi-object is called phi-polygon if its frontier is shaped by straight lines, rays or line segments (Bennell, Scheithauer, Stoyan, & Romanova, 2010; Chernov, Stoyan, & Romanova, 2010). We can classify phi-objects in primary (circle, regular and convex polygons) and composed, where composed objects can be described using union and intersection of primary objects. Basic objects were introduced to deal with irregular objects that allow continuous rotation. Mathematically, phi-objects are represented using linear, non-linear and piecewise inequalities, and the placement of a phi-object is defined by a translation vector and a rotation angle. The phi-function for two objects $\mathbf{P}^i$ and $\mathbf{P}^j$ returns a positive value (in particular, the Euclidean distance) between non-overlapping objects, for which we have the following definition.

Definition 2. Consider two phi-objects $\mathbf{P}^i$ and $\mathbf{P}^j$ with placement parameters defined by $\mathcal{O}_i = (\mathbf{v}^i, \theta_i)$ and $\mathcal{O}_j = (\mathbf{v}^j, \theta_j)$, respectively, where $\mathbf{v}^i$ is the translation vector and θ_i is the angle of rotation. The interaction between $\mathbf{P}^i(\theta_i)\oplus\mathbf{v}^i$ and $\mathbf{P}^j(\theta_j)\oplus\mathbf{v}^j$ is established by the phi-function ϕ^{ij}, which holds the following properties.

(i) $\phi^{ij} > 0$ if $(\mathbf{P}^i(\theta_i) \oplus \mathbf{v}^i) \cap (\mathbf{P}^j(\theta_j) \oplus \mathbf{v}^j) = \emptyset$
(ii) $\phi^{ij} = 0$ if $int(\mathbf{P}^i(\theta_i) \oplus \mathbf{v}^i) \cap int(\mathbf{P}^j(\theta_j) \oplus \mathbf{v}^j) = \emptyset$ and $fr(\mathbf{P}^i(\theta_i)\oplus\mathbf{v}^i)\cap fr(\mathbf{P}^j(\theta_j)\oplus\mathbf{v}^j) \neq \emptyset$
(iii) $\phi^{ij} < 0$ if $int(\mathbf{P}^i(\theta_i)\oplus\mathbf{v}^i)\cap int(\mathbf{P}^j(\theta_j)\oplus\mathbf{v}^j) \neq \emptyset$
(iv) ϕ^{ij} is continuous for all values of variables ($\mathbf{v}^i$, θ_i), ($\mathbf{v}^j$, θ_j),

where $int(\mathbf{P})$ and $fr(\mathbf{P})$ are the interior and the frontier of phi-object $\mathbf{P}$, respectively. The phi-function can be interpreted as an extension of the Minkowski sum, since the 0-level of phi-function described by $\phi^{ij} = 0$ coincides with the frontier of the no-fit polygon between two oriented objects (Stoyan, Novozhilova, & Kartashov, 1996). If the values of phi-function are equal to the Euclidean distance between objects for all placement parameters of non-overlapping objects, it is called normalized phi-function.

To illustrate the phi-function equations, we give an example of phi-function for a rectangle and a circle. Let $\mathbf{P}^1$ be a rectangle of width $2a$, length $2b$, and $\mathbf{v}^1 = (x_1, y_1)$, and $\mathbf{P}^2$ be the circle of radius r and $\mathbf{v}^2 = (x_2, y_2)$. Without loss of generality we consider no rotation for the rectangle. Fig. 6 illustrates the 0-level ($\phi^{12}(x, y) = 0$) of the normalized phi-function for the rectangle and the circle. In each point of the curve defined by $\phi^{12}(x, y) = 0$, the values of the normalized phi-function are equal to 0, it means that the rectangle and the circle are in contact. Outside the 0-level, the values of the normalized phi-function coincide with the Euclidean distance between these objects. The length of each arrow in Fig. 6 is equal to the Euclidean distance between the rectangle and the circle in different positions in the k-level ($\phi^{12} = \rho \geq r$). The regions defined by $\phi^{12} = 0$ and $\phi^{12} = \rho$ are described by four line segments and four curves. Let us define the following functions, where $(x, y) = (x_2 - x_1, y_2 - y_1)$.

$$\varphi_1(x, y) = x - a - r; \quad \varphi_2(x, y) = y - b - r;$$
$$\varphi_3(x, y) = -x - b - r; \quad \varphi_4(x, y) = -y - b - r;$$
$$\overline{\varphi}_1(x, y) = \sqrt{(x - a)^2 + (y - b)^2} - r;$$
$$\overline{\varphi}_2(x, y) = \sqrt{(x + a)^2 + (y - b)^2} - r;$$
$$\overline{\varphi}_3(x, y) = \sqrt{(x + a)^2 + (y + b)^2} - r;$$
$$\overline{\varphi}_4(x, y) = \sqrt{(x - a)^2 + (y + b)^2} - r;$$
$$\psi_1(x, y) = x + y - a - b - r; \quad \psi_2(x, y) = -x + y - a - b - r;$$
$$\psi_3(x, y) = -x - y - a - b - r; \quad \psi_4(x, y) = x - y - a - b - r.$$

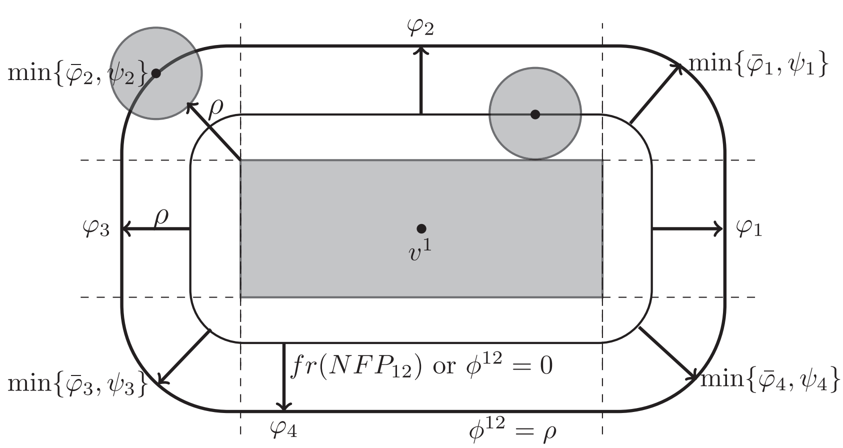


Fig. 6. Interaction between a rectangle and a circle.

Table 1
General overview of the MIP models.

Variable type	Model type	Geometric tool	Authors
$(x, y) \in \mathbb{R}^2$ Continuous (Section 4.1)	Piece oriented	NFP	Scheithauer and Terno (1993), Daniels et al. (1994) Dean (2002), Fischetti and Luzzi (2009), Alvarez-Valdes et al. (2013), Cherri et al. (2016b)
		trigonometry	Scheithauer and Terno (1993), Cherri et al. (2016b)
$(x, y) \in \mathbb{Z}^2$ Discrete (Section 4.2)	Board oriented	NFP	Toledo et al. (2013), Rodrigues and Toledo (2017)
		raster points	Baldacci et al. (2014)
$(x, y) \in \mathbb{R} \times \mathbb{Z}$ Semi-continuous (Section 4.3)	Piece oriented	NFP	Leao et al. (2016)

The normalized phi-function for a rectangle and a circle is given by:

$$\phi^{12}(x, y) = \max_{k=1,\dots,4}\{\max\{\varphi_k(x, y), \min\{\overline{\varphi}_k(x, y), \psi_k(x, y)\}\}\}.$$

This example and the interaction between other oriented objects using normalized phi-functions can be found in Bennell et al. (2010). For further details on phi-functions for objects that allow continuous rotation, we refer to Chernov et al. (2010), Stoyan and Romanova (2013) and Bennell, Scheithauer, Stoyan, Romanova, and Pankratov (2015).

Mathematical models defined in terms of phi-functions are given in Stoyan et al. (1996) for oriented objects. Models that considered continuous object rotations and allowable distances between objects are given in Chernov et al. (2010), Chernov, Stoyan, Romanova, and Pankratov (2012), Stoyan, Zlotnik, and Chugay (2012), Stoyan and Romanova (2013), Bennell et al. (2015), Stoyan, Pankratov, and Romanova (2016a) and Stoyan, Pankratov, and Romanova (2017). When a minimal distance (ρ_{ij}^{-}) between two objects is required, the normalized phi-function usually have radicals. To avoid that Chernov et al. (2010, 2012) introduced the concept of adjusted phi-function which preserves the sign of $\phi^{ij} - \rho_{ij}^{-}$, where ϕ^{ij} is the normalized phi-function. The development of some radical free adjusted phi-functions for basic types of objects can be found in Chernov et al. (2012). In Scheithauer (2018), phi-function tools and models for irregular packing problems are also reviewed. In addition, he provides an overview about different cutting and packing problems, which includes mathematical modeling, solution methods and upper and lower bounds.

Stoyan, Romanova, Pankratov, and Chugay (2015) and Stoyan, Pankratov, and Romanova (2016b) introduced the quasi phi-function technique which consists of a generalization of phi-functions by including auxiliary variables. The quasi phi-functions allow to model the relation between two objects that was unknown or complex when using phi-functions. Consider $O \subset \mathbb{R}^n$ as the set of auxiliary variables and $O' \in O$, a function $\phi\prime^{ij}(\mathcal{O}_i, \mathcal{O}_j, O\prime)$ is a quasi phi-function for objects i and j if $\max_{O\prime \in O} \phi\prime^{ij}(\mathcal{O}_i, \mathcal{O}_j, O\prime)$ is a phi-function for the objects.

To build a quasi phi-function, the authors consider a half-plane (for two-dimensional problems) or a half-space (for three-dimensional problems) denoted by π for which they define the phi-functions $\phi^{i\pi}(\mathcal{O}_i, \mathcal{O}_\pi)$ and $\phi^{j\pi^c}(\mathcal{O}_j, \mathcal{O}_\pi)$, where π^c is the complement of $int(\pi)$, i.e., $\pi^c = \mathbb{R}^2 \backslash int(\pi)$. Then, the function $\phi\prime^{ij}(\mathcal{O}_i, \mathcal{O}_j, O\prime) = \min\{\phi^{i\pi}(\mathcal{O}_i, \mathcal{O}_\pi), \phi^{j\pi^c}(\mathcal{O}_j, \mathcal{O}_\pi)\}$ is a quasi phi-function for objects i and j.

To exemplify the quasi phi-functions, consider two convex continuously translated and rotated polygons i and j that are described by their vertices $(\bar{x}_i^\nu, \bar{y}_i^\nu)$, $\nu = 1, \dots, \nu_i$. We denote the equation that describes the continuously translated and rotated half-plane (or half-space) by Ω_π. The phi-functions for objects i and π, and j and π^c are given by $\phi^{i\pi}(\mathcal{O}_i, \mathcal{O}_\pi) = \min_{1 \le \nu \le \nu_i} \Omega_p(\bar{x}_i^\nu, \bar{y}_i^\nu)$ and $\phi^{j\pi}(\mathcal{O}_j, \mathcal{O}_\pi) = \min_{1 \le \nu \le \nu_j}(-\Omega_p(\bar{x}_j^\nu, \bar{y}_j^\nu))$, respectively. Then the quasi phi-function for objects i and j is stated as follows.

$$\phi\prime^{ij}(\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_\pi) = \min\{\phi^{i\pi}(\mathcal{O}_i, \mathcal{O}_\pi), \phi^{j\pi}(\mathcal{O}_j, \mathcal{O}_\pi)\}.$$

This example and other examples of quasi phi-functions for regular and irregular 2D and 3D objects can be found in Stoyan et al. (2015) and Stoyan et al. (2016b). The normalized and adjusted concepts were also extended to quasi phi-functions. Mathematical models based on quasi phi-function were proposed in Stoyan et al. (2016b, 2015) and Romanova, Bennell, Stoyan, and Pankratov (2018).

4. Mixed-integer linear programming models

Most of the mathematical models proposed for nesting problems belong to the mixed-integer linear programming area. Here we classify these models according to the variables that define the positioning of the reference point of the pieces on the board, which can be continuous or integer variables, and the representation of pieces and boards. Continuous positioning models allow general layouts, while discrete models restrict the position of the reference point of pieces to a finite set of points on the board, or even apply the rasterization technique to describe pieces and boards. An intermediate approach between continuous and discrete models was also proposed in the literature. In this case, one axis is continuous and the second one is discretized. Then, this section is divided into continuous models, discrete models, and semi-continuous model.

We can also distinguish the models between piece oriented and board oriented. Piece oriented models ensure that the pieces do not overlap each other by using the geometry of the pieces and taking into account the position of points of the pieces on the board. Board oriented models take into account the geometry used to represent the board by verifying if a point or a region of the board is occupied or not. To deal with the geometry of the problem, most papers discussed in this section use no-fit polygon and inner-fit polygon. However, Scheithauer and Terno (1993) and Cherri et al. (2016b) also described models by using direct trigonometry, and Baldacci et al. (2014) described a model based on the raster point representation. In addition, most of the models were proposed for the irregular strip packing problem. Only Scheithauer and Terno (1993) and Baldacci et al. (2014) described mathematical models for the knapsack problem and Daniels et al. (1994) proposed a feasibility model, in which the boards are irregular in both problem types. Table 1 presents an overview of the models discussed in this section.

4.1. Continuous models

The continuous positioning models proposed in the literature use no-fit polygon or direct trigonometry to represent the problem. Sections 4.1.1 and 4.1.2 describe the placement constraints based on no-fit polygon and direct trigonometry, respectively. The mathematical models are summarized in Section 4.1.3.

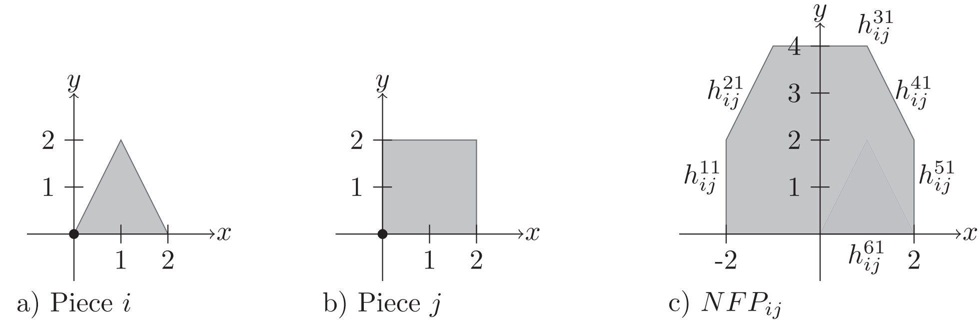


Fig. 7. Construction of the no-fit polygon constraints.

4.1.1. Constraints based on no-fit polygon

Most of the continuous mathematical models that use the no-fit polygon consider similar strategies to build the non-overlapping constraints. Scheithauer and Terno (1993), Daniels et al. (1994), Dean (2002), Fischetti and Luzzi (2009) and Alvarez-Valdes et al. (2013) decomposed the outer region of the no-fit polygon into a number of convex regions (polyhedrons), however, they applied different strategies to decompose it. On the other hand, Cherri et al. (2016b) considered the no-fit polygon decomposed into convex parts.

Let us consider the following data set in the decomposition of the outside region of no-fit polygons into convex regions.

- NFP_{ij}: the no-fit polygon between pieces i and j, where i is the fixed piece and j is the orbital piece.
- $\overline{NFP}_{ij}$: outside region of NFP_{ij}.
- k_{ij}: number of convex regions of $\overline{NFP}_{ij}$, $1 \le i < j \le m$.
- $\overline{NFP}_{ij}^{k}$: kth convex region of $\overline{NFP}_{ij}$, $k = 1, \ldots, k_{ij}$, $1 \le i < j \le m$.
- f_{ij}^{k}: number of edges that define the kth convex region $\overline{NFP}_{ij}^{k}$.

Apart from the strategy used to generate regions $\overline{NFP}_{ij}^{k}$, in this class of models the non-overlapping constraints are defined by the edges of each $\overline{NFP}_{ij}^{k}$, for which we consider the following variables.

- x_i: x-coordinate of the reference point of piece i on the board.
- y_i: y-coordinate of the reference point of piece i on the board.
- u_{ij}^{k}: is equal to 1 if the reference point of piece j is on convex region $\overline{NFP}_{ij}^{k}$, 0 otherwise.

Daniels et al.**'s constraints**

Consider two pieces i and j and region $\overline{NFP}_{ij}$, where piece i is fixed. To prevent pieces i and j from overlapping, we have to ensure that the reference point of piece j is on a convex region $\overline{NFP}_{ij}^{k}$. Then, it follows that all linear inequalities described by the edges of region $\overline{NFP}_{ij}^{k}$ must be satisfied, i.e.,

$$\alpha_{ij}^{kf}(x_j - x_i) + \beta_{ij}^{kf}(y_j - y_i) \le q_{ij}^{kf}, \quad f = 1, \ldots, f_{ij}^{k}.$$

To impose that piece j is in at least one region k, $k = 1, \ldots, k_{ij}$, we associate each region k with a binary variable u_{ij}^{k}. Therefore, the non-overlapping constraints are given as follows.

$$\alpha_{ij}^{kf}(x_j - x_i) + \beta_{ij}^{kf}(y_j - y_i) \le q_{ij}^{kf} + M(1 - u_{ij}^{k}),$$
$$1 \le i < j \le m, k = 1, \ldots, k_{ij}, f = 1, \ldots, f_{ij}^{k}, \tag{1}$$

$$\sum_{k=1}^{k_{ij}} u_{ij}^{k} = 1, \ \ 1 \le i < j \le m. \tag{2}$$

Further details about the development of constraints (1) and (2) can be found in Dean (2002), Li and Milenkovic (1995) and Cherri et al. (2016b). To illustrate the non-overlapping constraints, consider pieces i and j and NFP_{ij} given in Fig. 7. The reference point of the pieces and NFP_{ij} are at the origin of the coordinate system. NFP_{ij} is defined by six edges that describe six convex regions outside NFP_{ij}, i.e., $k_{ij} = 6$ and $f_{ij}^{k} = 1$, $k = 1, \ldots, 6$. The edges are numbered in clockwise direction and NFP_{ij} is given by: $h_{ij}^{11}(x, y) = -x - 2 \le 0$; $h_{ij}^{21}(x, y) = y - 2x - 6 \le 0$; $h_{ij}^{31}(x, y) = y - 4 \le 0$; $h_{ij}^{41}(x, y) = y + 2x - 6 \le 0$; $h_{ij}^{51}(x, y) = x - 2 \le 0$; $h_{ij}^{61}(x, y) = -y \le 0$.

If we consider that each convex region k is defined by each edge of NFP_{ij}, the non-overlapping constraints are given by:

$$x_j - x_i \le -2 + M(1 - u_{ij}^{1});$$
$$2(x_j - x_i) - (y_j - y_i) \le -6 + M(1 - u_{ij}^{2});$$
$$-(y_j - y_i) \le -4 + M(1 - u_{ij}^{3});$$
$$-2(x_j - x_i) - (y_j - y_i) \le -6 + M(1 - u_{ij}^{4});$$
$$-(x_j - x_i) \le -2 + M(1 - u_{ij}^{5});$$
$$y_j - y_i \le M(1 - u_{ij}^{6});$$
$$\sum_{k=1}^{6} u_{ij}^{k} = 1.$$

If the problem to be solved has non-convex boards, the inner-fit polygon is also given by a non-convex region. To ensure that the pieces are entirely inside the board, we can use constraints (1) and (2). In this case, the inner-fit polygon of a piece i is decomposed into k_{i0} convex regions, where each one is associated with a binary variable, u_{i0}^{k}, $k = 1, \ldots, k_{i0}$, to represent if a piece i is placed on region k. A region is described by a set of f_{i0}^{k} edges, consequently there are f_{i0}^{k} constraints associated with it.

$$\alpha_{i0}^{kf} x_i + \beta_{i0}^{kf} y_i \le q_{i0}^{kf} + M(1 - u_{i0}^{k}),$$
$$i = 1, \ldots, m, k = 1, \ldots, k_{i0}, f = 1, \ldots, f_{i0}^{k}, \tag{3}$$

$$\sum_{k=1}^{k_{i0}} u_{i0}^{k} = 1, \ \ i = 1, \ldots, m. \tag{4}$$

Constraints (1)–(4) were firstly used by Daniels et al. (1994) in the feasibility problem of packing a set of non-convex pieces into a non-convex board. Instead of presenting the mixed-integer programming model in a standard representation, the authors described it as an algorithm. However, they did not provide an algorithm to generate the convex regions outside the no-fit polygon. In this placement problem, it is known in advance that the given set of pieces fits into the board, but most of the practical applications consider a set of boards or an unlimited board. In the first case, it is necessary to solve many containment problems and matching problems to decide each set of pieces that fits into each board. However, if all decisions are made together better solutions can

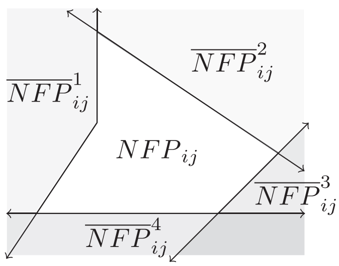


Fig. 8. Illustration of the regions outside NFP_{ij} according to Dean (2002).

be found in terms of saving raw material. In the second case, the inner-fit constraints must be modified and additional constraints are necessary as discussed in the following model.

Dean's constraints

Dean (2002) proposed a mathematical model for the strip packing problem. However, he adapted constraints (1) and (2) by considering a gap of fixed distance between two pieces, which is useful for some applications. Consider $|AB^{kf}_{ij}|$ as the length of edge f of $\overline{NFP}^k_{ij}$ and ρ_{ij} as a required distance between pieces i and j, constraints (1) are replaced by:

$$\alpha^{kf}_{ij}(x_j - x_i) + \beta^{kf}_{ij}(y_j - y_i) + |AB^{kf}_{ij}|\rho_{ij} \le q^{kf}_{ij} + M(1 - u^k_{ij}),$$
$$1 \le i < j \le m, k = 1, \dots, k_{ij}, f = 1, \dots, f^k_{ij}. \quad (5)$$

In Dean's model, the outer region of the no-fit polygon is divided into convex regions that are defined according to the edges of the no-fit polygon. The representation of the decomposition of the outer region of a NFP_{ij} is given in Fig. 8. Note that, the regions can intersect each other.

In the irregular strip packing problem, the inner-fit polygon constraints (3) and (4) are replaced by inner-fit rectangle constraints that are described by the enveloping rectangle of the pieces. Considering that the reference point of a piece i is at the bottommost and leftmost point of the enveloping rectangle, the constraints that ensures each piece is inside the board are given by:

$$y_i + w_i \le W, \quad i = 1, \dots, m. \quad (6)$$

The constraints that define the used length of the board (z), which must be minimized, are given by:

$$x_i + l_i \le z, \quad i = 1, \dots, m. \quad (7)$$

Fischetti & Luzzi's constraints

Fischetti and Luzzi (2009) also used constraints (1) and (2) and divided the outer region of the no-fit polygon into convex areas to solve irregular strip packing problems. However, they consider disjunctive areas, i.e., $\overline{NFP}^h_{ij} \cap \overline{NFP}^k_{ij} = \emptyset,\ h, k = 1, \dots, k_{ij},\ h \neq k$ and $\overline{NFP}_{ij} = \bigcup_{k=1}^{k_{ij}} \overline{NFP}^k_{ij}$. An example of a representation of $\overline{NFP}_{ij}$ is given in Fig. 9, but the authors did not provide a procedure to generate the edges that separate the convex regions (dotted edges).

The big M in integer programming models is well-known by its weak linear relaxations. To strengthen the linear programming relaxation, the authors applied the lifting technique to the big M, where the term $M(1 - u^k_{ij})$ is replaced by $\sum_{h=1}^{k_{ij}} \eta^{kfh}_{ij} u^h_{ij}$. Then, constraints (1) are rewritten as:

$$\alpha^{kf}_{ij}(x_j - x_i) + \beta^{kf}_{ij}(y_j - y_i) \le q^{kf}_{ij} + \sum_{h=1}^{k_{ij}} \eta^{kfh}_{ij} u^h_{ij},$$
$$1 \le i < j \le m, k = 1, \dots, k_{ij}, f = 1, \dots, f^k_{ij}.$$

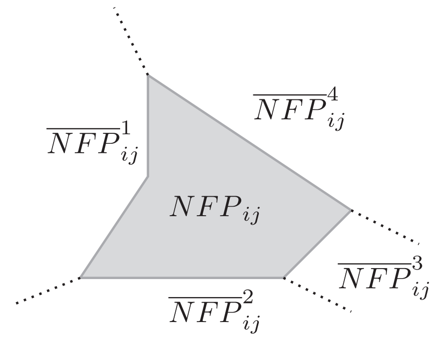


Fig. 9. Illustration of the regions outside NFP_{ij} according to Fischetti and Luzzi (2009).

Since $\sum_{h=1}^{k_{ij}} u^h_{ij} = 1$, then $q^{kf}_{ij} = q^{kf}_{ij} \sum_{h=1}^{k_{ij}} u^h_{ij}$. Defining $\delta^{kfh}_{ij} = q^{kf}_{ij} + \eta^{kfh}_{ij}$ it follows that:

$$\alpha^{kf}_{ij}(x_j - x_i) + \beta^{kf}_{ij}(y_j - y_i) \le \sum_{h=1}^{k_{ij}} \delta^{kfh}_{ij} u^h_{ij},$$
$$1 \le i < j \le m, k = 1, \dots, k_{ij}, f = 1, \dots, f^k_{ij}. \quad (8)$$

The equation is valid for δ^{kfh}_{ij} defined as:

$$\delta^{kfh}_{ij} = \max_{((x_j, y_j) - (x_i, y_i)) \in \overline{NFP}^k_{ij} \cap \mathcal{R}} \alpha^{kf}_{ij}(x_j - x_i) + \beta^{kf}_{ij}(y_j - y_i),$$
$$1 \le i < j \le m, k = 1, \dots, k_{ij}, f = 1, \dots, f^k_{ij},$$

where, $\mathcal{R}$ is a bounding box that is large enough and contains all feasible positioning of pieces i and j. The authors consider that $\mathcal{R}$ has width $2W$ and height $2\bar{L}$, where $\bar{L}$ is an upper bound on the length of the board.

Alvarez-Valdes et al.'s constraints

Alvarez-Valdes et al. (2013) applied the lifting technique to constraints (6) and (7) in order to tight the bounds on the positioning variables of the pieces. These constraints depend on the polyhedrons defined outside the no-fit polygon and the procedure used to determine them can be described in three steps. Firstly, for each point, segment and convex polygons inside of the no-fit polygon is defined the binary variable u^k_{ij}. The non-convex polygons inside the no-fit polygon are previously decomposed into convex polygons. Secondly, a variable u^k_{ij} is assigned to each concavity outside the no-fit polygon such that a variable is associated with a convex region. In the third step, horizontal slices are defined by the edges of the no-fit polygon, and for each one is also assigned a variable u^k_{ij}. Fig. 10(a) illustrates Steps 2 and 3 of the procedure applied to a NFP_{ij}. Step 2 gives the hatched and crosshatched areas associated with variables u^1_{ij} and u^2_{ij}. In Step 3, the remaining areas are defined and are associated with variables $u^3_{ij}, \dots, u^8_{ij}$.

In order to describe the lifting constraints, let us consider $\overline{X}^{ij}$ and $\underline{X}^{ij}$ the maximum and the minimum value of the x-coordinate of NFP_{ij}, respectively, and $\overline{Y}^{ij}$ and $\underline{Y}^{ij}$ the maximum and the minimum value of the y-coordinate, respectively. By decomposing the outer regions of the no-fit polygon according to the procedure previously described, for each $\overline{NFP}^k_{ij}$ we define $\overline{x}^k_{ij}$ and $\underline{x}^k_{ij}$ as the upper bound and the lower bound for variable x_j, and $\overline{y}^k_{ij}$ and $\underline{y}^k_{ij}$ as the upper bound and the lower bound for variable y_j. Let $VNFP_{ij}$ be the set of variables associated with NFP_{ij}. This set of variables can be decomposed according to the following index sets: $U_{ij} = \{k | u^k_{ij} \in VNFP_{ij}, \underline{y}^k_{ij} \ge 0\}$; $D_{ij} = \{k | u^k_{ij} \in VNFP_{ij}, \overline{y}^k_{ij} \le 0\}$; $R_{ij} = \{k | u^k_{ij} \in VNFP_{ij}, \underline{x}^k_{ij} \ge 0\}$; $L_{ij} = \{k | u^k_{ij} \in VNFP_{ij}, \overline{x}^k_{ij} \le 0\}$.

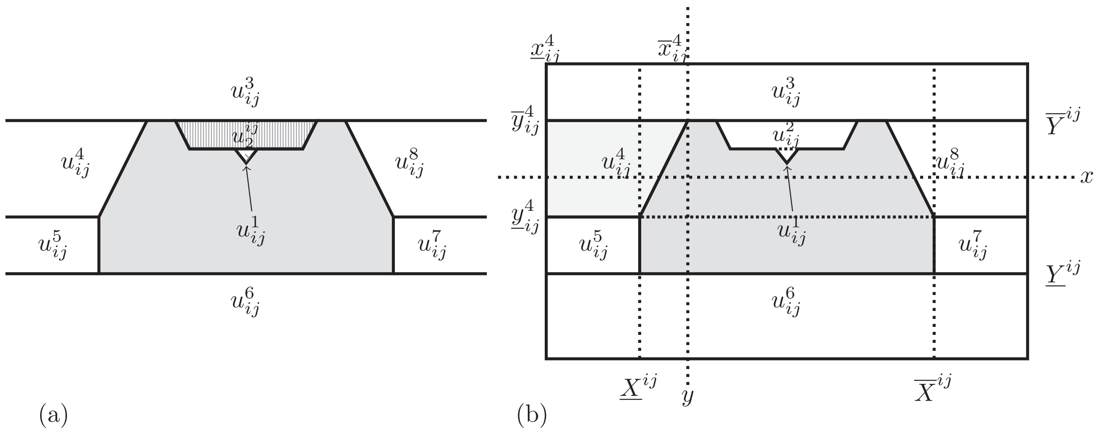


Fig. 10. (a) Representation of the decomposition procedure for the outside region of NFP_{ij}. (b) Definitions in the NFP_{ij}-coordinate system of Alvarez-Valdes et al. (2013).

Fig. 10(b) illustrates the coordinates $\overline{X}^{ij}$, $\underline{X}^{ij}$, $\overline{Y}^{ij}$ and $\underline{Y}^{ij}$ of the no-fit polygon. Each region $\overline{NFP}^k_{ij}$ is associated with a variable u^k_{ij}. We illustrate the upper and the lower bounds for variables x_j and y_j associated with variable u^4_{ij}: $\overline{x}^4_{ij}$, $\underline{x}^4_{ij}$, $\overline{y}^4_{ij}$ and $\underline{y}^4_{ij}$. The index sets of $VNFP_{ij}$ can be given by: $U_{ij}=\{1,2,3\}$, $D_{ij}=\{5,6,7\}$, $R_{ij}=\{7,8\}$ and $L_{ij}=\{4,5\}$.

If the reference point of piece j is above piece i, then a variable which index is in set U_{ij} will be equal to 1. Analogously, if the reference point is below, to the right or to the left, one variable which index is in set D_{ij}, R_{ij} or L_{ij} will be equal to 1, respectively. Then, the lifted constraints are given by:

$$\sum_{k\in R_{ij}} \underline{x}^k_{ij} u^k_{ij} \le x_j \le z - l_j - \sum_{k\in LS_{ij}} \lambda^k_{ij} u^k_{ij},\ 1\le i<j\le m, \tag{9}$$

$$\sum_{k\in R_{ij}} \underline{y}^k_{ij} u^k_{ij} \le y_j \le W - w_j - \sum_{k\in DS_{ij}} \mu^k_{ij} u^k_{ij},\ 1\le i<j\le m. \tag{10}$$

Constraints (9) are the lifted left-hand side and the lifted right-hand side bound constraint of a piece j. The lower bound on x_j is valid when piece j is on the right side of piece i, while the upper bound is valid if the reference point of piece j is on the left side. The coefficient λ^k_{ij} is given by $l_i-(\overline{x}^k_{ij}-\underline{X}^{ij})$ and set $LS_{ij}=\{k:\lambda^k_{ij}>0\}$. The lifted bottom and the lifted top bound constraints are given in (10). The lower bound on variable y_j is valid if piece j is above piece i, while the upper bound is valid if piece j is under piece i. In the upper bound calculus, the coefficients are $\mu^k_{ij}=w_i-(\overline{y}^k_{ij}-\underline{Y}^{ij})$ and set $DS_{ij}=\{k:\mu^k_{ij}>0\}$.

Scheithauer & Terno's constraints

To define the no-fit polygon, Scheithauer and Terno (1993) considered a sequence of the edges of NFP_{ij} between two neighboring vertices of its convex hull, where some half-planes associated with the edges have to be fulfilled. They defined the no-fit polygon using *and* and *or* relations. To illustrate that, let us consider the edges of the no-fit polygon in Fig. 11.

$NFP_{ij}=\{[(h^{ij}_{11}=-1.5x+y\le 0)\vee(h^{ij}_{12}=-x+1\le 0)]\wedge[h^{ij}_{21}=2x+3y-11\le 0]\wedge[h^{ij}_{31}=x-y-3\le 0]\wedge[h^{ij}_{41}=-y\le 0]\}$. Consequently,

$\overline{NFP}_{ij}=\{[(h^{ij}_{11}=-1.5x+y\ge 0)\wedge(h^{ij}_{12}=-x+1\ge 0)]\vee[h^{ij}_{21}=2x+3y-11\ge 0]\vee[h^{ij}_{31}=x-y-3\ge 0]\vee[h^{ij}_{41}=-y\ge 0]\}$.

In this case the outside region of NFP_{ij} is also divided into k_{ij} regions. The non-overlapping constraints are similar to (1) and (2), however, the authors reduced the number of variables without losing the optimal solution. For each pair of pieces i and j, there are k_{ij} binary variables to represent *and* and *or* conditions, where u^k_{ij} is equal to 1 if piece j is placed on region k, and 0 otherwise. To guarantee that at least one system of f^k_{ij} inequalities is satisfied, they used $\sigma_{ij}=\lceil\log_2(k_{ij})\rceil$ binary variables u^p_{ij}, $p=1,\dots,\sigma_{ij}$. The non-overlapping constraints are written with the binary coefficients (γ^{pk}_{ij}) of $2^{\sigma_{ij}}-k$, i.e., $\sum_{p=1}^{\sigma_{ij}}2^{\sigma_{ij}-p}\gamma^{pk}_{ij}=2^{\sigma_{ij}}-k$, $k=1,\dots,k_{ij}$. Thus, the non-overlapping constraints between two pieces i and j are written as:

$$h^{kf}_{ij}(x_j-x_i,y_j-y_i)+M\left(\sum_{p=1}^{\sigma_{ij}}\gamma^{pk}_{ij}+\sum_{p=1}^{\sigma_{ij}}(-1)^{\gamma^{pk}_{ij}}u^p_{ij}\right)\ge 0,$$
$$1\le i<j\le m,\ f=1,\dots,f^k_{ij},\ k=1,\dots,k_{ij}, \tag{11}$$

$$\sum_{p=1}^{\sigma_{ij}}2^{\sigma_{ij}-p}u^p_{ij}\ge 2^{\sigma_{ij}}-k_{ij},\ 1\le i<j\le m, \tag{12}$$

where $h^{kf}_{ij}(x,y)=\alpha^{kf}_{ij}x+\beta^{kf}_{ij}y-q^{kf}_{ij}$.

The main differences between constraints (1) and (2) and (11) and (12) are in the number of binary variables u^k_{ij}, and the terms that multiply coefficient M. Note that, the number of binary variables in Scheithauer & Terno's constraints is smaller than the number of variables in constraints (1) and (2), since $k_{ij}>\sigma_{ij}=\lceil\log_2(k_{ij})\rceil$, for all values of k_{ij}.

The polygons that represent the inner-fit polygon in irregular boards can be described by the union of convex polygons and constraints (11) and (12) can be easily adapted. Consider that the inner-fit polygon of piece i (IFP_i) is the union of k_{0i} convex polygons. Each convex polygon k has f^k_{0i} edges that are given by: $h^{kf}_{0i}(x_i,y_i)=\alpha^{kf}_{0i}x_i+\beta^{kf}_{0i}y_i-q^{kf}_{0i}$. Then, for piece i there are k_{0i} binary variables, where u^k_{0i} is equal to 1 if piece i is placed on region k, and 0 otherwise. Similarly to the non-overlapping constraints, the number of k_{0i} binary variables in the inner-fit polygon is reduced to $\sigma_{0i}=\lceil\log_2(k_{0i})\rceil$ variables u^p_{0i}, $p=1,\dots,\sigma_{0i}$. The inner-fit polygon constraints of piece i are written using the binary

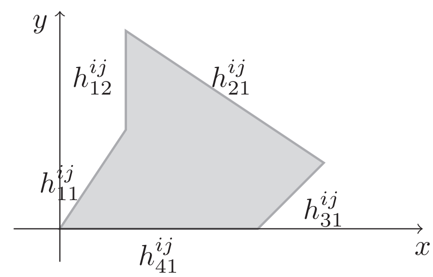


Fig. 11. No-fit polygon representation according to Scheithauer and Terno (1993).

coefficients of $2^{\sigma_{0i}} - k$, i.e., $\sum_{p=1}^{\sigma_{0i}} 2^{\sigma_{0i}-p}\gamma_{0i}^{pk} = 2^{\sigma_{0i}} - k$, $k = 1, \ldots, k_{0i}$. Considering the variable a_i that is equal to 1 if the piece is packed into the board, and 0 otherwise, the inner-fit polygon constraints are given as follows.

$$h_{0i}^{kf}(x_i, y_i) - M\left(\sum_{p=1}^{\sigma_{0i}} \gamma_{0i}^{pk} + \sum_{p=1}^{\sigma_{0i}} (-1)^{\gamma_{0i}^{pk}} u_{0i}^{p}\right) \le M(1 - a_i),$$
$$i = 1, \ldots, m, f = 1, \ldots, f_{0i}^{k}, k = 1, \ldots, k_{0i}, \quad (13)$$

$$\sum_{p=1}^{\sigma_{0i}} 2^{\sigma_{0i}-p} u_{0i}^{p} \ge 2^{\sigma_{0i}} - k_{0i},\ i = 1, \ldots, m. \quad (14)$$

Constraints of Cherri et al. **(2016b)**

Similarly to Scheithauer and Terno (1993), Cherri et al. (2016b) use the edges of the no-fit polygon to build the non-overlapping constraints. However, the authors decomposed the pieces into convex pieces and consequently the no-fit polygons are also described by convex parts. Then, a $NFP_{ij} = \bigcup_{p=1,\ldots,Q_{ij}} NFP_{ij}^{p}$, where Q_{ij} is the number of parts of NFP_{ij}. Pieces i and j do not overlap if the reference point of the fixed piece j is outside NFP_{ij}^{p} for all $p = 1, \ldots, Q_{ij}$. The non-overlapping constraints are ensured by forcing the reference point of piece j to be on the right side of an edge f, $f = 1, \ldots, f_{ij}^{p}$, of each NFP_{ij}^{p}, $p = 1, \ldots, Q_{ij}$. The development of these constraints are similar to the development of the non-overlapping constraints (1) and (2), and they are given by:

$$\alpha_{ij}^{p}(x_j - x_i) + \beta_{ij}^{p}(y_j - y_i) \le q_{ij}^{pf} + M(1 - u_{ij}^{pf}),$$
$$1 \le i < j \le m, p = 1, \ldots, Q_{ij}, f = 1, \ldots, f_{ij}^{p}, \quad (15)$$

$$\sum_{f=1}^{f_{ij}^{p}} u_{ij}^{pf} = 1,\ \ 1 \le i < j \le m, p = 1, \ldots, Q_{ij}. \quad (16)$$

The authors discussed some valid inequalities derived from the geometry feature of different parts of the no-fit polygon, ways of eliminating redundant variables u_{ij}^{pf}, breaking symmetry of the positioning variables x_i, and the estimation of the big M, which we do not explain here for brevity. In addition, they incorporate fixed rotations to pieces in the model. In this case, a replica for each piece orientation is added to the set of pieces. Consider Γ_s as the set of a piece s in all allowed orientations, $s = 1, \ldots, S$. Since the demand of the pieces is unitary, only one piece orientation from each set Γ_s is chosen as stated in constraints (17). Note that S is equal to m, but the authors defined it in a more general situation that a set Γ_s can include mutually exclusive pieces.

$$\sum_{i\in\Gamma_s} \delta_i = 1, \quad s = 1, \ldots, S. \quad (17)$$

To eliminate symmetries, constraints (16) are replaced by (18)–(20).

$$\sum_{f=1}^{f_{ij}^{p}} u_{ij}^{pf} \ge \delta_i + \delta_j - 1, \quad 1 \le i < j \le m,\ p = 1, \ldots, Q_{ij}, \quad (18)$$

$$\sum_{f=1}^{f_{ij}^{p}} u_{ij}^{pf} \le \delta_i, \quad 1 \le i < j \le m,\ p = 1, \ldots, Q_{ij}, \quad (19)$$

$$\sum_{f=1}^{f_{ij}^{p}} u_{ij}^{pf} \le \delta_j, \quad 1 \le i < j \le m,\ p = 1, \ldots, Q_{ij}. \quad (20)$$

4.1.2. Constraints based on direct trigonometry

The mathematical models of Scheithauer and Terno (1993) and Cherri et al. (2016b) based on direct trigonometry describe the non-overlapping constraints using the D-function. Scheithauer and Terno (1993) consider only convex pieces, while Cherri et al. (2016b) also consider non-convex pieces, but they decompose non-convex pieces into convex parts. First, consider that there is only one part in each piece (convex pieces).

Scheithauer & Terno's constraints

Two pieces i and j do not overlap if the vertices of piece j are on the right hand side of an oriented edge of piece i. Let (x_i^A, y_i^A) and (x_i^B, y_i^B) be two consecutive vertices of piece i that define the edge e, and (x_j^v, y_j^v) be the vertex v of piece j. The positioning of vertex v of piece j is given by $(x_j^v + x_j, y_j^v + y_j)$. The horizontal and the vertical relative position between this point and the positioning of the reference point of piece i, (x_i, y_i), can be represented as $(x_j^v + x_j - x_i, y_j^v + y_j - y_i)$. The vertex v of piece j is on the right hand side of edge defined by points (x_i^A, y_i^A) and (x_i^B, y_i^B) if:

$$(x_i^A - x_i^B)(y_i^A - (y_j^v + y_j - y_i)) - (y_i^A - y_i^B)(x_i^A - (x_j^v + x_j - x_i)) \le 0.$$

This inequality can be rewritten as:

$$\bar{\alpha}_i^e(x_j - x_i) + \bar{\beta}_i^e(y_j - y_i) - \bar{q}_{ij}^{ev} \le 0,$$

where $\bar{q}_{ij}^{ev} = (x_i^B - x_i^A)(y_i^A - y_j^v) + (y_i^A - y_i^B)(x_i^A - x_j^v)$, $\bar{\alpha}_i^e = (x_i^B - x_i^A)$, $\bar{\beta}_i^e = (y_i^A - y_i^B)$.

The value of $\bar{q}_{ij}^{ev}$ depends on the vertices of piece j, then this inequality is valid for $\bar{q}_{ij}^{e} = \max_{v=1,\ldots,v_j}\{\bar{q}_{ij}^{ev}\}$. The variables related to this constraint are defined as $\bar{u}_{ij}^{e}$, which is equal to 1 if piece j is on the right side of edge e of piece i, 0 otherwise. Therefore, the non-overlapping constraints in the model of Scheithauer and Terno (1993) are written as follows:

$$\bar{\alpha}_i^e(x_j - x_i) + \bar{\beta}_i^e(y_j - y_i) \le \bar{q}_{ij}^{e} + M(1 - \bar{u}_{ij}^{e}),$$
$$e = 1, \ldots, e_i, 1 \le i < j \le m, \quad (21)$$

$$\sum_{e=1}^{e_i} \bar{u}_{ij}^{e} + \sum_{e=1}^{e_j} \bar{u}_{ij}^{e} \ge 1,\ \ 1 \le i < j \le m. \quad (22)$$

For problems that consist of packing convex pieces into a convex board, the inner-fit polygon constraints can be defined similarly to the non-overlapping constraints, i.e., if a piece is packed, it must be on the left side of all edges of the board. Let 0 be the board index, and e_0 be the number of edges that defines the board. If a piece is inside the board, the following constraints must be satisfied:

$$\bar{\alpha}_0^e x_i + \bar{\beta}_0^e y_i \le \bar{q}_{0i}^{e}, \quad e = 1, \ldots, e_0.$$

Thus, the inner-fit polygon constraints are given by:

$$\bar{\alpha}_0^e x_i + \bar{\beta}_0^e y_i \le \bar{q}_{0i}^{e} + M(1 - a_i), \quad i = 1, \ldots, m, e = 1, \ldots, e_0. \quad (23)$$

Constraints of Cherri et al. **(2016b)**

The direct trigonometry model of Cherri et al. (2016b) is used to solve the irregular strip packing problem. Since the model considers pieces of any shape, to handle the complex geometry they decompose the pieces into convex parts. In addition, the authors adapt the model to consider fixed rotations of pieces. The non-overlapping constraints are similar to (21) and (22). However, two more indexes related to the parts of the pair of pieces i and j are added. Suppose that each piece i is decomposed into P_i parts. Each part of piece i cannot overlap each part of piece j, where the number of edges in each one is denoted by e_i^p. The non-overlapping

Table 2
Continuous positioning models.

Authors	Model n.	Problem	Objective	Constraints	Linear var.	Binary var.	Orientation
Scheithauer and Terno (1993)	M1.1	knapsack	(29)	(11)–(14)	x_i, y_i	a_i, u^k_{ij}, u^k_{0i}	fixed
Scheithauer and Terno (1993)	M1.2	knapsack	(29)	(21)–(23)	x_i, y_i	a_i, u^k_{ij}, u^k_{0i}	fixed
Daniels et al. (1994)	M2	feasibility	-	(1)–(4)	x_i, y_i	u^k_{ij}, u^k_{0i}	fixed
Dean (2002)	M3	strip packing	(30)	(2), (5)–(7)	z, x_i, y_i	u^k_{ij}	fixed
Fischetti and Luzzi (2009)	M4	strip packing	(31)	(2), (6)–(8)	z, x_i, y_i	u^h_{ij}	fixed
Alvarez-Valdes et al. (2013)	M5	strip packing	(30)	(2), (8)–(10)	z, x_i, y_i	u^h_{ij}, u^k_{ij}	fixed
Cherri et al. (2016b)	M6.1	strip packing	(30)	(6), (7), (15), (16)	z, x_i, y_i	u^{pf}_{ij}	fixed
Cherri et al. (2016b)	M6.2	strip packing	(30)	(6), (7), (15), (17)–(20)	z, x_i, y_i	u^{pf}_{ij}, δ_j	few rotations
Cherri et al. (2016b)	M6.3	strip packing	(30)	(6), (7), (24), (25)	z, x_i, y_i	$\bar{u}^{ers}_{ij}$	fixed
Cherri et al. (2016b)	M6.4	strip packing	(30)	(6), (7), (15), (24), (26)–(28)	z, x_i, y_i	$\bar{u}^{ers}_{ij}$	few rotations

constraints between pieces i and j are given by (24) and (25), where the variables $\bar{u}^e_{ij}$ are replaced by $\bar{u}^{ers}_{ij}$.

$$\bar{\alpha}^e_i(x_j - x_i) + \bar{\beta}^e_i(y_j - y_i) \le \bar{q}^{ers}_{ij} + M(1 - \bar{u}^{ers}_{ij}),$$
$$r = 1, \dots, P_i, s = 1, \dots, P_j, e = 1, \dots, e^p_i, \tag{24}$$

$$\sum_{e=1}^{e^p_i} \bar{u}^{ers}_{ij} + \sum_{e=1}^{e^p_j} \bar{u}^{ers}_{ji} = 1,\ r = 1, \dots, P_i, s = 1, \dots, P_j, 1 \le i < j \le m. \tag{25}$$

In Scheithauer and Terno (1993), constraints (25) are considered as inequalities, which means for this model that more than one edge can be active for the same pair of parts. However, optimal solutions are not lost if only one edge is active (equality constraints). To incorporate fixed rotation to the pieces and eliminate symmetries, constraints (25) are replaced by (17) and (26)–(28), where constraints (27) and (28) eliminate symmetries.

$$\sum_{e=1}^{e^p_i} \bar{u}^{ers}_{ij} + \sum_{e=1}^{e^p_j} \bar{u}^{ers}_{ji} \ge \delta_i + \delta_j - 1,$$
$$1 \le i < j \le m, r = 1, \dots, P_i, s = 1, \dots, P_j \tag{26}$$

$$\sum_{e=1}^{e^p_i} \bar{u}^{ers}_{ij} + \sum_{e=1}^{e^p_j} \bar{u}^{ers}_{ji} \le \delta_i, \quad 1 \le i < j \le m, r = 1, \dots, P_i, s = 1, \dots, P_j \tag{27}$$

$$\sum_{e=1}^{e^p_i} \bar{u}^{ers}_{ij} + \sum_{e=1}^{e^p_j} \bar{u}^{ers}_{ji} \le \delta_j, \quad 1 \le i < j \le m, r = 1, \dots, P_i, s = 1, \dots, P_j. \tag{28}$$

4.1.3. Complete continuous positioning models

Continuous positioning models were proposed for the knapsack problem, the strip packing problem and the feasibility problem. In knapsack problems, the objective function is to maximize the total profit of the packed pieces, which is given by:

$$\text{maximize} \quad \sum_{i=1}^{m} c_i a_i. \tag{29}$$

As the objective function in strip packing problems is to minimize the total length of the board used to pack all pieces, the objective is described as follows:

$$\text{minimize} \quad z. \tag{30}$$

A slightly different objective function was considered in Fischetti and Luzzi (2009) where the space among the pieces is also minimized, which is given as follows.

$$\text{minimize} \quad z + \varepsilon \sum_{i=1}^{m} (x_i + y_i). \tag{31}$$

The continuous positioning models proposed for nesting problems are summarized in Table 2. These mathematical models were solved by exact algorithms, except the models of Scheithauer and Terno (1993) for which no computational experiments were performed. The authors used commercial or free solvers and a branch-and-bound algorithm was also designed in Alvarez-Valdes et al. (2013). Despite the progress of computers and optimization solvers, only small problems can be optimally solved. More recent computational experiments for the strip packing problem considering pieces with fixed orientation and without holes showed that optimal solution was proved only for problems with up to 14 pieces within a time limit of 3600 seconds. In addition, a feasible solution was found for instances with up to 45 pieces. If pieces with holes and rotations of 0 and 180 degrees are considered, only instances with up to 10 pieces are optimally solved. A detailed comparison among the models of Fischetti and Luzzi (2009), Alvarez-Valdes et al. (2013) and Cherri et al. (2016b) when solved by CPLEX can be found in Alvarez-Valdes et al. (2013) and Cherri et al. (2016b).

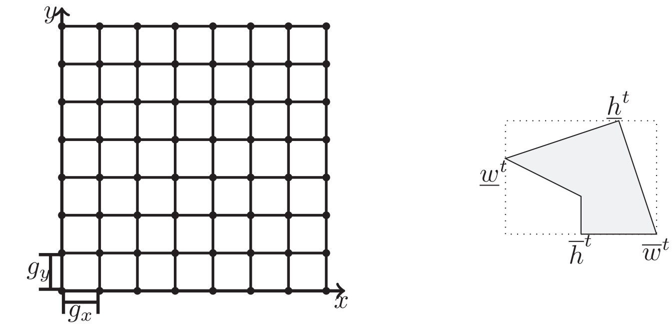
4.2. Discrete models

In this section, we discuss two types of discrete models that are based on the no-fit polygon and raster point methods. They were proposed for the strip packing problem, the knapsack problem and the bin packing problem.

4.2.1. Constraints based on no-fit polygon

The model of Toledo et al. (2013) consists of representing the board by a grid, where the reference point of the pieces can only be placed at the dots on the board. Fig. 12(a) illustrates a discretized board. The x-axis and the y-axis have resolution g_x and g_y, respectively. The dots represent where the reference point of the pieces can be placed. In this board representation, the inner-fit polygon of a piece is the set of points where its reference point can be placed such that it is entirely inside the board. Likewise, the no-fit polygon NFP_{ij} is the set of points where piece j cannot be positioned such that pieces i and j do not overlap.

In contrast to the previous models, the pieces are grouped according to their type, where pieces of type t have a demand q_t. The reference point of pieces of type t is (0, 0) and the vertices are recorded in the clockwise direction, where the list of vertices is denoted by $((x^t_1, y^t_1), (x^t_2, y^t_2), \dots, (x^t_{\nu_t}, y^t_{\nu_t}))$. The enveloping rectangle of pieces of type t is denoted by $\underline{w}^t, \overline{w}^t, \underline{h}^t, \overline{h}^t$, where $\underline{w}^t = \min_{\nu \in \{1,\dots,\nu_t\}}\{x^t_\nu\}$, $\overline{w}^t = \max_{\nu \in \{1,\dots,\nu_t\}}\{x^t_\nu\}$, $\underline{h}^t = \min_{\nu \in \{1,\dots,\nu_t\}}\{y^t_\nu\}$, $\overline{h}^t =$



(a) (b)

Fig. 12. (a) Dotted board representation. (b) Enveloping rectangle parameters of a piece in Toledo et al. (2013).

$\max_{\nu \in \{1,\ldots,\nu_t\}}\{y^t_\nu\}$. Fig. 12(b) shows the enveloping rectangle parameters of pieces of type t. For their mathematical formulation, let us consider the following sets.

- $\mathcal{T} = \{1, \ldots, T\}$: set of piece types.
- $\mathcal{C} = \{0, \ldots, C-1\}$: set of board columns, where $C = \lfloor L/g_x \rfloor + 1$ is the number of columns on the board.
- $\mathcal{R} = \{0, \ldots, R-1\}$: set of board rows, where $R = \lfloor W/g_x \rfloor + 1$ is the number of rows on the board.
- $\mathcal{D} = \{1, \ldots, D\}$: set of board dots, where $D = C \times R$ is the number of dots on the board.
- $\mathcal{IFP}_t$: the inner-fit polygon of piece $t \in \mathcal{T}$ ($\mathcal{IFP}_t \subset \mathcal{D}$).
- $\mathcal{NFP}^d_{t,u}$: the no-fit polygon between piece $u \in \mathcal{T}$ and $t \in \mathcal{T}$ when piece t is placed at $d \in \mathcal{IFP}_t$.

In this model, the variable related to the positioning of each piece is δ^d_t, which is equal to 1 if the reference point of a piece of type t is positioned at dot d, 0 otherwise. If the reference point of piece t is at dot $d = (c, r) \in (\mathcal{C} \times \mathcal{R})$, then the length of the board used to pack the piece is $cg_x + \overline{w}^t$. Since the problem solved is the strip packing problem, the inner-fit polygon constraints are written as:

$$(cg_x + \overline{w}^t)\delta^d_t \le z, \quad \forall t \in \mathcal{T}, \forall d \in \mathcal{IFP}_t. \tag{32}$$

In addition, it is necessary to consider an upper bound ($\bar{L}$) on the board length z to reduce the number of dots on the board. The non-overlapping constraints and the demand constraints are given respectively by:

$$\delta^e_u + \delta^d_t \le 1, \quad \forall t, u \in \mathcal{T}, \forall e \in \mathcal{NFP}^d_{t,u}, \forall d \in \mathcal{IFP}_t, \tag{33}$$

$$\sum_{d \in \mathcal{IFP}_t} \delta^d_t = q_t, \quad \forall t \in \mathcal{T}. \tag{34}$$

Rodrigues and Toledo (2017) proposed a clique covering model based on Toledo et al. (2013) and as stated by the authors it reduces the number of constraints, the number of nonzeros in the model and provide better lower bounds. In this case, the non-overlapping constraints are written according to an undirected conflict graph $G = (V, E)$. Variables δ^d_t correspond to vertices in V and an overlap between two pieces corresponds to an edge in E. Then, a feasible solution for the problem is a stable set in the graph. In order to define the new constraints, consider $K \subseteq V$ as a clique from G, $\mathcal{E}$ as an edge clique covering of G and $\mathcal{V}$ as a vertex clique covering of G. Constraints (32) and (33) are replaced by constraints (35) and (36), respectively.

$$\sum_{(d,t)\in K} (cg_x + \overline{w}^t)\delta^d_t \le z, \ \forall K \in \mathcal{V}, \tag{35}$$

$$\sum_{(d,t)\in K} \delta^d_t \le 1, \quad \forall K \in \mathcal{E}. \tag{36}$$

In constraints (35), only one vertex in a clique from the vertex clique covering can be chosen, then the number of constraints is reduced by summing all vertex in each clique. Any solution that satisfies constraints (36) is a stable set which ensures the non-overlapping between two pieces. Constraints (36) together with constraints (34) ensure a feasible solution to the strip packing problem. In addition, the authors proposed a set of valid inequalities and a method to calculate a stronger lower bound that are not presented here for brevity.

4.2.2. Constraints based on raster points

Baldacci et al. (2014) proposed a raster point model for irregular knapsack problems, where pieces and boards are irregular, have defects and different levels of quality. 90 and 180 degrees rotations of pieces were also considered. To geometrically represent the problem, the authors rounded down the board shape and rounded up the surface of pieces and defects. Let $\mathcal{D}$ denote the set of pixels of the board. The set of pixels in $\mathcal{D}$ occupied by a piece feasibly packed into the board is called configuration. For the mathematical model, they first determine all possible configurations for each piece of type $t \in T$ on the board (the index set is denoted by $\mathbb{C}_t$), which include different rotations. In addition, the index set of all possible configurations is given by $\mathbb{C} = \mathbb{C}_1 \cup \ldots \cup \mathbb{C}_T$. A configuration $l \in \mathbb{C}$ is a configuration of a piece t and is denoted by $C_l \subset \mathcal{D}$. The quality level of a pixel (x, y) is denoted by q_{xy}, which is equal to zero if it is a hole and q_{max} if it is of highest quality. Then the value of this configuration is $v_l = \sum_{(x,l)\in C_l} q_{xy}$. The variables in this model are denoted by ϵ_l and consist of deciding whether a piece t is cut from the board according to configuration $l \in \mathbb{C}_t$. Then, the knapsack problem consists of maximizing the value of the chosen configurations that is represented by:

$$\text{maximize} \quad \sum_{l \in \mathbb{C}} v_l \epsilon_l. \tag{37}$$

Consider $\mathbb{P}_{xy} = \{l \in \mathbb{C} : (x, y) \in C_l\}$ as the subset of configuration index $\mathbb{C}$ covering the pixel (x, y) of the board. The model must ensure that each pixel cannot be covered by more than one piece and the demand is not exceeded, which are represented by constraints (38) and (39), respectively.

$$\sum_{l \in \mathbb{P}_{xy}} \epsilon_l \le 1, \ \forall (x, y) \in D, \tag{38}$$

$$\sum_{l \in \mathbb{C}_l} \epsilon_l \le q_t, \ \forall t \in \mathcal{T}. \tag{39}$$

The authors used the irregular single knapsack model as a suproblem of the irregular multiple stock-size cutting stock problem. Consider that there are n different boards available and $\mathbb{L}_b$ is the number of cutting patterns for board P^{0b}, $b = 1, \ldots, n$. Let $\mathbb{L} = \mathbb{L}_1 \cup \ldots \cup \mathbb{L}_n$ be the index set of all cutting patterns. A cutting pattern $l \in \mathbb{L}$ has cost $c_l = \sum_{(x,y)\in \mathbb{L}_b} q^b_{xy}$ and p_{tl} is the number of pieces of type t in this cutting pattern. The problem consists of determining if a board cut according to a cutting pattern is used or not, where the variable is represented by φ_l, while minimizing the cost of the cutting patterns used. Then, the objective function is given by:

$$\text{minimize} \quad \sum_{l \in \mathbb{L}} c_l \varphi_l. \tag{40}$$

Since each board is unique, a board can be cut according to only one cutting pattern, which is represented in constraints (41). In addition, the demand must be met, which is ensured by constraints (42).

$$\sum_{l \in \mathbb{L}_b} \varphi_l \le 1, \ b = 1, \ldots, n, \tag{41}$$

$$\sum_{l \in \mathbb{L}} p_{tl} \varphi_l \ge q_t, \ \forall t \in \mathcal{T}. \tag{42}$$

Table 3
Discrete positioning models.

Authors	Model n.	Problem	Objective	Constraints	Linear var.	Binary var.	Orientation
Toledo et al. (2013)	M7	strip packing	(30)	(32)–(34)	z	δ_t^d	fixed
Rodrigues and Toledo (2017)	M8	strip packing	(30)	(34)–(36)	z	δ_t^d	fixed
Baldacci et al. (2014)	M9.1	knapsack	(37)	(38), (39)		ϵ_l	few rotations
Baldacci et al. (2014)	M9.2	bin packing	(40)	(41), (42)		φ_l	few rotations

4.2.3. Complete discrete positioning models

All the discrete positioning models proposed for nesting problems are summarized in Table 3.

The models of Toledo et al. (2013) and Rodrigues and Toledo (2017) were solved by CPLEX, while the model of Baldacci et al. (2014) was solved by a heuristic. A comparison between the models of Toledo et al. (2013) and Rodrigues and Toledo (2017) showed that the clique covering model provided better results. It proved the optimality for a strong heterogeneous instance with up to 25 pieces, while the model of Toledo et al. (2013) proved the optimality for instances with up to 21 pieces within a time limit of 3600 seconds. According to Baldacci et al. (2014) due to the huge number of variables, solving the irregular knapsack model by a MIP solver is impractical. Then, the authors used a Lagrangian relaxation of the irregular single knapsack model in a heuristic method. The master problem was also solved by a heuristic method. However, the performance of the models presented in this section are subject to the discretization used and the number of piece types.

4.3. Semi-continuous model

Leao et al. (2016) proposed a mathematical model for the strip packing problem that discretizes the board, the no-fit polygons and the inner-fit polygons in the horizontal direction. As the board has unlimited length, by discretizing the board in the y-axes, the number of stripes on the board is smaller than the number of stripes produced by a vertical discretization.

Consider that the board has S stripes, the same distance between the stripes on the board is applied to no-fit and inner-fit polygons (see Fig. 13). The reference point of the pieces must be placed only on the board stripes. Then, each piece i is associated with a continuous variable and a set of binary variables. The continuous variable x_i refers to the positioning of its reference point in the x-coordinate. A binary variable δ_i^{si} is equal to 1 if the reference point of piece i is on stripe si of the board, and 0 otherwise, for $0 \leq si \leq S-1$.

Analogously to the continuous positioning model, the pieces of the same type are individually treated. For each piece i, $r_i^{\min}$ and $r_i^{\max}$ are the minimum and the maximum x-coordinate, and $t_i^{\min}$, $t_i^{\max}$ are the minimum and the maximum y-coordinate, for $i = 1, \ldots, m$. These parameters define the inner-fit rectangle of a piece i, that are described by the following constraints:

$$x_i \geq -r_i^{min}, \qquad 1 \leq i \leq m, \tag{43}$$

$$\sum_{s=-t_i^{min}}^{S-t_i^{max}} \delta_i^s = 1, \qquad 1 \leq i \leq m. \tag{44}$$

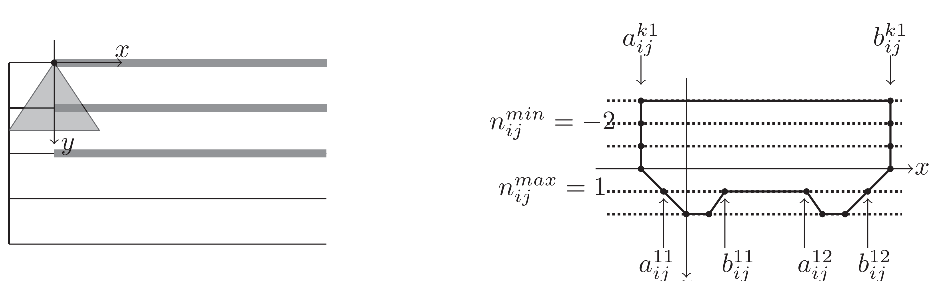
Constraints (44) also ensure that all pieces are packed on the board.

The origin of the coordinate system of no-fit polygon NFP_{ij} is defined at the reference point of piece i. The reference point of piece j can be placed only on the stripes outside the no-fit polygon. For each stripe, we must define feasible positions for variable x_j according to the no-fit polygon coordinates. The parameters associated with each no-fit polygon NFP_{ij} are:

- $n_{ij}^{\max}$: the maximum stripe of NFP_{ij};
- $n_{ij}^{\min}$: the minimum stripe of NFP_{ij};
- a_{ij}^{kc}: the minimum x-coordinate of stripe k in concavity c, $1 \leq i < j \leq m$, $n_{ij}^{\min} \leq k \leq n_{ij}^{\max}$, $c = 1, ..., C_{ij}^k$, $k = n_{ij}^{\min}, ..., n_{ij}^{\max}$;
- b_{ij}^{kc}: the maximum x-coordinate of stripe k in concavity c, $1 \leq i < j \leq m$, $n_{ij}^{\min} \leq k \leq n_{ij}^{\max}$, $c = 1, \ldots, C_{ij}^k$, $k = n_{ij}^{\min}, \ldots, n_{ij}^{\max}$.

Fig. 13(b) shows the parameters of NFP_{ij}, which has one concavity. The non-overlapping constraints are defined if the relative position in the y-coordinate of pieces i and j $(sj - si)$ is between n_{ij}^{min} and n_{ij}^{max}. In this case, when piece j is placed on stripe sj, we must ensure that x-coordinate of its reference point is on the left hand side of $a_{ij}^{(sj-si)c}$ or on the right hand side of $b_{ij}^{(sj-si)c}$ for each concavity c. To describe the non-overlapping constraints, we need the additional variable γ_{ij}^c that is equal to 1 if the reference point of piece i is on the right side of piece j in concavity c, 0 otherwise, $c = 1, \ldots, C_{ij}^k$, $k = n_{ij}^{\min}, \ldots, n_{ij}^{\max}$. The non-overlapping constraints are defined as follows.

$$\begin{aligned} & x_i \leq x_j - b_{ij}^{(s_j-s_i)c} + \gamma_{ij}^c M + (1-\delta_i^{s_i})M + (1-\delta_j^{s_j})M, \\ & 1 \leq i \leq N, \quad i+1 \leq j \leq N, \quad c = 1, \ldots, C_{ij}^{s_j-s_i}, -t_i^{min} \leq s_i \leq S - t_i^{max}, \\ & -t_j^{min} \leq s_j \leq S - t_j^{max}, n_{ij}^{min} \leq s_j - s_i \leq n_{ij}^{max}, \end{aligned} \tag{45}$$



(a) The inner-fit polygon of a piece. (b) Parameters of NFP_{ij}.

Fig. 13. Inner-fit polygon and no-fit polygon for the semi-continuous model (take from Leao et al. (2016)).

Table 4
Semi-continuous positioning model.

Authors	Model n.	Problem	Objective	Constraints	Linear var.	Binary var.	Orientation
Leao et al. (2016)	M10	strip packing	(30)	(43)–(47)	z, x_i	$\delta_i^{s_i}$, γ_{ij}^c	fixed

$$x_i \geq x_j - a_{ij}^{(s_j-s_i)c} - (1-\gamma_{ij}^c)M - (1-\delta_i^{s_i})M - (1-\delta_j^{s_j})M,$$
$$1 \leq i \leq N, \quad i+1 \leq j \leq m, \quad c = 1,\ldots,C_{ij}^{s_j-s_i}, \tag{46}$$
$$-t_i^{min} \leq s_i \leq S - t_i^{max}, \quad -t_j^{min} \leq s_j \leq S - t_j^{max},$$
$$n_{ij}^{min} \leq s_j - s_i \leq n_{ij}^{max}.$$

Constraints (45) will be active if the relative position between piece i and j is between $n_{ij}^{\min}$ and $n_{ij}^{\max}$ and piece j is on the left side of piece i, while constraints (46) will be active if piece j is on the right side of piece i. The length of the board that must be minimized is defined is defined by constraint (47).

$$z \geq x_i + r_i^{max}, \qquad 1 \leq i \leq m. \tag{47}$$

The summary of the semi-continuous positioning model is given in Table 4. This model was solved by CPLEX with a time limit of one hour, which found an optimal solution only for small problems (up to 10 pieces). However, it is possible to find a feasible solution for large instances, where the authors analyzed instances with up to 70 pieces.

4.4. Remarks

Integer linear programming models have also been used to solve simplified versions of irregular cutting problems. For example, Santoro and Lemos (2015) proposed a continuous positioning model to solve irregular strip packing problems where pieces are convex polygons with number of sides limited to 8. Litvinchev, Infante, and Ozuna (2015b) proposed a mathematical model to pack circular pieces in a rectangular board with fixed dimensions. However, by not limiting the distance metrics to the Euclidean one, the pieces can be ellipses, rectangles, rhombuses, octagons, etc. These ideas were extended in Litvinchev, Infante, and Ozuna (2015a) by including L-shaped pieces. Their model consists of discretizing the board into a set of dots where the reference point of the pieces can be placed.

Compaction and separation models have been used in heuristic methods to improve a given layout or even to obtain a feasible layout if there is overlapping among pieces. These models are usually continuous models, like those described in Section 4.1.1, in which binary variables indicating the relative position of the pieces are fixed and therefore a linear model with the real variables indicating the position of the pieces on the board has to be solved. These ideas were discussed in Li and Milenkovic (1995), Bennell and Dowsland (2001) and Gomes and Oliveira (2006), where some differences can occur in the objective function or in the introduction of artificial variables (Gomes & Oliveira, 2006). In addition, Li and Milenkovic (1993) proposed a compaction model inspired by the physic concept of translational motion of objects. Stoyan et al. (1996) proposed a linear programming model based on phi-function for compaction of polygonal pieces with fixed orientation. In this case, the non-overlapping constraints are similar to those based on no-fit polygon.

Matheuristics based on the MIP models have recently been proposed for nesting problems. Cherri, Carravilla, and Toledo (2016a) developed an iterative heuristic method based on the mathematical model of Toledo et al. (2013) for the strip packing problem. However, to reduce the number of dots on the board, they used irregular grids. The heuristic has also a compaction phase, where the compaction model of Gomes and Oliveira (2006) with additional constraints was solved. Martinez-Sykora, Alvarez-Valdes, Bennell, Ruiz, and Tamarit (2017) proposed a constructive algorithm that uses one-dimensional bin packing models and the mathematical model of Alvarez-Valdes et al. (2013) to solve two-dimensional irregular bin packing problems where pieces can be freely rotated.

5. Non-linear programming models

A characteristic of integer linear programming models is that fixed orientation or just a few rotations are allowed to the pieces. When continuous rotation is taken into account, the mathematical programming models become non-linear. The geometric tools used to deal with continuous rotation in the literature are phi-functions, quasi phi-functions and direct trigonometry (circle cover representation, D-function and separation lines).

5.1. Constraints based on phi-function and quasi phi-functions

Chernov et al. (2010), Stoyan et al. (2012), Stoyan and Romanova (2013), Stoyan et al. (2016a), Stoyan et al. (2017) and Stoyan et al. (2015), Stoyan et al. (2016b), Romanova et al. (2018) described mathematical models for nesting problems with continuous rotation of pieces based on phi-functions and quasi phi-functions, respectively. In Stoyan et al. (1996), the phi-function based model is linear as pieces are represented by half-spaces and have fixed orientation. The constraints in mathematical models that use max-min phi-functions can be linear, quadratic and piecewise inequalities. They represent the minimum and the maximum distance allowable between phi-objects and there are different ways to develop these expressions. The constraints in mathematical models based on quasi phi-functions are simpler than those based on phi-function. However, by adding auxiliary variables the size of the models increases. Tools and related papers have been described in Section 3.4.

For the general statement of the constraints based on phi-functions, let $\mathbf{P}_0$ be the phi-object that represents the board, $\mathbf{P}_0^c$ be the complement of $\mathbf{P}_0$, i.e., $\mathbf{P}_0^c = \mathbb{R}^2 \backslash \mathbf{P}_0$, and $\mathbf{P}_{\bar{0}} = cl(\mathbf{P}_0^c)$, where cl denotes the closure of the complement of $\mathbf{P}_0$. The inner containment of pieces on the board holds if:

$$\phi^{\bar{0}i} \geq 0, \quad i = 1,\ldots,m. \tag{48}$$

In other words, the phi-object $\mathbf{P}_i$ cannot overlap the phi-object $\mathbf{P}_0^c$. A limit on the positioning and rotation angles of pieces on the container can be ensured by considering an upper and a lower bound on the variables $\mathcal{O}_i$:

$$\underline{\mathcal{O}}_i \leq \mathcal{O}_i \leq \overline{\mathcal{O}}_i, \quad i = 1,\ldots,m. \tag{49}$$

The non-overlapping constraints are given by:

$$\phi^{ij} \geq 0, \quad 1 \leq i < j \leq m. \tag{50}$$

A suitable distance between a piece and the container walls, and between two pieces can also be taken into account. The distance constraints are written using normalized phi-functions and adjusted phi-functions, which are denoted by $\tilde{\phi}$. Let ρ_{ij}^- and ρ_{ij}^+ denote the minimum and the maximum distance between pieces i and j, where ρ_{0j}^- and ρ_{0j}^+ are the minimum and the maximum distance between piece j and the walls of the container. Then, the distance constraints are given by:

$$\rho_{0i}^- \leq \tilde{\phi}^{\bar{0}i} \leq \rho_{0i}^+, \; i = 1,\ldots,m, \tag{51}$$

$$\rho_{ij}^{-} \leq \tilde{\phi}^{ij} \leq \rho_{ij}^{+}, \;\; 1 \leq i < j \leq m. \tag{52}$$

Despite the difference between phi-functions and quasi phi-functions described in Section 3.4, the constraints in models based on quasi phi-functions can be generally stated as constraints (48)–(52). The objective function depends on the placement parameters of pieces, variable sizes of pieces, on metrical characteristics of the container as well as on auxiliary variables when quasi phi-functions are used. It can be generally stated as follows.

$$\text{minimize } F(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}m), \tag{53}$$

where $\mathcal{O}_0$ are the variables related to the board. In particular, the two-dimensional problem can minimize the length, the perimeter, the area of rectangular containers or the radius of circular containers (Stoyan et al., 2016a; 2017), while three-dimensional problems can minimize the used volume of the containers (Chernov et al., 2010; Romanova et al., 2018; Stoyan & Romanova, 2013; Stoyan et al., 2015). Stoyan et al. (2012) also considered a board with defects that is described as a multiply connected region.

5.2. Constraints based on direct trigonometry

By assuming continuous rotation of pieces, the mathematical models based on direct trigonometry become non-linear. Three types of trigonometry were used, which are circle covering representation, D-function and separation lines. Jones (2014), Rocha et al. (2014) and Rocha et al. (2016) proposed mathematical models for the irregular strip packing problem based on the circle covering representation of pieces. In Jones (2014), pieces are described by a set of inscribed circles, while in Rocha et al. (2016) they are described by complete circle covering representation. To ensure the non-overlapping between two pieces, circles of a piece i cannot overlap circles of a piece j. A piece is entirely inside the board if all circles are also inside the board. The D-function was used in Cherri et al. (2018) to define the non-overlapping constraints, while the separation lines were considered in Peralta et al. (2018).

Rocha et al.**'s constraints**

In the non-linear programming model described in Rocha et al. (2016, 2014), each piece is covered by a set of circles, where the center of circle 0 is the reference point of the piece. For each piece i, we define n_i as the number of circles, r_i^l as the radius of circle l, G_{li} as the distance between circle 0 and l and θ_i^{0l} is the initial angle of circle l. The decision variables related to piece i are x_i, y_i, θ_i, which define its position in x and y-axis, and its orientation, respectively. The non-overlapping constraints are given as follows:

$$\begin{aligned} &\left[(x_i + G_{li}\cos(\theta_i^{0l} + \theta_i)) - (x_j + G_{kj}\cos(\theta_j^{0k} + \theta_j))\right]^2 \\ &\quad + \left[(y_i + G_{li}\sin(\theta_i^{0l} + \theta_i)) - (y_j + G_{kj}\sin(\theta_j^{0k} + \theta_j))\right]^2 \\ &\geq (r_i^l + r_j^k)^2, \quad 1 \leq i < j \leq m, \\ &l = 1, \ldots, n_i, k = 1, \ldots, n_j. \end{aligned} \tag{54}$$

where $(x_i + G_{li}\cos(\theta_i^{0l} + \theta_i), y_i + G_{li}\sin(\theta_i^{0l} + \theta_i))$ is the position of the center of circle l in piece i rotated θ_i degrees. Constraints (55)–(58) ensure that all pieces are entirely inside the board.

$$x_i + G_{li}\cos(\theta_i^{0l} + \theta_i) + r_i^l \leq z, \;\; i = 1, \ldots, m, l = 1, \ldots, n_i, \tag{55}$$

$$-x_i - G_{li}\cos(\theta_i^{0l} + \theta_i) + r_i^l \leq 0, \;\; i = 1, \ldots, m, l = 1, \ldots, n_i, \tag{56}$$

$$y_i + G_{li}\sin(\theta_i^{0l} + \theta_i) + r_i^l \leq W, \; i = 1, \ldots, m, l = 1, \ldots, n_i, \tag{57}$$

$$-y_i - G_{li}\sin(\theta_i^{0l} + \theta_i) + r_i^l \leq 0, \; i = 1, \ldots., m, l = 1, \ldots, n_i, \tag{58}$$

$$x_i, y_i, \theta_i \in \mathbb{R}, \; i = 1, \ldots, m. \tag{59}$$

To find the circle covering representation of a piece, the authors used an iterative algorithm that operates in the medial axis of the polygon. A characteristic of circle covering models is the huge number of circles in piece representation and consequently the huge number of constraints. By summing all non-overlapping constraints, the authors reduced the number of constraints in the model.

Jone's constraints

There are a few differences between Jones' model and the previous one concerning the way they deal with the data set of pieces. In Jones (2014), the reference point of a piece is the center of the largest inscribed circle and is given by (0, 0). The center of the remaining circles inscribed in the piece is denoted by (a_i^l, b_i^l) (differently from the previous model where the authors used the distance G_{li}). Moreover, the author considered the vertices of the pieces, where a piece i has υ_i vertices with coordinates $(\bar{x}_i^\upsilon, \bar{y}_i^\upsilon)$. A major difference between both models is the elimination of trigonometry functions, where terms $\cos\theta_i$ and $\sin\theta_i$ are replaced by c_i and s_i and the following constraints are added to the model.

$$c_i^2 + s_i^2 = 1, \quad i = 1, \ldots, m. \tag{60}$$

Thus, the model is classified as a quadratic programming model. The coordinates of the center of circle l of piece i according to the rotation formula is $(c_i a_i^l - s_i b_i^l + x_i, s_i a_i^l + c_i b_i^l + y_i)$. Then, circle l of piece i does not overlap circle k of piece j if:

$$\begin{aligned} &(c_i a_i^l - s_i b_i^l + x_i - c_j a_j^k + s_j b_j^k - x_j)^2 \\ &\quad + (s_i a_i^l + c_i b_i^l + y_i - s_j a_j^k - c_j b_j^k - y_j)^2 \geq (r_i^l + r_j^k)^2, \\ &1 \leq i < j \leq m, \quad l = 1, \ldots, m_l, k = 1, \ldots, m_k. \end{aligned} \tag{61}$$

The inner fit polygon constraints are defined using the vertices of the pieces, where the vertex t of piece i is $(c_i\bar{x}_i^\upsilon - s_i\bar{y}_i^\upsilon + x_i, s_i\bar{x}_i^\upsilon + c_i\bar{y}_i^\upsilon + y_i)$. Then, these constraints are given as follows.

$$0 \leq c_i\bar{x}_i^\upsilon - s_i\bar{y}_i^\upsilon + x_i \leq z, \;\; i = 1, \ldots, m, \upsilon = 1, \ldots, \upsilon_i. \tag{62}$$

$$0 \leq s_i\bar{x}_i^\upsilon + c_i\bar{y}_i^\upsilon + y_i \leq W, \;\; i = 1, \ldots, m, \upsilon = 1, \ldots, \upsilon_i. \tag{63}$$

Constraints (64)–(66) define the domain of the variables. Constraint (67) limits the board length to an upper bound value and gives a breaking constraint ($s_1 \geq 0$) to avoid redundant solutions, where the angle of the first piece is in the interval [0, π].

$$-1 \leq c_i, s_i \leq +1, \;\; i = 1, \ldots, m, \tag{64}$$

$$r_i^1 \leq x_i \leq \bar{L} - r_i^1, \;\; i = 1, \ldots, m, \tag{65}$$

$$r_i^1 \leq y_i \leq W - r_i^1, \;\; i = 1, \ldots, m, \tag{66}$$

$$0 \leq z \leq \bar{L}, \quad s_1 \geq 0. \tag{67}$$

Constraints of Cherri et al. **(2018)**

Cherri et al. (2018) studied the strip packing problem with continuous rotation of pieces. The key feature of their mathematical model is the definition of the non-overlapping constraints using D-functions. To build the model, each non-convex piece i is decomposed into P_i parts. Let $(\bar{x}_{ir}^\upsilon, \bar{y}_{ir}^\upsilon)$ be the vertex υ of part r of piece i defined in relation to the reference point of piece i, $i = 1, \ldots, m$, $r = 1, \ldots, P_i$, $\upsilon = 1, \ldots, \upsilon_{ir}$. The position variables of each vertex υ of piece i are defined according to the following constraints:

$$x_{ir}^\upsilon = x_i + \bar{x}_i^\upsilon c_i + \bar{y}_i^\upsilon s_i, \;\; i = 1, \ldots, m, r = 1, \ldots, P_i, \upsilon = 1, \ldots, \upsilon_{ir}, \tag{68}$$

$$y_{ir}^\upsilon = y_i + \bar{x}_i^\upsilon c_i - \bar{y}_i^\upsilon s_i, \;\; i = 1, \ldots, m, r = 1, \ldots, P_i, \upsilon = 1, \ldots, \upsilon_{ir}, \tag{69}$$

Table 5
Non-linear programming models for problems with continuous rotation.

Authors	Model n.	Problem	Objective	Constraints	Linear var.	Binary var.
Chernov et al. (2010), Stoyan and Romanova (2013), Stoyan et al. (2016a, 2017); Stoyan et al. (2012)	M11	general	(53)	(48)–(52)	$\mathcal{O}_i$	
Stoyan et al. (2015, 2016b) and Romanova et al. (2018)	M11	general	(53)	(48)–(52)	$\mathcal{O}_i$ and auxiliary var.	
Rocha et al. (2016, 2014)	M12	strip packing	(30)	(54)–(59)	z, x_i, y_i, θ_i	
Jones (2014)	M13	strip packing	(30)	(60)–(67)	z, x_i, y_i, c_i, s_i	
Cherri et al. (2018)	M14	strip packing	(30)	(60), (64), (68)-(75)	z, x_i, y_i, x_i^v, y_i^v c_i, s_i	$\bar{u}_{ij}^{rsv}$
Peralta et al. (2018)	M15	strip packing	(30)	(74), (75), (76)–(80)	x_{ir}^v, y_{ir}^v, θ_i, x_{ij}^{rs}, y_{ij}^{rs}, θ_{ij}^{rs}	

$$x_{ir}^v, y_{ir}^v \in \mathbb{R},\ \ i = 1, \dots, m, r = 1, \dots, P_i, v = 1, \dots, v_{ir}. \tag{70}$$

To ensure that variables c_i and s_i are the sine and cosine of an angle θ_i, constraints (60) and (64) are also necessary.

The non-overlapping constraints are similar to constraints (24) and (25) of Cherri et al. (2016b). However, here the pieces can be freely rotated and are given by:

$$(x_{ir}^A - x_{ir}^B)(y_{ir}^A - y_{js}^v) - (y_{ir}^A - y_{ir}^B)(x_{ir}^A - x_{js}^v) \le M(1 - \bar{u}_{ij}^{rsA}),\\ i, j = 1, \dots, m, i \neq j, r = 1, \dots, P_i, s = 1, \dots, P_j,\\ A, B = 1, \dots, v_{ir}, v = 1, \dots, v_{js}, \tag{71}$$

$$\sum_{A=1}^{v_{is}} \bar{u}_{ij}^{rsA} + \sum_{v=1}^{v_{jr}} \bar{u}_{ji}^{rsv} = 1,\ \ i, j = 1, \dots, m, i \le j,\\ r = 1, \dots, P_i, s = 1, \dots, P_j, \tag{72}$$

$$\bar{u}_{ij}^{rsv} \in \{0, 1\},\ \ i, j = 1, \dots, m, r = 1, \dots, P_i,\\ s = 1, \dots, P_j, v = 1, ..., v_{is}. \tag{73}$$

The following constraints ensure that the pieces are inside the strip, where constraints (74) define the length of the board.

$$0 \le x_{ir}^v \le z,\ \ i = 1, \dots, m, r = 1, \dots, P_i, v = 1, \dots, v_{ir}, \tag{74}$$

$$0 \le y_{ir}^v \le W,\ \ i = 1, \dots, m, r = 1, \dots, P_i, v = 1, \dots, v_{ir}. \tag{75}$$

To improve the performance of the optimization solver, Cherri et al. (2018) provide an estimation of the big M and symmetry breaking constraints.

Peralta et al.**'s constraints**

Peralta et al. (2018) also proposed a model for the irregular strip packing problem and decomposed non-convex pieces into convex parts. The non-overlapping constraints are stated using separation lines. In order to define these constraints, consider c_{ij}^{rs} and d_{ij}^{rs} as:

$$c_{ij}^{rs} = \frac{(\bar{x}^{k+1} - \bar{x}^k)\sin\theta_{ij}^{rs} + (\bar{y}^{k+1} - \bar{y}^k)\cos\theta_{ij}^{rs}}{(\bar{x}^{k+1} - \bar{x}^k)\cos\theta_{ij}^{rs} + (\bar{y}^k - \bar{y}^{k+1})\sin\theta_{ij}^{rs}}$$

and

$$d_{ij}^{rs} = y_{ij}^{rs} - c_{ij}^{rs} x_{ij}^{rs},$$

where $(\bar{x}^k, \bar{y}^k)$ and $(\bar{x}^{k+1}, \bar{y}^{k+1})$ are two consecutive vertices used to define the separation line which can be vertices from one part of pieces i or j. Variable θ_{ij}^{rs} and $(x_{ij}^{rs}, y_{ij}^{rs})$ are the rotation angle and the translation parameters of the separation line between parts r and s in piece i and piece j, respectively. Then, the non-overlapping constraints are given by:

$$y_{ir}^v - c_{ij}^{rs} x_{ir}^v - d_{ij}^{rs} \le 0,\ \ i, j = 1, \dots, m, i \neq j, r = 1, \dots, P_i,\\ s = 1, \dots, P_j, v = 1, \dots, v_{ir}, \tag{76}$$

$$y_{js}^v - c_{ij}^{rs} x_{js}^v - d_{ij}^{rs} \ge 0,\ \ i, j = 1, \dots, m, i \neq j, r = 1, \dots, P_i,\\ s = 1, \dots, P_j, v = 1, \dots, v_{js}, \tag{77}$$

where x_{ir}^v and y_{ir}^v are similar to the variables defined in (68)–(70). However they do not replace $\cos\theta_i$ and $\sin\theta_i$ by c_i and s_i, respectively, i.e.:

$$x_{ir}^v = x_i + \bar{x}_i^v \cos\theta_i + \bar{y}_i^v \sin\theta_i,\\ i = 1, \dots, m, r = 1, \dots, P_i, v = 1, \dots, v_{ir}, \tag{78}$$

$$y_{ir}^v = y_i + \bar{x}_i^v \cos\theta_i - \bar{y}_i^v \sin\theta_i,\\ i = 1, \dots, m, r = 1, \dots, P_i, v = 1, \dots, v_{ir}. \tag{79}$$

Therefore, the variables of the model are

$$x_{ir}^v, y_{ir}^v, \theta_i, x_{ij}^{rs}, y_{ij}^{rs}, \theta_{ij}^{rs} \in \mathbb{R}, i, j = 1, \dots, m, i \neq j,\\ r = 1, \dots, P_i, s = 1, \dots, P_j v = 1, \dots, v_{ir}. \tag{80}$$

5.3. Complete non-linear programming models

The non-linear programming models proposed for nesting problems are summarized in Table 5.

Solving the non-linear programming models is much harder than solving the MIP models presented in Section 4. Then, some authors combined heuristics and solution of subproblems derived from the proposed models. To solve non-linear programming models based on phi-functions and quasi phi-functions, Chernov et al. (2010), Stoyan et al. (2012), Stoyan et al. (2015), Stoyan et al. (2016a), Stoyan et al. (2016b) and Romanova et al. (2018) described algorithms that search for a local optimal solution. The methods consist of finding a set of starting solutions and for each solution an iterative local minimum search is performed. The iterative procedure reduces the original non-linear programming model, that is quadratic to the number of variables, to a sequence of non-linear programming subproblems of considerably smaller dimension with a smaller number of non-linear inequalities (O(m)).

To find the inscribed circles, Jones (2014) proposed a method that iteratively inserts circles to pieces according to the overlap that appears in the solution of the model provided by a branch-and-bound search. Wang, Hanselman, and Gounaris (2018) proposed a customized branch-and-bound method, where the core of the algorithm consists of solving a linear relaxation of this mathematical model, which improved the results obtained by Jones

(2014). Wang et al. (2018) compared both strategies and their method solved to optimality few instances with up to five pieces, while Jones (2014) optimally solved instances with up to four pieces.

Attempts to solve the complete mathematical models using commercial or free solvers have been made in Cherri et al. (2018); Rocha et al. (2016) and Peralta et al. (2018). However, the optimization solver used by Peralta et al. (2018) demands a starting solution, which was obtained by a bottom-left algorithm. Rocha et al. (2016) and Peralta et al. (2018) presented results only for medium size and large instances (15 to 45 and 15 to 300 pieces, respectively) for which they succeeded in finding feasible solutions within an hour. Cherri et al. (2018) solved instances with up to five pieces and found the optimal solution for one instance with three pieces, for which the running time was limited to seven hours.

5.4. Remarks

Non-linear programming models have also been discussed in the literature to obtain feasible layouts in irregular strip packing problem. These models can be used in heuristic methods that allow to visit infeasible solutions. Imamichi, Yagiura, and Nagamochi (2009) proposed an unconstrained non-linear programming model, where the objective function is based on Minkowski sum. The objective of their model is to minimize the total depth penetration and protrusion. In a previous study, Imamichi and Nagamochi (2007) proposed an unconstrained non-linear programming model that followed the same idea of the previous one. However, the authors represented two and three-dimensional pieces by circles and spheres and the objective is to minimize depth penetration and protrusion of circles and spheres that belong to different pieces.

Kallrath (2009) analyzed two problems, the case in which one rectangular board is enough to pack all pieces and the case in which more than one rectangular board is necessary to pack all pieces. The first case is modeled as a non-convex non-linear programming problem and the problem considered is the open dimension problem with upper and lower bounds on the board width and length. The second case is modeled as a non-convex mixed integer non-linear programming problem. In this case, two types of problems are considered: the previous open dimension problem and the problem of packing pieces into boards of fixed dimensions. In both types of problem, free rotation of pieces are allowed and the non-overlapping constraints are addressed using separation lines. Moreover, he developed symmetry breaking constraints to speed up solution methods in optimization solvers.

6. Constraint programming models

In constraint programming, the problems are described by mathematical constraints (such as equalities and inequalities) and symbolic constraints, where the constraint satisfaction problem declares the relation among the variables. The constraints work as a procedure that allows to direct the search for a solution and relies on the problem structure (see Bockmayr & Hooker, 2005). They can be implemented in objected-oriented, rule-based, logic-based and imperative languages.

Solving a constraint programming model consists of assigning values to the variables, which must have a finite domain. This process is known in the literature as labeling and can result in a new problem or an infeasible problem. Then, the search for feasible solutions is performed on a search tree, where the algorithm is analogous to branch-and-bound algorithms and consists of branching and backtracking schemes. During the search, the infeasible values for the variables are removed (filtering) and inference methods are applied over consistent arcs. This process is known as propagation and can be strengthened by the usage of global constraints. A global constraint gathers the relation among variables in a single constraint. Symbolic constraints usually have a small number of variables and connecting them can benefit the search for a solution. As the domain of the variables in constraint programming is finite, only discrete positioning space for the pieces was considered in irregular cutting and packing problems.

Constraint programming was used to solve irregular cutting and packing problems in Ribeiro, Carravilla, and Oliveira (1999), Carravilla et al. (2003), Ribeiro and Carravilla (2009) and Cherri et al. (2017). All models are based on the no-fit polygon concept.

Carravilla et al. **and Ribeiro & Carravilla's constraints**

In Carravilla et al. (2003), constraint logic programming was used to solve the irregular strip packing problem. In a previous paper, Ribeiro et al. (1999) considered constraint logic programming to solve the problem with convex pieces. Constraint logic programming is an extension of logic programming, where constraints are allowed in the body of clauses. The variables are the placement of the reference point of pieces, where $(x_i, y_i) \in \mathbb{Z}^2$. Then, the domain of the variables is defined according to the inner-fit rectangle, which imposes lower and upper bounds on each integer variable x_i and y_i, and it is represented as follows.

$$doDomains(W, \bar{L}, r_i^{min}, \bar{L} - r_i^{max}, t_i^{min}, W - t_i^{max}, x_i, y_i), \tag{81}$$

where $\bar{L}$ is an upper bound on the length of the board.

To define the non-overlapping constraints, the non-convex no-fit polygons are decomposed into convex subpolygons (parts), which generates inner edges in the no-fit polygon. Then, the non-overlapping constraints impose that the reference point of the pieces must be on the left side of at least one edges of all non-fit polygon parts or on the frontier of the no-fit polygon. They can be represented by

$$doConstraintsNFP(x_i, y_i, x_j, y_j, NFP_{ij}). \tag{82}$$

The authors added the following symmetric breaking constraints to narrow the positioning of pieces i and j of the same type.

$$x_i \le x_j \tag{83}$$

$$\text{If } (x_i = x_j) \text{ Then } (y_i \le y_j). \tag{84}$$

As in Carravilla et al. (2003), Ribeiro and Carravilla (2009) decomposed the no-fit polygons into convex polygons. However, to set the non-overlapping constraints in the constraint logic programming model, they used reified constraints to label the edges of the no-fit polygons as “true” or “false”, where the inner edges are defined as false. The reified constraints describe the edges of the subpolygon and define a list of binary values, while the global constraint sum ensures that at least one constraint related to an edge of the subpolygons is satisfied.

According to the authors this strategy showed poor performance in terms of propagation and backtracking. To tackle this problem, they proposed a operationally global constraint named “outside”, which deals with overlapping among pieces. In addition, a special backtracking was used to cooperate with the new global constraint. A global constraint is activated for each pair of pieces when a piece and the x-coordinate of the second one are already positioned, which reduces the domain of the y-coordinate of the second piece. Carravilla et al. (2003) reported some computational experiments, where a feasible solution was found for an instance with 43 pieces and proved the optimality for an instance with seven convex pieces in less than one minute.

Cherri et al.**'s constraints**

Cherri et al. (2017) proposed two types of constraint programming models and adapted them to all variants of irregular cutting and packing problems. The problems investigated are placement problem, identical item placement problem, constrained placement problem, knapsack problem, one open dimension problem, cutting stock problem, bin backing problem and two open dimension problem. Both types of models are based on the model of Toledo et al. (2013), where the board is represented by a grid and defined by a set of dots D. Moreover, each piece of type t has a fixed number of rotations (R_t). The inner-fit polygon of each piece of type t at rotation r is given by a set of dots denoted by IFP_{tr}. For each piece of type t and t' at rotation r and r', respectively, they define the no-fit polygon $NFP_{tr}^{t'r'}$.

The first model type considers binary representation, where the variables of the model are δ_{tr}^{d} that is equal to 1 if a piece of type t at rotation r is placed at dot d, and 0 otherwise. If a piece of type t is placed at dot d, then by using the no-fit polygon $NFP_{tr}^{t'r'}$ we can define the set of points where piece of type t' cannot be placed, which is represented by $\Phi_{trd}^{t'r'}$. Then, the non-overlapping constraints are given by:

$$\text{If } (\delta_{tr}^{d} = 1) \text{ Then } (\delta_{t'r'}^{d'} = 0), t = 1, \ldots, T, t' = 1, \ldots, T,$$
$$r = 1, \ldots, R_t, r' = 1, \ldots, R_{t'}, d \in IFP_{tr}, d' \in \Phi_{trd}^{t'r'}. \tag{85}$$

The second model type considers an integer representation, where each piece of type t at a rotation r is mapped into a single integer number that is given by $n_{tr} = (t-1) \times R^{max} + r$, where $R^{max} = \max_t\{R_t : t = 1, \ldots, T\}$. Then, the domain of the variables, denoted by ν_d, $d \in D$, is given by $\{n_{tr} : t = 1, \ldots, T, r = 1, \ldots, R_t, d \in IFP_{tr}\} \cup \{0\}$. The assignment of each variable to a single value ensures that a point is assigned only to one piece. In the non-overlapping constraints, if a piece of type t at rotation r is placed at dot d ($\nu_d = n_{tr}$), the value $n_{t'r'}$ is removed from the domain of all dots d' in $NFP_{tr}^{t'r'}$, i. e.,

$$\text{If } (\nu_d = n_{tr}) \text{ Then } (n_{t'r'} \notin \nu_{d'}), t = 1, \ldots, T, t' = 1, \ldots, T,$$
$$r = 1, \ldots, R_t, r' = 1, \ldots, R_{t'}, d \in IFP_{tr}, d' \in \Phi_{trd}^{t'r'}. \tag{86}$$

For this representation, the authors proposed a global constraint named "NoOverlap" in order to produce an effective propagation. The global constraint is defined using a constraint propagator and consists of setting the value of a subset of variables $\nu_{d'}$ to zero each time a piece t at rotation r is positioned at dot d. This set of dots is given by $\Psi_{trd} = \{d' : d' \in \bigcap_{t' \in T, r' \in R_{t'}} \Phi_{trd}^{t'r'}\}$. In addition, it eliminates $n_{t'd'}$ possible values for each variable $\nu_{d'}$, $d' \in \Phi_{trd}^{t'r'}/\Psi_{trd}$. Then, the NoOverlap constraint is represented as follows.

$$\text{NoOverlap}\left(\{\nu_d : \Psi_{trd}, \Phi_{trd}^{t'r'}/\Psi_{trd}\}\right). \tag{87}$$

Apart from the problem to be solved using the first or second constraint programming model type, the non-overlapping constraints are the same for all problems. The difference among the models is mainly in the objective function and the domain of the variables. This last one is defined according to the inner-fit polygon. Moreover, in the integer model one of the two possibilities of the non-overlapping constraints must be chosen. The authors used the binary representation, the integer representation and the integer representation with global constraints to solve all variants of irregular cutting and packing problems. For this purpose, they used CPLEX with a time limit of 3600 seconds and instances with up to 99 pieces. The integer representation with global constraints showed the best performance for all analyzed instances and all problem types, however, optimality is hard to prove. For most instances, the binary and integer representations exceeded the limit of memory.

7. Future research directions

Important progress has been made in the last years in modeling and optimally solving nesting problems, and despite significant differences among the proposed models, research along this direction is still rather limited. There is still plenty of room for the incorporation of innovative methods and techniques and important characteristics of the nesting problem have also been scarcely or not at all addressed.

In what concerns solution methods there are clear research opportunities regarding:

- **Tighter lower bounds –** All models proposed in the literature have very weak relaxations. In particular, most mixed integer linear programming models use big-M constraints, which are known by their weak linear relaxation. Then, researchers have to deal with them one way or another and a possible research direction is the use of indicator constraints. Moreover, weak relaxations hinder an efficient application of any tree-search based algorithm (e.g. branch-and-bound). The development of good (tight) lower bounds for this problem would probably be the major breakthrough this field could see.
- **Valid inequalities –** Valid inequalities have already been proposed for some models, however the development of specialized cuts based on the geometric characteristics of the pieces, and its incorporation in branch-and-cut methods, is still an important research opportunity.
- **Decomposition methods –** All models proposed in the literature have been solved monolithically. However, in many other hard combinatorial optimization problems, decomposition methods (e.g. Benders decomposition, Lagrangian relaxation) have been successfully developed, bringing to a different level the size of the problems solved till proven optimality. The development of decomposition approaches for nesting problems is also an open research avenue.
- **Clustering of the pieces –** In nesting problems where pieces are demanded with a significant multiplicity (several copies of each piece type), it is usual to observe in layouts generated by human experts the clustering of pieces in some different groups, which are then replicated along the layout. This hierarchical resolution of the problem in two levels, a first one where clusters of pieces are formed and a second one where appropriate clusters are selected and placed on the board, could allow the resolution of much bigger problems. Both levels could be tackled by mathematical programming models.
- **Hybridization of mathematical programming with constraint programming –** The resolution of mathematical programming nesting models by commercial solvers faces important difficulties to quickly phantomize partial solutions that will lead to unfeasible layouts. On the other hand, constraint programming proved to be very fast in checking solution feasibility. As in other hard combinatorial optimization problems, the hybridization of mathematical programming with constraint programming is a relevant research direction.
- **Matheuristics –** Although this review focused on the exact solution of nesting problems based on mathematical models, there are also important research opportunities in the development of heuristics based on mathematical programming models, i.e. methods fitting the solution framework usually designated as matheuristics.

From the problem side, there are also still important challenges. Very relevant problem characteristics have seldom or never been addressed in the literature:

- **Continuous rotation of the pieces –** Almost all approaches consider that the pieces may be placed on the board in an

Table 6
General organization of nesting mathematical models.

Variable type	Model type	Geometric tool	Piece/board representation	MIP	LP	NLP[1]	CP
$(x, y) \in \mathbb{R}^2$	piece oriented	NFP	polygon	Scheithauer and Terno (1993) Daniels et al. (1994) Dean (2002) Fischetti and Luzzi (2009) Alvarez-Valdes et al. (2013) Cherri et al. (2016b)[2]	–	–	–
		trigonometry	polygon	Scheithauer and Terno (1993) Cherri et al. (2016b)[2]	–	Jones (2014) Rocha et al. (2016) Cherri et al. (2018) Peralta et al. (2018)	–
		phi-function	phi-object	–	Stoyan et al. (1996)	Chernov et al. (2010) Stoyan and Romanova (2013) Stoyan et al. (2012) Stoyan et al. (2016a) Stoyan et al. (2017)	–
		quasi phi-function	phi-object	–	–	Stoyan et al. (2015) Stoyan et al. (2016b) Romanova et al. (2018)	–
$(x, y) \in \mathbb{Z}^2$	piece oriented	NFP	polygon	–	–	–	Ribeiro et al. (1999) Carravilla et al. (2003) Ribeiro and Carravilla (2009)
	board oriented	NFP	polygon/dots	Toledo et al. (2013) Rodrigues and Toledo (2017)	–	–	Cherri et al. (2017)[3]
		raster points	pixels	Baldacci et al. (2014)[3]	–	–	–
$(x, y) \in \mathbb{R} \times \mathbb{Z}$	piece oriented	NFP	polygon	Leao et al. (2016)	–	–	–

[1] continuous rotation.
[2] no rotation and a finite number of rotations.
[3] a finite number of rotations.

orientation chosen from a discrete set of feasible orientations. However, in some important real-world applications any orientation is feasible. Continuous rotation of the pieces has been considered by some authors, but there is plenty of room for innovative research on this topic.

- **Irregular board –** In several applications (e.g. leather cutting) the board has an irregular (not rectangular) shape. Existing approaches that tackled this characteristic use a raster representation for the pieces and board. However, raster representations are memory-consuming and have precision problems. Solving nesting problems with the irregular board represented by polygons, with mathematical programming models is innovative research.
- **Quality regions –** Still concerning the board characteristics, being able to deal with different quality regions is required in some real-world applications. Mainly when dealing with natural materials, the board may not only present defects (regions that can not be used to cut pieces) but also may present regions with different characteristics, usually designated as quality regions. Some pieces may be cut only from some quality regions. Once again, only models using raster representations of the pieces and the board have dealt with quality regions. Using more accurate polygonal representations in mathematical models would move forward the knowledge frontier in this field.
- **Three-dimensional nesting problems –** Finally, what can be considered the next frontier in the exact solution of nesting problems with mathematical models is the three-dimensional nesting problem, aka three-dimensional irregular packing problem. The development of 3D printing technologies brings new real challenges and research opportunities.

8. Summary

In this paper a review of mathematical models for the solution of nesting problems is presented. As nesting problems are cutting and packing problems characterized by the irregular shape of the pieces to be cut, not surprisingly this review starts by presenting the geometric representation approaches for the pieces, from raster representations to phi-functions. The following section concerns mixed-integer linear programming models. In order to clarify and highlight the differences and similarities between them, all models were rewritten to use the same notation. At the same time, to avoid repetition, only the constraints that are different in each model are discussed. Section 5 is devoted to non-linear programming models. Non-linearities may arise both from the geometric representation of the irregular pieces (e.g. describing each piece as a cover of circles) and from the consideration of continuous rotations for the pieces, i.e. that pieces may be placed on the board in any orientation. Finally, constrained programming models for nesting problems are discussed in Section 6. Constrained programming based approaches prove to be very fast in checking (in)feasibility but can only be used together with discrete representations of the board. Building on the analysis of how and what has been approached in this field, future research directions are proposed in the last section.

Summarizing, the mathematical models proposed for nesting problems can be distinguished according to the type of formulation (mixed-integer linear, non-linear and constraint programming), the type of variables (continuous or discrete), the geometric representation of the pieces (raster points, direct trigonometry, no-fit polygon and phi-function), or the model type (piece oriented and board oriented). In what concerns the type of problem solved, e.g. irregular open dimension problem, irregular placement problem or irregular knapsack problem, the mathematical representations can be easily adapted from one type to another, or even extended for the multi-board problem types, as the irregular cutting-stock problem and the irregular bin-packing problem. Table 6 organizes the mathematical models for nesting problems discussed in this paper according to the characteristics listed above.

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