IFT - UNESP
INSTITUTO DE FÍSICA TEÓRICA

TESE DE DOUTORAMENTO IFT–T.000/99

Superstring compactifications with manifest spacetime supersymmetry

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Fevereiro de 2025

To my father Jaime,

whose strength and unwavering support continue to inspire me.

In loving memory.

Agradecimentos

Quero deixar meus agradecimentos e menções às pessoas que foram fundamentais na minha vida durante o meu doutorado. Seria injusto deixar de frisar a importância que elas tiveram nesta trajetória, especialmente no documento que resume minhas realizações científicas ao longo de todos esses anos.

Primeiramente, gostaria de expressar minha profunda gratidão à minha família pelo apoio incondicional ao longo desta jornada. À minha mãe Adriana, pelo carinho, paciência, compreensão e suporte inabalável, além dos inúmeros conselhos sábios que sempre me guiaram. Ao meu irmão Jaime Filho, pelo apoio constante, pela admiração mútua e pelas nossas frequentes discussões sobre tudo e todos, que tanto enriqueceram minha perspectiva. Além disso, pelas incontáveis experiências compartilhadas, que carrego comigo desde o momento de seu nascimento. Ao meu pai, que, embora não esteja mais entre nós, foi fundamental na construção do meu caráter e sempre acreditou no meu potencial, seus ensinamentos permanecem comigo. A ele, dedico esta tese.

Agradeço ao meu orientador, Nathan Berkovits, por ter me aceitado como seu estudante e por responder a várias perguntas sobre física e, sobretudo, teoria das supercordas. Ao Matthias Gaberdiel, por ter me concedido a oportunidade de passar um ano do meu doutorado na ETH Zurich, e pelas constantes discussões sobre supercordas em AdS_3. Além disso, gostaria de agradecer aos meus colaboradores Nathan Berkovits, Karim Benakli, Matheus Lize e Wenqi Ke, que também são responsáveis pelas conquistas desta tese. Agradeço também aos demais professores do IFT-Unesp e ETH Zurich por terem contribuído para a minha formação. Em particular, gostaria de agradecer às secretárias Beatrix, Jandira e Stefanie, pelo efetivo auxílio nas demandas institucionais.

Além do mais, gostaria de expressar minha sincera gratidão aos amigos e colegas que compartilharam comigo esta caminhada, seja nos desafios da pesquisa ou nos momentos de descontração: Bob, Beat, Bruno^2, Daniel, Dennis^2, Eggon, Egor, ETH Italian crew, Fera, Gabriel^2, Isa, Ivan, João, Johannes, Juan, Julian, Kiarash, Konstantin, Martijn, Max, Nadine, Rafa, Rodrigo, Sibylle, Sylvain, Ulisses, Victor, Vinícius, Vit, assim como a tantos outros cuja contribuição foi essencial ao longo desta jornada, ainda que seus nomes possam me escapar no momento.

Em especial, quero expressar minha gratidão aos meus amigos Enzo Solis Gonzalez, Felipe Zeni, Júlio Vieira, Lucas Martins, Matheus Lize e Raul Palma, pelas tantas discussões e momentos de descontração. Ao Enzo Solis Gonzalez e Matheus Lize, pelos treinos de perna. Ao Priesley (O Príncipe), por sempre ter razão desde 2019. Ao Carlisson, pelas discussões esclarecedoras. Ao João Caetano, pelos intermináveis debates sobre Brasil e economia. Ao Maycol, por ser meu freguês no truco gaúcho. Um destaque especial também vai para os excepcionais Dani Arrufat e Sam Bird. Nos divertimos muito juntos e espero poder compartilhar novos momentos em breve.

Por fim, agradeço à FAPESP pelo apoio financeiro concedido através dos processos 2022/14599-0 e 2023/00015-0.

"No man ever steps in the same river twice,

for it's not the same river and he's not the same man."

— Heraclitus

Resumo

Esta tese explora compactificações da supercorda em espaços de quatro e seis dimensões. A teoria é quantizada utilizando o formalismo híbrido supersimétrico para a supercorda, ao mesmo tempo em que se utilizam métodos de teoria conforme em duas dimensões. Após o desenvolvimento da descrição híbrida em quatro dimensões acoplada a um campo eletromagnético, constrói-se uma ação de teoria de campos da supercorda para os primeiros estados massivos — tanto no superspaço quanto em termos dos campos componentes. A ação inclui um campo massivo de spin-3/2 e um campo massivo de spin-2 propagando na presença de um campo de gauge $\mathrm{U}(1)$ não nulo.

A quantização da supercorda compactificada em um Calabi-Yau de duas dimensões complexas pode ser alcançada usando a descrição híbrida de seis dimensões. No entanto, o formalismo híbrido permite que apenas metade das oito SUSYs do superespaço $d = 6$ $\mathcal{N} = 1$ estejam manifestas. Superamos essa limitação e estendemos o formalismo de forma que todas as SUSYs do espaço-tempo possam ser manifestas. O operador BRST, estados físicos e uma prescrição de amplitude são explicitamente construídos.

Em seguida, estudamos a supercorda Tipo IIB em um espaço $\mathrm{AdS}_3 \times \mathrm{S}^3$. Utilizando o formalismo híbrido de Berkovits-Vafa-Witten para o caso de fluxo NS-NS puro, calculamos uma amplitude supersimétrica de três pontos de operadores de vértice half-BPS inseridos na fronteira de AdS_3. O cálculo é realizado em termos das variáveis covariantes de $\mathrm{PSU}(1,1|2)$. Encontramos que integrar os campos fermiônicos da supercorda na integral de caminho gera o vielbein do espaço-tempo, que codifica explicitamente que o grupo conforme na fronteira é identificado com o grupo de simetria do interior de AdS.

A supercorda em $\mathrm{AdS}_3 \times \mathrm{S}^3$ pode ser descrita por uma mistura de fluxos de três-forma auto-duais NS-NS e R-R. Construímos uma ação de folha de mundo para a supercorda em $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ com fluxos mistos que é manifestamente invariante sob transformações de $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$. Quantizamos covariantemente o modelo, demonstrando sua invariância conforme em um loop. Terminamos mostrando como é possível relacionar a descrição supersimétrica com a ação de folha de mundo de Berkovits-Vafa-Witten em $\mathrm{AdS}_3 \times \mathrm{S}^3$ com fluxos mistos.

Abstract

This thesis explores the superstring compactified to four- and six-dimensional backgrounds. The theory is quantized using the spacetime supersymmetric hybrid formalism for the superstring while leveraging two-dimensional worldsheet methods. After the four-dimensional hybrid description is developed in an electromagnetic background, a superstring field theory action for the first massive states is constructed — both in superspace and in terms of the components fields. The action includes a massive spin-3/2 and a massive spin-2 field propagating in the presence of a constant non-zero $\mathrm{U}(1)$ gauge field.

Quantization of the superstring compactified to a Calabi-Yau twofold can be achieved using the six-dimensional hybrid description. However, the hybrid formalism allows only half of the eight $d = 6$ $\mathcal{N} = 1$ SUSYs manifest. We overcome this issue and extend the formalism such that all spacetime SUSYs can be made manifest. The BRST operator, physical states and a scattering amplitude prescription are explicitly constructed.

We then study the Type IIB superstring in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background. By making use of the Berkovits-Vafa-Witten hybrid formalism for the pure NS-NS flux case, we compute a supersymmetric three-point amplitude of half-BPS vertex operators inserted on the AdS_3 boundary. The computation is performed using the $\mathrm{PSU}(1,1|2)$-covariant variables. It is found that integrating out the fermionic worldsheet fields in the path integral gives rise to the target-space vielbein, which explicitly encodes that the conformal group on the boundary is identified with the symmetry group of the AdS bulk.

The superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$ can be supported by a mixture of NS-NS and R-R self-dual three-form flux. We construct a manifestly $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$-invariant sigma-model action for the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with mixed flux. The model is then covariantly quantized and proven to be conformal invariant at the one-loop level. We conclude by showing how one can relate the supersymmetric description with the Berkovits-Vafa-Witten $\mathrm{AdS}_3 \times \mathrm{S}^3$ worldsheet action with mixed flux.

Contents

Chapter 1

Overview of this thesis

In this thesis, chapters 2, 3, and 4 present original work along with concise introductions to each discussed topic. Chapter 2 deals with the open superstring field theory of the four-dimensional hybrid formalism in a constant $\mathrm{U}(1)$ background, which exhibits manifest $d = 4$ $\mathcal{N} = 1$ spacetime supersymmetry. Chapter 3 covers the six-dimensional hybrid formalism in a flat background, along with its extension that manifests all spacetime supersymmetries of $d = 6$ $\mathcal{N} = 1$ superspace. Chapter 4 explores the superstring in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background and scattering amplitudes thereof, both using the six-dimensional hybrid formalism, described by the supergroup $\mathrm{PSU}(1,1|2)$, and from its supersymmetric extension, which exhibits manifest $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ invariance. We conclude in Chapter 5 by reviewing the main findings contained in the thesis and end up with a discussion on the possible future research directions. At last, several appendices provide additional technical details.

Let us elaborate further on the structure of this thesis. In Chapter 2, we start by motivating the problem and providing comments on the relation between the hybrid and RNS variables. After this, we give a concise review of the four-dimensional hybrid formalism for the superstring, stating its key properties and presenting the relevant variables in terms of the oscillator modes of free fields. We also show how to compute the equations of motion and the open superstring field theory action in this formalism.

Once the free theory is explained, we move on to charged superstrings in a constant electromagnetic background, and start by solving the equations of motion and boundary conditions for the bosonic worldsheet fields, and then expressing the solution in terms of modes. The mode expansions for the fermionic worldsheet fields are then found by imposing that the left and right-moving superconformal generators coincide at the boundary. We also write down the worldsheet action for the charged superstring in the hybrid formalism, and describe how the superconformal generators of the previous section are modified in this case.

Having established the hybrid formalism for the interacting case, we compute the string field action for the massless sector of the open superstring compactified to four dimensions, and show that the results are consistent with earlier computations done for the massless sector from bosonic string field theory. We then provide one of the main results of this chapter, namely, the computation of the $d = 4\ \mathcal{N} = 1$ superspace action for the first massive compactification-independent states of the charged open superstring in a constant electromagnetic background. When on-shell, the first massive states describe 12 bosonic and 12 fermionic complex degrees of freedom, including a charged massive spin-3/2 and spin-2 fields.

After expanding the superspace action in terms of the component fields, and removing the unphysical degrees of freedom by gauge-fixing, we write the component Lagrangian describing the first massive states of the superstring in a constant $\mathrm{U}(1)$ background. Moreover, we also write the equations of motion and constraints derived from the component Lagrangian, characterizing the physical degrees of freedom. Chapter 2 is based on the publications [1] [2] [3] [4].

Chapter 3 covers the study of the superstring compactified to six flat directions. The chapter starts explaining to the reader why, in the six-dimensional hybrid formalism, only half of the eight $d = 6\ \mathcal{N} = 1$ SUSYs can be made manifest. Furthermore, before delving into more technical details, the potential solution to address this limitation is outlined. Building on the previous discussion, the six-dimensional hybrid formalism in a flat background is reviewed, while specifying the worldsheet action and massless vertex operators according to our conventions. Next, it is shown how one can describe the hybrid formalism with the addition of the remaining fermionic coordinates θ^α of $d = 6\ \mathcal{N} = 1$ superspace. The latter comes alongside fermionic first-class constraints D_α, such that the gauge symmetry generated by these constraints can be used to gauge away the new variables.

Besides the fermionic fields θ^α, unconstrained bosonic ghost-fields λ^α, and its conjugate momenta, are added to the worldsheet action in such a way that the total central charge of the stress-tensor vanishes. With the addition of the non-minimal variables, a new manifestly spacetime supersymmetric BRST operator G^+ is then defined and a supersymmetric unintegrated vertex operator U is constructed, as well as its integrated version W. It is then shown that BRST invariance of U implies the $d = 6$ SYM equations of motion in superspace. With both integrated and unintegrated vertex operators at our disposal, a tree-level scattering amplitude prescription is given which shares many similarities to the $d = 10$ non-minimal pure spinor formalism one. Chapter 3 is based on the publication [5].

Chapter 4 discusses the superstring propagating in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ target-space.

We start off by introducing the main motivation for exploring the superstring in AdS, namely, the AdS/CFT correspondence. Particularly, we also spell out the reasoning behind studying the $AdS_3 \times S^3$ background. In order to fix our notation and conventions for the worldsheet theory, we then provide a thorough exposition of the hybrid formalism in $AdS_3 \times S^3$ with pure NS-NS three-form flux, which is given by a WZW model of $PSU(1,1|2)_k$, where k labels the amount of flux in the background. The physical state conditions of the formalism are then solved, while defining half-BPS vertex operators in terms of a fermionic zero-mode coordinate θ^α.

In addition, after performing a similarity transformation along the AdS_3 boundary direction, we define the worldsheet fields and vertex operators depending on $\mathbf{x} \in \partial AdS_3$, and introduce the vielbein field $E_A{}^B(\mathbf{x})$. Finally, a $PSU(1,1|2)$-covariant three-point amplitude for vertex operators inserted on the AdS3 boundary is computed, and it is shown that integrating out the fermionic fields θ^α in the path integral implies the appearance of $E_A{}^B(\mathbf{x})$ in the kinematic factor. We further validate our results through comparison with the RNS formalism.

Since the hybrid formalism only preserves half of the spacetime supersymmetries manifest, we now turn to the study of the extended hybrid formalism in an $AdS_3 \times S^3$ target-space. This formalism was discussed in a flat background in Chapter 3 and manifestly preserves all spacetime supersymmetries. The superstring in $AdS_3 \times S^3$ can be supported by a mixture of NS-NS and R-R flux. We first write an ansatz for the Type IIB sigma-model action in a general six-dimensional background. After identifying the background superfields and spelling out in detail our conventions for the $PSU(1,1|2) \times PSU(1,1|2)$ Lie superalgebra, we construct the worldsheet action for the superstring in $AdS_3 \times S^3 \times T^4$ with mixed NS-NS and R-R self-dual three-form flux, and then argue how this sigma-model action can be derived: either by substituting the values for the background superfields, or via a perturbative analysis from the integrated vertex operator around flat $d = 6$ spacetime.

Subsequently, one-loop conformal invariance of the model is proven by using the covariant background field method. It is then shown how one can relate the manifestly $PSU(1,1|2) \times PSU(1,1|2)$ worldsheet action with the $AdS_3 \times S^3$ hybrid formalism with mixed flux, which has the supergroup $PSU(1,1|2)$ as the target-superspace. Chapter 4 is based on the publications [6] [7].

To wrap up, we conclude in Chapter 5, where we highlight the main results contained in each chapter and examine possible directions for future study. The appendices include supporting material which serves to assist the readers seeking technical details not covered in the main text.

Chapter 2

Higher-spin states of the superstring in an electromagnetic background

In this chapter, using the manifestly spacetime supersymmetric hybrid formalism for open superstring field theory, we construct a superspace action for the charged first massive states of the superstring in a constant electromagnetic background. The physical degrees of freedom of the action include a massive spin-3/2 and a massive spin-2 field. The hybrid formalism has the advantage over the RNS formalism of manifest $d = 4$ $\mathcal{N} = 1$ SUSY so that the spin-2 and spin-3/2 fields are combined into a single superfield and there is no need for picture-changing or spin fields.

Subsequently, the interacting superspace action for the charged massive states is developed in components, providing a Lagrangian for the physical component fields after the large gauge symmetry of the string field theory description is fixed. The resulting equations of motion describe the propagation of charged spin-3/2 and spin-1/2 fields on the one hand, and spin-2, spin-1, and spin-0 on the other. In the absence of an electromagnetic background, the Rarita-Schwinger and Fierz-Pauli Lagrangians are retrieved for spin-3/2 and 2, respectively. Furthermore, the Lagrangian derived does not suffer from the loss of causality problem occurring in the minimal coupling approach.

2.1 Introduction

Constructing consistent effective field theory actions for higher-spin fields is a challenging task, as recognized in the pioneering work of Dirac [8] and soon after by Fierz and Pauli [9]. A significant obstacle arises when one tries to couple massive higher-spin fields to a constant electromagnetic background. Johnson and Sudarshan found that relativistic covariance of the theory is lost upon quantization when massive spin-3/2 fields are minimally coupled to a constant electromagnetic background [10]. In subsequent works, Velo and Zwanziger showed that minimally coupled actions for spin-3/2 and spin-2 fields already exhibit inconsistencies at the classical level, including faster-than-light behavior and the propagation of a wrong number of degrees of freedom [11] [12].

Superstring theory is known to be a consistent theory of quantum gravity, containing an infinite number of both bosons and fermions in its spectrum as a result of the string's different oscillation modes. These oscillations include massive states of arbitrary spin, where the mass squared is proportional to the inverse of the fundamental string coupling α'. In particular, the open superstring spectrum contains a massless $\mathrm{U}(1)$ gauge field, and its first excited level includes a massive spin-3/2 and a massive spin-2 field. For that reason, string field theory presents itself as a natural candidate for deriving effective actions for massive higher-spin particles in a constant electromagnetic background.

Using bosonic open string field theory, Argyres and Nappi constructed a consistent Lagrangian for a charged massive spin-2 field in $d = 26$ dimensions [13] [14]. Under dimensional reduction, it was shown in [15] that there is no propagating spin-1 state and one gets in four dimensions a theory of a coupled system of charged massive spin-2 and spin-0 fields.

In this chapter, we will generalize the Argyres and Nappi result to the supersymmetric case using open superstring field theory, which includes both the massive fermionic spin-3/2 and the bosonic spin-2 states. Although one can in principle use the Ramond-Neveu-Schwarz (RNS) formalism of open superstring field theory to perform these computations, we will instead use the four-dimensional hybrid formalism of open superstring field theory for two reasons. Firstly, the hybrid formalism has manifest $d = 4$ $\mathcal{N} = 1$ spacetime supersymmetry, which will allow us to combine the spin-3/2 and spin-2 fields into a single $d = 4$ superfield and compute the Lagrangian and equations of motion in superspace. Secondly, the hybrid formalism avoids the complicated picture-changing operators and spin fields, which are necessary in the Ramond-Neveu-Schwarz formalism to describe the spin-2 and spin-3/2 states.

The hybrid description of the superstring consists of a field redefinition from the gauge-fixed RNS superstring into a set of Green-Schwarz-like variables, allowing spacetime supersymmetry to be made manifest. This can be achieved in either two dimensions [16], four dimensions [17], six dimensions [18],[1] or in a

[1]See chapters 3 and 4 for a description of the six-dimensional hybrid formalism in a flat and in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background, respectively.

U(5) subgroup of the ten-dimensional super-Poincaré group [19]. In this chapter, our focus will be the construction down to four-dimensional spacetime.

The field redefinition maps the free gauge-fixed RNS fields to the free hybrid fields, which split into a compactification-independent part, describing the four-dimensional Minkowski spacetime, and a compactification-dependent part. The compactification-dependent fields describe a $c = 9$ $\mathcal{N} = 2$ superconformal field theory (SCFT) and, hence, can correspond to any Calabi-Yau manifold with three complex dimensions. In addition, the SCFT describing the compactification-dependent variables decouples from the four-dimensional fields, i.e., it has no poles with the $c = -3$ $\mathcal{N} = 2$ generators of the four-dimensional part. In this setting, the critical $c = 15$ $\mathcal{N} = 1$ RNS superstring is described as a critical $\mathcal{N} = 2$ string with central $c = 6$.

To better clarify the presentation for readers familiar with the RNS description, let us first recall some basic features of the RNS superstring — and its field content — before transitioning to the hybrid worldsheet variables. When certain aspects seem obscure, it can be helpful to know how to translate between the RNS and hybrid expressions. The following discussion might also be useful to the development of a general intuition about the hybrid description of the superstring.

In the gauge-fixed RNS formalism with ten uncompactified directions, the matter fields $\{\partial x^M, \psi^M\}$, $M = \{0 \text{ to } 9\}$, satisfy a $c = 15$ $\mathcal{N} = 1$ SCA where the generators are given by

$$T_{\mathrm{m}} = -\frac{1}{2}\partial x^M \partial x_M - \frac{1}{2}\psi^M \partial \psi_M \,, \tag{2.1a}$$

$$G_{\mathrm{m}} = i\psi^M \partial x_M \,, \tag{2.1b}$$

and the ghost fields $\{b, c, \beta, \gamma\}$ satisfy a $c = -15$ $\mathcal{N} = 1$ SCA with[2]

$$T_{\mathrm{gh}} = -2b\partial c - \partial b c - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\partial\beta\gamma \,, \tag{2.2a}$$

$$G_{\mathrm{gh}} = b\gamma - 2\partial\beta c - 3\beta\partial c \,. \tag{2.2b}$$

However, as we mentioned above, one might also describe the superstring as a twisted $c = 6$ $\mathcal{N} = 2$ string. This can be done by bosonizing $\beta = e^{-\phi}\partial\xi$ and $\gamma = \eta e^{\phi}$, and working in the large Hilbert space — allowing for the inclusion of

[2] One can check that the algebra relations follow from the free-field OPEs in Appendix E, together with our normal-ordering prescription detailed in Appendix F. Additionally, our conventions for the $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCAs are spelled out in Appendix H.

the ξ zero-mode in our formulas. In terms of the RNS variables, the twisted $c = 6$ $\mathcal{N} = 2$ generators take the form

$$T = -\frac{1}{2}\partial x^M \partial x_M - \frac{1}{2}\psi^M \partial \psi_M - 2b\partial c - \partial bc - \frac{1}{2}\partial\phi\partial\phi - \partial^2\phi - \eta\partial\xi\,, \tag{2.3a}$$

$$G^+ = cT_{x,\psi,\phi,\eta,\xi} + bc\partial c - \frac{1}{2}\eta e^{\phi} G_{\mathrm{m}} - \frac{1}{4} b\eta\partial\eta e^{2\phi} + \partial^2 c - \partial(\eta\xi c)\,, \tag{2.3b}$$

$$G^- = b\,, \tag{2.3c}$$

$$J = -bc + \eta\xi\,, \tag{2.3d}$$

where the supercurrent G^+ is the usual $\mathcal{N} = 1$ BRST current with the addition of a suitable total derivative in the large Hilbert space.

Furthermore, any twisted $c = 6$ $\mathcal{N} = 2$ SCA can be extended to a twisted small $c = 6$ $\mathcal{N} = 4$ SCA, see Appendix H. The remaining $\mathcal{N} = 4$ generators are given by

$$\widetilde{G}^+ = \eta\,, \tag{2.4a}$$

$$\widetilde{G}^- = \xi T_{x,\psi,\phi} + \frac{1}{2} be^{\phi} G_{\mathrm{m}} - \frac{1}{4} b\partial b\eta e^{2\phi} - \xi b\partial c - \partial(bc\xi) + \partial^2\xi\,, \tag{2.4b}$$

$$J^{++} = c\eta\,, \tag{2.4c}$$

$$J^{--} = -b\xi\,. \tag{2.4d}$$

Note that the $\mathrm{U}(1)$ current J is equal the ghost current minus the picture current, i.e., $J = j_{ghost} - j_{picture}$ where $j_{ghost} = -bc - \partial\phi$ and $j_{picture} = -\eta\xi - \partial\phi$.

In four-dimensional compactifications of the superstring, it is convenient to split the RNS fields into a four-dimensional contribution and a decoupled six-dimensional part

$$\{x^m, \psi^m, b, c, \eta, \xi, \phi\} \quad \oplus \quad \{x^i, \overline{x}_i, \psi^i_{\mathrm{RNS}}, \overline{\psi}_i^{\mathrm{RNS}}\}\,, \tag{2.5}$$

where $m = \{0 \text{ to } 3\}$ labels the four-dimensional spacetime directions, and $i = \{1 \text{ to } 3\}$ denotes the complex three-dimensional Calabi-Yau directions $\mathcal{M}_6$. After a field redefinition to the four-dimensional hybrid formalism, the RNS field content can be mapped to the following free worldsheet variables [17]

$$\underbrace{\{x^m, \rho, p_\alpha, \theta^\alpha, \overline{p}^{\dot{\alpha}}, \overline{\theta}_{\dot{\alpha}}\}}_{\mathbb{R}^{1,3}} \quad \oplus \quad \underbrace{\{x^i, \overline{x}_i, \psi^i, \overline{\psi}_i\}}_{\mathcal{M}_6}\,, \tag{2.6}$$

where $\alpha, \dot{\alpha} = \{1, 2\}$ are the four-dimensional spinor indices.

Note that in the four-dimensional part, one has five bosons $\{x^m, \phi\}$ and eight fermions $\{\psi^m, b, c, \eta, \xi\}$ in the RNS formalism. This is the same number of degrees of freedom as in the hybrid description after including the chiral boson ρ, namely, five bosons $\{x^m, \rho\}$ and the eight fermions $\{p_\alpha, \theta^\alpha, \overline{p}^{\dot{\alpha}}, \overline{\theta}_{\dot{\alpha}}\}$. Equivalently, for the Calabi-Yau directions, one has three xs and three ψs for both descriptions. In particular, the fermionic fields for the Calabi-Yau directions $\{\psi^i, \overline{\psi}_i\}$ are a twisted version of the RNS ones $\{\psi^i_{\text{RNS}}, \overline{\psi}_i^{\text{RNS}}\}$.

As opposed to the RNS formalism, states and operators constructed with integer powers of the free worldsheet fields (2.6) and $e^{n\rho}$, where n is an integer, are automatically GSO-projected [20]. Therefore, these operators have no branch cuts with the spacetime SUSY generators, which is an advantage of the hybrid description since these branch cuts imply that one has to sum over spin structures in RNS.

2.2 Hybrid formalism in a flat four-dimensional spacetime

In what follows, we will further elaborate on the worldsheet variables (2.6) and superconformal generators of the four-dimensional hybrid formalism in detail. The worldsheet action, relevant fields and their main properties are presented, as well as a formulation in terms of oscillator modes for the case without a background U(1) gauge field. For further details, we refer to the original works [17] [18] [21] and [22].

Our conventions for the worldsheet theory follow [23] and we are using $\alpha' = \frac{1}{2}$ when the string constant is omitted. For manipulations with sigma matrices and "dotted"/"undotted" spinor indices, we utilize the conventions of ref. [24], for example, $x_{\alpha\dot{\alpha}} = \sigma^m_{\alpha\dot{\alpha}} x_m$, $\overline{\sigma}^{\dot{\alpha}\alpha}_m \sigma^m_{\beta\dot{\beta}} = -2\delta^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}}$, $(\psi\chi) = \psi^\alpha \chi_\alpha$, $(\overline{\psi}\overline{\chi}) = \overline{\psi}_{\dot{\alpha}} \overline{\chi}^{\dot{\alpha}}$, $(\sigma^m \overline{\sigma}^n + \sigma^n \overline{\sigma}^m)_\alpha^{\ \beta} = -2\eta^{mn} \delta^\beta_\alpha$, etc.

2.2.1 Worldsheet action and superconformal generators

The Euclidean worldsheet action of the four-dimensional spacetime part consists of four bosons x^m, $m = 0$ to 3, with two pairs of left-moving canonically conjugate Weyl fermions $\{p^\alpha, \theta_\beta\}$ and $\{\overline{p}_{\dot{\alpha}}, \overline{\theta}^{\dot{\beta}}\}$ ($\alpha, \dot{\alpha} = \{1, 2\}$) having conformal weight $(1,0)$ each, and a chiral boson ρ. We also have the right-moving variables which will be denoted by a "hat".

In conformal gauge, the action is given by

$$S_0 = \frac{1}{2\pi}\int d^2z\left\{\frac{1}{\alpha'}\partial x^m\overline{\partial}x_m + p^\alpha\overline{\partial}\theta_\alpha + \overline{p}_{\dot{\alpha}}\overline{\partial}\overline{\theta}^{\dot{\alpha}} + \widehat{p}^\alpha\partial\widehat{\theta}_\alpha + \widehat{\overline{p}}_{\dot{\alpha}}\partial\widehat{\overline{\theta}}^{\dot{\alpha}} + \frac{1}{2}\partial\rho\overline{\partial}\rho + \frac{1}{2}\partial\widehat{\rho}\overline{\partial}\widehat{\rho}\right\} + S_6\,, \tag{2.7}$$

where $\eta^{mn} = \mathrm{diag}(-,+,+,+)$, $\partial \equiv \partial_z$ and $\overline{\partial} \equiv \partial_{\overline{z}}$. Middle alphabet letters such as m,n,p will be used to denote four-dimensional spacetime indices throughout this work.

In the strip, the Euclidean coordinates take the standard values: $0 \leq \sigma \leq \pi$ and $-\infty < \tau < \infty$, where $z = e^{-iw}$ with $w = \sigma + i\tau$. In the action, to go from the plane to the strip, one just substitutes $d^2z = 2d\sigma d\tau$, $\partial = \partial_w$ and $\overline{\partial} = \partial_{\overline{w}}$, where $\partial_w = \frac{1}{2}(\partial_\sigma - i\partial_\tau)$ and $\partial_{\overline{w}} = \frac{1}{2}(\partial_\sigma + i\partial_\tau)$. As is commonly done, we will use a bar to denote complex conjugation of z and w, but this should not be confused with $\overline{p}_{\dot{\alpha}}$ and $\overline{\theta}^{\dot{\alpha}}$ which are left-moving variables.

The worldsheet fields in the hybrid formalism are related to those in the gauge-fixed RNS description by a field redefinition [17]. The internal six dimensional matter part of the action, S_6, is the same as in RNS. Without loss of generality, we will suppress the right-moving fields in the rest of this paper. In the free case, they are related to the left-moving ones in the usual way by the boundary conditions, for example, $p^\alpha(z) = \widehat{p}^\alpha(\overline{z})$, $\theta^\alpha(z) = \widehat{\theta}^\alpha(\overline{z})$ at $\mathrm{Im}\{z\} = 0$ and x^m satisfies Neumann boundary conditions.

We group the RNS matter variables for the internal directions $\{x^\mu, \psi^\mu\}$, $\mu = 4$ to 9, into a $\mathbf{3}$ and $\overline{\mathbf{3}}$ of $\mathrm{SU}(3)$ and denote these variables by $\{x^k, \overline{x}_k, \psi^k, \overline{\psi}_k\}$ with indices j,k,l running from 1 to 3. The description that we shall use corresponds to an uncompactified superstring if x^j takes values on $\mathbb{R}^6$, or to a toroidally-compactified superstring if x^j takes values on T^6. The free field OPEs in the complex plane for the four-dimensional part are

$$p^\alpha(y)\theta_\beta(z) \sim \frac{\delta^\alpha_\beta}{y-z}\,, \qquad \overline{p}_{\dot{\alpha}}(y)\overline{\theta}^{\dot{\beta}}(z) \sim \frac{\delta^{\dot{\beta}}_{\dot{\alpha}}}{y-z}\,, \tag{2.8a}$$

$$\rho(y)\rho(z) \sim \log(y-z)\,, \quad x^m(y)x^n(z) \sim -\frac{\alpha'}{2}\eta^{mn}(\log|y-z|^2 + \log|y-\overline{z}|^2)\,, \tag{2.8b}$$

And for the internal part, we have

$$\psi^j(y)\overline{\psi}_k(z) \sim -\frac{\delta^j_k}{y-z}\,, \quad x^j(y)\overline{x}_k(z) \sim -\frac{\alpha'}{2}\delta^j_k(\log|y-z|^2 + \log|y-\overline{z}|^2)\,, \tag{2.9a}$$

$$H_C(y)H_C(z) \sim -3\log(y-z)\,, \tag{2.9b}$$

where we defined $i\psi^k\overline{\psi}_k = \partial H_C$ through bosonization.

The action is invariant under four-dimensional spacetime supersymmetry generated by

$$Q_\alpha = \oint (p_\alpha - i\sqrt{\frac{2}{\alpha'}}\overline{\theta}^{\dot{\alpha}}\partial x_{\alpha\dot{\alpha}} + \frac{1}{2}\overline{\theta}^2\partial\theta_\alpha)\,, \tag{2.10a}$$

$$\overline{Q}^{\dot{\alpha}} = \oint (\overline{p}^{\dot{\alpha}} - i\sqrt{\frac{2}{\alpha'}}\theta_\alpha\partial x^{\dot{\alpha}\alpha} + \frac{1}{2}\theta^2\partial\overline{\theta}^{\dot{\alpha}})\,, \tag{2.10b}$$

where $\oint \equiv \frac{1}{2\pi i}\oint dz$, which satisfy the usual supersymmetry algebra

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = -2i\sqrt{\frac{2}{\alpha'}}\oint \partial x_{\alpha\dot{\alpha}}\,, \tag{2.11a}$$

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0\,. \tag{2.11b}$$

Note that the four-dimensional supersymmetry generators commute with all worldsheet fields related to the compactified directions. Relevant supersymmetric combinations of the worldsheet variables are

$$d_\alpha = -p_\alpha - i\sqrt{\frac{2}{\alpha'}}\overline{\theta}^{\dot{\alpha}}\partial x_{\alpha\dot{\alpha}} + \overline{\theta}^2\partial\theta_\alpha - \frac{1}{2}\theta_\alpha\partial\overline{\theta}^2\,, \tag{2.12a}$$

$$\overline{d}_{\dot{\alpha}} = -\overline{p}_{\dot{\alpha}} + i\sqrt{\frac{2}{\alpha'}}\theta^\alpha\partial x_{\alpha\dot{\alpha}} + \theta^2\partial\overline{\theta}_{\dot{\alpha}} - \frac{1}{2}\overline{\theta}_{\dot{\alpha}}\partial\theta^2\,, \tag{2.12b}$$

$$\Pi_{\alpha\dot{\alpha}} = \sqrt{\frac{2}{\alpha'}}\partial x_{\alpha\dot{\alpha}} + 2i\partial\theta_\alpha\overline{\theta}_{\dot{\alpha}} + 2i\partial\overline{\theta}_{\dot{\alpha}}\theta_\alpha\,, \tag{2.12c}$$

with the following OPEs

$$d_\alpha(y)\Pi_{\beta\dot{\beta}}(z) \sim \frac{4i\epsilon_{\alpha\beta}\partial\overline{\theta}_{\dot{\beta}}}{y-z}\,, \quad \overline{d}_{\dot{\alpha}}(y)\Pi_{\beta\dot{\beta}}(z) \sim \frac{-4i\epsilon_{\dot{\alpha}\dot{\beta}}\partial\theta_\beta}{y-z}\,, \tag{2.13a}$$

$$d_\alpha(z)\overline{d}_{\dot{\alpha}}(y) \sim \frac{2i\Pi_{\alpha\dot{\alpha}}}{y-z}\,. \tag{2.13b}$$

The hermiticity conditions are defined as $(p^\alpha)^\dagger = -\overline{p}^{\dot\alpha}$, $(\theta^\alpha)^\dagger = \overline{\theta}^{\dot\alpha}$, $(\partial x_{\alpha\dot\alpha})^\dagger = -\partial x_{\alpha\dot\alpha}$, $(d_\alpha)^\dagger = -\overline{d}_{\dot\alpha}$, $(\partial\rho)^\dagger = -(2\partial\rho - \partial H_C)$ and $(\partial H_C)^\dagger = -(3\partial\rho - 2\partial H_C)$[3]. Note that $(\partial\theta^\alpha)^\dagger = -\partial\overline{\theta}^{\dot\alpha}$ using the standard CFT rule for a primary field ϕ of conformal weight h on the plane, namely, $[\phi(z)]^\dagger = \phi^\dagger(\overline{z}^{-1})\overline{z}^{-2h}$.

From the worldsheet fields, one can form the generators of a twisted small $\mathcal{N} = 4$ algebra

$$T = T_4 + T_6\,, \qquad J = J_4 + J_6\,, \tag{2.14a}$$

$$G^\pm = G_4^\pm + G_6^\pm\,, \qquad \widetilde{G}^\pm = \widetilde{G}_4^\pm + \widetilde{G}_6^\pm\,, \tag{2.14b}$$

$$J^{++} = e^{-i\rho+iH_C}\,, \qquad J^{--} = e^{i\rho-iH_C}\,, \tag{2.14c}$$

where T is the stress tensor and

$$\begin{aligned}
T_4 &= \frac{1}{4}\Pi^{\dot\alpha\alpha}\Pi_{\alpha\dot\alpha} + \partial\theta^\alpha d_\alpha + \partial\overline{\theta}_{\dot\alpha}\overline{d}^{\dot\alpha} + \frac{1}{2}\partial\rho\partial\rho - \frac{i}{2}\partial^2\rho\,, & J_4 &= -i\partial\rho\,,\\
T_6 &= -\frac{2}{\alpha'}\partial\overline{x}_k\partial x^k - \partial\psi^k\overline{\psi}_k\,, & J_6 &= -\psi^k\overline{\psi}_k = i\partial H_C\,,\\
G_4^+ &= \frac{1}{2\sqrt{8}}e^{i\rho}d^2\,, & G_4^- &= -\frac{1}{2\sqrt{8}}e^{-i\rho}\overline{d}^2\,,\\
\widetilde{G}_4^+ &= -\frac{1}{2\sqrt{8}}e^{-2i\rho+iH_C}\overline{d}^2\,, & \widetilde{G}_4^- &= -\frac{1}{2\sqrt{8}}e^{2i\rho-iH_C}d^2\,,\\
G_6^+ &= \sqrt{\frac{2}{\alpha'}}\partial\overline{x}_j\psi^j\,, & G_6^- &= \sqrt{\frac{2}{\alpha'}}\partial x^j\overline{\psi}_j\,,\\
\widetilde{G}_6^+ &= \frac{1}{2}\sqrt{\frac{2}{\alpha'}}e^{-i\rho}\epsilon_{jkl}\partial x^j\psi^k\psi^l\,, & \widetilde{G}_6^- &= -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}e^{i\rho}\epsilon^{ijk}\partial\overline{x}_i\overline{\psi}_j\overline{\psi}_k\,.
\end{aligned}$$

As one can see, the small $\mathcal{N} = 4$ algebra includes four supercurrents $\{G^\pm, \widetilde{G}^\pm\}$ and three spin-1 currents $\{J, J^{++}, J^{--}\}$, which generate an $SU(2)$ algebra. A few OPEs these generators satisfy are

$$G_4^+(y)G_4^-(z) \sim -\frac{1}{(y-z)^3} + \frac{J_4}{(y-z)^2} + \frac{T_4}{(y-z)}\,, \tag{2.15a}$$

$$G_6^+(y)G_6^-(z) \sim \frac{3}{(y-z)^3} + \frac{J_6}{(y-z)^2} + \frac{T_6}{(y-z)}\,, \tag{2.15b}$$

$$G_4^+(y)\widetilde{G}_4^+(z) \sim -\frac{1}{(y-z)^2}e^{-i\rho+iH_C} + \frac{1}{(y-z)}\partial e^{-i\rho}e^{iH_C}\,, \tag{2.15c}$$

$$G_6^+(y)\widetilde{G}_6^+(z) \sim \frac{3}{(y-z)^2}e^{-i\rho+iH_C} + \frac{1}{(y-z)}e^{-i\rho}\partial e^{iH_C}\,, \tag{2.15d}$$

[3] We do not discuss here the unusual complex conjugation properties defined for $\partial\rho$ and ∂H_C and recommend [17] and [22] for an explanation, where it is referred as the $\sim$ conjugation.

$$J^{\pm\pm}(y)G^{\mp}(z) \sim \mp\frac{\widetilde{G}^{\pm}}{y-z}\,, \tag{2.15e}$$

$$\left\{\oint G^{+}(z), \oint \widetilde{G}^{+}(z)\right\} = 0\,. \tag{2.15f}$$

Some comments are in order. First, note that the generators $\{G^{\pm}, T, J\}$ form a twisted $c = 6$ $\mathcal{N} = 2$ superconformal field theory (SCFT). This SCFT splits in two parts: one describing the four-dimensional spacetime $\{G_4^{\pm}, T_4, J_4\}$ as a twisted $c = -3$ $\mathcal{N} = 2$ SCFT and the other describing the six dimensional internal part $\{G_6^{\pm}, T_6, J_6\}$ as a twisted $c = 9$ $\mathcal{N} = 2$ SCFT. The small $\mathcal{N} = 4$ algebra above is then constructed by adding the currents J^{++} and J^{--} to form the SU(2) triplet $\{J, J^{++}, J^{--}\}$.

One should observe that $\{G_4^{\pm}, T_4, J_4\}$ and $\{G_6^{\pm}, T_6, J_6\}$ decouple from each other, i.e., they have non-singular OPEs between them. Consequently, the six dimensional background can be replaced by any Calabi-Yau background described by an $\mathcal{N} = 2$ SCFT. Another important fact is that, in the twisted case (considered in this work), the TT OPE has no conformal anomaly so one can use topological methods to compute the spectrum of correlation functions [18] without the need of introducing superconformal ghosts.

We also define the ρ-charge of an operator $\mathcal{O}$ as the single pole in the OPE of J_4 with $\mathcal{O}$, and the "Calabi-Yau"-charge (CY-charge) as the single pole in the OPE of J_6 with $\mathcal{O}$. Properties of the generators and the hybrid variables for the twisted case are summarized in the following tables

	Weight	CY-charge	ρ-charge
ψ^j	0	1	0
$\overline{\psi}_j$	1	-1	0
$e^{in\rho}$	$\frac{-n(n+1)}{2}$	0	n
e^{iH_C}	0	3	0
e^{-iH_C}	3	-3	0

Table 2.1: Conformal weight, CY-charge and ρ-charge in the twisted case.

Generator	Weight
$G^+, \widetilde{G}^+, J$	1
$G^-, \widetilde{G}^-, J^{--}, T$	2
J^{++}	0

Table 2.2: Conformal weight of the twisted small $\mathcal{N} = 4$ generators.

2.2.2 Free field oscillator expansions

In this work, we will need the description in terms of oscillator modes of the worldsheet fields. Considering first the free case, the oscillator expansions in the complex plane for the four-dimensional variables are

$$p_\alpha(z) = \sum_N \frac{p_{\alpha N}}{z^{N+1}}, \quad \overline{p}_{\dot\alpha}(z) = \sum_N \frac{\overline{p}_{\dot\alpha N}}{z^{N+1}}, \tag{2.16a}$$

$$\theta_\alpha(z) = \sum_N \frac{\theta_{\alpha N}}{z^N}, \quad \overline{\theta}_{\dot\alpha} = \sum_N \frac{\overline{\theta}_{\dot\alpha N}}{z^N}, \tag{2.16b}$$

$$x^m = x_0^m - i\alpha' p^m \log|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{N\in\mathbb{Z}-\{0\}} \frac{\alpha_N^m}{N}(z^{-N}+\overline{z}^{-N}), \tag{2.16c}$$

where $\alpha_0^m = (2\alpha')^{1/2}p^m$, x^m satisfies Neumann boundary conditions and capital middle alphabet letters, such as M and N, are used to denote the oscillator numbers. We also used that $\phi(z) = \sum_N \frac{\phi_N}{z^{N+h}}$ for a primary field $\phi(z)$ of conformal weight h.

The hermiticity properties for the modes are $(\theta_N^\alpha)^\dagger = \overline{\theta}^{\dot\alpha}_{-N}$, $(p_N^\alpha)^\dagger = -\overline{p}^{\dot\alpha}_{-N}$ and $(\alpha_N^m)^\dagger = \alpha^m_{-N}$, with the commutation relations

$$[\alpha_M^m, \alpha_N^n] = M\delta_{M+N,0}\eta^{mn}, \quad \{p_M^\beta, \theta_{\alpha N}\} = \delta_\alpha^\beta \delta_{M+N,0}, \quad \{\overline{p}_{\dot\beta M}, \overline{\theta}_N^{\dot\alpha}\} = \delta_{\dot\beta}^{\dot\alpha}\delta_{M+N,0}, \tag{2.17}$$

giving the OPEs (2.8).

The supersymmetric variables (2.12) in terms of the free field oscillators have the mode expansions

$$d_{\alpha N} = -p_{\alpha N} - \sum_R (\sigma^m\overline{\theta}_R)_\alpha \alpha_{mN-R} - \sum_{R,S}(N-2R-S)(\overline{\theta}_R\overline{\theta}_S)\theta_{\alpha N-R-S}, \tag{2.18a}$$

$$\overline{d}_{\dot\alpha N} = -\overline{p}_{\dot\alpha N} + \sum_R (\theta_R\sigma^m)_{\dot\alpha}\alpha_{mN-R} - \sum_{R,S}(N-2R-S)(\theta_R\theta_S)\overline{\theta}_{\dot\alpha N-R-S}, \tag{2.18b}$$

$$\Pi_{\alpha\dot\alpha N} = -i\sigma^m_{\alpha\dot\alpha}\alpha_{mN} + 2i\sum_R (N-2R)\theta_{\alpha R}\overline{\theta}_{\dot\alpha N-R}, \tag{2.18c}$$

where

$$d_\alpha(z) = \sum_N \frac{d_{\alpha N}}{z^{N+1}}, \quad \overline{d}_{\dot\alpha}(z) = \sum_N \frac{\overline{d}_{\dot\alpha N}}{z^{N+1}}, \quad \Pi_{\alpha\dot\alpha}(z) = \sum_N \frac{\Pi_{\alpha\dot\alpha N}}{z^{N+1}}.$$

And these modes satisfy a set of commutation relations

$$[d_{\alpha N}, \Pi_{\beta\dot\beta M}] = 4i\epsilon_{\alpha\beta}\partial\overline{\theta}_{\dot\beta N+M}, \quad [\overline{d}_{\dot\alpha N}, \Pi_{\beta\dot\beta M}] = -4i\epsilon_{\dot\alpha\dot\beta}\partial\theta_{\beta N+M}, \tag{2.19a}$$

$$\{d_{\alpha M}, \partial\theta_N^\beta\} = -N\delta_\alpha^\beta\delta_{M+N,0}, \quad \{\overline{d}_M^{\dot\alpha}, \partial\overline{\theta}_{\dot\beta N}\} = -N\delta_{\dot\beta}^{\dot\alpha}\delta_{M+N,0}, \tag{2.19b}$$

$$\{d_{\alpha N}, \overline{d}_{\dot{\alpha} M}\} = 2i\Pi_{\alpha\dot{\alpha} M+N}\,, \tag{2.19c}$$

with $\partial\theta^{\alpha}_{N} \equiv \oint dz\, z^{N}\partial\theta^{\alpha}(z)$ and $\partial\overline{\theta}_{\dot{\alpha}N} \equiv \oint dz\, z^{N}\partial\overline{\theta}_{\dot{\alpha}}(z)$.

The Virasoro generators of the four-dimensional part are defined as

$$T_4(z) = \sum_N \frac{L_N}{z^{N+2}} + \frac{1}{2}\partial\rho\partial\rho(z) - \frac{i}{2}\partial^2\rho(z)\,, \tag{2.20a}$$

$$L_M = \sum_N \left(\frac{1}{4}\Pi^{\dot{\alpha}\alpha}_{N}\Pi_{\alpha\dot{\alpha}-N+M} + \partial\theta^{\alpha}_{N}d_{\alpha-N+M} + \partial\overline{\theta}_{\dot{\alpha}N}\overline{d}^{\dot{\alpha}}_{-N+M}\right). \tag{2.20b}$$

Notice that the mode L_0 has no normal ordering constant. And we will write

$$d^2(z) = \sum_N \frac{d^2_N}{z^{N+2}}\,, \quad \overline{d}^2(z) = \sum_N \frac{\overline{d}^2_N}{z^{N+2}}\,,$$

with $d^2_N = \sum_{M\in\mathbb{Z}} d^{\alpha}_{N+M}d_{\alpha-M}$ and $\overline{d}^2_N = \sum_{M\in\mathbb{Z}} \overline{d}_{\dot{\alpha}N+M}\overline{d}^{\dot{\alpha}}_{-M}$.

We remark that when commuting or anti-commuting with functions of θ_0 and $\overline{\theta}_0$,

$$d_{0\alpha} = -p_{\alpha 0} - (\sigma^m\overline{\theta}_0)_{\alpha}\alpha_{0m} + \dots\,, \quad \overline{d}_{0\dot{\alpha}} = -\overline{p}_{\dot{\alpha}0} + (\theta_0\sigma^m)_{\dot{\alpha}}\alpha_{0m} + \dots\,, \tag{2.21}$$

act as the usual derivatives D_{α} and $\overline{D}_{\dot{\alpha}}$ of [24]. To make contact with that notation, one can use the replacements $p_{\alpha 0} \to -\frac{\partial}{\partial\theta^{\alpha}}$, $\overline{p}_{\dot{\alpha}0} \to \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}$, $\theta^{\alpha}_0 \to \theta^{\alpha}$, $\overline{\theta}^{\dot{\alpha}}_0 \to \overline{\theta}^{\dot{\alpha}}$ and $\alpha^m_0 = -i\partial^m$. Note that $(D^{\alpha})^{\dagger} = \overline{D}^{\dot{\alpha}}$ in [24], while $(d^{\alpha}_0)^{\dagger} = -\overline{d}^{\dot{\alpha}}_0$. This is not contradictory because d^{α}_0 and $\overline{d}^{\dot{\alpha}}_0$ act through commutators/anticommutators.

2.2.3 The string field Φ and superspace action

It was shown in [21] that the string field Φ of the manifestly SO$(1,3)$ super-Poincaré invariant open superstring field theory can be written as $\Phi = \Phi_{-1} + \Phi_0 + \Phi_1$ modulo exact terms in $G^+/\widetilde{G}^+$, with the field Φ_n carrying n units of ρ-charge and $-n$ units of CY-charge. The linearized equations of motion for Φ are

$$\widetilde{G}^+_4 G^+_4\Phi_{-1} + \widetilde{G}^+_4 G^+_6\Phi_0 + \widetilde{G}^+_4\widetilde{G}^+_6\Phi_1 = 0\,, \tag{2.22a}$$

$$(\widetilde{G}^+_6 G^+_6 + \widetilde{G}^+_4 G^+_4)\Phi_0 + \widetilde{G}^+_4 G^+_6\Phi_1 + \widetilde{G}^+_6 G^+_4\Phi_{-1} = 0\,, \tag{2.22b}$$

$$G^+_4 G^+_6\Phi_{-1} + G^+_4\widetilde{G}^+_6\Phi_0 + G^+_4\widetilde{G}^+_4\Phi_1 = 0\,, \tag{2.22c}$$

When we write $G_4^+\mathcal{O}$ instead of $G_4^+(z)\mathcal{O}$ we mean taking the contour integral of G_4^+ around $\mathcal{O}$, i.e., $\oint dz\, G_4^+(z)\mathcal{O}$, and similarly for the other generators.

These equations of motion are invariant under the linearized gauge transformations

$$\delta\Phi_{-1} = G_4^+\Lambda_{-2} + G_6^+\Lambda_{-1} + \widetilde{G}_6^+\Lambda_0 + \widetilde{G}_4^+\Lambda_1, \tag{2.23a}$$
$$\delta\Phi_0 = G_4^+\Lambda_{-1} + G_6^+\Lambda_0 + \widetilde{G}_6^+\Lambda_1 + \widetilde{G}_4^+\Lambda_2, \tag{2.23b}$$
$$\delta\Phi_1 = G_4^+\Lambda_0 + G_6^+\Lambda_1 + \widetilde{G}_6^+\Lambda_2 + \widetilde{G}_4^+\Lambda_3. \tag{2.23c}$$

where the gauge parameter Λ_n carries n units of ρ-charge and $-n-1$ units of CY-charge. In the situation that we will encounter, Φ_{-1} and Φ_1 will be algebraically gauged away. Due to this, only (2.22b) will contribute to the quadratic superspace action

$$S = \langle \Phi_0(\widetilde{G}_6^+G_6^+ + \widetilde{G}_4^+G_4^+)\Phi_0\rangle\,, \tag{2.24}$$

which is evaluated as a two-point CFT correlation function on the plane with the normalization $\langle e^{-i\rho+iH_C}(\theta_0\theta_0)(\overline{\theta}_0\overline{\theta}_0)\rangle = 1$.

As an example, consider the four-dimensional massless sector of the open superstring which is independent of the compactification. Since there is nothing we can write with conformal weight zero at zero momentum for Φ_1 and Φ_{-1}, one finds that $\Phi_0 = V(x^m, \theta_0^\alpha, \overline{\theta}_0^{\dot\alpha})$ and $\Phi_1 = \Phi_{-1} = 0$ where V is the standard real vector superfield for the four-dimensional super-Maxwell multiplet. Then, schematically, the quadratic superspace action and linearized gauge transformations are

$$S = \langle V\widetilde{G}_4^+G_4^+V\rangle\,, \quad \delta V = G_4^+\Lambda_{-1} + \widetilde{G}_4^+\Lambda_2\,.$$

2.3 Superstrings in a constant electromagnetic background field

This section describes the quantization of charged open superstrings in a constant electromagnetic background. Besides the usual coupling to the Lorentz current, a new boundary term S_b is added to the worldsheet action, coupling the spacetime fermionic worldsheet variables to the background U(1) gauge field in a non-minimal fashion. Expressions for the oscillator modes and also for the small $\mathcal{N} = 4$ generators which generalize the free case are obtained. Charged open bosonic strings were studied in [25], [13], [14] and, more recently, in [26].

2.3.1 Worldsheet action and boundary conditions

To couple the superstring to a constant background gauge field, we employ the hybrid formalism in terms of oscillator modes for the four-dimensional variables. However, the chiral boson ρ will continue to be described using the free field OPEs in our treatment. This will preserve the four-dimensional spacetime supersymmetry and the gauge symmetry of our superstring field theory description. More importantly, our treatment will also preserve the form of the small $\mathcal{N}=4$ algebra, so that the reasoning used in constructing the formalism in the free case will also hold for the interacting case.

We will consider an open string with total charge $Q = q_0 + q_\pi$ and, as usual, the constant electromagnetic field strength F_{mn} couples to the charges q_0 and q_π at the ends of the string by the conserved current associated with Lorentz transformations[4]

$$J_\tau^{mn} = -\frac{i}{\alpha'}\partial_\tau x^{[m}x^{n]} - \frac{1}{2}\Big[(p\sigma^{mn}\theta) + (\overline{p}\overline{\sigma}^{mn}\overline{\theta}) - (\widehat{p}\sigma^{mn}\widehat{\theta}) - (\widehat{\overline{p}}\overline{\sigma}^{mn}\widehat{\overline{\theta}})\Big], \tag{2.25}$$

obtained by varying the worldsheet action with

$$\delta x^m = \omega^{mn}x_n, \quad \delta\theta_\alpha = -\frac{1}{2}\omega_{mn}\sigma^{mn}{}_\alpha{}^\beta\theta_\beta, \quad \delta\overline{\theta}_{\dot\alpha} = \frac{1}{2}\omega_{mn}\overline{\sigma}^{mn\dot\beta}{}_{\dot\alpha}\overline{\theta}_{\dot\beta}, \tag{2.26a}$$

$$\delta p_\alpha = -\frac{1}{2}\omega_{mn}\sigma^{mn}{}_\alpha{}^\beta p_\beta, \quad \delta\overline{p}_{\dot\alpha} = \frac{1}{2}\omega_{mn}\overline{\sigma}^{mn\dot\beta}{}_{\dot\alpha}\overline{p}_{\dot\beta}, \tag{2.26b}$$

as well as for the right-moving fields. The variables ω_{mn} represent the parameters responsible for Lorentz transformations and the matrices σ^{mn}, $\overline{\sigma}^{mn}$ generate the spinor representations of the Lorentz group and are defined as in the Appendix of [24].

The interaction term in the Euclidean action is then

$$S_{int} = \frac{\alpha'}{2}\int d\tau F_{mn}\Big\{-\frac{i}{\alpha'}\partial_\tau x^m x^n - \frac{1}{2}\Big[(p\sigma^{mn}\theta) + (\overline{p}\overline{\sigma}^{mn}\overline{\theta}) - (\widehat{p}\sigma^{mn}\widehat{\theta}) - (\widehat{\overline{p}}\overline{\sigma}^{mn}\widehat{\overline{\theta}})\Big]\Big\}, \tag{2.27}$$

and the total worldsheet action is given by

$$S = S_0 + q_0 S_{int}\big|_{\sigma=0} + q_\pi S_{int}\big|_{\sigma=\pi}. \tag{2.28}$$

[4]For a generic rank-2 tensor T, we use the conventions $T_{[mn]} = \frac{1}{2}(T_{mn} - T_{nm})$ and $T_{(mn)} = \frac{1}{2}(T_{mn} + T_{nm})$.

Bosonic worldsheet fields

We start by analyzing the known case of the bosonic variables. The equation of motion and boundary conditions for the bosonic worldsheet fields are

$$\partial_\tau^2 x^m + \partial_\sigma^2 x^m = 0\,, \tag{2.29a}$$

$$\partial_\sigma x^n + 2\pi i\alpha' q_0 \partial_\tau x_m F^{mn} = 0\,, \quad \sigma = 0\,, \tag{2.29b}$$

$$\partial_\sigma x^n - 2\pi i\alpha' q_\pi \partial_\tau x_m F^{mn} = 0\,, \quad \sigma = \pi\,. \tag{2.29c}$$

This set of equations has a general solution [13]

$$x(w) = \frac{\widehat{x}}{2} - \left(\frac{1}{QF}\right)\widehat{p} + \sqrt{\frac{\alpha'}{2}}\sum_N \left(\frac{1}{-\varepsilon - iN}\right)\Big[e^{(-\varepsilon-iN)(-i\tau+\sigma)+\gamma} + e^{(-\varepsilon-iN)(-i\tau-\sigma)-\gamma}\Big]\alpha_N\,, \tag{2.30}$$

where we are using a matrix notation for the spacetime indices, for example, $(F^{-1})\widehat{p} = (F^{-1})^{mn}\widehat{p}_n$, and the real antisymmetric spacetime matrices ε, γ and γ' are defined as

$$\varepsilon = \frac{1}{\pi}(\gamma + \gamma')\,, \tag{2.31a}$$

$$\gamma = \operatorname{arctanh}(2\pi\alpha' q_0 F)\,, \tag{2.31b}$$

$$\gamma' = \operatorname{arctanh}(2\pi\alpha' q_\pi F)\,. \tag{2.31c}$$

Canonical quantization implies for the coefficients $\widehat{x}^{\,m}$, $\widehat{p}^{\,m}$ and α_N^m the following commutators

$$[\alpha_M^m, \alpha_N^n] = (M - i\varepsilon)^{mn}\delta_{M+N,0}\,, \quad [\widehat{x}^{\,m}, \widehat{p}^{\,n}] = i\eta^{mn}\,. \tag{2.32}$$

As the notation suggests, $\widehat{x}^{\,m}$ and $\widehat{p}^{\,m}$ have the interpretation of physical position and momentum, as is easily verified analyzing the point particle limit. This interpretation also justifies the factor of $\frac{1}{2}$ in $\widehat{x}^{\,m}$, which is explained in [13]. In particular, one could expect that α_0^m would be the covariant derivative for the charged string, but note that $[\alpha_0^m, \alpha_0^n] = -i\varepsilon^{mn}$, which is not the commutation relation of a covariant derivative, namely, $[\mathcal{D}_m, \mathcal{D}_n] = iQF_{mn}$. However, suppose we construct a real spacetime matrix $\mathcal{M}$ satisfying

$$\mathcal{M}\mathcal{M}^T = \frac{\varepsilon}{QF}\,. \tag{2.33}$$

One can then define

$$\alpha_0 = \mathcal{M}(p - \frac{1}{2}QFx) = -i\mathcal{M}\mathcal{D} \equiv -i\mathfrak{D}\,, \tag{2.34}$$

where $\mathcal{D}_n = (\partial - \frac{i}{2}QFx)_n$ is the covariant derivative with $A_m = -\frac{1}{2}F_{mn}x^n$. We then obtain

$$[\alpha_{0m}, \alpha_{0r}] = -\mathcal{M}_{mn}\mathcal{M}_{rs}[\mathcal{D}^n, \mathcal{D}^s] = -iQ(\mathcal{M}F\mathcal{M}^T)_{mr} = -iQ\left(\frac{\varepsilon}{QF}F\right)_{mr} = -i\varepsilon_{mr}\,, \tag{2.35}$$

as desired.

In the neutral string limit, $Q \to 0$, (2.33) implies that

$$\mathcal{M}\mathcal{M}^T \to (1 - (\pi q_0 F)^2)^{-1}\,. \tag{2.36}$$

Consequently, it is consistent for small F_{mn} to define

$$\mathcal{M} \to (1 + \pi q_0 F)^{-1}\,. \tag{2.37}$$

Moreover, we will assume that ε, $\mathcal{M}$ and γ commute as matrices, which is justified given that each can be expressed in a power series in F and, as such, can be put in block skew-diagonal form by a suitable Lorentz transformation.

Fermionic worldsheet fields

Now we turn to the fermionic worldsheet fields. As in the bosonic case, the equations of motion are unaffected by boundary contributions. The subtle part is to find solutions that satisfy the non-trivial boundary conditions. We will now argue that the boundary conditions implied by the interaction term of (2.28) are inconsistent, and that preservation of the worldsheet superconformal invariance will require additional interaction terms for the fermionic worldsheet fields.

In the strip, at $\sigma = \pi$, the boundary condition from (2.28) that the fermionic fields would need to satisfy is

$$
\begin{aligned}
&p^{\alpha}\delta\theta_{\alpha}+\overline{p}_{\dot{\alpha}}\delta\overline{\theta}^{\dot{\alpha}}+\widehat{p}^{\alpha}\delta\widehat{\theta}_{\alpha}+\widehat{\overline{p}}_{\dot{\alpha}}\delta\widehat{\overline{\theta}}^{\dot{\alpha}}\\
&-\frac{\pi\alpha' q_{\pi}}{2}\left[\delta(pF\cdot\sigma\theta)+\delta(\overline{p}F\cdot\overline{\sigma}\overline{\theta})-\delta(\widehat{p}F\cdot\sigma\widehat{\theta})-\delta(\widehat{\overline{p}}F\cdot\overline{\sigma}\widehat{\overline{\theta}})\right]=0\,,
\end{aligned}
\tag{2.38}
$$

where $F\cdot\sigma \equiv F_{mn}\sigma^{mn}$, and the boundary condition at $\sigma=0$ would have a similar form but with the replacement $q_{\pi}\to -q_0$. One would then need to find an expression for the worldsheet fields that satisfies (2.38), but it turns out that a solution of the form (suppressing anti-chiral fields)

$$
\widehat{p}^{\alpha}=ap^{\alpha}+bF_{mn}(p\sigma^{mn})^{\alpha},\quad \widehat{\theta}^{\alpha}=c\theta^{\alpha}+dF_{mn}(\theta\sigma^{mn})^{\alpha}\,, \tag{2.39}
$$

cannot be found for any non-trivial value of the coefficients (a,b,c,d). This suggests that we need to modify the boundary term in the action for the fermionic fields.

To solve this problem, we will ignore the worldsheet action for a moment and start with a requirement that fixes the boundary conditions of the fermionic fields. It will then be realized that a new boundary contribution S_b, in addition to the terms in equation (2.27), needs to be added to the action for these boundary conditions to be satisfied.

Note that in [27] the equations of motion for the supersymmetric Born-Infeld theory were obtained by demanding that $\widehat{\mathcal{Q}}=\mathcal{Q}$ at the boundary, where $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ are the left and right-moving BRST operators. Following the same logic, we want to impose boundary conditions for the fermionic worldsheet fields such that the left and right-moving small $\mathcal{N}=4$ superconformal generators coincide at $\sigma=0$ and $\sigma=\pi$. This will be accomplished by first looking at the terms in the superconformal generators that have ∂x^m, which is already fixed by (2.30). To be more concrete, from (2.30), we have (recall that $z=e^{-iw}$, with $w=\sigma+i\tau$)

$$
\partial_w x^m(w)=-\sqrt{\frac{\alpha'}{2}}\sum_N\left[e^{(-\varepsilon-iN)(-\sigma-i\tau)-\gamma}\right]^m{}_n\,\alpha_N^n\,, \tag{2.40a}
$$

$$
\partial_{\overline{w}} x^m(\overline{w})=\sqrt{\frac{\alpha'}{2}}\sum_N\left[e^{(-\varepsilon-iN)(\sigma-i\tau)+\gamma}\right]^m{}_n\,\alpha_N^n\,. \tag{2.40b}
$$

Therefore, at the boundary, we obtain the following relations

$$\partial_{\overline{w}} x^m(\overline{w})\Big|_{\sigma=0} = -[e^{2\gamma}]^m{}_n \partial_w x^n(w)\Big|_{\sigma=0}\,, \tag{2.41a}$$

$$\partial_{\overline{w}} x^m(\overline{w})\Big|_{\sigma=\pi} = -[e^{-2\gamma'}]^m{}_n \partial_w x^n(w)\Big|_{\sigma=\pi}\,. \tag{2.41b}$$

Notice that the boundary conditions relating $\overline{\partial} x^m$ to ∂x^m resemble a Lorentz transformation given by choosing $\omega = 2\gamma$ at $\sigma = 0$ and $\omega = -2\gamma'$ at $\sigma = \pi$ in view of the exponentiated form of equations (2.26). Besides that, we know that the small $\mathcal{N} = 4$ superconformal generators are composed of Lorentz invariant terms, with the left-moving ones containing the worldsheet fields $\{\partial x^m, p_\alpha, \theta_\alpha, \overline{p}_{\dot\alpha}, \overline{\theta}_{\dot\alpha}\}$ and the right-moving ones containing $\{\overline{\partial} x^m, \widehat{p}_\alpha, \widehat{\theta}_\alpha, \widehat{\overline{p}}_{\dot\alpha}, \widehat{\overline{\theta}}_{\dot\alpha}\}$. So to obtain the intended boundary conditions relating the left and right-moving superconformal generators in the interacting case, namely,

$$\widehat{G}^{\pm}(\overline{z}) = G^{\pm}(z)\,, \qquad \widehat{\widetilde{G}}^{\pm}(\overline{z}) = \widetilde{G}^{\pm}(z)\,, \qquad \widehat{T}(\overline{z}) = T(z)\,, \tag{2.42}$$

at $\mathrm{Im}\{z\} = 0$, one can relate the left and right-moving fermionic worldsheet fields by a condition resembling a Lorentz transformation given with the same parameter ω_{mn} as in (2.41), i.e., at $\sigma = 0$

$$\widehat{p}_\alpha(\overline{w}) = -[e^{-\gamma\cdot\sigma}]_\alpha{}^\beta p_\beta(w)\,, \qquad \widehat{\overline{p}}_{\dot\alpha}(\overline{w}) = -[e^{\gamma\cdot\overline{\sigma}}]^{\dot\beta}{}_{\dot\alpha} \overline{p}_{\dot\beta}(w)\,, \tag{2.43a}$$

$$\widehat{\theta}_\alpha(\overline{w}) = [e^{-\gamma\cdot\sigma}]_\alpha{}^\beta \theta_\beta(w)\,, \qquad \widehat{\overline{\theta}}_{\dot\alpha}(\overline{w}) = [e^{\gamma\cdot\overline{\sigma}}]^{\dot\beta}{}_{\dot\alpha} \overline{\theta}_{\dot\beta}(w)\,. \tag{2.43b}$$

And at $\sigma = \pi$

$$\widehat{p}_\alpha(\overline{w}) = -[e^{\gamma'\cdot\sigma}]_\alpha{}^\beta p_\beta(w)\,, \qquad \widehat{\overline{p}}_{\dot\alpha}(\overline{w}) = -[e^{-\gamma'\cdot\overline{\sigma}}]^{\dot\beta}{}_{\dot\alpha} \overline{p}_{\dot\beta}(w)\,, \tag{2.44a}$$

$$\widehat{\theta}_\alpha(\overline{w}) = [e^{\gamma'\cdot\sigma}]_\alpha{}^\beta \theta_\beta(w)\,, \qquad \widehat{\overline{\theta}}_{\dot\alpha}(\overline{w}) = [e^{-\gamma'\cdot\overline{\sigma}}]^{\dot\beta}{}_{\dot\alpha} \overline{\theta}_{\dot\beta}(w)\,, \tag{2.44b}$$

where $(\gamma\cdot\sigma)_\beta^{\ \alpha} \equiv \gamma_{mn}\sigma^{mn}{}_\beta{}^\alpha$, $(\gamma\cdot\overline{\sigma})^{\dot\alpha}{}_{\dot\beta} \equiv \gamma_{mn}\overline{\sigma}^{mn\dot\alpha}{}_{\dot\beta}$ and similarly for γ'. We stress that whenever the letter σ ($\overline{\sigma}$) is accompanied by a "dot", we mean the Lorentz generator σ^{mn} ($\overline{\sigma}^{mn}$) and not the worldsheet variable σ. We hope the different index structure helps to avoid any confusion.

By the same reasoning, the mode expansions of the fermionic fields should take the following form

$$p_\alpha(w) = -i\sum_N \left[e^{(-iN+\frac{1}{2}\varepsilon\cdot\sigma)(-\sigma-i\tau)+\frac{1}{2}\gamma\cdot\sigma}\right]_\alpha^{\ \beta} p_{\beta N}\,, \tag{2.45a}$$

$$\theta_\alpha(w) = \sum_N \left[e^{(-iN+\frac{1}{2}\varepsilon\cdot\sigma)(-\sigma-i\tau)+\frac{1}{2}\gamma\cdot\sigma}\right]_\alpha^{\ \beta} \theta_{\beta N}\,, \tag{2.45b}$$

$$\overline{p}_{\dot\alpha}(w) = -i\sum_N \left[e^{(-iN-\frac{1}{2}\varepsilon\cdot\overline{\sigma})(-\sigma-i\tau)-\frac{1}{2}\gamma\cdot\overline{\sigma}}\right]^{\dot\beta}_{\ \dot\alpha} \overline{p}_{\dot\beta N}\,, \tag{2.45c}$$

$$\overline{\theta}_{\dot\alpha}(w) = \sum_N \left[e^{(-iN-\frac{1}{2}\varepsilon\cdot\overline{\sigma})(-\sigma-i\tau)-\frac{1}{2}\gamma\cdot\overline{\sigma}}\right]^{\dot\beta}_{\ \dot\alpha} \overline{\theta}_{\dot\beta N}\,, \tag{2.45d}$$

so that the exponential factors of the background are not present in the Lorentz invariant terms of the small $\mathcal{N} = 4$ superconformal generators. Note that $(\varepsilon\cdot\sigma)_\beta^{\ \alpha} \equiv \varepsilon_{mn}\sigma^{mn}{}_\beta{}^\alpha$ and $(\varepsilon\cdot\overline{\sigma})^{\dot\alpha}_{\ \dot\beta} \equiv \varepsilon_{mn}\overline{\sigma}^{mn\dot\alpha}{}_{\dot\beta}$.

In short, one can say that Lorentz invariance fixed the form of the fermionic worldsheet fields in the interacting case. The appearance of a Lorentz transformation in the boundary conditions linking the left and right-moving fields is not surprising from the point of view of an open string with endpoints of charge q_0 and q_π attached to a D-brane. In this setting, T-duality can be used to show that the tilt of the D-brane in spacetime is related to the field strength F, consequently, the boundary conditions express the fact that the coordinates become rotated by the gauge field in the dual description.

Using equations (2.43a) to (2.44b) in (2.38), one can show that the following term should be added to the action for it to imply our desired boundary conditions

$$\begin{aligned} S_b = q_0\frac{\alpha'}{8}\int d\tau F_{mn}\Big[&-(\widehat{p}\{e^{-\gamma\cdot\sigma},\sigma^{mn}\}\theta) + (p\{e^{\gamma\cdot\sigma},\sigma^{mn}\}\widehat{\theta}) \\ &-(\widehat{\overline{p}}\{e^{-\gamma\cdot\overline{\sigma}},\overline{\sigma}^{mn}\}\overline{\theta}) + (\overline{p}\{e^{\gamma\cdot\overline{\sigma}},\overline{\sigma}^{mn}\}\widehat{\overline{\theta}})\Big]\Big|_{\sigma=0} \\ + q_\pi\frac{\alpha'}{8}\int d\tau F_{mn}\Big[&-(\widehat{p}\{e^{\gamma'\cdot\sigma},\sigma^{mn}\}\theta) + (p\{e^{-\gamma'\cdot\sigma},\sigma^{mn}\}\widehat{\theta}) \\ &-(\widehat{\overline{p}}\{e^{\gamma'\cdot\overline{\sigma}},\overline{\sigma}^{mn}\}\overline{\theta}) + (\overline{p}\{e^{-\gamma'\cdot\overline{\sigma}},\overline{\sigma}^{mn}\}\widehat{\overline{\theta}})\Big]\Big|_{\sigma=\pi}\,. \end{aligned} \tag{2.46}$$

Up to the addition of trivial terms whose interaction term vanishes, the contribution S_b is the unique symmetric combination in $(p_\alpha, \theta_\alpha, \overline{p}_{\dot\alpha}, \overline{\theta}_{\dot\alpha})$ and $(\widehat{p}_\alpha, \widehat{\theta}_\alpha, \widehat{\overline{p}}_{\dot\alpha}, \widehat{\overline{\theta}}_{\dot\alpha})$ that we can add to the action to obtain the boundary conditions of (2.43a) to (2.44b).

So gathering expressions (2.7), (2.27) and (2.46), we can write the total worldsheet action for the charged open superstring in the hybrid formalism coupled to a constant electromagnetic background as

$$S = S_0 + q_0 S_{int}\big|_{\sigma=0} + q_\pi S_{int}\big|_{\sigma=\pi} + S_b\,. \tag{2.47}$$

2.3.2 Superconformal generators and commutation relations

We will present here the description of the superconformal generators in terms of oscillator modes, which is central for the computations contained in future sections.

Canonical quantization of the fermionic worldsheet variables implies the same commutation relations as in the free case for the fermionic modes

$$\{p^\beta_M, \theta_{\alpha N}\} = \delta^\beta_\alpha \delta_{M+N,0}\,, \qquad \{\overline{p}_{\dot\beta M}, \overline{\theta}^{\dot\alpha}_N\} = \delta^{\dot\alpha}_{\dot\beta}\delta_{M+N,0}\,. \tag{2.48}$$

The boundary conditions and mode expansions determined that the Lorentz invariant terms present in the superconformal generators, such as $d^2(z)$, are holomorphic and independent of exponential factors of the background gauge field. This makes it straightforward to obtain the modes of $\{G^\pm, \widetilde{G}^\pm, T\}$ through the usual method of contour integration, applying the doubling trick to consider only the left-moving variables defined in the whole plane.

One starts with the mode expansions in the plane

$$\partial x^m = -i\sqrt{\frac{\alpha'}{2}}\sum_N [z^{-i\varepsilon - N - 1} e^{-\gamma}]^m{}_n \alpha^n_N\,, \tag{2.49a}$$

$$p_\alpha(z) = \sum_N [z^{-N-\frac{i}{2}\varepsilon\cdot\sigma} e^{\frac{1}{2}\gamma\cdot\sigma}]_\alpha{}^\beta p_{\beta N}\,, \tag{2.49b}$$

$$\theta_\alpha(z) = \sum_N [z^{-N-\frac{i}{2}\varepsilon\cdot\sigma} e^{\frac{1}{2}\gamma\cdot\sigma}]_\alpha{}^\beta \theta_{\beta N}\,, \tag{2.49c}$$

$$\overline{p}_{\dot\alpha}(z) = \sum_N [z^{-N+\frac{i}{2}\varepsilon\cdot\overline{\sigma}} e^{-\frac{1}{2}\gamma\cdot\overline{\sigma}}]^{\dot\beta}{}_{\dot\alpha}\overline{p}_{\dot\beta N}\,, \tag{2.49d}$$

$$\overline{\theta}_{\dot\alpha}(z) = \sum_N [z^{-N+\frac{i}{2}\varepsilon\cdot\overline{\sigma}} e^{-\frac{1}{2}\gamma\cdot\overline{\sigma}}]^{\dot\beta}{}_{\dot\alpha}\overline{\theta}_{\dot\beta N}\,. \tag{2.49e}$$

Then, as an example, substituting equations (2.49) in $d^2(z)$ and $\overline{d}^2(z)$, where d_α

and $\bar{d}_{\dot\alpha}$ were given in (2.12), we obtain

$$d^2(z) = \sum_N \frac{d_N^2}{z^{N+2}}\,, \qquad\qquad \bar{d}^2(z) = \sum_N \frac{\bar{d}_N^2}{z^{N+2}}\,, \tag{2.50}$$

which have the same form as in the free case, with $d_N^2 = \sum_{M\in\mathbb{Z}} d^\alpha_{N+M} d_{\alpha -M}$ and $\bar{d}_N^2 = \sum_{M\in\mathbb{Z}} \bar{d}_{\dot\alpha N+M}\bar{d}^{\dot\alpha}_{-M}$. And the modes of the supersymmetric variables can be read off by substituting the expressions (2.49) in (2.12)

$$\begin{aligned}
d_{\alpha N} &= -p_{\alpha N} - \sum_R (\sigma^m \bar\theta_R)_\alpha \alpha_{m N-R} - \sum_{RS}(N-2R-S)(\bar\theta_R\bar\theta_S)\theta_{\alpha N-R-S} \\
&\quad - \frac{i}{2}\sum_{RS}(\bar\theta_R\bar\theta_S)(\varepsilon\cdot\sigma)_\alpha^{\ \beta}\theta_{\beta N-R-S}\,, && \text{(2.51a)}\\
\bar{d}_{\dot\alpha N} &= -\bar{p}_{\dot\alpha N} + \sum_R (\theta_R\sigma^m)_{\dot\alpha}\alpha_{m N-R} - \sum_{RS}(N-2R-S)(\theta_R\theta_S)\bar\theta_{\dot\alpha N-R-S} \\
&\quad + \frac{i}{2}\sum_{RS}(\theta_R\theta_S)(\varepsilon\cdot\bar\sigma)^{\dot\beta}_{\ \dot\alpha}\bar\theta_{\dot\beta N-R-S}\,, && \text{(2.51b)}\\
\Pi_{\alpha\dot\alpha N} &= -i\sigma^m_{\alpha\dot\alpha}\alpha_{mN} + 2i\sum_R(N-2R)\theta_{\alpha R}\bar\theta_{\dot\alpha N-R} + \sum_R(\varepsilon\cdot\sigma)_\alpha^{\ \beta}\theta_{\beta R}\bar\theta_{\dot\alpha N-R} \\
&\quad + \sum_R(\varepsilon\cdot\bar\sigma)^{\dot\beta}_{\ \dot\alpha}\theta_{\alpha R}\bar\theta_{\dot\beta N-R}\,, && \text{(2.51c)}\\
\partial\theta_{\alpha N} &= -N\theta_{\alpha N} - \frac{i}{2}(\varepsilon\cdot\sigma)_\alpha^{\ \beta}\theta_{\beta N}\,, \quad \partial\bar\theta_{\dot\alpha N} = -N\bar\theta_{\dot\alpha N} + \frac{i}{2}(\varepsilon\cdot\bar\sigma)^{\dot\beta}_{\ \dot\alpha}\bar\theta_{\dot\beta N}\,. && \text{(2.51d)}
\end{aligned}$$

Note that $\partial\theta_{\alpha 0} = \partial\bar\theta_{\dot\alpha 0} = 0$ when $\varepsilon = 0$. The supersymmetric modes of the interacting fields satisfy

$$\begin{aligned}
&[d_{\alpha N}, \Pi_{\beta\dot\beta M}] = 4i\epsilon_{\alpha\beta}\partial\bar\theta_{\dot\beta N+M}\,, \quad [\bar{d}_{\dot\alpha N}, \Pi_{\beta\dot\beta M}] = -4i\epsilon_{\dot\alpha\dot\beta}\partial\theta_{\beta N+M}\,, && \text{(2.52a)}\\
&\{d_{\alpha M}, \partial\theta^\beta_N\} = -N\delta_\alpha^\beta\delta_{M+N,0} + \frac{i}{2}(\varepsilon\cdot\sigma)_\alpha^{\ \beta}\delta_{M+N,0}\,, && \text{(2.52b)}\\
&\{\bar{d}^{\dot\alpha}_M, \partial\bar\theta_{\dot\beta N}\} = -N\delta^{\dot\alpha}_{\dot\beta}\delta_{M+N,0} + \frac{i}{2}(\varepsilon\cdot\bar\sigma)^{\dot\alpha}_{\ \dot\beta}\delta_{M+N,0}\,, && \text{(2.52c)}\\
&\{d_{\alpha N}, \bar{d}_{\dot\alpha M}\} = 2i\Pi_{\alpha\dot\alpha M+N}\,. && \text{(2.52d)}
\end{aligned}$$

These commutation relations follow from equation (2.51) by using

$$[\alpha^m_M, \alpha^n_N] = (M - i\varepsilon)^{mn}\delta_{M+N,0}\,, \quad \{p^\beta_M, \theta_{\alpha N}\} = \delta_\alpha^\beta\delta_{M+N,0}\,, \quad \{\bar{p}_{\dot\beta M}, \bar\theta^{\dot\alpha}_N\} = \delta^{\dot\alpha}_{\dot\beta}\delta_{M+N,0}\,, \tag{2.53}$$

and preserve the same structure as the commutation relations in the free case

(2.19). Observe that, from (2.51), the introduction of the background modified the "super derivatives" zero modes by a term proportional to ε

$$d_{\alpha 0} f(\theta_0, \overline{\theta}_0) = \left[-p_{\alpha 0} - (\sigma^m \overline{\theta}_0)_\alpha \alpha_{0m} - \frac{i}{2}(\overline{\theta}_0 \overline{\theta}_0)(\varepsilon \cdot \sigma)_\alpha^{\ \beta} \theta_{\beta 0} \right] f(\theta_0, \overline{\theta}_0) \,, \tag{2.54a}$$

$$\overline{d}_{\dot\alpha 0} f(\theta_0, \overline{\theta}_0) = \left[-\overline{p}_{\dot\alpha 0} + (\theta_0 \sigma^m)_{\dot\alpha} \alpha_{0m} + \frac{i}{2}(\theta_0 \theta_0)(\varepsilon \cdot \overline{\sigma})^{\dot\beta}_{\ \dot\alpha} \overline{\theta}_{\dot\beta 0} \right] f(\theta_0, \overline{\theta}_0) \,, \tag{2.54b}$$

$$\Pi_0^m f(\theta_0, \overline{\theta}_0) = \left[-i\alpha_0^m + \frac{i}{2}\epsilon^{mnrs}\varepsilon_{rs}(\theta_0 \sigma_n \overline{\theta}_0) \right] f(\theta_0, \overline{\theta}_0) \,, \tag{2.54c}$$

where ϵ^{mnpq} is the four-dimensional Levi-Civita symbol with $\epsilon_{0123} = -1$. Equation (2.54) can be seen as a generalization of (2.34) from bosonic strings to the supersymmetric counterpart in four dimensions.

Of course, the superconformal generators have the same form as in the non-interacting case

$$G_4^+(z) = \frac{1}{2\sqrt{8}} e^{i\rho}(z) \sum_N \frac{d_N^2}{z^{N+2}} \,, \quad G_4^-(z) = -\frac{1}{2\sqrt{8}} e^{-i\rho}(z) \sum_N \frac{\overline{d}_N^2}{z^{N+2}} \,, \tag{2.55a}$$

$$\widetilde{G}_4^+(z) = -\frac{1}{2\sqrt{8}} e^{-2i\rho + iH_C}(z) \sum_N \frac{\overline{d}_N^2}{z^{N+2}} \,, \quad \widetilde{G}_4^-(z) = -\frac{1}{2\sqrt{8}} e^{2i\rho - iH_C}(z) \sum_N \frac{d_N^2}{z^{N+2}} \,, \tag{2.55b}$$

$$T_4(z) = \sum_N \frac{L_N}{z^{N+2}} + \frac{1}{2}\partial\rho\partial\rho(z) - \frac{i}{2}\partial^2\rho(z) \,, \tag{2.55c}$$

$$L_M = \sum_N \left(\frac{1}{4}\Pi_N^{\dot\alpha\alpha}\Pi_{\alpha\dot\alpha -N+M} + \partial\theta_N^\alpha d_{\alpha -N+M} + \partial\overline{\theta}_{\dot\alpha N}\overline{d}^{\dot\alpha}_{-N+M} \right) , \tag{2.55d}$$

using (2.51) for the supersymmetric modes. Also, using the fact that $L_1 L_{-1}|0\rangle - L_{-1}L_1|0\rangle = 2L_0|0\rangle$, one finds that L_0 does not acquire a normal ordering constant when the background field is nonzero. In the bosonic string case [13], a nonvanishing normal ordering constant was found to be proportional to ε^2. The vanishing of the normal ordering constant in our analysis is a consequence that our description preserves spacetime supersymmetry.

For later use, we also define the constant matrices

$$\Delta_\alpha^{\ \beta} \equiv \{d_{\alpha 1}, \partial\theta^\beta_{-1}\} = \delta_\alpha^\beta + \frac{i}{2}(\varepsilon \cdot \sigma)_\alpha^{\ \beta} \,, \quad \overline{\Delta}^{\dot\alpha}_{\ \dot\beta} \equiv \{\overline{d}^{\dot\alpha}_1, \partial\overline{\theta}_{\dot\beta -1}\} = \delta^{\dot\alpha}_{\dot\beta} + \frac{i}{2}(\varepsilon \cdot \overline{\sigma})^{\dot\alpha}_{\ \dot\beta} \,. \tag{2.56}$$

Identities which will be useful for future computations of commutation relations between the modes can be found in Appendix A together with our conventions for sigma matrices and spinorial indices.

2.4 Massless spin-1 multiplet in a constant electromagnetic background

Using the description of the hybrid formalism coupled to a constant U(1) background gauge field developed in the last section, we calculate in this section the action for the four-dimensional super-Maxwell multiplet of the charged open superstring.

2.4.1 Equations of motion and superspace action

As we saw in Section 2.2.3, the vertex operator for the compactification-independent massless states of the open superstring has vanishing J charge and weight zero at zero momentum and is described by a superfield V. In the case of charged strings, we need to allow the vector superfield V describing the super-Maxell multiplet to be complex. As usual, V can be expanded in terms of θ_0^α and $\overline{\theta}_0^{\dot{\alpha}}$

$$V(x^m,\theta_0^\alpha,\overline{\theta}_0^{\dot{\alpha}}) = \phi + i(\theta_0\chi_1) - i(\overline{\theta}_0\overline{\chi}_2) + i(\theta_0\theta_0)M_1 - i(\overline{\theta}_0\overline{\theta}_0)M_2^* - (\theta_0\sigma^m\overline{\theta}_0)A_m \\ - i(\overline{\theta}_0\overline{\theta}_0)(\theta_0\psi_1) + i(\theta_0\theta_0)(\overline{\theta}_0\overline{\psi}_2) + \frac{1}{2}(\theta_0\theta_0)(\overline{\theta}_0\overline{\theta}_0)D\,, \tag{2.57}$$

where (ϕ, A_m, D) are complex. The equation of motion and gauge transformations for V are

$$G_4^+\widetilde{G}_4^+V = 0\,, \qquad \delta V = G_4^+\Lambda_{-1} + \widetilde{G}_4^+\Lambda_2\,, \tag{2.58}$$

where Λ_{-1} and Λ_2 can be written as $\Lambda_{-1} = \sqrt{8}e^{-i\rho}i\xi$ and $\Lambda_2 = \sqrt{8}e^{2i\rho-iH_c}i\zeta$, with ξ and ζ carrying no J charge and having conformal weight zero, i.e., they are complex vector superfields and functions of x^m, θ_0^α and $\overline{\theta}_0^{\dot{\alpha}}$.

From (2.55), the equation of motion and gauge transformations in terms of the supersymmetric modes read

$$\frac{1}{2}d_1^2\overline{d}_{-1}^2V - d_0^2\overline{d}_0^2V = 0\,, \qquad \delta V = \frac{i}{2}d_0^2\xi - \frac{i}{2}\overline{d}_0^2\zeta\,. \tag{2.59}$$

We can express it in terms of zero modes using that $d_1^2\overline{d}_{-1}^2V = (2d_0^2\overline{d}_0^2 - 2d_0^\alpha\overline{d}_0^2d_{\alpha0} -$

$32d_0^\alpha \partial\theta_{\alpha 0})V$. The equation of motion then simplifies to

$$d_0^\alpha \bar{d}_0^2 d_{\alpha 0} V + 16\partial\theta_0^\alpha d_{\alpha 0} V = 0\,. \tag{2.60}$$

The only non-trivial gauge transformation comes from ζ, and one sees that (2.60) is gauge invariant by noting that $d_0^\alpha \bar{d}_0^2 d_{\alpha 0}\delta V = 8id_0^\alpha \partial\theta_{\alpha 0}\bar{d}_0^2\zeta$.

In the free case, $\varepsilon = 0$, d_0^α reduces to the usual super derivatives (2.21) and $\partial\theta_0^\alpha = 0$. So we recover the super-Maxell equation of motion $d_0^\alpha \bar{d}_0^2 d_{\alpha 0} V = 0$, or $D^\alpha \bar{D}^2 D_\alpha V = 0$ in the notation of [24].

Equation (2.60) comes from the action given by evaluating $\langle V\widetilde{G}_4^+ G_4^+ V\rangle$, which we write in $\mathcal{N} = 1$ superspace as

$$S = \frac{1}{16}\int d^4x\; p_0^2\bar{p}_0^2 \Big[V^\dagger (d_0^\alpha \bar{d}_0^2 d_{\alpha 0} + 16\partial\theta_0^\alpha d_{\alpha 0})V\Big]\,. \tag{2.61}$$

To get the expression in terms of components, one can use the gauge transformations to go to the WZ gauge in which the only nonzero components of V and $V^\dagger$ are

$$V = -(\theta_0\sigma^m\bar{\theta}_0)A_m - i(\bar{\theta}_0\bar{\theta}_0)(\theta_0\psi_1) + i(\theta_0\theta_0)(\bar{\theta}_0\bar{\psi}_2) + \frac{1}{2}(\theta_0\theta_0)(\bar{\theta}_0\bar{\theta}_0)D\,, \tag{2.62a}$$

$$V^\dagger = -(\theta_0\sigma^m\bar{\theta}_0)A_m^* + i(\theta_0\theta_0)(\bar{\theta}_0\bar{\psi}_1) - i(\bar{\theta}_0\bar{\theta}_0)(\theta_0\psi_2) + \frac{1}{2}(\theta_0\theta_0)(\bar{\theta}_0\bar{\theta}_0)D^*\,, \tag{2.62b}$$

where we also wrote $V^\dagger$ to emphasize that V is a complex superfield for the charged superstring.

Expanding the oscillator modes in (2.60) using (2.51), we obtain the equations of motion for the components

$$D = 0\,, \tag{2.63a}$$

$$\mathfrak{D}_m(\bar{\sigma}^m\psi_1)^{\dot{\alpha}} = 0\,, \tag{2.63b}$$

$$\mathfrak{D}_m(\sigma^m\bar{\psi}_2)_\alpha = 0\,, \tag{2.63c}$$

$$\mathfrak{D}^2 A^m - \mathfrak{D}^m\mathfrak{D}^n A_n + 2i\varepsilon^{mn}A_n = 0\,, \tag{2.63d}$$

with gauge transformations

$$\delta A_m = \mathfrak{D}_m a\,, \tag{2.64a}$$

$$\delta D = \delta\psi_{1\alpha} = \delta\bar{\psi}_2^{\dot{\alpha}} = 0\,, \tag{2.64b}$$

where a is an arbitrary gauge parameter and recall that $[\mathfrak{D}_m, \mathfrak{D}_n] = -[\alpha_{0m}, \alpha_{0n}] = i\varepsilon_{mn}$. These equations of motion are obtained by varying the action

$$S = \frac{1}{2}\int d^4x\Big[A^*_m\left(\mathfrak{D}^2 A^m - \mathfrak{D}^m\mathfrak{D}^n A_n + 2i\varepsilon^{mn}A_n\right) - i(\overline{\psi}_1\overline{\sigma}^m\mathfrak{D}_m\psi_1) \\ - i(\psi_2\sigma^m\mathfrak{D}_m\overline{\psi}_2) + D^*D\Big]\,. \tag{2.65}$$

Note that when the background is zero ($\varepsilon = 0$), the above action becomes two decoupled actions for the super-Maxwell multiplet

$$S = \int d^4x\Big[-\frac{1}{4}F_1^2 - \frac{i}{2}(\overline{\psi}_1\overline{\sigma}^m\partial_m\psi_1) + \frac{1}{2}D_1^2 + (1\leftrightarrow 2)\Big]\,, \tag{2.66}$$

where $A = A_1 + iA_2$, $D = D_1 + iD_2$ and $F_{Imn} = \partial_m A_{In} - \partial_n A_{Im}$ ($I = 1,2$).

If we perform the substitution $A_m \to (\mathcal{M}A)_m$, (2.63d) and the gauge transformation for A_m can be put in the form given by [13]

$$(\mathcal{D}\cdot\frac{\varepsilon}{QF}\cdot\mathcal{D})A^m - \mathcal{D}^m(\mathcal{D}\cdot\frac{\varepsilon}{QF}\cdot A) + 2i(\varepsilon A)^m = 0\,,\qquad \delta A_m = \mathcal{D}_m a\,. \tag{2.67}$$

Observe that the vector field A_m remains massless in the presence of the background. This is due to the normal ordering constant being absent in the Virasoro algebra of the four-dimensional part of the superstring (2.55), an effect of our supersymmetric description. This can be contrasted with the results found in [13], where A_m acquires a mass term proportional to ε^2, a consequence of the shift in the normal ordering constant by the same amount. Nevertheless, one can check that the difference between equation (2.67) and equation (3.5) of [13] has vanishing gauge variation, so both results are consistent. Notice when comparing (2.67) with (3.5) of [13] that there is a sign difference in the term with no derivatives because we define the commutator $[\mathcal{D}_m, \mathcal{D}_n] = iQF_{mn}$, whereas $[\mathcal{D}_m, \mathcal{D}_n] = -iQF_{mn}$ in [13].

When $q_\pi \to -q_0$, or the neutral string limit, one has

$$\Big[\partial\cdot(1-\pi^2q_0^2F^2)^{-1}\cdot\partial\Big]A_1^m - \partial^m\Big[\partial\cdot(1-\pi^2q_0^2F^2)^{-1}\cdot A_1\Big] = 0\,, \tag{2.68}$$

and

$$[(1+\pi q_0 F)^{-1}]_{mn}\overline{\sigma}^m\partial^n\psi_1 = 0\,, \tag{2.69}$$

where we used (2.36) and (2.37). Similar expressions hold for A_{2m} and ψ_2. Equations (2.68) and (2.69) can be obtained by varying the supersymmetric Born-Infeld action [28] [29]

$$S_{superBI} = \int d^4x \Big[-\det\Big(\eta_{mn} + q_0 \pi F_{1mn} - 2(\overline{\psi}_1 \sigma_m \partial_n \psi_1) \Big) \Big]^{1/2} . \tag{2.70}$$

More precisely, as shown in [13], equation (2.68) can be obtained by expanding the field strength around a constant background in the equations of motion coming from (2.70).

2.5 Superspace action of the first massive state of the superstring in a constant electromagnetic background

Repeating the steps of the last section, we compute here the superspace action for the first massive states of the charged open superstring compactified to four dimensions and coupled to a constant electromagnetic background. This action is non-polynomial in F_{mn} and describes a massive complex spin-2 multiplet and two massive complex scalar multiplets, which are the compactification-independent states in four dimensions preserving $\mathcal{N} = 1$ supersymmetry. The case without an electromagnetic background was studied in [30].

2.5.1 String field/vertex operator

Since we are ignoring compactification-dependent contributions, fields that depend on the internal directions j of the Calabi-Yau are not allowed and the most general complex string field having conformal weight +1 at zero momentum and $(\text{mass})^2 = \frac{1}{\alpha'} = 2$ is

$$\Phi_0 = \varphi - (\partial\rho - \partial H_C)B + i(\partial H_C - 3\partial\rho)C \,, \tag{2.71a}$$
$$\Phi_1 = \sqrt{8} e^{i\rho} \overline{\psi}_j \partial x^j \widetilde{A} \,, \tag{2.71b}$$
$$\Phi_{-1} = \sqrt{8} e^{-i\rho} \psi^j \partial \overline{x}_j A \,, \tag{2.71c}$$
$$\varphi \equiv d^{\alpha}_{-1} W_{1\alpha} - \overline{d}_{\dot\alpha -1} \overline{W}{}^{\dot\alpha}_2 + i\Pi^m_{-1} V_m + \partial\theta^{\alpha}_{-1} V_{1\alpha} - \partial\overline{\theta}_{\dot\alpha -1} \overline{V}{}^{\dot\alpha}_2 \,, \tag{2.71d}$$

where φ is a superfield annihilated by modes > 1 and is a general linear combination of the four-dimensional supersymmetric worldsheet variables of conformal weight $+1$. Although Φ_1 and Φ_{-1} do depend on the Calabi-Yau metric, we will show that they can be gauged away algebraically, so this doesn't contradict the fact that Φ is independent of the specific form of the compactification. The quantities A, $\widetilde{A}$, B, C, W_1^α, $\overline{W}_2^{\dot\alpha}$, V_1^α, $\overline{V}_2^{\dot\alpha}$ and V_m are usual $\mathcal{N}=1$ superfields which depend only on the zero modes of $(x,\theta,\overline{\theta})$, i.e., they are superfields annihilated by modes ≥ 1. The minus sign in front of $\partial\overline{\theta}_{\dot\alpha-1}$ and $\overline{d}_{\dot\alpha-1}$ is a consequence of the hermiticity conditions $(\partial\theta_\alpha)^\dagger = -\partial\overline{\theta}_{\dot\alpha}$ and $(d_\alpha)^\dagger = -\overline{d}_{\dot\alpha}$.

2.5.2 Gauge transformations

We first look at (2.23a), (2.23b) and (2.23c). In our case, $\Lambda_0 = \Lambda_1 = 0$ and we consider

$$\delta\Phi_{-1} = G_6^+\Lambda_{-1}\,, \tag{2.72a}$$
$$\delta\Phi_0 = G_4^+\Lambda_{-1} + \widetilde{G}_4^+\Lambda_2\,, \tag{2.72b}$$
$$\delta\Phi_1 = \widetilde{G}_6^+\Lambda_2\,. \tag{2.72c}$$

The Calabi-Yau-independent gauge parameters are

$$\Lambda_{-1} = \sqrt{8}e^{-i\rho}(\lambda + \partial\rho F + \partial H_C K)\,, \tag{2.73a}$$
$$\Lambda_2 = \sqrt{8}e^{2i\rho - iH_C}\left[\omega + (2\partial\rho - \partial H_C)\widetilde{F} + (3\partial\rho - 2\partial H_C)\widetilde{K}\right], \tag{2.73b}$$

with

$$\lambda \equiv 2i(d^\alpha_{-1}C_{1\alpha} - \overline{d}_{\dot\alpha-1}\overline{E}_2^{\dot\alpha} + \partial\theta^\alpha_{-1}B_{1\alpha} - \partial\overline{\theta}_{\dot\alpha-1}\overline{H}_2^{\dot\alpha} + i\Pi^m_{-1}B_m)\,,$$
$$\omega \equiv 2i(d^\alpha_{-1}E_{1\alpha} - \overline{d}_{\dot\alpha-1}\overline{C}_2^{\dot\alpha} + \partial\theta^\alpha_{-1}H_{1\alpha} - \partial\overline{\theta}_{\dot\alpha-1}\overline{B}_2^{\dot\alpha} + i\Pi^m_{-1}B^*_m)\,,$$

where ω and λ are annihilated by modes >1 and $F, K, C_{1\alpha}, \overline{C}_2^{\dot\alpha}, B_{1\alpha}, \overline{B}_2^{\dot\alpha}, E_{1\alpha}, \overline{E}_2^{\dot\alpha}, H_{1\alpha}, \overline{H}_2^{\dot\alpha}$ and B_m are superfields depending only on zero modes.

Calculating the gauge transformations, we find

$$\delta\varphi = \frac{1}{2}d_0^2\lambda - \frac{1}{2}\overline{d}_0^2\omega + \frac{1}{2}d_{-1}^2F + \frac{1}{2}\overline{d}_{-1}^2\widetilde{F}\,, \tag{2.74a}$$
$$\delta iB = \frac{1}{4}\overline{d}_1^2\omega - \frac{1}{4}d_1^2\lambda + \frac{i}{2}\overline{d}_0^2\widetilde{F} + \frac{i}{2}d_0^2F + \frac{3i}{4}\overline{d}_0^2\widetilde{K} + \frac{3i}{4}d_0^2K\,, \tag{2.74b}$$
$$\delta C = -\frac{1}{4}\overline{d}_1^2\omega - \frac{1}{4}d_1^2\lambda - \frac{i}{2}\overline{d}_0^2\widetilde{F} + \frac{i}{2}d_0^2F - \frac{i}{4}\overline{d}_0^2\widetilde{K} + \frac{i}{4}d_0^2K\,, \tag{2.74c}$$
$$\delta A = -iK, \tag{2.74d}$$

$$\delta\widetilde{A} = i\widetilde{K}\,. \tag{2.74e}$$

From (2.74), we see that the superfields appearing in Φ_1 and Φ_{-1} can be gauged away algebraically using the K and $\widetilde{K}$ gauge parameters, as we anticipated in Section 2.2.3. Some useful relations to check gauge invariance that are obvious from (2.74) are

$$\begin{aligned}
\delta_\lambda iB &= \delta_\lambda C\,, & \delta_\omega iB &= -\delta_\omega C\,,\\
\delta_F iB &= \delta_F C\,, & \delta_{\widetilde{F}} iB &= -\delta_{\widetilde{F}} C\,,\\
\delta_K iB &= 3\delta_K C\,, & \delta_{\widetilde{K}} iB &= -3\delta_{\widetilde{K}} C\,.
\end{aligned}$$

Following [30], we focus on a subset of the gauge parameters. The reason for this is that $C_{1\alpha}$, $\overline{C}_2^{\dot\alpha}$, B_m, F, $H_{1\alpha}$ and $\overline{H}_2^{\dot\alpha}$ can be ignored, being parameters of Λ_{-1} and Λ_2 that can be obtained from a state exact in $G^+/\widetilde{G}^+$. After using the explicit form of φ, λ and ω with the commutation relations for the supersymmetric modes, we obtain

$$\begin{aligned}
\delta W_{1\alpha} &= 2i\Delta_\alpha{}^\beta B_{1\beta} + 4\Pi_{\alpha\dot\alpha 0}\overline{E}_2^{\dot\alpha} - i\overline{d}_0^2 E_{1\alpha}\,, && (2.75a)\\
\delta\overline{W}_2^{\dot\alpha} &= -2i\overline{\Delta}^{\dot\alpha}{}_{\dot\beta}\overline{B}_2^{\dot\beta} - 4\Pi_0^{\dot\alpha\alpha}E_{1\alpha} + id_0^2\overline{E}_2^{\dot\alpha}\,, && (2.75b)\\
\delta V^m &= -4i\sigma^m_{\alpha\dot\alpha}\overline{d}_0^{\dot\alpha}E_1^\alpha - 4i\sigma^m_{\alpha\dot\alpha}d_0^\alpha\overline{E}_2^{\dot\alpha}\,, && (2.75c)\\
\delta\overline{V}_2^{\dot\alpha} &= -i\overline{d}_0^2\overline{B}_2^{\dot\alpha} - 16i\overline{E}_2^{\dot\alpha}\,, && (2.75d)\\
\delta V_{1\alpha} &= id_0^2 B_{1\alpha} + 16iE_{1\alpha}\,, && (2.75e)\\
\delta(iB - C) &= \overline{d}_{\dot\alpha 0}(\delta\overline{W}_2^{\dot\alpha} - id_0^2\overline{E}_2^{\dot\alpha})\,, && (2.75f)\\
\delta(iB + C) &= -d_0^\alpha(\delta W_{1\alpha} + i\overline{d}_0^2 E_{1\alpha})\,, && (2.75g)
\end{aligned}$$

where $\Delta_\alpha^\beta = \{d_{\alpha 1}, \partial\theta^\beta_{-1}\} = \delta_\alpha^\beta + \frac{i}{2}(\varepsilon\cdot\sigma)_\alpha{}^\beta$ and $\overline{\Delta}^{\dot\alpha}_{\dot\beta} = \{\overline{d}_1^{\dot\alpha}, \partial\overline{\theta}_{\dot\beta -1}\} = \delta^{\dot\alpha}_{\dot\beta} + \frac{i}{2}(\varepsilon\cdot\overline{\sigma})^{\dot\alpha}{}_{\dot\beta}$.

Equations (2.75a) and (2.75b) imply that the superfields W_1^α ($\overline{W}_2^{\dot\alpha}$) can be algebraically gauge fixed to zero by choosing an appropriate B_1^α ($\overline{B}_2^{\dot\alpha}$), therefore, we can consistently take $W_1^\alpha = \overline{W}_2^{\dot\alpha} = 0$ in the action. Imposing $W_1^\alpha = \overline{W}_2^{\dot\alpha} = \delta W_1^\alpha = \delta\overline{W}_2^{\dot\alpha} = 0$, the gauge transformations become

$$\begin{aligned}
\delta V^m &= -4i\sigma^m_{\alpha\dot\alpha}\overline{d}_0^{\dot\alpha}E_1^\alpha - 4i\sigma^m_{\alpha\dot\alpha}d_0^\alpha\overline{E}_2^{\dot\alpha}\,, && (2.76a)\\
\overline{\Delta}^{\dot\alpha}{}_{\dot\beta}\delta\overline{V}_2^{\dot\beta} &= 2\overline{d}_0^2\Pi_0^{\dot\alpha\alpha}E_{1\alpha} - \frac{i}{2}\overline{d}_0^2 d_0^2\overline{E}_2^{\dot\alpha} - 16i\overline{\Delta}^{\dot\alpha}{}_{\dot\beta}\overline{E}_2^{\dot\beta}\,, && (2.76b)\\
\Delta_\alpha{}^\beta\delta V_{1\beta} &= -2d_0^2\Pi_{\alpha\dot\alpha 0}\overline{E}_2^{\dot\alpha} + \frac{i}{2}d_0^2\overline{d}_0^2 E_{1\alpha} + 16i\Delta_\alpha{}^\beta E_{1\beta}\,, && (2.76c)
\end{aligned}$$

$$i\delta B = -\frac{i}{2}\left(\bar{d}_{\dot{\alpha}0} d_0^2 \bar{E}_2^{\dot{\alpha}} + d_0^{\alpha} \bar{d}_0^2 E_{1\alpha}\right), \tag{2.76d}$$

$$\delta C = \frac{i}{2}\left(\bar{d}_{\dot{\alpha}0} d_0^2 \bar{E}_2^{\dot{\alpha}} - d_0^{\alpha} \bar{d}_0^2 E_{1\alpha}\right). \tag{2.76e}$$

2.5.3 Equations of motion and superspace action

Equations (2.22a) and (2.22c) give

$$\bar{d}_0^2 d_0^2 A + 2\bar{d}_0^2 (iB - C) = 0\,, \tag{2.77a}$$

$$d_0^2 \bar{d}_0^2 \widetilde{A} - 2d_0^2 (iB + C) = 0\,. \tag{2.77b}$$

Equation (2.22b) is more complicated to evaluate, it implies that

$$\begin{gathered}(d_{-1}^2 \bar{d}_1^2 - 2d_0^2 \bar{d}_0^2 + d_1^2 \bar{d}_{-1}^2)\varphi \\ +(\bar{d}_0^2 d_{-1}^2 - 2\bar{d}_{-1}^2 d_0^2 + d_0^2 \bar{d}_{-1}^2 - 2d_{-1}^2 \bar{d}_0^2) iB \\ -(-3\bar{d}_0^2 d_{-1}^2 + 6\bar{d}_{-1}^2 d_0^2 + 3d_0^2 \bar{d}_{-1}^2 - 6d_{-1}^2 \bar{d}_0^2) C = 0\,,\end{gathered} \tag{2.78a}$$

$$\begin{gathered}(-d_0^2 \bar{d}_1^2 - d_1^2 \bar{d}_0^2 + d_2^2 \bar{d}_{-1}^2)\varphi \\ +(\{d_0^2, \bar{d}_0^2\} - 64) iB \\ +(-3[d_0^2, \bar{d}_0^2] C) - 48(d_0^2 A - \bar{d}_0^2 \widetilde{A}) = 0\,,\end{gathered} \tag{2.78b}$$

$$\begin{gathered}(3d_0^2 \bar{d}_1^2 - 3d_1^2 \bar{d}_0^2 + d_2^2 \bar{d}_{-1}^2)\varphi \\ -[d_0^2, \bar{d}_0^2] iB \\ -(64 i d_0^{\alpha} \Pi_{\alpha\dot{\alpha}0} \bar{d}_0^{\dot{\alpha}} - 19 d_0^2 \bar{d}_0^2 - 3\bar{d}_0^2 d_0^2 - 64) C \\ +16(\bar{d}_0^2 \widetilde{A} + d_0^2 A) = 0\,,\end{gathered} \tag{2.78c}$$

where, after acting with the generators on the string field Φ_0, one obtains terms proportional to J^{++}, $J^{++}(\partial\rho - \partial H_C)$ and $J^{++}(\partial H_C - 3\partial\rho)$ which correspond respectively to the three equations above. This form of the equations is particularly useful to check gauge invariance using (2.74). Some helpful relations between the modes can be found in Appendix A.

The task now is to eliminate operators with mode numbers ≥ 1 by using (2.52). Equations (2.78b) and (2.78c) can be expressed entirely in terms of zero modes (the overall oscillator number is zero), so that they only give one independent relation. On the other hand, each term in equation (2.78a) has overall oscillator mode -1, consequently, (2.78a) will give us one independent equation for each of the supersymmetric modes $\partial\theta^{\alpha}_{-1}$, Π^{m}_{-1}, d^{α}_{-1}, etc. Note that in the gauge $W^{\alpha}_{1} = 0$, the terms proportional to $\partial\theta^{\alpha}_{-1}$ do not contribute to the superspace action. The same holds for terms proportional to $\partial\overline{\theta}_{\dot{\alpha}-1}$. Of course, to evaluate the CFT correlator corresponding to the superspace action, one needs to consider the equations of motion together with the appropriate factors of J^{++}, $J^{++}(\partial\rho - \partial H_C)$ and $J^{++}(\partial H_C - 3\partial\rho)$.

From now on, to proceed in the computation of the action, we take $A = \widetilde{A} = W^{\alpha}_{1} = \overline{W}^{\dot{\alpha}}_{2} = 0$. In this gauge,

$$d_{\alpha 1}\varphi = -2(\sigma^{m}\partial\overline{\theta}_{0})_{\alpha}V_{m} + \Delta_{\alpha}{}^{\beta}V_{1\beta}\,, \tag{2.79a}$$

$$\overline{d}^{\dot{\alpha}}_{1}\varphi = -2(\overline{\sigma}^{m}\partial\theta_{0})^{\dot{\alpha}}V_{m} - \overline{\Delta}^{\dot{\alpha}}{}_{\dot{\beta}}\overline{V}^{\dot{\beta}}_{2}\,, \tag{2.79b}$$

$$\Pi_{\alpha\dot{\alpha}1}\varphi = -i\sigma^{n}_{\alpha\dot{\alpha}}(\eta_{nm} - i\varepsilon_{nm})V^{m}\,, \tag{2.79c}$$

$$\partial\theta_{\alpha 1}\varphi = \partial\overline{\theta}^{\dot{\alpha}}_{1}\varphi = 0\,. \tag{2.79d}$$

For simplicity, we also perform the redefinitions

$$-2(\sigma^{m}\partial\overline{\theta}_{0})_{\alpha}V_{m} + \Delta_{\alpha}{}^{\beta}V_{1\beta} \rightarrow U_{1\alpha}\,, \tag{2.80a}$$

$$2(\overline{\sigma}^{m}\partial\theta_{0})^{\dot{\alpha}}V_{m} + \overline{\Delta}^{\dot{\alpha}}{}_{\dot{\beta}}\overline{V}^{\dot{\beta}}_{2} \rightarrow \overline{U}^{\dot{\alpha}}_{2}\,. \tag{2.80b}$$

The gauge transformations for $U_{1\alpha}$ and $\overline{U}^{\dot{\alpha}}_{2}$ are then

$$\delta U_{1\alpha} = -16i\partial\overline{\theta}^{\dot{\alpha}}_{0}(\overline{d}_{\dot{\alpha}0}E_{1\alpha} + d_{\alpha 0}\overline{E}_{2\dot{\alpha}}) - 2d^{2}_{0}\Pi_{\alpha\dot{\alpha}0}\overline{E}^{\dot{\alpha}}_{2} + \frac{i}{2}d^{2}_{0}\overline{d}^{2}_{0}E_{1\alpha} + 16i\Delta_{\alpha}{}^{\beta}E_{1\beta}\,, \tag{2.81a}$$

$$\delta\overline{U}^{\dot{\alpha}}_{2} = 16i\partial\theta_{\alpha 0}(\overline{d}^{\dot{\alpha}}_{0}E^{\alpha}_{1} + d^{\alpha}_{0}\overline{E}^{\dot{\alpha}}_{2}) + 2\overline{d}^{2}_{0}\Pi^{\dot{\alpha}\alpha}_{0}E_{1\alpha} - \frac{i}{2}\overline{d}^{2}_{0}d^{2}_{0}\overline{E}^{\dot{\alpha}}_{2} - 16i\overline{\Delta}^{\dot{\alpha}}{}_{\dot{\beta}}\overline{E}^{\dot{\beta}}_{2}\,, \tag{2.81b}$$

if we require that $\delta W^{\alpha}_{1} = \delta\overline{W}^{\dot{\alpha}}_{2} = 0$ so that W^{α}_{1} and $\overline{W}^{\dot{\alpha}}_{2}$ remain zero.

Computing the CFT two-point function $\langle\Phi^{\dagger}_{0}(\widetilde{G}^{+}_{6}G^{+}_{6} + \widetilde{G}^{+}_{4}G^{+}_{4})\Phi_{0}\rangle$, one finds that the string field theory action in $\mathcal{N} = 1$ superspace for the first massive compactification-independent fields of the charged open superstring coupled to a constant electromagnetic background is

$$
\begin{aligned}
S = -\frac{1}{16}\int d^4x\, p_0^2\bar{p}_0^2 \Big\{ & V_n^\dagger(\eta^{nm} - i\varepsilon^{nm})\Big[-\{d_0^2, \bar{d}_0^2\}V_m + 16\Pi_0^n\Pi_{n0}V_m - 32(\eta_{mp} \\
& - i\varepsilon_{mp})V^p - 32\Big((\partial\bar{\theta}_0\bar{d}_0)V_m + (\partial\theta_0 d_0)V_m\Big) + 8\bar{\sigma}_m^{\dot{\alpha}\alpha}\Big(d_{\alpha 0}\overline{U}_{2\dot{\alpha}} - \bar{d}_{\dot{\alpha}0}U_{1\alpha}\Big)B \\
& + 32\Pi_{m0} + 24\bar{\sigma}_m^{\dot{\alpha}\alpha}[\bar{d}_{\dot{\alpha}0}, d_{\alpha 0}]C\Big] + U_2^\alpha\Big[-8\sigma^n_{\alpha\dot{\alpha}}(\eta_{nm} - i\varepsilon_{nm})\bar{d}_0^{\dot{\alpha}}V^m + 4\bar{d}_{\dot{\alpha}0}d_{\alpha 0}\overline{U}_2^{\dot{\alpha}} \\
& - 4\bar{d}_0^2U_{1\alpha} + d_{\alpha 0}\bar{d}_0^2(-2iB + 18C) + \partial\theta_{\alpha 0}(-32iB - 96C) - 48i\Pi_{\alpha\dot{\alpha}0}\bar{d}_0^{\dot{\alpha}}C\Big] \\
& - \overline{U}_{1\dot{\alpha}}\Big[-8\bar{\sigma}^{n\dot{\alpha}\alpha}(\eta_{nm} - i\varepsilon_{nm})d_{\alpha 0}V^m + 4d_0^2\overline{U}_2^{\dot{\alpha}} - 4d_0^\alpha\bar{d}_0^{\dot{\alpha}}U_{1\alpha} - \bar{d}_0^{\dot{\alpha}}d_0^2(2iB + 18C) \\
& + \partial\bar{\theta}_0^{\dot{\alpha}}(-32iB + 96C) + 48i\Pi_0^{\dot{\alpha}\alpha}d_{\alpha 0}C\Big] + B^\dagger\Big[-32\Pi_0^n(\eta_{nm} - i\varepsilon_{nm})V^m \\
& + (\{d_0^2, \bar{d}_0^2\} - 64)B + 3i[d_0^2, \bar{d}_0^2]C - i\Big(2d_0^2\bar{d}_{\dot{\alpha}0} + 32\partial\bar{\theta}_{\dot{\alpha}0}\Big)\overline{U}_2^{\dot{\alpha}} + i\Big(2\bar{d}_0^2d_0^\alpha + 32\partial\theta_0^\alpha\Big)\times \\
& \times U_{1\alpha}\Big] + 3C^\dagger\Big[-8\bar{\sigma}^{n\dot{\alpha}\alpha}[d_{\alpha 0}, \bar{d}_{\dot{\alpha}0}](\eta_{nm} - i\varepsilon_{nm})V^m - \Big(6d_0^\alpha\bar{d}_0^2 + 8i\Pi_0^{\dot{\alpha}\alpha}\bar{d}_{\dot{\alpha}0}\Big)U_{1\alpha} \\
& - \Big(6\bar{d}_{\dot{\alpha}0}d_0^2 + 8i\Pi_{\alpha\dot{\alpha}0}d_0^\alpha\Big)\overline{U}_2^{\dot{\alpha}} - [d_0^2, \bar{d}_0^2]iB \\
& - \Big(-11\{d_0^2, \bar{d}_0^2\} + 128\Pi_0^n\Pi_{n0} - 256\partial\bar{\theta}_{\dot{\alpha}0}\bar{d}_0^{\dot{\alpha}} - 256\partial\theta_0^\alpha d_{\alpha 0} - 64\Big)C\Big]\Big\}\,, \qquad (2.82)
\end{aligned}
$$

with the equations of motion

$$
\begin{aligned}
& -\{d_0^2, \bar{d}_0^2\}V_m + 16\Pi_0^n\Pi_{n0}V_m - 32(\eta_{mn} - i\varepsilon_{mn})V^n - 32\Big[(\partial\bar{\theta}_0\bar{d}_0)V_m + (\partial\theta_0 d_0)V_m\Big] \\
& + 8\bar{\sigma}_m^{\dot{\alpha}\alpha}(d_{\alpha 0}\overline{U}_{2\dot{\alpha}} - \bar{d}_{\dot{\alpha}0}U_{1\alpha}) + 32\Pi_{m0}B + 24\sigma_{m\alpha\dot{\alpha}}[\bar{d}_0^{\dot{\alpha}}, d_0^\alpha]C = 0\,, \qquad (2.83a)
\end{aligned}
$$

$$
\begin{aligned}
& -8\sigma^n_{\alpha\dot{\alpha}}(\eta_{nm} - i\varepsilon_{nm})\bar{d}_0^{\dot{\alpha}}V^m + 4\bar{d}_{\dot{\alpha}0}d_{\alpha 0}\overline{U}_2^{\dot{\alpha}} - 4\bar{d}_0^2U_{1\alpha} + d_{\alpha 0}\bar{d}_0^2(-2iB + 18C) \\
& + \partial\theta_{\alpha 0}(-32iB - 96C) - 48i\Pi_{\alpha\dot{\alpha}0}\bar{d}_0^{\dot{\alpha}}C = 0\,, \qquad (2.83b)
\end{aligned}
$$

$$
\begin{aligned}
& -8\bar{\sigma}^{n\dot{\alpha}\alpha}(\eta_{nm} - i\varepsilon_{nm})d_{\alpha 0}V^m + 4d_0^2\overline{U}_2^{\dot{\alpha}} - 4d_0^\alpha\bar{d}_0^{\dot{\alpha}}U_{1\alpha} - \bar{d}_0^{\dot{\alpha}}d_0^2(2iB + 18C) \\
& + \partial\bar{\theta}_0^{\dot{\alpha}}(-32iB + 96C) + 48i\Pi_0^{\dot{\alpha}\alpha}d_{\alpha 0}C = 0\,, \qquad (2.83c)
\end{aligned}
$$

$$
\begin{aligned}
& -32i\Pi_0^n(\eta_{nm} - i\varepsilon_{nm})V^m + (\{d_0^2, \bar{d}_0^2\} - 64)iB - 3[d_0^2, \bar{d}_0^2]C \\
& + (2d_0^2\bar{d}_{\dot{\alpha}0} + 32\partial\bar{\theta}_{\dot{\alpha}0})\overline{U}_2^{\dot{\alpha}} - (2\bar{d}_0^2d_0^\alpha + 32\partial\theta_0^\alpha)U_{1\alpha} = 0\,, \qquad (2.83d)
\end{aligned}
$$

$$
\begin{aligned}
&- 8\overline{\sigma}^{n\dot{\alpha}\alpha}[d_{\alpha 0}, \overline{d}_{\dot{\alpha}0}](\eta_{nm} - i\varepsilon_{nm})V^m \\
&- \left(6d_0^\alpha \overline{d}_0^2 + 8i\Pi_0^{\dot{\alpha}\alpha}\overline{d}_{\dot{\alpha}0}\right) U_{1\alpha} - \left(6\overline{d}_{\dot{\alpha}0}d_0^2 + 8i\Pi_{\alpha\dot{\alpha}0}d_0^\alpha\right) \overline{U}_2^{\dot{\alpha}} \\
&- [d_0^2, \overline{d}_0^2]iB - \left(-11\{d_0^2, \overline{d}_0^2\} + 128\Pi_0^n\Pi_{n0} - 256\partial\overline{\theta}_{\dot{\alpha}0}\overline{d}_0^{\dot{\alpha}} - 256\partial\theta_0^\alpha d_{\alpha 0} - 64\right) C = 0\,.
\end{aligned}
\tag{2.83e}
$$

Using (2.83b), (2.83c), (2.83d), (2.83e) and (2.77), one can show that

$$
\begin{aligned}
B = &-\frac{i}{32}\left[[d_0^2, \overline{d}_0^2]C + \overline{d}_{\dot{\alpha}}d_0^2\overline{U}_2^{\dot{\alpha}} - d_0^\alpha\overline{d}_0^2U_{1\alpha}\right] + \frac{i}{2}(\partial\theta_0 d_0)(3C + iB) \\
&- \frac{i}{2}(\partial\overline{\theta}_0\overline{d}_0)(3C - iB)\,,
\end{aligned}
\tag{2.84}
$$

and

$$
\begin{aligned}
64(\Pi_0^n\Pi_{n0} + 1)C = 6\{d_0^2, \overline{d}_0^2\}C - 2\overline{d}_{\dot{\alpha}0}d_0^2\overline{U}_2^{\dot{\alpha}} - 2d_0^\alpha\overline{d}_0^2U_{1\alpha} \\
+ 32(\partial\theta_0 d_0)(-iB + C) + 32(\partial\overline{\theta}_0\overline{d}_0)(iB + C)\,.
\end{aligned}
\tag{2.85}
$$

Equations (2.84) and (2.85) generalize (3.8) and (3.9) from [30] for the uncharged and non-interacting case. At the level of the equations of motion, one can also gauge-fix $U_{1\alpha} = \overline{U}_2^{\dot{\alpha}} = 0$ by the gauge transformations (2.81a) and (2.81b). Note that using our conventions for the supersymmetric variables, the quadratic action of [30] for the non-interacting case is

$$
\begin{aligned}
S_{free} = -\frac{1}{16}\int d^4x\; p_0^2\overline{p}_0^2\Big\{ &V^m\Big[- \{d_0^2, \overline{d}_0^2\}V_m + 16\Pi_0^n\Pi_{n0}V_m - 32V_m \\
&+ 16\overline{\sigma}_m^{\dot{\alpha}\alpha}(d_{\alpha 0}\overline{V}_{\dot{\alpha}} - \overline{d}_{\dot{\alpha}0}V_\alpha) + 64\Pi_{m0}B + 48\overline{\sigma}_m^{\dot{\alpha}\alpha}[\overline{d}_{\dot{\alpha}0}, d_{\alpha 0}]C\Big] \\
&+ V^\alpha\Big[8\overline{d}_{\dot{\alpha}0}d_{\alpha 0}\overline{V}^{\dot{\alpha}} - 4\overline{d}_0^2V_\alpha + 2d_{\alpha 0}\overline{d}_0^2(-2iB + 18C) - 96i\Pi_{\alpha\dot{\alpha}0}\overline{d}_0^{\dot{\alpha}}C\Big] \\
&+ \overline{V}_{\dot{\alpha}}\Big[- 4d_0^2\overline{V}^{\dot{\alpha}} + 2\overline{d}_0^{\dot{\alpha}}d_0^2(2iB + 18C) - 96i\Pi_0^{\dot{\alpha}\alpha}d_{\alpha 0}C\Big] \\
&+ B\Big[\{d_0^2, \overline{d}_0^2\}B - 64B + 6i[d_0^2, \overline{d}_0^2]C\Big] + 3C\Big[11\{d_0^2, \overline{d}_0^2\}C - 128\Pi_0^n\Pi_{n0}C \\
&+ 64C\Big]\Big\}\,.
\end{aligned}
\tag{2.86}
$$

2.6 Spin-3/2 and spin-2 charged massive states in a constant electromagnetic background

With the superspace action for the massive states of the superstring in an electromagnetic background at our disposal (2.82), we can now expand the superfields in components, eliminte the pure gauge degrees of freedom and write an effective action for the massive spin-3/2 and massive spin-2 fields. Due to the complexity of the terms in the action and the huge number of auxiliary fields, this is a tedious and long task which was explained thoroughly in ref. [2].

In this section, we will summarize the main results and skip most of the technical the deatils. Hence, we will write the action and equations of motion for the spin-3/2 and spin-2 fields after numerous simplifications, discussed in [2], have been implemented.

As noted in [14] for the case of the open bosonic string, and as it appears through our manipulations in this work, the consitency of the Lagrangian (2.82) and the derivation of the equations of motion make use of the anti-symmetric property of ϵ_{mn}, but nowhere does the explicit dependence of ϵ_{mn} on F_{mn} intervene. Therefore, our analysis continues to be valid if we take everywhere the limit of quantum field theory $\epsilon_{mn} \to QF_{mn}$ and $\mathfrak{D}_m \to D_m$.

Following this, to ease the presentation, our working configuration can be reformulated, independently of a stringy framework or not, as follows: the superfields are charged under the U(1) of the elctromagnetic background, to which we associate a covariant derivative $\mathfrak{D}_m$, whose commutator gives a constant anti-symmetric tensor ϵ_{mn}, hereafter referred to as the "electromagnetic field strength", an obvious abuse of language. For convenience, we will assume that $\{V_m, \mathcal{B}, \mathcal{C}, U_{1\alpha}, \overline{U}_2^{\dot{\alpha}}\}$ carry a positive unit charge, so their conjugates are negatively charged. It is then easy to verify that the action (2.82) is U(1)-invariant. For the covariant derivative we have $[\mathfrak{D}_m, \mathfrak{D}_n] = iq\epsilon_{mn}$, with $q = \pm 1$. For example, given a positively charged superfield component ϕ, we have

$$[\mathfrak{D}_m, \mathfrak{D}_n]\phi = i\epsilon_{mn}\phi\,, \qquad\qquad [\mathfrak{D}_m, \mathfrak{D}_n]\phi^* = -i\epsilon_{mn}\phi^*\,. \tag{2.87}$$

2.6.1 The complex superfields

The expansion into components of the superfields reads:

$$\begin{aligned}
V_m &= C_m + i(\theta\chi_{1m}) - i(\overline{\theta}\overline{\chi}_{2m}) + i(\theta\theta)M_{1m} - i(\overline{\theta\theta})\overline{M}_{2m} + (\theta\sigma^n\overline{\theta})h_{mn} \\
&\quad + i(\theta\theta)(\overline{\theta}\overline{\lambda}_{1m}) - i(\overline{\theta\theta})(\theta\lambda_{2m}) + (\theta\theta)(\overline{\theta\theta})D_m\,, && (2.88a)\\
\mathcal{B} &= \varphi + i(\theta\gamma_1) - i(\overline{\theta}\overline{\gamma}_2) + i(\theta\theta)N_1 - i(\overline{\theta\theta})\overline{N}_2 + (\theta\sigma^m\overline{\theta})c_m \\
&\quad + i(\theta\theta)(\overline{\theta}\overline{\rho}_1) - i(\overline{\theta\theta})(\theta\rho_2) + (\theta\theta)(\overline{\theta\theta})G\,, && (2.88b)\\
\mathcal{C} &= \phi + i(\theta\xi_1) - i(\overline{\theta}\overline{\xi}_2) + i(\theta\theta)M_1 - i(\overline{\theta\theta})\overline{M}_2 + (\theta\sigma^m\overline{\theta})a_m \\
&\quad + i(\theta\theta)(\overline{\theta}\overline{\psi}_1) - i(\overline{\theta\theta})(\theta\psi_2) + (\theta\theta)(\overline{\theta\theta})D\,, && (2.88c)\\
U_{1\alpha} &= v_{1\alpha} + \theta_\alpha s_1 - (\sigma^{mn}\theta)_\alpha s_{1mn} + (\sigma^m\overline{\theta})_\alpha w_{1m} + (\theta\theta)\eta_{1\alpha} + (\overline{\theta\theta})\zeta_{1\alpha} + (\theta\sigma^m\overline{\theta})r_{1m\alpha} \\
&\quad + (\theta\theta)(\sigma^m\overline{\theta})_\alpha q_{1m} + (\overline{\theta\theta})\theta_\alpha t_1 - (\overline{\theta\theta})(\sigma^{mn}\theta)_\alpha t_{1mn} + (\theta\theta)(\overline{\theta\theta})\mu_{1\alpha}\,, && (2.88d)\\
\overline{U}_1^{\dot{\alpha}} &= \overline{v}_1^{\dot{\alpha}} + \overline{\theta}^{\dot{\alpha}}\overline{s}_1 - (\overline{\sigma}^{mn}\overline{\theta})^{\dot{\alpha}}\overline{s}_{1mn} - (\overline{\sigma}^m\theta)^{\dot{\alpha}}\overline{w}_{1m} + (\overline{\theta\theta})\overline{\eta}_1^{\dot{\alpha}} + (\theta\theta)\overline{\zeta}_1^{\dot{\alpha}} + (\theta\sigma^m\overline{\theta})\overline{r}_{1m}^{\dot{\alpha}} \\
&\quad - (\overline{\theta\theta})(\overline{\sigma}^m\theta)^{\dot{\alpha}}\overline{q}_{1m} + (\theta\theta)\overline{\theta}^{\dot{\alpha}}\overline{t}_1 - (\theta\theta)(\overline{\sigma}^{mn}\overline{\theta})^{\dot{\alpha}}\overline{t}_{1mn} + (\theta\theta)(\overline{\theta\theta})\overline{\mu}_1^{\dot{\alpha}}\,, && (2.88e)\\
U_{2\alpha} &= v_{2\alpha} + \theta_\alpha s_2 - (\sigma^{mn}\theta)_\alpha s_{2mn} + (\sigma^m\overline{\theta})_\alpha w_{2m} + (\theta\theta)\eta_{2\alpha} + (\overline{\theta\theta})\zeta_{2\alpha} + (\theta\sigma^m\overline{\theta})r_{2m\alpha} \\
&\quad + (\theta\theta)(\sigma^m\overline{\theta})_\alpha q_{2m} + (\overline{\theta\theta})\theta_\alpha t_2 - (\overline{\theta\theta})(\sigma^{mn}\theta)_\alpha t_{2mn} + (\theta\theta)(\overline{\theta\theta})\mu_{2\alpha}\,, && (2.88f)\\
\overline{U}_2^{\dot{\alpha}} &= \overline{v}_2^{\dot{\alpha}} + \overline{\theta}^{\dot{\alpha}}\overline{s}_2 - (\overline{\sigma}^{mn}\overline{\theta})^{\dot{\alpha}}\overline{s}_{2mn} - (\overline{\sigma}^m\theta)^{\dot{\alpha}}\overline{w}_{2m} + (\overline{\theta\theta})\overline{\eta}_2^{\dot{\alpha}} + (\theta\theta)\overline{\zeta}_2^{\dot{\alpha}} + (\theta\sigma^m\overline{\theta})\overline{r}_{2m}^{\dot{\alpha}} \\
&\quad - (\overline{\theta\theta})(\overline{\sigma}^m\theta)^{\dot{\alpha}}\overline{q}_{2m} + (\theta\theta)\overline{\theta}^{\dot{\alpha}}\overline{t}_2 - (\theta\theta)(\overline{\sigma}^{mn}\overline{\theta})^{\dot{\alpha}}\overline{t}_{2mn} + (\theta\theta)(\overline{\theta\theta})\overline{\mu}_2^{\dot{\alpha}}\,, && (2.88g)
\end{aligned}$$

where the gauge parameter superfields $E_{1\alpha}$, $E_{2\alpha}$ are given by

$$\begin{aligned}
E_{1\alpha} &= \Lambda_{1\alpha} + \theta_\alpha\Lambda_2 - (\sigma^{mn}\theta)_\alpha\Lambda_{2mn} + (\sigma^m\overline{\theta})_\alpha\Lambda_{3m} + (\theta\theta)\Lambda_{4\alpha} + (\overline{\theta\theta})\Lambda_{5\alpha} + (\theta\sigma^m\overline{\theta})\Lambda_{6m\alpha} \\
&\quad + (\theta\theta)(\sigma^m\overline{\theta})_\alpha\Lambda_{7m} + (\overline{\theta\theta})\theta_\alpha\Lambda_8 - (\overline{\theta\theta})(\sigma^{mn}\theta)_\alpha\Lambda_{8mn} + (\theta\theta)(\overline{\theta\theta})\Lambda_{9\alpha}\,, && (2.89a)\\
\overline{E}_1^{\dot{\alpha}} &= \overline{\Lambda}_1^{\dot{\alpha}} + \overline{\theta}^{\dot{\alpha}}\overline{\Lambda}_2 - (\overline{\sigma}^{mn}\overline{\theta})^{\dot{\alpha}}\overline{\Lambda}_{2mn} - (\overline{\sigma}^m\theta)^{\dot{\alpha}}\overline{\Lambda}_{3m} + (\overline{\theta\theta})\overline{\Lambda}_4^{\dot{\alpha}} + (\theta\theta)\overline{\Lambda}_5^{\dot{\alpha}} + (\theta\sigma^m\overline{\theta})\overline{\Lambda}_{6m}^{\dot{\alpha}} \\
&\quad - (\overline{\theta\theta})(\overline{\sigma}^m\theta)^{\dot{\alpha}}\overline{\Lambda}_{7m} + (\theta\theta)\overline{\theta}^{\dot{\alpha}}\overline{\Lambda}_8 - (\theta\theta)(\overline{\sigma}^{mn}\overline{\theta})^{\dot{\alpha}}\overline{\Lambda}_{8mn} + (\theta\theta)(\overline{\theta\theta})\overline{\Lambda}_9^{\dot{\alpha}}\,, && (2.89b)\\
E_{2\alpha} &= \Upsilon_{1\alpha} + \theta_\alpha\Upsilon_2 - (\sigma^{mn}\theta)_\alpha\Upsilon_{2mn} + (\sigma^m\overline{\theta})_\alpha\Upsilon_{3m} + (\theta\theta)\Upsilon_{4\alpha} + (\overline{\theta\theta})\Upsilon_{5\alpha} + (\theta\sigma^m\overline{\theta})\Upsilon_{6m\alpha} \\
&\quad + (\theta\theta)(\sigma^m\overline{\theta})_\alpha\Upsilon_{7m} + (\overline{\theta\theta})\theta_\alpha\Upsilon_8 - (\overline{\theta\theta})(\sigma^{mn}\theta)_\alpha\Upsilon_{8mn} + (\theta\theta)(\overline{\theta\theta})\Upsilon_{9\alpha}\,, && (2.89c)\\
\overline{E}_2^{\dot{\alpha}} &= \overline{\Upsilon}_1^{\dot{\alpha}} + \overline{\theta}^{\dot{\alpha}}\overline{\Upsilon}_2 - (\overline{\sigma}^{mn}\overline{\theta})^{\dot{\alpha}}\overline{\Upsilon}_{2mn} - (\overline{\sigma}^m\theta)^{\dot{\alpha}}\overline{\Upsilon}_{3m} + (\overline{\theta\theta})\overline{\Upsilon}_4^{\dot{\alpha}} + (\theta\theta)\overline{\Upsilon}_5^{\dot{\alpha}} + (\theta\sigma^m\overline{\theta})\overline{\Upsilon}_{6m}^{\dot{\alpha}} \\
&\quad - (\overline{\theta\theta})(\overline{\sigma}^m\theta)^{\dot{\alpha}}\overline{\Upsilon}_{7m} + (\theta\theta)\overline{\theta}^{\dot{\alpha}}\overline{\Upsilon}_8 - (\theta\theta)(\overline{\sigma}^{mn}\overline{\theta})^{\dot{\alpha}}\overline{\Upsilon}_{8mn} + (\theta\theta)(\overline{\theta\theta})\overline{\Upsilon}_9^{\dot{\alpha}}\,. && (2.89d)
\end{aligned}$$

We also denote the dual field by $\widetilde{\epsilon}^{mn}$ with

$$\widetilde{\epsilon}^{mn} = \frac{1}{2}\varepsilon^{mnpq}\epsilon_{pq}\,, \qquad \epsilon^{mn} = -\frac{1}{2}\varepsilon^{mnpq}\widetilde{\epsilon}_{pq}\,. \tag{2.90}$$

Obviously, the sum $(\epsilon_{mn} + i\widetilde{\epsilon}_{mn})$ is self-dual. Some useful identities related to the field strengths are

$$\epsilon_{mn}\widetilde{\epsilon}^{mk} = \frac{1}{4}\delta_n{}^k\epsilon_{ab}\widetilde{\epsilon}^{ab}\,, \qquad \epsilon_{mn}\epsilon^{mk} - \widetilde{\epsilon}_{mn}\widetilde{\epsilon}^{mk} = \frac{1}{2}\delta_n{}^k\epsilon_{ab}\epsilon^{ab}\,. \tag{2.91}$$

2.6.2 Charged massive bosons

After using the gauge transformations described in Section 2.5.2, and performing suitable field redefinitions, most of the component fields can be eliminated from the superfields (2.88). In what follows, we list all component fields which will remain and, therefore, describe the physical degrees of freedom in the Lagrangian for the bosonic fields $\mathcal{L}_B$ derived from the superspace action (2.82).

For the components in $\mathcal{B}$, the physical degrees of freedom are contained in the fieds $\{c_m, N_1, N_2\}$. For the components in $\mathcal{C}$, the physical degrees of freedom are contained $\{a_m, M_1, M_2\}$. For the components in V_m, the physical degrees of freedom are contained in h_{mn}, where only the symmetric part survives after using the gauge transformations and field redefinitions. For the components in $\mathcal{U}_{1\alpha}$ and $\overline{\mathcal{U}}_2^{\dot\alpha}$, all of them can be removed, either by using the gauge transformations or by field redefinitions. In what follows, we label the trace of h_{mn} by h, and assume that the antisymmetric part of h_{mn} has been gauged to zero. The details regarding the gauge transformations and field redefinitions are presented in ref. [2].

The Lagrangian $\mathcal{L}_B$ can be separated in two separately gauge invariant parts $\mathcal{L}_1$ and $\mathcal{L}_2$. This means that one can write $\mathcal{L}_B = \mathcal{L}_1 + \mathcal{L}_2$. In terms of the components, the first part reads

$$\mathcal{L}_1 = \overline{\mathcal{M}}_1\Big(-2+\mathfrak{D}^2\Big)\mathcal{M}_1 + \overline{\mathcal{N}}_1\Big(-2+\mathfrak{D}^2\Big)\mathcal{N}_1\,, \tag{2.92}$$

where we defined the complex scalars

$$\mathcal{M}_1 = M_1 + \overline{M}_2\,, \qquad \mathcal{M}_2 = i(M_1 - \overline{M}_2)\,,$$
$$\mathcal{N}_1 = N_1 + \overline{N}_2\,, \qquad \mathcal{N}_2 = i(N_1 - \overline{N}_2)\,,$$

and performed a field redefinition to eliminate $\mathcal{M}_2$ and $\mathcal{N}_2$.

The gauge invariant part $\mathcal{L}_2$ is more complicated, and also more interesting, since it contains the massive spin-2 field h_{mn}. After performing field redefinitions and using the gauge transformations, and introducing the definitions

$$(\epsilon\epsilon) = \epsilon^{mn}\epsilon_{mn}\,, \qquad\qquad (\epsilon\widetilde{\epsilon}) = \epsilon^{mn}\widetilde{\epsilon}_{mn}\,, \tag{2.93}$$

it is possible to write the Lagrangian $\mathcal{L}_2$ as a deformed Fierz-Pauli Lagrangian [2] [3]

$$\begin{aligned}
\mathcal{L}_2 = {} & \overline{\mathcal{C}}^m\mathfrak{D}^2\mathcal{C}_m + \mathfrak{D}^m\overline{\mathcal{C}}_m\mathfrak{D}^n\mathcal{C}_n - 2\overline{\mathcal{C}}^m t(\eta_{mn} - i\epsilon_{mn})\mathcal{C}^n \\
& + \left[\overline{A}\mathfrak{D}_m - \frac{i}{2}\widetilde{\epsilon}_{mb}\overline{B}\mathfrak{D}^b + \frac{1}{8}(\epsilon\widetilde{\epsilon})\overline{B}\mathfrak{D}_m - \frac{1}{2}\varepsilon_{mabc}\overline{\mathcal{H}}^{bc}\mathfrak{D}^a - \frac{i}{2}\widetilde{\epsilon}_{ma}\overline{\mathcal{H}}^{ba}\mathfrak{D}_b + \frac{i}{2}\widetilde{\epsilon}_{mb}\overline{\mathcal{H}}\mathfrak{D}^b\right] \\
& \times \mathcal{A}^{mn}\left[\mathfrak{D}_n A + \frac{i}{2}\widetilde{\epsilon}_{nl}\mathfrak{D}^l B + \frac{1}{8}(\epsilon\widetilde{\epsilon})\mathfrak{D}_n B - \frac{1}{2}\varepsilon_{nlpq}\mathfrak{D}^l\mathcal{H}^{pq} + \frac{i}{2}\widetilde{\epsilon}_{nl}\mathfrak{D}_p\mathcal{H}^{pl}\right. \\
& \left. - \frac{i}{2}\widetilde{\epsilon}_{nl}\mathfrak{D}^l\mathcal{H}\right] - 2\overline{A}A + \overline{B}\left(\mathfrak{D}^2 - 2\right)B - \frac{1}{2}\epsilon_{mn}\epsilon^{mk}\overline{B}\mathfrak{D}^n\mathfrak{D}_k B \\
& + \frac{1}{2}\left[i\left(\mathfrak{D}^n\overline{\mathcal{H}}_{nm}\epsilon^{mk}\mathfrak{D}_k B\right) - \frac{1}{2}(\epsilon\epsilon)\overline{\mathcal{H}}B + \text{h.c.}\right] + \frac{1}{2}\overline{\mathcal{H}}_{(mn)}\mathfrak{D}^2 h^{mn} \\
& + \frac{1}{2}\mathfrak{D}^n\overline{\mathcal{H}}_{mn}\mathfrak{D}_k h^{mk} + \frac{1}{2}\mathfrak{D}^n\overline{\mathcal{H}}_{nm}\mathfrak{D}_k\mathcal{H}^{km} + \frac{1}{2}\left(\overline{\mathcal{H}}^{mn}\mathfrak{D}_m\mathfrak{D}_n h + \text{h.c.}\right) \\
& - 2\overline{\mathcal{H}}^{(mn)}\mathcal{H}_{(mn)} + \overline{\mathcal{H}}^{(mn)}h_{mn} - \frac{1}{2}\overline{\mathcal{H}}\left(\mathfrak{D}^2 - 2\right)h \\
& + \left(\overline{\mathcal{H}}^{[mn]} + \frac{1}{2}i\epsilon^{mn}\overline{B}\right)\left(\mathcal{H}_{[mn]} - \frac{1}{2}i\epsilon_{mn}B\right),
\end{aligned} \tag{2.94}$$

where

$$\mathcal{H}_{mn} = (\eta_{mk} - i\epsilon_{mk})h^k{}_n\,, \qquad\qquad \mathcal{H} = h\,, \tag{2.95}$$

and

$$\mathcal{C}_m = (\eta_{mn} - i\epsilon_{mn})C^n\,. \tag{2.96}$$

It is important to note that the Lagrangian (2.94) contais two complex massive bosons A and B, which were not present in the superfield epansion (2.88). However, these complex bosons correspond to the massive vectors a_m and c_m, respectively. More precisely, it can be shown from the equations of motion that a_m and c_m describe one complex degree of freedom each. As a consequence, the Lagrangian for a_m can be converted to a dual Lagrangian for the complex scalar A — after the addition of an auxiliary field A to the a_m Lagrangian and performing a suitable field redefinition. The same holds true for the c_m Lagrangian. The details are further explained in ref. [2].

The equations of motion coming from $\mathcal{L}_1$ are straightforward to obtain

$$\Big(\mathfrak{D}^2-2\Big)\mathcal{M}_1=0\,,\qquad\qquad \Big(\mathfrak{D}^2-2\Big)\mathcal{N}_1=0\,. \tag{2.97}$$

Deriving the equations of motion from $\mathcal{L}_2$ in a transparent form is a tedious exercise. Here, we will only quote the result. To decouple the massive spin-2 on-shell, one first defines the traceless and symmetric tensor $\mathfrak{h}_{mn}$ as

$$\begin{aligned}
\mathfrak{h}_{mn} &= \frac{4}{3}h_{mn}-\frac{i}{2}\Big(\epsilon_m{}^k h_{kn}+\epsilon_{nk}h^k{}_m\Big)-\frac{1}{6}\Big(\mathfrak{D}_m\mathfrak{D}_k h^k{}_n+\mathfrak{D}_n\mathfrak{D}_k h^k{}_m\Big)\\
&-\frac{i}{4}\eta_{mn}\Big(\epsilon^{kl}\mathfrak{D}^p\mathfrak{D}_l h_{kp}\Big)+\frac{i}{4}\Big(\epsilon_{mk}\mathfrak{D}^l\mathfrak{D}^k h_{nl}+\epsilon_{nk}\mathfrak{D}^l\mathfrak{D}^k h_{ml}-\epsilon_{mk}\mathfrak{D}_l\mathfrak{D}_n h^{kl}\\
&-\epsilon_{nk}\mathfrak{D}_l\mathfrak{D}_m h^{kl}\Big)+\frac{1}{2-\epsilon\epsilon}\bigg[\frac{1}{6}(\epsilon\epsilon)\Big(\mathfrak{D}_m\mathfrak{D}_n B+\mathfrak{D}_n\mathfrak{D}_m B\Big)+\frac{i}{2}\Big(\epsilon_{mk}\mathfrak{D}^k\mathfrak{D}_n B\\
&+\epsilon_{nk}\mathfrak{D}^k\mathfrak{D}_m B\Big)-\Big(\frac{1}{6}+\frac{1}{8}\epsilon\epsilon\Big)(\epsilon\epsilon)\eta_{mn}B+\frac{1}{2}(1-\epsilon\epsilon)\epsilon_m{}^k\epsilon_{kn}B\bigg]\\
&+\frac{1}{2+\epsilon\epsilon}\bigg[-\frac{i}{2}\Big(\widetilde{\epsilon}_{mk}\mathfrak{D}^k\mathfrak{D}_n A+\widetilde{\epsilon}_{nk}\mathfrak{D}^k\mathfrak{D}_m A\Big)+\frac{5}{8}(\epsilon\widetilde{\epsilon})\eta_{mn}A\\
&-\frac{1}{4}(\epsilon\widetilde{\epsilon})\Big(\mathfrak{D}_m\mathfrak{D}_n A+\mathfrak{D}_n\mathfrak{D}_m A\Big)+\Big(\widetilde{\epsilon}_{mk}\epsilon_{ln}\mathfrak{D}^k\mathfrak{D}^l A+\widetilde{\epsilon}_{nk}\epsilon_{lm}\mathfrak{D}^k\mathfrak{D}^l A\Big)\bigg]\,,
\end{aligned} \tag{2.98}$$

which leads to decoupled equations of motion and constraints,

$$\Big(\mathfrak{D}^2-2\Big)\mathfrak{h}_{mn}+2i\Big(\epsilon_m{}^k\mathfrak{h}_{nk}+\epsilon_n{}^k\mathfrak{h}_{mk}\Big)=0\,, \tag{2.99a}$$

$$\mathfrak{D}^n\mathfrak{h}_{mn}=0\,, \tag{2.99b}$$

$$\mathfrak{h}=0\,, \tag{2.99c}$$

$$\mathfrak{D}^2\mathcal{C}_m-2(\eta_{mn}-i\epsilon_{mn})\mathcal{C}^n-\mathfrak{D}_m\mathfrak{D}_n\mathcal{C}^n=0\,, \tag{2.99d}$$

$$\mathfrak{D}^m\mathcal{C}_m=0\,, \tag{2.99e}$$

$$\Big(\mathfrak{D}^2-2\Big)A=0\,, \tag{2.99f}$$

$$\Big(\mathfrak{D}^2-2\Big)B=0\,. \tag{2.99g}$$

As in the Argyres-Nappi Lagrangian, a simple analysis of the equations of motion and constraints confirms causal propagation of the spin-2. Note that, at the first massive level, the bosonic sector of the superstring has 12 complex degrees of freedom: 5 from the symmetric traceless spin-2 $\mathfrak{h}_{mn}$, 3 from the massive vector $\mathcal{C}_m$, and the remaining 4 from the massive complex scalars $\{\mathcal{M}_1,\mathcal{N}_1,A,B\}$.

2.6.3 Charged massive fermions

As usual, the discussion for the fermions is a mirror of the bosonic one. Similarly as in Section 2.6.2, we begin by stating which components of the superfield (2.88) carry the physical fermionic degrees o freedom, i.e., the ones that remain after performing field redefinitions and using the gauge transformations of the superstring field theory description. Once this is completed, we will write the Lagrangian $\mathcal{L}_F$ for the physical fermionic component fields from the interacting superspace action (2.82).

For the components in $\mathcal{B}$, the physical degrees of freedom are contained in the fieds $\{\gamma_1, \overline{\gamma}_2\}$. For the components in $\mathcal{C}$, the physical degrees of freedom are contained $\{\overline{\psi}_1, \psi_2\}$. For the components in V_m, the physical degrees of freedom are contained in $\{\chi_{1m}, \overline{\chi}_{2m}, \overline{\lambda}_{1m}, \lambda_{2m}\}$, which describe the massive spin-3/2 degrees of freedom. For the components in $\mathcal{U}_{1\alpha}$ and $\overline{\mathcal{U}}_2^{\dot{\alpha}}$, all of them can be removed, either by using the gauge transformations or by field redefinitions, see ref. [2].

As with h_{mn} and C_m in the bosonic sector, the fermions in the V_m superfield will also appear contracted with one or two $(1 - i\epsilon)$ factors in the Lagrangian. In order to make the formulas more concise, we define spinors with bold symbols

$$\overline{\boldsymbol{\lambda}}_{1m} = (\eta_{mn} - i\epsilon_{mn})\overline{\lambda}_1^n\,, \qquad \boldsymbol{\lambda}_{2m} = (\eta_{mn} - i\epsilon_{mn})\lambda_2^n\,, \tag{2.100a}$$

$$\boldsymbol{\chi}_{1m} = (\eta_{mn} - i\epsilon_{mn})\chi_1^n\,, \qquad \overline{\boldsymbol{\chi}}_{2m} = (\eta_{mn} - i\epsilon_{mn})\overline{\chi}_2^n\,. \tag{2.100b}$$

The resulting Lagrangian $\mathcal{L}_F$ for the fermionic components of the superspace action (2.82) then reads

$$\begin{aligned}\mathcal{L}_F = &-\frac{i}{2}\Big[2\Big(\lambda_1^m\sigma^n\mathfrak{D}_n\overline{\boldsymbol{\lambda}}_{1m}\Big) + \Big(\overline{\boldsymbol{\chi}}_{1m}\overline{\sigma}^n\sigma^k\overline{\sigma}^m\mathfrak{D}_k\boldsymbol{\chi}_{1n}\Big)\Big] - \sqrt{2}\Big[(\boldsymbol{\lambda}_1^m\boldsymbol{\chi}_{1m}) + \text{h.c.}\Big]\\ &+\Big[-\frac{i}{4}\Big(\psi_1\sigma^m\mathfrak{D}_m\overline{\psi}_1\Big) + 2i\Big(\gamma_1\sigma^m\mathfrak{D}_m\overline{\gamma}_1\Big)\Big] + \Big[\frac{3}{\sqrt{2}}\Big(\boldsymbol{\chi}_1^m\sigma_{mn}\mathfrak{D}^n\psi_1\Big)\\ &-\frac{1}{2\sqrt{2}}\Big(\boldsymbol{\chi}_1^m\mathfrak{D}_m\psi_1\Big) - \frac{i}{2}\Big(\boldsymbol{\lambda}_1^m\sigma_m\overline{\psi}_1\Big) - 2i\Big(\overline{\boldsymbol{\chi}}_1^m\overline{\sigma}_m\gamma_1\Big) - \sqrt{2}\Big(\boldsymbol{\lambda}_1^m\mathfrak{D}_m\gamma_1\Big) + \text{h.c.}\Big]\\ &+\Big[\frac{1}{\sqrt{2}}\Big(\psi_1\gamma_1\Big) + \text{h.c.}\Big] + \Big(1 \leftrightarrow 2\Big) - \Big[\frac{1}{2}\overline{\boldsymbol{\chi}}_1^m(\epsilon\cdot\overline{\sigma})\overline{\sigma}_m\gamma_1\\ &+\frac{1}{2}\boldsymbol{\chi}_2^m(\epsilon\cdot\sigma)\sigma_m\overline{\gamma}_2 + \text{h.c.}\Big]\,.\end{aligned} \tag{2.101}$$

Analogously to the bosonic case, we can find decoupled equations of motion for the spin-3/2 and spin-1/2 fields. For that, we define

$$\overline{\lambda}'_{1m} = \overline{\lambda}_{1m} + \frac{i}{2\sqrt{2}}[1 - i(\epsilon\cdot\overline{\sigma})]\overline{\sigma}_m\gamma_1 - \frac{1}{2}[\eta_{mn} - i(\epsilon_{mn} + i\tilde{\epsilon}_{mn})]\mathfrak{D}^n\overline{\psi}_1\,, \tag{2.102a}$$

$$\chi'_{1m} = \chi_{1m} + \frac{1}{2\sqrt{2}}(\epsilon\cdot\sigma)\sigma_m\overline{\psi}_1\,, \tag{2.102b}$$

which leads to the following equations of motion and constraints for the spin-3/2 fields

$$\sigma^n\mathfrak{D}_n\overline{\lambda}'_{1m} = -\sqrt{2}(\eta_{mn} - i\epsilon_{mn})\chi_1'^n\ , \tag{2.103a}$$

$$\overline{\sigma}^n\mathfrak{D}_n\chi'_{1m} = -\sqrt{2}\overline{\lambda}_{1m}\,, \tag{2.103b}$$

$$\mathfrak{D}^m\chi'_{1m} = 0\,, \tag{2.103c}$$

$$\mathfrak{D}^m\overline{\lambda}'_{1m} = -\frac{\sqrt{2}}{4}\overline{\sigma}^m(\epsilon\cdot\sigma)\chi'_{1m}\,, \tag{2.103d}$$

$$\overline{\sigma}^m\chi'_{1m} = 0\,, \tag{2.103e}$$

$$\sigma^m\overline{\lambda}'_{1m} = 0\,, \tag{2.103f}$$

as well as the Dirac equations for the spin-1/2 fields

$$i\overline{\sigma}^m\mathfrak{D}_m\gamma_1 = -\sqrt{2}\overline{\psi}_1\,, \tag{2.104a}$$

$$i\sigma^m\mathfrak{D}_m\overline{\psi}_1 = -\sqrt{2}\gamma_1\,. \tag{2.104b}$$

Note that we also have a similar set of equations of motion for the fermionic fields with index 2. In addition, $\{\chi'_{1m}, \overline{\lambda}'_{1m}\}$ describe 4 complex on-shell degrees of freedom, and $\{\gamma_1, \overline{\psi}_1\}$ describe 2 complex on-shell degrees of freedom. Together with the fermionic fields with index 2, we indeed have the 12 complex degrees of freedom as in the bosonic counterpart. Of course, this ir already expected by spacetime SUSY.

Even though the spin-3/2 and spin-1/2 fields appear coupled at the level of the Lagrangian (2.101), we have shown that it is possible to find a decoupled system of equations of motion and constraints derived which are from (2.101).

For the bosonic sector, we were able to write the Lagrangian for the charged fields as a deformed Fier-Pauli Lagragian (2.94). The analogue for the fermionic case $\mathcal{L}_F$ is to find a field redefinition such that the Lagrangian (2.101) reduces to a "Rarita-Schwinger plus Dirac" Lagrangian when the electromagnetic fields vanishes. Indeed, it is possible to find such a field redefinition [2]. Here, we will only quote the result.

The resulting expressions are very long. Therefore, to make them easier to read, we will separate the Lagrangian into three parts:

$$\mathcal{L}_F = \mathcal{L}_{\mathrm{RSD}} + \mathcal{L}_{\mathrm{km}} + \mathcal{L}_{\mathrm{coupl}}\,, \tag{2.105}$$

where $\mathcal{L}_{\mathrm{RSD}}$ consists in a sum of Rarita-Schwinger and Dirac Lagrangians. $\mathcal{L}_{\mathrm{km}}$ are the corrections of the kinetic and mass terms due to the electromagnetic background, and they vanish when $\epsilon = 0$. $\mathcal{L}_{\mathrm{coupl}}$ contains only new couplings between the spin-3/2 and spin-1/2 fields that are induced by the electromagnetic field. It also vanishes when $\epsilon = 0$.

The first part takes the expected simple form

$$\begin{aligned}\mathcal{L}_{\mathrm{RSD}} = & -\frac{1}{2}\varepsilon^{mnkl}\Big(\lambda_{1m}\sigma_n\mathfrak{D}_k\overline{\lambda}_{1l}\Big) + \frac{1}{2}\varepsilon_{mnkl}\Big(\overline{\chi}_1^m\overline{\sigma}^n\mathfrak{D}^k\chi_1^l\Big) - \sqrt{2}\Big[(\lambda_1^m\sigma_{mn}\chi_1^n) + \text{h.c.}\Big] \\ & - \frac{1}{2}i\Big(\psi_1\sigma^m\mathfrak{D}_m\overline{\psi}_1\Big) - \frac{1}{2}i\Big(\overline{\gamma}_1\overline{\sigma}^m\mathfrak{D}_m\gamma_1\Big) - \left[\frac{1}{\sqrt{2}}(\psi_1\gamma_1) + \text{h.c.}\right] + (1\leftrightarrow 2)\,,\end{aligned} \tag{2.106}$$

and it is the Lagrangian that was historically first considered and lead to the issues discussed in the introduction.

The new contribution to the kinetic and mass terms reads

$$\begin{aligned}\mathcal{L}_{\mathrm{km}} = & -\frac{1}{2}i\tilde{\epsilon}^{mn}\Big(\psi_1\sigma_m\mathfrak{D}_n\overline{\psi}_1\Big) - \frac{1}{2}\epsilon^{mk}G_{kn}\Big(\psi_1\sigma_l\mathfrak{D}_m\mathfrak{D}^l\mathfrak{D}^n\overline{\psi}_1\Big) \\ & - \frac{i}{2}\psi_1(\epsilon\cdot\sigma)\sigma^m(\epsilon\cdot\overline{\sigma})\mathfrak{D}_m\overline{\psi}_1 + \frac{1}{2}\varepsilon^{mnkl}(\epsilon_{mp} - i\tilde{\epsilon}_{mp})(\epsilon_{lq} + i\tilde{\epsilon}_{lq})\psi_1\sigma_n\mathfrak{D}^p\mathfrak{D}_k\mathfrak{D}^q\overline{\psi}_1 \\ & - i\epsilon^{mk}\epsilon_{kn}\Big(\psi_1\sigma_m\mathfrak{D}^n\overline{\psi}_1\Big) - \frac{1}{2}\epsilon^{pm}G_{mk}(\epsilon^{kl} + i\tilde{\epsilon}^{kl})(\epsilon_{pq} - i\tilde{\epsilon}_{pq})\psi_1\sigma_n\mathfrak{D}^q\mathfrak{D}^n\mathfrak{D}_l\overline{\psi}_1 \\ & + i\epsilon_{mk}G^{kl}\epsilon_{ln}\psi_1\sigma^p\mathfrak{D}^m\mathfrak{D}_p\mathfrak{D}^n\overline{\psi}_1 - \frac{i}{2}(\epsilon\epsilon)\tilde{\epsilon}^{mn}\psi_1\sigma_n\mathfrak{D}_m\overline{\psi}_1 + \frac{i}{2}(\epsilon\tilde{\epsilon})\epsilon^{mn}\psi_1\sigma_n\mathfrak{D}_m\overline{\psi}_1 \\ & + \frac{1}{4}\epsilon_{mk}G^{kn}\Big(\overline{\gamma}_1\overline{\sigma}^m\sigma^l\overline{\sigma}_n\mathfrak{D}_l\gamma_1\Big) - \frac{1}{4}\Big[i\epsilon_{mk}G^{kn}\overline{\gamma}_1\overline{\sigma}^m(\epsilon\cdot\sigma)\sigma^l\overline{\sigma}_n\mathfrak{D}_l\gamma_1 + \text{h.c.}\Big] \\ & - \frac{1}{4}\epsilon_{mk}G^{kn}\overline{\gamma}_1\overline{\sigma}^m(\epsilon\cdot\sigma)\sigma^l(\epsilon\cdot\overline{\sigma})\overline{\sigma}_n\mathfrak{D}_l\gamma_1 - i\tilde{\epsilon}^{mn}\overline{\gamma}_1\overline{\sigma}_n\mathfrak{D}_m\gamma_1 - \frac{i}{2}(\epsilon\epsilon)\overline{\gamma}_1\overline{\sigma}^m\mathfrak{D}_m\gamma_1 \\ & - \frac{i}{2}\overline{\gamma}_1(\epsilon\cdot\overline{\sigma})\overline{\sigma}^m(\epsilon\cdot\sigma)\mathfrak{D}_m\gamma_1 + \Big[- \frac{i}{2}\psi_1(\epsilon\cdot\sigma)\gamma_1 + \frac{1}{\sqrt{2}}\psi_1\sigma^m(\epsilon\cdot\overline{\sigma})\overline{\sigma}^n(\epsilon\cdot\sigma)\mathfrak{D}_m\mathfrak{D}_n\gamma_1 \\ & + \frac{i}{2\sqrt{2}}\epsilon_{mk}G^{kn}\overline{\gamma}_1\overline{\sigma}^m\sigma^l\mathfrak{D}_l\mathfrak{D}_n\overline{\psi}_1 - \frac{1}{2\sqrt{2}}(\epsilon\epsilon + i\epsilon\tilde{\epsilon})(\psi_1\gamma_1) - \frac{1}{2\sqrt{2}}(\epsilon\epsilon - i\epsilon\tilde{\epsilon})\Big(\psi_1\mathfrak{D}^2\gamma_1\Big) \\ & + \frac{1}{\sqrt{2}}\epsilon_{mk}G^{kn}\overline{\gamma}_1\overline{\sigma}^m(\epsilon\cdot\sigma)\sigma^l\mathfrak{D}_l\mathfrak{D}_n\overline{\psi}_1 + \frac{i}{4\sqrt{2}}\epsilon_{mk}G^{kn}(\epsilon\epsilon - i\epsilon\tilde{\epsilon})\psi_1\sigma^l\overline{\sigma}_n\mathfrak{D}^m\mathfrak{D}_l\gamma_1\end{aligned}$$

$$
\begin{aligned}
&+\text{h.c.}\Big] - \frac{1}{4}\epsilon_{mn}\Big[\Big(\overline{\chi}_1^m\overline{\sigma}^n\sigma^k\overline{\sigma}^l\mathfrak{D}_k\chi_{1l}\Big) + \text{h.c.}\Big] - \frac{1}{4}\epsilon^{mk}G_{kn}\Big(\overline{\chi}_1^l\overline{\sigma}_l\sigma_p\overline{\sigma}_q\mathfrak{D}_m\mathfrak{D}^p\mathfrak{D}^n\chi_1^q\Big) \\
&+\frac{1}{2}\epsilon_{mk}\Big(\lambda_1^m\sigma^n\mathfrak{D}_n\overline{\lambda}_1^k\Big) + \frac{1}{4\sqrt{2}}\Big[i\Big(\lambda_1^m\Sigma_{mn}\chi_1^n\Big) - 2i\epsilon_{mn}\Big(\lambda_1^n\sigma^k\overline{\sigma}^l\mathfrak{D}_k\mathfrak{D}^m\chi_{1l}\Big) \\
&+\text{h.c.}\Big] + (1\leftrightarrow 2, \epsilon\leftrightarrow -\epsilon)\,,
\end{aligned}
\tag{2.107}
$$

while the new couplings between spin-1/2 and spin-3/2 fields are

$$
\begin{aligned}
\mathcal{L}_{\text{coupl}} = {}& \sqrt{2}i\lambda_1^m\sigma_{mn}(\epsilon\cdot\sigma)\mathfrak{D}^n\gamma_1 - \frac{i}{\sqrt{2}}(\epsilon_{mn} - i\tilde{\epsilon}_{mn})\Big(\lambda_1^m\mathfrak{D}^n\gamma_1\Big) \\
&+\frac{1}{2\sqrt{2}}i\epsilon_{mn}\Big(\lambda_1^n\sigma^k\overline{\sigma}^m\mathfrak{D}_k\gamma_1\Big) + \frac{1}{2\sqrt{2}}\epsilon_{mn}\lambda_1^n\sigma^k(\epsilon\cdot\overline{\sigma})\overline{\sigma}^m\mathfrak{D}_k\gamma_1 + \frac{1}{2}\epsilon^{mn}\Big(\lambda_{1m}\sigma_n\overline{\psi}_1\Big) \\
&-\frac{i}{4}(\epsilon\epsilon + i\epsilon\tilde{\epsilon})\Big(\lambda_1^m\sigma_m\overline{\psi}_1\Big) + \frac{i}{2}\varepsilon^{mnkl}(\epsilon_{lq} + i\tilde{\epsilon}_{lq})\lambda_{1m}\sigma_n\mathfrak{D}_k\mathfrak{D}^q\overline{\psi}_1 \\
&+\frac{1}{2}\epsilon_{mk}(\eta^{kl} - i\epsilon^{kl} + \tilde{\epsilon}^{kl})\Big(\lambda_1^m\sigma^n\mathfrak{D}_n\mathfrak{D}_l\overline{\psi}_1\Big) + \frac{1}{4}\overline{\chi}_1^m(\epsilon\cdot\overline{\sigma})\overline{\sigma}_m\gamma_1 \\
&-\frac{3}{2}\overline{\chi}_1^m\overline{\sigma}_m(\epsilon\cdot\sigma)\gamma_1 + \frac{1}{4}\epsilon^{mk}G_{kn}\Big(\overline{\chi}_1^l\overline{\sigma}_l\sigma_p\overline{\sigma}^n\mathfrak{D}_m\mathfrak{D}^p\gamma_1\Big) - \frac{i}{4\sqrt{2}}\Big(\psi_1\Sigma^{mn}\mathfrak{D}_m\chi_{1n}\Big) \\
&-\frac{i}{\sqrt{2}}\overline{\chi}_1^m\overline{\sigma}_m(\epsilon\cdot\sigma)\sigma_n\mathfrak{D}^n\overline{\psi}_1 - \frac{i}{4}\epsilon^{mk}G_{kn}\overline{\chi}_1^l\overline{\sigma}_l\sigma_p(\epsilon\cdot\overline{\sigma})\overline{\sigma}^n\mathfrak{D}_m\mathfrak{D}^p\gamma_1 \\
&+\frac{i}{4}\overline{\chi}_1^m(\epsilon\cdot\sigma)\overline{\sigma}_m(\epsilon\cdot\sigma)\gamma_1 + \frac{i}{2}(\epsilon\epsilon)\Big(\overline{\chi}_1^m\overline{\sigma}_m\gamma_1\Big) \\
&+\frac{1}{8\sqrt{2}}(\epsilon\epsilon + i\epsilon\tilde{\epsilon})\Big(\overline{\chi}_1^m\overline{\sigma}_m\sigma_n\mathfrak{D}^n\overline{\psi}_1\Big) - \frac{1}{4\sqrt{2}}\overline{\chi}_1^m(\epsilon\cdot\overline{\sigma})\overline{\sigma}_m(\epsilon\cdot\sigma)\sigma_n\mathfrak{D}^n\overline{\psi}_1 \\
&+\frac{1}{4\sqrt{2}}\overline{\chi}_1^m\overline{\sigma}_m(\epsilon\cdot\sigma)\sigma_n(\epsilon\cdot\overline{\sigma})\mathfrak{D}^n\overline{\psi}_1 + \frac{i}{2\sqrt{2}}\epsilon^{mk}G_{kn}\Big(\overline{\chi}_1^l\overline{\sigma}_l\sigma_p\mathfrak{D}_m\mathfrak{D}^p\mathfrak{D}^n\overline{\psi}_1\Big) \\
&-\frac{1}{2\sqrt{2}}(\epsilon_{mn} - i\tilde{\epsilon}_{mn})G^{nk}\epsilon_{kl}\psi_1\sigma^p\overline{\sigma}^q\mathfrak{D}^m\mathfrak{D}_p\mathfrak{D}^l\chi_{1q} + \text{h.c.} + (1\leftrightarrow 2, \epsilon\leftrightarrow -\epsilon)\,,
\end{aligned}
\tag{2.108}
$$

In these expressions, we introduced the notation

$$
\Sigma_{mn} \equiv \sigma_m\overline{\sigma}_n(\epsilon\cdot\sigma) - \sigma_m(\epsilon\cdot\overline{\sigma})\overline{\sigma}_n - (\epsilon\cdot\sigma)\sigma_m\overline{\sigma}_n\,,
\tag{2.109}
$$

and

$$
G_{mn} \equiv (\eta_{mn} - \mathrm{i}\epsilon_{mn})^{-1}\,.
\tag{2.110}
$$

Chapter 3

The superstring in a flat six-dimensional background

We begin by introducing the six-dimensional hybrid formalism for the superstring, establishing our conventions for the worldsheet and target space variables. In order to preserve more supersymmetries manifestly in the six-dimensional spacetime, the formalism is extended by adding $d = 6$ $\mathcal{N} = 1$ superspace variables and unconstrained bosonic ghosts to the worldsheet theory. A manifestly spacetime supersymmetric vertex operator U is then constructed. BRST invariance of U is shown to imply the SYM equations of motion in $d = 6$ $\mathcal{N} = 1$ superspace. Finally, it is shown that spacetime supersymmetric scattering amplitudes can be computed in a similar manner as in the non-minimal pure spinor formalism.

3.1 Introduction

As of today, superstring theory is the only known mathematically consistent quantum theory of gravity that can, in principle, accommodate the much studied and successful standard model of particle physics. Despite the fact that quantum consistency requires the superstring to have ten spacetime directions, from a practical and experimental point of view, it is self-evident that the most interesting case for its study comes from backgrounds where one has four uncompactified and six compactified directions of the ten-dimensional spacetime. As was the case in the discussion carried out in Chapter 2.

At the same time, the lack of tractability of the theory in a general target space, and the dualities connecting the different mathematical formulations of string theory and quantum field theory [31], have made the understanding of solvable compactifications of the superstring an active area of ongoing research since its discovery more than half a century ago [32] [33].

The study of superstring compactifications can have applications both to the existing theories of physics and to areas of pure mathematics [34] [35] [36] [37] [38]. The most notable cases of study are compactifications on the so-called Calabi-Yau backgrounds, which are complex and compact Kähler manifolds with a Hermitian metric and vanishing first Chern class [38]. As an additional bonus, these conditions turn out to imply a supersymmetric spacetime for the superstring to propagate.

Although spacetime supersymmetry has not been observed experimentally, it is a powerful tool for simplifying difficult calculations and holds significant phenomenological value [24] [39]. Moreover, another important motivation for exploring $d \neq 4$ superstring compactifications — to be discussed extensively in Chapter 4 — comes from the intriguing AdS/CFT duality which relates the superstring propagating in a $d+1$-dimensional Anti de-Sitter background to a conformal field theory living on the d-dimensional AdS boundary [40].

In this chapter, our focus will be on compactifications of the superstring on Calabi-Yau spaces of complex dimension two. More specifically, we will be studying the superstring on $\mathbb{R}^6 \times \mathcal{M}_4$, where $\mathcal{M}_4$ can be either K3 or T^4, by using the spacetime supersymmetric six-dimensional hybrid formalism [18]. As in compactifications to four-dimensional spacetime, the six-dimensional hybrid formalism consists in a field redefinition of the gauge-fixed Ramond-Neveu-Schwarz (RNS) superstring into a set of Green-Schwarz-like (GS) variables, allowing spacetime supersymmetry to be made manifest.

The hybrid formalism stands in contrast to the more conventional GS [41] and RNS formalisms of the superstring [20]. Even though the GS superstring has manifest spacetime SUSY, quantization becomes difficult due to the lack of manifest Lorentz covariance in the light-cone gauge, and computations of scattering amplitudes from the GS superstring remain a challenging task. Despite the fact that the RNS superstring is quantizable in a Lorentz-covariant manner, spacetime supersymmetry is not manifest, and the theory has an infinite number of SUSY charges related by picture-changing [20]. In addition, as opposed to the GS-action, quantization is straightforward since the hybrid action is quadratic in a flat background. Additionally, the hybrid description enjoys an $\mathcal{N}=4$ superconformal symmetry, which can be used to compute n-point multiloop superstring amplitudes from a topological prescription [18].

Unlike the four-dimensional hybrid formalism discussed in Chapter 2, which is described in terms of standard $d=4$ $\mathcal{N}=1$ superspace variables [24], the six-dimensional hybrid formalism does not include the standard $d=6$ $\mathcal{N}=1$

superspace variables $\{x^{\underline{a}}, \theta^{\alpha j}\}$ as fundamental worldsheet fields [42] [43], where $\underline{a} = \{0 \text{ to } 5\}$, $\alpha = \{1 \text{ to } 4\}$ and $j = \{1,2\}$. As we will presently elaborate, the reason for this is that there are not enough fundamental degrees of freedom in the gauge-fixed RNS description, which implies that the fermionic coordinates $\theta^{\alpha j}$ cannot be constructed as free worldsheet fields. Consequently, one can only make half of the $d = 6$ $\mathcal{N} = 1$ SUSYs manifest in the six-dimensional hybrid formalism.

Let us develop further the discussion above. For simplicity, we only consider the open string or holomorphic sector. As we have alluded to, in order to exhibit $d = 6$ $\mathcal{N} = 1$ SUSY manifest for the superstring compactified in $\mathbb{R}^6 \times \mathcal{M}_4$,[1] one would like to have the superspace coordinates $\{x^{\underline{a}}, \theta^{\alpha j}\}$ as fundamental worldsheet variables. In $d = 6$ $\mathcal{N} = 1$ superspace descriptions, these coordinates transform in a geometric manner under the spacetime SUSY charge $Q_{\alpha j}$

$$\delta x^{\underline{a}} = \frac{i}{2}\epsilon_{jk}\varepsilon^{\alpha j}\sigma^{\underline{a}}_{\alpha\beta}\theta^{\beta k}\,, \tag{3.1a}$$

$$\delta\theta^{\alpha j} = \varepsilon^{\alpha j}\,, \tag{3.1b}$$

where $\varepsilon^{\alpha j}$ is the constant fermionic parameter of the transformation. Moreover, the SUSY generators satisfy the usual six-dimensional algebra

$$\{Q_{\alpha j}, Q_{\beta k}\} = -\epsilon_{jk}P_{\alpha\beta}\,, \tag{3.2}$$

where the antisymmetric symbol takes the values $\epsilon_{12} = \epsilon^{21} = 1$ and $P_{\alpha\beta}$ is the momentum operator, antisymmetric in the spinor indices.[2]

With the aim of quantizing the superstring with manifest six-dimensional spacetime SUSY, and having the usual superspace coordinates as fundamental free worldsheet fields, a reasonable starting point is to first try to construct $d = 6$ $\mathcal{N} = 1$ SUSY generators from the gauge-fixed RNS worldsheet variables. Out of the sixteen SUSY generators of the RNS formalism, we must choose eight of them to be matched with $Q_{\alpha j}$ given in eqs. (3.2).

Since the SUSY generators in RNS carry picture-charge [20], for the purpose of having a closed SUSY algebra one has to choose four generators in the $-\frac{1}{2}$-picture $q_\alpha^{-\frac{1}{2}}$ and four in the $+\frac{1}{2}$-picture $q_\alpha^{\frac{1}{2}}$, i.e., we can write

$$Q^{\text{hyb}}_{\alpha 1} = q_\alpha^{-\frac{1}{2}}\,, \tag{3.3a}$$

[1] Strictly speaking, we take $\mathcal{M}_4 = \mathrm{T}^4$, since K3 only preserves half of the $d = 6$ $\mathcal{N} = 1$ SUSYs.
[2] Our conventions for the six-dimensional Pauli matrices are spelled out in Appendix B.

$$Q^{\text{hyb}}_{\alpha 2} = q^{\frac{1}{2}}_{\alpha} \,, \tag{3.3b}$$

such that they close to the momentum operator in the zero-picture

$$\{Q^{\text{hyb}}_{\alpha j}, Q^{\text{hyb}}_{\beta k}\} = -i\epsilon_{jk} \oint \partial x_{\alpha\beta} \,, \tag{3.4}$$

as desired.

Furthermore, the six superspace bosons $x^{\underline{a}}$ in (3.1) can be chosen out of the ten xs from the RNS variables. Consequently, one is left to find eight fermionic free worldsheet fields $\theta^{\alpha j}$ such that (3.1) is satisfied, i.e., $\theta^{\alpha j}$ transform by a constant translation under the SUSY charges, namely, $Q^{\text{hyb}}_{\alpha j}\theta^{\beta k} = \delta^{\beta}_{\alpha}\delta^{k}_{j}$. However, despite the fact that it is possible to identify $\theta^{\alpha 1}$ and $\theta^{\alpha 2}$ meeting this criteria, one finds that

$$\theta^{\alpha 2} \sim \theta^{\alpha 1} e^{-iH^{\text{RNS}}_{C} - 2\phi}\xi c \,, \tag{3.5}$$

as a result, one cannot choose the eight $\theta^{\alpha j}$ to be free fields. In the equation above, $i\partial H^{\text{RNS}}_{C}$ is the U(1)-current for the twisted $\mathcal{N} = 2$ superconformal algebra (SCA) describing the compactified directions, ξ is the fermionic ghost coming from the bosonization of the $\{\beta, \gamma\}$-ghosts, and c is the c-ghost of the RNS description [20].

From the aforementioned discussion, we conclude that only four fermionic superspace coordinates, say $\theta^{\alpha 1}$, can be defined from the free gauge-fixed RNS worldsheet variables. This implies that only half of the eight spacetime supersymmetries of $d = 6$ $\mathcal{N} = 1$ superspace will be manifest in the six-dimensional hybrid formalism. With that in mind, and for simplicity of the notation, we will write $\theta^{\alpha 1} = \theta^{\alpha}$ for the hybrid superspace fermionic fields. In particular, since $Q^{\text{hyb}}_{\alpha 1}\theta^{\beta} = \delta^{\beta}_{\alpha}$, choosing

$$Q^{\text{hyb}}_{\alpha 1} = \oint p_{\alpha} \,, \tag{3.6}$$

where p_{α} is the conjugate momentum of θ^{α}, accounts for the remaining fermionic degrees of freedom in the gauge-fixed RNS description.

On top of the eight fermionic worldsheet fields $\{p_{\alpha}, \theta^{\alpha}\}$, the field redefinition from RNS to the six-dimensional hybrid formalism yields two additional chiral bosons, ρ and σ, to the compactification-independent worldsheet variables. In the RNS formalism, the four dimensional compactification part is described by four bosons $\{x^{I}, \overline{x}_{I}\}$ and four fermions $\{\psi^{I}_{\text{RNS}}, \overline{\psi}^{\text{RNS}}_{I}\}$, which satisfy a $c = 6$ $\mathcal{N} = 2$

SCA and where $I = 1, 2$. Under the field redefinition, the fermions $\{\psi^I_{\text{RNS}}, \overline{\psi}^{\text{RNS}}_I\}$ get mapped to the twisted fermions $\{\psi^I, \overline{\psi}_I\}$ with conformal weight zero and one, respectively. As a consequence, the $c = 6$ $\mathcal{N} = 2$ SCA becomes twisted, and the bosonic fields $\{x^I, \overline{x}_I\}$ stay untouched.

Schematically, the field redefinition takes the gauge-fixed free RNS variables

$$\{x^{\underline{a}}, \psi^{\underline{a}}, b, c, \eta, \xi, \phi\} \quad \oplus \quad \{x^I, \overline{x}_I, \psi^I_{\text{RNS}}, \overline{\psi}^{\text{RNS}}_I\}\,, \tag{3.7}$$

and maps to the hybrid formalism free worldsheet fields

$$\underbrace{\{x^{\underline{a}}, p_\alpha, \theta^\alpha, \sigma, \rho\}}_{\mathbb{R}^{1,5}} \quad \oplus \quad \underbrace{\{x^I, \overline{x}_I, \psi^I, \overline{\psi}_I\}}_{\mathcal{M}_4}\,, \tag{3.8}$$

where the ghosts $\{\phi, \xi, \eta\}$ come from the bosonization of the $\{\beta, \gamma\}$-ghosts, namely, $\beta = e^{-\phi}\partial\xi$ and $\gamma = \eta e^{\phi}$. In addition, the chiral boson σ in (3.8) comes from the bosonization $b = e^{-i\sigma}$ and $c = e^{i\sigma}$.

When counting the degrees of freedom, note that we have six xs and two chiral bosons in RNS, same number as the bosonic fields in the hybrid formalism $\{x^{\underline{a}}, \sigma, \rho\}$, and the eight RNS fermions $\{\psi^{\underline{a}}, b, c, \eta, \xi\}$ match exactly with the eight fermionic variables $\{p_\alpha, \theta^\alpha\}$ since $\alpha = \{1 \text{ to } 4\}$. For the compactification-dependent part, note also that we have four fermions and four bosons in both descriptions. It is important to be aware that the field redefinition is engineered in a way that the free set of RNS fields (3.7) is mapped to the free worldsheet fields (3.8).

To be more specific, the field redefinition takes the following form for the six-dimensional part

$$p_\alpha = e^{-\frac{\phi}{2}} e^{-\frac{i}{2}H^{\text{RNS}}_C} S_\alpha\,, \qquad \theta^\alpha = S^\alpha e^{\frac{i}{2}H^{\text{RNS}}_C} e^{\frac{\phi}{2}}\,, \qquad \rho = -2\phi + i\chi - iH^{\text{RNS}}_C\,, \tag{3.9}$$

and for the compactification-dependent fermions it reads

$$\psi^I = e^{-i\chi+\phi}\psi^I_{\text{RNS}}\,, \qquad\qquad \overline{\psi}_I = \overline{\psi}^{\text{RNS}}_I e^{-\phi+i\chi}\,, \tag{3.10}$$

where $i\partial H^{\text{RNS}}_C = \psi^I_{\text{RNS}}\overline{\psi}^{\text{RNS}}_I$, $\{S_\alpha, S^\alpha\}$ are the spin-fields for the six-dimensional spacetime and χ is a chiral boson coming from the bosonization of $\{\eta, \xi\}$ as $\eta = e^{-i\chi}$ and $\xi = e^{i\chi}$. Note further that this intricate field redefinition implies that the SUSY charge (3.3b) comes in a "non-standard" form, i.e., it depends on the chiral bosons ρ and σ, namely,

$$Q^{\text{hyb}}_{\alpha 2} = \oint \left(e^{-\rho - i\sigma} p_\alpha + i\partial x_{\alpha\beta}\theta^\beta \right). \tag{3.11}$$

Therefore, since the six-dimensional hybrid formalism has four of the θ coordinates of superspace as fundamental worldsheet variables [43], it is clear that only half of the eight $d = 6$ $\mathcal{N} = 1$ SUSYs can be made manifest, i.e., the ones generated by the charge (3.6). To overcome this issue and make $d = 6$ $\mathcal{N} = 1$ manifest, ref. [44] introduced four more θ coordinates, along with their conjugate momenta, as fundamental worldsheet fields together with four fermionic first-class constraints D_α. In such a way that the gauge symmetry generated by these constraints can be used to gauge away the new variables. Therefore, when $D_\alpha = 0$, one recovers the hybrid description.

Under these circumstances, the constraint $D_\alpha = 0$ has to be imposed "by hand", which means that identifying the usual $d = 6$ $\mathcal{N} = 1$ superfields in the vertex operator is not feasible in practice. Consequently, it is unclear where each of the component fields sits in the vertex before using $D_\alpha = 0$ and making contact with the usual six-dimensional hybrid description. In addition, this also implies a major obstacle for defining scattering amplitudes with vertex operators depending on eight θs.

In Section 3.3 of this chapter, we will show that, after relaxing the harmonic constraint D_α, ghost number one supersymmetric unintegrated vertex operators U can be written in terms of $d = 6$ $\mathcal{N} = 1$ superfields. In addition, BRST invariance of U will be shown to imply the $d = 6$ super-Yang-Mills (SYM) equations of motion in superspace [43] [42]. Besides the fermionic fields θ, unconstrained bosonic ghost-fields λ^α, and its conjugate momenta, will be added to the worldsheet action in such a way that the total central charge of the stress-tensor vanishes.

The BRST current of the theory will take the form $G^+_{\text{hyb}} - \lambda^\alpha D_\alpha$, where G^+_{hyb} is the positively charged $\mathcal{N} = 2$ supercurrent of the hybrid formalism in supersymmetric notation, and the term $-\lambda^\alpha D_\alpha$ is responsible for the relaxation of the constraint $D_\alpha = 0$. The ghost-number current will be defined in terms of the $\text{U}(1)$ current of the hybrid $\mathcal{N} = 2$ algebra. Furthermore, as in the non-minimal pure spinor formalism [45], non-minimal/topological variables will be introduced to the BRST current in order to define supersymmetric scattering amplitude computations with a suitable regulator $\mathcal{R}$. We will end up by using the amplitude prescription to compute a three-point amplitude of $d = 6$ SYM states.

3.2 Hybrid formalism in a flat six-dimensional background

In this section, we review the worldsheet variables and the physical state conditions of the hybrid formalism for the superstring in a flat six-dimensional background. Novel results include identity (3.17) and the computation of eq. (3.38) taking care of the normal-ordering contributions. This description will serve as the starting point for Section 3.3.

3.2.1 Worldsheet action and superconformal generators

After performing a field redefinition of the gauge-fixed RNS variables [18] [46], the worldsheet fields of the six-dimensional part consist of six conformal weight zero bosons $x^{\underline{a}}$, $\underline{a} = \{0 \text{ to } 5\}$, and a canonically conjugate left-moving pair of fermions $\{p_\alpha, \theta^\alpha\}$ of conformal weight one and zero, respectively, together with its right-moving part $\{\widehat{p}_{\widehat{\alpha}}, \widehat{\theta}^{\widehat{\alpha}}\}$, where $\alpha, \widehat{\alpha} = \{1 \text{ to } 4\}$.

In a flat six-dimensional background the worldsheet action in conformal gauge takes the form[3]

$$S = \int d^2z \left(\frac{1}{2} \partial x^{\underline{a}} \overline{\partial} x_{\underline{a}} + p_\alpha \overline{\partial} \theta^\alpha + \widehat{p}_{\widehat{\alpha}} \partial \widehat{\theta}^{\widehat{\alpha}} \right) + S_{\rho,\sigma} + S_C \,, \tag{3.12}$$

where $\frac{\partial}{\partial z} = \partial$, $\frac{\partial}{\partial \overline{z}} = \overline{\partial}$, $S_{\rho,\sigma}$ is the part of the action characterizing the chiral bosons ρ and σ, as well as their anti-chiral counterparts, to be defined by their OPEs and stress-tensor below, and S_C corresponds to the four-dimensional compactification variables. These variables can be taken to be any $c = 6$ $\mathcal{N} = 2$ superconformal field theory describing the compactification manifold, which can be either K3 or T^4 [46].

For the Type-IIB (Type-IIA) superstring, an up α index and an up (down) $\widehat{\alpha}$ index transform as a Weyl spinor of SU(4), a down α index and a down (up) $\widehat{\alpha}$ index transform as an anti-Weyl spinor of SU(4). In this case, note that Weyl and anti-Weyl spinors are not related by complex conjugation. Also, we will only discuss the open string part of the worldsheet theory in what follows.

To define physical states, one needs to supplement the action (3.12) with the

[3] The OPEs between our fundamental worldsheet fields $\{p_\alpha, \theta^\alpha, \rho, \sigma\}$ are given by eqs. (3.31), with the replacement of $\alpha j \to \alpha$.

twisted $c = 6\,\mathcal{N} = 2$ constraints [18]

$$T_{\text{hyb}} = -\frac{1}{2}\partial x^{\underline{a}}\partial x_{\underline{a}} - p_\alpha \partial\theta^\alpha - \frac{1}{2}\partial\rho\partial\rho - \frac{1}{2}\partial\sigma\partial\sigma + \frac{3}{2}\partial^2(\rho + i\sigma) + T_C\,, \tag{3.13a}$$

$$G^+_{\text{hyb}} = -(p)^4 e^{-2\rho - i\sigma} + \frac{i}{2} p_\alpha p_\beta \partial x^{\alpha\beta} e^{-\rho} - \frac{1}{2}\partial x^{\underline{a}}\partial x_{\underline{a}} e^{i\sigma} - p_\alpha\partial\theta^\alpha e^{i\sigma}$$
$$- \frac{1}{2}\partial(\rho + i\sigma)\partial(\rho + i\sigma)e^{i\sigma} + \frac{1}{2}\partial^2(\rho + i\sigma)e^{i\sigma} + G^+_C\,, \tag{3.13b}$$

$$G^-_{\text{hyb}} = e^{-i\sigma} + G^-_C\,, \tag{3.13c}$$

$$J_{\text{hyb}} = \partial(\rho + i\sigma) + J_C\,, \tag{3.13d}$$

where $(p)^4 = \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta} p_\alpha p_\beta p_\gamma p_\delta$, $x_{\alpha\beta} = \sigma^{\underline{a}}_{\alpha\beta} x_{\underline{a}}$ and $\sigma^{\underline{a}}_{\alpha\beta}$ are the six-dimensional Pauli matrices, which are 4×4 antisymmetric in the spinor indices. Our conventions for the six-dimensional Pauli matrices are detailed in Appendix B.

Note that $\{T_C, G^\pm_C, J_C\}$ represent a twisted $c = 6\,\mathcal{N} = 2$ superconformal field theory describing the compactification manifold, so that $\{T_{\text{hyb}} - T_C, G^\pm_{\text{hyb}} - G^\pm_C, J_{\text{hyb}} - J_C\}$ describe a $c = 0\,\mathcal{N} = 2$ superconformal algebra (SCA). The generators $\{T_C, G^\pm_C, J_C\}$ have no poles with the six-dimensional worldsheet variables and no poles with the chiral bosons $\{\rho, \sigma\}$. For the closed string, we also have the right-moving piece of the above algebra.

The operators $e^{m\rho + ni\sigma}$ are conformal tensors and have conformal weight $\frac{1}{2}(-m^2 + 3m + n^2 - 3n)$. The definition of normal-ordering used in eqs. (3.13), and in the rest of this work, is presented in Appendix D. In particular, notice that we can write the first two terms in the second line of (3.13b) in a more compact form as

$$-\frac{1}{2}\partial(\rho + i\sigma)\partial(\rho + i\sigma)e^{i\sigma} + \frac{1}{2}\partial^2(\rho + i\sigma)e^{i\sigma} = \left(\partial e^{-\rho - i\sigma}, e^{\rho + 2i\sigma}\right), \tag{3.14}$$

using our normal-ordering prescription.

Correspondingly, any twisted $c = 6\,\mathcal{N} = 2$ SCA (3.13) can be extended to a twisted small $c = 6\,\mathcal{N} = 4$ SCA [18] by adding two bosonic currents and two supercurrents, as detailed in Appendix H. The additional $\mathcal{N} = 4$ superconformal generators in the six-dimensional hybrid formalism are

$$\widetilde{G}^+_{\text{hyb}} = e^\rho J^{++}_C - e^{\rho + i\sigma}\widetilde{G}^+_C\,, \tag{3.15a}$$

$$\widetilde{G}^-_{\text{hyb}} = \Big(- (p)^4 e^{-3\rho - 2i\sigma} + \frac{i}{2} p_\alpha p_\beta \partial x^{\alpha\beta} e^{-2\rho - i\sigma} - \frac{1}{2}\partial x^{\underline{a}}\partial x_{\underline{a}} e^{-\rho} - p_\alpha \partial\theta^\alpha e^{-\rho}$$
$$- \left(\partial e^{-\rho - i\sigma}, e^{i\sigma}\right)\Big) J^{--}_C + e^{-\rho - i\sigma}\widetilde{G}^-_C\,, \tag{3.15b}$$

$$J^{++}_{\rm hyb} = -e^{\rho+i\sigma} J^{++}_{C}\,, \tag{3.15c}$$

$$J^{--}_{\rm hyb} = e^{-\rho-i\sigma} J^{--}_{C}\,, \tag{3.15d}$$

where $\{\widetilde{G}^{\pm}_{C}, J^{\pm\pm}_{C}\}$, that together with $\{T_C, G^{\pm}_{C}, J_C\}$, form a twisted small $c = 6$ $\mathcal{N} = 4$ SCA which has no poles with the six-dimensional worldsheet variables and also no poles with the chiral bosons $\{\rho, \sigma\}$.

The spacetime supersymmetry charges in the six-dimensional hybrid formalism are given by [46]

$$Q^{\rm hyb}_{\alpha 1} = \oint p_\alpha\,, \qquad\qquad Q^{\rm hyb}_{\alpha 2} = \oint \left(e^{-\rho-i\sigma} p_\alpha + i\partial x_{\alpha\beta}\theta^\beta\right), \tag{3.16}$$

and satisfy the spacetime SUSY algebra $\{Q^{\rm hyb}_{\alpha 1}, Q^{\rm hyb}_{\alpha 2}\} = -i\oint \partial x_{\alpha\beta}$. Note that the charge $Q^{\rm hyb}_{\alpha 2}$ has the presence of the $\{\rho,\sigma\}$-ghosts and, for that reason, it is called the "non-standard" supersymmetry generator [46].

The superconformal generators (3.13) are manifestly invariant under the SUSY charge $Q^{\rm hyb}_{\alpha 1}$. Invariance under $Q^{\rm hyb}_{\alpha 2}$ is difficult to check for the supercurrent $G^{+}_{\rm hyb}$. However, the latter can be made manifest by noting that one can write the supercurrent as

$$G^{+}_{\rm hyb} = -\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta} Q^{\rm hyb}_{\alpha 2} Q^{\rm hyb}_{\beta 2} Q^{\rm hyb}_{\gamma 2} Q^{\rm hyb}_{\delta 2} e^{2\rho+3i\sigma} + G^{+}_{C}\,, \tag{3.17}$$

which is a property that also holds in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background including the normal-ordering contributions, see [6] and Section 4.2.

Note that we are denoting operators defined throughout this section with the subscript/superscript "hyb", so as to not cause confusion with the generators to be introduced in Section 3.3.

3.2.2 Physical states

Following refs. [46] [6], physical states $V_{\rm hyb}$ of the theory are defined to satisfy the equation of motion[4]

$$(G^{+}_{\rm hyb})_0 (\widetilde{G}^{+}_{\rm hyb})_0 V_{\rm hyb} = 0\,, \tag{3.18}$$

[4]For a holomorphic operator $\mathcal{O}$ with conformal dimension h, $(\mathcal{O})_r$ is defined by the usual mode expansion in the plane, namely, $\mathcal{O}(z) = \sum_r (\mathcal{O})_r z^{-r-h}$.

so that the vertex operator V_{hyb} is defined up to the gauge transformation

$$\delta V_{\text{hyb}} = (G^+_{\text{hyb}})_0 \Lambda + (\widetilde{G}_{\text{hyb}})_0 \Omega \,, \tag{3.19}$$

for some Λ and some Ω. Moreover, it is consistent to impose the additional gauge-fixing conditions

$$(G^-_{\text{hyb}})_0 V_{\text{hyb}} = (\widetilde{G}^-_{\text{hyb}})_0 V_{\text{hyb}} = (T_{\text{hyb}})_0 V_{\text{hyb}} = (J_{\text{hyb}})_0 V_{\text{hyb}} = 0 \,. \tag{3.20}$$

As an example, let us consider the massless compactification-independent states in six dimensions for the open superstring. The most general vertex operator with conformal weight zero and no poles with the U(1)-current J_{hyb} has the form

$$V_{\text{hyb}} = \sum_{n=0}^{\infty} V_n e^{n(\rho + i\sigma)} \,. \tag{3.21}$$

The conditions of no double poles or higher with G^-_{hyb} and with $\widetilde{G}^-_{\text{hyb}}$ imply that $V_n = 0$ for $n \geq 2$ and $n \leq -2$, respectively. From the remaining equations coming from $(\widetilde{G}^-_{\text{hyb}})_0 V_{\text{hyb}} = 0$, together with the gauge transformations (3.19), one can gauge-fix V_{hyb} to the form

$$V_{\text{hyb}} = V_1 e^{\rho + i\sigma} + V_0 \,, \tag{3.22}$$

where

$$V_1 = \theta^\alpha \chi_{\alpha 2} + \frac{i}{2} (\theta \sigma^{\underline{a}} \theta) a_{\underline{a}} - (\theta^3)_\alpha \psi^{\alpha 2} \,, \tag{3.23a}$$

$$V_0 = \theta^\alpha \chi_{\alpha 1} \,, \tag{3.23b}$$

with $\psi^{\alpha j} = \epsilon^{jk} i \partial^{\alpha\beta} \chi_{\beta k}$ the gluino and $a_{\underline{a}}$ the gluon. The two-dimensional Levi-Civita symbol takes the values $\epsilon_{12} = \epsilon^{21} = 1$. Even though $\chi_{\alpha j}$ is not gauge-invariant, we have that $\delta \psi^{\alpha j} = 0$ under a gauge transformation. In our conventions, we are using

$$(\theta^3)_\alpha = \frac{1}{6} \epsilon_{\alpha\beta\gamma\delta} \theta^\beta \theta^\gamma \theta^\delta \,, \qquad (\theta^4) = \frac{1}{24} \epsilon_{\alpha\beta\gamma\delta} \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta \,, \tag{3.24}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol with $\epsilon_{1234} = 1$.

The superfield V_1 satisfies the equation of motion $\partial^{\alpha\beta} \nabla_\alpha \nabla_\beta V_1 = 0$ which, together with the gauge transformations, imply that the component fields obey

$$\partial^{\underline{a}}\partial_{\underline{a}}a_{\underline{b}} = \partial^{\underline{a}}a_{\underline{a}} = 0\,, \qquad \delta a_{\underline{a}} = \partial_{\underline{a}}\lambda\,, \tag{3.25a}$$
$$\partial_{\alpha\beta}\psi^{\beta j} = 0\,, \qquad \delta\psi^{\alpha j} = 0\,, \tag{3.25b}$$

for some λ. The gauge transformation of $a_{\underline{a}}$ comes from choosing $\Lambda = (\theta)^4\lambda$ in (3.19).

Eqs. (3.25) are the field content of $d = 6$ super-Yang-Mills (SYM). It is also important to note that all degrees of freedom are contained in the superfield V_1 of eq. (3.23a). In Section 3.3, we will see how one can describe superstring vertex operators for the SYM states in terms of the usual superfields of $d = 6$ $\mathcal{N} = 1$ superspace [43].

Furthermore, by considering (3.21) in the gauge where $(G^+_{\text{hyb}})_0(\widetilde{G}^+_{\text{hyb}})_0 V_{\text{hyb}} = 0$, we find that the integrated vertex operator for the open superstring compactification-independent massless states is

$$\begin{aligned} W_{\text{hyb}} &= \int (G^+_{\text{hyb}})_0 (G^-_{\text{hyb}})_{-1} V_{\text{hyb}} \\ &= \int \left(-e^{-\rho - i\sigma} p_\alpha (\nabla^3)^\alpha - \frac{i}{2}\partial x^{\alpha\beta}\nabla_\alpha\nabla_\beta + i p_\alpha \partial^{\alpha\beta}\nabla_\beta \right) V_1 + p_\alpha(\nabla^3)^\alpha V_2\,. \end{aligned} \tag{3.26}$$

3.2.3 Six-dimensional hybrid formalism with harmonic-like constraints

Even though the six-dimensional hybrid formalism presented above preserves manifest SO(1,5) Lorentz invariance, only half of the eight supersymmetries of $d = 6$ $\mathcal{N} = 1$ superspace are manifest, i.e., act geometrically in the target superspace. This can be observed by the fact that only four left-moving θs are present in the worldsheet action (3.12) as fundamental fields.

However, one can proceed as in ref. [44] and add four more left-moving θs and four right-moving $\widehat{\theta}$s, as well as their conjugate momenta as fundamental worldsheet variables to the action. This doubling of fermionic degrees of freedom can be accomplished by appending the index $j = \{1, 2\}$ to $\{p_\alpha, \theta^\alpha\}$, so that we end up with

$$S = \int d^2z \left(\frac{1}{2}\partial x^{\underline{a}}\overline{\partial} x_{\underline{a}} + p_{\alpha j}\overline{\partial}\theta^{\alpha j} + \widehat{p}_{\widehat{\alpha} j}\partial\widehat{\theta}^{\widehat{\alpha} j} \right) + S_{\rho,\sigma} + S_C\,, \tag{3.27}$$

where $p_{\alpha j} = \epsilon_{jk} p^k_\alpha$, $\epsilon_{12} = -\epsilon^{12} = 1$, $\epsilon_{jk}\epsilon^{kl} = \delta^l_j$ and repeated indices are summed over.

Consequently, eq. (3.27) is invariant under the $d = 6$ $\mathcal{N} = 1$ spacetime supersymmetry transformations generated by the charge

$$Q_{\alpha j} = \oint \left(p_{\alpha j} - \frac{i}{2}\epsilon_{jk}\partial x_{\alpha\beta}\theta^{\beta k} - \frac{1}{24}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{jk}\epsilon_{lm}\theta^{\beta k}\theta^{\gamma l}\partial\theta^{\delta m} \right), \tag{3.28}$$

and which satisfy the $d = 6$ $\mathcal{N} = 1$ SUSY algebra

$$\{Q_{\alpha j}, Q_{\beta k}\} = -i\epsilon_{jk} \oint \partial x_{\alpha\beta} \,. \tag{3.29}$$

For the closed string, we also have a left- and a right-moving supersymmetry generator $Q_{\alpha j}$ and $\widehat{Q}_{\widehat{\alpha} j}$, respectively. These charges then generate the $d = 6$ $\mathcal{N} = 2$ supersymmetry and, hence, for Type II strings the amount of SUSY is doubled.

Beyond that, it is convenient to construct extensions of the worldsheet fields $\{p_{\alpha j}, \partial x^{\underline{a}}\}$ that are invariant under the transformations generated by (3.28). One can easily check that this is achieved by the following on-shell spacetime supersymmetric — or just supersymmetric — worldsheet variables

$$d_{\alpha j} = p_{\alpha j} + \frac{i}{2}\epsilon_{jk}\partial x_{\alpha\beta}\theta^{\beta k} + \frac{1}{8}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{jk}\epsilon_{lm}\theta^{\beta k}\theta^{\gamma l}\partial\theta^{\delta m} \,, \tag{3.30a}$$

$$\Pi^{\underline{a}} = \partial x^{\underline{a}} - \frac{i}{2}\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}\theta^{\alpha j}\partial\theta^{\beta k} \,. \tag{3.30b}$$

The six-dimensional worldsheet fields in (3.27) have the following singularities in their OPEs

$$p_{\alpha j}(y)\theta^{\beta k}(z) \sim \delta^k_j \delta^\beta_\alpha (y-z)^{-1} \,, \tag{3.31a}$$

$$\partial x^{\underline{a}}(y)\partial x^{\underline{b}}(z) \sim -\eta^{\underline{ab}}(y-z)^{-2} \,, \tag{3.31b}$$

$$\rho(y)\rho(z) \sim -\log(y-z) \,, \tag{3.31c}$$

$$\sigma(y)\sigma(z) \sim -\log(y-z) \,, \tag{3.31d}$$

where $\eta^{\underline{ab}} = \mathrm{diag}(-,+,+,+,+,+)$ and, in turn, eqs. (3.31) can be used to show that the supersymmetric variables (3.30) satisfy

$$d_{\alpha j}(y)d_{\beta k}(z) \sim (y-z)^{-1} i\epsilon_{jk}\Pi_{\alpha\beta}(z) \,, \tag{3.32a}$$

$$d_{\alpha j}(y)\Pi^{\underline{a}}(z) \sim -(y-z)^{-1} i\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}\partial\theta^{\beta k}(z) \,, \tag{3.32b}$$

$$\Pi^{\underline{a}}(y)\Pi^{\underline{b}}(z) \sim -(y-z)^{-2}\eta^{\underline{ab}}\,, \tag{3.32c}$$

$$d_{\alpha j}(y)\partial\theta^{\beta k}(z) \sim (y-z)^{-2}\delta_j^k\delta_\alpha^\beta\,. \tag{3.32d}$$

Also, notice the following ordering effect using the OPEs of the fundamental fields

$$\oint \frac{dy}{y-z}\,\Pi_{\alpha\beta}(y)d_{\gamma j}(z) = \frac{3i}{4}\epsilon_{jk}\epsilon_{\alpha\beta\gamma\delta}\partial^2\theta^{\delta k}(z)\,, \tag{3.33a}$$

$$\oint \frac{dy}{y-z}\,d_{\gamma j}(y)\Pi_{\alpha\beta}(z) = -\frac{i}{4}\epsilon_{jk}\epsilon_{\alpha\beta\gamma\delta}\partial^2\theta^{\delta k}(z)\,. \tag{3.33b}$$

As we have argued, the worldsheet action (3.27) is invariant under the $d=6$ $\mathcal{N}=1$ spacetime supersymmetry transformations, however, in order to preserve the description of the original six-dimensional hybrid superstring, one must include a set of constraints which reduce the action (3.27) to (3.12). This can be accomplished by the fermionic first-class constraints [44]

$$D_\alpha = d_{\alpha 2} - e^{-\rho-i\sigma}d_{\alpha 1}\,, \tag{3.34}$$

and since

$$[D_\alpha, \theta^{\beta 2}] = \delta_\alpha^\beta\,, \tag{3.35}$$

one can use (4.180) to gauge-fix (3.27) to (3.12). Therefore, working with the action (3.27) and the harmonic constraint D_α, it is possible to manifestly preserve all of the $d=6$ $\mathcal{N}=1$ supersymmetries.[5]

In this case, the $\mathcal{N}=2$ constraints (3.13) are modified and can be written in a manifestly spacetime supersymmetric form as [44]

$$\begin{aligned} T_{\text{hyb}} &= -\frac{1}{2}\Pi^{\underline{a}}\Pi_{\underline{a}} - d_{\alpha 1}\partial\theta^{\alpha 1} - e^{-\rho-i\sigma}d_{\alpha 1}\partial\theta^{\alpha 2} - \frac{1}{2}\partial\rho\partial\rho - \frac{1}{2}\partial\sigma\partial\sigma \\ &\quad + \frac{3}{2}\partial^2(\rho+i\sigma) + T_C\,, \end{aligned} \tag{3.36a}$$

$$\begin{aligned} G^+_{\text{hyb}} &= -(d_1)^4 e^{-2\rho-i\sigma} + \frac{i}{2}d_{\alpha 1}d_{\beta 1}\Pi^{\alpha\beta}e^{-\rho} + d_{\alpha 1}\partial\theta^{\alpha 2}\partial(\rho+i\sigma)e^{-\rho} + d_{\alpha 1}\partial^2\theta^{\alpha 2}e^{-\rho} \\ &\quad - \frac{1}{2}\Pi^{\underline{a}}\Pi_{\underline{a}}e^{i\sigma} - d_{\alpha 1}\partial\theta^{\alpha 1}e^{i\sigma} - \frac{1}{2}\partial(\rho+i\sigma)\partial(\rho+i\sigma)e^{i\sigma} \\ &\quad + \frac{1}{2}\partial^2(\rho+i\sigma)e^{i\sigma} + G^+_C\,, \end{aligned} \tag{3.36b}$$

[5] We note in passing that there exists a similarity transformation with the property $e^S d_{\alpha 2}e^{-S} = D_\alpha$ where $S = \theta^{\alpha 2}d_{\alpha 1}e^{-\rho-i\sigma} - \frac{i}{2}\theta^{\alpha 2}\theta^{\beta 2}\Pi_{\alpha\beta}e^{-\rho-i\sigma} + (\theta^2)^3_\alpha\partial\theta^{\alpha 1}e^{-\rho-i\sigma} + \frac{1}{2}(\theta^2)^4\partial(\rho+i\sigma)e^{-2\rho-2i\sigma}$.

$$G^-_{\text{hyb}} = e^{-i\sigma} + G^-_C\,, \tag{3.36c}$$
$$J_{\text{hyb}} = \partial(\rho + i\sigma) + J_C\,, \tag{3.36d}$$

which, as in the previous section, still obey a twisted $c = 6$ $\mathcal{N} = 2$ SCA, and we defined $(d_1)^4 = \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta} d_{\alpha 1} d_{\beta 1} d_{\gamma 1} d_{\delta 1}$. Let us mention that when one gauge fix $\theta^{\alpha 2} = 0$, the constraints (3.36) reduce to the ones compatible with the action (3.12), i.e., eqs. (3.13). Note that the stress-tensor T_{hyb} is the expected stress tensor when $D_\alpha = 0$, because

$$-d_{\alpha 1}\partial\theta^{\alpha 1} - e^{-\rho - i\sigma} d_{\alpha 1}\partial\theta^{\alpha 2} = -d_{\alpha j}\partial\theta^{\alpha j} + D_\alpha \partial\theta^{\alpha 2}\,. \tag{3.37}$$

It is also important to be aware that the $\mathcal{N} = 2$ algebra is preserved independently of how one chooses to gauge-fix the local symmetry generated by D_α. This is because the form of the $\mathcal{N} = 2$ generators (3.36) was chosen so that they have no poles with the harmonic-like constraint (4.180). The non-trivial part in showing this is for the generator G^+_{hyb}, nonetheless, it becomes manifest by noting the property that one can write G^+_{hyb} as

$$G^+_{\text{hyb}} = -\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}[D_\alpha, \{D_\beta, [D_\gamma, \{D_\delta, e^{2\rho + 3i\sigma}\}]\}] + G^+_C\,, \tag{3.38}$$

where the graded bracket $[D_\alpha, \mathcal{O}\}(z) = \oint dy\, D_\alpha(y)\mathcal{O}(z)$ denotes the simple pole in the OPE between D_α and $\mathcal{O}$.

The details of the calculation establishing eq. (3.38) are given in Appendix F. To the knowledge of the author, this is the first time that eq. (3.38) is proven considering the normal-ordering contributions. Note also the similarity between identities (3.38) and (3.17).

For the massless compactification-independent sector of the open superstring, the vertex operator now reads

$$V_{\text{hyb}} = \sum_{n=-\infty}^{\infty} V_n(x, \theta) e^{n(\rho + i\sigma)}\,, \tag{3.39}$$

which takes the same form as in eq. (3.21), but now $V_n(x, \theta)$ depends on the zero modes of $\{x^{\underline{a}}, \theta^{\alpha j}\}$. Therefore, it contains the eight fermionic θ coordinates of $d = 6$ $\mathcal{N} = 1$ superspace. Of course, contrasting with the hybrid formalism of the previous section, in the present case the physical states V_{hyb} also have to be annihilated by D_α.

It is interesting to note what is the effect of imposing the constraint D_α for $V_{\rm hyb}$ of eq. (3.39). To do that, let us first define the new superspace variables

$$\theta^{\alpha-} = \frac{1}{2}\big(\theta^{\alpha 2} - e^{\rho+i\sigma}\theta^{\alpha 1}\big)\,, \qquad \theta^{\alpha+} = \frac{1}{2}\big(\theta^{\alpha 1} + e^{-\rho-i\sigma}\theta^{\alpha 2}\big)\,. \tag{3.40}$$

The condition that $V_{\rm hyb}$ has no poles with D_α implies that

$$\big(\nabla_{\alpha 2} - e^{-\rho-i\sigma}\nabla_{\alpha 1}\big)V_{\rm hyb} = 0\,, \tag{3.41}$$

where $\nabla_{\alpha j} = \frac{\partial}{\partial\theta^{\alpha j}} - \frac{i}{2}\epsilon_{jk}\theta^{\beta k}\partial_{\alpha\beta}$ is the zero mode of $d_{\alpha j}$ acting on $V_{\rm hyb}$. By defining $x'^{\underline{a}} = x^{\underline{a}}$ and then doing the shift

$$x'^{\underline{a}} + i\theta^{\alpha-}\theta^{\beta+}\sigma^{\underline{a}}_{\alpha\beta} \mapsto x^{\underline{a}}\,, \tag{3.42}$$

we learn that $V_{\rm hyb}$ is independent of $\theta^{\alpha-}$ and, for that reason, it is a function of only the zero modes of $\{x^{\underline{a}}, \theta^{\alpha+}\}$. As a consequence, after identifying $\theta^{\alpha+} = \theta^\alpha$ the component fields of $V_{\rm hyb}$ in (3.39) can be related to the component fields of $V_{\rm hyb}$ in (3.21), therefore, we recover the usual six-dimensional description of the vertex operator in Section 3.2.1. Nonetheless, the identification of the component fields is only possible after imposing $D_\alpha = 0$.

From eqs. (3.36), one can construct the remaining twisted small $c = 6$ $\mathcal{N} = 4$ generators, and in the gauge where $(G^+_{\rm hyb})_0(\widetilde{G}^+_{\rm hyb})_0 V_{\rm hyb} = 0$ the integrated vertex operator for the massless compactification-independent states of the open superstring now reads [44]

$$\begin{aligned}
W_{\rm hyb} &= \int (G^+_{\rm hyb})_0(G^-_{\rm hyb})_{-1}V_{\rm hyb}\\
&= \int \bigg[\Big(-\frac{1}{6}e^{-\rho-i\sigma}\epsilon^{\alpha\beta\gamma\delta}d_{\alpha 1}\nabla_{\beta 1}\nabla_{\gamma 1}\nabla_{\delta 1} - \frac{i}{2}\Pi^{\alpha\beta}\nabla_{\alpha 1}\nabla_{\beta 1}\\
&\quad + i d_{\alpha 1}\partial^{\alpha\beta}\nabla_{\beta 1} - \partial\theta^{\alpha 2}\nabla_{\alpha 1}\Big)V_1 + \frac{1}{6}\epsilon^{\alpha\beta\gamma\delta}d_{\alpha 1}\nabla_{\beta 1}\nabla_{\gamma 1}\nabla_{\delta 1}V_2\bigg]\\
&= \int \bigg[\frac{1}{6}\epsilon^{\alpha\beta\gamma\delta}\Big(d_{\alpha 2}\nabla_{\beta 1}\nabla_{\gamma 1}\nabla_{\delta 2} - d_{\alpha 1}\nabla_{\beta 2}\nabla_{\gamma 2}\nabla_{\delta 1}\Big)\\
&\quad + \frac{i}{4}\Pi^{\alpha\beta}[\nabla_{\alpha 1}, \nabla_{\beta 2}] - \frac{1}{2}\partial\theta^{\alpha 1}\nabla_{\alpha 1} + \frac{1}{2}\partial\theta^{\alpha 2}\nabla_{\alpha 2}\bigg]V_0\,,
\end{aligned} \tag{3.43}$$

where the supersymmetric derivative $\nabla_{\alpha j}$ satisfy the algebra $\{\nabla_{\alpha j}, \nabla_{\beta k}\} = -i\epsilon_{jk}\partial_{\alpha\beta}$. To arrive at the last equality in (3.43), we subtracted a total derivative and used the relations implied by the constraint (4.180), namely, $\nabla_{\alpha 1}V_1 = -\nabla_{\alpha 2}V_0$ and

$\nabla_{\alpha 1} V_2 = \nabla_{\alpha 2} V_1$.

3.3 Extended hybrid formalism

In spite of the fact that we have described the worldsheet action, $\mathcal{N} = 2$ constraints and compactification-independent vertex operators while preserving manifest $d = 6$ $\mathcal{N} = 1$ supersymmetry in Section 3.2.3, it remains unclear what are the rules to compute correlation functions using the superconformal generators and vertex operators depending on all eight θ coordinates of $d = 6$ $\mathcal{N} = 1$ superspace.

In addition, it is not evident if there is a relation between the vertex operator (3.39) and the superfields appearing in superspace descriptions of $d = 6$ SYM [43] [42]. As a consequence, one cannot identify what each component of the superfield (3.39) corresponds to before using the constraint $D_\alpha = 0$ to make contact with (3.21), which depends on only half of the θs. One of the purposes of this section is to clarify and understand how one can overcome these drawbacks by relaxing the constraint $D_\alpha = 0$ in the definition of physical states.

3.3.1 Worldsheet variables

To the worldsheet theory (3.27), we introduce a bosonic spinor λ^α of conformal weight zero and its conjugate momenta w_α of conformal weight one. As we will momentarily see, the ghost λ^α will be responsible for relaxing the constraint D_α. We also include the non-minimal variables $\{\overline{\lambda}_\alpha, r_\alpha\}$ [45] of conformal weight zero, as well as their conjugate momenta $\{\overline{w}^\alpha, s^\alpha\}$ of conformal weight one. The fields $\{s^\alpha, r_\alpha\}$ are worldsheet fermions and $\{\overline{w}^\alpha, \overline{\lambda}_\alpha\}$ bosons.

The worldsheet action now takes the form

$$S = \int d^2z \left(\frac{1}{2} \partial x^{\underline{a}} \overline{\partial} x_{\underline{a}} + p_{\alpha j} \overline{\partial} \theta^{\alpha j} + w_\alpha \overline{\partial} \lambda^\alpha + s^\alpha \overline{\partial} r_\alpha + \overline{w}^\alpha \overline{\partial} \overline{\lambda}_\alpha \right.$$
$$\left. + \widehat{p}_{\widehat{\alpha} j} \partial \widehat{\theta}^{\widehat{\alpha} j} + \widehat{w}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}} + \widehat{s}^{\widehat{\alpha}} \partial \widehat{r}_{\widehat{\alpha}} + \widehat{\overline{w}}^{\widehat{\alpha}} \partial \widehat{\overline{\lambda}}_{\widehat{\alpha}} \right) + S_{\rho,\sigma} + S_C \,, \tag{3.44}$$

where the "hatted" fields are right-moving and, for simplicity, will be ignored in what follows. The singularities in the OPEs of the new variables are

$$w_\alpha(y) \lambda^\beta(z) \sim -\delta_\alpha^\beta (y - z)^{-1} \,, \tag{3.45a}$$
$$\overline{w}^\alpha(y) \overline{\lambda}_\beta(z) \sim -\delta_\beta^\alpha (y - z)^{-1} \,, \tag{3.45b}$$

$$s^\alpha(y) r_\beta(z) \sim \delta^\alpha_\beta (y-z)^{-1}\,, \tag{3.45c}$$

and, unlike in [45], $\{\lambda^\alpha, \overline{\lambda}_\alpha, r_\alpha\}$ are not constrained. Note further that, as opposed to the worldsheet action (3.27), the stress-tensor of (3.44) has vanishing central charge.

3.3.2 Extended twisted $c = 6$ $\mathcal{N} = 2$ generators

With these additional variables, it is still possible to construct superconformal generators satisfying a twisted $c = 6$ $\mathcal{N} = 2$ SCA as in Section 3.2.

In this case, we have

$$T = T_{\text{hyb}} - D_\alpha \partial\theta^{\alpha 2} - w_\alpha \partial\lambda^\alpha - \overline{w}^\alpha \partial\overline{\lambda}_\alpha - s^\alpha \partial r_\alpha\,, \tag{3.46a}$$

$$G^+ = G^+_{\text{hyb}} - \lambda^\alpha D_\alpha - \overline{w}^\alpha r_\alpha\,, \tag{3.46b}$$

$$G^- = G^-_{\text{hyb}} + w_\alpha \partial\theta^{\alpha 2} + s^\alpha \partial\overline{\lambda}_\alpha\,, \tag{3.46c}$$

$$J = J_{\text{hyb}} - w_\alpha \lambda^\alpha - s^\alpha r_\alpha\,, \tag{3.46d}$$

where $\{T_{\text{hyb}}, G^+_{\text{hyb}}, G^-_{\text{hyb}}, J_{\text{hyb}}\}$ are the $c = 6$ $\mathcal{N} = 2$ generators of eqs. (3.36). Note that T is now the usual stress-tensor, because the terms added in (3.46a) to T_{hyb} precisely cancel the atypical contribution in (3.37). Explicitly, we now have

$$\begin{aligned} T = &-\frac{1}{2}\Pi^{\underline{a}}\Pi_{\underline{a}} - d_{\alpha j}\partial\theta^{\alpha j} - w_\alpha \partial\lambda^\alpha - \overline{w}^\alpha \partial\overline{\lambda}_\alpha - s^\alpha \partial r_\alpha \\ &- \frac{1}{2}\partial\rho\partial\rho - \frac{1}{2}\partial\sigma\partial\sigma + \frac{3}{2}\partial^2(\rho + i\sigma) + T_C\,. \end{aligned} \tag{3.47}$$

Of course, the superconformal generator G^+ continues to be nilpotent. This is easy to see from that fact that G^+_{hyb} has no poles with itself, no poles with D_α and the constraint D_α is first-class.

It is important to comment on the significance of each of the contributions appearing in the fermionic generator G^+. The zero mode of G^+_{hyb} is related to the BRST operator Q_{RNS} of the RNS formalism in the gauge where $\theta^{\alpha 2} = 0$, this follows from the fact that the hybrid variables are related to the gauge-fixed RNS variables through a field redefinition [46].

The term $-\lambda^\alpha D_\alpha$ in G^+ is necessary for the reason that we are relaxing the constraint D_α. As a consequence, the condition $D_\alpha = 0$ from (4.180) does not need to be imposed "by hand" in our definition of physical states from now on (see Section 3.3.3). The last contribution, $-\overline{w}^\alpha r_\alpha$, is the non-minimal/topological term

[45], and it implies that the cohomology of $(G^+)_0$ is independent of $\{\overline{w}^\alpha, \overline{\lambda}_\alpha, s^\alpha, r_\alpha\}$ through the usual quartet argument. This term is required in order to get a $c = 6$ $\mathcal{N} = 2$ SCA and it will play a key role in defining a spacetime supersymmetric prescription for scattering amplitude computations in Section 3.3.4.

We should emphasize that even though we have an $\mathcal{N} = 2$ SCA with critical central charge ($c = 6$) in eqs. (3.46), the physical states of the superstring cannot be defined as $\mathcal{N} = 2$ primaries like in the hybrid formalism [17]. The reason for this is because, by the quartet machanism, the cohomology of $(G^+)_0$ is guaranteed to be independent of the non-minimal/topological variables [45]. However, this mechanism has nothing to say about the primaries of the $\mathcal{N} = 2$ algebra, i.e., if the they are preserved or not after the worldsheet theory is modified. Therefore, when studying vertex operators of the superstring, one must look for states in the cohomology of $(G^+)_0$.

As an additional observation, let us sketch a direct way to arrive at the supercurrent (3.46b) from the six-dimensional hybrid formalism: by adding non-minimal variables and performing a suitable similarity transformation. Start with G^+_{hyb} in eq. (3.17) and add the non-minimal variables $\{p_{\alpha 2}, \theta^{\alpha 2}, w_\alpha, \lambda^\alpha, \overline{w}^\alpha, \overline{\lambda}_\alpha, s^\alpha, r_\alpha\}$, so that the supercurrent becomes

$$G^{+\prime} = G^+_{\text{hyb}} - \lambda^\alpha p_{\alpha 2} - \overline{w}^\alpha r_\alpha \,. \tag{3.48}$$

Then, after performing the similarity transformation $e^{R_2} e^{R_1} G^{+\prime} e^{-R_1} e^{-R_2} \to G^{+\prime}$ where $R_1 = -Q^{\text{hyb}}_{\alpha 2} \theta^{\alpha 2}$ and $R_2 = -\frac{i}{2} \partial x_{\alpha\beta} \theta^{\alpha 1} \theta^{\alpha 2}$, one learns that $G^{+\prime} = G^+$ in (3.46b) up to terms proportional to $\theta^{\alpha 2}$.[6] This procedure is similar to the construction adopted in refs. [47] [48] in relating the RNS formalism with the pure spinor formalism. Moreover, we also learn that $e^{R_2} e^{R_1} p_{\alpha 2} e^{-R_1} e^{-R_2} = e^{R_2}(p_{\alpha 2} - Q^{\text{hyb}}_{\alpha 2}) e^{-R_2} = D_\alpha$ up to terms proportional to the non-minimal variable $\theta^{\alpha 2}$. The charge $Q^{\text{hyb}}_{\alpha 2}$ was defined in eq. (3.16), therefore, we conclude that the constraint D_α is related to the "non-standard" SUSYs of the hybrid formalism.

Starting from the $d = 10$ pure spinor formalism, there have been other approaches to describe the superstring in a six-dimensional background with manifest $d = 6$ $\mathcal{N} = 1$ supersymmetry [49] [50] [51] [52]. In these works, the non-minimal variables are absent but the ghosts $\{w_\alpha, \lambda^\alpha\}$ usually appear from the decomposition of the $d = 10$ pure spinor $\lambda^{\overline{\alpha}}$, $\overline{\alpha} = \{1 \text{ to } 16\}$, in terms of $SO(1,5)$

[6] Since the BRST operator G^+ is supersymmetric, one can consider an additional similarity transformation to restore the missing $\theta^{\alpha 2}$ terms, analogously as in ref. [47].

spinors. Particularly, in ref. [51] a BRST operator of the form $Q_{\text{PS}} = \oint \lambda^\alpha D_\alpha$ was proposed as the dimensional reduction of the BRST operator in the $d = 10$ pure spinor formalism. In ref. [52], it was also considered adding the ghosts $\{w_\alpha, \lambda^\alpha\}$ to the superconformal generators (3.36). The advantage of the approach detailed below is that we will be able to explicit write a BRST invariant superstring vertex operator in terms of $d = 6$ $\mathcal{N} = 1$ superfields and the manifest spacetime supersymmetric worldsheet variables.

3.3.3 Massless compactification-independent vertex operators

We consider the compactification-independent physical states with conformal weight zero at zero momentum for the open superstring or holomorphic sector, so that we are seeking for a vertex operator U which describes the $d = 6$ SYM multiplet. We will start by specifying what are the physical state conditions the vertex operator has to fulfill. Then write the vertex in terms of $d = 6$ $\mathcal{N} = 1$ superfields depending on the eight θ coordinates. After that, it will be shown that BRST invariance of U reproduces the on-shell $d = 6$ SYM equations in superspace.

Since we have a nilpotent BRST charge $(G^+)_0$, we can require physical unintegrated vertex operators U to be ghost-number-one states in the cohomology of $(G^+)_0$. Without loss of generality, the ghost-number current is defined to be the $\mathrm{U}(1)$ generator of the $\mathcal{N} = 2$ algebra, eq. (3.46d). Moreover, the stress-tensor T has vanishing conformal anomaly, it is then consistent to require U to be a conformal weight zero primary field as well. When these conditions are satisfied, and given the fact that $\{(G^+)_0, (G^-)_0\} = (T)_0$, the superconformal generator $(G^-)_0$ has to annihilate the state U, which means that U is in the covariant Lorenz gauge [53]. The latter condition is analogous to the $b_0 = 0$ constraint in bosonic string theory.

For the compactification-independent massless sector of the open superstring, the manifestly spacetime supersymmetric ghost-number-one unintegrated vertex operator U in the cohomology of $(G^+)_0$ takes the form

$$
\begin{aligned}
U = {} & -\lambda^\alpha\big(A_{\alpha 2} - A_{\alpha 1} e^{-\rho - i\sigma}\big) + \big(\partial\theta^{\alpha 1} A_{\alpha 1} + \Pi^{\underline{a}} A_{\underline{a}} + d_{\alpha 1} W^{\alpha 1}\big) e^{i\sigma} - d_{\alpha 1} W^{\alpha 2} \partial(i\sigma) e^{-\rho} \\
& - \partial\theta^{\alpha 2} A_{\alpha 1} \partial(\rho + i\sigma) e^{-\rho} + i d_{\alpha 1} \Pi^{\alpha\beta} A_{\beta 1} e^{-\rho} - \frac{i}{2} d_{\alpha 1} d_{\beta 1} A^{\alpha\beta} e^{-\rho} - \partial^2\theta^{\alpha 2} A_{\alpha 1} e^{-\rho} \\
& + \partial d_{\alpha 1}\big(W^{\alpha 2} - i\partial^{\alpha\beta} A_{\beta 1}\big) e^{-\rho} + \frac{i}{2} \partial\Pi^{\alpha\beta} \nabla_{\alpha 1} A_{\beta 1} e^{-\rho} + d_{\alpha 1}\big(- 2W^{\alpha 2} \\
& + i\partial^{\alpha\beta} A_{\beta 1}\big) \partial\rho e^{-\rho} - \frac{i}{2} \Pi^{\alpha\beta} \nabla_{\alpha 1} A_{\beta 1} \partial\rho e^{-\rho} - \frac{i}{4} \partial^{\alpha\beta} \nabla_{\alpha 1} A_{\beta 1} \partial^2 e^{-\rho} + (d_1^3)^\alpha A_{\alpha 1} e^{-2\rho - i\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^{\alpha\beta\gamma\delta}\Big(-\frac{1}{4}d_{\alpha1}d_{\beta1}\nabla_{\gamma1}A_{\delta1}\partial e^{-2\rho-i\sigma}-\frac{1}{4}\partial(d_{\alpha1}d_{\beta1})\nabla_{\gamma1}A_{\delta1}e^{-2\rho-i\sigma}\\
&-\frac{1}{12}\partial^2 d_{\alpha1}\nabla_{\beta1}\nabla_{\gamma1}A_{\delta1}e^{-2\rho-i\sigma}-\frac{1}{6}\partial d_{\alpha1}\nabla_{\beta1}\nabla_{\gamma1}A_{\delta1}\partial e^{-2\rho-i\sigma}\\
&-\frac{1}{12}d_{\alpha1}\nabla_{\beta1}\nabla_{\gamma1}A_{\delta1}\partial^2 e^{-2\rho-i\sigma}\Big)+\frac{1}{4}(\nabla_1^3)^\alpha A_{\alpha1}\frac{1}{6}\partial^3 e^{-2\rho-i\sigma}\,,
\end{aligned}
\tag{3.49}
$$

where $A_{\underline{a}}$ is the superspace gauge field, $W^{\alpha j}$ is the superspace spinor field-strength and $F_{\underline{ab}}$ is the superspace field-strength.[7] The first components of the superfields $\{A_{\underline{a}}, W^{\alpha j}, F_{\underline{ab}}\}$ are the gluon, the gluino and the gluon field-strength, respectively. These superfields are defined in terms of the superspace gauge field $A_{\alpha j}$. In linearized form, we have

$$A_{\underline{a}} = -\frac{i}{4}\epsilon^{jk}\sigma_{\underline{a}}^{\alpha\beta}(\nabla_{\alpha j}A_{\beta k}+\nabla_{\beta k}A_{\alpha j})\,, \tag{3.50a}$$

$$W^{\alpha j} = \frac{i}{3}\epsilon^{jk}\sigma^{\underline{a}\alpha\beta}(\partial_{\underline{a}}A_{\beta k}-\nabla_{\beta k}A_{\underline{a}})\,, \tag{3.50b}$$

$$F_{\underline{ab}} = \partial_{\underline{a}}A_{\underline{b}}-\partial_{\underline{b}}A_{\underline{a}}\,. \tag{3.50c}$$

It is easy to see that U is annihilated by $(G^-)_0$ and so we have $\partial^{\underline{a}}A_{\underline{a}}=0$, which is the usual Lorenz gauge condition. The non-trivial part is showing that BRST invariance of U implies the linearized $d=6$ SYM equations of motion [43] [42]

$$(\sigma^{\underline{abc}})^{\alpha\beta}(\nabla_{\alpha j}A_{\beta k}+\nabla_{\beta k}A_{\alpha j})=0\,, \tag{3.51a}$$

$$\nabla_{\alpha j}W^{\beta k}+\frac{i}{2}\delta_j^k(\sigma_{\underline{ab}})^\beta{}_\alpha F^{\underline{ab}}=0\,, \tag{3.51b}$$

where $(\sigma^{\underline{abc}})^{\alpha\beta}=\frac{i}{3!}(\sigma^{[\underline{a}}\sigma^{\underline{b}}\sigma^{\underline{c}]})^{\alpha\beta}$ is the symmetric anti-self-dual three-form and $(\sigma_{\underline{ab}})^\beta{}_\alpha=\frac{i}{2}(\sigma^{[\underline{a}}\sigma^{\underline{b}]})^\beta{}_\alpha$ is the generator of Lorentz transformations.

The calculation leading to (3.51) is straightforward but tedious. It involves taking care of various normal-ordering contributions. Let us briefly outline at which steps some of the above equations can be obtained. For example, eq. (3.51a) comes from the terms with $\lambda^\alpha\lambda^\beta$ in $(G^+)_0U$, and eq. (3.51b) can be obtained by the terms proportional to $\lambda^\alpha d_{\beta1}e^{i\sigma}$, $\lambda^\alpha\partial d_{\beta1}e^{-\rho}$, $\lambda^\alpha d_{\beta1}\partial(i\sigma)e^{-\rho}$ and $\lambda^\alpha\partial^2 d_{\beta1}e^{-2\rho-i\sigma}$.

Note further that U in (3.49) is defined up to a gauge transformation $\delta U=(G^+)_0\Lambda$ for some conformal weight zero and U(1)-charge zero gauge parameter Λ, and U is also annihilated by $(\widetilde{G}^+_{\text{hyb}})_0$ of (3.15a), a condition that will become more

[7] See Appendix K for a review of $d=6$ $\mathcal{N}=1$ super-Yang-Mills.

clear when we write the amplitude prescription (3.60) in the following section.[8] Taking Λ to be a function of the zero modes of $\{x^{\underline{a}}, \theta^{\alpha j}\}$, we have that

$$\delta U = -\lambda^{\alpha}\left(\nabla_{\alpha 2}\Lambda - \nabla_{\alpha 1}\Lambda e^{-\rho - i\sigma}\right) + \left(\partial\theta^{\alpha 1}\nabla_{\alpha 1}\Lambda + \Pi^{\underline{a}}\partial_{\underline{a}}\Lambda\right)e^{i\sigma} + \dots\,, \tag{3.52}$$

which precisely reproduces the gauge transformations (K.3) of the $d = 6$ $\mathcal{N} = 1$ superspace description, i.e., $\delta A_{\alpha j} = \nabla_{\alpha j}\Lambda$ and $\delta A_{\underline{a}} = \partial_{\underline{a}}\Lambda$.

For scattering amplitude computations, vertex operators in integrated form are necessary. As we have an $\mathcal{N} = 2$ SCA (3.46), it is straightforward to define integrated vertex operators. They are given by

$$W = \int (G^{-})_{-1} U\,, \tag{3.53}$$

which, for the compactification-independent massless sector of the open superstring, takes the simple form

$$W = \int \left(\partial\theta^{\alpha j} A_{\alpha j} + \Pi^{\underline{a}} A_{\underline{a}} + d_{\alpha 1} W^{\alpha 1} + d_{\alpha 1} e^{-\rho - i\sigma} W^{\alpha 2}\right). \tag{3.54}$$

Note that only the first four terms in (3.49) contribute to the integrated vertex W. Not surprisingly, the integrated vertex (3.54) has a similar structure as in the first equality of eq. (3.43).

The gauge transformations of W are given by $\delta W = (G^{+})_0 \Omega^{-}$ for some conformal weight one and U(1)-charge minus one gauge parameter Ω^{-}. Taking $\Omega^{-} = -w_{\alpha} W^{\alpha 2}$, which is annihilated by $(\widetilde{G}^{+}_{\text{hyb}})_0$, we can write W as

$$\begin{aligned} W = \int \Bigg(& \partial\theta^{\alpha j} A_{\alpha j} + \Pi^{\underline{a}} A_{\underline{a}} + d_{\alpha j} W^{\alpha j} - \frac{i}{2} N_{\underline{ab}} F^{\underline{ab}} \\ & - \frac{i}{2} w_{\alpha} d_{\beta 1} d_{\gamma 1} \partial^{\beta\gamma} W^{\alpha 2} e^{-\rho} + w_{\alpha} \Pi^{\underline{a}} \partial_{\underline{a}} W^{\alpha 2} e^{i\sigma} \Bigg)\,, \end{aligned} \tag{3.55}$$

where $N_{\underline{ab}} = w_{\alpha} (\sigma_{\underline{ab}})^{\alpha}{}_{\beta} \lambda^{\alpha}$.

From an argument concerning the level of the Lorentz currents in the RNS and pure spinor formalisms, the first line of (3.55) takes the form conjectured in ref. [54, footnote 3] to be the correct integrated vertex operator for the massless sector of the open superstring compactified to six dimensions.

[8] When translated to the RNS variables, the condition $(\widetilde{G}^{+}_{\text{hyb}})_0 U = 0$ is equivalent as saying that U lives in the small Hilbert space, i.e., it is annihilated by the η_0-ghost [20]. See also Appendix E.

3.3.4 Tree-level scattering amplitudes

In Section 3.2, we introduced an unintegrated vertex operator V_{hyb} with zero $\text{U}(1)$-charge, eq. (3.21). When on-shell, this vertex operator was shown to describe $d = 6$ SYM. Moreover, one can show that there exists a gauge choice where (3.21) can be taken to be an $\mathcal{N} = 2$ superconformal primary field with respect to the SCA (3.13) [55].

In terms of V_{hyb}, the tree-level three-point amplitude prescription for the massless states in the hybrid formalism of Section 3.2.1 is [17]

$$\Big\langle V_{\text{hyb}}(z_1)\big((\widetilde{G}^+_{\text{hyb}})_0 V_{\text{hyb}}\big)(z_2)U_{\text{hyb}}(z_3)\Big\rangle\,, \tag{3.56}$$

where $\langle e^{3\rho+3i\sigma}J_C^{++}(\theta)^4\rangle = 1$ with $(\theta)^4 = \frac{1}{24}\epsilon_{\alpha\beta\gamma\delta}\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta$ and we defined $U_{\text{hyb}} = (G^+_{\text{hyb}})_0 V_{\text{hyb}}$. It is interesting to note that, in some gauge choice, U_{hyb} in (3.56) looks very similar to U in (3.49), at least in the ghost structure when we take $\lambda^\alpha = 0$. However, since vertex operators only depend on four θ coordinates, they do not have a simple transformation rule under all spacetime SUSYs.

We can try to use the elements of the hybrid formalism outlined in the paragraph above to formulate a prescription for calculating scattering amplitudes in terms of the superconformal generators (3.46) and the vertex operators in (3.49) and (3.54), which are constructed from the manifestly spacetime supersymmetric variables. In this setting, recall that the eight supersymmetry generators are given by (3.28), as opposed to the ghost-dependent SUSYs (3.16) in the six-dimensional hybrid description.

In view of that, it is tempting to conjecture that U can be written as $U = (G^+)_0 V$ for some V which is also an $\mathcal{N} = 2$ primary field with respect to the SCA (3.46). Unfortunately, we could not accomplish this much and find a V with both of these properties. Nonetheless, it is possible to find a conformal weight zero and $\text{U}(1)$-charge zero field V such that $U = (G^+)_0 V$ and, as we will see, this is enough to define a consistent tree-level scattering amplitude prescription.

Consider

$$V(z) = \oint \frac{dy}{y-z}\big(-(\theta^1)^4 e^{2\rho+i\sigma}\big)(y)U(z)\,, \tag{3.57}$$

and note that $(G^+)_0 V = U$ by using the fact that $(G^+)_0$ annihilates U and the property

$$(G^+)_0\Big(-(\theta^1)^4 e^{2\rho+i\sigma}\Big) = 1\,. \tag{3.58}$$

Explicitly, the field V is given by

$$\begin{aligned} V = & -\lambda^\alpha(\theta^1)^4 A_{\alpha 2} e^{2\rho+i\sigma} + (\theta^1)^3_\alpha W^{\alpha 1} e^{2\rho+2i\sigma} + \Big(\frac{i}{2}\theta^{\alpha 1}\theta^{\beta 1} A_{\alpha\beta} \\ & - \frac{i}{2}(\theta^1)^4 \partial^{\alpha\beta}\nabla_{\alpha 1} A_{\beta 1}\Big) e^{\rho+i\sigma} + \theta^{\alpha 1} A_{\alpha 1} + \frac{1}{2}\theta^{\alpha 1}\theta^{\beta 1}\nabla_{\alpha 1} A_{\beta 1} \\ & - \frac{1}{6}\theta^{\alpha 1}\theta^{\beta 1}\theta^{\gamma 1}\nabla_{\alpha 1}\nabla_{\beta 1} A_{\gamma 1} + \frac{1}{4}(\theta^1)^4 (\nabla^3_1)^\alpha A_{\alpha 1}\,, \end{aligned} \tag{3.59}$$

where $(\theta^1)^4 = \frac{1}{24}\epsilon_{\alpha\beta\gamma\delta}\theta^{\alpha 1}\theta^{\beta 1}\theta^{\gamma 1}\theta^{\delta 1}$ and $(\theta^1)^3_\alpha = \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta}\theta^{\beta 1}\theta^{\gamma 1}\theta^{\delta 1}$. Note that V has a different ghost structure than (3.22).

In close analogy with (3.56), the spacetime supersymmetric tree-level three-point amplitude is defined as

$$\mathcal{A}_3 = \int [d\lambda][d\overline{\lambda}] d^4 r d^8\theta\, \mathcal{R}\Big\langle V(z_1)\big((\widetilde{G}^+_{\text{hyb}})_0 V\big)(z_2) U(z_3)\Big\rangle\,, \tag{3.60}$$

where V is given by (3.59), G^+_{hyb} is given by (3.15a) and $U = (G^+)_0 V$ is the ghost number one vertex operator in eq. (3.49). We also define $\langle e^{3\rho+3i\sigma} J^{++}_C\rangle = 1$, due to the anomaly in the U(1) current.

Since the bosonic variables λ^α and $\overline{\lambda}_\alpha$ are non-compact, a regularization factor $\mathcal{R} = \exp((G^+)_0\chi)$ needs to be introduced. We will take $\chi = \overline{\lambda}_\alpha\theta^{\alpha 2}$ [45], so that one finds

$$\mathcal{R} = \exp\big(-\lambda^\alpha\overline{\lambda}_\alpha + r_\alpha\theta^{\alpha 2}\big)\,. \tag{3.61}$$

For simplicity, the integration over the $x^{\underline{a}}$ zero modes is being ignored, since it is done in the standard manner [23]. Given that the expression inside brackets is BRST invariant and $\mathcal{R} = 1 + (G^+)_0(\ldots)$, the amplitude (3.60) is independent of χ as long as χ is annihilated by $(G^+_{\text{hyb}})_0$.

Despite the asymmetric appearance, the amplitude (3.60) is symmetric in the three insertions. This is easy to see by noting that $(\widetilde{G}^+_{\text{hyb}})_0 U = (\widetilde{G}^+_{\text{hyb}})_0\chi = 0$ and $\{(\widetilde{G}^+_{\text{hyb}})_0, (G^+)_0\} = 0$. As long as one chooses χ such that $(\widetilde{G}^+_{\text{hyb}})_0\chi = 0$, the amplitude (3.60) will be independent of the choice of χ. Since the $(\widetilde{G}^+_{\text{hyb}})_0$ cohomology is trivial one can even choose χ to be exact.

From the $c = 6$ $\mathcal{N} = 2$ SCA (3.46), it is straightforward to use the procedure outlined in Appendix H and construct the remaining generators of the small $c = 6$ $\mathcal{N} = 4$ SCA. In such a case, one could have thought that it would be possible to define the amplitude (3.60) with the superconformal generator $\widetilde{G}^+$ of the $\mathcal{N} = 4$ algebra associated with (3.46) instead of $\widetilde{G}^+_{\text{hyb}}$ in (3.15a). However, it turns out that an amplitude defined in this way would give a vanishing result. The reason for this is that $\widetilde{G}^+$ involves an overall factor containing $\delta^4(r)$,[9] but we already have the four zero modes of r_α and $\theta^{\alpha 2}$ coming from the regulator $\mathcal{R}$. The issue arising when trying to use $\widetilde{G}^+$ in our prescription might be related to the fact that physical states of the superstring cannot be defined as $\mathcal{N} = 2$ primaries with respect to the algebra (3.46).

The amplitude (3.60) is gauge-invariant under $\delta V = (G^+)_0\Lambda + (G^+_{\text{hyb}})_0\Omega$. Since U satisfies $(G^+_{\text{hyb}})_0 U = 0$ and $U = (G^+)_0 V$, we have that V obeys the equation $(G^+_{\text{hyb}})_0 (G^+)_0 V = 0$, which is invariant under the gauge transformation $\delta V = (G^+)_0\Lambda + (G^+_{\text{hyb}})_0\Omega$ for any $\{\Omega, \Lambda\}$.

The amplitude (3.60) is supersymmetric. Although the regulator is not manifestly spacetime supersymmetric, its spacetime supersymmetry transformation under the generators (3.28) is BRST trivial, and hence vanishes inside the amplitude expression (3.60). Moreover, the vertex operator U is written in terms of the supersymmetric worldsheet variables, and we have shown that the amplitude is symmetric in the three insertions.

In order to check the consistency of our proposal, let us compute the three-point amplitude involving three massless states (3.59). To simplify the analysis, we will consider the three gluon amplitude $\mathcal{A}_{BBB}$, so that we can effectively put the gluino to zero in the $d = 6$ SYM superfields (see eqs. (K.21)). In this particular case, we have that $(\theta^1)^3_\alpha W^{\alpha 1} = 0$ in (3.59). Furthermore, the non-zero contributions to (3.60) can be determined by looking at which terms have the right amount of ghost insertions to saturate the background charge of the $\{\rho, \sigma\}$ ghosts, we are then left with the following worldsheet correlator

$$\begin{aligned}\mathcal{A}_{BBB} = \int [d\lambda][d\overline{\lambda}]d^4r d^8\theta\, \mathcal{R}\Big\langle \left(\frac{i}{2}\theta^{\alpha 1}\theta^{\beta 1} A^{(1)}_{\alpha\beta} e^{\rho + i\sigma}\right)(z_1) \times \\ \times \left(\frac{i}{2}\theta^{\gamma 1}\theta^{\delta 1} A^{(2)}_{\gamma\delta} e^{2\rho + i\sigma} J^{++}_C\right)(z_2) \left(\Pi^{\underline{a}} A^{(3)}_{\underline{a}} + d_{\sigma 1} W^{(3)\sigma 1}\right) e^{i\sigma}(z_3)\Big\rangle + (2 \leftrightarrow 3)\,,\end{aligned} \tag{3.62}$$

[9] This is easier to see in the bosonized form of $\{w_\alpha, \lambda^\alpha, s^\alpha, r_\alpha\}$.

and, after using $\mathrm{SL}(2,\mathbb{R})$ invariance to choose $z_1 = \infty$, $z_2 = 1$ and $z_3 = 0$, it easy to see that

$$\mathcal{A}_{BBB} = -i\big((a_1 \cdot a_2)(k_2 \cdot a_3) + (a_1 \cdot a_3)(k_1 \cdot a_2) + (a_2 \cdot a_3)(k_3 \cdot a_1)\big) + (2 \leftrightarrow 3)\,, \tag{3.63}$$

which gives the sought after result, as expected. Since U describes the $d = 6$ SYM multiplet, and by invariance under $d = 6$ $\mathcal{N} = 1$ supersymmetry transformations, we can conclude that our prescription also reproduces the expected answer for the three-point amplitude involving one gluon and two gluinos $\mathcal{A}_{BFF}$.

It is then elementary to generalize (3.60) to the case where we have n super-Yang-Mills multiplets

$$\mathcal{A}_n = \int [d\lambda][d\overline{\lambda}]d^4r d^8\theta\, \mathcal{R}\Big\langle V(z_1)\big((\widetilde{G}^+_{\text{hyb}})_0 V\big)(z_2)U(z_3)\prod_{m=4}^{n}\int dz_m (G^-)_{-1}U(z_m)\Big\rangle\,, \tag{3.64}$$

where $\{z_1, z_2, z_3\}$ can be chosen arbitrarily by $\mathrm{SL}(2,\mathbb{R})$ invariance. As we have only described vertex operators for the massless compactification-independent states, just scattering of $d = 6$ SYM multiplets was considered, however, the tree-level prescription should also apply to massive compactification-independent states.

Chapter 4

The superstring in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background

This chapter deals with the superstring in an $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$ background, where $\mathcal{M}_4$ can be either K3 or T^4. We first introduce the hybrid formalism in $\mathrm{AdS}_3 \times \mathrm{S}^3$ with pure NS-NS three-form flux. Subsequently, the computation of a $\mathrm{PSU}(1,1|2)$-covariant three-point amplitude for half-BPS vertex operators inserted on the AdS_3 boundary is presented, as well as its relation with the analogous computation from the RNS formalism. It is found that integrating out the fermionic worldsheet fields in the path integral gives rise to the target-space vielbein, which explicitly encodes that the conformal group on the boundary is identified with the symmetry group of the AdS bulk.

From the extended six-dimensional hybrid formalism, which was developed in Section 3.3 in a flat background, a quantizable and manifestly $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$-invariant action for the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with mixed NS-NS and R-R self-dual three-form flux is constructed. This action is the analogue of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor action for the $\mathrm{AdS}_3 \times \mathrm{S}^3$ case. The model is then quantized and proven to be conformal invariant at the one-loop level. We conclude by showing how one can relate the supersymmetric description with the worldsheet action of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ hybrid formalism with mixed flux.

4.1 Introduction

Superstring theory is not only an attractive physical description for being a mathematically consistent framework where quantum mechanics and general relativity can coexist, but it also exhibits many fascinating properties. Among these is the occurrence of intriguing and formidable dualities, some of which relate its different formulations, while others challenge conventional intuition and imply that a quantum theory of gravity in some spacetime can be equivalent to a quantum field theory without gravity, residing in a spacetime of different dimensionality.

The manifestation of a duality in physics lies in the realization that two distinct mathematical formulations describe the same physical observables or processes, in other words, both descriptions are said to be quantum equivalent even though they can appear dramatically different at a microscopic level. For example, the five known superstring theories Type I, Type IIA and IIB, and the two heterotic superstrings, are connected by a web of dualities such as T-duality, S-duality and U-duality [56]. In addition, it is believed that the five ten-dimensional superstrings are different perturbative limits of one underlying 11-dimensional theory, so-called M-theory [57] [58].

The domain of superstring dualities is not only restricted to connect two distinct types of string theories, what's even more impressive, there are dualities where a superstring theory propagating in an Anti de-Sitter background can be quantum equivalent to a four-dimensional gauge theory. At first sight, this seems like a surprising claim since string theory contains Einstein's gravity and is only quantum consistent in $d = 10$ spacetime dimensions, while the gauge theory lives in a four-dimensional flat spacetime and does not contain a graviton in its spectrum.

The most well known example of a duality of this type is between the Type IIB superstring in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background and the maximally supersymmetric four-dimensional $\mathcal{N} = 4$ SYM theory with gauge group $\mathrm{U(N)}$ [40]. In this case, the string theory is characterized by the string coupling g_s and the dimensionless AdS_5 radius in string units R_{AdS_5}, while the gauge theory depends on the Yang-Mills coupling g_{YM} and the rank of the gauge group N. It is also a common practice to express the Yang-Mills coupling as the 't Hooft coupling $\lambda = g^2_{\mathrm{YM}} N$ [59]. According to the duality, the parameters of the two theories are related as

$$g_s \sim \frac{\lambda}{N}, \qquad\qquad R^2_{\mathrm{AdS}_5} \sim \sqrt{\lambda}\,, \tag{4.1}$$

up to constant factors.

This duality is known as the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, given that $\mathcal{N} = 4$ SYM is a conformal field theory (CFT). Besides the parameters aforementioned, the identification between the two sides can be further extended by matching the global symmetries and scaling dimensions of the dual CFT to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ isometries and energies of the string side, respectively.

Although many checks have been performed over the last 25 years, the correspondence is still a conjecture. This is mainly because the duality is of the strong/weak type, when the 't Hooft coupling is large $\lambda \gg 1$, the string tension is large $R_{\mathrm{AdS}_5} \gg 1$. To put it differently, the classical or supergravity regime of the string theory $R_{\mathrm{AdS}_5} \gg 1$ is mapped to the strongly coupled regime of the $\mathcal{N} = 4$ SYM $\lambda \gg 1$. Conversely, the perturbative regime of the gauge theory $\lambda \ll 1$ is mapped to the non-perturbative — or tensionless – regime of the string theory, i.e., small R_{AdS_5}.[1] Therefore, the perturbative domains of both sides correspond to the non-perturbative ones in the dual theory, making it difficult to perform explicit calculations and verify the equivalence of the conjecture in all necessary cases. Nonetheless, note that one can use the perturbative domain of one theory to obtain predictions for inaccessible regimes of the dual theory, a key property that has driven extensive research in the field.

Based on the discussion so far, a crucial advancement for having a fine grained understanding of the duality relating the superstring in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background to the four-dimensional $\mathcal{N} = 4$ SYM theory is achieving enough control over the worldsheet description — or string side — such that first principles calculations can be performed. An elementary effort in this direction is the construction of quantizable worldsheet actions with $\mathrm{AdS}_5 \times \mathrm{S}^5$ as the target-space. However, the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background is supported by a non-zero amount of Ramond-Ramond (R-R) flux, which turns out to be one of the main barriers obstructing a worldsheet description via the conventional formalisms of the superstring.

For example, in the Ramond-Neveu-Schwarz (RNS) formalism, constructing a worldsheet action in the presence of R-R fields remains a complicated task, because vertex operators for the R-R sector break worldsheet supersymmetry [60]. Conversely, despite being feasible to construct actions in R-R backgrounds from the Green-Schwarz (GS) superstring [61], the quantization procedure is limited due to obstacles when imposing the light-cone gauge condition [62] [63].

Despite the fact that there is a quantizable sigma-model action for the superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ from the pure spinor formalism [64], the R-R fields break the holomorphic/anti-holomorphic factorization of the worldsheet theory. As a result, the powerful complex methods of two-dimensional CFT [65] cannot be applied in a straightforward manner, and progress towards vertex operators and superstring amplitude computations in AdS_5 turned out to be rather slow since the advent of

[1] Note that the radius is dimensionless in our conventions, it is expressed in string units or in terms of the string length $\sqrt{\alpha'}$.

the AdS/CFT correspondence [40].

In view of that, and with the hope of further exploring the inner workings of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality purely from a string theory perspective, a promising route for investigation is to attempt superstring computations in simpler AdS target-spaces, where the worldsheet theory is under better control. This can serve as a powerful guide for calculations in the more challenging, and also interesting, instance of the AdS/CFT correspondence, namely, the one relating the superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ with the $\mathcal{N} = 4$ SYM quantum field theory.

In particular, there exists an AdS target-space where holomorphic and anti-holomorphic factorization of the worldsheet description is still preserved. This is the case for the Type IIB superstring in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background in the absence of R-R fields, i.e., with pure Neveu-Schwarz-Neveu-Schwarz (NS-NS) self-dual three-form flux turned on. The holomorphic structure helps in the tractability of the theory. As a consequence, this particular example fits well to be a primary candidate for the understanding of quantitative features of covariant descriptions of the superstring in AdS. The latter remark will be further explored in this thesis.

Starting in Section 4.2, we shall cover a spacetime supersymmetric formulation of the Type IIB superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$, namely, the hybrid formalism for the superstring [18] [46]. The sigma-model action of the hybrid string has the supergroup $\mathrm{PSU}(1,1|2)$ as the target-superspace and it can accommodate a mixture of both NS-NS and R-R constant three-form flux. In the pure NS-NS case, the worldsheet theory is given by a $\mathrm{PSU}(1,1|2)_k$ WZW model where k labels the amount of NS-NS flux and is quantized [46]. Along with the worldsheet action, the hybrid formalism enjoys a small $\mathcal{N} = 4$ superconformal symmetry, and scattering amplitudes are computed according to the $\mathcal{N} = 4$ topological prescription [18].

Next, we will make use of the pure NS-NS hybrid formalism in $\mathrm{AdS}_3 \times \mathrm{S}^3$ with k units of three-form flux and compute a three-point amplitude for half-BPS vertex operators inserted at a position $\mathbf{x}$ on the AdS_3 boundary. This will be done in a manifestly $\mathrm{PSU}(1,1|2)$-covariant fashion, i.e., using the spacetime supersymmetric worldsheet variables of the hybrid description. The computation will be carried out after defining curved worldsheet fields by making use of the vielbein field

$$E_A{}^B(\mathbf{x}) = \delta_A^B + \mathbf{x} f_{+A}{}^B - 2\mathbf{x}^2 \eta_{A+} \delta_+^B \,,$$

where $A \in \mathrm{PSU}(1,1|2)$ Lie-superalgebra.

The vertex operators depend on a fermionic coordinate θ^α. As an outcome, we will show that integrating out the fermions $\theta^\alpha(\mathbf{x})$ in the path integral gives rise to $E_A{}^B(\mathbf{x})$, which encodes that the conformal group on the boundary corresponds to the symmetry group of the AdS_3 bulk [66]. Specifically, the fermionic zero-mode integration takes the following form

$$\int d^4\theta\, \theta^\alpha(\mathbf{x}_4)\theta^\beta(\mathbf{x}_3)\theta^\gamma(\mathbf{x}_2)\theta^\delta(\mathbf{x}_1) \\ = \epsilon^{\rho\sigma\mu\nu} E_{\rho 1}{}^{\alpha 1}(-\mathbf{x}_4) E_{\sigma 1}{}^{\beta 1}(-\mathbf{x}_3) E_{\mu 1}{}^{\gamma 1}(-\mathbf{x}_2) E_{\nu 1}{}^{\delta 1}(-\mathbf{x}_1)\,.$$

Since spacetime supersymmetric superstring scattering amplitudes in curved backgrounds have been hardly ever investigated, this construction can have some important applications. In the first place, it provides intuition for what happens after the worldsheet fermions are integrated out in a general AdS background amplitude computation. Secondly, it gives insights about what the correct amplitude prescription in the more interesting case of $\mathrm{AdS}_5 \times \mathrm{S}^5$ target-space might be. There have been significant works over the last years on trying to understand superstring vertex operators [67] [68], and the correct amplitude measure for the fermionic fields θ^α in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor formalism [69] [70].

Additionally, we should emphasize that the $\mathrm{AdS}_3 \times \mathrm{S}^3$ background can be supported by a mixture of NS-NS and R-R self-dual three-form flux. Since progress in understanding the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring worldsheet at the quantum level is hindered by the presence of R-R fields, studying the analogue of the quantizable pure spinor formalism for the $\mathrm{AdS}_3 \times \mathrm{S}^3$ case might be a useful toy model. For example, it can be a valuable alternative for developing computational techniques that might also work for the higher-dimensional background. The target-space for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor superstring is given by $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$, and the equivalent description for $\mathrm{AdS}_3 \times \mathrm{S}^3$ has the supergroup $\frac{\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)}$ as the target-space, so that all sixteen supersymmetries are manifest in the worldsheet action.

The dual CFT for the $\mathrm{AdS}_3 \times \mathrm{S}^3$ background for arbitrary values of the NS-NS flux f_{NS} and the R-R flux $\mathrm{f_{RR}}$ is not known. However, at $k = \frac{1}{f_{NS}^2} = 1$ units of NS-NS flux and absence of R-R flux,[2] the dual CFT has been recently identified [71] [72] using the six-dimensional hybrid formalism in AdS_3 [46] [6], and many other detailed checks from both sides of the duality have been performed [73] [74]

[2]We apologize for sometimes calling the NS-NS flux both by k and f_{NS}. As one can see from eq. (4.2), these quantities are indeed related in our conventions. In fact, k is the level of the WZ-term in the action and f_{NS} the value of the constant three-form in our superstring vertex operator (see Section 4.7.4).

[75] [76]. Note that in the $\mathrm{AdS}_3 \times \mathrm{S}^3$ target-space with mixed flux, the inverse of the AdS_3 radius is given by f which is defined as

$$f^2 = f_{RR}^2 + f_{NS}^2 \,, \qquad\qquad k = \frac{f_{NS}}{f^3} \,, \tag{4.2}$$

and where k determines the level of the Wess-Zumino coupling.

In Section 4.7, employing the manifestly spacetime supersymmetric formalism, developed in Section 3.3 for the flat case, we will construct a quantizable worldsheet action for the mixed NS-NS and R-R flux $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background with the super-coset $\frac{\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)}$ as the target-superspace, and show that it remains conformal invariant at the one-loop level in Section 4.8. Thus, proving that the background supergravity superfields satisfy the on-shell conditions [77]. In addition to the hybrid variables [46] [18], the sigma-model contains eight superspace fermionic coordinates plus their conjugate momenta and eight unconstrained bosonic spinors $\{\lambda^\alpha, \widehat{\lambda}^{\widehat{\alpha}}\}$ plus their conjugate momenta $\{w_\alpha, \widehat{w}_{\widehat{\alpha}}\}$. These bosonic ghosts play a similar role of the pure spinor variables in the $d = 6$ case [54] [51]. The relation between the $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$-covariant description and the hybrid formalism of Section 4.2 will be explained in detail in Section 4.9.

The construction presented in Section 4.7 may serve several purposes. The $\frac{\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)}$ super-coset formulation provides an analogue of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor action [64] in a lower-dimensional setting. This suggests that reformulating the vertex operators of [78] and the amplitudes computed in [6] [79] — originally obtained using the hybrid formalism in AdS_3 — in terms of $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$-covariant variables could offer new insights into the appropriate amplitude measure for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case [70].

One can also take the vanishing R-R flux limit in our worldsheet action, as a consequence, what remains is a pure NS-NS model with the super-coset $\frac{\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)}$ as the target-superspace. Therefore, it also provides a new superstring description which, at $k = 1$ units of NS-NS flux, has the AdS/CFT duality under good control. In particular, it is known from a string theory correlator how a twistorial incidence relation emerges from the worldsheet variables [74].

4.2 Hybrid formalism in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background

In this section, we review the hybrid formalism in $\mathrm{AdS}_3 \times \mathrm{S}^3$ with pure NS-NS self-dual three-form flux while defining our notation for the worldsheet theory.

4.2.1 Worldsheet action

The hybrid description [18] [46] of the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}_4$, where $\mathcal{M}_4$ is either K3 or T^4, can be divided into a "compactification-independent" and a "compactification-dependent" part. The compactification-independent sector describes $\mathrm{AdS}_3 \times \mathrm{S}^3$. It consists in a $\mathrm{PSU}(1,1|2)_k$ WZW model together with a $c = 28$ chiral boson ρ and the $c = -26$ chiral boson σ. The compactification-dependent sector is composed of a twisted $c = 6$ $\mathcal{N} = 2$ superconformal field theory (SCFT) describing the four-dimensional manifold $\mathcal{M}_4$. One also has the right-moving counterpart of each of these sectors. Since the worldsheet theory enjoys a holomorphic/anti-holomorphic factorization, the right-movers will mostly be ignored for simplicity and clarity of the presentation.

The worldsheet action for the hybrid superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$ is given by

$$\begin{aligned} S = \frac{1}{2}k \int d^2z \, \mathrm{sTr}\left(g^{-1}\partial g g^{-1}\overline{\partial} g\right) - \frac{i}{2}k \int_{\mathcal{B}} \mathrm{sTr}\left(g^{-1}dgg^{-1}dgg^{-1}dg\right) \\ + S_{\rho,\sigma} + S_C \, , \end{aligned} \tag{4.3}$$

where S_C is the action for the compactification directions containing four bosons and four fermions. The latter is defined by the twisted $c = 6$ $\mathcal{N} = 2$ SCFT it describes. $S_{\rho,\sigma}$ is the action for the chiral bosons $\{\rho, \sigma\}$, which is defined by the following OPE's for these fields

$$\rho(y)\rho(z) \sim -\log(y-z)\, , \tag{4.4a}$$

$$\sigma(y)\sigma(z) \sim -\log(y-z)\, . \tag{4.4b}$$

The first line of eq. (4.3) describes a $\mathrm{PSU}(1,1|2)_k$ WZW model. As a $\mathrm{PSU}(1,1|2)$ representative, one can take the group element

$$g = e^{Z^A T_A}\, , \qquad\qquad Z^A = \{\theta^{\alpha j}, x^{\underline{a}}\}\, , \tag{4.5}$$

where $A = \{\alpha j, \underline{a}\}$ is a tangent space index and labels the supercoordinates, and T_A are the generators of $\mathrm{PSU}(1,1|2)$ Lie superalgebra. The algebra generators satisfy the commutation relations

$$[T_A, T_B\} = f_{AB}{}^C T_C\, , \qquad\qquad [T_A, T_B\} = T_A T_B - (-)^{|A||B|} T_B T_A\, , \tag{4.6}$$

whose structure constants are given by

$$f_{\alpha j\,\beta k}{}^{\underline{a}} = i\sqrt{2}\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}\,, \quad f_{\underline{a}\,\alpha j}{}^{\beta k} = i\sqrt{2}\sigma_{\underline{a}\alpha\gamma}\widehat{\delta}^{\gamma\beta}\delta^k_j\,, \quad f_{\underline{a}\,\underline{b}}{}^{\underline{c}} = \sqrt{2}(\sigma_{\underline{ab}}{}^{\underline{c}})_{\alpha\beta}\widehat{\delta}^{\alpha\beta}\,, \tag{4.7}$$

and where

$$(\sigma_{\underline{abc}})_{\alpha\beta} = \frac{i}{3!}(\sigma_{[\underline{a}}\sigma_{\underline{b}}\sigma_{\underline{c}]})_{\alpha\beta}\,, \qquad \widehat{\delta}^{\alpha\beta} = 2\sqrt{2}(\sigma^{012})^{\alpha\beta}\,. \tag{4.8}$$

Note that in the equation above we anti-symmetrize with square brackets and without dividing by the number of terms. In our notation, $\underline{a} = \{0 \text{ to } 5\}$ is an $\mathrm{SO}(1,5)$ vector index, $\alpha = \{1 \text{ to } 4\}$ is a fundamental $\mathrm{SU}(4)$ index and $j = \{1,2\}$ is an $\mathrm{SU}(2)$ index.

The four by four anti-symmetric matrices $\sigma_{\underline{a}\alpha\beta}$ are the $\mathrm{SO}(1,5)$ Pauli matrices which obey the Dirac algebra

$$\sigma^{\underline{a}\alpha\beta}\sigma^{\underline{b}}_{\alpha\gamma} + \sigma^{\underline{b}\alpha\beta}\sigma^{\underline{a}}_{\alpha\gamma} = \eta^{\underline{ab}}\delta^{\beta}_{\gamma}\,, \qquad \sigma^{\underline{a}\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\sigma^{\underline{a}}_{\gamma\delta}\,, \tag{4.9}$$

where $\eta_{\underline{ab}}$ is the usual mostly plus metric of the six-dimensional flat Minkowski background. In addition, the symbol $\widehat{\delta}_{\alpha\beta}$ and its inverse $\widehat{\delta}^{\alpha\beta}$ satisfy some interesting properties, namely,

$$\sigma^{a}_{\alpha\beta} = (\widehat{\delta}\sigma^{a}\widehat{\delta})_{\alpha\beta}\,, \qquad \sigma^{a'}_{\alpha\beta} = -(\widehat{\delta}\sigma^{a'}\widehat{\delta})_{\alpha\beta}\,, \qquad \widehat{\delta}^{\alpha\beta}\widehat{\delta}_{\beta\gamma} = \delta^{\alpha}_{\gamma}\,, \tag{4.10}$$

where we write $a = \{0,1,2\}$ for the AdS_3 directions and we write $a' = \{3,4,5\}$ for the S^3 directions. Supplementary identities for the six-dimensional Pauli matrices are given in Appendix B.

The action (4.3) is invariant under global left and right $\mathrm{PSU}(1,1|2)$ transformations of g, i.e.,

$$\mathrm{PSU}(1,1|2)_{\mathrm{L}} \times \mathrm{PSU}(1,1|2)_{\mathrm{R}}\,. \tag{4.11}$$

In particular, for the pure NS-NS case we are considering, this symmetry is actually enhanced to a local $g(z) \times g(\bar{z})$ symmetry acting on g as

$$g \to g_L(z) g g_R^{-1}(\bar{z})\,, \tag{4.12}$$

where $g_L(z)(g_R(\bar{z}))$ can be any holomorphic (anti-holomorphic) map from the worldsheet to $\mathrm{PSU}(1,1|2)$.

The supertrace over the $\mathrm{PSU}(1,1|2)$ generators defines the metric

$$\mathrm{sTr}(T_A T_B) = \eta_{AB}\,, \qquad \eta^{AB}\eta_{BC} = \delta^A_C\,, \tag{4.13}$$

whose non-zero components are

$$\mathrm{sTr}(T_{\underline{a}} T_{\underline{b}}) = \eta_{\underline{ab}}\,, \qquad \mathrm{sTr}(T_{\alpha j} T_{\beta k}) = \epsilon_{jk}\widehat{\delta}_{\alpha\beta}\,, \tag{4.14}$$

where $\epsilon_{12} = \epsilon^{21} = 1$ is the anti-symmetric tensor and $\widehat{\delta}_{\alpha\beta} = 2\sqrt{2}(\sigma^{012})_{\alpha\beta}$ is the symmetric matrix which enables one to contract spinor indices in an $\mathrm{SO}(1,2) \times \mathrm{SO}(3)$ invariant manner.

For an object X_A transforming in the representation A, we raise and lower tangent-space indices according to $X^A = \eta^{AB} X_B$ and $X_A = \eta_{AB} X^B$. Of course, the same rules apply for the structure constants f_{ABC}, which are graded anti-symmetric in the 1-2 and 1-3 indices.

From the fundamental field g appearing in the worldsheet action, we define the left-currents by

$$dgg^{-1} = J^A_L T_A\,, \tag{4.15}$$

and the right-currents by

$$g^{-1}dg = J^A_R T_A\,, \tag{4.16}$$

where we write $J^A_L = \{S^{\alpha j}_L, K^{\underline{a}}_L\}$ and $J^A_R = \{S^{\alpha j}_R, K^{\underline{a}}_R\}$. Note that the left-currents are right-invariant and the right-currents are left-invariant under global $\mathrm{PSU}(1,1|2)$ transformations. Although somewhat confusing, the latter statement is in fact correct.

The enhanced symmetry (4.12) of the WZW model imply that the $(1,0)$ left-currents are purely holomorphic and the $(0,1)$ right-currents anti-holomorphic, i. e.,

$$\overline{\partial}(\partial g g^{-1}) = 0\,, \qquad \partial(g^{-1}\overline{\partial} g) = 0\,. \tag{4.17}$$

Therefore, for simplicity of the notation, we will just write $J^A_{L\,z} = J^A$ and $J^A_{R\,\overline{z}} = \overline{J}^A$, so that the components read[3]

$$J_A = \{S_{\alpha j}, K_{\underline{a}}\}\,, \qquad \overline{J}_A = \{\overline{S}_{\alpha j}, \overline{K}_{\underline{a}}\}\,. \tag{4.18}$$

In addition, from the worldsheet action (4.3) and after rescaling the currents by k^{-1} and $k \to 2k$, one can show that the current algebra between the left-currents is

$$J_A(y)J_B(z) \sim -\frac{2k}{(y-z)^2}\eta_{AB} + \frac{1}{(y-z)}f_{AB}{}^{C}J_C\,. \tag{4.19}$$

The current algebra between the anti-holomorphic right-currents can be derived from (4.19) by using the symmetry of the worldsheet action (4.3) under $z \leftrightarrow \overline{z}$ and $g \leftrightarrow g^{-1}$.

4.2.2 Superconformal generators

The hybrid superstring description in $\mathrm{AdS}_3 \times \mathrm{S}^3$ enjoys a twisted $c = 6$ $\mathcal{N} = 2$ superconformal symmetry generated by [46]

$$T = T_{\mathrm{PSU}} - \frac{1}{2}\partial\rho\partial\rho - \frac{1}{2}\partial\sigma\partial\sigma + \frac{3}{2}\partial^2(\rho + i\sigma) + T_C\,, \tag{4.20a}$$

$$\begin{aligned} G^+ = &-\frac{1}{4k}(S_1)^4 e^{-2\rho - i\sigma} - \frac{1}{2k}\left(\frac{i}{2\sqrt{2}}S_{\alpha 1}S_{\beta 1}K^{\alpha\beta} + \widehat{\delta}^{\alpha\beta}S_{\alpha 1}\partial S_{\beta 1}\right)e^{-\rho} \\ &+ T_{\mathrm{PSU}}e^{i\sigma} + \left(\partial e^{-\rho - i\sigma}, e^{\rho + 2i\sigma}\right) + G_C^+\,, \end{aligned} \tag{4.20b}$$

$$G^- = e^{-i\sigma} + G_C^-\,, \tag{4.20c}$$

$$J = \partial(\rho + i\sigma) + J_C\,, \tag{4.20d}$$

where $(S_1)^4 = \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}S_{\alpha 1}S_{\beta 1}S_{\gamma 1}S_{\delta 1}$, $K^{\alpha\beta} = \sigma^{\underline{a}\alpha\beta}K_{\underline{a}}$.

The $\mathrm{PSU}(1,1|2)_k$ stress-tensor is given by

$$\begin{aligned} T_{\mathrm{PSU}} &= -\frac{1}{4k}J_A J_B \eta^{AB} \\ &= -\frac{1}{4k}\left(K_{\underline{a}}K_{\underline{b}}\eta^{\underline{ab}} + S_{\alpha j}S_{\beta k}\eta^{\alpha j \beta k}\right)\,, \end{aligned} \tag{4.21}$$

and the generators $\{G_C^{\pm}, T_C\}$ obey a twisted $c = 6$ $\mathcal{N} = 2$ superconformal algebra (SCA) for the compactification directions and have no poles with the $\{\rho, \sigma\}$-ghosts and no poles with the matter currents.

[3] Writing the fermionic currents as $S_{\alpha j}$ and the bosonic ones as $K_{\underline{a}}$ turns out to give a more transparent notation for the $\mathcal{N} = 4$ generators that we define in Section 4.2.2.

Eqs. (4.20) need to be supplemented with a normal-ordering prescription for the $\mathrm{PSU}(1,1|2)_k$ currents. The normal-ordering is not commutative. We normal-order the currents according to

$$(J_A J_B)(z) = \oint dy\,(y-z)^{-1} J_A(y) J_B(z)\,, \tag{4.22a}$$

$$J_A J_B = (-)^{|A||B|} J_B J_A + f_{AB}{}^{C} \partial J_C\,. \tag{4.22b}$$

The normal-ordering is also not associative. In our conventions, we normal-order from right to left so that $J_A J_B J_C = (J_A(J_B J_C))$.

Consequently, the ordering is not important in T_{PSU} because of the contraction with the metric, but it is important in the second term of the supercurrent G^+. In terms of modes, the normal-ordering is in agreement with the property

$$(J_A J_B)_0 V = (-)^{|A||B|} \nabla_B \nabla_A V\,, \tag{4.23}$$

for a $\mathrm{PSU}(1,1|2)$ primary field V and with ∇_A the zero-mode of J_A.

The $c = 6$ $\mathcal{N} = 2$ SCA (4.20) can be readily verified from the OPEs (4.4) and (4.19). The more complicated properties to check come from the supercurrent G^+. As shown in ref. [46], one way to fix the form of the superconformal generator G^+ is by demanding the naive generalization from flat to curved space to be invariant under the "non-standard" spacetime supersymmetries generated by $Q_{\alpha 2}$. Another involved consistency condition to prove of the algebra generators (4.20) is checking that the OPE of G^+ with itself is regular.

Instead of demonstrating term by term that G^+ commutes with $Q_{\alpha 2}$ and that $G^+(y)G^+(z) \sim 0$, we take a simpler route. Note that it is possible to write the supercurrent as

$$G^+ = -\frac{1}{4k}\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta} Q_{\alpha 2} Q_{\beta 2} Q_{\gamma 2} Q_{\delta 2} e^{2\rho + 3i\sigma} + G^+_C\,, \tag{4.24}$$

which makes manifest its invariance under the non-standard SUSYs generated by the charge

$$Q_{\alpha 2} = \oint \left(S_{\alpha 1} e^{-\rho - i\sigma} + S_{\alpha 2}\right), \tag{4.25}$$

and also makes manifest the nilpotence property. Identity (4.24) is proved in Appendix G.

In the hybrid formalism, the standard spacetime supersymmetry generator is

$$Q_{\alpha 1} = \oint S_{\alpha 1}\,, \tag{4.26}$$

and so we have the desired spacetime SUSY algebra

$$\{Q_{\alpha j}, Q_{\beta k}\} = f_{\alpha j\,\beta k}{}^{\underline{a}} \oint K_{\underline{a}}\,. \tag{4.27}$$

When mapped to the RNS description, $Q_{\alpha 1}$ and $Q_{\alpha 2}$ correspond to the spacetime SUSY generators in the $-\frac{1}{2}$ and $\frac{1}{2}$ picture, respectively.

Let us emphasize that the reason for calling $Q_{\alpha 1}$ as the standard SUSY comes from the fact that it is ghost-independent, and so acts in a similar form as the supersymmetry generator of conventional superspace descriptions [43].

In what follows, we will write the zero-modes of the $\mathrm{PSU}(1,1|2)_k$ currents as $\nabla_A = \oint J_A$. More specifically, we define

$$\nabla_{\underline{a}} = \oint K_{\underline{a}}\,, \qquad\qquad \nabla_{\alpha j} = \oint S_{\alpha j}\,. \tag{4.28}$$

This notation is convenient, since half the spacetime supersymmetries in the hybrid superstring $Q_{\alpha j}$ act different than the zero modes of the $\mathrm{PSU}(1,1|2)_k$ SUSY currents $S_{\alpha j}$. The latter is a consequence of the presence of the $\{\rho,\sigma\}$-ghosts in the four SUSYs $Q_{\alpha 2}$ of eq. (4.25).

Any twisted $c = 6$ $\mathcal{N} = 2$ SCA can be extended to a twisted small $c = 6$ $\mathcal{N} = 4$ SCA [18]. In addition to the generators (4.20), the remaining $\mathcal{N} = 4$ generators of the hybrid formalism take the form

$$\widetilde{G}^{+} = e^{\rho} J_C^{++} - e^{\rho+i\sigma}\widetilde{G}_C^{+}\,, \tag{4.29a}$$

$$\begin{aligned}\widetilde{G}^{-} = \bigg[&- \frac{1}{4k}(S_1)^4 e^{-3\rho-2i\sigma} + \frac{1}{2k}\left(\frac{i}{2\sqrt{2}} S_{\alpha 1} S_{\beta 1} K^{\alpha\beta} + \widehat{\delta}^{\alpha\beta} S_{\alpha 1}\partial S_{\beta 1}\right) e^{-2\rho-i\sigma} \\ &+ T_{\mathrm{PSU}} e^{-\rho} - \left(\partial e^{-\rho-i\sigma}, e^{i\sigma}\right)\bigg] J_C^{--} + e^{-\rho-i\sigma}\widetilde{G}_C^{-}\,,\end{aligned} \tag{4.29b}$$

$$J^{++} = -e^{\rho+i\sigma} J_C^{++}\,, \tag{4.29c}$$

$$J^{--} = e^{-\rho-i\sigma} J_C^{--}\,. \tag{4.29d}$$

The generators $\{G_C^{\pm}, T_C\}$ together with $\{\widetilde{G}_C^{\pm}, J_C^{\pm\pm}\}$ form a twisted small $c = 6$ $\mathcal{N} = 4$ SCA for the compactification directions which has no poles with the $\{\rho,\sigma\}$-ghosts and no poles with the matter currents. Their explicit form is not needed in this work.

Remember that we are only discussing the holomorphic part, and hence one also has a right-moving twisted small $c = 6$ $\mathcal{N} = 4$ SCA. We display our conventions for the twisted $\mathcal{N} = 2$ SCA and twisted small $\mathcal{N} = 4$ SCA in Appendix H.

4.2.3 Physical state conditions

Physical states $\mathcal{V}$ of the hybrid superstring are defined to satisfy the following constraints [46]

$$G_0^+ \widetilde{G}_0^+ \mathcal{V} = 0\,, \qquad\qquad G_0^- \mathcal{V} = \widetilde{G}_0^- \mathcal{V} = T_0 \mathcal{V} = J_0 \mathcal{V} = 0\,, \tag{4.30}$$

and the state $\mathcal{V}$ is determined up to the gauge transformation

$$\delta \mathcal{V} = G_0^+ \Lambda + \widetilde{G}_0^+ \Omega + \widetilde{G}_0^- \widetilde{G}_0^+ \Sigma\,, \tag{4.31}$$

where $\{\Lambda, \Omega\}$ are annihilated by $\{G_0^-, \widetilde{G}_0^-, T_0\}$ and Σ is annihilated by $\{G_0^-, T_0\}$. For a holomorphic operator $\mathcal{O}$ of conformal weight h, the notation $\mathcal{O}_n$ means the pole of order $n + h$.

Let us pause and comment about our gauge-fixing conditions. The first equation in (4.30) can be translated to the standard physical state condition of the RNS formalism $Q_{\mathrm{RNS}} V_{\mathrm{RNS}} = 0$, where V_{RNS} lives in the small hilbert space and is related to $\mathcal{V}$ as $\mathcal{V} = \xi V_{\mathrm{RNS}}$ (see Section 4.6 and Appendix E). The constraint $T_0 \mathcal{V} = 0$ is the usual mass-shell condition in string theory. When translated to RNS, the constraint $J_0 \mathcal{V} = 0$ is equivalent as saying that the ghost- minus the picture-number of V_{RNS} is equal to one, as always happens for a physical RNS state [20].

The additional constraints $G_0^- \mathcal{V} = \widetilde{G}_0^- \mathcal{V} = 0$ in eqs. (4.30) are convenient to further eliminate auxiliary degrees of freedom and imply a covariant gauge choice, e.g., they are equivalent to the Lorenz gauge condition for the open string sector [46] [24] [55] [1]. As we will see in Section 4.3.2, it is also possible to define a Lorenz-type gauge in the $\mathrm{AdS}_3 \times \mathrm{S}^3$ hybrid formalism which turns out to be suited for performing amplitude computations.

Note that in this formalism one of the candidates for the integrated vertex operator takes the form

$$\int G_0^+ G_{-1}^- \mathcal{V}\,, \tag{4.32}$$

which corresponds to a vertex operator in the same picture as $\mathcal{V}$ when translating to the RNS language [46]. However, the important difference, when compared to the RNS formalism, is that $\mathcal{V}$ carries states both from the Ramond and Neveu-Schwarz sectors. In the RNS description, the Ramond states carry half-integer picture and the NS states carry integer picture. In fact, this is a crucial feature of the hybrid formalism. It treats Ramond and Neveu-Schwarz sectors in the same footing, since it only uses worldsheet variables of integer conformal weight.

4.2.4 Amplitude prescription

The prescription to compute n-point tree-level scattering amplitudes is given by

$$\mathcal{A}_n = \left\langle \mathcal{V}^3(z_3)\widetilde{G}_0^+\mathcal{V}^{(2)}(z_2)\left(\prod_{m=4}^{n}\int dz_m\, G_{-1}^- G_0^+ \mathcal{V}^{(m)}(z_m)\right) G_0^+\mathcal{V}^{(1)}(z_1)\right\rangle, \tag{4.33}$$

where $\mathcal{V}^{(n)}$ is the vertex operator satisfying the physical state conditions (4.30) and gauge transformations (4.31), and we are choosing $z_1 = 0$, $z_2 = 1$ $z_3 = \infty$ by $\mathrm{SL}(2,\mathbb{C})$ invariance. Note that the contribution from the right-movers is also being suppressed in $\mathcal{A}_n$.

In eq. (4.33), the zero-mode integration over the fermions is done by generalizing the flat space prescription, i.e.,

$$\begin{aligned}\int d^4\theta &= \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\nabla_{\delta 1}\nabla_{\gamma 1}\nabla_{\beta 1}\nabla_{\alpha 1} \\ &= (\nabla_1)^4\,, \end{aligned}\tag{4.34}$$

where $\nabla_{\alpha 1}$ is the standard spacetime SUSY charge in $\mathrm{AdS}_3 \times \mathrm{S}^3$, see eq. (4.26). After integrating out the non-zero modes, the amplitude (4.33) can always be expressed in terms of $\nabla_{\underline{a}}$, the standard SUSY charge $\nabla_{\alpha 1}$ and the fermonic coordinate θ^α (see eq. (4.37) below). In particular, note that the measure (4.34) is invariant under both $\nabla_{\underline{a}}$ and $\nabla_{\alpha 1}$, so that the usual "integration by parts" is well defined.

The chiral bosons $\{\rho,\sigma\}$ carry a non-zero amount of background charge, as can be seen from eq. (4.20a). Therefore, the tree-level amplitude is non-zero only when the path integral insertions contribute with the factor $e^{3\rho+3i\sigma}$ in eq. (4.33). In addition, since the compactification generators $\{T_C, G_C^{\pm}, J_C\}$ obey a twisted $c = 6\ \mathcal{N} = 2$ SCA, one also needs the insertion of J_C^{++}. So that, in total, one gets $e^{3\rho+3i\sigma} J_C^{++}$ for the chiral bosons. In the amplitude (4.33), the factor of J_C^{++} comes from the term $\widetilde{G}_0^+ \mathcal{V}$.

In this work, we will sometimes use definitions such as

$$(\theta^3)_\alpha = \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta}\theta^\beta\theta^\gamma\theta^\delta\,, \qquad (\theta)^4 = \frac{1}{24}\epsilon_{\alpha\beta\gamma\delta}\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta\,, \tag{4.35a}$$

$$(\nabla_1^3)^\alpha = \frac{1}{6}\epsilon^{\alpha\beta\gamma\delta}\nabla_{\beta 1}\nabla_{\gamma 1}\nabla_{\delta 1}\,, \qquad (\nabla_1)^4 = \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\nabla_{\alpha 1}\nabla_{\beta 1}\nabla_{\gamma 1}\nabla_{\delta 1}\,. \tag{4.35b}$$

4.3 Vertex operators in the hybrid formalism

This section deals with half-BPS vertex operators for the superstring in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background with pure NS-NS three-form flux. After introducing the zero-mode variable θ^α, we will define the concept of a superfield in our superstring description for this background. Subsequently, the form of the half-BPS vertex operators will be determined by solving the constraints presented in Section 4.2.3.

For simplicity, we will consider vertex operators with no spectral flow [80]. Let us also emphasize that we will be working from the $\mathrm{PSU}(1,1|2)$ supergroup perspective in all stages of our development. For the readers not interested in the technical details, the gauge-fixed vertex operator is given in eq. (4.49).

4.3.1 Superfields in $\mathrm{AdS}_3 \times \mathrm{S}^3$

For the compactification-independent massless sector, the condition $T_0\mathcal{V} = 0$ (see eqs. (4.30)) imply that the vertex operator $\mathcal{V}$ in $\mathrm{AdS}_3 \times \mathrm{S}^3$ transforms as a primary under the $\mathrm{PSU}(1,1|2)_k$ currents, i.e.,

$$J_A(y)\mathcal{V}(z) \sim (y-z)^{-1}\nabla_A\mathcal{V}(z)\,. \tag{4.36}$$

In particular, this implies that $\mathcal{V}$ has a pole with the fermionic current $S_{\alpha j}$.

Since the standard SUSYs have the simple form (4.26), similar as in flat space, it is convenient to define $\mathcal{V}$ to be a superfield expanded in terms of a fermionic coordinate θ^α which transforms as a Weyl spinor and is conjugate to $Q_{\alpha 1}$. More precisely, we define the superspace fermionic variable θ^α by the property[4]

$$\nabla_{\alpha 1}\theta^\beta = \delta_\alpha^\beta \,. \tag{4.37}$$

Therefore, when we speak of a superfield in $\mathrm{AdS}_3 \times \mathrm{S}^3$, we will be referring to a state which transforms as a primary under the $\mathrm{PSU}(1,1|2)_k$ currents and which has a finite expansion in terms of the fermionic coordinate θ^α. Note further that θ^α is not trivially related to the group manifold coordinates $\theta^{\alpha j}$ in (4.5), i.e., $\theta^\alpha \neq \theta^{\alpha 1}$ and $\theta^\alpha \neq \theta^{\alpha 2}$.

From the definition (4.37), we deduce that the remaining zero-modes of the $\mathrm{PSU}(1,1|2)_k$ currents satisfy

$$\nabla_{\underline{a}}\theta^\alpha = f_{\beta 1 \underline{a}}{}^{\alpha 1}\theta^\beta \,, \tag{4.38a}$$

$$\nabla_{\alpha 2}\theta^\beta = \frac{1}{2} f_{\alpha 2\,\gamma 1}{}^{\underline{a}} f_{\delta 1\,\underline{a}}{}^{\beta 1}\theta^\gamma\theta^\delta \,. \tag{4.38b}$$

Since $\mathcal{V} = \mathcal{V}(\theta)$, eq. (4.37) also implies that the component fields of $\mathcal{V}$ are annihilated by $Q_{\alpha 1} = \nabla_{\alpha 1}$. In addition, we also have the expected property $T_0\theta^\alpha = 0$, as can be easily checked.

In the flat background hybrid formalism, the vertex operator $\mathcal{V}$ for the massless compactification-independent sector is a superfield depending on a fermionic coordinate θ^α and $\mathcal{V}$ has a simple pole with the standard spacetime supersymmetry current p_α, since

$$p_\alpha(y)\theta^\beta(z) \sim \delta_\alpha^\beta (y-z)^{-1} \,, \tag{4.39}$$

where $\{p_\alpha, \theta^\beta\}$ are holomorphic fermionic fundamental worldsheet fields of conformal weight one and zero, respectively. Therefore, one can view the definition (4.37) as a consequence of (4.36) and the generalization of the definition of $\mathcal{V}$ from the flat to the curved $\mathrm{AdS}_3 \times \mathrm{S}^3$ spacetime. With the difference that θ^α is not (a priori) a fundamental worldsheet coordinate in our description in terms of $g \in \mathrm{PSU}(1,1|2)$ in (4.3). Nevertheless, it is consistent to think of θ^α as a fermionic zero-mode in the supergroup description and satisfying properties (4.37) and (4.38).

Regardless of that, it turns out that in a pure NS-NS $\mathrm{AdS}_3 \times \mathrm{S}^3$ background the coordinate θ^α can be viewed as a fundamental holomorphic worldsheet field. This

[4] Recall that in our notation $Q_{\alpha 1} = \nabla_{\alpha 1}$, see eqs. (4.26) and (4.28).

hinges on the fact that the $\mathrm{PSU}(1,1|2)_k$ current algebra (4.19) can be expressed in terms of a $\mathrm{SU}(1,1)_{k+2} \times \mathrm{SU}(2)_{k-2}$ current algebra plus the eight free fermions (4.39) [46].

Indeed, let $\underline{a} = \{a, a'\}$ where $a = \{0,1,2\}$ and $a' = \{3,4,5\}$ label the AdS_3 and S^3 directions, respectively. We denote the $\mathrm{SU}(1,1)_{k+2} \times \mathrm{SU}(2)_{k-2}$ currents by $\mathcal{J}_{\underline{a}}$. The $\mathrm{SU}(1,1)_{k+2}$ current algebra reads

$$\mathcal{J}_a(y)\mathcal{J}_b(z) \sim -\eta_{ab}\frac{2(k+2)}{(y-z)^2} + \frac{1}{(y-z)} f_{a\,b}{}^{c}\,\mathcal{J}_c\,, \tag{4.40}$$

and the $\mathrm{SU}(2)_{k-2}$ reads

$$\mathcal{J}_{a'}(y)\mathcal{J}_{b'}(z) \sim -\eta_{a'b'}\frac{2(k-2)}{(y-z)^2} + \frac{1}{(y-z)} f_{a'\,b'}{}^{c'}\,\mathcal{J}_{c'}\,, \tag{4.41}$$

where $f_{012} = f_{345} = -2$. Then, by defining

$$S_{\alpha 1} = p_\alpha\,, \tag{4.42a}$$

$$S_{\alpha 2} = -2k\widehat{\delta}_{\alpha\beta}\partial\theta^\beta + f_{\alpha 2\,\beta 1}{}^{\underline{a}}\left(\mathcal{J}_{\underline{a}} + \frac{1}{2} f_{\underline{a}\,\gamma 1}{}^{\delta 1} p_\delta \theta^\gamma\right)\theta^\beta\,, \tag{4.42b}$$

$$K_{\underline{a}} = \mathcal{J}_{\underline{a}} + f_{\underline{a}\,\alpha 1}{}^{\beta 1} p_\beta \theta^\alpha\,, \tag{4.42c}$$

we recover the $\mathrm{PSU}(1,1|2)_k$ current algebra (4.19), as we wanted to show. The bosonic curents $\mathcal{J}_{\underline{a}}$ are the usual decoupled currents which appear in the RNS description [81] [82].

We should mention that in the hybrid description the eight free fermions $\{p_\alpha, \theta^\beta\}$ come from a field redefinition involving the six $\psi^{\underline{a}}$'s and the bosonized form of the $\{\beta,\gamma\}$-ghosts of the RNS formalism [46]. As a corollary of this observation, one knows from the beginning that $\theta^\alpha(z)$ is holomorphic in a pure NS-NS $\mathrm{AdS}_3 \times \mathrm{S}^3$ background.

Using eqs. (4.42), one can readily check that the relations (4.37) and (4.38) are reproduced. Except when comparing with RNS in Section 4.6, we will not use the explicit form of the currents (4.42) in terms of the free fields $\{p_\alpha, \theta^\beta\}$. This means that we will be working from the supergroup perspective and hence with the currents constructed from $g \in \mathrm{PSU}(1,1|2)$. However, it will be assumed eq. (4.37), which naturally follows from the generalization of a "superspace coordinate" from the flat to the curved $\mathrm{AdS}_3 \times \mathrm{S}^3$ background in the hybrid formalism.

In this case, let us emphasize that θ^α is a fermionic zero-mode (Schrödinger operator) which is responsible for building up our physical states in a covariant fashion. Therefore, it will not be necessary to know how it depends on $g(z,\bar{z})$ in this work, and so this reasoning should also generalize to $\mathrm{AdS}_3 \times \mathrm{S}^3$ when turning on a constant R-R three-form flux [83].

4.3.2 Vertex operators for the massless states

Now, we will determine the gauge-fixed half-BPS vertex operators by analyzing the physical state conditions of Section 4.2.3. As usual, we concentrate on the holomorphic part of the theory.

For the massless compactification-independent states (i.e., states of conformal weight zero at zero momentum) of the Type IIB superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$, the condition that the vertex operator $\mathcal{V}$ should have no single poles with J imply that it takes the form

$$\mathcal{V} = \sum_n e^{n(\rho+i\sigma)} V_n \, . \tag{4.43}$$

Demanding $\mathcal{V}$ to have no double poles or higher with G^- and no double poles or higher with $\widetilde{G}^-$ imply that $V_n = 0$ for $n \geq 2$ and $V_n = 0$ for $n \leq -2$, respectively. Moreover, the condition $\widetilde{G}_0^- \mathcal{V} = 0$ also gives the following constraints for the remaining superfields $\{V_{-1}, V_0, V_1\}$

$$\nabla_{\alpha 1} V_{-1} = 0 \, , \tag{4.44a}$$

$$\frac{i}{2\sqrt{2}} \sigma_{\underline{a}}^{\alpha\beta} \nabla_{\alpha 1} \nabla_{\beta 1} V_0 - \nabla_{\underline{a}} V_{-1} = 0 \, , \tag{4.44b}$$

$$(\nabla_1)^4 V_1 = 0 \, , \tag{4.44c}$$

$$(\nabla_1^3)^\alpha V_1 + \Big(- i\sqrt{2} \nabla^{\alpha\beta} \nabla_{\beta 1} + 2\widehat{\delta}^{\alpha\beta} \nabla_{\beta 1} \Big) V_0 - 2\widehat{\delta}^{\alpha\beta} \nabla_{\beta 2} V_{-1} = 0 \, , \tag{4.44d}$$

$$\nabla^{\alpha\beta} \nabla_{\alpha 1} \nabla_{\beta 1} V_1 = 0 \, . \tag{4.44e}$$

Let us now determine what are the physical states by analyzing eqs. (4.44) together with the gauge transformations (4.31) for the remaining superfields. From eq. (4.44a) we learn that V_{-1} has no components proportional to θ^α. By taking $\Sigma = 4ke^{\rho+i\sigma}(\theta)^4 V_{-1}$ in (4.31), we see that V_{-1} can be gauged away. Therefore, eq. (4.44b) implies that $V_0 = v_0 + \theta^\alpha \chi_{\alpha 1}$ for some $\{v_0, \chi_{\alpha 1}\}$. Actually, the component v_0 can be removed by taking $\Omega = -e^\rho J_C^{++} v_0$ in the gauge transformations (4.31). Therefore, we conclude that one can gauge-fix V_0 to the form

$$V_0 = \theta^\alpha \chi_{\alpha 1} \,. \tag{4.45}$$

We now turn to analyze the components of the superfield V_1, the most important part of the vertex operator $\mathcal{V}$. Firstly, eq. (4.44c) implies that V_1 has no $(\theta)^4$ component. Now, consider the gauge transformation given by

$$\Lambda = 2\sqrt{2}ke^{2\rho+i\sigma}\xi \,, \tag{4.46a}$$

$$\xi = -\frac{i}{2}(\theta\sigma_{\underline{a}}\theta)\omega^{\underline{a}} + i(\theta^3)_\alpha \tau^\alpha + (\theta)^4 \lambda \,, \tag{4.46b}$$

for some $\{\omega^{\underline{a}}, \tau^\alpha, \lambda\}$. From (4.31), one finds

$$\begin{aligned} \delta V_1 &= -\frac{i}{2}\nabla^{\alpha\beta}\nabla_{\alpha 1}\nabla_{\beta 1}\xi \\ &= \nabla^{\underline{a}}\omega_{\underline{a}} + \theta^\alpha \nabla_{\alpha\beta}\tau^\beta + \frac{i}{2}(\theta\sigma^{\underline{a}}\theta)\nabla_{\underline{a}}\lambda \,. \end{aligned} \tag{4.47}$$

Using the gauge parameter $\omega^{\underline{a}}$, one can gauge away the first component of V_1. As a result, we can gauge-fix the superfield V_1 to the following form

$$V_1 = \theta^\alpha \chi_{\alpha 2} + \frac{i}{2}(\theta\sigma_{\underline{a}}\theta)a^{\underline{a}} - (\theta^3)_\alpha \psi^{\alpha 2} \,. \tag{4.48}$$

In addition, eq. (4.44d) implies that $\psi^{\alpha 2} = (i\sqrt{2}\nabla^{\alpha\beta} - 2\widehat{\delta}^{\alpha\beta})\chi_{\beta 1}$. As a consequence, all the degrees of freedom are contained in the superfield V_1. In view of that and for later convenience, we define $V_1 = V$. Therefore, the gauge-fixed vertex operator (4.43) takes the form

$$\mathcal{V} = e^{\rho+i\sigma}V + V_0 \,, \tag{4.49}$$

where

$$V = \theta^\alpha \chi_{\alpha 2} + \frac{i}{2}(\theta\sigma_{\underline{a}}\theta)a^{\underline{a}} - (\theta^3)_\alpha \psi^{\alpha 2} \,, \tag{4.50a}$$

$$V_0 = \theta^\alpha \chi_{\alpha 1} \,. \tag{4.50b}$$

Note further that the superfield V satisfies the equation of motion $\nabla^{\alpha\beta}\nabla_{\alpha 1}\nabla_{\beta 1}V = 0$ (see eq. (4.44e)).

All component fields obey the mass-shell condition $\nabla^{\underline{a}}\nabla_{\underline{a}} = 0$. It is also convenient to define the gauge invariant "fermions"

$$\psi^{\alpha j} = \epsilon^{jk}(i\sqrt{2}\nabla^{\alpha\beta} - 2\widehat{\delta}^{\alpha\beta})\chi_{\beta k}\,. \tag{4.51}$$

Note that these fermions satisfy the "Dirac-like" equation

$$D_{\alpha\beta}\psi^{\beta j} = 0\,, \tag{4.52}$$

in curved space, where $D_{\alpha\beta} = \sigma^{\underline{a}}_{\alpha\beta} D_{\underline{a}}$.

For an object X_A transforming in the representation A of $\mathrm{PSU}(1,1|2)$, we define the covariant derivative as

$$D_{\underline{a}} X_B = \nabla_{\underline{a}} X_B - \frac{1}{2} f_{\underline{a}\,B}{}^C X_C\,. \tag{4.53}$$

In fact, one can show that $D_{\alpha\beta}\psi^{\beta j} = \nabla_{\alpha\beta}\psi^{\beta j}$ by using the explicit form of the structure constants (4.7).

In terms of the RNS formalism language, the components of V proportional to $(\theta\sigma_{\underline{a}}\theta)$ are states from the NS-sector and the components proportional to θ^α and $(\theta^3)_\alpha$ are states from the R-sector.

Although we are only discussing the holomorphic part of the theory for simplicity, the identification of the equations of motion derived from the string constraints (4.30) with the supergravity field equations in $\mathrm{AdS}_3 \times \mathrm{S}^3$ was elaborated in ref. [84].

Since the fermionic variables θ^α are charged under the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ bosonic subgroup of $\mathrm{PSU}(1,1|2)$, we can also relate the components of the superfield in (4.50) with the Maldacena-Ooguri vertex operators described in terms of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ quantum numbers [80] [85]. We refer to Appendix I.2 for this description.

4.4 Curved worldsheet fields in the hybrid desecription

For the purpose of computing $\mathrm{PSU}(1,1|2)$-covariant superstring scattering amplitudes with vertex operators being functions of the spacetime boundary positions, we will introduce worldsheet fields depending on the boundary AdS_3 coordinates $\mathbf{x}$. This will be done by performing a similarity transformation in the

∇_+ direction with parameter the complex coordinate $\mathbf{x}$ and where

$$\nabla_+ = -\frac{i}{2}\oint (K_1 + iK_2)\,, \tag{4.54}$$

is the translation generator along the AdS_3 boundary or, equivalently, in the dual CFT [81]. Naturally, the vielbein field $E_A{}^B(\mathbf{x})$ will emerge in our description.

When writing a field $\mathcal{O}$ without any labels, it means that it only depends on the worldsheet coordinates z. Therefore, it is inserted in the position $\mathbf{x} = 0$ in the boundary.[5] For an operator function of any $\mathbf{x} \in \partial\mathrm{AdS}_3$, we will write $\mathcal{O}(\mathbf{x}, z)$ — or simply $\mathcal{O}(\mathbf{x})$ — which is equivalent to $e^{\mathbf{x}\nabla_+}\mathcal{O}e^{-\mathbf{x}\nabla_+}$. As we will presently see, there are only a finite number of terms that contribute in this similarity transformation for our fundamental worldsheet variables.

Vertex operators translated by the generator (4.54) were used in refs. [73] [74] to match worldsheet correlators at $k = 1$ units of NS-NS flux with the dual two-dimensional CFT correlators [86] [87].

4.4.1 Similarity transformation and the vielbein

Consider the holomorphic $\mathrm{PSU}(1,1|2)_k$ currents (4.18), the effect of introducing dependence on the boundary AdS_3 coordinates $\mathbf{x}$ is given by

$$\begin{aligned} J_A(\mathbf{x}, z) &= e^{\mathbf{x}\nabla_+} J_A(z) e^{-\mathbf{x}\nabla_+} \\ &= J_A(z) + \mathbf{x} f_{+A}{}^B J_B(z) + \frac{\mathbf{x}^2}{2} f_{+A}{}^B f_{+B}{}^C J_C(z)\,, \end{aligned} \tag{4.55}$$

since

$$f_{+A}{}^B f_{+B}{}^C = -4\delta_+^C \eta_{A+}\,. \tag{4.56}$$

This means that we can write

$$J_A(\mathbf{x}, z) = E_A{}^B(\mathbf{x}) J_B(z)\,, \tag{4.57}$$

where

$$E_A{}^B(\mathbf{x}) = \delta_A^B + \mathbf{x} f_{+A}{}^B - 2\mathbf{x}^2 \eta_{A+} \delta_+^B\,, \tag{4.58}$$

[5] I would like to thank Lucas Martins and Dennis Zavaleta for discussions regarding this point.

and so the matrix ${E_A}^B(\mathbf{x})$ converts a flat worldsheet field to a curved one (in spacetime).

Therefore, we will take our freedom and call the quadratic matrix ${E_A}^B(\mathbf{x})$ the vielbein field [24]. In particular, note from (4.58) that ${E_A}^B(\mathbf{x})$ has a finite number of terms and at most quadratic in $\mathbf{x}$. Since $J_A = \{S_{\alpha j}, K_{\underline{a}}\}$, in our usual notation, we simply write

$$S_{\alpha j}(\mathbf{x}) = {E_{\alpha j}}^{\beta k}(\mathbf{x}) S_{\beta k}\,, \qquad\qquad K_{\underline{a}}(\mathbf{x}) = {E_{\underline{a}}}^{\underline{b}}(\mathbf{x}) K_{\underline{b}}\,. \tag{4.59}$$

In supergravity descriptions, the vielbein field ${E_A}^B(\mathbf{x})$ carries a lower Einstein index and an upper Lorentz (or structure group) index [24], and ${E_A}^B(\mathbf{x})$ is not written with an explicit spacetime dependence of $\mathbf{x}$. In this work, we will not differentiate between Einstein and Lorentz indices. However, this should yield no confusion, since for any object $\mathcal{O}$ depending on $\mathbf{x}$ we will explicit write $\mathcal{O}(\mathbf{x})$.

The vertex operator (4.49) in the $\mathbf{x}$-basis $\mathcal{V}(\mathbf{x}, z)$ is

$$\mathcal{V}(\mathbf{x}, z) = e^{\mathbf{x}\nabla_+} \mathcal{V}(z) e^{-\mathbf{x}\nabla_+}\,. \tag{4.60}$$

therefore, from (4.36), the action of the spacetime dependent $\mathrm{PSU}(1,1|2)_k$ currents J_A is given by

$$J_A(\mathbf{x}_1, y)\mathcal{V}(\mathbf{x}_2, z) \sim (y-z)^{-1}\Big(\nabla_A \mathcal{V} + \mathbf{x}_{12} {f_{+A}}^B \nabla_B \mathcal{V} - 2\mathbf{x}_{12}^2 \eta_{A+} \nabla_+ \mathcal{V}\Big)(\mathbf{x}_2, z)\,, \tag{4.61}$$

where $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2$. In our formulas, we shall also write

$$\begin{aligned} J_A(\mathbf{x}_1, y)\mathcal{V}(\mathbf{x}_2, z) &\sim (y-z)^{-1}\big(\nabla_A(\mathbf{x}_{12})\mathcal{V}\big)(\mathbf{x}_2, z) \\ &= (y-z)^{-1} {E_A}^B(\mathbf{x}_{12})(\nabla_B \mathcal{V})(\mathbf{x}_2, z)\,, \end{aligned} \tag{4.62}$$

to simplify the notation.

4.4.2 Curved fermionic coordinates

As we discussed in the beginning of Section 4.3, the vertex $\mathcal{V}$ in eq. (4.49) is a superfield in our superstring description, i.e., it is a function of the fermionic zero-mode variable θ^α. In order to compute amplitudes involving the vertex $\mathcal{V}$ inserted in the AdS_3 boundary, we need to specify what are the analogues of the superspace coordinates θ^α when we introduce dependence on the spacetime coordinate $\mathbf{x}$.

As before, in the $\mathbf{x}$-basis, we have that

$$\theta^\alpha(\mathbf{x}) = e^{\mathbf{x}\nabla_+}\theta^\alpha e^{-\mathbf{x}\nabla_+}\,, \tag{4.63}$$

and eq. (4.38a) implies

$$\begin{aligned}\theta^\alpha(\mathbf{x}) &= E_{\beta 1}{}^{\alpha 1}(-\mathbf{x})\theta^\beta \\ &= \left(\delta^\alpha_\beta + \mathbf{x} f_{\beta 1\, +}{}^{\alpha 1}\right)\theta^\beta \\ &= \theta^\alpha - \mathbf{x} i\sqrt{2}(\theta\sigma_+\widehat{\delta})^\alpha\,.\end{aligned} \tag{4.64}$$

Therefore, from (4.37), one finds that the action of the standard SUSYs on $\theta^\alpha(\mathbf{x})$ is

$$\begin{aligned}\nabla_{\alpha 1}\theta^\beta(\mathbf{x}) &= E_{\alpha 1}{}^{\beta 1}(-\mathbf{x}) \\ &= \delta^\beta_\alpha - \mathbf{x} i\sqrt{2}(\sigma_+\widehat{\delta})_\alpha{}^\beta\,,\end{aligned} \tag{4.65}$$

where $E_{\alpha 1}{}^{\beta 1}(\mathbf{x})$ is the vielbein field of eq. (4.58).

From the last property, together with eq. (4.34), we then have determined a way to integrate the curved worldhsheet fermions $\theta^\alpha(\mathbf{x})$ in a tree-level amplitude computation. The answer is given in terms of the vielbein, namely,[6]

$$\begin{aligned}&\int d^4\theta\, \theta^\alpha(\mathbf{x}_4)\theta^\beta(\mathbf{x}_3)\theta^\gamma(\mathbf{x}_2)\theta^\delta(\mathbf{x}_1) \\ &\quad = \epsilon^{\rho\sigma\mu\nu} E_{\rho 1}{}^{\alpha 1}(-\mathbf{x}_4)E_{\sigma 1}{}^{\beta 1}(-\mathbf{x}_3)E_{\mu 1}{}^{\gamma 1}(-\mathbf{x}_2)E_{\nu 1}{}^{\delta 1}(-\mathbf{x}_1)\,.\end{aligned} \tag{4.67}$$

If desired, the expression above can be explicitly evaluated using the definition (4.58), one finds

$$\begin{aligned}&\int d^4\theta\, \theta^\alpha(\mathbf{x}_4)\theta^\beta(\mathbf{x}_3)\theta^\gamma(\mathbf{x}_2)\theta^\delta(\mathbf{x}_1) \\ &\quad = \epsilon^{\alpha\beta\gamma\delta} + \frac{i}{\sqrt{2}}\Big(- \mathbf{x}_4\widehat{\delta}^{\alpha[\beta}\sigma_+^{\gamma\delta]} + \mathbf{x}_3\widehat{\delta}^{\beta[\alpha}\sigma_+^{\gamma\delta]} - \mathbf{x}_2\widehat{\delta}^{\gamma[\alpha}\sigma_+^{\beta\delta]} + \mathbf{x}_1\widehat{\delta}^{\delta[\alpha}\sigma_+^{\beta\gamma]}\Big)\end{aligned}$$

[6] To perform calculations, it is actually easier to use the less condensed but more practical notation of a curved delta-function for the spinorial vielbein, i.e.,

$$E_{\alpha 1}{}^{\beta 1}(-\mathbf{x}) = \delta^\beta_\alpha(\mathbf{x})\,. \tag{4.66}$$

$$- 2\sigma_+^{\alpha\beta}\sigma_+^{\gamma\delta}(\mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_3\mathbf{x}_4) + 2\sigma_+^{\alpha\gamma}\sigma_+^{\beta\delta}(\mathbf{x}_1\mathbf{x}_3 + \mathbf{x}_2\mathbf{x}_4) - 2\sigma_+^{\alpha\delta}\sigma_+^{\beta\gamma}(\mathbf{x}_1\mathbf{x}_4 + \mathbf{x}_2\mathbf{x}_3)\,, \tag{4.68}$$

hence, only terms up to quadratic-order in $\mathbf{x}$ appear when integrating out the fermionic zero-modes $\theta^\alpha(\mathbf{x})$'s.

4.4.3 Some properties of the vielbein

We have explicitly shown how flat worldsheet fields can be made dependent on the boundary AdS_3 coordinates $\mathbf{x}$. One of the key ideas is the presence of the spacetime dependent matrix (4.58), which naturally appears in our superstring description after performing a similarity transformation in the direction ∇_+ with parameter $\mathbf{x}$.

For the purpose of carrying out computations, it is useful to state some of the identities satisfied by $E_A{}^B(\mathbf{x})$. One can show that

$$E_A{}^B(\mathbf{x})E_B{}^C(-\mathbf{x}) = \delta_A^C\,. \tag{4.69}$$

and, note also

$$E_{\alpha 1}{}^{\gamma 1}(-\mathbf{x})\sigma_{\underline{a}\gamma\delta}E_{\beta 1}{}^{\delta 1}(-\mathbf{x}) = E_{\underline{a}}{}^{\underline{b}}(\mathbf{x})\sigma_{\underline{b}\alpha\beta}\,, \tag{4.70a}$$

$$E_{\gamma 1}{}^{\alpha 1}(-\mathbf{x})\sigma_{\underline{a}}^{\gamma\delta}E_{\delta 1}{}^{\beta 1}(-\mathbf{x}) = E_{\underline{a}}{}^{\underline{b}}(-\mathbf{x})\sigma_{\underline{b}}^{\alpha\beta}\,, \tag{4.70b}$$

and that

$$e^{\mathbf{x}\nabla_+}(\theta\sigma_{\underline{a}}\theta)e^{-\mathbf{x}\nabla_+} = E_{\underline{a}}{}^{\underline{b}}(\mathbf{x})(\theta\sigma_{\underline{b}}\theta)\,, \tag{4.71}$$

hence, $E_A{}^B(\mathbf{x})$ transforms a "flat Pauli matrix" to a "curved one" in spacetime, as is expected for a vielbein [24].

In particular, the vielbein field with bosonic indices $E_{\underline{a}}{}^{\underline{b}}(\mathbf{x})$ satisfy

$$E_{\underline{a}}{}^{\underline{b}}(\mathbf{x}) = E^{\underline{b}}{}_{\underline{a}}(-\mathbf{x})\,, \tag{4.72a}$$

$$E_{\underline{a}}{}^{\underline{c}}(\mathbf{x}_i)E_{\underline{b}}{}^{\underline{d}}(\mathbf{x}_j)\eta_{\underline{cd}} = E_{\underline{ab}}(\mathbf{x}_{ij})\,, \tag{4.72b}$$

where we are denoting $E_{\underline{ab}}(\mathbf{x}) = \eta_{\underline{bc}}E_{\underline{a}}{}^{\underline{c}}(\mathbf{x})$.

We also have that

$$[E_{\underline{a}}{}^{\underline{c}}(\mathbf{x})\partial_{\underline{c}}, E_{\underline{b}}{}^{\underline{d}}(\mathbf{x})\partial_{\underline{d}}] = c_{\underline{ab}}{}^{\underline{c}}(\mathbf{x})E_{\underline{c}}{}^{\underline{d}}(\mathbf{x})\partial_{\underline{d}}\,, \tag{4.73}$$

where

$$c_{\underline{ab}}{}^{\underline{c}}(\mathbf{x}) = \delta^{+}_{[\underline{a}|} f_{+\,|\underline{b}]}{}^{\underline{c}} + \mathbf{x} f_{+\,[\underline{a}|}{}^{+} f_{+\,|\underline{b}]}{}^{\underline{c}} - 2\mathbf{x}^2 \eta_{+[\underline{a}|} f_{+\,|\underline{b}]}{}^{\underline{c}}\,, \tag{4.74}$$

and we used $\partial_{\underline{a}} \mathbf{x} = \delta^{+}_{\underline{a}}$.

As a consequence, one identifies

$$E_{+}{}^{+}\partial_{+} = \partial_{+}\,, \qquad E_{3}{}^{+}\partial_{+} = -\mathbf{x}\partial_{+}\,, \qquad E_{-}{}^{+}\partial_{+} = \mathbf{x}^2\partial_{+}\,, \tag{4.75}$$

as the generators of infinitesimal two-dimensional conformal transformations. In effect, eqs. (4.75) highlight that the conformal group acting on the boundary corresponds to the symmetry group of the bulk AdS_3 spacetime [66]. Hence, we found a standard property of the AdS/CFT correspondence folklore via a first-principles superstring theory calculation.[7] This observation might give important hints towards the correct description of superstring vertex operators in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [69] [70].

For the purpose of computing scattering amplitudes, we also define the curved structure constants

$$f_{\underline{abc}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = E_{\underline{a}}{}^{\underline{d}}(\mathbf{x}_1) E_{\underline{b}}{}^{\underline{e}}(\mathbf{x}_2) E_{\underline{c}}{}^{\underline{f}}(\mathbf{x}_3) f_{\underline{def}}\,, \tag{4.76}$$

which only depend on the distance $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, as can be easily seen from the explicit expression

$$\begin{aligned} f_{\underline{abc}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = {} & f_{\underline{abc}} + 4\Big(\mathbf{x}_{12}\eta_{\underline{c}+}\eta_{\underline{ab}} - \mathbf{x}_{13}\eta_{\underline{b}+}\eta_{\underline{ac}} + \mathbf{x}_{23}\eta_{\underline{a}+}\eta_{\underline{bc}}\Big) \\ & - 2\Big(\mathbf{x}_{12}\mathbf{x}_{13}\eta_{\underline{a}+} f_{+\underline{bc}} - \mathbf{x}_{12}\mathbf{x}_{23}\eta_{\underline{b}+} f_{+\underline{ca}} + \mathbf{x}_{13}\mathbf{x}_{23}\eta_{\underline{c}+} f_{+\underline{ab}}\Big) \\ & + 8\mathbf{x}_{12}\mathbf{x}_{13}\mathbf{x}_{23}\eta_{\underline{a}+}\eta_{\underline{b}+}\eta_{\underline{c}+}\,. \end{aligned} \tag{4.77}$$

Furthermore, this means that the curved structure constants (4.76) are invariant under a constant shift of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, i.e., they satisfy

$$f_{\underline{abc}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = f_{\underline{abc}}(\mathbf{x}_{14}, \mathbf{x}_{24}, \mathbf{x}_{34})\,, \tag{4.78}$$

for any $\mathbf{x}_4$.

[7] Recall that the conformal group in AdS_3 is $\mathrm{SO}(2,2) \cong \mathrm{SU}(1,1)_{\mathrm{L}} \times \mathrm{SU}(1,1)_{\mathrm{R}}$ and we are only displaying the holomorphic part of the worldsheet theory.

4.5 Amplitude computation from the hybrid formalism

In Section 4.2, we introduced the worldsheet action for the $\mathrm{AdS}_3 \times \mathrm{S}^3$ hybrid formalism together with the OPEs satisfied by the fundamental fields: the $\{\rho, \sigma\}$-ghosts and the $\mathrm{PSU}(1,1|2)_k$ currents. We also defined constraints that determine the physical states in a suitable gauge choice and wrote a tree-level scattering amplitude prescription. In particular, the fermionic measure of integration was described in terms of the standard spacetime SUSYs $\nabla_{\alpha 1}$.

In Sections 4.3 and 4.4, after introducing the zero-mode fermionic coordinate θ^α, we determined the gauge-fixed vertex operators for the half-BPS states. Additionally, we showed how vertex operators inserted at $\mathbf{x} = 0$ can be translated to an arbitrary position $\mathbf{x}$ in the AdS_3 boundary by the means of a similarity transformation and using the vielbein $E_A{}^B(\mathbf{x})$.

That being said, we have collected enough information to calculate tree-level $\mathrm{PSU}(1,1|2)$-covariant scattering amplitudes for half-BPS vertex operators in AdS_3. For this reason, the content of this section is to exemplify how these tools can be used in practice by computing a three-point amplitude and highlighting some new features present in this covariant approach.

4.5.1 Three-point amplitude in AdS_3

Following the prescription (4.33), the three-point amplitude for the half-BPS vertex operator $\mathcal{V}$ in (4.49) and (4.60) is given by

$$\mathcal{A}_3 = \left\langle \mathcal{V}^{(3)}(\mathbf{x}_3, z_3)\big(\widetilde{G}_0^+ \mathcal{V}^{(2)}\big)(\mathbf{x}_2, z_2)\big(G_0^+ \mathcal{V}^{(1)}\big)(\mathbf{x}_1, z_1)\right\rangle , \tag{4.79}$$

where

$$\mathcal{V}(\mathbf{x}, z) = e^{\rho + i\sigma} V(\mathbf{x}, z) , \tag{4.80a}$$

$$\big(\widetilde{G}_0^+ \mathcal{V}\big)(\mathbf{x}, z) = e^{2\rho + i\sigma} J_C^{++} V(\mathbf{x}, z) , \tag{4.80b}$$

$$\begin{aligned}\big(G_0^+ \mathcal{V}\big)(\mathbf{x}, z) = -\frac{1}{2k} e^{i\sigma} \bigg[&\frac{i}{2\sqrt{2}} \Big(K^{\alpha\beta} \nabla_{\alpha 1} \nabla_{\beta 1} V + 2 S_{\alpha 1} \nabla^{\alpha\beta} \nabla_{\beta 1} V \Big) \\ &- \widehat{\delta}^{\alpha\beta} S_{\alpha 1} \nabla_{\beta 1} V \bigg] (\mathbf{x}, z) .\end{aligned} \tag{4.80c}$$

In writing eqs. (4.80), we are ignoring terms in $\mathcal{V}$ and $\left(G_0^+ \mathcal{V}\right)$ in (4.80c) that do not contribute to the correlator: either due to the $\{\rho, \sigma\}$-ghosts background charge saturation or because it is a total derivative (and hence a null state in the CFT). For completeness, gauge-invariance of (4.80) is shown in Appendix J.

To simplify the notation, let us denote $(\mathbf{x}_i, z_i) = (\mathbf{i})$ in (4.79). The upper index in $\mathcal{V}^{(1)}$ is there to label the state, similarly for $\{\mathcal{V}^{(2)}, \mathcal{V}^{(3)}\}$. After integrating out the $\{\rho, \sigma\}$-ghosts, "integrating by parts" to eliminate the explicit z dependence and using the equation of motion $\nabla^{\alpha\beta}\nabla_{\alpha 1}\nabla_{\beta 1} V = 0$, the amplitude (4.79) reads[8]

$$\begin{aligned}
\mathcal{A}_3 = \frac{1}{2k}\frac{1}{\sqrt{2}}\bigg[\frac{i}{2}\bigg(&\Big\langle V^{(3)}(\mathbf{3})\nabla^{\alpha\beta}V^{(2)}(\mathbf{2})\nabla_{\alpha 1}\nabla_{\beta 1}V^{(1)}(\mathbf{1})\Big\rangle \\
&+ \Big\langle \nabla^{\alpha\beta}V^{(3)}(\mathbf{3})\nabla_{\alpha 1}\nabla_{\beta 1}V^{(2)}(\mathbf{2})V^{(1)}(\mathbf{1})\Big\rangle \\
&+ \Big\langle \nabla_{\alpha 1}\nabla_{\beta 1}V^{(3)}(\mathbf{3})V^{(2)}(\mathbf{2})\nabla^{\alpha\beta}V^{(1)}(\mathbf{1})\Big\rangle\bigg) \\
&+ 2\sqrt{2}\widehat{\delta}^{\alpha\beta}\Big\langle V^{(3)}(\mathbf{3})\nabla_{\alpha 1}V^{(2)}(\mathbf{2})\nabla_{\beta 1}V^{(1)}(\mathbf{1})\Big\rangle\bigg], \qquad (4.81)
\end{aligned}$$

where we wrote it in the more symmetric form. For the latter, we used the identity

$$\begin{aligned}
i\Big\langle & V^{(3)}(\mathbf{3})\nabla_{\alpha 1}(x_1)V^{(2)}(\mathbf{2})\left(\nabla_{\beta 1}\nabla^{\alpha\beta}V^{(1)}\right)(\mathbf{1})\Big\rangle \\
&= \frac{i}{2}\Big\langle \nabla^{\alpha\beta}V^{(3)}(\mathbf{3})\nabla_{\alpha 1}\nabla_{\beta 1}V^{(2)}(\mathbf{2})V^{(1)}(\mathbf{1})\Big\rangle \\
&+ \frac{i}{2}\Big\langle \nabla_{\alpha 1}\nabla_{\beta 1}V^{(3)}(\mathbf{3})V^{(2)}(\mathbf{2})\nabla^{\alpha\beta}V^{(1)}(\mathbf{1})\Big\rangle. \qquad (4.82)
\end{aligned}$$

The last term of eq. (4.81) is not present in the flat space calculation and, therefore, it corresponds to a curvature correction.

We should underscore the fact that only the holomorphic part of the scattering amplitude $\mathcal{A}_3$ is being written. As in any closed string calculation where holomorphic/anti-holomorphic factorization takes place [88], one needs to multiply eq. (4.81) with the corresponding right-moving contribution to get the complete answer. Strictly speaking, this means that the amplitude (4.81) is $\mathrm{PSU}(1,1|2)_\mathrm{L} \times \mathrm{PSU}(1,1|2)_\mathrm{R}$-covariant. In particular, we remarked in our discussion of the hybrid formalism in Section 4.2 that the $\mathrm{PSU}(1,1|2)_\mathrm{L}$ currents are

[8] We have checked that this partial integration produces the same answer before and after integrating over the worldsheet fermions. The reason for this is that the fermionic measure is invariant under $\nabla_{\underline{a}}$ and $\nabla_{\alpha 1}$, which are the zero-modes appearing in the vertices.

purely holomorphic and the $\mathrm{PSU}(1,1|2)_\mathrm{R}$ currents purely anti-holomorphic.

4.5.2 Integrating out the fermions

Eq. (4.81) gives a $\mathrm{PSU}(1,1|2)$-covariant expression for the three-point amplitude of half-BPS states in $\mathrm{AdS}_3 \times \mathrm{S}^3$. Let us now illustrate how the integration over the curved fermionic worldsheet variables (4.67) can be implemented with an example.

For simplicity, we will take $V^{(i)} = \frac{i}{2}(\theta\sigma^{\underline{a}}\theta)a_{\underline{a}\,i}$, so that only states from the NS-sector are being considered. After integrating out the θ's using the prescription (4.67), the amplitude (4.81) for the NS states becomes

$$\begin{aligned}\mathcal{A}_3^{\mathrm{NS}} = -\frac{1}{2k}\frac{1}{\sqrt{2}}E_{\underline{a}}{}^{\underline{d}}(\mathbf{x}_1)E_{\underline{b}}{}^{\underline{e}}(\mathbf{x}_2)E_{\underline{c}}{}^{\underline{f}}(\mathbf{x}_3)\Big[&\eta_{\underline{de}}a_3^{\underline{c}}(\mathbf{3})a_2^{\underline{b}}(\mathbf{2})\left(D_{\underline{f}}(-\mathbf{x}_1)a_1^{\underline{a}}\right)(\mathbf{1})\\ &+\eta_{\underline{ef}}a_3^{\underline{c}}(\mathbf{3})\left(D_{\underline{d}}(-\mathbf{x}_2)a_2^{\underline{b}}\right)(\mathbf{2})a_1^{\underline{a}}(\mathbf{1})+\eta_{\underline{df}}\left(D_{\underline{e}}(-\mathbf{x}_3)a_3^{\underline{c}}\right)(\mathbf{3})a_2^{\underline{b}}(\mathbf{2})a_1^{\underline{a}}(\mathbf{1})\\ &-\frac{1}{2}f_{\underline{def}}a_3^{\underline{c}}(\mathbf{3})a_2^{\underline{b}}(\mathbf{2})a_1^{\underline{a}}(\mathbf{1})\Big]\,,\end{aligned} \tag{4.83}$$

which one can write in the more compact form as

$$\begin{aligned}\mathcal{A}_3^{\mathrm{NS}} = -\frac{1}{2k}\frac{1}{\sqrt{2}}\Big[&E_{\underline{ab}}(\mathbf{x}_{12})a_3^{\underline{c}}(\mathbf{3})a_2^{\underline{b}}(\mathbf{2})\left(D_{\underline{c}}(\mathbf{x}_{31})a_1^{\underline{a}}\right)(\mathbf{1})\\ &+E_{\underline{bc}}(\mathbf{x}_{23})a_3^{\underline{c}}(\mathbf{3})\left(D_{\underline{a}}(\mathbf{x}_{12})a_2^{\underline{b}}\right)(\mathbf{2})a_1^{\underline{a}}(\mathbf{1})\\ &+E_{\underline{ac}}(\mathbf{x}_{13})\left(D_{\underline{b}}(\mathbf{x}_{23})a_3^{\underline{c}}\right)(\mathbf{3})a_2^{\underline{b}}(\mathbf{2})a_1^{\underline{a}}(\mathbf{1})\\ &-\frac{1}{2}f_{\underline{abc}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)a_3^{\underline{c}}(\mathbf{3})a_2^{\underline{b}}(\mathbf{2})a_1^{\underline{a}}(\mathbf{1})\Big]\,,\end{aligned} \tag{4.84}$$

where $D_{\underline{a}}(\mathbf{x}_{12}) = E_{\underline{a}}{}^{\underline{b}}(\mathbf{x}_{12})D_{\underline{b}}$ and $f_{\underline{abc}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ are the covariant derivative (4.53) and the structure constants with curved indices (4.77). We are also using that

$$E_{\underline{a}}{}^{\underline{c}}(\mathbf{x}_i)E_{\underline{b}}{}^{\underline{d}}(\mathbf{x}_j)\eta_{\underline{cd}} = E_{\underline{ab}}(\mathbf{x}_{ij})\,. \tag{4.85}$$

The design of the amplitude (4.84) begs for an interpretation. As we have alluded to below eq. (4.75), the spacetime vielbein field $E_{\underline{a}}{}^{\underline{b}}(\mathbf{x})$ encodes that a conformal transformation in the AdS_3 boundary corresponds to a rotation in the AdS_3 bulk. In particular, this can be seen by the observation that the object $E_a{}^{+}\partial_{+}$ generates infinitesimal Möbius transformations along $\partial\mathrm{AdS}_3$.

Moreover, from eq. (4.84), one can explicitly deduce that the consequence of integrating out the fermionic worldsheet fields in the correlator was the appearance of the vielbein field $E_{\underline{a}}{}^{\underline{b}}(\mathbf{x})$. In other worlds, the tangent space vector indices were "rotated" by the matrix $E_{\underline{a}}{}^{\underline{b}}(\mathbf{x})$. This rotation also affected the indices of the structure constants $f_{\underline{abc}}$ which became $f_{\underline{abc}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of (4.76).

Let us point out that what is left in (4.84) is the kinematic factor of the three-point amplitude written in terms of the component fields from the NS-sector. Using the vertex operator (4.50a), one can similarly write the kinematic factor involving the states from the R-sector.

In addition, by conformal invariance in the worldsheet and target-space, the z and $\mathbf{x}$ dependence of the amplitude is completely fixed [85, eq. (2.13)]. More precisely, the amplitude is independent of z, and the $\mathbf{x}$ dependence is determined by the $\mathrm{SL}(2,\mathbb{R})$ spin j_i of the insertions in (4.79). So that it takes the general form [85]

$$\mathcal{A}_3 \sim \mathbf{x}_{12}^{j_3-j_1-j_2}\mathbf{x}_{13}^{j_2-j_1-j_3}\mathbf{x}_{23}^{j_1-j_2-j_3}\,. \tag{4.86}$$

In Appendix I.2, we give a brief explanation on how the $\mathrm{SL}(2,\mathbb{R})$ spin j_i for the fermionic coordinates and component fields can be derived from our worldsheet variables. In particular, note that the variables θ^α in the vertex carry a non-zero charge, see eqs. (I.8).

4.6 Comparison between hybrid and RNS

Up to now, in the calculations displayed throughout this work, we have used the hybrid description written in terms of the supergroup variable g (or the $\mathrm{PSU}(1,1|2)$ currents) as in the worldsheet action (4.3). Even the definition of the fermionic zero-mode variable θ^α in Section 4.3.1 could be motivated in this formulation, which is the best suited for the study of the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$ since it generalizes to the case where a non-zero amount of R-R self-dual three-form flux is turned on [46].

That being the case, the hybrid formalism in $\mathrm{AdS}_3 \times \mathrm{S}^3$ with pure NS-NS three-form flux can also be written in terms of bosonic currents $\mathcal{J}_{\underline{a}}$ and free fermions $\{p_\alpha, \theta^\alpha\}$. As was mentioned above eqs. (4.42), this hinges on the fact that the matter part of the RNS formalism in the pure NS-NS $\mathrm{AdS}_3 \times \mathrm{S}^3$ target-space is given in terms of the bosonic currents $\mathcal{J}_{\underline{a}}$ and the six free fermions $\psi_{\underline{a}}$ [81] [82]

[89]. Therefore, the free RNS fermions $\psi_{\underline{a}}$ plus $\{\beta, \gamma\}$-ghosts are related to the free fermions $\{p_\alpha, \theta^\alpha\}$ in (4.39).

In this section, we will further explore this correspondence between hybrid and RNS variables to compare the vertex operators and amplitude computation for the NS-sector states in Section 4.5 with the analogous calculation in the RNS formalism. Achieving the same result with a different method should give further support to our construction.

4.6.1 From hybrid to RNS variables

Let us identify, in RNS language, the contributions to the three-point amplitude (4.79) for the NS-sector states. In terms of the RNS variables, the hybrid formalism worldsheet fields can be expressed as [18] [46]

$$S_{\alpha 1} = e^{-\frac{\phi}{2}} e^{-\frac{i}{2} H_C^{\mathrm{RNS}}} S_\alpha \,, \qquad \theta^\alpha = S^\alpha e^{\frac{i}{2} H_C^{\mathrm{RNS}}} e^{\frac{\phi}{2}} \,, \tag{4.87a}$$

$$e^\rho = e^{-2\phi + i\chi - iH_C^{\mathrm{RNS}}} \,, \qquad J_C^{++} = e^{-2i\chi + 2\phi + iH_C^{\mathrm{RNS}}} \,, \tag{4.87b}$$

where S_α is the spin-field for the six-dimensional part and the boson H_C^{RNS} defines the spin-field for the compactified directions.[9]

The field $e^{i\sigma}$ in the six-dimensional hybrid formalism is the c-ghost of the RNS description in bosonized form and $e^{-i\sigma}$ the b-ghost. Similarly, the chiral bosons $\{\phi, \chi\}$ come from the superconformal ghosts

$$\beta = e^{-\phi} \partial \xi = e^{-\phi} \partial e^{i\chi} \,, \qquad \gamma = \eta e^{\phi} = e^{-i\chi} e^{\phi} . \tag{4.88}$$

Note that the RNS variables obey the usual OPEs

$$H_C^{\mathrm{RNS}}(y) H_C^{\mathrm{RNS}}(z) \sim -2 \log(y-z) \,, \qquad \sigma(y)\sigma(z) \sim -\log(y-z) \,, \tag{4.89a}$$

$$\phi(y)\phi(z) \sim -\log(y-z) \,, \qquad \chi(y)\chi(z) \sim -\log(y-z) \,. \tag{4.89b}$$

Consequently, in terms of the RNS description, we have that

$$\mathcal{V} = \xi V_{\mathrm{hyb}}^{-1} \,, \tag{4.90a}$$

$$\widetilde{G}_0^+ \mathcal{V} = V_{\mathrm{hyb}}^{-1} \,, \tag{4.90b}$$

$$G_0^+ \mathcal{V} = V_{\mathrm{hyb}}^{0} \,, \tag{4.90c}$$

[9] We apologyze for using the letter S both for the RNS spin-field S_α and for the PSU$(1,1|2)$ fermionic currents $S_{\alpha j}$. Since the currents also carry an SU(2) index, this notation is unambiguous.

where

$$V^{-1}_{\text{hyb}} = c\psi_{\underline{a}} e^{-\phi} a^{\underline{a}}\,, \tag{4.91a}$$

$$V^{0}_{\text{hyb}} = -\frac{1}{2k}\frac{1}{\sqrt{2}}c\left(K_{\underline{a}}a^{\underline{a}} + \psi^{\underline{a}}\psi^{\underline{b}}\nabla_{\underline{a}}a_{\underline{b}}\right), \tag{4.91b}$$

with V^{-1}_{hyb} and V^{0}_{hyb} being vertex operators in the -1 and zero picture for the NS-sector massless states, respectively. The $\mathrm{SU}(1,1)_{k+2} \times \mathrm{SU}(2)_{k-2}$ current $\mathcal{J}_{\underline{a}}$ decouples from the fermions and is defined in eqs. (4.40) and (4.41). We emphasize that no excitations in the compactified directions are being considered.

To get to eqs. (4.91), we used the following identifications between the RNS and hybrid fermionic fields in the vertex operators

$$f_{\underline{a}\,\alpha 1}{}^{\beta 1} S_{\beta 1}\theta^{\alpha} = \frac{1}{2} f_{\underline{abc}}\psi^{\underline{c}}\psi^{\underline{b}}\,, \tag{4.92a}$$

$$(\sigma^{\underline{bc}})^{\beta}{}_{\alpha} S_{\beta 1}\theta^{\alpha} = -i\psi^{\underline{b}}\psi^{\underline{c}}\,, \tag{4.92b}$$

$$K_{\underline{a}} = \mathcal{J}_{\underline{a}} + \frac{1}{2} f_{\underline{abc}}\psi^{\underline{c}}\psi^{\underline{b}}\,. \tag{4.92c}$$

4.6.2 RNS formalism in $\mathrm{AdS}_3 \times \mathrm{S}^3$

Of course, one can arrive at the vertex operators (4.91) directly from the RNS description of $\mathrm{AdS}_3 \times \mathrm{S}^3$, which is given by a bosonic $\mathrm{SU}(1,1)_{k+2} \times \mathrm{SU}(2)_{k-2}$ current algebra plus six free fermions $\psi_{\underline{a}}$. Needless to say, one should consider the GSO projected theory in order to eliminate the tachyons in RNS [20] [81]. This comes in contrast with the hybrid description, in which the physical states are automatically GSO projected [17].

In terms of the RNS variables, the bosonic currents of $\mathrm{PSU}(1,1|2)_k$ are the same as in (4.92c) and read

$$K_{\underline{a}} = \mathcal{J}_{\underline{a}} + \frac{1}{2} f_{\underline{abc}}\psi^{\underline{c}}\psi^{\underline{b}}\,, \tag{4.93}$$

where the currents $\mathcal{J}_{\underline{a}}$ are defined in eqs. (4.40) and (4.41) and $\psi_{\underline{a}}$ are the six free worldsheet fermions of the RNS formalism satisfying the usual OPE relation

$$\psi_{\underline{a}}(y)\psi_{\underline{b}}(z) \sim (y-z)^{-1}\eta_{\underline{ab}}\,. \tag{4.94}$$

Under the $\mathrm{PSU}(1,1|2)$ bosonic currents, the fermions transform in the adjoint representation

$$K_{\underline{a}}(y)\psi_{\underline{b}}(z) \sim (y-z)^{-1} f_{\underline{ab}}{}^{\underline{c}}\psi_{\underline{c}}\,. \tag{4.95}$$

Recall that the structure constants are defined in eqs. (4.7).

The currents $\mathcal{J}_{\underline{a}}$ have no poles with the fermions $\psi_{\underline{a}}$ and the $\mathcal{N}=1$ supercurrent of the RNS formalism for the $\mathrm{AdS}_3 \times \mathrm{S}^3$ part is

$$G_6 = \frac{i}{\sqrt{2k}}\left(\mathcal{J}^{\underline{a}}\psi_{\underline{a}} + \frac{1}{6} f_{\underline{abc}}\psi^{\underline{c}}\psi^{\underline{b}}\psi^{\underline{a}}\right), \tag{4.96}$$

that, together with the stress-tensor,

$$T_6 = -\frac{1}{4k}\mathcal{J}_{\underline{a}}\mathcal{J}_{\underline{b}}\eta^{\underline{ab}} - \frac{1}{2}\psi_{\underline{a}}\partial\psi_{\underline{b}}\eta^{\underline{ab}}\,, \tag{4.97}$$

generate a $c=9$ $\mathcal{N}=1$ SCA. The four bosons and four fermions for the compactification directions generate a $c=6$ $\mathcal{N}=1$ SCA whose supercurrent and stress-tensor we denote by G_C^{RNS} and T_C^{RNS}, respectively. In total, one has the usual matter $c=15$ $\mathcal{N}=1$ SCA of the RNS description

$$T_{\mathrm{m}}(y)T_{\mathrm{m}}(z) \sim \frac{\frac{c}{2}}{(y-z)^4} + \frac{2T_{\mathrm{m}}(z)}{(y-z)^2} + \frac{\partial T_{\mathrm{m}}(z)}{(y-z)}\,, \tag{4.98a}$$

$$T_{\mathrm{m}}(y)G_{\mathrm{m}}(z) \sim \frac{\frac{3}{2}G_{\mathrm{m}}(z)}{(y-z)^2} + \frac{\partial G_{\mathrm{m}}(z)}{(y-z)}\,, \tag{4.98b}$$

$$G_{\mathrm{m}}(y)G_{\mathrm{m}}(z) \sim \frac{\frac{2}{3}c}{(y-z)^3} + \frac{2T_{\mathrm{m}}(z)}{(y-z)}\,, \tag{4.98c}$$

with generators $\{G_{\mathrm{m}} = G_6 + G_C^{\mathrm{RNS}}, T_{\mathrm{m}} = T_6 + T_C^{\mathrm{RNS}}\}$.

The NS-sector massless unintegrated vertex operators in the -1 and zero picture are

$$V_{\mathrm{RNS}}^{-1} = c\psi_{\underline{a}}e^{-\phi}a^{\underline{a}}\,, \tag{4.99a}$$

$$\begin{aligned} V_{\mathrm{RNS}}^{0} &= -\frac{1}{2k}\frac{1}{\sqrt{2}}c\left(\mathcal{J}_{\underline{a}}a^{\underline{a}} + \frac{1}{2}f_{\underline{abc}}\psi^{\underline{c}}\psi^{\underline{b}}a^{\underline{a}} + \psi^{\underline{a}}\psi^{\underline{b}}(\mathcal{J}_{\underline{a}})_0 a_{\underline{b}}\right) \\ &= -\frac{1}{2k}\frac{1}{\sqrt{2}}c\left(K_{\underline{a}}a^{\underline{a}} + \psi^{\underline{a}}\psi^{\underline{b}}(\mathcal{J}_{\underline{a}})_0 a_{\underline{b}}\right), \end{aligned} \tag{4.99b}$$

where $(\mathcal{J}_{\underline{a}})_0$ is the zero-mode of the current $\mathcal{J}_{\underline{a}}$ and $K_{\underline{a}}$ is defined in eq. (4.93). Up to a constant $V^0_{\mathrm{RNS}} = ZV^{-1}_{\mathrm{RNS}}$, where

$$\begin{aligned} Z &= 2Q_{\mathrm{RNS}}e^{i\chi} \\ &= G_{\mathrm{m}}e^{\phi} + b\partial e^{-i\chi}e^{2\phi} + \frac{1}{2}\partial\big(be^{-i\chi}e^{2\phi}\big) + 2c\partial e^{i\chi}\,, \end{aligned} \tag{4.100}$$

is the picture-changing operator [20]. The BRST operator in the RNS formalism is[10]

$$\begin{aligned} Q_{\mathrm{RNS}} &= \oint j_{\mathrm{BRST}} \\ &= \oint \Big(c\big(T_{\mathrm{m}} + T_{\phi,\chi}\big) + bc\partial c - \frac{1}{2}e^{-i\chi+\phi}G_{\mathrm{m}} + \frac{1}{4}be^{-2i\chi+2\phi} + \partial^2 c + \partial\big(\partial(i\chi)c\big)\Big)\,. \end{aligned} \tag{4.101}$$

Since the zero-mode o $\mathcal{J}_{\underline{a}}$ acts on $a_{\underline{a}}$ as the zero-mode of $K_{\underline{a}}$. We can write V_0^{RNS} in the form

$$V_0^{\mathrm{RNS}} = -\frac{1}{2k}\frac{1}{\sqrt{2}}c\Big(K_{\underline{a}}a^{\underline{a}} + \psi^{\underline{a}}\psi^{\underline{b}}\nabla_{\underline{a}}a_{\underline{b}}\Big)\,, \tag{4.102}$$

which precisely matches the vertex operator (4.91b) found by the field redefinition from the hybrid formalism.

4.6.3 Three-point amplitude in RNS variables

We can now use the tools developed in this section to compute the three-point amplitude (4.84) for the NS-sector states inserted in the AdS_3 boundary directly in terms of the RNS formalism prescription.

As before, for fields depending on the boundary coordinates, we have

$$\begin{aligned} \psi_{\underline{a}}(\mathbf{x}) &= e^{\mathbf{x}\nabla_+}\psi_{\underline{a}}e^{-\mathbf{x}\nabla_+} \\ &= E_{\underline{a}}{}^{\underline{b}}(\mathbf{x})\psi_{\underline{b}}\,, \end{aligned} \tag{4.103}$$

where $E_{\underline{a}}{}^{\underline{b}}(\mathbf{x})$ is given by (4.58). Hence, the fundamental OPEs read

$$\psi_{\underline{a}}(\mathbf{x}_i,y)\psi_{\underline{b}}(\mathbf{x}_j,z) \sim (y-z)^{-1}E_{\underline{a}\underline{b}}(\mathbf{x}_{ij})\,, \tag{4.104a}$$

[10] The option for the total derivative added in the BRST current j_{BRST} is chosen such that the double pole between j_{BRST} and b is given by the ghost- minus the picture-current. For that reason, j_{BRST} gets mapped to the $\mathcal{N}=2$ superconformal generator G^+ (4.20b).

$$K_{\underline{a}}(\mathbf{x}_i, y)\psi_{\underline{b}}(\mathbf{x}_j, z) \sim (y-z)^{-1} E_{\underline{a}}{}^{\underline{c}}(\mathbf{x}_i) E_{\underline{b}}{}^{\underline{d}}(\mathbf{x}_j) f_{\underline{cd}}{}^{\underline{e}} \psi_{\underline{e}} \,. \tag{4.104b}$$

Considering the vertex operators (4.99), the three-point amplitude for the NS-sector states becomes

$$\begin{aligned}
\mathcal{A}_3^{\mathrm{NS,RNS}} &= \left\langle V_{-1}^{\mathrm{RNS}}(\mathbf{3}) V_{-1}^{\mathrm{RNS}}(\mathbf{2}) V_0^{\mathrm{RNS}}(\mathbf{1}) \right\rangle \\
&= -\frac{1}{2k}\frac{1}{\sqrt{2}}\Big[E_{\underline{ab}}(\mathbf{x}_{12}) a_3^{\underline{c}}(\mathbf{3}) a_2^{\underline{b}}(\mathbf{2}) \left(D_{\underline{c}}(\mathbf{x}_{31}) a_1^{\underline{a}} \right)(\mathbf{1}) \\
&+ E_{\underline{bc}}(\mathbf{x}_{23}) a_3^{\underline{c}}(\mathbf{3}) \left(D_{\underline{a}}(\mathbf{x}_{12}) a_2^{\underline{b}} \right)(\mathbf{2}) a_1^{\underline{a}}(\mathbf{1}) \\
&+ E_{\underline{ac}}(\mathbf{x}_{13}) \left(D_{\underline{b}}(\mathbf{x}_{23}) a_3^{\underline{c}} \right)(\mathbf{3}) a_2^{\underline{b}}(\mathbf{2}) a_1^{\underline{a}}(\mathbf{1}) \\
&- \frac{1}{2} f_{\underline{abc}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) a_3^{\underline{c}}(\mathbf{3}) a_2^{\underline{b}}(\mathbf{2}) a_1^{\underline{a}}(\mathbf{1}) \Big] \,,
\end{aligned} \tag{4.105}$$

which precisely matches (4.84), as we wanted to show. This calculation gives further support for our construction using the supergroup variables and the fermionic zero-mode coordinates $\theta^\alpha(\mathbf{x})$ in the hybrid formalism.

Lastly, let us mention that, under the field redefinition (4.87), the RNS tree-level zero-mode integration gets mapped to the hybrid measure of Section 4.2.4 only if one works in the large Hilbert space, namely,

$$\xi c \partial c \partial^2 c e^{-2\phi} \sim e^{3\rho + 3i\sigma} J_C^{++}(\theta)^4 \,. \tag{4.106}$$

4.7 The superstring in the mixed flux $\mathrm{AdS}_3 \times \mathrm{S}^3$ background with manifest $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$

The superstring compactified on T^4 and propagating in $\mathrm{AdS}_3 \times \mathrm{S}^3$ can be described by a mixture of NS-NS and R-R self-dual three-form flux [46]. In this section, after writing a general ansatz for the sigma-model action, we will begin by identifying the background superfields appearing in the theory in Section 4.7.1, and then write the worldsheet action for the mixed flux $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background in Section 4.7.2. Subsequently, it will be shown how to derive the sigma-model action in Section 4.7.3 by substituting the values of the background superfields. We will further confirm the latter result via a perturbative analysis in Section 4.7.4.

4.7.1 Type IIB worldsheet action in a six-dimensional curved background

In order to identify the background superfields and before delving into the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ target-space, let us start by discussing the worldsheet action in an arbitrary curved six-dimensional background. A reasonable guess for the general form of the action can be inferred from the structure of the integrated vertex operator (3.55) (see also (4.144)) [90]

$$
\begin{aligned}
S = \int d^2z \, \Big(& \frac{1}{2} J^{\underline{b}} \overline{J}^{\underline{a}} \eta_{\underline{ab}} + J^B \overline{J}^A B_{AB} + d_{\alpha j} \overline{J}^{\alpha j} + \widehat{d}_{\widehat{\alpha} j} J^{\widehat{\alpha} j} + d_{\alpha j} \widehat{d}_{\widehat{\beta} k} F^{\alpha j \widehat{\beta} k} \\
& + N_{\underline{ab}} \widehat{d}_{\widehat{\beta} k} C^{\widehat{\beta} k \, \underline{ab}} + \widehat{N}_{\underline{ab}} d_{\alpha j} \widehat{C}^{\alpha j \, \underline{ab}} + w_\alpha \overline{\nabla} \lambda^\alpha + \widehat{w}_{\widehat{\alpha}} \nabla \widehat{\lambda}^{\widehat{\alpha}} - \frac{1}{4} R^{\underline{abcd}} N_{\underline{ab}} \widehat{N}_{\underline{cd}} \Big) \\
& + S_{\rho,\sigma} + S_C \,,
\end{aligned} \tag{4.107}
$$

where $S_{\rho,\sigma}$ is the action for the chiral bosons of the six-dimensional hybrid formalism and S_C is the action for the four-dimensional compactification manifold of T^4 [46]. In writing eq. (4.107), we are considering only constant deformations in the R-R superfield-strength $F^{\alpha j \widehat{\beta} k}$ and in the superfields $\{C^{\widehat{\beta} k \, \underline{ab}}, \widehat{C}^{\alpha j \, \underline{ab}}, R^{\underline{abcd}}\}$, so that the $\{\rho, \sigma\}$-ghosts decouple in the integrated vertex operator. This assumption will be enough for writing a consistent worldsheet action in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$.

Similarly as in the six-dimensional hybrid formalism, it is possible that higher-order terms in $F^{\alpha j \widehat{\beta} k}$ appear in (4.107) (see [46] eq. (8.39)) which couple the $\{\rho, \sigma\}$-ghosts to the matter and the $\{\lambda^\alpha, w_\alpha\}$ ghost variables in this case. We will not be concerned in determining them since, as we will see, this gives a consistent worldsheet action for the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$. Additionally, our result will be related to the hybrid description in Section 4.9.

In eq. (4.107), the worldsheet fields $\{J^A_z, J^A_{\overline{z}}\}$ are the pullback of the target space super-vielbein $J^A = dZ^M E_M{}^A$, where $Z^M = \{x^m, \theta^{\mu j}, \widehat{\theta}^{\widehat{\mu} j}\}$ are the curved supercoordinates. The indice $M = \{m, \mu j, \widehat{\mu} j\}$ labels the curved superspace indices and $A = \{\underline{a}, \alpha j, \widehat{\alpha} j\}$ labels the tangent superspace indices. As usual, we will write $J^A_z = J^A$, $J^A_{\overline{z}} = \overline{J}^A$, $\nabla_z = \nabla$ and $\nabla_{\overline{z}} = \overline{\nabla}$ to simplify the notation, we hope the context of the equation is enough for not causing confusion with the corresponding one-forms. The inverse of the super-vielbein matrix $E_M{}^A$ is denoted as $E_A{}^M$ and it is responsible for connecting curved and flat indices [24].

Since in a curved background the separation between left- and right-movers is lost, we use a “hat” on top of the worldsheet variables which are purely anti-holomorphic in flat target-space. Also, we are interested in writing (4.107) in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background and so we can ignore the Fradkin-Tseytlin term which couples the dilaton to the worldsheet curvature. The reason for this is that the dilaton is constant in $\mathrm{AdS}_3 \times \mathrm{S}^3$ and, therefore, it will contribute the usual coupling constant dependence in scattering amplitudes. Moreover, given that there is no $p_{\alpha j}$ and $\widehat{p}_{\widehat{\alpha} j}$ in (4.107), we can treat $d_{\alpha j} = p_{\alpha j} + \ldots$ and $\widehat{d}_{\widehat{\alpha} j} = \widehat{p}_{\widehat{\alpha} j} + \ldots$ as independent variables.

The covariant derivatives ∇ and $\overline{\nabla}$ are defined using the pullback of the spin-connections $\Omega_\alpha{}^\beta = dZ^M \Omega_{M\alpha}{}^\beta$ and $\widehat{\Omega}_{\widehat{\alpha}}{}^{\widehat{\beta}} = dZ^M \widehat{\Omega}_{M\widehat{\alpha}}{}^{\widehat{\beta}}$. Their action on the ghosts $\{\lambda^\alpha, \widehat{\lambda}^{\widehat{\alpha}}\}$ is

$$\overline{\nabla}\lambda^\alpha = \overline{\partial}\lambda^\alpha + \lambda^\beta \overline{\Omega}_\beta{}^\alpha \,, \qquad \nabla\widehat{\lambda}^{\widehat{\alpha}} = \partial\widehat{\lambda}^{\widehat{\alpha}} + \widehat{\lambda}^{\widehat{\beta}} \widehat{\Omega}_{\widehat{\beta}}{}^{\widehat{\alpha}} \,. \tag{4.108}$$

We covariantize superspace derivatives acting on "un-hatted" and "hatted" spinor indices using $\Omega_\alpha{}^\beta$ and $\widehat{\Omega}_{\widehat{\alpha}}{}^{\widehat{\beta}}$, respectively. In general, the action of the covariant derivative one-form ∇ on a q-form Y^A is defined by

$$\nabla Y^A = dY^A + Y^B \Omega_B{}^A \,, \tag{4.109}$$

where $\Omega_B{}^A = dZ^M \Omega_{MB}{}^A$ is the connection one-form.

The background superfields $\{B_{AB}, F^{\alpha j \widehat{\beta} k}, C^{\widehat{\alpha} j\, \underline{ab}}, \widehat{C}^{\alpha j\, \underline{ab}}\}$ are functions of the zero-modes of $\{x^{\underline{a}}, \theta^{\alpha j}, \widehat{\theta}^{\widehat{\alpha} j}\}$. More specifically, the superfield B_{AB} is the superspace two-form potential and the lowest component of the superfield $B_{\underline{ab}}$ is the NS-NS two-form $b_{\underline{ab}}$. The lowest component of $F^{\alpha j \widehat{\beta} k}$ is the R-R field-strength $f^{\alpha j \widehat{\beta} k}$, the lowest component of $R^{\underline{abcd}}$ is related to the Riemann curvature and the lowest components of $C^{\widehat{\alpha} j\, \underline{ab}}$ and $\widehat{C}^{\alpha j\, \underline{ab}}$ are related to the gravitini and dilatini [90]. The worldsheet fields $N_{\underline{ab}} = w_\alpha (\sigma_{\underline{ab}})^\alpha{}_\beta \lambda^\beta$ and $\widehat{N}_{\underline{ab}} = \widehat{w}_{\widehat{\alpha}} (\sigma_{\underline{ab}})^{\widehat{\alpha}}{}_{\widehat{\beta}} \widehat{\lambda}^{\widehat{\beta}}$ are the Lorentz currents for the bosonic ghosts $\{w_\alpha, \lambda^\beta, \widehat{w}_{\widehat{\alpha}}, \widehat{\lambda}^{\widehat{\beta}}\}$.

One way to accomplish writing the action (4.107) in an $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background with mixed NS-NS and R-R three-form flux is to explicitly substitute the values for the background superfields appearing in (4.107) in the presence of a constant R-R field-strength $f^{\alpha j \widehat{\beta} k}$ and a suitable NS-NS two-form $b_{\underline{ab}}$ such that the supergravity constraints are satisfied [90].

Equivalently, one can start with the superstring propagating in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with pure R-R flux. In the presence of a constant R-R three-form flux parametrized by f_{RR}, the lowest component of the background superfield $F^{\alpha j \widehat{\beta} k}$ is non-zero and invertible, consequently, the worldsheet variables $d_{\alpha j}$ and $\widehat{d}_{\widehat{\alpha} j}$ can be integrated out from eq. (4.107). The result is a sigma-model with a supermanifold as a target-space, where the six-dimensional part is described by the superspace coordinates $\{x^{\underline{a}}, \theta^{\alpha j}, \widehat{\theta}^{\widehat{\alpha} j}\}$ plus ghosts. In the latter case, turning on a constant NS-NS three-form flux parametrized by f_{NS} corresponds to adding a Wess-Zumino (WZ) term to the pure R-R three-form flux worldsheet action in $\mathrm{AdS}_3 \times \mathrm{S}^3$.

In particular, this strategy was used in [46] from the six-dimensional hybrid formalism to describe the mixed flux action from the supergroup $\mathrm{PSU}(1,1|2)$. In Section 4.7.3, we will show that starting with a constant R-R three-form flux sigma-model with suitable rescalings of the currents, integrating out the worldsheet fields $d_{\alpha j}$ and $\widehat{d}_{\widehat{\alpha} j}$ in (4.107), modifying the two-form potential $B_{\alpha j \widehat{\beta} k}$ to accomodate the mixed flux background, and adding a WZ term corresponding to turning on the NS-NS three-form, one obtains a description of the superstring in an $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background with mixed NS-NS and R-R three-form flux constructed from the group element $g \in \frac{\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2) \times \mathrm{SO}(3)}$. Moreover, after taking the limit $f_{RR} \to 0$, it is found a description of the pure NS-NS model with the super-coset $\frac{\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2) \times \mathrm{SO}(3)}$ as the target superspace. The latter is the analogue of the WZW model of $\mathrm{PSU}(1,1|2)$ found in [46] for the superstring in AdS_3.

4.7.2 The sigma-model on the supergroup

As was pointed out in refs. [91] [44], the Type IIB superstring compactified on T^4 and propagating in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background with pure R-R flux can be described by the super-coset

$$\frac{\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2) \times \mathrm{SO}(3)}, \tag{4.110}$$

whose bosonic part is $\frac{\mathrm{SO}(2,2) \times \mathrm{SO}(4)}{\mathrm{SO}(1,2) \times \mathrm{SO}(3)} = \frac{\mathrm{SO}(2,2)}{\mathrm{SO}(1,2)} \times \frac{\mathrm{SO}(4)}{\mathrm{SO}(3)} = \mathrm{AdS}_3 \times \mathrm{S}^3$. Furthermore, in this background, the super-vielbein $J^A = dZ^M E_M{}^A$ and the connection one-form $J^{[\underline{ab}]}$ can be identified with the left-invariant one-forms [91]

$$J^{\underline{A}} = (g^{-1} dg)^{\underline{A}}, \tag{4.111}$$

where $\underline{A} = \{[\underline{ab}], A\}$ and $g(x, \theta, \widehat{\theta})$ takes values in the supercoset $\frac{\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2) \times \mathrm{SO}(3)}$.

Note that the index $\underline{A} = \{[\underline{ab}], \alpha j, \underline{a}, \widehat{\alpha} j\}$, so that it ranges over the 12 bosonic and the 16 fermionic generators $T_{\underline{A}} = \{T_{[\underline{ab}]}, T_{\alpha j}, T_{\underline{a}}, T_{\widehat{\alpha} j}\}$ of the Lie superalgebra of

$\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$. More precisely, indices $[\underline{ab}]$ correspond to the $\mathrm{SO}(1,2) \times \mathrm{SO}(3)$ generators, $\underline{a} = \{0 \text{ to } 5\}$ to the translation generators and $\alpha, \widehat{\alpha} = \{1 \text{ to } 4\}$ together with $j = \{1,2\}$ to the supersymmetry generators. In particular, $\underline{a} = \{a, a'\}$ with $a = \{0,1,2\}$ corresponding to the AdS_3 directions and $a' = \{3,4,5\}$ to the S^3 directions. Consequently, the isotropy group generators split as $T_{[\underline{ab}]} = \{T_{[ab]}, T_{[a'b']}\}$.

The generators $T_{\underline{A}}$ of the Lie superalgebra of $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ satisfy the graded Lie-bracket

$$[T_{\underline{A}}, T_{\underline{B}}\} = i f_{\underline{AB}}{}^{\underline{C}} T_{\underline{C}} \,, \qquad [T_{\underline{A}}, T_{\underline{B}}\} = T_{\underline{A}} T_{\underline{B}} - (-)^{|\underline{A}||\underline{B}|} T_{\underline{B}} T_{\underline{A}} \,, \tag{4.112}$$

where we define $|\underline{A}| = 0$ if it corresponds to a bosonic and $|\underline{A}| = 1$ if it corresponds to a fermionic indice. The non-vanishing structure constants $f_{\underline{AB}}{}^{\underline{C}}$ of $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ are

$$f_{\alpha j\, \beta k}{}^{\underline{a}} = -\sigma^{\underline{a}}_{\alpha\beta} \epsilon_{jk} \,, \qquad f_{\widehat{\alpha} j\, \widehat{\beta} k}{}^{\underline{a}} = -\sigma^{\underline{a}}_{\widehat{\alpha}\widehat{\beta}} \epsilon_{jk} \,, \tag{4.113a}$$

$$f_{\beta k\, \underline{a}}{}^{\widehat{\alpha} j} = -\widehat{\delta}^{\widehat{\alpha}\gamma} \sigma_{\underline{a}\gamma\beta} \delta^j_k \,, \qquad f_{\widehat{\beta} k\, \underline{a}}{}^{\alpha j} = -\widehat{\delta}^{\alpha\widehat{\gamma}} \sigma_{\underline{a}\widehat{\gamma}\widehat{\beta}} \delta^j_k \,, \tag{4.113b}$$

$$f_{\alpha j\, \widehat{\beta} k}{}^{[ab]} = i (\sigma^{ab})_\alpha{}^\gamma \widehat{\delta}_{\gamma\widehat{\beta}} \epsilon_{jk} \,, \qquad f_{\alpha j\, \widehat{\beta} k}{}^{[a'b']} = -i (\sigma^{a'b'})_\alpha{}^\gamma \widehat{\delta}_{\gamma\widehat{\beta}} \epsilon_{jk} \,, \tag{4.113c}$$

$$f_{[\underline{ab}]\, \alpha k}{}^{\beta j} = i (\sigma_{\underline{ab}})_\alpha{}^\beta \delta^j_k \,, \qquad f_{[\underline{ab}]\, \widehat{\alpha} k}{}^{\widehat{\beta} j} = i (\sigma_{\underline{ab}})_{\widehat{\alpha}}{}^{\widehat{\beta}} \delta^j_k \,, \tag{4.113d}$$

$$f_{c\, d}{}^{[ab]} = \delta^a_{[c} \delta^b_{d]} \,, \qquad f_{c'\, d'}{}^{[a'b']} = -\delta^{a'}_{[c'} \delta^{b'}_{d']} \,, \tag{4.113e}$$

$$f_{[\underline{cd}]\, [\underline{ef}]}{}^{[\underline{ab}]} = \frac{1}{2} \left(\eta_{\underline{e}[\underline{c}} \delta^{[\underline{a}}_{\underline{d}]} \delta^{\underline{b}]}_{\underline{f}} + \eta_{\underline{f}[\underline{d}} \delta^{[\underline{a}}_{\underline{c}]} \delta^{\underline{b}]}_{\underline{e}} \right) , \qquad f_{[\underline{bc}]\, \underline{d}}{}^{\underline{a}} = \eta_{\underline{d}[\underline{b}} \delta^{\underline{a}}_{\underline{c}]} \,, \tag{4.113f}$$

where $\widehat{\delta}^{\alpha\widehat{\beta}} = 2\sqrt{2} (\sigma^{012})^{\alpha\widehat{\beta}}$, $(\sigma^{\underline{abc}})^{\alpha\beta} = \frac{i}{3!} (\sigma^{[\underline{a}} \sigma^{\underline{b}} \sigma^{\underline{c}]})^{\alpha\beta}$, $(\sigma^{\underline{ab}})_\alpha{}^\beta = \frac{i}{2} (\sigma^{[\underline{a}} \sigma^{\underline{b}]})_\alpha{}^\beta$ and we anti-symmetrize with square brackets without dividing by the number of terms, e.g., $\delta^a_{[c} \delta^b_{d]} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c$. Similarly, symmetrization is denoted with round brackets. Note that the matrix $\widehat{\delta}^{\alpha\widehat{\beta}}$ enables one to contract an α index with a $\widehat{\beta}$ index in an $\mathrm{SO}(1,2) \times \mathrm{SO}(3)$ invariant manner. Detailed information about the Pauli matrices $\sigma^{\underline{a}}_{\alpha\beta}$ and its properties can be found in Appendix B.

We choose the representative of the super-coset $\frac{\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2) \times \mathrm{SO}(3)}$ as

$$g = e^{x^{\underline{a}} T_{\underline{a}} + \theta^{\alpha j} T_{\alpha j} + \widehat{\theta}^{\widehat{\beta} k} T_{\widehat{\beta} k}} \,, \tag{4.114}$$

and one can check, from the definition of the left-invariant one-forms

$$g^{-1}dg = J^{\underline{A}} T_{\underline{A}}\,, \tag{4.115}$$

that in the flat space limit of (4.113) one obtains $J^{\alpha j} = d\theta^{\alpha j}$, $J^{\underline{a}} = \Pi^{\underline{a}}$ and $J^{\widehat{\beta}k} = d\widehat{\theta}^{\widehat{\beta}k}$, which are the super-vielbeins in a flat six-dimensional background (3.30), as desired [92].

Global $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ transformations are defined to act on the coset representative g from the left and gauge transformations from the isotropy group $\mathrm{SO}(1,2) \times \mathrm{SO}(3)$ are defined to act on the coset representative from the right. Therefore, under a combined global and a local transformation, we write

$$g \to e^{\Sigma} g e^{\Omega}\,, \tag{4.116}$$

where e^{Σ} corresponds to a global $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ and e^{Ω} to a local $\mathrm{SO}(1,2) \times \mathrm{SO}(3)$ transformation of g. It is then manifest that the left-invariant currents (4.115) are invariant under global transformations. On the other hand, under a local transformation of the isotropy group, we have that

$$\delta J^{[\underline{ab}]} = \omega^{[\underline{cd}]} J^{[\underline{ef}]} i f_{[\underline{ef}]\,[\underline{cd}]}{}^{[\underline{ab}]} + d\omega^{[\underline{ab}]}\,, \tag{4.117a}$$

$$\delta J^{A} = \omega^{[\underline{ab}]} J^{B} i f_{B\,[\underline{ab}]}{}^{A}\,, \tag{4.117b}$$

and the ghosts transform according to

$$\delta\lambda^{\alpha} = -\omega^{[\underline{ab}]}(\sigma_{\underline{ab}})^{\alpha}{}_{\beta}\lambda^{\beta}\,, \qquad \delta\widehat{\lambda}^{\widehat{\alpha}} = -\omega^{[\underline{ab}]}(\sigma_{\underline{ab}})^{\widehat{\alpha}}{}_{\widehat{\beta}}\widehat{\lambda}^{\widehat{\beta}}\,, \tag{4.118a}$$

$$\delta w_{\alpha} = \omega^{[\underline{ab}]} w_{\beta}(\sigma_{\underline{ab}})^{\beta}{}_{\alpha}\,, \qquad \delta\widehat{w}_{\widehat{\alpha}} = \omega^{[\underline{ab}]}\widehat{w}_{\widehat{\beta}}(\sigma_{\underline{ab}})^{\widehat{\beta}}{}_{\widehat{\alpha}}\,, \tag{4.118b}$$

where $\Omega = \omega^{[\underline{ab}]} T_{[\underline{ab}]}$ in (4.116).

Another important property is that the left-invariant one-forms satisfy the Maurer-Cartan equations

$$dJ^{\underline{C}} = -\frac{i}{2} f_{AB}{}^{\underline{C}} J^{\underline{B}} J^{\underline{A}}\,, \tag{4.119}$$

Here, $d = dz\partial + d\bar{z}\bar{\partial}$, $J^{\underline{A}} = dzJ^{\underline{A}} + d\bar{z}\bar{J}^{\underline{A}}$ and we use the same conventions when working with differential forms as in [24], in particular, we omit the wedge product symbol in (4.119) and in the subsequent discussions. Further properties of

$\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ are discussed in Appendix L.

Having identified the super-vielbeins with the left-invariant currents of the super-coset $\frac{\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)}$ and introduced the supergroup $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$, we are now in a position to write the worldsheet action (4.107) in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background for the Type IIB superstring compactified on T^4 with mixed constant NS-NS and R-R three-form flux. The worldsheet action takes the form

$$\begin{aligned} S = \frac{1}{f^2} \int d^2z \bigg(&\frac{1}{2} J^{\underline{b}} \overline{J}^{\underline{a}} \eta_{\underline{ab}} + \epsilon_{jk} \widehat{\delta}_{\alpha\widehat{\beta}} J^{\widehat{\beta}k} \overline{J}^{\alpha j} + w_\alpha \overline{\nabla} \lambda^\alpha \\ &+ \widehat{w}_{\widehat{\alpha}} \nabla \widehat{\lambda}^{\widehat{\alpha}} - \eta^{[\underline{ab}][\underline{cd}]} N_{\underline{ab}} \widehat{N}_{\underline{cd}} \bigg) + \frac{i}{f^2} S_{\mathrm{WZ}} + S_{\rho,\sigma} + S_C \,, \end{aligned} \tag{4.120}$$

where $\eta^{[\underline{ab}][\underline{cd}]} = \frac{1}{2}\{\eta^{a[c}\eta^{d]b}, -\eta^{a'[c'}\eta^{d']b'}\}$ is the inverse of the $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ metric (see eqs. (L.10)), the covariant derivatives are

$$\overline{\nabla} \lambda^\alpha = \overline{\partial} \lambda^\alpha + \overline{J}^{[\underline{ab}]} (\sigma_{\underline{ab}})^\alpha{}_\beta \lambda^\beta \,, \qquad \nabla \widehat{\lambda}^{\widehat{\alpha}} = \partial \widehat{\lambda}^{\widehat{\alpha}} + J^{[\underline{ab}]} (\sigma_{\underline{ab}})^{\widehat{\alpha}}{}_{\widehat{\beta}} \widehat{\lambda}^{\widehat{\beta}} \,, \tag{4.121}$$

and the Wess-Zumino term is given by

$$S_{\mathrm{WZ}} = - \int_{\mathcal{B}} \frac{1}{6} J^C J^B J^A H_{ABC} \,, \tag{4.122}$$

with[11]

$$H_{\alpha j \, \beta k \, \underline{a}} = \frac{i}{2} \big(2 - \tfrac{f_{RR}}{f}\big) \epsilon_{jk} \sigma_{\underline{a} \alpha\beta} \,, \qquad H_{\widehat{\alpha} j \, \widehat{\beta} k \, \underline{a}} = -\frac{i}{2} \big(2 - \tfrac{f_{RR}}{f}\big) \epsilon_{jk} \sigma_{\underline{a} \widehat{\alpha}\widehat{\beta}} \,, \tag{4.123a}$$

$$H_{\underline{abc}} = \frac{f_{NS}}{f} (\sigma_{\underline{abc}})_{\alpha\widehat{\beta}} \widehat{\delta}^{\alpha\widehat{\beta}} \,, \qquad H_{\alpha j \, \widehat{\beta} k \, \underline{a}} = \frac{i}{2} \frac{f_{NS}}{f} \epsilon_{jk} \sigma_{\underline{a} \alpha\widehat{\beta}} \,. \tag{4.123b}$$

The details about the derivation of the sigma-model (4.120) can be found in Sections 4.7.3 and 4.7.4 below.

In eq. (4.122), the integration is carried over a three-manifold $\mathcal{B}$ whose boundary is the worldsheet. As in (4.107), $S_{\rho,\sigma}$ is the action for the chiral bosons of the six-dimensional hybrid formalism, which remain free fields, and S_C is the action representing the compactification directions. The constant f is the inverse of the AdS_3 radius and is given by $f = \sqrt{f_{RR}^2 + f_{NS}^2}$, where f_{NS} and f_{RR} parametrize the NS-NS and R-R self-dual three-form flux, respectively. We shall also parametrize

[11] Note that $H_{012} = H_{345}$ and hence it is self-dual. Moreover, the constants H_{ABC}s are graded anti-symmetric in the 1-2 and 2-3 indices, while the $f_{\underline{ABC}}$s are graded anti-symmetric in the 1-2 and 1-3 indices. See eqs. (L.9) for our conventions.

the NS-NS flux by the constant $k = f_{NS} f^{-3}$. Note that $H = dB$, where B is the two-form potential. The three-form $H_{\alpha j\, \widehat{\beta} k\, \underline{a}}$ in (4.122) is necessary for S_{WZ} to be closed (see Section 4.7.3), its origin will be further clarified via a perturbative derivation in Section 4.7.4.

As is elaborated in Appendix L, the Lie superalgebra $\mathfrak{g}$ of $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ admits a $\mathbb{Z}_4$-automorphism, so that it can be decomposed as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3\,, \tag{4.124}$$

and, as a consequence, one can split the left-invariant currents (4.115) according to their $\mathbb{Z}_4$-grade

$$J = J^0 + J^1 + J^2 + J^3\,, \tag{4.125}$$

where $J^0 = J^{[ab]} T_{[\underline{ab}]}$, $J^1 = J^{\alpha j} T_{\alpha j}$, $J^2 = J^{\underline{a}} T_{\underline{a}}$ and $J^3 = J^{\widehat{\alpha} j} T_{\widehat{\alpha} j}$. Using the Maurer-Cartan equations (4.119) and the supertrace over the generators (L.7), the world-sheet action (4.120) can be written in a more symmetric form

$$\begin{aligned} S = \frac{1}{f^2} \int d^2 z\ \mathrm{sTr} \Big[& \frac{1}{2} J^2 \bar{J}^2 + \frac{1}{2} \Big(\bar{J}^1 J^3 + J^1 \bar{J}^3 \Big) + w \overline{\nabla} \lambda \\ & + \widehat{w} \nabla \widehat{\lambda} - N \widehat{N} \Big] - \frac{i}{f^2} \int_{\mathcal{B}} \Big(\mathcal{H}_{\mathrm{NS}} + \mathcal{H}_{\mathrm{RR}} \Big) + S_{\rho,\sigma} + S_C\,, \end{aligned} \tag{4.126}$$

where we defined

$$\begin{aligned} &\lambda = \lambda^{\alpha j} T_{\alpha j}\,, && w = w_{\alpha j} \delta^{\widehat{\alpha}\widehat{\beta}} \epsilon^{jk} T_{\widehat{\beta} k}\,, && \lambda^{\alpha j} = \frac{1}{\sqrt{2}} \{ \lambda^{\alpha}, \lambda^{\alpha} \}\,, && w_{\alpha j} = \frac{1}{\sqrt{2}} \{ w_{\alpha}, w_{\alpha} \}\,, \\ &\widehat{\lambda} = \widehat{\lambda}^{\widehat{\alpha} j} T_{\widehat{\alpha} j}\,, && \widehat{w} = \widehat{w}_{\widehat{\alpha} j} \epsilon^{jk} \delta^{\widehat{\alpha}\beta} T_{\beta k}\,, && \widehat{\lambda}^{\widehat{\alpha} j} = \frac{1}{\sqrt{2}} \{ \widehat{\lambda}^{\widehat{\alpha}}, \widehat{\lambda}^{\widehat{\alpha}} \}\,, && \widehat{w}_{\widehat{\alpha} j} = \frac{1}{\sqrt{2}} \{ \widehat{w}_{\widehat{\alpha}}, \widehat{w}_{\widehat{\alpha}} \}\,, \\ &\overline{\nabla} \lambda = \bar{\partial} \lambda + [\bar{J}^0, \lambda]\,, && N = -\{ w, \lambda \}\,, && \nabla \widehat{\lambda} = \partial \widehat{\lambda} + [J^0, \widehat{\lambda}]\,, && \widehat{N} = -\{ \widehat{w}, \widehat{\lambda} \}\,, \end{aligned}$$

and

$$\mathcal{H}_{\mathrm{NS}} = \frac{1}{6} \Big(J^{\underline{c}} J^{\underline{b}} J^{\underline{a}} H_{\underline{abc}} + 6 J^{\underline{a}} J^{\widehat{\beta} k} J^{\alpha j} H_{\alpha j\, \widehat{\beta} k\, \underline{a}} \Big)\,, \tag{4.127a}$$

$$\mathcal{H}_{\mathrm{RR}} = -\frac{1}{2} \frac{f_{RR}}{f}\, \mathrm{sTr} \Big(J^2 J^1 J^1 - J^2 J^3 J^3 \Big)\,, \tag{4.127b}$$

so that $\mathcal{H}_{\mathrm{NS}}$ is proportional to the amount of NS-NS flux f_{NS} and $\mathcal{H}_{\mathrm{RR}}$ to the amount of R-R flux f_{RR}.

It is important to note that the three-form $\mathcal{H}_{\mathrm{NS}}$ has $\mathbb{Z}_4$-grade equal to two, therefore it cannot be written as a supertrace over $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ in terms of the currents in (4.125), given that the supertrace must be $\mathbb{Z}_4$-invariant. On the other hand, the three-form $\mathcal{H}_{\mathrm{RR}}$ is exact (see eq. (4.140)) and hence can be written under a two-dimensional integral over the worldsheet, it is the WZ term that also appears in the $\mathrm{AdS}_2 \times \mathrm{S}^2$ and $\mathrm{AdS}_5 \times \mathrm{S}^5$ worldsheet actions with pure R-R flux [91].

In addition, the sigma-model (4.126) has a $\mathbb{Z}_2$-symmetry under the exchange of holomorphic and anti-holomorphic worldsheet coordinates, flipping the grading of the left-invariant fermionic currents (i.e., $J^1 \leftrightarrow J^3$), and further redefining $f_{NS} \to -f_{NS}$. When $f_{NS} = 0$, this is the $\mathbb{Z}_2$-symmetry ejoyed by the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor sigma-model, which corresponds to eq. (4.126) with $S_{\rho,\sigma} = S_C = 0$, $\mathcal{H}_{\mathrm{NS}} = 0$ and so $f = f_{RR}$. Of course, in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case, the $\mathbb{Z}_4$-coset of interest is $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times \mathrm{SO}(5)}$, the left-invariant currents $J \in \mathrm{PSU}(2,2|4)$, λ and $\widehat{\lambda}$ are replaced by $d = 10$ pure spinor variables, and the supertrace is taken over the $\mathrm{PSU}(2,2|4)$ Lie superalgebra generators, see [64, eq. (2.1)].

As an important observation, note that in the limit $f_{RR} \to 0$ we have $f^{-2} = f_{NS}^{-2}$ and $f^{-1} f_{NS} = 1$, consequently, one obtains the pure NS-NS model with the super-coset $\frac{\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)}$ as the target-superspace, whose worldsheet action is given by

$$
\begin{aligned}
S = \frac{1}{f_{NS}^2} \int d^2 z \ \mathrm{sTr} \Big[& \frac{1}{2} J^2 \bar{J}^2 + \frac{1}{2}\left(\bar{J}^1 J^3 + J^1 \bar{J}^3 \right) \\
& + w \overline{\nabla} \lambda + \widehat{w} \nabla \widehat{\lambda} - N \widehat{N} \Big] - \frac{i}{f_{NS}^2} \int_B \mathcal{H}_{\mathrm{NS}} + S_{\rho,\sigma} + S_C \, .
\end{aligned}
\tag{4.128}
$$

In comparison to the six-dimensional hybrid formalism in an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background [46], the model (4.120) has all 16 supersymmetries of $\mathcal{N} = 2$ six-dimensional superspace manifest, whereas in [46] only half of the 16 supersymmetries were manifest. Notice that the price one pays for this is the presence of additional ghosts among the worldsheet variables. We should emphasize that even the matter sector of the action (4.120) is different from the one in the Green-Schwarz formulation [93] [94] [95] [96], for the reason that (4.120) contains a kinetic term for the fermions which breaks the Kappa-symmetry. The same distinction already appears when comparing the matter sector of the Green-Schwarz superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [61] with the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor description [64], or when comparing the $\mathrm{AdS}_2 \times \mathrm{S}^2$ Green-Schwarz with the $\mathrm{AdS}_2 \times \mathrm{S}^2$ hybrid action [91].

4.7.3 Derivation from the supergravity constraints

In the presence of a constant R-R three-form flux parametrized by f_{RR}, the R-R field-strength $f^{\alpha j \widehat{\beta} k}$ is proportional to f_{RR}, the two-form potential $B_{\alpha j \widehat{\beta} k}$ is proportional to $f_{RR}^{-1} = f^{-1}$ and the superfield $R^{\underline{abcd}}$ is proportional to $f_{RR}^2 = f^2$, where f is the inverse of the AdS_3 radius. It is convenient to rescale the background fields in (4.107) as

$$B_{\alpha j \widehat{\beta} k} \to f_{RR}^{-1} B_{\alpha j \widehat{\beta} k}\,, \qquad F^{\alpha j \widehat{\beta} k} \to f_{RR} F^{\alpha j \widehat{\beta} k}\,, \qquad R^{\underline{abcd}} \to f^2 R^{\underline{abcd}}\,, \tag{4.129}$$

and the worldsheet fields as [46] [54]

$$J^{\underline{a}} \to f^{-1} J^{\underline{a}}\,, \qquad \overline{J}^{\underline{a}} \to f^{-1} \overline{J}^{\underline{a}}\,, \qquad d_{\alpha j} \to f_{RR}^{-\frac{1}{2}} f^{-1} d_{\alpha j}\,, \tag{4.130a}$$
$$\widehat{d}_{\widehat{\beta} k} \to f_{RR}^{-\frac{1}{2}} f^{-1} \widehat{d}_{\widehat{\beta} k}\,, \qquad J^{\alpha j} \to f_{RR}^{\frac{1}{2}} f^{-1} J^{\alpha j}\,, \qquad \overline{J}^{\alpha j} \to f_{RR}^{\frac{1}{2}} f^{-1} \overline{J}^{\alpha j}\,, \tag{4.130b}$$
$$J^{\widehat{\beta} k} \to f_{RR}^{\frac{1}{2}} f^{-1} J^{\widehat{\beta} k}\,, \qquad \overline{J}^{\widehat{\beta} k} \to f_{RR}^{\frac{1}{2}} f^{-1} \overline{J}^{\widehat{\beta} k}\,, \qquad \lambda^{\alpha} \to f^{-1} \lambda^{\alpha}\,, \tag{4.130c}$$
$$\widehat{\lambda}^{\widehat{\alpha}} \to f^{-1} \widehat{\lambda}^{\widehat{\alpha}}\,, \qquad w_{\alpha} \to f^{-1} w_{\alpha}\,, \qquad \widehat{w}_{\widehat{\alpha}} \to f^{-1} \widehat{w}_{\widehat{\alpha}}\,, \tag{4.130d}$$

so that the action gets an overall factor of $f_{RR}^{-2} = f^{-2}$. Therefore, working with the worldsheet action (4.107) with a factor of f^{-2} in front is equivalent as treating the superfields $\{B_{\alpha j \widehat{\beta} k}, F^{\alpha j \widehat{\beta} k}, R^{\underline{abcd}}\}$ in (4.107) to be independent of f_{RR}, this observation will make the formulas below more transparent. In eqs. (4.130), it is important to note that even though $f_{RR} = f$ in a pure R-R background, we explicitly wrote the factors of the inverse of the AdS_3 radius f, in this way, the rescalings have a natural generalization when turning on an NS-NS three-form flux parametrized by f_{NS}, where the inverse of the AdS_3 radius becomes $f = \sqrt{f_{RR}^2 + f_{NS}^2}$ [46].

In the pure R-R flux $\mathrm{AdS}_3 \times \mathrm{S}^3$ background, the non-vanishing background superfields in the action (4.107) take the values [44] [54]

$$F^{\alpha j \widehat{\beta} k} = -\epsilon^{jk} \widehat{\delta}^{\alpha \widehat{\beta}}\,, \tag{4.131a}$$
$$B_{\alpha j \widehat{\beta} k} = B_{\widehat{\beta} k \alpha j} = -\frac{1}{4} \epsilon_{jk} \widehat{\delta}_{\alpha \widehat{\beta}}\,, \tag{4.131b}$$
$$R^{\underline{abcd}} = 4 \eta^{[\underline{ab}][\underline{cd}]}\,, \tag{4.131c}$$

From the torsion constraints

$$T_{\alpha j \underline{a}}{}^{\widehat{\beta} k} = -i f_{\alpha j \gamma l \underline{a}} F^{\gamma l \widehat{\beta} k}\,, \qquad T_{\widehat{\alpha} j \underline{a}}{}^{\beta k} = i f_{\widehat{\alpha} j \widehat{\gamma} l \underline{a}} F^{\beta k \widehat{\gamma} l}\,, \tag{4.132}$$

and the definition of the three-form with flat indices $H_{ABC} = \frac{1}{2}\nabla_{[A}B_{BC]} + \frac{1}{2}T_{[AB|}{}^{D}B_{D|C]}$, we obtain the desired supergravity constraints

$$H_{\alpha j\,\beta k\,\underline{a}} = \frac{i}{2}\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}\,, \qquad\qquad H_{\widehat{\alpha} j\,\widehat{\beta} k\,\underline{a}} = -\frac{i}{2}\epsilon_{jk}\sigma_{\underline{a}\widehat{\alpha}\widehat{\beta}}\,. \tag{4.133}$$

Besides that, using the definition of the curvature two-form $R_B{}^A$ (see Appendix C.2), one can check that choosing the connection one-form as $\Omega_A{}^B = i f_{[\underline{ab}]\,B}{}^{A} J^{[\underline{ab}]}$ agrees with $R^{\underline{abcd}}$ in eq. (4.131c).

The superfield $F^{\alpha j\,\widehat{\beta} k}$ in (4.131a) is invertible, therefore, we can integrate $d_{\alpha j}$ and $\widehat{d}_{\widehat{\beta} k}$ in the action via the equations of motion

$$d_{\alpha j} = \epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}} J^{\widehat{\beta} k}\,, \qquad\qquad \widehat{d}_{\widehat{\alpha} j} = \epsilon_{jk}\widehat{\delta}_{\widehat{\alpha}\beta}\overline{J}^{\beta k}\,. \tag{4.134}$$

Consequently, the $\mathrm{AdS}_3 \times \mathrm{S}^3$ worldsheet action (4.107) in a pure R-R background takes the form [54]

$$\begin{aligned} S = \frac{1}{f^2}\int d^2z \Bigg[& \frac{1}{2}J^{\underline{b}}\overline{J}^{\underline{a}}\eta_{\underline{ab}} + \frac{3}{4}\epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}} J^{\widehat{\beta} k}\overline{J}^{\alpha j} + \frac{1}{4}\epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}}\overline{J}^{\widehat{\beta} k} J^{\alpha j} \\ & + w_\alpha \overline{\nabla}\lambda^\alpha + \widehat{w}_{\widehat{\alpha}}\nabla\widehat{\lambda}^{\widehat{\alpha}} - \eta^{[\underline{ab}][\underline{cd}]} N_{\underline{ab}}\widehat{N}_{\underline{cd}} \Bigg] + S_{\rho,\sigma} + S_C\,, \end{aligned} \tag{4.135}$$

where $f = f_{RR}$ in (4.135) and is the inverse of the AdS_3 radius.

For the Type IIB superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$, one can also turn on a constant NS-NS three-form flux $H_{\underline{abc}}$. We can include in (4.135) the interaction corresponding to this field by constructing a Wess-Zumino term from a $\frac{\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)}$-invariant and closed three-form $\mathcal{H}_{\mathrm{NS}}$. Locally, this closed three-form must describe a first-order deformation of flat six-dimensional spacetime by the NS-NS field $b_{\underline{ab}}$. Up to a constant, the closed three-form $\mathcal{H}_{\mathrm{NS}}$ satisfying these properties is unique and given by (4.127a), which we repeat below for completeness

$$\mathcal{H}_{\mathrm{NS}} = \frac{1}{6}\left(J^{\underline{c}} J^{\underline{b}} J^{\underline{a}} H_{\underline{abc}} + 6 J^{\underline{a}} J^{\widehat{\beta} k} J^{\alpha j} H_{\alpha j\,\widehat{\beta} k\,\underline{a}} \right), \tag{4.136}$$

where $H_{\underline{abc}} = \frac{f_{NS}}{f}(\sigma_{\underline{abc}})_{\alpha\widehat{\beta}}\widehat{\delta}^{\alpha\widehat{\beta}}$ and $H_{\alpha j\,\widehat{\beta} k\,\underline{a}} = \frac{i}{2}\frac{f_{NS}}{f}\epsilon_{jk}\sigma_{\underline{a}\alpha\widehat{\beta}}$ with f_{NS} parametrizing the amount of NS-NS flux. One can check that (4.136) is closed by using the Maurer-Cartan equations (4.119).

In view of that, it is natural to think that the worldsheet action describing the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with mixed flux consists in taking the inverse of the

AdS_3 radius as $f = \sqrt{f_{RR}^2 + f_{NS}^2}$ and adding to the action (4.135) the term

$$-\frac{i}{f^2}\int_{\mathcal{B}} \mathcal{H}_{\mathrm{NS}}\,, \tag{4.137}$$

where the integration is carried over a three-manifold $\mathcal{B}$ whose boundary is the worldsheet. Nevertheless, this doesn't work as expected. Performing this modification will spoil one-loop conformal invariance of eq. (4.135) and, hence, what is obtained does not correspond to a consistent sigma-model for the superstring propagating in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$. As we will presently see, for conformal invariance to be preserved in the mixed flux AdS_3 background, one also needs to modify the superspace two-form $B_{\alpha j\,\widehat{\beta}k}$ in (4.131b) besides adding (4.137) to eq. (4.135).

The situation is then a bit different from what happens in the six-dimensional hybrid formalism in $\mathrm{AdS}_3 \times \mathrm{S}^3$ with mixed NS-NS and R-R three-form flux [46]. In that case, the target-space is the supergroup $\mathrm{PSU}(1,1|2)$ and turning on a constant NS-NS flux, by starting from the pure R-R $\mathrm{AdS}_3 \times \mathrm{S}^3$ worldsheet action, corresponds to just adding the integral of a $\mathrm{PSU}(1,1|2)$ closed three-form to the sigma-model. So that no further modification of the terms already present in the action is necessary. On the other hand, in the description of the Green-Schwarz superstring with target-space the super-coset $\frac{\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)}$, it was already observed that the naive Wess-Zumino term in the fermionic left-invariant currents should be modified for the preservation of one-loop conformal invariance and integrability of the model [96].

Accordingly, to obtain a consistent worldsheet action in $\mathrm{AdS}_3 \times \mathrm{S}^3$ in the presence of mixed NS-NS and R-R three-form flux, we start with the general form (4.107), perform the rescalings (4.129) and (4.130), and modify the two-form $B_{\alpha j\,\widehat{\beta}k}$ so that the background superfields of eqs. (4.131) now take the form

$$F^{\alpha j\,\widehat{\beta}k} = -\epsilon^{jk}\widehat{\delta}^{\alpha\widehat{\beta}}\,, \tag{4.138a}$$

$$B_{\alpha j\,\widehat{\beta}k} = B_{\widehat{\beta}k\,\alpha j} = -\frac{1}{4}\big(2 - \tfrac{f_{RR}}{f}\big)\epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}}\,, \tag{4.138b}$$

$$R^{\underline{abcd}} = 4\eta^{[\underline{ab}][\underline{cd}]}\,. \tag{4.138c}$$

Integrating out $d_{\alpha j}$ and $\widehat{d}_{\widehat{\beta}k}$ as before and adding the NS-NS deformation (4.137), the resulting sigma-model for the superstring propagating in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ is then given by

$$S = \frac{1}{f^2}\int d^2z \left[\frac{1}{2}J^{\underline{b}}\overline{J}^{\underline{a}}\eta_{\underline{ab}} + \epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}}J^{\widehat{\beta}k}\overline{J}^{\alpha j} - (2 - \tfrac{f_{RR}}{f})\frac{1}{4}\epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}}\left(J^{\widehat{\beta}k}\overline{J}^{\alpha j} - \overline{J}^{\widehat{\beta}k}J^{\alpha j}\right)\right.$$
$$\left. + w_\alpha\overline{\nabla}\lambda^\alpha + \widehat{w}_{\alpha h}\nabla\widehat{\lambda}^{\widehat{\alpha}} - \eta^{[\underline{ab}][\underline{cd}]}N_{\underline{ab}}\widehat{N}_{\underline{cd}}\right] - \frac{i}{f^2}\int_{\mathcal{B}}\mathcal{H}_{\mathrm{NS}} + S_{\rho,\sigma} + S_C\,, \qquad (4.139)$$

where $f = \sqrt{f_{RR}^2 + f_{NS}^2}$ is the inverse of the AdS_3 radius. To recover eq. (4.120), one just needs to note that we can write

$$\begin{aligned}
&-\frac{1}{f^2}(2 - \tfrac{f_{RR}}{f})\int d^2z\,\frac{1}{4}\epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}}\left(J^{\widehat{\beta}k}\overline{J}^{\alpha j} - \overline{J}^{\widehat{\beta}k}J^{\alpha j}\right)\\
&\quad= \frac{1}{f^2}(2 - \tfrac{f_{RR}}{f})\frac{i}{4}\int_{\mathcal{B}} d\left(\epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}}J^{\widehat{\beta}k}J^{\alpha j}\right)\\
&\quad= -\frac{i}{f^2}\int_{\mathcal{B}}\frac{1}{2}\left(J^{\underline{a}}J^{\beta k}J^{\alpha j}H_{\alpha j\,\beta k\,\underline{a}} + J^{\underline{a}}J^{\widehat{\beta}k}J^{\widehat{\alpha}j}H_{\widehat{\alpha}j\,\widehat{\beta}k\,\underline{a}}\right), \qquad (4.140)
\end{aligned}$$

where $H_{\alpha j\,\beta k\,\underline{a}}$ and $H_{\widehat{\alpha}j\,\widehat{\beta}k\,\underline{a}}$ are given by (4.123).[12] Eq. (4.140) is the Wess-Zumino term of ref. [91] which appears in quantizable super-coset descriptions of the $\mathrm{AdS}_2 \times \mathrm{S}^2$ and $\mathrm{AdS}_5 \times \mathrm{S}^5$ backgrounds as well.

The reason for choosing $B_{\alpha j\,\widehat{\beta}k}$ as in eq. (4.138b) hinges on the fact that in the $f_{NS} \to 0$ limit, i.e., $f = f_{RR}$ we recover the pure R-R worldsheet action (4.135). Alongside that, the choice (4.138b) for the two-form potential is required for one-loop conformal invariance of the action (4.120) (see Section 4.8), which is known to be compatible with background superfields satisfying the supergravity equations of motion [77].

Note that the constraints (4.133) in the mixed flux case are

$$H_{\underline{a}\,\alpha j\,\beta k} = \frac{i}{2}(2 - \tfrac{f_{RR}}{f})\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}\,, \qquad H_{\underline{a}\,\widehat{\alpha}j\,\widehat{\beta}k} = -\frac{i}{2}(2 - \tfrac{f_{RR}}{f})\epsilon_{jk}\sigma_{\underline{a}\widehat{\alpha}\widehat{\beta}}\,, \qquad (4.141)$$

and so they have the desired form in both limits: $f_{NS} \to 0$ and $f_{RR} \to 0$ which are consistent $\mathrm{AdS}_3 \times \mathrm{S}^3$ backgrounds for the superstring. Note further that, without loss of generality, one can take $f_{RR} \geq 0$.

[12] In eq. (4.140), to get from the first to the second line we used that $J^{\alpha j} \wedge J^{\widehat{\beta}k} = d\sigma^0 \wedge d\sigma^1 \epsilon^{IJ} J_J^{\alpha j} J_I^{\widehat{\beta}k}$, $d^2z = 2d\sigma^0 d\sigma^1$ and $\epsilon^{z\overline{z}} = -2i$. To get from the second to the last line we used the Maurer-Cartan equations. In our conventions, the Euclidean wordsheet coordinates $\sigma^I = \{\sigma^0, \sigma^1\}$ are related to the complex coordinates as $z = \sigma^0 - i\sigma^1$ and $\overline{z} = \sigma^0 + i\sigma^1$ (see Appendix C.1).

4.7.4 Perturbative derivation

In the previous section, the $\mathrm{AdS}_3 \times \mathrm{S}^3$ action (4.120) was justified by substituting the values for the background superfields in (4.107). The latter can be inferred from the worldsheet action in a general ten-dimensional background [90], or by covariantizing the massless closed superstring integrated vertex operator (4.144) with respect to target-space super-reparametrization invariance. Below, we will further confirm our result and show how one can derive (4.120) via a perturbative analysis starting from the integrated vertex operator.

Firstly, note that up to cubic-order in the worldsheet fields, the supertrace term in eq. (4.126) is

$$\frac{1}{f^2}\int d^2z\ \mathrm{sTr}\left[\frac{1}{2}J^2\overline{J}^2+\frac{1}{2}\left(\overline{J}^1J^3+J^1\overline{J}^3\right)+w\overline{\nabla}\lambda+\widehat{w}\nabla\widehat{\lambda}-N\widehat{N}\right]$$
$$=\frac{1}{f^2}\int d^2z\left(\frac{1}{2}\partial x^{\underline{b}}\overline{\partial}x^{\underline{a}}\eta_{\underline{ab}}+\epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}}\partial\widehat{\theta}^{\widehat{\beta}k}\overline{\partial}\theta^{\alpha j}+w_\alpha\overline{\partial}\lambda^\alpha+\widehat{w}_{\widehat{\alpha}}\partial\widehat{\lambda}^{\widehat{\alpha}}\right), \qquad (4.142)$$

and the three-form in eq. (4.126) is given by

$$-\frac{i}{f^2}\int_{\mathcal{B}}\left(\mathcal{H}_{\mathrm{NS}}+\mathcal{H}_{\mathrm{RR}}\right)$$
$$=\frac{1}{f^2}\int d^2z\left\{\frac{f_{NS}}{f}\left[\frac{1}{3}x^{\underline{c}}\overline{\partial}x^{\underline{b}}\partial x^{\underline{a}}(\sigma_{\underline{abc}})_{\alpha\widehat{\beta}}\widehat{\delta}^{\alpha\widehat{\beta}}+\frac{i}{2}\epsilon_{jk}\left(\partial x_{\alpha\widehat{\beta}}\overline{\partial}\widehat{\theta}^{\widehat{\beta}k}\theta^{\alpha j}-\overline{\partial}x_{\alpha\widehat{\beta}}\partial\widehat{\theta}^{\widehat{\beta}k}\theta^{\alpha j}\right)\right]\right.$$
$$+\frac{f_{RR}}{f}\left[\frac{i}{4}\epsilon_{jk}\left(\overline{\partial}x_{\alpha\beta}\partial\theta^{\beta k}\theta^{\alpha j}-\partial x_{\alpha\beta}\overline{\partial}\theta^{\beta k}\theta^{\alpha j}\right)+\frac{i}{4}\epsilon_{jk}\left(\partial x_{\widehat{\alpha}\widehat{\beta}}\overline{\partial}\widehat{\theta}^{\widehat{\beta}k}\widehat{\theta}^{\widehat{\alpha}j}\right.\right.$$
$$\left.\left.\left.-\overline{\partial}x_{\widehat{\alpha}\widehat{\beta}}\partial\widehat{\theta}^{\widehat{\beta}k}\widehat{\theta}^{\widehat{\alpha}j}\right)\right]\right\}, \qquad (4.143)$$

where $x_{\alpha\widehat{\beta}}=x^{\underline{a}}\sigma_{\underline{a}\alpha\widehat{\beta}}$. Let us see if we can reproduce the above results by doing a perturbative analysis starting from flat space.

The linearized deformation around the flat background is given by the integrated vertex operator $\int W_{\mathrm{SG}}$. For the case of the closed superstring, the integrated vertex operator can be obtained as the left-right product of the open superstring vertex operator in eq. (3.55). In the analysis of this section, we want to confirm that the worldsheet action in eq. (4.120) corresponds to turning on the NS-NS and the R-R three-form flux up to cubic-order in the worldsheet fields.

Consider the integrated vertex operator for the compactification-independent massless sector of the Type IIB superstring which reads

$$W_{\mathrm{SG}} = \overline{\partial}\widehat{\theta}^{\widehat{\beta}k}\partial\theta^{\alpha j}A_{\alpha j\,\widehat{\beta}k} + \partial\theta^{\alpha j}\overline{\Pi}^{\underline{a}}A_{\underline{a}\,\alpha j} + \overline{\partial}\widehat{\theta}^{\widehat{\beta}k}\Pi^{\underline{a}}A_{\underline{a}\,\widehat{\beta}k} + \Pi^{\underline{b}}\overline{\Pi}^{\underline{a}}A_{\underline{ab}} + d_{\alpha j}\overline{\partial}\widehat{\theta}^{\widehat{\beta}k}E_{\widehat{\beta}k}{}^{\alpha j}$$
$$+ d_{\alpha j}\overline{\Pi}^{\underline{a}}E_{\underline{a}}{}^{\alpha j} + \widehat{d}_{\widehat{\beta}k}\partial\theta^{\alpha j}E_{\alpha j}{}^{\widehat{\beta}k} + \widehat{d}_{\widehat{\beta}k}\Pi^{\underline{a}}E_{\underline{a}}{}^{\widehat{\beta}k} + d_{\alpha j}\widehat{d}_{\widehat{\beta}k}F^{\alpha j\,\widehat{\beta}k} - \frac{i}{2}N_{\underline{ab}}\overline{\Pi}^{\underline{c}}\Omega_{\underline{c}}{}^{\underline{ab}}$$
$$- \frac{i}{2}\widehat{N}_{\underline{ab}}\Pi^{\underline{c}}\widehat{\Omega}_{\underline{c}}{}^{\underline{ab}} + (\dots)\,, \tag{4.144}$$

where the $d = 6\,\mathcal{N} = 2$ superfields

$$\{A_{\alpha j\,\widehat{\beta}k}, A_{\underline{a}\,\alpha j}, A_{\underline{a}\,\widehat{\beta}k}, A_{\underline{ab}}, E_{\widehat{\beta}k}{}^{\alpha j}, E_{\underline{a}}{}^{\alpha j}, E_{\alpha j}{}^{\widehat{\beta}k}, E_{\underline{a}}{}^{\widehat{\beta}k}, F^{\alpha j\,\widehat{\beta}k}, \Omega_{\underline{abc}}, \widehat{\Omega}_{\underline{abc}}\}\,, \tag{4.145}$$

are functions of the zero-modes of $\{x^{\underline{a}}, \theta^{\alpha j}, \widehat{\theta}^{\widehat{\alpha}j}\}$ and the terms in $(\dots)$ do not contribute to the analysis below since, for example, they involve

$$-\frac{i}{2}N_{\underline{ab}}\overline{\partial}\widehat{\theta}^{\widehat{\beta}k}\Omega_{\widehat{\beta}k}{}^{\underline{ab}} - \frac{i}{2}\widehat{N}_{\underline{ab}}\partial\theta^{\alpha j}\widehat{\Omega}_{\alpha j}{}^{\underline{ab}}\,, \tag{4.146}$$

which is zero up to cubic-order in the worldsheet variables for constant NS-NS and R-R three-form flux. Moreover, the other terms in $(\dots)$ identically vanish for these constant fluxes. Some of the remaining contributions to W_{SG} are written in eq. (M.2).

The $d = 6$ Type IIB supergravity spectrum is described by the bi-spinor superfield $A_{\alpha j\,\widehat{\beta}k}$ [97], which satisfy the following linearized equations of motion

$$(\sigma^{\underline{abc}})^{\alpha\beta}\left(D_{\alpha j}A_{\beta k\,\widehat{\gamma}k} + D_{\beta k}A_{\alpha j\,\widehat{\gamma}l}\right) = 0\,, \tag{4.147a}$$
$$(\sigma^{\underline{abc}})^{\widehat{\alpha}\widehat{\beta}}\left(D_{\widehat{\alpha}j}A_{\gamma l\,\widehat{\beta}k} + D_{\widehat{\beta}k}A_{\gamma l\,\widehat{\alpha}j}\right) = 0\,, \tag{4.147b}$$

and gauge invariances

$$\delta A_{\alpha j\,\widehat{\beta}k} = D_{\alpha j}\widehat{\Omega}_{\widehat{\beta}k} + D_{\widehat{\beta}k}\Omega_{\alpha j}\,, \tag{4.148}$$

where

$$(\sigma^{\underline{abc}})^{\alpha\beta}\left(D_{\alpha j}\Omega_{\beta k} + D_{\beta k}\Omega_{\alpha j}\right) = 0\,, \tag{4.149a}$$
$$(\sigma^{\underline{abc}})^{\widehat{\alpha}\widehat{\beta}}\left(D_{\widehat{\alpha}j}\widehat{\Omega}_{\widehat{\beta}k} + D_{\widehat{\beta}k}\widehat{\Omega}_{\widehat{\alpha}j}\right) = 0\,, \tag{4.149b}$$

for the superfields $\{\Omega_{\alpha j}, \widehat{\Omega}_{\widehat{\alpha}j}\}$ functions of the zero-modes of $\{x^{\underline{a}}, \theta^{\alpha j}, \widehat{\theta}^{\widehat{\alpha}j}\}$, $D_{\alpha j} =$

$\frac{\partial}{\partial \theta^{\alpha j}} - \frac{i}{2}\epsilon_{jk}\theta^{\beta k}\partial_{\alpha\beta}$ and $D_{\widehat{\alpha} j} = \frac{\partial}{\partial \widehat{\theta}^{\widehat{\alpha} j}} - \frac{i}{2}\epsilon_{jk}\widehat{\theta}^{\widehat{\beta}k}\partial_{\widehat{\alpha}\widehat{\beta}}$. The remaining superfields appearing in (4.144) are the $d = 6$ $\mathcal{N} = 2$ linearized supergravity connections and field-strengths. They are defined in terms of $A_{\alpha j\,\widehat{\beta}k}$ according to the equations in Appendix M, where it is also written the remaining equations obtained from BRST invariance of the integrated vertex operator.

Considering a linear perturbation of the flat six-dimensional model to the $\mathrm{AdS}_3 \times \mathrm{S}^3$ background with mixed three-form flux amounts to turning on the NS-NS two-form $b_{\underline{ab}}$ and the R-R field-strength $f^{\alpha j\,\widehat{\beta}k}$. In this case, there exists a gauge such that the non-zero components of the superfield $A_{\alpha j\,\widehat{\beta}k}$ are

$$\begin{aligned} A_{\beta k\,\widehat{\gamma} l} = {} & \frac{1}{32}(\sigma^{\underline{b}}\theta_k)_\beta(\sigma^{\underline{a}}\widehat{\theta}_l)_{\widehat{\gamma}}\eta_{\underline{ab}} - \frac{1}{4}(\sigma^{\underline{b}}\theta_k)_\beta(\sigma^{\underline{a}}\widehat{\theta}_l)_{\widehat{\gamma}} b_{\underline{ab}} - \frac{1}{32}(\sigma_{\underline{d}}\theta_k)_\beta(\theta^j\sigma^{\underline{abd}}\theta_j)(\sigma^{\underline{c}}\widehat{\theta}_l)_{\widehat{\gamma}}\partial_{[\underline{a}}b_{\underline{b}]\underline{c}} \\ & - \frac{1}{32}(\sigma_{\underline{d}}\widehat{\theta}_l)_{\widehat{\gamma}}(\widehat{\theta}^j\sigma^{\underline{abd}}\widehat{\theta}_j)(\sigma^{\underline{c}}\theta_k)_\beta\partial_{[\underline{a}}b_{\underline{b}]\underline{c}} + \frac{1}{9}\epsilon_{\beta\gamma\delta\sigma}\epsilon_{\widehat{\gamma}\widehat{\delta}\widehat{\sigma}\widehat{\rho}}\theta^\gamma_k\widehat{\theta}^{\widehat{\delta}}_l\theta^\delta_m\widehat{\theta}^{\widehat{\sigma}}_n f^{\sigma m\,\widehat{\rho} n} + (\ldots)\,, \end{aligned} \tag{4.150}$$

where $\theta^\alpha_j = \epsilon_{jk}\theta^{\alpha k}$ and $\widehat{\theta}^{\widehat{\alpha}}_j = \epsilon_{jk}\widehat{\theta}^{\widehat{\alpha}k}$. Note that the first term in (4.150) corresponds to a total derivative in the integrated vertex, it was added so that we can reproduce exactly the coefficients appearing in eq. (4.143). The contributions in eq. (4.150) denoted by $(\ldots)$ involve at least second-order derivatives of $b_{\underline{ab}}$ or first-order derivatives of $f^{\alpha j\,\widehat{\beta}k}$ and hence vanish for a constant NS-NS and R-R three-form flux.

Explicitly, in the $\mathrm{AdS}_3 \times \mathrm{S}^3$ background, we have that

$$b_{\underline{ab}} = \frac{1}{3} f_{NS}\widehat{\delta}^{\alpha\widehat{\beta}}(\sigma_{\underline{abc}})_{\alpha\widehat{\beta}}x^{\underline{c}}\,, \qquad f^{\alpha j\,\widehat{\beta}k} = -f_{RR}\widehat{\delta}^{\alpha\widehat{\beta}}\epsilon^{jk}\,, \tag{4.151}$$

and using eqs. (M.1) we can express all the superfields in the integrated vertex (4.144) in terms of $A_{\alpha j\,\widehat{\beta}k}$ and, therefore, in terms of the fields (4.151). Up to first-order in the worldsheet variables $\{x^{\underline{a}}, \theta^{\alpha j}, \widehat{\theta}^{\widehat{\alpha} j}\}$, the $d = 6$ $\mathcal{N} = 2$ superfields are given by

$$A_{\underline{a}\,\alpha j} = \frac{i}{16}(\sigma_{\underline{a}}\theta_j)_\alpha\,, \qquad A_{\underline{a}\,\widehat{\beta}k} = \frac{i}{16}(\sigma_{\underline{a}}\widehat{\theta}_k)_{\widehat{\beta}}\,, \tag{4.152a}$$

$$A_{\underline{ab}} = -\frac{1}{8}\eta_{\underline{ab}} + \frac{1}{3} f_{NS}\widehat{\delta}^{\alpha\widehat{\beta}}(\sigma_{\underline{abc}})_{\alpha\widehat{\beta}}x^{\underline{c}}\,, \qquad E_{\widehat{\beta}k}{}^{\alpha j} = 0\,, \tag{4.152b}$$

$$E_{\underline{a}}{}^{\beta k} = \frac{2}{3} i f_{NS}\widehat{\delta}^{\beta\widehat{\gamma}}\sigma_{\underline{a}\widehat{\gamma}\delta}\theta^{\delta k} + i f_{RR}\widehat{\delta}^{\beta\widehat{\gamma}}\sigma_{\underline{a}\widehat{\gamma}\widehat{\delta}}\widehat{\theta}^{\widehat{\delta}k}\,, \qquad E_{\alpha j}{}^{\widehat{\beta}k} = 0\,, \tag{4.152c}$$

$$E_{\underline{a}}{}^{\widehat{\gamma} l} = -\frac{2}{3} i f_{NS}\widehat{\delta}^{\widehat{\gamma}\delta}\sigma_{\underline{a}\delta\widehat{\sigma}}\widehat{\theta}^{\widehat{\sigma} l} + i f_{RR}\widehat{\delta}^{\widehat{\gamma}\delta}\sigma_{\underline{a}\delta\sigma}\theta^{\sigma l}\,, \qquad F^{\alpha j\,\widehat{\beta}k} = -f_{RR}\epsilon^{jk}\widehat{\delta}^{\alpha\widehat{\beta}}\,, \tag{4.152d}$$

$$\Omega_{\underline{abc}} = -\frac{2}{3} f_{NS} (\sigma_{\underline{abc}})_{\alpha\widehat{\beta}} \widehat{\delta}^{\alpha\widehat{\beta}}\,, \qquad \widehat{\Omega}_{\underline{abc}} = \frac{2}{3} f_{NS} (\sigma_{\underline{abc}})_{\alpha\widehat{\beta}} \widehat{\delta}^{\alpha\widehat{\beta}}\,. \tag{4.152e}$$

Consequently, up to cubic-order in the worldsheet fields and after rescaling $\theta^{\alpha j} \to f_{RR}^{\frac{1}{2}} \theta^{\alpha j}$ and $\widehat{\theta}^{\widehat{\alpha} j} \to f_{RR}^{\frac{1}{2}} \widehat{\theta}^{\widehat{\alpha} j}$, the linearly perturbed worldsheet action is

$$\begin{aligned} S_{\text{flat}} + \int W_{\text{SG}} = \int d^2 z \Bigg[& \frac{1}{2} \partial x^{\underline{b}} \bar{\partial} x^{\underline{a}} \eta_{\underline{ab}} \left(1 - \frac{1}{4}\right) + \epsilon_{jk} \widehat{\delta}_{\alpha\widehat{\beta}} \partial \widehat{\theta}^{\widehat{\beta} k} \bar{\partial} \theta^{\alpha j} + w_\alpha \bar{\partial} \lambda^\alpha + \widehat{w}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}} \\ & + \frac{1}{3} f_{NS} x^{\underline{c}} \partial x^{\underline{b}} \bar{\partial} x^{\underline{a}} (\sigma_{\underline{abc}})_{\alpha\widehat{\beta}} \widehat{\delta}^{\alpha\widehat{\beta}} - \frac{2}{3} i \epsilon_{jk} f_{NS} \left(\bar{\partial} x_{\alpha\widehat{\beta}} \partial \widehat{\theta}^{\widehat{\beta} k} \theta^{\alpha j} + \partial x_{\alpha\widehat{\beta}} \widehat{\theta}^{\widehat{\beta} k} \bar{\partial} \theta^{\alpha j} \right) \\ & + \frac{i}{2} f_{RR} \epsilon_{jk} \left(\bar{\partial} x_{\widehat{\alpha}\widehat{\beta}} \widehat{\theta}^{\widehat{\beta} k} \partial \widehat{\theta}^{\widehat{\alpha} j} + \partial x_{\alpha\beta} \theta^{\beta k} \bar{\partial} \theta^{\alpha j} \right) + \frac{i}{3} f_{NS} (\sigma^{\underline{abc}})_{\alpha\widehat{\beta}} \delta^{\alpha\widehat{\beta}} \Big(\bar{\partial} x_{\underline{c}} N_{\underline{ab}} \\ & - \partial x_{\underline{c}} \widehat{N}_{\underline{ab}} \Big) \Bigg] + S_{\rho,\sigma} + S_C\,, \end{aligned} \tag{4.153}$$

and by further rescaling $x^{\underline{a}} \to \frac{2}{\sqrt{3}} x^{\underline{a}}$, $f_{NS} \to \frac{3\sqrt{3}}{8} f_{NS}$ and $f_{RR} \to \frac{\sqrt{3}}{2} f_{RR}$, we finally get

$$\begin{aligned} S_{\text{flat}} + \int W_{\text{SG}} = \int d^2 z \Bigg[& \frac{1}{2} \partial x^{\underline{b}} \bar{\partial} x^{\underline{a}} \eta_{\underline{ab}} + \epsilon_{jk} \widehat{\delta}_{\alpha\widehat{\beta}} \partial \widehat{\theta}^{\widehat{\beta} k} \bar{\partial} \theta^{\alpha j} + w_\alpha \bar{\partial} \lambda^\alpha + \widehat{w}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}} \\ & + \frac{1}{3} f_{NS} x^{\underline{c}} \partial x^{\underline{b}} \bar{\partial} x^{\underline{a}} (\sigma_{\underline{abc}})_{\alpha\widehat{\beta}} \widehat{\delta}^{\alpha\widehat{\beta}} + \frac{i}{2} f_{NS} \epsilon_{jk} \left(\partial x_{\alpha\widehat{\beta}} \bar{\partial} \widehat{\theta}^{\widehat{\beta} k} \theta^{\alpha j} - \bar{\partial} x_{\alpha\widehat{\beta}} \partial \widehat{\theta}^{\widehat{\beta} k} \theta^{\alpha j} \right) \\ & + \frac{i}{2} f_{RR} \epsilon_{jk} \left(\bar{\partial} x_{\widehat{\alpha}\widehat{\beta}} \widehat{\theta}^{\widehat{\beta} k} \partial \widehat{\theta}^{\widehat{\alpha} j} + \partial x_{\alpha\beta} \theta^{\beta k} \bar{\partial} \theta^{\alpha j} \right) + \frac{i}{4} f_{NS} (\sigma^{\underline{abc}})_{\alpha\widehat{\beta}} \delta^{\alpha\widehat{\beta}} \Big(\bar{\partial} x_{\underline{c}} N_{\underline{ab}} \\ & - \partial x_{\underline{c}} \widehat{N}_{\underline{ab}} \Big) \Bigg] + S_{\rho,\sigma} + S_C\,, \end{aligned} \tag{4.154}$$

where we integrated by parts and ignored terms proportional to $\partial\bar{\partial}x$, $\partial\bar{\partial}\theta$ and $\partial\bar{\partial}\widehat{\theta}$, which can be removed by redefining x, $\widehat{\theta}$ and θ. After rescaling all the worldsheet fields by f^{-1}, the action (4.154) reproduces all terms appearing in eqs. (4.142) and (4.143), except for the contributions involving the ghost currents $\{N_{\underline{ab}}, \widehat{N}_{\underline{ab}}\}$, which appear in (4.154) but are absent in (4.142) and (4.143). Nevertheless, this fact can be easily remedied by shifting the ghosts in (4.154) as

$$\lambda^\alpha \to \lambda^\alpha - \frac{i}{4} f_{NS} (\sigma^{\underline{abc}})_{\beta\widehat{\gamma}} \delta^{\beta\widehat{\gamma}} x_{\underline{c}} (\sigma_{\underline{ab}} \lambda)^\alpha\,, \tag{4.155a}$$

$$\widehat{\lambda}^{\widehat{\alpha}} \to \widehat{\lambda}^{\widehat{\alpha}} + \frac{i}{4} f_{NS} (\sigma^{\underline{abc}})_{\beta\widehat{\gamma}} \delta^{\beta\widehat{\gamma}} x_{\underline{c}} (\sigma_{\underline{ab}} \widehat{\lambda})^{\widehat{\alpha}}\,, \tag{4.155b}$$

and then removing additional terms of cubic-order proportional to $\bar{\partial}\lambda^\alpha$ and $\partial\widehat{\lambda}^{\widehat{\alpha}}$ by also redefining w_α and $\widehat{w}_{\widehat{\alpha}}$.

Therefore, our perturbative analysis in eq. (4.154) replicates the worldsheet action (4.120) up to cubic-order in the worldsheet variables. In addition, note that to put the contributions proportional to f_{RR} in (4.154) in the same form as the ones appearing in eq. (4.143), one can again integrate by parts and eliminate all terms proportional to $\partial\bar{\partial}x$, $\partial\bar{\partial}\theta$ and $\partial\bar{\partial}\widehat{\theta}$ by suitable field redefinitions.

Thus, we have confirmed that the deformed action (4.120) corresponds to turning on the NS-NS two-form $b_{\underline{ab}}$ and a constant R-R field-strength $f^{\alpha j \widehat{\beta} k}$, as presented in eqs. (4.151). For this purpose, it was enough to consider the deformation (4.144), given that the remaining terms in $(\ldots)$ that can appear in the integrated vertex do not contribute to our perturbative analysis.

4.8 One-loop conformal invariance of the super-coset sigma-model

In this section, we will check that conformal invariance of the classical action (4.120) is preserved at the one-loop level in the sigma-model perturbation theory. To accomplish that, the divergent part of the quantum effective action will be computed using the covariant background field method [91] [98] [99] [100] and shown that it vanishes. Therefore, the beta function is zero at one-loop.

Let us first point out that for the Green-Schwarz superstring in the mixed flux $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ background it was shown that there is no divergence in the one-loop effectve action for the terms proportional to the classical bosonic currents $\{J^{[\underline{ab}]}, J^{\underline{a}}\}$ in ref. [96]. There, it was found that after gauge-fixing Kappa-symmetry transformations the UV divergent contribution involving the classical currents $J^{\underline{a}}$ is proportional to the Killing form of $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ [96, eq. (7.15)] (or, equivalently, to the second Casimir (4.159)) and hence vanishes. Since we are employing a covariant framework in this paper, we don't have to deal with the subtleties arising from gauge-fixing Kappa-symmetry.

For the purpose of covariantly quantizing our theory, we will make use of the covariant background field method, which consists in expanding the coset element g as

$$g = g_{\mathrm{cl}} e^{fX}, \tag{4.156}$$

where g_{cl} is the classical field and X parametrizes the quantum fluctuations. By using the gauge transformations (4.116), we can take $X \in \mathfrak{g}\backslash\mathfrak{g}_0$ so that

$$
\begin{aligned}
X &= X^A T_A \\
&= X^1 + X^2 + X^3 \,,
\end{aligned} \tag{4.157}
$$

as a consequence, the left-invariant one-form J expanded around the classical configuration $g_{\rm cl}$ is given by

$$
\begin{aligned}
J &= e^{-fX} J_{\rm cl} e^{fX} + e^{-fX} d e^{fX} \\
&= J_{\rm cl} + f\big(dX + [J_{\rm cl}, X]\big) + \frac{1}{2} f^2\big([dX + [J_{\rm cl}, X], X]\big) + \mathcal{O}(f^3) \,.
\end{aligned} \tag{4.158}
$$

For simplicity, the subscript in $J_{\rm cl}$ coming from eq. (4.158) will be dropped in the rest of this section, so that it is understood that all left-invariant one-forms $J^{\underline{A}}$ correspond to classical fields in the formulas below.

Let us make a few important observations before expanding the sigma-model (4.120) in powers of the quantum fluctuations. When substituting (4.158) into (4.120) there will be terms independent of X^A which are quadratic in the background currents $J^{\underline{A}}$, these make up the classical action $S_{\rm cl}$. There will also be terms which are linear in the fluctuations X^A and these do not contribute to the effective action. Therefore, we will be concerned with the terms quadratic in the fluctuations X^A which are the necessary ones for calculating the one-loop beta function. Note that we will only examine UV divergences in this section, given that infrared effects are expected to vanish when summing up the perturbation series [101]. By power counting, the UV divergent contributions must involve one classical current of conformal weight $(1,0)$ and one of conformal weight $(0,1)$.

As was mentioned below eq. (4.126), when $f_{NS} = 0$, the sigma-model (4.120) takes the same form as the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor worldsheet action. Concerning the latter, the divergent contributions to the one-loop effective action from the matter and ghost part were shown to be proportional to the second Casimir $C_2(\mathrm{PSU}(2,2|4))$ [91] [99] and given that $C_2(\mathrm{PSU}(2,2|4)) = 0$, the pure spinor action in $\mathrm{AdS}_5 \times \mathrm{S}^5$ was proved to be one-loop conformal invariant. Therefore, from the fact that the computation performed in refs. [91] and [99] only uses properties of the target-space supergroup, and from

$$
C_2(\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)) = 0 \,, \tag{4.159}
$$

one already knows that the worldsheet action (4.120) is conformal invariant at the one-loop level when $f_{NS} = 0$. Eq. (4.159) can be readily checked from the definition of the second Casimir

$$f_{\underline{A}\,\underline{C}}{}^{\underline{D}} f_{\underline{B}\,\underline{D}}{}^{\underline{C}} (-)^{|\underline{D}|} = \frac{1}{4}\eta_{\underline{A}\underline{B}} C_2(\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2))\,. \tag{4.160}$$

With the above observations, we can anticipate some aspects of the one-loop calculation to be done below. Taking into account the relation between the inverse AdS_3 radius (f) and the fluxes $\frac{f_{RR}^2}{f^2} = 1 - \frac{f_{NS}^2}{f^2}$, one then concludes that the divergent contributions in the computation of the one-loop effective action for the model (4.120) can be of order $\mathcal{O}(1)$ or proportional to $\frac{f_{NS}}{f}$, and that the terms of $\mathcal{O}(1)$ shall cancel by the mechanism (4.159). The reason for this is that when $f_{NS} = 0$ the action has the same form as in refs. [91] [99]. In the following, we will perform the computation with the factors of $\frac{f_{RR}}{f}$ and $\frac{f_{NS}}{f}$ coming from (4.120) explicitly written and, only in the end, substitute $\frac{f_{RR}^2}{f^2} = 1 - \frac{f_{NS}^2}{f^2}$ to show that the divergent part vanishes.

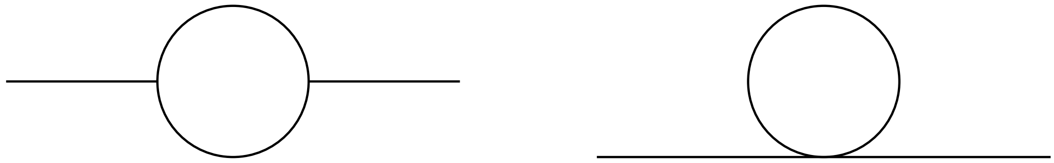
In particular, since the divergent contributions proportional to the classical currents

$$\{J^{[\underline{ab}]}\overline{J}^{[\underline{cd}]}, J^{[\underline{ab}]}\widehat{N}_{\underline{cd}}, \overline{J}^{[\underline{ab}]}N_{\underline{cd}}, N_{\underline{ab}}\widehat{N}_{\underline{cd}}\}\,, \tag{4.161}$$

do not involve $\frac{f_{RR}}{f}$ neither $\frac{f_{NS}}{f}$ at any stage of the computation, but only factors of order $\mathcal{O}(1)$, we already know that they vanish [99]. Furthermore, because contractions between the quantum fluctuations of the ghosts $\{w_\alpha, \lambda^\alpha, \widehat{w}_{\widehat{\alpha}}, \widehat{\lambda}^{\widehat{\alpha}}\}$ only contribute to these $\mathcal{O}(1)$ factors, we can focus on the divergences coming from integrating over the fluctuations appearing in the background expansion of the left-invariant currents $J^{\underline{A}}$.

More precisely, the divergences of the one-loop effective action from integrating over the quantum fluctuations of the ghosts are of $\mathcal{O}(1)$ and proportional to the classical fields $\{N_{\underline{ab}}, \widehat{N}_{\underline{cd}}\}$, consequently, they will cancel against $\mathcal{O}(1)$ contributions coming from integrating over the fluctuations in the expansion of $J^{[\underline{ab}]}$ and $\overline{J}^{[\underline{ab}]}$ in (4.120) [99].

The contributions quadratic in the fluctuations will be separated into a kinetic term S_{kin}, a term involving the fermionic currents S_{ferm} and a term involving the bosonic currents S_{bos}. Additionally, we will not bother writing terms that appear when expanding the ghosts currents $\{N_{\underline{ab}}, \widehat{N}_{\underline{ab}}\}$ in quantum fluctuations, in view



(a) (b)

Figure 4.1: One-loop diagrams contributing to the effective action. The external lines consist of the classical currents.

of our argument in the paragraph above. Therefore, S_{ferm} and S_{bos} will comprise all terms quadratic in the quantum fluctuations containing $X^B \nabla X^A$, which contribute to diagrams of the type shown in fig. 4.1a, and terms proportional to $X^B X^A$ where $X^A(y) X^B(z) \neq 0$ (see eq. (4.163)), which contribute to diagrams of the type shown in fig. 4.1b.

Expanding (4.120) to quadratic order in the quantum fluctuations is a long exercise, especially because the three-dimensional integral over the three-form $\mathcal{H}_{\text{NS}}$ needs to be written as a two-dimensional integral over the worldsheet. This can be accomplished by using the Maurer-Cartan eqs. (4.119) and the identity $\nabla^2 X^A = X^B R_B{}^A$ (see eqs. (C.11)), the final result is written in Appendix N. On the other hand, the expansion of the remaining terms is straightforward, since one can use the supertrace representation of the worldsheet action (4.126) to ease the task, the result is represented in eq. (N.2). After plugging (4.158) into (4.120), one finds that the kinetic term for the X^A's is

$$S_{\text{kin}} = \int d^2 z \ \text{sTr} \left(\frac{1}{2} \partial X^2 \overline{\partial} X^2 + \overline{\partial} X^1 \partial X^3 \right), \tag{4.162}$$

which gives the following propagator for the fluctuations

$$X^A(y) X^B(z) \sim -\eta^{BA} \log |y-z|^2 \, . \tag{4.163}$$

The terms involving the fermionic left-invariant currents that can give a non-zero contribution to the one-loop beta function are

$$\begin{aligned} S^{(1)}_{\text{ferm}} = \int d^2 z \, \Big\{ & \frac{1}{8} \Big[(2 + \tfrac{f_{RR}}{f}) \overline{\nabla} X^{\underline{a}} X^{\beta k} - (2 + 3\tfrac{f_{RR}}{f}) \overline{\nabla} X^{\beta k} X^{\underline{a}} \Big] J^{\alpha j} f_{\beta k \, \underline{a} \, \alpha j} \\ & + \frac{1}{8} \Big[(2 + \tfrac{f_{RR}}{f}) \nabla X^{\underline{b}} X^{\widehat{\gamma} l} - (2 + 3\tfrac{f_{RR}}{f}) \nabla X^{\widehat{\gamma} l} X^{\underline{b}} \Big] \overline{J}^{\widehat{\beta} k} i f_{\widehat{\gamma} l \, \underline{b} \, \widehat{\beta} k} \\ & + \frac{1}{8} \Big[(2 - \tfrac{f_{RR}}{f}) \nabla X^{\underline{a}} X^{\gamma l} - (2 - 3\tfrac{f_{RR}}{f}) \nabla X^{\gamma l} X^{\underline{a}} \Big] \overline{J}^{\alpha j} i f_{\gamma l \, \underline{a} \, \alpha j} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{8}\Big[(2-\tfrac{f_{RR}}{f})\overline{\nabla}X^{\underline{a}}X^{\hat{\delta}m}-(2-3\tfrac{f_{RR}}{f})\overline{\nabla}X^{\hat{\delta}m}X^{\underline{a}}\Big]J^{\hat{\beta}k}if_{\hat{\delta}m\,\underline{a}\,\hat{\beta}k}\\
&-\frac{1}{2}\Big(X^{\underline{b}}\nabla X^{\gamma l}+3\nabla X^{\underline{b}}X^{\gamma l}\Big)\bar{J}^{\hat{\beta}k}H_{\gamma l\,\hat{\beta}k\,\underline{b}}+\frac{1}{2}\Big(X^{\underline{a}}\overline{\nabla}X^{\hat{\delta}m}\\
&+3\overline{\nabla}X^{\underline{a}}X^{\hat{\delta}m}\Big)J^{\alpha j}H_{\alpha j\,\hat{\delta}m\,\underline{a}}+\frac{1}{2}\Big(X^{\underline{a}}\overline{\nabla}X^{\alpha j}+3\overline{\nabla}X^{\underline{a}}X^{\alpha j}\Big)J^{\hat{\beta}k}H_{\alpha j\,\hat{\beta}k\,\underline{a}}\\
&-\frac{1}{2}\Big(X^{\underline{a}}\nabla X^{\hat{\beta}k}+3\nabla X^{\underline{a}}X^{\hat{\beta}k}\Big)\bar{J}^{\alpha j}H_{\alpha j\,\hat{\beta}k\,\underline{a}}\Big\}, \qquad (4.164)
\end{aligned}
$$

and

$$
\begin{aligned}
S^{(2)}_{\text{ferm}}=\int d^2z\,\Big\{&\frac{1}{2}\frac{f_{RR}}{f}X^{\underline{b}}X^{\underline{a}}\bar{J}^{\hat{\beta}k}J^{\alpha j}f_{\hat{\beta}k\,\underline{a}}{}^{\gamma l}f_{\gamma l\,\underline{b}\,\alpha j}-\frac{1}{4}X^{\hat{\delta}m}X^{\gamma l}\bar{J}^{\hat{\beta}k}J^{\alpha j}\Big[\frac{f_{RR}}{f}f_{\hat{\beta}k\,\hat{\delta}m}{}^{\underline{a}}f_{\underline{a}\,\gamma l\,\alpha j}\\
&+(2-\tfrac{f_{RR}}{f})f_{\hat{\beta}k\,\gamma l}{}^{[\underline{cd}]}f_{[\underline{cd}]\,\hat{\delta}m\,\alpha j}\Big]-\frac{1}{2}\frac{f_{RR}}{f}X^{\underline{b}}X^{\underline{a}}J^{\hat{\beta}k}\bar{J}^{\alpha j}f_{\hat{\beta}k\,\underline{a}}{}^{\gamma l}f_{\gamma l\,\underline{b}\,\alpha j}\\
&-\frac{1}{4}X^{\hat{\delta}m}X^{\gamma l}J^{\hat{\beta}k}\bar{J}^{\alpha j}\Big[(2+\tfrac{f_{RR}}{f})f_{\hat{\beta}k\,\gamma l}{}^{[\underline{ab}]}f_{[\underline{ab}]\,\hat{\delta}m\,\alpha j}-\frac{f_{RR}}{f}f_{\hat{\beta}k\,\hat{\delta}m}{}^{\underline{a}}f_{\underline{a}\,\gamma l\,\alpha j}\Big]\\
&+\frac{1}{4}\Big[\Big(J^{\gamma l}\bar{J}^{\delta m}-\bar{J}^{\gamma l}J^{\delta m}\Big)if_{\delta m\,\gamma l}{}^{\underline{a}}-\Big(J^{\hat{\gamma}l}\bar{J}^{\hat{\delta}m}-\bar{J}^{\hat{\gamma}l}J^{\hat{\delta}m}\Big)if_{\hat{\delta}m\,\hat{\gamma}l}{}^{\underline{a}}\Big]X^{\hat{\beta}k}X^{\alpha j}H_{\alpha j\,\hat{\beta}k\,\underline{a}}\\
&+\frac{1}{2}\Big[\Big(J^{\gamma l}\bar{J}^{\alpha j}-\bar{J}^{\gamma l}J^{\alpha j}\Big)if_{\gamma l\,\underline{b}}{}^{\hat{\beta}k}+\Big(J^{\hat{\gamma}l}\bar{J}^{\hat{\beta}k}-\bar{J}^{\hat{\gamma}l}J^{\hat{\beta}k}\Big)if_{\hat{\gamma}l\,\underline{b}}{}^{\alpha j}\Big]X^{\underline{b}}X^{\underline{a}}H_{\alpha j\,\hat{\beta}k\,\underline{a}}\\
&+\frac{1}{4}(2-\tfrac{f_{RR}}{f})\Big(J^{\hat{\delta}m}\bar{J}^{\gamma l}-\bar{J}^{\hat{\delta}m}J^{\gamma l}\Big)X^{\hat{\beta}k}X^{\alpha j}R_{\gamma l\,\hat{\delta}m\,\alpha j\,\hat{\beta}k}\Big\}, \qquad (4.165)
\end{aligned}
$$

where the last term in (4.165) comes from using using $\nabla^2 X^A = X^B R_B{}^A$ after integrating by parts in the kinetic term (4.162) dressed up with the connections. In total, we will write $S_{\text{ferm}} = S^{(1)}_{\text{ferm}} + S^{(2)}_{\text{ferm}}$.

The terms involving the bosonic currents that can give a non-zero contribution to the one-loop beta function are

$$
\begin{aligned}
S_{\text{bos}}=\int d^2z\,\Big[&\frac{1}{4}(1+\tfrac{f_{RR}}{f})\overline{\nabla}X^{\beta k}X^{\alpha j}J^{\underline{a}}if_{\alpha j\,\beta k\,\underline{a}}+\frac{1}{4}(1-\tfrac{f_{RR}}{f})\overline{\nabla}X^{\hat{\beta}k}X^{\hat{\alpha}j}J^{\underline{a}}if_{\hat{\alpha}j\,\hat{\beta}k\,\underline{a}}\\
&+\frac{1}{4}(1-\tfrac{f_{RR}}{f})\nabla X^{\beta k}X^{\alpha j}\bar{J}^{\underline{a}}if_{\alpha j\,\beta k\,\underline{a}}+\frac{1}{4}(1+\tfrac{f_{RR}}{f})\nabla X^{\hat{\beta}k}X^{\hat{\alpha}j}\bar{J}^{\underline{a}}if_{\hat{\alpha}j\,\hat{\beta}k\,\underline{a}}\\
&-\frac{1}{2}J^{\underline{c}}X^{\underline{b}}\overline{\nabla}X^{\underline{a}}H_{\underline{abc}}+\frac{1}{2}\bar{J}^{\underline{c}}X^{\underline{b}}\nabla X^{\underline{a}}H_{\underline{abc}}+\frac{1}{2}J^{\underline{a}}\Big(\overline{\nabla}X^{\hat{\beta}k}X^{\alpha j}-X^{\hat{\beta}k}\overline{\nabla}X^{\alpha j}\Big)H_{\alpha j\,\hat{\beta}k\,\underline{a}}\\
&+\frac{1}{2}\bar{J}^{\underline{a}}\Big(X^{\hat{\beta}k}\nabla X^{\alpha j}-\nabla X^{\hat{\beta}k}X^{\alpha j}\Big)H_{\alpha j\,\hat{\beta}k\,\underline{a}}-\frac{1}{4}\frac{f_{RR}}{f}X^{\hat{\beta}k}X^{\alpha j}\bar{J}^{\underline{b}}J^{\underline{a}}\Big(f_{\underline{b}\,\hat{\beta}k}{}^{\gamma l}f_{\gamma l\,\alpha j\,\underline{a}}\\
&+f_{\underline{b}\,\alpha j}{}^{\hat{\gamma}l}f_{\hat{\gamma}l\,\hat{\beta}k\,\underline{a}}\Big)-\frac{1}{2}X^{\underline{d}}X^{\underline{c}}\bar{J}^{\underline{b}}J^{\underline{a}}f_{\underline{b}\,\underline{c}}{}^{[\underline{ef}]}f_{[\underline{ef}]\,\underline{d}\,\underline{a}}+\frac{1}{2}X^{\underline{d}}\overline{\nabla}X^{\underline{c}}if_{\underline{c}\,\underline{d}}{}^{[\underline{ab}]}N_{\underline{ab}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2} X^{\underline{d}} \nabla X^{\underline{c}} i f_{\underline{c}\,\underline{d}}{}^{[ab]} \widehat{N}_{ab} + \frac{1}{2}\Big(X^{\widehat{\beta}k} \overline{\nabla} X^{\alpha j} + X^{\alpha j} \overline{\nabla} X^{\widehat{\beta}k} \Big) i f_{\alpha j\,\widehat{\beta}k}{}^{[ab]} N_{ab} \\
&+ \frac{1}{2}\Big(X^{\widehat{\beta}k} \nabla X^{\alpha j} + X^{\alpha j} \nabla X^{\widehat{\beta}k} \Big) i f_{\alpha j\,\widehat{\beta}k}{}^{[ab]} \widehat{N}_{ab} + i\bar{\partial} X_{\underline{b}} X_{\underline{a}} J^{[ab]} + i\partial X_{\underline{b}} X_{\underline{a}} \overline{J}^{[ab]} \\
&- 2 X_{\underline{d}} X_{\underline{c}} J^{[db]} \overline{J}^{[ca]} \eta_{ab} + J^{[ab]} X^{\widehat{\beta}k} \bar{\partial} X^{\alpha j} i f_{[ab]\,\widehat{\beta}k\,\alpha j} - \overline{J}^{[ab]} \partial X^{\widehat{\beta}k} X^{\alpha j} i f_{[ab]\,\alpha j\,\widehat{\beta}k} \\
&- J^{[ab]} \overline{J}^{[cd]} X^{\widehat{\gamma}l} X^{\delta m} f_{[ab]\,\widehat{\gamma}l\,\alpha j} f_{[cd]\,\delta m}{}^{\alpha j} + \frac{1}{4}\big(2 - \tfrac{f_{RR}}{f}\big) X^{\widehat{\beta}k} X^{\alpha j} J^{\underline{b}} \overline{J}^{\underline{a}} R_{\underline{a}\,\underline{b}\,\alpha j\,\widehat{\beta}k} \Bigg], \qquad (4.166)
\end{aligned}
$$

where the last seven terms in eq. (4.166) come from the connections that dress the kinetic term (4.162).

We now have everything in place to calculate the divergent part of the one-loop effective action. We denote functional integration over the fluctuations by angled brackets $\langle \dots \rangle$, and start by considering the contributions proportional to the classical fermionic left-invariant currents (4.164) and (4.165), so that we have

$$
\begin{aligned}
&\left\langle e^{-S_{\text{kin}} - S_{\text{ferm}}} \right\rangle \Big|_{\text{1PI/one-loop}} \\
&\overset{\text{div.}}{=} \int d^2 z\, \overline{J}^{\widehat{\beta}k}(z) J^{\alpha j}(z) \log|0|^2 \left[\frac{3}{4} \frac{f_{RR}}{f} f_{\widehat{\beta}k}{}^{\underline{a}\,\gamma l} f_{\gamma l\,\underline{a}\,\alpha j} + \frac{1}{4}\big(2 - \tfrac{f_{RR}}{f}\big) f_{\widehat{\beta}k}{}^{\widehat{\gamma}l\,[ab]} f_{[ab]\,\widehat{\gamma}l\,\alpha j} \right] \\
&+ \int d^2 z \int d^2 y\, \overline{J}^{\widehat{\beta}k}(z) J^{\alpha j}(y) \Bigg\{ |y-z|^{-2} \frac{1}{32} \big(2 + \tfrac{f_{RR}}{f}\big)\big(2 + 3\tfrac{f_{RR}}{f}\big) f_{\gamma l\,\underline{a}\,\alpha j} f^{\gamma l\,\underline{a}}{}_{\widehat{\beta}k} \\
&- \delta^{(2)}(y-z) \log|y-z|^2 \frac{1}{64} \left[\big(2 + \tfrac{f_{RR}}{f}\big)^2 + \big(2 + 3\tfrac{f_{RR}}{f}\big)^2 \right] f_{\gamma l\,\underline{a}\,\alpha j} f^{\gamma l\,\underline{a}}{}_{\widehat{\beta}k} \\
&+ \frac{3}{2} |y-z|^{-2} H_{\alpha j}{}^{\gamma l\,\underline{a}} H_{\gamma l\,\widehat{\beta}k\,\underline{a}} + \frac{5}{2} \delta^{(2)}(y-z) \log|y-z|^2 H_{\alpha j}{}^{\gamma l\,\underline{a}} H_{\gamma l\,\widehat{\beta}k\,\underline{a}} \Bigg\} \\
&+ (\dots) \\
&= 0\,. \qquad (4.167)
\end{aligned}
$$

In eq. (4.167), we wrote all possible divergent terms proportional to $\overline{J}^{\widehat{\beta}k} J^{\alpha j}$ and in $(\dots)$ are the remaining contributions with two fermionic background currents.

It is not difficult to understand why eq. (4.167) vanishes. First, note that each individual term there proportional to $\overline{J}^{\widehat{\beta}k} J^{\alpha j}$ is identically zero, this is because

$$
f_{\gamma l\,\underline{a}\,\alpha j} f^{\gamma l\,\underline{a}}{}_{\widehat{\beta}k} = 0\,, \qquad f_{\widehat{\beta}k}{}^{\widehat{\gamma}l\,[ab]} f_{[ab]\,\widehat{\gamma}l\,\alpha j} = 0\,, \qquad H_{\alpha j}{}^{\gamma l\,\underline{a}} H_{\gamma l\,\widehat{\beta}k\,\underline{a}} = 0\,. \qquad (4.168)
$$

Similarly, by observing the structure of eqs. (4.164) and (4.165) and using $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ symmetry, one can easily convince oneself that the other possible divergent terms in $(\dots)$ involving the background currents $\{J^{\widehat{\beta}k}\overline{J}^{\alpha j}, \overline{J}^{\beta k} J^{\alpha j}, \overline{J}^{\widehat{\beta}k} J^{\widehat{\alpha}j}\}$

can only be proportional to (4.168) or to the following other combinations of the structure constants and $H_{\alpha j\,\widehat{\beta} k\,\underline{a}}$

$$f_{\alpha j}{}^{\underline{a}\,\widehat{\gamma} l} H_{\beta k\,\widehat{\gamma} l\,\underline{a}} = 0\,, \qquad f_{\alpha j\,\beta k}{}^{\underline{a}} H^{\widehat{\gamma} l}{}_{\widehat{\gamma} l\,\underline{a}} = 0\,, \tag{4.169a}$$

$$f_{\widehat{\alpha} j}{}^{\underline{a}\,\gamma l} H_{\gamma l\,\widehat{\beta} k\,\underline{a}} = 0\,, \qquad f_{\widehat{\alpha} j\,\widehat{\beta} k}{}^{\underline{a}} H^{\widehat{\gamma} l}{}_{\widehat{\gamma} l\,\underline{a}} = 0\,, \tag{4.169b}$$

and, hence, there is no divergence in the one-loop effective action coming from diagrams with two classical fermionic left-invariant currents as external lines.

We have seen that the cancelation of divergences proportional to the background fermionic currents in the one-loop effective action (4.167) does not impose any constraints in the relative coefficients of the sigma-model (4.120), for the reason that all possible divergent terms individually vanish. The situation is quite different for terms involving two classical bosonic currents, in particular for the ones with $J^{\underline{b}}\overline{J}^{\underline{a}}$. As we will presently see, this contribution will imply a non-trivial relation between the relative coefficients in the worldsheet action (4.120).

Consider the general form of the functional integral after integrating over the quantum fluctuations

$$\begin{aligned} &\left\langle e^{-S_{\mathrm{kin}}-S_{\mathrm{bos}}} \right\rangle\Big|_{\mathrm{1PI/one-loop}} \\ &\overset{\mathrm{div.}}{=} \int d^2z \, \log|0|^2 \Big[J^{\underline{a}}\overline{J}^{\underline{b}}(z) \mathcal{C}^{(1)}_{\underline{ab}} + \left(J^{\underline{c}}\overline{J}^{[\underline{ba}]} - \overline{J}^{\underline{c}}J^{[\underline{ba}]} \right)(z) \mathcal{C}^{(2)}_{\underline{abc}} \\ &+ \eta^{[\underline{ab}][\underline{de}]} \left(J^{\underline{c}}\widehat{N}_{\underline{de}} - \overline{J}^{\underline{c}}N_{\underline{de}} \right)(z) \mathcal{C}^{(3)}_{\underline{abc}} \Big] + (\ldots)\,, \end{aligned} \tag{4.170}$$

where the terms in $(\ldots)$ above are of $\mathcal{O}(1)$ and correspond to contributions proportional to the classical fields (4.161), hence, after adding to (4.170) the piece coming from integrating over the quantum fluctuations of the ghost-currents $\{N_{\underline{ab}}, \widehat{N}_{\underline{ab}}\}$, these terms sum up to zero by the property (4.159), as we argued above. In order to prove one-loop conformal invariance, it remains to show that the coefficients

$$\mathcal{C}^{(1)}_{\underline{ab}}\,,\ \mathcal{C}^{(2)}_{\underline{abc}}\,,\ \mathcal{C}^{(3)}_{\underline{abc}}\,, \tag{4.171}$$

vanish. Note that in (4.170) we are also anticipating the antisymmetry in the exchange of z and $\overline{z}$ in the classical fields multiplying $\mathcal{C}^{(2)}_{\underline{abc}}$ and $\mathcal{C}^{(3)}_{\underline{abc}}$, since these coefficients are proportional to f_{NS}.

The divergent terms involving $J^{\underline{a}}\overline{J}^{\underline{b}}$ are

$$
\begin{aligned}
&\int d^2z \, \log|0|^2 J^{\underline{a}}\overline{J}^{\underline{b}}(z)\mathcal{C}^{(1)}_{\underline{ab}} \\
&\quad = \int d^2z \, \log|0|^2 J^{\underline{a}}\overline{J}^{\underline{b}}(z)\left[\frac{1}{4}\frac{f_{RR}}{f}\left(f_{\underline{b}}{}^{\widehat{\beta}k\,\widehat{\gamma}l}f_{\widehat{\gamma}l\,\widehat{\beta}k\,\underline{a}} - f_{\underline{b}}{}^{\alpha j\,\gamma l}f_{\gamma l\,\alpha j\,\underline{a}}\right) - \frac{1}{2}f_{\underline{b}}{}^{\underline{c}\,[\underline{ef}]}f_{[\underline{ef}]\,\underline{c}\,\underline{a}}\right] \\
&\quad + \int d^2z \int d^2y \, J^{\underline{a}}(y)\overline{J}^{\underline{b}}(z)\left(|y-z|^{-2} - \delta^{(2)}(y-z)\log|y-z|^2\right)\times \\
&\quad \times \left\{\frac{1}{16}f_{\alpha j\,\beta k\,\underline{a}}f^{\beta k\,\alpha j}{}_{\underline{b}}\left[(1+\tfrac{f_{RR}}{f})^2 + (1-\tfrac{f_{RR}}{f})^2\right] + \frac{1}{4}H_{\underline{ade}}H^{\underline{de}}{}_{\underline{b}}\left(-1+\tfrac{1}{2}\right)\right\} \\
&\quad = \int d^2z \, \log|0|^2 J^{\underline{a}}\overline{J}^{\underline{b}}(z)\left[-\frac{1}{2}f_{\underline{b}}{}^{\underline{c}\,[\underline{ef}]}f_{[\underline{ef}]\,\underline{c}\,\underline{a}} - \frac{1}{4}f_{\alpha j\,\beta k\,\underline{a}}f^{\beta k\,\alpha j}{}_{\underline{b}}(1+\tfrac{f_{RR}^2}{f^2})\right. \\
&\quad \left. + \frac{1}{4}H_{\underline{ade}}H^{\underline{de}}{}_{\underline{b}}\right] \\
&\quad = \int d^2z \, \log|0|^2 J^{\underline{a}}\overline{J}^{\underline{b}}(z)\frac{1}{2}f_{\alpha j\,\beta k\,\underline{a}}f^{\beta k\,\alpha j}{}_{\underline{b}}\left(1 - \frac{1}{2} - \frac{1}{2}\frac{f_{RR}^2}{f^2} - \frac{1}{2}\frac{f_{NS}^2}{f^2}\right) \\
&\quad = 0\,,
\end{aligned}
\tag{4.172}
$$

where we used $f^2 = f_{RR}^2 + f_{NS}^2$ and therefore $\mathcal{C}^{(1)}_{\underline{ab}} = 0$. To arrive at eq. (4.172), we also needed

$$
f_{\underline{b}}{}^{\underline{c}\,[\underline{ef}]}f_{[\underline{ef}]\,\underline{c}\,\underline{a}} = -f_{\alpha j\,\beta k\,\underline{a}}f^{\beta k\,\alpha j}{}_{\underline{b}}\,, \qquad H_{\underline{ade}}H^{\underline{de}}{}_{\underline{b}} = -\frac{f_{NS}^2}{f^2}f_{\alpha j\,\beta k\,\underline{a}}f^{\beta k\,\alpha j}{}_{\underline{b}}\,,
\tag{4.173}
$$

and

$$
\int d^2y \, |y-z|^{-2} \overset{\text{div.}}{=} -\log|0|^2\,.
\tag{4.174}
$$

The second contribution to the divergent terms involving the bosonic currents is

$$
\begin{aligned}
&\int d^2z \, \log|0|^2\left(J^{\underline{c}}\overline{J}^{[\underline{ba}]} - \overline{J}^{\underline{c}}J^{[\underline{ba}]}\right)(z)\mathcal{C}^{(2)}_{\underline{abc}} \\
&\quad = -\int d^2z \, \log|0|^2 iH_{\underline{abc}}\left(J^{\underline{c}}\overline{J}^{[\underline{ba}]} - \overline{J}^{\underline{c}}J^{[\underline{ba}]}\right)(z)\left(1 - \frac{1}{2} - \frac{1}{2}\right) \\
&\quad + \int d^2z \int d^2y \left(|y-z|^{-2} - \delta^{(2)}(y-z)\log|y-z|^2\right)\left(J^{\underline{c}}(y)\overline{J}^{[\underline{ba}]}(z)\right. \\
&\quad \left. - \overline{J}^{\underline{c}}(y)J^{[\underline{ba}]}(z)\right)iH_{\underline{abc}}\left(-\frac{1}{2} + \frac{1}{2}\right) \\
&\quad = 0\,,
\end{aligned}
\tag{4.175}
$$

where the first numerical factor inside the round brackets comes from integrating over the bosonic fluctuations, and the remaining factors from integrating over the fermionic ones, and we also used that

$$f_{[\underline{ab}]}{}^{\widehat{\beta}k\,\alpha j} H_{\alpha j\,\widehat{\beta}k\,\underline{c}} = -H_{\underline{abc}}\,, \tag{4.176}$$

consequently, $\mathcal{C}^{(2)}_{\underline{abc}} = 0$.

Finally, the third contribution is given by

$$\begin{aligned}
&\int d^2z \, \log|0|^2 \eta^{[\underline{ab}][\underline{de}]} \left(J^{\underline{c}} \widehat{N}_{\underline{de}} - \overline{J}^{\underline{c}} N_{\underline{de}} \right)(z) \mathcal{C}^{(3)}_{\underline{abc}} \\
&\quad = \int d^2y \int d^2z \left(|y-z|^{-2} - \delta^{(2)}(y-z) \log|y-z|^2 \right) \times \\
&\quad \times \eta^{[\underline{ab}][\underline{de}]} \left(J^{\underline{c}}(y) \widehat{N}_{\underline{de}}(z) - \overline{J}^{\underline{c}}(y) N_{\underline{de}}(z) \right) i H_{\underline{abc}} \left(-\frac{1}{2} + \frac{1}{2} \right) \\
&\quad = 0\,,
\end{aligned} \tag{4.177}$$

where the first $\frac{1}{2}$ inside the round brackets comes from integrating over the bosonic fluctuations and the second from integrating over the fermionic ones, and so we have $\mathcal{C}^{(3)}_{\underline{abc}} = 0$. Therefore,

$$\left\langle e^{-S_{\text{kin}} - S_{\text{bos}}} \right\rangle \Big|_{\text{1PI/one-loop}} \overset{\text{div.}}{=} 0\,, \tag{4.178}$$

as we wanted to prove.

Taking together the absence of divergences proportional to the classical currents (4.161) and the results (4.167) and (4.178), we have shown that the worldsheet action (4.120) is conformally invariant at the one-loop level for any value of f_{NS} and f_{RR} or, equivalently, k and f. Since this fact is known to correspond as on-shell background supergravity fields, we have further confirmed that the NS-NS deformation (4.137), alongside with the choice (4.138), is a consistent solution for the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with mixed NS-NS and R-R three-form flux.

4.9 Relation of the super-coset description with the hybrid formalism

In ref. [44], it was shown how to relate a worldsheet action in the pure R-R flux case from the $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ supergroup to the Berkovits-Vafa-Witten

$\mathrm{AdS}_3 \times \mathrm{S}^3$ sigma-model, which is written in terms of the $\mathrm{PSU}(1,1|2)$ variables [46]. In this section, we will generalize this result and show that the mixed NS-NS and R-R flux description (4.120) can be gauge-fixed to the hybrid formalism [46] in a similar fashion. This will provide additional validation of the results presented in this paper.

Firstly, note that the term $-\lambda^\alpha D_\alpha$ in the supercurrent (3.46b) is responsible for relaxing the contraint D_α, this is the primary reason for the introduction of the bosonic ghosts $\{w_\alpha, \lambda^\alpha\}$ [5].[13] In order to make contact with the worldsheet action in the mixed-flux $\mathrm{AdS}_3 \times \mathrm{S}^3$ hybrid formalism [46], we will proceed as in ref. [44] and consider imposing the constraint $D_\alpha = 0$ "by hand", which means that we can effectively drop the ghosts from our expressions.

Therefore, let us ignore the $\{w_\alpha, \lambda^\alpha\}$-ghosts and rewrite the worldsheet action (4.120) with a first-order kinetic term for the fermions

$$S = \frac{1}{f^2}\int d^2z\Big(\frac{1}{2}J^{\underline{b}}\overline{J}^{\underline{a}}\eta_{\underline{ab}} - \frac{1}{4}\epsilon_{jk}\widehat{\delta}_{\alpha\widehat{\beta}}(2-\tfrac{f_{RR}}{f})\left(J^{\widehat{\beta}k}\overline{J}^{\alpha j} - \overline{J}^{\widehat{\beta}k}J^{\alpha j}\right) + d_{\alpha j}\overline{J}^{\alpha j} + \widehat{d}_{\widehat{\alpha}j}J^{\widehat{\alpha}j} - d_{\alpha j}\widehat{d}_{\widehat{\beta}k}\widehat{\delta}^{\alpha\widehat{\beta}}\epsilon^{jk}\Big) - \frac{i}{f^2}\int_{\mathcal{B}}\mathcal{H}_{\mathrm{NS}} + S_{\rho,\sigma} + S_C\,, \tag{4.179}$$

and which is now subject to the constraints [44]

$$D_\alpha = d_{\alpha 2} - e^{-\rho - i\sigma}d_{\alpha 1} = 0\,, \qquad D_{\widehat{\alpha}} = \widehat{d}_{\widehat{\alpha}1} + e^{-\overline{\rho}-i\overline{\sigma}}\widehat{d}_{\widehat{\alpha}2} = 0\,. \tag{4.180}$$

To recover (4.120) one just needs to plug the auxiliary equations of motion for $d_{\alpha j}$ and $\widehat{d}_{\widehat{\alpha}j}$ in (4.179).

The sigma-model action (4.179) is written in terms of the left-invariant currents $g^{-1}dg$ with g defined in eq. (4.114). In order to gauge-fix to the hybrid string, we define the new fermionic coordinates

$$\theta^{\alpha j} = \frac{1}{\sqrt{2}}\{\theta^{\alpha 1} - \widehat{\theta}^{\widehat{\alpha}1}, -\theta^{\alpha 2} + \widehat{\theta}^{\widehat{\alpha}2}\}\,, \quad \theta'^{\alpha j} = \frac{1}{\sqrt{2}}\{\theta^{\alpha 1} + \widehat{\theta}^{\widehat{\alpha}1}, -\theta^{\alpha 2} - \widehat{\theta}^{\widehat{\alpha}2}\}\,, \tag{4.181}$$

so that the group element g can be parametrized as

$$g = GHG'H'\,, \tag{4.182}$$

where $= e^{\theta^{\alpha j}\mathcal{T}_{\alpha j}}$, $H = e^{x^{\underline{a}}\mathcal{T}_{\underline{a}}}$, $G' = e^{\theta'^{\alpha j}\mathcal{T}'_{\alpha j}}$ and $H' = e^{x^{\underline{a}}\mathcal{T}'_{\underline{a}}}$.

The generators $\{\mathcal{T}_{\widetilde{A}}, \mathcal{T}'_{\widetilde{A}}\}$, $\widetilde{A} = \{\alpha j, \underline{a}\}$, generate two decoupled $\mathrm{PSU}(1,1|2)$

[13] Of course, we are ignoring the additional topological variables in our discussion.

Lie superalgebras and they can be constructed in terms of the T_A's in (4.112) according to

$$\mathcal{T}_{\underline{a}} = \frac{1}{\sqrt{2}}\left(T_{\underline{a}} - \frac{i}{2}(\sigma_{\underline{a}}{}^{\underline{bc}})_{\gamma\widehat{\delta}}\widehat{\delta}^{\gamma\widehat{\delta}}T_{[\underline{bc}]}\right), \qquad \mathcal{T}'_{\underline{a}} = \frac{1}{\sqrt{2}}\left(T_{\underline{a}} + \frac{i}{2}(\sigma_{\underline{a}}{}^{\underline{bc}})_{\gamma\widehat{\delta}}\widehat{\delta}^{\gamma\widehat{\delta}}T_{[\underline{bc}]}\right), \tag{4.183a}$$

$$\mathcal{T}_{\alpha 1} = \frac{1}{\sqrt{2}}(T_{\alpha 1} - T_{\widehat{\alpha}1}), \qquad \mathcal{T}'_{\alpha 1} = \frac{1}{\sqrt{2}}(T_{\alpha 1} + T_{\widehat{\alpha}1}), \tag{4.183b}$$

$$\mathcal{T}_{\alpha 2} = -\frac{1}{\sqrt{2}}(T_{\alpha 2} - T_{\widehat{\alpha}2}), \qquad \mathcal{T}'_{\alpha 2} = -\frac{1}{\sqrt{2}}(T_{\alpha 2} + T_{\widehat{\alpha}2}), \tag{4.183c}$$

consequently, the commutation relations take the form[14]

$$\{\mathcal{T}_{\alpha j}, \mathcal{T}_{\beta k}\} = \sqrt{2}i\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}\mathcal{T}_{\underline{a}}, \qquad \{\mathcal{T}'_{\alpha j}, \mathcal{T}'_{\beta k}\} = \sqrt{2}i\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}\mathcal{T}'_{\underline{a}}, \tag{4.184a}$$

$$[\mathcal{T}_{\underline{a}}, \mathcal{T}_{\alpha j}] = \sqrt{2}i\sigma_{\underline{a}\alpha\gamma}\widehat{\delta}^{\gamma\beta}\mathcal{T}_{\beta j}, \qquad [\mathcal{T}'_{\underline{a}}, \mathcal{T}'_{\alpha j}] = -\sqrt{2}i\sigma_{\underline{a}\alpha\gamma}\widehat{\delta}^{\gamma\beta}\mathcal{T}'_{\beta j}, \tag{4.184b}$$

$$[\mathcal{T}_{\underline{a}}, \mathcal{T}_{\underline{b}}] = \sqrt{2}(\sigma_{\underline{ab}}{}^{\underline{c}})_{\alpha\beta}\widehat{\delta}^{\alpha\beta}\mathcal{T}_{\underline{c}}, \qquad [\mathcal{T}'_{\underline{a}}, \mathcal{T}'_{\underline{b}}] = -\sqrt{2}(\sigma_{\underline{ab}}{}^{\underline{c}})_{\alpha\beta}\widehat{\delta}^{\alpha\beta}\mathcal{T}'_{\underline{c}}. \tag{4.184c}$$

Furthermore, the supertrace reads

$$\mathrm{sTr}(\mathcal{T}_{\underline{a}}\mathcal{T}_{\underline{b}}) = \eta_{\underline{ab}}, \qquad \mathrm{sTr}(\mathcal{T}'_{\underline{a}}\mathcal{T}'_{\underline{b}}) = \eta_{\underline{ab}}, \tag{4.185a}$$

$$\mathrm{sTr}(\mathcal{T}_{\alpha j}\mathcal{T}_{\beta k}) = \eta_{\alpha j\,\beta k} = \epsilon_{jk}\widehat{\delta}_{\alpha\beta}, \qquad \mathrm{sTr}(\mathcal{T}'_{\alpha j}\mathcal{T}'_{\beta k}) = \eta_{\alpha j\,\beta k} = -\epsilon_{jk}\widehat{\delta}_{\alpha\beta}. \tag{4.185b}$$

Thus, the left-invariant one forms can be written in the following form

$$g^{-1}dg = H^{-1}dH + H^{-1}G^{-1}dGH + H'^{-1}dH' + H'^{-1}G'^{-1}dG'H', \tag{4.186}$$

which implies that one can write the currents $J^{\underline{A}}$ as

$$J^{\alpha 1} = \frac{1}{\sqrt{2}}(S^{\alpha 1} + S'^{\alpha 1}), \qquad J^{\alpha 2} = -\frac{1}{\sqrt{2}}(S^{\alpha 2} - S'^{\alpha 2}), \tag{4.187a}$$

$$J^{\widehat{\alpha}1} = \frac{1}{\sqrt{2}}(-S^{\alpha 1} + S'^{\alpha 1}), \qquad J^{\widehat{\alpha}2} = \frac{1}{\sqrt{2}}(S^{\alpha 2} - S'^{\alpha 2}), \tag{4.187b}$$

$$J^{\underline{a}} = \frac{1}{\sqrt{2}}(K^{\underline{a}} + K'^{\underline{a}}), \qquad J^{[\underline{ab}]} = -\frac{i}{2\sqrt{2}}(\sigma^{\underline{abc}})_{\alpha\widehat{\beta}}\widehat{\delta}^{\alpha\widehat{\beta}}(K_{\underline{c}} - K'_{\underline{c}}), \tag{4.187c}$$

[14] After redefining $\mathcal{T}'_{\alpha j} \to i\mathcal{T}'_{\alpha j}$ and $\mathcal{T}'_{\underline{a}} \to -\mathcal{T}'_{\underline{a}}$, both PSU(1,1|2) algebras in (4.184) will take the same form.

where we defined the left-invariant currents

$$S^{\alpha j} = (H^{-1}G^{-1}dGH)^{\alpha j}\,, \qquad K^{\underline{a}} = (H^{-1}dH)^{\underline{a}} + (H^{-1}G^{-1}dGH)^{\underline{a}}\,, \tag{4.188a}$$
$$S'^{\alpha j} = (H'^{-1}G'^{-1}dG'H')^{\alpha j}\,, \quad K'^{\underline{a}} = (H'^{-1}dH')^{\underline{a}} + (H'^{-1}G'^{-1}dG'H')^{\underline{a}}\,. \tag{4.188b}$$

Using the $\mathrm{SO}(1,2) \times \mathrm{SO}(3)$ gauge-symmetry $\delta g = g\omega'^{\underline{a}}\mathcal{T}'_{\underline{a}}$ of the worldsheet action (4.179) for some $\omega'^{\underline{a}}(x)$, we can gauge $H' = 1$. And using the eight fermionic constraints (4.180), we can gauge $\theta'^{\alpha j}$ to zero, so that $G' = 1$.

Consequently, in this gauge the "primed" currents vanish and, from eqs. (4.187), the sigma-model action (4.179) takes the form

$$\begin{aligned} S = \frac{1}{f^2}\int d^2z\,\frac{1}{2}\Big[&\frac{1}{2}K^{\underline{b}}\overline{K}^{\underline{a}}\eta_{\underline{ab}} + d_{\alpha 1}\Big(\overline{S}^{\alpha 1} + e^{-\rho-i\sigma}\overline{S}^{\alpha 2}\Big) \\ &+ \widehat{d}_{\widehat{\alpha}2}\Big(S^{\alpha 2} - e^{-\overline{\rho}-i\overline{\sigma}}S^{\alpha 1}\Big) + \widehat{\delta}^{\alpha\widehat{\beta}}d_{\alpha 1}\widehat{d}_{\widehat{\beta}2}\Big(1 + e^{-\rho-i\sigma}e^{\overline{-\rho-i\sigma}}\Big)\Big] \\ &+ S_{\rho,\sigma} + S_C - \frac{i}{12\sqrt{2}}k\int_{\mathcal{B}}\Big(K^{\underline{c}}K^{\underline{b}}K^{\underline{a}}(\sigma_{\underline{abc}})_{\alpha\widehat{\beta}}\widehat{\delta}^{\alpha\widehat{\beta}} + K^{\underline{a}}S^{\beta k}S^{\alpha j}3i\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}\Big)\,, \end{aligned} \tag{4.189}$$

where we used the constraints (4.180) to solve for $d_{\alpha 1}$ and $\widehat{d}_{\widehat{\alpha}2}$, and also rescaled $d_{\alpha 1} \to \frac{1}{\sqrt{2}}d_{\alpha 1}$ and $\widehat{d}_{\widehat{\alpha}2} \to \frac{1}{\sqrt{2}}\widehat{d}_{\widehat{\alpha}2}$ to arrive at eq. (4.189). Note that the terms proportional to $B_{\alpha j\,\widehat{\beta}k}$ vanish in the gauge $H' = G' = 1$.

We now integrate out $d_{\alpha 1}$ and $\widehat{d}_{\widehat{\alpha}2}$ to obtain

$$\begin{aligned} S = \frac{1}{f^2}\int d^2z\,\Big[&\frac{1}{2}K^{\underline{b}}\overline{K}^{\underline{a}}\eta_{\underline{ab}} + \frac{1}{2}\epsilon_{jk}\widehat{\delta}_{\alpha\beta}S^{\beta k}\overline{S}^{\alpha j} + \Big(1 + \frac{1}{4}\frac{f_{RR}^2}{f^2}e^{\phi}e^{\overline{\phi}}\Big)^{-1}\widehat{\delta}_{\alpha\beta}\Big(\frac{1}{2}\frac{f_{RR}}{f}e^{\phi}S^{\alpha 2}\overline{S}^{\beta 2} \\ &- \frac{1}{2}\frac{f_{RR}}{f}e^{\overline{\phi}}S^{\alpha 1}\overline{S}^{\beta 1} + S^{\alpha 1}\overline{S}^{\beta 2} - \overline{S}^{\alpha 1}S^{\beta 2}\Big)\Big] + ik\int_{\mathcal{B}}\frac{1}{2\sqrt{2}}\Big(K^{\underline{c}}K^{\underline{b}}K^{\underline{a}}(\sigma_{\underline{abc}})_{\alpha\widehat{\beta}}\widehat{\delta}^{\alpha\widehat{\beta}} \\ &+ K^{\underline{a}}S^{\beta k}S^{\alpha j}3i\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}\Big) + S_{\rho,\sigma} + S_C\,, \end{aligned} \tag{4.190}$$

with $e^{\phi} = e^{-\rho-i\sigma}$ and $e^{\overline{\phi}} = e^{-\overline{\rho}-i\overline{\sigma}}$. We also rescaled $f^{-2} \to 2f^{-2}$, $k \to -6k$ and $\{e^{\phi}, e^{\overline{\phi}}\} \to \frac{1}{2}\frac{f_{RR}}{f}\{e^{\phi}, e^{\overline{\phi}}\}$ to arrive at (4.190).

Given that $g = GH$ and[15]

$$g^{-1}dg = J_R^{\widetilde{A}}\mathcal{T}_{\widetilde{A}}\,, \tag{4.191}$$

[15]The subscript R in the currents $J_R^{\widetilde{A}}$ indicates that these are the Noether currents from the right $\mathrm{PSU}(1,1|2)$ transformations, see ref. [6] for further details.

for $J_R^{\widetilde{A}} = \{S^{\alpha j}, K^{\underline{a}}\}$, we can write the worldsheet action in the following form

$$S = \frac{1}{f^2} S_0 + ik S_{\mathrm{WZ}} + \frac{1}{f^2} S_1 + S_{\rho,\sigma} + S_C \,, \tag{4.192}$$

where

$$S_0 = \frac{1}{2} \int d^2 z \ \mathrm{sTr} \left(g^{-1} \partial g g^{-1} \overline{\partial} g \right), \tag{4.193a}$$

$$S_{\mathrm{WZ}} = -\frac{1}{2} \int_{\mathcal{B}} \mathrm{sTr} \left(g^{-1} dg g^{-1} dg g^{-1} dg \right), \tag{4.193b}$$

$$S_1 = \int d^2 z \left(1 + \frac{1}{4} \frac{f_{RR}^2}{f^2} e^{\phi} e^{\overline{\phi}} \right)^{-1} \widehat{\delta}_{\alpha\beta} \left(\frac{1}{2} \frac{f_{RR}}{f} e^{\phi} S^{\alpha 2} \overline{S}^{\beta 2} - \frac{1}{2} \frac{f_{RR}}{f} e^{\overline{\phi}} S^{\alpha 1} \overline{S}^{\beta 1} \right.$$
$$\left. + S^{\alpha 1} \overline{S}^{\beta 2} - \overline{S}^{\alpha 1} S^{\beta 2} \right). \tag{4.193c}$$

Eq. (4.192) is precisely the worldsheet action for the hybrid superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$ with mixed NS-NS and R-R three-form flux, as we wanted to show. Similarly as eq. (4.120), this action was also proved to be conformal invariant at one-loop for any k and f [46].

Chapter 5

Conclusion

We emphasize the main findings contained in each chapter and also comment on potential directions for future research.

5.1 Summary

Throughout this thesis, we have studied compactifications of the superstring down to four- and six-dimensional target-spaces while preserving manifest spacetime supersymmetry. The discussion and development of the theory was conducted from the worldsheet perspective, which focuses on the two-dimensional CFT nature of the sigma-model action. Particularly, our findings include results for the superstring propagating in a four-dimensional spacetime with a background $\mathrm{U}(1)$ gauge field, and for the superstring compactified to a flat and to an $\mathrm{AdS}_3 \times \mathrm{S}^3$ six-dimensional background.

In Chapter 2, we computed consistent Lagrangians and equations of motion for massive spin-3/2 and spin-2 fields in an electromagnetic background using superstring field theory. First, we showed how to couple the hybrid formalism for the open superstring to a constant electromagnetic background, and derived expressions for the worldsheet variables in terms of the oscillator modes. We then computed the open superstring field theory action for the compactification-independent massless sector in a constant $\mathrm{U}(1)$ background. Perfect agreement was found with previous calculations from bosonic string theory. After that, we constructed the open superstring field theory action in $d = 4$ $\mathcal{N} = 1$ superspace for the first massive compactification-independent states in a constant $\mathrm{U}(1)$ background.

Following that, the superstring field theory action in superspace for the massive states was expanded in components. The pure gauge degrees of freedom were eliminated and, consequently, what was left is a Lagrangian containing only the physical fields. It was shown that the Lagrangian describes 12 complex bosonic and 12 complex fermionic degrees of freedom on-shell, including a massive spin-3/2 and a massive spin-2 field. Even though the action has couplings of the higher-spin excitations with the lower-spin ones, at the level of the equations of motion the spin-3/2 and spin-2 fields decouple.

In Chapter 3, we described how to extend the six-dimensional hybrid formalism in a flat background such that all SUSYs of $d = 6$ $\mathcal{N} = 1$ superspace can be made manifest, including vertex operators and a tree-level amplitude prescription. First, we reviewed the six-dimensional hybrid formalism in a flat background. Then, we explained how four more θ coordinates can be added as worldsheet variables, followed by the inclusion of the harmonic constraint D_α.

After relaxing the harmonic first-class constraint D_α — by defining a new BRST operator G^+ — vertex operators and a tree-level scattering amplitude prescription were constructed while preserving manifest spacetime supersymmetry. Specifically, it was shown that BRST invariance of the vertex operator implies the $d = 6$ SYM equations of motion in $\mathcal{N} = 1$ superspace. Furthermore, we confirmed that the three-point amplitude of SYM states is reproduced.

In Chapter 4, we studied the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3$. A supersymmetric three-point amplitude of half-BPS vertex operators inserted on the AdS_3 boundary was computed. After that, we constructed a sigma-model action for the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with mixed flux and all SUSYs manifest, and proved that the model is quantum consistent at the one-loop level.

We started the chapter introducing the six-dimensional hybrid formalism in $\mathrm{AdS}_3 \times \mathrm{S}^3$ and, after explaining the technical details involved, wrote our main result — a $\mathrm{PSU}(1,1|2)$-covariant three-point amplitude for half-BPS states inserted on the AdS_3 boundary — whose coordinates were labelled by $\mathbf{x}$. As a corollary, we found that the kinematic factor gets dressed with the vielbein field ${E_A}^B(\mathbf{x})$ after the worldsheet fermions are integrated out in the path integral. In addition, we saw the compelling fact of the conformal group on the boundary being identified with the symmetry group of the AdS_3 bulk by explicitly analyzing the form of ${E_A}^B(\mathbf{x})$, which naturally appears in our covariant superstring description. It was also found that the results agree with the RNS formalism answer.

The hybrid formalism for the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ has only half of the eight spacetime supersymmetries manifest. Using the extended hybrid formalism, we constructed a quantizable and $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$-invariant worldsheet action for the superstring in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with mixed NS-NS and R-R three-form flux. We proved that this description is conformal invariant at the one-loop level using the covariant background field method. For that to be the case, it was necessary that the NS-NS flux f_{NS} and the R-R flux f_{RR} were connected to the inverse AdS_3 radius f, similarly as in the GS superstring. Additionally, we have shown how this model can be related to the Berkovits-Vafa-Witten hybrid formalism with mixed flux, which further validated our results.

5.2 Outlook

The results discussed in this thesis contribute to the understanding of superstring compactifications to flat and curved backgrounds, in addition to enhancing the understanding of manifest spacetime supersymmetry in the superstring. Since our findings are grounded in a detailed construction of superstring descriptions in regimes that are very little explored from the worldsheet side, this work can serve as a starting point for future investigation of a wide range of topics, as well as a reference for original computational methods.

The construction of the open superstring field theory in an electromagnetic background presented in Chapter 2 is completely general and should be valid for any massive state of the superstring. Superstring worldsheet calculations often encounter obstacles coming from states in the Ramond sector. Since the hybrid formalism of the superstring preserves manifest $d = 4$ $\mathcal{N} = 1$ SUSY, an interesting direction of study would be to determine the equations of motion and constraints satisfied by the fermions of arbitrary mass and spin propagating in an electromagnetic background. In addition, the framework developed in Chapter 2 can be applied to beyond the quadratic order of the string field theory action, allowing one to compute higher-order corrections to the open superstring field theory in a $U(1)$ background.

For compactifications down to a six-dimensional background, the formulation developed in Chapter 3 with manifest $d = 6$ $\mathcal{N} = 1$ SUSY might be a fruitful avenue for a further exploration of manifest spacetime supersymmetry in the superstring. An important open problem is to understand the precise relation of the spacetime supersymmetric formalism with the $d = 6$ pure spinor description of the superstring [49] [50] [51].[1] Progress in this direction might also have applications to the origin of the $d = 10$ pure spinor formalism [54]. In addition, $d = 6$ $\mathcal{N} = 1$ supersymmetry can be formulated in harmonic superspace. Since the relation between harmonic superspace and ordinary superspace is well understood, the

[1] A six-dimensional pure spinor $\lambda^{\alpha j}$ is defined to satisfy the constraint $\epsilon_{jk}\lambda^{\alpha}\sigma^{a}_{\alpha\beta}\lambda^{\beta k} = 0$.

formalism developed in Chapter 3 can offer important hints towards a description of the superstring using harmonic superspace coordinates as fundamental worldsheet fields. In particular, the $\{\rho, \sigma\}$-ghosts of the hybrid formalism should play the role of the harmonic variables in the superstring description.

Finally, the $\mathrm{AdS}_3 \times \mathrm{S}^3$ framework discussed in Chapter 4 can have applications ranging from superstring amplitudes in AdS to insights in worldsheet descriptions in the presence of R-R flux. The construction of the vertex operators inserted on the AdS_3 boundary provides all the necessary elements for the exploration of higher-point tree-level scattering amplitudes in $\mathrm{AdS}_3 \times \mathrm{S}^3$ with manifest spacetime supersymmetry. We gave a zero-mode prescription for the fermionic worldsheet fields with vertex operators depending on the AdS_3 boundary coordinates $\mathbf{x}$. The remaining ingredient is the integrated vertex operator for the half-BPS states. The holomorphic part of the integrated vertex reads

$$\int G_0^+ G_{-1}^- \mathcal{V} = \int \frac{1}{2k} \left[\frac{1}{\sqrt{2}} \left(\frac{i}{2} K^{\alpha\beta} \nabla_{\alpha 1} \nabla_{\beta 1} + i S_{\alpha 1} \nabla^{\alpha\beta} \nabla_{\beta 1} \right) - \widehat{\delta}^{\alpha\beta} S_{\alpha 1} \nabla_{\beta 1} \right] V \,, \quad (5.1)$$

where we only wrote the terms that contribute to the tree-level amplitudes of half-BPS states.

The sigma-model in $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with mixed flux from the super-coset $\frac{\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,2) \times \mathrm{SO}(3)}$ is the analogue of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor worldsheet action for the lower dimensional Anti-de Sitter spacetime, as it contains bosonic ghosts λ^α and $\widehat{\lambda}^{\widehat{\alpha}}$. It can also be viewed as the "supersymmetrization" of the Berkovits-Vafa-Witten description of AdS_3 from the hybrid formalism [46], where only eight of the sixteen spacetime supersymmetries were manifest in the action. Therefore, it is natural to think that it is feasible to derive new understandings for the worldsheet description of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor superstring, and consequently the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, by studying the lower dimensional counterpart in an equivalent framework.

Particularly, it would be interesting to determine in what manner the vielbein field $E_A{}^B(\mathbf{x})$ in AdS_3 emerges from the super-coset variables. Additionally, one could also look at how the $\mathrm{AdS}_3 \times \mathrm{S}^3$ twistors [72] [74] fit into the super-coset formulation, an advancement which could have important applications for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ pure spinor formalism and its tensionless limit [69] [70] [92]. A twistor formulation of string theory in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background has been proposed [102], which shares the same features as the $\mathrm{AdS}_3 \times \mathrm{S}^3$ twistors.

As a final remark, the findings presented in this thesis may serve as a valuable tool for an enhanced understanding of spacetime supersymmetry in the superstring. They might also provide a groundwork for future first-principles worldsheet calculations in AdS backgrounds, where spacetime supersymmetry plays a pivotal role.

Appendix A

Results in spinor and oscillator algebra in $d = 4$

Some relations and definitions about the supersymmetric modes that were used in the calculations of Chapter 2 are given below. For a more complete list of identities and conventions for the sigma matrices and spinorial indices, see the appendices of [24].

$$(\chi^{\alpha}\psi_{\alpha}) = (\chi\psi)\,, \qquad (\overline{\chi}_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}}) = (\overline{\chi}\overline{\psi})\,, \qquad (\chi\psi)^{\dagger} = (\overline{\chi}\overline{\psi})\,. \tag{A.1}$$

$$\theta_0^{\alpha}\theta_{\beta 0} = \frac{1}{2}\delta^{\alpha}_{\beta}(\theta_0\theta_0)\,, \qquad \overline{\theta}_{\dot{\alpha}0}\overline{\theta}_0^{\dot{\beta}} = \frac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}(\overline{\theta}_0\overline{\theta}_0)\,. \tag{A.2}$$

$$[p_{\alpha 0}, (\theta_0\theta_0)] = -2\theta_{\alpha 0}\,, \qquad [\overline{p}_{\dot{\alpha}0}\,, (\overline{\theta}_0\overline{\theta}_0)] = -2\overline{\theta}_{\dot{\alpha}0}\,, \tag{A.3a}$$

$$[(p_0 p_0), (\theta_0\theta_0)] = -4 - 4(\theta_0 p_0)\,, \qquad [(\overline{p}_0\overline{p}_0), (\overline{\theta}_0\overline{\theta}_0)] = -4 - 4(\overline{\theta}_0\overline{p}_0)\,. \tag{A.3b}$$

$$\alpha_{0n}\alpha_{0p}(\overline{\sigma}^n\sigma^p)^{\dot{\alpha}}{}_{\dot{\beta}} = -\alpha_{0n}\alpha_0^n\delta^{\dot{\alpha}}_{\dot{\beta}} - i\varepsilon_{np}(\overline{\sigma}^{np})^{\dot{\alpha}}{}_{\dot{\beta}}\,, \tag{A.4a}$$

$$\alpha_{0n}\alpha_{0p}(\sigma^n\overline{\sigma}^p)_{\alpha}{}^{\beta} = -\alpha_{0n}\alpha_0^n\delta^{\beta}_{\alpha} - i\varepsilon_{np}(\sigma^{np})_{\alpha}{}^{\beta}\,. \tag{A.4b}$$

$$[d_n^{\alpha}, \Pi_{\alpha\dot{\alpha}m}] = 8i\partial\overline{\theta}_{\dot{\alpha}m+n}\,, \qquad [\overline{d}_n^{\dot{\alpha}}, \Pi_{\alpha\dot{\alpha}m}] = -8i\partial\theta_{\alpha m+n}\,. \tag{A.5}$$

$$
\begin{aligned}
[d_{\alpha m}, \overline{d}_{\dot\alpha n}\overline{d}^{\dot\alpha}_n] &= 4i\Pi_{\alpha\dot\alpha m+n}\overline{d}^{\dot\alpha}_n + 16\partial\theta_{\alpha m+2n} \\
&= 4i\overline{d}^{\dot\alpha}_n\Pi_{\alpha\dot\alpha m+n} - 16\partial\theta_{\alpha m+2n}\,, \qquad \text{(A.6a)}\\
[\overline{d}_{\dot\alpha m}, d^{\alpha}_n d_{\alpha n}] &= -4i\Pi_{\alpha\dot\alpha m+n}d^{\alpha}_n + 16\partial\overline{\theta}_{\dot\alpha m+2n} \\
&= -4id^{\alpha}_n\Pi_{\alpha\dot\alpha m+n} - 16\partial\overline{\theta}_{\dot\alpha m+2n}\,. \qquad \text{(A.6b)}
\end{aligned}
$$

$$
d^{\alpha}_0\Pi_{\alpha\dot\alpha 0}\overline{d}^{\dot\alpha}_0 = -\overline{d}^{\dot\alpha}_0\Pi_{\alpha\dot\alpha 0}d^{\alpha}_0 - 4i\Pi^n_0\Pi_{n0} + 8i\partial\theta^{\alpha}_0 d_{\alpha 0} + 8i\partial\overline{\theta}_{\dot\alpha 0}\overline{d}^{\dot\alpha}_0\,. \qquad \text{(A.7)}
$$

$$
\begin{aligned}
[d^{\alpha}_0 d_{\alpha 0}, \overline{d}_{\dot\alpha 0}\overline{d}^{\dot\alpha}_0] &= -8i\overline{d}^{\dot\alpha}_0\Pi_{\alpha\dot\alpha 0}d^{\alpha}_0 + 16\Pi^n_0\Pi_{n0} - 32\partial\overline{\theta}_{\dot\alpha 0}\overline{d}^{\dot\alpha}_0 - 32\partial\theta^{\alpha}_0 d_{\alpha 0} \\
&= 8id^{\alpha}_0\Pi_{\alpha\dot\alpha 0}\overline{d}^{\dot\alpha}_0 - 16\Pi^n_0\Pi_{n0} + 32\partial\overline{\theta}_{\dot\alpha 0}\overline{d}^{\dot\alpha}_0 + 32\partial\theta^{\alpha}_0 d_{\alpha 0} \\
&= 4i\Pi_{\alpha\dot\alpha 0}[d^{\alpha}_0, \overline{d}^{\dot\alpha}_0] + 32d^{\alpha}_0\partial\theta_{\alpha 0} - 32\partial\overline{\theta}_{\dot\alpha 0}\overline{d}^{\dot\alpha}_0\,. \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
d_{\alpha 0}f(\theta_0,\overline{\theta}_0) &= \big[-p_{\alpha 0} - (\sigma^m\overline{\theta}_0)_\alpha\alpha_{0m} - \frac{i}{2}(\overline{\theta}_0\overline{\theta}_0)(\varepsilon\cdot\sigma)_\alpha^{\ \beta}\theta_{\beta 0}\big]f(\theta_0,\overline{\theta}_0)\,, \qquad \text{(A.9a)}\\
\overline{d}_{\dot\alpha 0}f(\theta_0,\overline{\theta}_0) &= \big[-\overline{p}_{\dot\alpha 0} + (\theta_0\sigma^m)_{\dot\alpha}\alpha_{0m} + \frac{i}{2}(\theta_0\theta_0)(\varepsilon\cdot\overline{\sigma})^{\dot\beta}_{\ \dot\alpha}\overline{\theta}_{\dot\beta 0}\big]f(\theta_0,\overline{\theta}_0), \qquad \text{(A.9b)}\\
\Pi^m_0 f(\theta_0,\overline{\theta}_0) &= \big[-i\alpha^m_0 + \frac{i}{2}\epsilon^{mrsn}\varepsilon_{rs}(\theta_0\sigma_n\overline{\theta}_0)\big]f(\theta_0,\overline{\theta}_0)\,. \qquad \text{(A.9c)}
\end{aligned}
$$

$$
\begin{aligned}
d^{\alpha}_0 d_{\alpha 0}f(\theta_0,\overline{\theta}_0) &= \big[p^2_0 - 2(\overline{\theta}_0\overline{\sigma}^m)^\alpha\alpha_{0m}p_{\alpha 0} + (\overline{\theta}_0\overline{\theta}_0)\alpha^n_0\alpha_{0n} - i(\overline{\theta}_0\overline{\theta}_0)(\varepsilon\cdot\sigma)_\beta^{\ \alpha}\theta^\beta_0 p_{\alpha 0}\big]f(\theta_0,\overline{\theta}_0)\,, \\
&\qquad \text{(A.10a)}\\
\overline{d}_{\dot\alpha 0}\overline{d}^{\dot\alpha}_0 f(\theta_0,\overline{\theta}_0) &= \big[\overline{p}^2_0 - 2(\overline{\sigma}^m\theta_0)^{\dot\alpha}\alpha_{0m}\overline{p}_{\dot\alpha 0} + (\theta_0\theta_0)\alpha^n_0\alpha_{0n} - i(\theta_0\theta_0)(\varepsilon\cdot\overline{\sigma})^{\dot\beta}_{\ \dot\alpha}\overline{\theta}_{\dot\beta 0}\overline{p}^{\dot\alpha}_0\big]f(\theta_0,\overline{\theta}_0)\,, \\
&\qquad \text{(A.10b)}\\
\Pi^m_0\Pi_{m0}f(\theta_0,\overline{\theta}_0) &= \big[-\alpha^m_0\alpha_{0m} + \alpha_{0m}\epsilon^{mrsn}\varepsilon_{rs}(\theta_0\sigma_n\overline{\theta}_0) - \frac{1}{2}\varepsilon_{rs}\varepsilon^{rs}(\theta_0\theta_0)(\overline{\theta}_0\overline{\theta}_0)\big]f(\theta_0,\overline{\theta}_0)\,. \\
&\qquad \text{(A.10c)}
\end{aligned}
$$

$$
\{d_{\alpha 0}, \partial\theta^\beta_0\} = \frac{i}{2}(\varepsilon\cdot\sigma)_\alpha^{\ \beta}\,, \qquad \{\overline{d}^{\dot\alpha}_0, \partial\overline{\theta}_{\dot\beta 0}\} = \frac{i}{2}(\varepsilon\cdot\overline{\sigma})^{\dot\alpha}_{\ \dot\beta}\,. \qquad \text{(A.11)}
$$

$$[d_0^2, \partial\theta_0^\beta] = i(\varepsilon\cdot\sigma)_\alpha{}^\beta d_0^\alpha\,, \qquad [\overline{d}_0^2, \partial\overline{\theta}_{\dot\beta 0}] = i(\varepsilon\cdot\overline{\sigma})^{\dot\alpha}{}_{\dot\beta}\overline{d}_{\dot\alpha 0}\,. \tag{A.12}$$

$$\Delta_\alpha{}^\beta = \delta_\alpha^\beta + \frac{i}{2}(\varepsilon\cdot\sigma)_\alpha{}^\beta\,, \qquad \overline{\Delta}^{\dot\alpha}{}_{\dot\beta} = \delta_{\dot\beta}^{\dot\alpha} + \frac{i}{2}(\varepsilon\cdot\overline{\sigma})^{\dot\alpha}{}_{\dot\beta}\,. \tag{A.13}$$

$$\begin{aligned}
(-d_0^2\overline{d}_1^2 - d_1^2\overline{d}_0^2 + d_2^2\overline{d}_{-1}^2)\varphi &= (-\overline{d}_1^2 d_0^2 + \overline{d}_2^2 d_{-1}^2 - \overline{d}_0^2 d_1^2)\varphi\,, &\text{(A.14a)}\\
(3d_0^2\overline{d}_1^2 - 3d_1^2\overline{d}_0^2 + d_2^2\overline{d}_{-1}^2)\varphi &= (-3\overline{d}_0^2 d_1^2 + 3\overline{d}_1^2 d_0^2 - \overline{d}_2^2 d_{-1}^2)\varphi\,, &\text{(A.14b)}\\
(d_{-1}^2\overline{d}_1^2 - 2d_0^2\overline{d}_0^2 + d_1^2\overline{d}_{-1}^2)\varphi &= (\overline{d}_{-1}^2 d_1^2 + \overline{d}_1^2 d_{-1}^2 - 2\overline{d}_0^2 d_0^2)\varphi\,, &\text{(A.14c)}\\
\overline{d}_1^2 d_{-1}^2 B &= d_1^2\overline{d}_{-1}^2 B - 2[d_0^2, \overline{d}_0^2]B\,, &\text{(A.14d)}\\
(64id_0^\alpha\Pi_{\alpha\dot\alpha 0}\overline{d}_0^{\dot\alpha} - 19d_0^2\overline{d}_0^2 - 3\overline{d}_0^2 d_0^2)C &= \big[-11\{d_0^2, \overline{d}_0^2\} + 128\Pi_0^n\Pi_{n0} - 256(\partial\overline{\theta}_0\overline{d}_0)\\
&\quad - 256(\partial\theta_0 d_0)\big]C\,. &\text{(A.14e)}
\end{aligned}$$

$$\begin{aligned}
d_0^2\overline{d}_1^2\varphi &= 2(d_0d_0)(\overline{d}_0\overline{d}_1)\varphi\,, &\text{(A.15a)}\\
d_1^2\overline{d}_0^2\varphi &= (8id_0^\alpha\Pi_{\alpha\dot\alpha 0}\overline{d}_1^{\dot\alpha} + 2d_0^\alpha(\overline{d}_0\overline{d}_0)d_{\alpha 1} + 8id_0^\alpha\overline{d}_0^{\dot\alpha}\Pi_{\alpha\dot\alpha 1} - 32d_0^\alpha\partial\theta_{\alpha 1})\varphi\,, &\text{(A.15b)}\\
d_2^2\overline{d}_{-1}^2\varphi &= (-16\Pi_{\alpha\dot\alpha 0}\Pi_1^{\dot\alpha\alpha} - 8i\Pi_{\alpha\dot\alpha 0}\overline{d}_0^{\dot\alpha}d_1^\alpha + 8id_0^\alpha\Pi_{\alpha\dot\alpha 0}\overline{d}_1^{\dot\alpha} + 8id_0^\alpha\overline{d}_0^{\dot\alpha}\Pi_{\alpha\dot\alpha 1}\\
&\quad - 32\partial\overline{\theta}_{\dot\alpha 0}\overline{d}_1^{\dot\alpha} - 32\overline{d}_{\dot\alpha 0}\partial\overline{\theta}_1^{\dot\alpha} - 64d_0^\alpha\partial\theta_{\alpha 1})\varphi\,, &\text{(A.15c)}\\
d_{-1}^2\overline{d}_1^2\varphi &= 4(d_{-1}d_0)(\overline{d}_0\overline{d}_1)\varphi\,, &\text{(A.15d)}\\
d_1^2\overline{d}_{-1}^2\varphi &= (8id_0^\alpha\Pi_{\alpha\dot\alpha 0}\overline{d}_0^{\dot\alpha} + 8i\Pi_{\alpha\dot\alpha -1}d_0^\alpha\overline{d}_1^{\dot\alpha} - 16\Pi_{\alpha\dot\alpha -1}\Pi_1^{\dot\alpha\alpha} - 8i\Pi_{\alpha\dot\alpha -1}\overline{d}_0^{\dot\alpha}d_1^\alpha\\
&\quad - 8i\overline{d}_{-1}^{\dot\alpha}d_0^\alpha\Pi_{\alpha\dot\alpha 1} - 4\overline{d}_{\dot\alpha -1}d_0^\alpha\overline{d}_0^{\dot\alpha}d_{\alpha 1} + 8id_{-1}^\alpha\Pi_{\alpha\dot\alpha 0}\overline{d}_1^{\dot\alpha} + 8id_{-1}^\alpha\overline{d}_0^{\dot\alpha}\Pi_{\alpha\dot\alpha 1}\\
&\quad - 64d_{-1}^\alpha\partial\theta_{\alpha 1} - 64\partial\overline{\theta}_{\dot\alpha -1}\overline{d}_1^{\dot\alpha})\varphi\,, &\text{(A.15e)}\\
d_0^2\overline{d}_0^2\varphi &= (d_0^\alpha d_{\alpha 0}\overline{d}_{\dot\alpha 0}\overline{d}_0^{\dot\alpha} + 2\overline{d}_{\dot\alpha -1}d_0^\alpha d_{\alpha 0}\overline{d}_1^{\dot\alpha} + 2d_{-1}^\alpha\overline{d}_{\dot\alpha 0}\overline{d}_0^{\dot\alpha}d_{\alpha 1} + 8id_{-1}^\alpha\Pi_{\alpha\dot\alpha 0}\overline{d}_1^{\dot\alpha}\\
&\quad + 8id_{-1}^\alpha\overline{d}_0^{\dot\alpha}\Pi_{\alpha\dot\alpha 1} + 8i\Pi_{\alpha\dot\alpha -1}d_0^\alpha\overline{d}_1^{\dot\alpha} - 32\partial\overline{\theta}_{\dot\alpha -1}\overline{d}_1^{\dot\alpha} - 32d_{-1}^\alpha\partial\theta_{\alpha 1})\varphi\,. &\text{(A.15f)}
\end{aligned}$$

$$d_0^2\overline{d}_0^2 B = (d_0d_0)(\overline{d}_0\overline{d}_0)B\,, \tag{A.16a}$$

$$d_1^2\overline{d}_{-1}^2 B = 8i d_0^\alpha \Pi_{\alpha\dot{\alpha}0}\overline{d}_0^{\dot{\alpha}} B\,, \tag{A.16b}$$

$$d_{-1}^2\overline{d}_0^2 B = 2d_{-1}^\alpha d_{\alpha 0}\overline{d}_{\dot{\alpha}0}\overline{d}_0^{\dot{\alpha}} B\,, \tag{A.16c}$$

$$d_0^2\overline{d}_{-1}^2 B = (2\overline{d}_{\dot{\alpha}-1}d_0^\alpha d_{\alpha 0}\overline{d}_0^{\dot{\alpha}} + 8i d_{-1}^\alpha \Pi_{\alpha\dot{\alpha}0}\overline{d}_0^{\dot{\alpha}} + 8i\Pi_{\alpha\dot{\alpha}-1}d_0^\alpha\overline{d}_0^{\dot{\alpha}} - 32\partial\overline{\theta}_{\dot{\alpha}-1}\overline{d}_0^{\dot{\alpha}})B\,, \tag{A.16d}$$

$$\overline{d}_{-1}^2 d_0^2 B = 2\overline{d}_{\dot{\alpha}-1}\overline{d}_0^{\dot{\alpha}} d_0^\alpha d_{\alpha 0} B\,, \tag{A.16e}$$

$$\overline{d}_0^2 d_{-1}^2 B = (2d_{-1}^\alpha \overline{d}_{\dot{\alpha}0}\overline{d}_0^{\dot{\alpha}} d_{\alpha 0} + 8i\overline{d}_{\dot{\alpha}-1}\Pi_0^{\dot{\alpha}\alpha}d_{\alpha 0} + 8i\Pi_{\alpha\dot{\alpha}-1}\overline{d}_0^{\dot{\alpha}}d_0^\alpha - 32\partial\theta_{-1}^\alpha d_{\alpha 0})B\,. \tag{A.16f}$$

Appendix B

Six-dimensional Pauli matrices

B.1 Definitions

The Lorentz group $\mathrm{SO}(1,5)$ is locally isomorphic to SU(4) and, under this identification, spinors of $\mathrm{SO}(1,5)$ transform as $\mathbf{4}$'s or $\mathbf{4}'$'s of SU(4). By definition, Weyl spinors transform as a $\mathbf{4}$ and are denoted by an upper lower case greek index ranging from 1 to 4. Anti-Weyl spinors transform as a $\mathbf{4}'$ and are denoted by a down lower case greek index ranging from 1 to 4. All other representations of $\mathrm{SO}(1,5)$ can be built from tensor products of $\mathbf{4}$'s and $\mathbf{4}'$'s. The following tensor products are of particular importance

$$\mathbf{4}\otimes\mathbf{4}\simeq\mathbf{6}\oplus\mathbf{10}_-\,, \tag{B.1a}$$

$$\mathbf{4}'\otimes\mathbf{4}'\simeq\mathbf{6}\oplus\mathbf{10}_+\,, \tag{B.1b}$$

$$\mathbf{4}\otimes\mathbf{4}'\simeq\mathbf{1}\oplus\mathbf{15}\,, \tag{B.1c}$$

where $\mathbf{1}$ denotes the singlet representation, $\mathbf{6}$ is antisymmetric in the spinor indices and denotes the vector representation, $\mathbf{10}_-$ and $\mathbf{10}_+$ are symmetric and correspond to anti-self-dual and self-dual three-forms, respectively,[1] and the traceless representation $\mathbf{15}$ is a two-form.

The SO(1,5) Pauli matrices are defined as

$$\begin{aligned}
\sigma^0_{\alpha\beta}&=\frac{1}{\sqrt{2}}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}, & \sigma^1_{\alpha\beta}&=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & \sigma^1\\ -\sigma^1 & 0\end{pmatrix},\\
\sigma^2_{\alpha\beta}&=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -\sigma^2\\ -\sigma^2 & 0\end{pmatrix}, & \sigma^3_{\alpha\beta}&=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & \sigma^3\\ -\sigma^3 & 0\end{pmatrix},\\
\sigma^4_{\alpha\beta}&=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\mathbb{1}\\ i\mathbb{1} & 0\end{pmatrix}, & \sigma^5_{\alpha\beta}&=\frac{1}{\sqrt{2}}\begin{pmatrix}\sigma^2 & 0\\ 0 & -\sigma^2\end{pmatrix},
\end{aligned} \tag{B.2}$$

[1] Note that $(\sigma_{012})^{\alpha\beta}=-(\sigma^{345})^{\alpha\beta}$ and $(\sigma_{012})_{\alpha\beta}=(\sigma^{345})_{\alpha\beta}$ (see eqs. (B.9)).

where the $\boldsymbol{\sigma}$-matrices are the usual SU(2) Pauli matrices

$$\boldsymbol{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \boldsymbol{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \boldsymbol{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{B.3}$$

The σ-matrices are antisymmetric and satisfy the algebra

$$\sigma^{\underline{a}\alpha\beta}\sigma^{\underline{b}}_{\alpha\gamma} + \sigma^{\underline{b}\alpha\beta}\sigma^{\underline{a}}_{\alpha\gamma} = \eta^{\underline{ab}}\delta^{\beta}_{\gamma} , \tag{B.4}$$

where $\eta^{\underline{ab}} = \mathrm{diag}(-,+,+,+,+,+)$, $\underline{a} = \{0 \text{ to } 5\}$, is the six-dimensional Minkowski metric and we define

$$\sigma^{\underline{a}\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\sigma^{\underline{a}}_{\gamma\delta} , \tag{B.5}$$

which are given by

$$\begin{aligned} \sigma^{0\alpha\beta} &= \frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{\sigma}^2 & 0 \\ 0 & \boldsymbol{\sigma}^2 \end{pmatrix} , \qquad & \sigma^{1\alpha\beta} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \boldsymbol{\sigma}^1 \\ -\boldsymbol{\sigma}^1 & 0 \end{pmatrix} , \\ \sigma^{2\alpha\beta} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^2 & 0 \end{pmatrix} , \qquad & \sigma^{3\alpha\beta} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \boldsymbol{\sigma}^3 \\ -\boldsymbol{\sigma}^3 & 0 \end{pmatrix} , \\ \sigma^{4\alpha\beta} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix} , \qquad & \sigma^{5\alpha\beta} &= \frac{1}{\sqrt{2}}\begin{pmatrix} -\boldsymbol{\sigma}^2 & 0 \\ 0 & \boldsymbol{\sigma}^2 \end{pmatrix} . \end{aligned} \tag{B.6}$$

It is convenient to introduce the unitary matrix B, also known as an intertwiner,

$$B_{\alpha}{}^{\beta} = -(B^*)_{\alpha}{}^{\beta} = \begin{pmatrix} \boldsymbol{\sigma}^2 & 0 \\ 0 & \boldsymbol{\sigma}^2 \end{pmatrix} , \qquad (B^*)_{\alpha}{}^{\beta}B_{\beta}{}^{\gamma} = -\delta_{\alpha}^{\gamma} , \tag{B.7}$$

so that

$$(\sigma^{\underline{a}}_{\alpha\beta})^* = (B)_{\alpha}{}^{\gamma}(B)_{\beta}{}^{\delta}\sigma^{\underline{a}}_{\gamma\delta} . \tag{B.8}$$

We also define

$$(\sigma^{\underline{ab}})_{\alpha}{}^{\beta} = \frac{i}{2}(\sigma^{[\underline{a}}\sigma^{\underline{b}]})_{\alpha}{}^{\beta} , \tag{B.9a}$$

$$(\sigma^{\underline{abc}})^{\alpha\beta} = \frac{i}{3!}(\sigma^{[\underline{a}}\sigma^{\underline{b}}\sigma^{\underline{c}]})^{\alpha\beta} , \tag{B.9b}$$

where we anti-symmetrize/symmetrize without dividing by the number of terms.

B.2 Identities

The Lorentz generators satisfy the commutators

$$[\sigma_{\underline{a}}, \sigma_{\underline{bc}}] = -i\eta_{\underline{a}[\underline{b}}\sigma_{\underline{c}]}\,, \tag{B.10a}$$

$$\begin{aligned}[\sigma_{\underline{ab}}, \sigma_{\underline{cd}}] &= \frac{i}{2}\Big(\eta_{\underline{c}[\underline{a}}\delta^{[\underline{e}}_{\underline{b}]}\delta^{\underline{f}]}_{\underline{d}} + \eta_{\underline{d}[\underline{b}}\delta^{[\underline{e}}_{\underline{a}]}\delta^{\underline{f}]}_{\underline{c}}\Big)\sigma_{\underline{ef}} \\ &= i\eta_{\underline{c}[\underline{a}}\sigma_{\underline{b}]\underline{d}} - i\eta_{\underline{d}[\underline{a}}\sigma_{\underline{b}]\underline{d}}\,.\end{aligned} \tag{B.10b}$$

Some useful identities are

$$\sigma^{\underline{a}}_{\alpha\beta}\sigma^{\underline{b}}_{\gamma\delta}\eta_{\underline{ab}} = \epsilon_{\alpha\beta\gamma\delta}\,, \tag{B.11a}$$

$$\sigma^{\underline{a}\alpha\beta}\sigma^{\underline{b}}_{\alpha\gamma}\eta_{\underline{ab}} = 3\delta^{\beta}_{\gamma}\,, \tag{B.11b}$$

$$\sigma^{\underline{a}\alpha\beta}\sigma^{\underline{b}}_{\alpha\beta} = 2\eta^{\underline{ab}}\,, \tag{B.11c}$$

$$\sigma^{\underline{a}\alpha\beta}\sigma^{\underline{b}}_{\gamma\delta}\eta_{\underline{ab}} = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\beta}_{\gamma}\delta^{\alpha}_{\delta}\,, \tag{B.11d}$$

$$\epsilon^{\alpha\beta\rho\sigma}\epsilon_{\gamma\delta\rho\sigma} = 2(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\beta}_{\gamma}\delta^{\alpha}_{\delta})\,, \tag{B.11e}$$

$$\epsilon^{\alpha\beta\gamma\delta}\sigma_{\underline{a}\delta\sigma} = -\frac{1}{2}\delta^{[\alpha}_{\sigma}\sigma^{\beta\gamma]}_{\underline{a}}\,, \tag{B.11f}$$

$$(\sigma^{\underline{ab}})^{\alpha}{}_{\beta}(\sigma^{\underline{cd}})^{\beta}{}_{\alpha} = \eta^{\underline{a}[\underline{c}}\eta^{\underline{d}]\underline{b}}\,, \tag{B.11g}$$

$$\eta_{\underline{ac}}\eta_{\underline{bd}}(\sigma^{\underline{ab}})_{\beta}{}^{\alpha}(\sigma^{\underline{cd}})_{\gamma}{}^{\delta} = -\frac{1}{2}\delta^{\alpha}_{\beta}\delta^{\delta}_{\gamma} + 2\delta^{\alpha}_{\gamma}\delta^{\delta}_{\beta}\,, \tag{B.11h}$$

$$(\sigma_{\underline{a}}\sigma_{\underline{b}}\sigma_{\underline{c}}\sigma_{\underline{d}})_{\alpha}{}^{\alpha} = \eta_{\underline{ab}}\eta_{\underline{cd}} + \eta_{\underline{ad}}\eta_{\underline{bc}} - \eta_{\underline{ac}}\eta_{\underline{bd}}\,, \tag{B.11i}$$

$$(\sigma_{\underline{abc}})_{\gamma\delta}\sigma^{\underline{c}}_{\alpha\beta} = -\frac{i}{2}\sigma_{[\underline{a}|\alpha(\gamma|}\sigma_{|\underline{b}]|\delta)\beta}\,, \tag{B.11j}$$

$$(\sigma^{\underline{ab}})^{\gamma}{}_{\delta}(\sigma_{\underline{abc}})_{\alpha\beta} = -\sigma_{\underline{c}\delta(\alpha}\delta^{\gamma}_{\beta)}\,, \tag{B.11k}$$

where $\epsilon_{1234} = 1$.

Note further that the antisymmetric tensors $\epsilon_{\alpha\beta\gamma\delta}$ and ϵ_{jk} satisfy the Schouten identities

$$\delta^{\sigma}_{[\alpha}\epsilon_{\beta\gamma\delta\rho]} = 0\,, \tag{B.12a}$$

$$\epsilon_{j[k}\epsilon_{lm]} = 0\,, \tag{B.12b}$$

and, in addition, we have

$$\epsilon^{jk}\epsilon_{lm} = -(\delta^{j}_{l}\delta^{k}_{m} - \delta^{k}_{l}\delta^{j}_{m})\,. \tag{B.13}$$

Some additional trace identities are

$$
\begin{aligned}
&(\sigma^{\underline{a}}\sigma^{\underline{b}}\sigma^{\underline{c}}\sigma^{\underline{d}}\sigma^{\underline{e}}\sigma^{\underline{f}})^{\alpha}{}_{\alpha} \\
&\quad = -\frac{1}{2}\epsilon^{\underline{abcdef}} + \frac{1}{2}\eta^{\underline{ad}}\eta^{\underline{e}[\underline{b}}\eta^{\underline{c}]\underline{f}} - \frac{1}{2}\eta^{\underline{bd}}\eta^{\underline{e}[\underline{a}}\eta^{\underline{c}]\underline{f}} + \frac{1}{2}\eta^{\underline{cd}}\eta^{\underline{e}[\underline{a}}\eta^{\underline{b}]\underline{f}} \\
&\quad + \frac{1}{2}\eta^{\underline{bc}}\big(-\eta^{\underline{ad}}\eta^{\underline{ef}} + \eta^{\underline{ae}}\eta^{\underline{df}} - \eta^{\underline{af}}\eta^{\underline{de}}\big) + \frac{1}{2}\eta^{\underline{ac}}\big(\eta^{\underline{bd}}\eta^{\underline{ef}} \\
&\quad - \eta^{\underline{be}}\eta^{\underline{df}} + \eta^{\underline{bf}}\eta^{\underline{de}}\big) + \frac{1}{2}\eta^{\underline{ab}}\big(-\eta^{\underline{cd}}\eta^{\underline{ef}} + \eta^{\underline{ce}}\eta^{\underline{df}} - \eta^{\underline{cf}}\eta^{\underline{de}}\big)\,,
\end{aligned}
\tag{B.14}
$$

and

$$
(\sigma_{\underline{abc}})^{\alpha\beta}(\sigma_{\underline{def}})_{\beta\alpha} = \frac{1}{2}\epsilon_{\underline{abcdef}} - \frac{1}{2}\eta_{[\underline{a}|\underline{d}}\eta_{|\underline{b}|\underline{e}}\eta_{|\underline{c}]\underline{f}}\,,
\tag{B.15}
$$

where $\epsilon_{012345} = -\epsilon^{012345} = 1$.

Some supplementary identities are

$$
(\sigma_{\underline{a}}{}^{\underline{bc}})_{\alpha\hat{\beta}}\hat{\delta}^{\alpha\hat{\beta}}(\sigma_{\underline{d}}{}^{\underline{ef}})_{\gamma\hat{\delta}}\hat{\delta}^{\gamma\hat{\delta}}\eta_{[\underline{bc}][\underline{ef}]} = -4\eta_{\underline{ad}}\,,
\tag{B.16a}
$$

$$
-f_{\alpha j\,\beta k}{}^{\underline{a}}(\sigma_{\underline{a}}{}^{\underline{bc}})_{\gamma\hat{\delta}}\hat{\delta}^{\gamma\hat{\delta}} = 2if_{\alpha j\,\hat{\beta}k}{}^{[\underline{bc}]}\,,
\tag{B.16b}
$$

$$
(\sigma_{\underline{a}}{}^{\underline{cd}})_{\alpha\hat{\beta}}\hat{\delta}^{\alpha\hat{\beta}}(\sigma_{\underline{b}}{}^{\underline{ef}})_{\gamma\hat{\delta}}\hat{\delta}^{\gamma\hat{\delta}}f_{[\underline{cd}]\,[\underline{ef}]}{}^{[\underline{gh}]} = -4f_{\underline{a}\,\underline{b}}{}^{[\underline{gh}]}\,,
\tag{B.16c}
$$

$$
(\sigma_{\underline{ab}}{}^{\underline{c}})_{\alpha\hat{\beta}}\hat{\delta}^{\alpha\hat{\beta}}(\sigma_{\underline{c}}{}^{\underline{de}})_{\gamma\hat{\delta}}\hat{\delta}^{\gamma\hat{\delta}} = -2f_{\underline{a}\,\underline{b}}{}^{[\underline{de}]}\,,
$$

$$
(\sigma_{\underline{bc}})^{\alpha}{}_{\beta}\hat{\delta}^{\beta\hat{\gamma}}\sigma_{\underline{a}\hat{\gamma}\alpha} = (\sigma_{\underline{abc}})_{\alpha\hat{\beta}}\hat{\delta}^{\alpha\hat{\beta}}\,,
\tag{B.16d}
$$

where the symbols in eqs. (B.16) are defined in Section 4.7.2.

Appendix C

Some useful conventions

C.1 Worldsheet

Except for Chapter 2, where our conventions are detailed in the main text, the remaining chapters use the conventions displayed below for the worldsheet theory.

The Euclidean worldsheet coordinates are labeled by $\sigma^I = \{\sigma^0, \sigma^1\}$ with metric given by $g_{IJ} = \mathrm{diag}(1,1)$ and we define the components of the antisymmetric tensor ϵ^{IJ} according to $\epsilon_{01} = \epsilon^{10} = 1$. As usual, the measure is written as $d^2\sigma = d\sigma^0 d\sigma^1$.

In terms of the of the Euclidean coordinates, we can form the complex variables on the cylinder $\{z, \bar{z}\}$, which are defined by

$$z = \sigma^0 - i\sigma^1\,, \qquad \bar{z} = \sigma^0 + i\sigma^1\,, \tag{C.1}$$

such that

$$\partial = \frac{1}{2}(\partial_0 + i\partial_1)\,, \qquad \bar{\partial} = \frac{1}{2}(\partial_0 - i\partial_1)\,. \tag{C.2}$$

The metric components are

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}\,, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0\,, \quad g^{z\bar{z}} = g^{\bar{z}z} = 2\,, \quad g^{zz} = g^{\bar{z}\bar{z}} = 0\,, \tag{C.3}$$

and the antisymmetric tensor components take the following form

$$\epsilon^{z\bar{z}} = -2i\,, \qquad \epsilon_{z\bar{z}} = -\frac{i}{2}\,. \tag{C.4}$$

In this case, we write $d^2z = 2d^2\sigma = -idzd\bar{z} = 2d\sigma^0 d\sigma^1$.

It is also useful to think of the worldsheet as a plane. The map from the cylinder to the plane is given by

$$z = e^{\sigma^0 - i\sigma^1}\,, \qquad\qquad \bar{z} = e^{\sigma^0 + i\sigma^1}\,, \tag{C.5}$$

and, without loss of generality, we will call the plane coordinates by z and $\bar{z}$ as well. The reason for this is that the form of conformal invariant expressions written in terms of the complex cylinder coordinates is equivalent as the ones written in terms of the plane coordinates. In the plane coordinates, lines of constant σ^0 are mapped to circles around the origin, the infinite past becomes $z = 0$ and the infinite future becomes $z = \infty$. Current conservation reads $\partial j_{\bar{z}} + \bar{\partial} j_z = 0$ and the associated Noether charge Q takes the nice form

$$Q = \oint dz j_z + \oint d\bar{z} j_{\bar{z}}\,. \tag{C.6}$$

When evaluating contour integrals, we use the convention

$$\oint dz \frac{1}{z} = \oint d\bar{z} \frac{1}{\bar{z}} = 1\,, \tag{C.7}$$

so that annoying factors of 2π are absent in most expressions as, e.g., in the worldsheet action and in the identities

$$\bar{\partial}(y - z)^{-1} = -\delta^{(2)}(y - z)\,, \qquad \partial(\bar{y} - \bar{z})^{-1} = -\delta^{(2)}(y - z)\,, \tag{C.8a}$$

$$\bar{\partial}(y - z)^{-2} = \partial_y \delta^{(2)}(y - z)\,, \qquad \partial(\bar{y} - \bar{z})^{-2} = \bar{\partial}_y \delta^{(2)}(y - z)\,. \tag{C.8b}$$

When working with differential forms we use the same conventions as [24]. In particular, the two-dimensional integral over the one-forms Δ and Σ is given by

$$\begin{aligned} \int \Delta\Sigma &= \int d^2\sigma\, \epsilon^{IJ} \Delta_J \Sigma_I \\ &= i \int d^2 z \left(\Delta\bar{\Sigma} - \bar{\Delta}\Sigma \right)\,, \end{aligned} \tag{C.9}$$

and the exterior derivative acts as

$$\begin{aligned} d\Sigma &= d\sigma^I d\sigma^J \partial_J \Sigma_I \\ &= -d\sigma^0 d\sigma^1 \epsilon^{IJ} \partial_J \Sigma_I\,. \end{aligned} \tag{C.10}$$

C.2 Supergeometry

Following ref. [24], we define the super-vielbein as $J^A = dZ^M E_M{}^A$, where $Z^M = \{x^m, \theta^{\mu j}, \widehat{\theta}^{\widehat{\mu} j}\}$ are the curved supercoordinates. The quantities $A = \{\underline{a}, \alpha j, \widehat{\alpha} j\}$ and $M = \{m, \mu j, \widehat{\mu} j\}$ label the tangent and the curved superspace indices, respectively. The connection one form is defined as $\Omega_B{}^A = dZ^M \Omega_{MB}{}^A$.

The action of the covariant derivative one-form ∇ on a q-form Y^A is

$$\nabla Y^A = dY^A + Y^B \Omega_B{}^A\,, \qquad \nabla^2 Y^A = Y^B R_B{}^A\,, \tag{C.11}$$

and we define the torsion two-form T^A and the connection two-form $R_B{}^A$ as

$$T^A = \nabla J^A\,, \tag{C.12a}$$

$$R_B{}^A = d\Omega_B{}^A + \Omega_B{}^C \Omega_C{}^A\,, \tag{C.12b}$$

where $T^A = \frac{1}{2} J^C J^B T_{BC}{}^A$ and $R_B{}^A = \frac{1}{2} J^D J^C R_{CDB}{}^A$.

For the Type IIB superstring in the $AdS_3 \times S^3$ background considered in Section 4.7, the torsions and curvatures can be nicely written in terms of the structure constants (4.113) as

$$T_{AB}{}^C = -i f_{AB}{}^C\,, \quad R_{AB}{}^{[\underline{ab}]} = -i f_{AB}{}^{[\underline{ab}]}\,, \quad R_{CDB}{}^A = f_{CD}{}^{[\underline{ab}]} f_{[\underline{ab}]B}{}^A\,, \tag{C.13}$$

where we are using that $\Omega_B{}^A = i f_{[\underline{ab}]\,B}{}^A J^{[\underline{ab}]}$ and $R^{[\underline{ab}]} = dJ^{[\underline{ab}]} + \frac{i}{2} J^{[\underline{ef}]} J^{[\underline{cd}]} f_{[\underline{cd}]\,[\underline{ef}]}{}^{[\underline{ab}]}$ to relate $R_{AB}{}^{[\underline{ab}]}$ with $R_{CDB}{}^A$. If desired, one can properly normalize eqs. (C.13) by rescalings of the super-vielbeins and of the connections.

Furthermore, from the three-form $H = dB$ one obtains the flat-index equation

$$H_{ABC} = \frac{1}{2} \nabla_{[A} B_{BC]} + \frac{1}{2} T_{[AB|}{}^D B_{D|C]}\,, \tag{C.14}$$

which follows from (C.11) and the definitions $H = \frac{1}{6} J^C J^B J^A H_{ABC}$ and $B = \frac{1}{2} J^B J^A B_{AB}$. Note that B_{AB} is graded anti-symmetric and H_{ABC} is graded anti-symmetric in the 1-2 and 2-3 indices.

Appendix D

Normal-ordering prescription

The normal-ordered product of the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ is denoted by $(\mathcal{O}_1\mathcal{O}_2)$, which is defined as

$$(\mathcal{O}_1\mathcal{O}_2)(z) = \oint \frac{dx}{x-z}\mathcal{O}_1(x)\mathcal{O}_2(z)\,. \tag{D.1}$$

This prescription consists in subtracting the poles evaluated at the point of the second entry. By convention, when nothing is specified, our expressions are normal-ordered from the right, e.g., $\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3 ... \mathcal{O}_n = (\mathcal{O}_1(\mathcal{O}_2(\mathcal{O}_3(...\mathcal{O}_n)...)))$. Also, whenever we are dealing with derivatives of exponentials, such as $\partial^2 e^{\rho}$, the ordering is always done with the exponential on the right, so that $\partial^2 e^{\rho} = (\partial\rho((\partial\rho)e^{\rho})) + ((\partial^2\rho)e^{\rho})$. Putting the exponential on the rightmost position agrees with the usual conformal-normal-ordering [23] when dealing with free fields.

Schematically, note that in terms of the definition in eq. (D.1), we have [65]

$$(\mathcal{O}_1(\mathcal{O}_2\mathcal{O}_3))(z) = \oint \frac{dx}{x-z}\,\mathcal{O}_1(x)(\mathcal{O}_2\mathcal{O}_3)(z) = \oint \frac{dx}{x-z}\oint \frac{dy}{y-z}\,\mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\,. \tag{D.2}$$

Appendix E

Some comments on the RNS superstring

We comment on the physical state conditions of the RNS superstring in bosonized form, i.e., working with the $\{\phi, \eta, \xi\}$-CFT.

E.1 Large Hilber space and picture-changing

Let us recall that the matter part of the RNS superstring action in conformal gauge corresponding to an uncompactified ten-dimensional manifold is given by [103]

$$S_{\rm m} = \int d^2z\, \frac{1}{2}\Big(\partial x^M \bar{\partial} x_M + \psi^M \bar{\partial}\psi_M + \widehat{\psi}^M \partial \widehat{\psi}_M\Big)\,, \tag{E.1}$$

where $M = \{0, \ldots, 9\}$ and $\eta^{MN} = \mathrm{diag}(-1, 1, \ldots, 1)$ if we are in Lorentz signature. Note that the anti-holomorphic (left-moving) fields are denoted with a "hat" and, for simplicity, we will only discuss the holomorphic — or open string — part of the theory below. The fields x^M have conformal weight zero and ψ^M conformal weight $\frac{1}{2}$. From the RNS worldsheet fields, one can form $c = 15$ $\mathcal{N} = 1$ superconformal generators

$$T_{\rm m} = -\frac{1}{2}\partial x^M \partial x_M - \frac{1}{2}\psi^M \partial \psi_M\,, \tag{E.2a}$$

$$G_{\rm m} = i\psi^M \partial x_M\,. \tag{E.2b}$$

The ghost part of the RNS action, which comes from gauge-fixing the $\mathcal{N} = 1$ worldsheet superconformal invariance, reads

$$S_{\rm gh} = \int d^2z\, \Big(b\bar{\partial}c + \beta\bar{\partial}\gamma + \widehat{b}\partial\widehat{c} + \widehat{\beta}\partial\widehat{\gamma}\Big)\,, \tag{E.3}$$

where the fermionic ghosts $\{b, c\}$ have conformal weights 2 and -1, and the bosonic ghosts $\{\beta, \gamma\}$ have conformal weights $\frac{3}{2}$ and $-\frac{1}{2}$, respectively. The $c = -15$

$\mathcal{N}=1$ superconformal generators corresponding to the ghost part are given by

$$T_{\mathrm{gh}} = -2b\partial c - \partial b c - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\partial\beta\gamma\,, \tag{E.4a}$$

$$G_{\mathrm{gh}} = b\gamma - 2\partial\beta c - 3\beta\partial c\,. \tag{E.4b}$$

In total, the gauge-fixed RNS worldsheet action is

$$S_{\mathrm{RNS}} = S_{\mathrm{m}} + S_{\mathrm{gh}}\,. \tag{E.5}$$

In addition, it follows that the fundamental fields satisfy the following OPEs

$$\partial x^M(y)\partial x^N(z) \sim -\eta^{MN}(y-z)^{-2}\,, \qquad b(y)c(z) \sim (y-z)^{-1}\,, \tag{E.6a}$$

$$\psi^M(y)\psi^N(z) \sim \eta^{MN}(y-z)^{-1}\,, \qquad \beta(y)\gamma(z) \sim -(y-z)^{-1}\,. \tag{E.6b}$$

Associated to the gauge-fixed action (E.5) there is a BRST charge

$$\begin{aligned} Q_{\mathrm{RNS}} &= \oint j_{\mathrm{BRST}} \\ &= \oint \left(cT_{x,\psi,\beta,\gamma} + bc\partial c - \frac{1}{2}\gamma G_{\mathrm{m}} - \frac{1}{4}b\gamma^2 + \frac{3}{2}\partial^2 c \right)\,, \end{aligned} \tag{E.7}$$

where j_{BRST} is called the BRST current and $T_{x,\psi,\beta,\gamma}$ is the combined stress-tensor of all the fields mentioned. Note that $T_{x,\psi,\beta,\gamma}$ has central charge 26 and the total derivative term $\frac{3}{2}\partial^2 c$ is added to make the BRST current a primary.

Let us now bosonize $\beta = e^{-\phi}\partial\xi$ and $\gamma = \eta e^{\phi}$, so that the BRST charge takes the form

$$Q_{\mathrm{RNS}} = \oint \left(cT_{x,\psi,\phi,\eta,\xi} + bc\partial c - \frac{1}{2}\eta e^{\phi} G_{\mathrm{m}} - \frac{1}{4}b\eta\partial\eta e^{2\phi} + \partial^2 c - \partial(\eta\xi c) \right)\,, \tag{E.8}$$

where we wrote eq. (E.7) in terms of the bosonized $\{\beta,\gamma\}$-ghosts and added the total derivative $-\frac{1}{2}\partial^2 c - \partial(\eta\xi c)$ to it compared to eq. (E.4). Note that this total derivative added to (E.7) lives in the large Hilbert space — it includes the ξ zero-mode — and that the BRST current in (E.8) still transforms as a tensor. This total derivative was added so that the BRST current (E.8) corresponds to the supercurrent G^+ of eqs. (2.3). The latter observation has no particular consequence on the discussion below.

From the rules of the BRST procedure [20] [23] [104], we define the physical states of the superstring to be GSO projected and ghost-number minus picture-number one vertex operators U which are independent of the ξ zero-mode — this is the small Hlber space condition — and which belong to the cohomology of Q_{RNS}. The condition of being GSO-projected can be realized by considering states which have no square-root cuts in the OPE with the spacetime supersymmetry generator $Q_\alpha^{-\frac{1}{2}}$.

By defining the ghost-number current as

$$j_{\text{ghost}} = -bc - \beta\gamma = -bc - \partial\phi\,, \tag{E.9}$$

and the picture-number current as

$$j_{\text{picture}} = -\eta\xi - \partial\phi = i\partial\chi - \partial\phi\,, \tag{E.10}$$

we have that the ghost-number and the picture-number operators are given by

$$N_{\text{ghost}} = \oint j_{\text{ghost}}\,, \qquad\qquad N_{\text{picture}} = \oint j_{\text{picture}}\,, \tag{E.11}$$

where we used $\eta = e^{-i\chi}$ and $\xi = e^{i\chi}$.

Consequently, the second condition on the physical states reads

$$N_{\text{ghost}} - N_{\text{picute}} = 1\,, \tag{E.12}$$

where the charges $\{N_{\text{ghost}}, N_{\text{picture}}\}$ were defined in eqs. (E.11). Moreover, the last two conditions can be implemented by demanding

$$\eta_0 U = Q_{\text{RNS}} U = 0\,, \tag{E.13}$$

where η_0 is the zero-mode of η. Therefore, U is subject to an equivalence relation, or gauge transformation,

$$\delta U = Q_{\text{RNS}}\Lambda\,, \tag{E.14}$$

for any Λ in the small Hilbert space.

It is interesting to elaborate on why the small Hilbert space constraint is important when working with the $\{\phi, \eta, \xi\}$-CFT. If the requirement $\eta_0 U = \eta_0 \Lambda = 0$ is relaxed, one could then take

$$\Lambda = -4c\xi\partial\xi e^{-2\phi} U \quad \Rightarrow \quad U = Q_{\text{RNS}}\Lambda\,, \tag{E.15}$$

for any U in the cohomology of Q_{RNS}. As a consequence, allowing Λ to live in the large Hilbert space implies a trivial cohomology. Note that eq. (E.15) follows from

$$Q_{\text{RNS}}\big(-4c\xi\partial\xi e^{-2\phi}\big) = 1\,, \tag{E.16}$$

and one then says that the operator $c\xi\partial\xi e^{-2\phi}$ trivializes the cohomology.[1]

Now, consider a physical state U in the cohomology of Q_{RNS}, note that the state $V = \xi U$ has the same energy as U, but it is not in the BRST cohomology of the RNS formalism

$$\begin{aligned} Q_{\text{RNS}}V &= Q_{\text{RNS}}(\xi U) \\ &= (Q_{\text{RNS}}\xi)U \neq 0\,, \end{aligned} \tag{E.18}$$

but note further that

$$\begin{aligned} 0 &= Q^2_{\text{RNS}}V \\ &= \frac{1}{2}Q_{\text{RNS}}(ZU)\,, \end{aligned} \tag{E.19}$$

where we defined $Z = 2Q_{\text{RNS}}\xi$.

In addition, if $ZU = Q_{\text{RNS}}\Lambda$ for some Λ in the small Hilbert space, we have that $U = Q_{\text{RNS}}(Y\Lambda)$, where Y is the inverse of Z, i.e., $ZY = 1$. Therefore, the state ZU is in the cohomology of Q_{RNS}. Moreover,

$$\eta_0(ZU) = 0\,, \tag{E.20}$$

so that ZU belongs to the small Hilbert space and ZU has $N_{\text{ghost}} - N_{\text{picture}} = 1$, since Z has $N_{\text{ghost}} - N_{\text{picture}} = 0$. Taking into account that Q_{RNS} commutes with the spacetime SUSY generator, we see that ZU satisfies all the conditions to be a physical state of the superstring.

The operator Z is called the picture-changing operator and Y is known as the

[1]It is possible to allow Λ to live in the large Hilbert space by considering an additional equivalence relation for U, namely,

$$\delta U = \eta_0\Omega\,, \tag{E.17}$$

for some Ω. This is put to practice in the hybrid formalism, e.g., see eq (3.19).

inverse picture-changing operator. Explicitly, they are given by

$$Z = G_{\mathrm{m}}e^{\phi} + b\partial\eta e^{2\phi} + \frac{1}{2}\partial\big(b\eta e^{2\phi}\big) + 2c\partial\xi\,, \tag{E.21a}$$

$$Y = 2c\partial\xi e^{-2\phi}\,, \tag{E.21b}$$

and they satisfy

$$ZY = 1\,, \qquad\qquad Q_{\mathrm{RNS}}Z = Q_{\mathrm{RNS}}Y = 0\,. \tag{E.22}$$

Of course, Z has $N_{\mathrm{picture}} = 1$ and Y has $N_{\mathrm{picture}} = -1$.

The existence of the operators $\{Z, Y\}$ implies that each physical vertex operator is represented by an infinite number of physical states. Indeed, if U is in the BRST cohomology, we have that

$$Q_{\mathrm{RNS}}(Z^m Y^n U) = 0\,, \tag{E.23}$$

and

$$\delta(Z^m Y^n U) = Q_{\mathrm{RNS}}\Lambda \quad \Rightarrow \quad \delta U = Q_{\mathrm{RNS}}(Y^m Z^n \Lambda)\,, \tag{E.24}$$

thus $Z^m Y^n U$ is also in the cohomology for any $m, n \geq 0$.

One should note that the cohomology is only non-trivial when U has conformal weight zero. In agreement with the old covariant quantization approach [103]. To see that, suppose U is in the cohomology of Q_{RNS} and has conformal weight h, then one can write $U = \frac{1}{h}Q_{\mathrm{RNS}}(b_0 U)$. And so we have arrived at a contradiction. Therefore, the non-trivial states in the cohomology of Q_{RNS} must have conformal weight zero.

In addition, we also learn that if U is in the BRST cohomology,

$$\begin{aligned} 0 &= b_0(Q_{\mathrm{RNS}}U) \\ &= (T_{\mathrm{RNS}})_0 U - Q_{\mathrm{RNS}}(b_0 U) \\ &= -Q_{\mathrm{RNS}}(b_0 U)\,, \end{aligned} \tag{E.25}$$

as a consequence, there is a consistent gauge choice for the physical states U by demanding U to have no double poles with the b-ghost, i.e.,

$$b_0 U = 0\,. \tag{E.26}$$

This condition is known as the Siegel gauge [105].

Eq. (E.26) imply that one can ignore states which have double poles with the b-ghost when solving the cohomology. If U is the vertex operator representing the gluon a_M, the condition $b_0 U = 0$ gives the well known covariant Lorenz gauge constraint $\partial^M a_M = 0$.

Appendix F

The supercurrent G^+ in a flat background

From eq. (3.38), we have that

$$G^+_{\text{hyb}} = -\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}[D_\alpha, \{D_\beta, [D_\gamma, \{D_\delta, e^{2\rho+3i\sigma}\}]\}] + G^+_C\,, \tag{F.1}$$

where $D_\alpha = d_{\alpha 2} - e^{-\rho-i\sigma}d_{\alpha 1}$ and the graded bracket $[D_\alpha, \mathcal{O}\}$ denotes the single pole in the OPE between D_α and $\mathcal{O}$. In the following, we evaluate each of the four contributions separately.

First contribution.

$$\begin{aligned}\oint dy\, D_\delta(y) e^{2\rho+3i\sigma}(z) &= \oint dy\, (d_{\delta 2} - d_{\delta 1}e^{-\rho-i\sigma})(y)e^{2\rho+3i\sigma}(z)\\ &= -(d_{\delta 1}e^{\rho+2i\sigma})(z)\,.\end{aligned} \tag{F.2}$$

The term appearing in (F.2) comes from the single pole in the OPE between $(d_{\delta 1}e^{-\rho-i\sigma})(y)$ and $e^{2\rho+3i\sigma}(z)$.

Second contribution.

$$\begin{aligned}-\oint dy\, D_\gamma(y)(d_{\delta 1}e^{\rho+2i\sigma})(z) &= -\oint dy\, (d_{\gamma 2} - d_{\gamma 1}e^{-\rho-i\sigma})(y)(d_{\delta 1}e^{\rho+2i\sigma})(z)\\ &= i(\Pi_{\gamma\delta}e^{\rho+2i\sigma})(z) - (d_{\gamma 1}d_{\delta 1}e^{i\sigma})(z)\,.\end{aligned} \tag{F.3}$$

The first term in (F.3) comes from the single pole in the OPE of $d_{\gamma 2}(y)$ and $(d_{\delta 1}e^{\rho+2i\sigma})(z)$. The second term comes from the single pole in the OPE between $(d_{\gamma 1}e^{-\rho-i\sigma})(y)$ and $(d_{\delta 1}e^{\rho+2i\sigma})(z)$.

Third contribution. Now we need to compute $\oint dy\, D_\beta(y)\Big(i(\Pi_{\gamma\delta}e^{\rho+2i\sigma})(z)$ $-(d_{\gamma 1}d_{\delta 1}e^{i\sigma})(z)\Big)$, which is most easily obtained by calculating the relevant terms independently. We have that

$$
\begin{aligned}
&\oint dy\, d_{\beta 2}(y)(-1)(d_{\gamma 1}d_{\delta 1}e^{i\sigma})(z)\\
&\quad = -\oint dy\, d_{\beta 2}(y)\oint \frac{dx}{x-z}\, d_{\gamma 1}(x)(d_{\delta 1}e^{i\sigma})(z)\\
&\quad = -\oint dy \oint \frac{dx}{x-z}\Big[-i(y-x)^{-1}\Pi_{\beta\gamma}(x)(d_{\delta 1}e^{i\sigma})(z)\\
&\quad - d_{\gamma 1}(x)\Big(-i(y-z)^{-1}(\Pi_{\beta\delta}e^{i\sigma})(z)\Big)\Big]\\
&\quad = i(\Pi_{\beta\gamma}(d_{\delta 1}e^{i\sigma}))(z) - i(d_{\gamma 1}(\Pi_{\beta\delta}e^{i\sigma}))(z)\\
&\quad = i(d_{\delta 1}(\Pi_{\beta\gamma}e^{i\sigma}))(z) - i(d_{\gamma 1}(\Pi_{\beta\delta}e^{i\sigma}))(z) + \epsilon_{\epsilon\beta\gamma\delta}(\partial^2\theta^{\epsilon 2}e^{i\sigma})(z)\,,
\end{aligned} \tag{F.4}
$$

where we used that $([\Pi_{\beta\gamma}, d_{\delta 1}]) = \oint dy\,\Big(\Pi_{\beta\gamma}(y)d_{\delta 1}(z) - d_{\delta 1}(y)\Pi_{\beta\gamma}(z)\Big) = -i\epsilon_{\epsilon\beta\gamma\delta}\partial^2\theta^{\epsilon 2}(z)$ according to eqs. (3.33). We also need

$$
\oint dy\, d_{\beta 2}(y) i(\Pi_{\gamma\delta}e^{\rho+2i\sigma})(z) = \epsilon_{\epsilon\beta\gamma\delta}(\partial\theta^{\epsilon 1}e^{\rho+2i\sigma})(z)\,. \tag{F.5}
$$

And

$$
\oint dy\,(d_{\beta 1}e^{-\rho-i\sigma})(y)(d_{\gamma 1}d_{\delta 1}e^{i\sigma})(z) = (d_{\beta 1}d_{\gamma 1}d_{\delta 1}e^{-\rho})(z)\,. \tag{F.6}
$$

And lastly, we have

$$
\begin{aligned}
&\oint dy\,(-i)(d_{\beta 1}e^{-\rho-i\sigma})(y)(\Pi_{\gamma\delta}e^{\rho+2i\sigma})(z)\\
&\quad = -i\oint dy\,(d_{\beta 1}e^{-\rho-i\sigma})(y)\oint \frac{dx}{x-z}\,\Pi_{\gamma\delta}(x)e^{\rho+2i\sigma}(z)\\
&\quad = -i\oint dy \oint \frac{dx}{x-z}\Big(i(y-x)^{-1}\epsilon_{\epsilon\beta\gamma\delta}(\partial\theta^{\epsilon 2}e^{-\rho-i\sigma})(x)e^{\rho+2i\sigma}(z)\\
&\quad - \Pi_{\gamma\delta}(x)(y-z)^{-1}(d_{\beta 1}e^{i\sigma})(z)\Big)\\
&\quad = \epsilon_{\epsilon\beta\gamma\delta}((\partial\theta^{\epsilon 2}e^{-\rho-i\sigma})e^{\rho+2i\sigma})(z) + i(\Pi_{\gamma\delta}(d_{\beta 1}e^{i\sigma}))(z)\\
&\quad = \epsilon_{\epsilon\beta\gamma\delta}(\partial\theta^{\epsilon 2}(\partial(\rho+i\sigma)e^{i\sigma}))(z) + i(d_{\beta 1}(\Pi_{\gamma\delta}e^{i\sigma}))(z)\,,
\end{aligned} \tag{F.7}
$$

where it was used that $\epsilon_{\epsilon\beta\gamma\delta}((\partial\theta^{\epsilon 2}e^{-\rho-i\sigma})e^{\rho+2i\sigma}) = \epsilon_{\epsilon\beta\gamma\delta}(\partial\theta^{\epsilon 2}(\partial(\rho+i\sigma)e^{i\sigma})) - \epsilon_{\epsilon\beta\gamma\delta}(\partial^2\theta^{\epsilon 2}e^{i\sigma})$ and $i(\Pi_{\gamma\delta}(d_{\beta 1}e^{i\sigma})) = i(d_{\beta 1}(\Pi_{\gamma\delta}e^{i\sigma})) + \epsilon_{\epsilon\beta\gamma\delta}(\partial^2\theta^{\epsilon 2}e^{i\sigma})$ to go from the third to the last line in the computation of (F.7).

Gathering eqs. (F.4)–(F.7), we have

$$
\begin{aligned}
&\oint dy\, D_\beta(y)\Big(i(\Pi_{\gamma\delta}e^{\rho+2i\sigma})(z) - (d_{\gamma 1}d_{\delta 1}e^{i\sigma})(z)\Big)\\
&\quad = (d_{\beta 1}d_{\gamma 1}d_{\delta 1}e^{-\rho})(z) + \epsilon_{\epsilon\beta\gamma\delta}(\partial\theta^{\epsilon 1}e^{\rho+2i\sigma})(z) + \epsilon_{\epsilon\beta\gamma\delta}(\partial^2\theta^{\epsilon 2}e^{i\sigma})(z)\\
&\quad + \epsilon_{\epsilon\beta\gamma\delta}(\partial\theta^{\epsilon 2}(\partial(\rho+i\sigma)e^{i\sigma}))(z) + i(d_{\beta 1}(\Pi_{\gamma\delta}e^{i\sigma}))(z) + i(d_{\delta 1}(\Pi_{\beta\gamma}e^{i\sigma}))(z)\\
&\quad - i(d_{\gamma 1}(\Pi_{\beta\delta}e^{i\sigma}))(z)\,.
\end{aligned}
\tag{F.8}
$$

Fourth contribution. According to eq. (F.1), to obtain G^+_{hyb}, we still need to act with $-\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\oint dy\, D_\alpha(y)$ in eq. (F.8). We get that

$$
\begin{aligned}
&-\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\oint dy\, D_\alpha(y)(d_{\beta 1}d_{\gamma 1}d_{\delta 1}e^{-\rho})(z)\\
&\quad = -\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\oint dy\,\Big(d_{\alpha 2}(y)\oint\frac{dx}{x-z}\,d_{\beta 1}(x)(d_{\gamma 1}d_{\delta 1}e^{-\rho})(z)\\
&\quad - (d_{\alpha 1}e^{-\rho-i\sigma})(y)(d_{\beta 1}d_{\gamma 1}d_{\delta 1}e^{-\rho})(z)\Big)\\
&\quad = -\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\oint dy\oint\frac{dx}{x-z}\Big[-i(y-x)^{-1}\Pi_{\alpha\beta}(x)(d_{\gamma 1}d_{\delta 1}e^{-\rho})(z)\\
&\quad - d_{\beta 1}(x)\Big(-i(y-z)^{-1}(\Pi_{\alpha\gamma}(d_{\delta 1}e^{-\rho}))(z) + i(y-z)^{-1}(d_{\gamma 1}(\Pi_{\alpha\delta}e^{-\rho}))(z)\Big)\\
&\quad + (y-z)^{-1}(d_{\alpha 1}d_{\beta 1}d_{\gamma 1}d_{\delta 1}e^{-2\rho-i\sigma})(z)\Big]\\
&\quad = -\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\Big(-i(\Pi_{\alpha\beta}(d_{\gamma 1}(d_{\delta 1}e^{-\rho})))(z) + i(d_{\beta 1}(\Pi_{\alpha\gamma}(d_{\delta 1}e^{-\rho})))(z)\\
&\quad - i(d_{\beta 1}(d_{\gamma 1}(\Pi_{\alpha\delta}e^{-\rho})))(z)\Big) - e^{-2\rho-i\sigma}(d_1)^4(z)\\
&\quad = -e^{-2\rho-i\sigma}(d_1)^4(z) + \frac{i}{4}(e^{-\rho}(d_{\alpha 1}(d_{\beta 1}\Pi^{\alpha\beta})))(z) + \frac{3}{4}(e^{-\rho}d_{\alpha 1}\partial^2\theta^{\alpha 2})(z)\,,
\end{aligned}
\tag{F.9}
$$

where $(d_1)^4 = \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}d_{\alpha 1}d_{\beta 1}d_{\gamma 1}d_{\delta 1}$ and, to get the last line, we used that $-i(e^{-\rho}(d_{\alpha 1}(\Pi^{\alpha\beta}d_{\beta 1}))) = -i(e^{-\rho}(d_{\alpha 1}(d_{\beta 1}\Pi^{\alpha\beta}))) - 3(e^{-\rho}d_{\alpha 1}\partial^2\theta^{\alpha 2})$ and $-i(e^{-\rho}(\Pi^{\alpha\beta}(d_{\alpha 1}d_{\beta 1}))) = -i(e^{-\rho}(d_{\alpha 1}(d_{\beta 1}\Pi^{\alpha\beta}))) - 6(e^{-\rho}d_{\alpha 1}\partial^2\theta^{\alpha 2})$.

The next terms are

$$
\begin{aligned}
&-\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\oint dy\, D_\alpha(y)\epsilon_{\epsilon\beta\gamma\delta}(\partial\theta^{\epsilon 1}e^{\rho+2i\sigma})(z)\\
&\quad = \frac{1}{4}\oint dy\,(d_{\alpha 1}e^{-\rho-i\sigma})(y)(\partial\theta^{\alpha 1}e^{\rho+2i\sigma})(z)\\
&\quad = \frac{1}{4}\oint dy\oint\frac{dx}{x-z}\Big(4(y-x)^{-1}\partial e^{-\rho-i\sigma}(x)e^{\rho+2i\sigma}(z)\\
&\quad + \partial\theta^{\alpha 1}(x)(y-z)^{-1}(d_{\alpha 1}e^{i\sigma})(z)\Big)
\end{aligned}
$$

$$= -\frac{1}{4}(d_{\alpha 1}(\partial\theta^{\alpha 1}e^{i\sigma}))(z) - \frac{1}{2}(\partial(\rho+i\sigma)(\partial(\rho+i\sigma)e^{i\sigma}))(z) + \frac{1}{2}(\partial^2(\rho+i\sigma)e^{i\sigma})(z)\,. \tag{F.10}$$

$$\begin{aligned} &-\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\oint dy\, D_\alpha(y)\epsilon_{\epsilon\beta\gamma\delta}(\partial^2\theta^{\epsilon 2}e^{i\sigma})(z) \\ &\quad = \frac{1}{4}\oint dy\,(d_{\alpha 1}e^{-\rho-i\sigma})(y)(\partial^2\theta^{\alpha 2}e^{i\sigma})(z) \\ &\quad = \frac{1}{4}(d_{\alpha 1}\partial^2\theta^{\alpha 2}e^{-\rho})(z)\,. \end{aligned} \tag{F.11}$$

$$\begin{aligned} &-\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\oint dy\, D_\alpha(y)\epsilon_{\epsilon\beta\gamma\delta}(\partial\theta^{\epsilon 2}(\partial(\rho+i\sigma)e^{i\sigma}))(z) \\ &\quad = \frac{1}{4}\oint dy\,(d_{\alpha 1}e^{-\rho-i\sigma})(y)(\partial\theta^{\alpha 2}(\partial(\rho+i\sigma)e^{i\sigma}))(z) \\ &\quad = \frac{1}{4}(d_{\alpha 1}(\partial\theta^{\alpha 2}(\partial(\rho+i\sigma)e^{-\rho})))(z)\,. \end{aligned} \tag{F.12}$$

When contracted with $-\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}$, the last three terms of (F.8) amount to $-\frac{i}{4}((d_{\beta 1}\Pi^{\alpha\beta})e^{i\sigma})$. Therefore, we are left with the expression

$$\begin{aligned} &-\frac{i}{4}\oint dy\, D_\alpha(y)((d_{\beta 1}\Pi^{\alpha\beta})e^{i\sigma})(z) \\ &\quad = -\frac{i}{4}\oint dy\,(d_{\alpha 2}-d_{\alpha 1}e^{-\rho-i\sigma})(y)((d_{\beta 1}\Pi^{\alpha\beta})e^{i\sigma})(z) \\ &\quad = -\frac{i}{4}\oint dy\,(d_{\alpha 2}-d_{\alpha 1}e^{-\rho-i\sigma})(y)\oint\frac{dx}{x-z}(d_{\beta 1}\Pi^{\alpha\beta})(x)e^{i\sigma}(z) \\ &\quad = -\frac{i}{4}\oint dy\oint\frac{dx}{x-z}\Big[\Big(-2i(y-x)^{-1}\Pi^m\Pi_m(x) - 3i(y-x)^{-1}(d_{\alpha 1}\partial\theta^{\alpha 1})(x)\Big)\times \\ &\quad \times e^{i\sigma}(z) \\ &\quad - 3i(y-x)^{-1}(d_{\alpha 1}\partial\theta^{\alpha 2}e^{-\rho-i\sigma})(x)e^{i\sigma}(z) - (d_{\beta 1}\Pi^{\alpha\beta})(x)(y-z)^{-1}(e^{-\rho}d_{\alpha 1})(z)\Big] \\ &\quad = -\frac{1}{2}(\Pi^{\underline{a}}\Pi_{\underline{a}}e^{i\sigma})(z) - \frac{3}{4}(d_{\alpha 1}\partial\theta^{\alpha 1}e^{i\sigma})(z) - \frac{3}{4}((d_{\alpha 1}\partial\theta^{\alpha 2}e^{-\rho-i\sigma})e^{i\sigma})(z) \\ &\quad + \frac{i}{4}((d_{\beta 1}\Pi^{\alpha\beta})(e^{-\rho}d_{\alpha 1}))(z) \\ &\quad = -\frac{1}{2}(\Pi^{\underline{a}}\Pi_{\underline{a}}e^{i\sigma})(z) - \frac{3}{4}(d_{\alpha 1}\partial\theta^{\alpha 1}e^{i\sigma})(z) + \frac{3}{4}(d_{\alpha 1}(\partial\theta^{\alpha 2}(\partial(\rho+i\sigma)e^{-\rho})))(z) \\ &\quad + \frac{i}{4}(e^{-\rho}(d_{\alpha 1}(d_{\beta 1}\Pi^{\alpha\beta})))(z)\,. \end{aligned} \tag{F.13}$$

To obtain the last line we used that $((d_{\alpha 1}\partial\theta^{\alpha 2}e^{-\rho-i\sigma})e^{i\sigma}) = (\partial(d_{\alpha 1}\partial\theta^{\alpha 2})e^{-\rho})$ $-(d_{\alpha 1}(\partial\theta^{\alpha 2}(\partial(\rho+i\sigma)e^{-\rho})))$ and $((d_{\beta 1}\Pi^{\alpha\beta})(e^{-\rho}d_{\alpha 1})) = (e^{-\rho}(d_{\alpha 1}(d_{\beta 1}\Pi^{\alpha\beta})))$ $-3i(\partial(d_{\alpha 1}\partial\theta^{\alpha 2})e^{-\rho})$.

Gathering eqs. (F.9)–(F.13), we obtain our final expression

$$
\begin{aligned}
G^+_{\text{hyb}} = &-(d_1)^4 e^{-2\rho-i\sigma} + \frac{i}{2} d_{\alpha 1} d_{\beta 1}\Pi^{\alpha\beta}e^{-\rho} + d_{\alpha 1}\partial\theta^{\alpha 2}\partial(\rho+i\sigma)e^{-\rho} + d_{\alpha 1}\partial^2\theta^{\alpha 2}e^{-\rho} \\
&- \frac{1}{2}\Pi^{\underline{a}}\Pi_{\underline{a}}e^{i\sigma} - d_{\alpha 1}\partial\theta^{\alpha 1}e^{i\sigma} - \frac{1}{2}\partial(\rho+i\sigma)\partial(\rho+i\sigma)e^{i\sigma} \\
&+ \frac{1}{2}\partial^2(\rho+i\sigma)e^{i\sigma} + G^+_C\,,
\end{aligned} \tag{F.14}
$$

where we have dropped the normal-ordering brackets.

Appendix G

The supercurrent G^+ in $\mathrm{AdS}_3 \times \mathrm{S}^3$

Let us prove eq. (4.24), namely,

$$G^+ = -\frac{1}{4k}\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}Q_{\alpha 2}Q_{\beta 2}Q_{\gamma 2}Q_{\delta 2}e^{2\rho+3i\sigma} + G^+_C\,, \tag{G.1}$$

where $Q_{\alpha 2} = \oint \left(S_{\alpha 1}e^{-\rho-i\sigma} + S_{\alpha 2}\right)$. Using the current algebra (4.19), we start by noting that

$$\begin{aligned}
Q_{\delta 2}e^{2\rho+3i\sigma} &= S_{\delta 1}e^{\rho+2i\sigma}\,, &\text{(G.2a)}\\
Q_{\gamma 2}Q_{\delta 2}e^{2\rho+i\sigma} &= -S_{\gamma 1}S_{\delta 1}e^{i\sigma} - i\sqrt{2}K_{\gamma\delta}e^{\rho+2i\sigma}\,, &\text{(G.2b)}\\
Q_{\beta 2}Q_{\gamma 2}Q_{\delta 2}e^{2\rho+i\sigma} &= -S_{\beta 1}S_{\gamma 1}S_{\delta 1}e^{-\rho} + i\sqrt{2}\big(K_{\beta\gamma}S_{\delta 1} - S_{\gamma 1}K_{\beta\delta} + K_{\gamma\delta}S_{\beta 1}\big)e^{i\sigma}\\
&\quad - 2\epsilon_{\beta\gamma\delta\rho}\widehat{\delta}^{\rho\sigma}\left[\left(S_{\sigma 1}e^{-\rho-i\sigma}, e^{\rho+2i\sigma}\right) + S_{\sigma 2}e^{\rho+2i\sigma}\right]. &\text{(G.2c)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&-\frac{1}{4k}\frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}Q_{\alpha 2}Q_{\beta 2}Q_{\gamma 2}Q_{\delta 2}e^{2\rho+3i\sigma}\\
&\quad = -\frac{1}{4k}(S_1)^4e^{-2\rho-i\sigma} - \frac{1}{2k}\left(\frac{i}{2\sqrt{2}}S_{\alpha 1}S_{\beta 1}K^{\alpha\beta} + \widehat{\delta}^{\alpha\beta}S_{\alpha 1}\partial S_{\beta 1}\right)e^{-\rho}\\
&\quad + T_{\mathrm{PSU}}e^{i\sigma} + \left(\partial e^{-\rho-i\sigma}, e^{\rho+2i\sigma}\right), \qquad \text{(G.3)}
\end{aligned}$$

implying we can write eq. (4.20b) as (4.24), as we wanted to show.

Appendix H

$\mathcal{N} = 1, \mathcal{N} = 2$ and small $\mathcal{N} = 4$ superconformal algebras

We present the general structure of $\mathcal{N} = 1, \mathcal{N} = 2$ and small $\mathcal{N} = 4$ superconformal algebras, as well as their twisted counterparts. We do not try to address questions such as when and how these algebras can be realized.

H.1 $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal algebras

An $\mathcal{N} = 1$ superconformal algebra with central charge c is given by a conformal weight two stress-tensor T and a conformal weight $\frac{3}{2}$ supercurrent G satisfying

$$T(y)T(z) \sim \frac{\frac{c}{2}}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{(y-z)}, \tag{H.1a}$$

$$T(y)G(z) \sim \frac{\frac{3}{2}G(z)}{(y-z)^2} + \frac{\partial G(z)}{(y-z)}, \tag{H.1b}$$

$$G(y)G(z) \sim \frac{\frac{2}{3}c}{(y-z)^3} + \frac{2T(z)}{(y-z)}. \tag{H.1c}$$

The $\mathcal{N} = 2$ superconformal algebra with central charge c satisfied by the generators $\{J, G^+, G^-, T\}$ is given by

$$T(y)T(z) \sim \frac{\frac{c}{2}}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{(y-z)}, \tag{H.2a}$$

$$G^+(y)G^-(z) \sim \frac{\frac{c}{3}}{(y-z)^3} + \frac{J(z)}{(y-z)^2} + \frac{T(z) + \frac{1}{2}\partial J(z)}{(y-z)}, \tag{H.2b}$$

$$T(y)G^\pm(z) \sim \frac{\frac{3}{2}G^\pm(z)}{(y-z)^2} + \frac{\partial G^\pm(z)}{(y-z)}, \tag{H.2c}$$

$$T(y)J(z) \sim \frac{J(z)}{(y-z)^2} + \frac{\partial J(z)}{(y-z)}, \tag{H.2d}$$

$$J(y)J(z) \sim \frac{\frac{c}{3}}{(y-z)^2}\,, \tag{H.2e}$$

$$J(y)G^{\pm}(z) \sim \pm\frac{G^{\pm}(z)}{(y-z)}\,. \tag{H.2f}$$

Here, T has conformal weight 2, $G^{\pm}$ has conformal weight $\frac{3}{2}$ and J has conformal weight 1.

Equivalently, in tems of the modes, the $\mathcal{N}=2$ SCA reads

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{c}{12}(m^3-m)\delta_{m,-n}\,, \tag{H.3a}$$

$$\{G^+_r, G^-_s\} = L_{r+s} + \tfrac{1}{2}(r-s)J_{r+s} + \tfrac{c}{6}(r^2-\tfrac{1}{4})\delta_{r,-s}\,, \tag{H.3b}$$

$$[L_m, G^{\pm}_r] = (\tfrac{1}{2}m-r)G^{\pm}_{m+r}\,, \tag{H.3c}$$

$$[L_m, J_n] = -nJ_{m+n}\,, \tag{H.3d}$$

$$[J_m, J_n] = \tfrac{c}{3}m\delta_{m,-n}\,, \tag{H.3e}$$

$$[J_m, G^{\pm}_r] = \pm G^{\pm}_{m+r}\,. \tag{H.3f}$$

H.2 Twisted $\mathcal{N}=2$ superconformal algebra

To construct an $\mathcal{N}=2$ twisted theory, we modify the stress-tensor T by adding $+\frac{1}{2}\partial J$ to it, so that

$$T + \frac{1}{2}\partial J \mapsto T\,, \tag{H.4}$$

and one can see that the dimension of every field in the theory is modified by $-\frac{1}{2}$ its U(1)-charge, which is generated by J. In particular, looking at the structure of the algebra (H.2), we see that the conformal weight of G^+ gets shifted to 1, that of G^- gets shifted to 2 and the conformal weight of the rest of the generators stay untouched. More importantly, the shift in the stress-tensor (H.4) results in the vanishing of the conformal anomaly in the TT OPE, so that the twisted stress-tensor is a primary. In contrast, there appears a triple pole in the TJ OPE proportional to the central charge c.

With the above considerations, we can now write the twisted $\mathcal{N}=2$ superconformal algebra with central charge c satisfied by the twisted generators $\{J, G^+, G^-, T\}$[1]

$$T(y)T(z) \sim \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{(y-z)}\,, \tag{H.5a}$$

[1]Here, T is the shifted stress-tensor of (H.4).

$$G^+(y)G^-(z) \sim \frac{\frac{c}{3}}{(y-z)^3} + \frac{J(z)}{(y-z)^2} + \frac{T(z)}{(y-z)}\,, \tag{H.5b}$$

$$T(y)G^+(z) \sim \frac{G^+(z)}{(y-z)^2} + \frac{\partial G^+(z)}{(y-z)}\,, \tag{H.5c}$$

$$T(y)G^-(z) \sim \frac{2G^-(z)}{(y-z)^2} + \frac{\partial G^-(z)}{(y-z)}\,, \tag{H.5d}$$

$$T(y)J(z) \sim -\frac{\frac{c}{3}}{(y-z)^3} + \frac{J(z)}{(y-z)^2} + \frac{\partial J(z)}{(y-z)}\,, \tag{H.5e}$$

$$J(y)J(z) \sim \frac{\frac{c}{3}}{(y-z)^2}\,, \tag{H.5f}$$

$$J(y)G^\pm(z) \sim \pm\frac{G^\pm(z)}{(y-z)}\,. \tag{H.5g}$$

H.3 Small and twisted small $\mathcal{N}=4$ superconformal algebras

A small $\mathcal{N}=4$ superconformal algebra consists of a conformal weight 2 generator T, four conformal weight $\frac{3}{2}$ fermionic currents $\{G^\pm, \widetilde{G}^\pm\}$ and three conformal weight 1 bosonic currents $\{J, J^{++}, J^{--}\}$ forming an $\mathfrak{su}(2)_{\frac{c}{6}}$ current algebra. In the description that we are using, it is convenient to build the small $\mathcal{N}=4$ SCA by starting with the $\mathcal{N}=2$ SCA in Appendix H.1 and lifting the $\mathfrak{u}(1)_{\frac{c}{6}}$ to an $\mathfrak{su}(2)_{\frac{c}{6}}$ current algebra. To do that, one adds to the generators $\{J, G^+, G^-, T\}$ the conformal weight 1 bosonic currents J^{++} and J^{--} of U(1) charge ± 2, respectively, satisfying the OPES

$$J(y)J^{\pm\pm}(z) \sim \pm 2\frac{J^{\pm\pm}(z)}{(y-z)}\,, \tag{H.6a}$$

$$J^{++}(y)J^{--}(z) \sim \frac{\frac{c}{6}}{(y-z)^2} + \frac{J(z)}{(y-z)}\,. \tag{H.6b}$$

Note that the level of the $\mathfrak{su}(2)$ current algebra is fixed by the Jacobi identities and the level of the $\mathfrak{u}(1)$ current algebra. On top of that, for the algebra to close, we also need to add two fermionic generators $\widetilde{G}^\pm$ and, in addition to the non-regular OPEs in eq. (H.2), we also have

$$J^{\pm\pm}(y)G^\mp(z) \sim \mp\frac{\widetilde{G}^\pm(z)}{(y-z)}\,, \tag{H.7a}$$

$$J^{\pm\pm}(y)\widetilde{G}^{\mp}(z) \sim \pm\frac{G^{\pm}(z)}{(y-z)}, \tag{H.7b}$$

$$G^{+}(y)\widetilde{G}^{+}(z) \sim \frac{2J^{++}(z)}{(y-z)^2} + \frac{\partial J^{++}(z)}{(y-z)}, \tag{H.7c}$$

$$\widetilde{G}^{-}(y)G^{-}(z) \sim \frac{2J^{--}(z)}{(y-z)^2} + \frac{\partial J^{--}(z)}{(y-z)}, \tag{H.7d}$$

$$\widetilde{G}^{+}(y)\widetilde{G}^{-}(z) \sim \frac{\frac{c}{3}}{(y-z)^3} + \frac{J(z)}{(y-z)^2} + \frac{T(z)+\frac{1}{2}\partial J(z)}{(y-z)}, \tag{H.7e}$$

$$T(y)J^{\pm\pm}(z) \sim \frac{J^{\pm\pm}(z)}{(y-z)^2} + \frac{\partial J^{\pm\pm}(z)}{(y-z)}, \tag{H.7f}$$

$$T(y)\widetilde{G}^{\pm}(z) \sim \frac{\frac{3}{2}\widetilde{G}^{\pm}(z)}{(y-z)^2} + \frac{\partial \widetilde{G}^{\pm}(z)}{(y-z)}. \tag{H.7g}$$

Therefore, we say that the generators $\{J, J^{\pm\pm}, G^{\pm}, \widetilde{G}^{\pm}, T\}$ form a small $\mathcal{N}=4$ SCA with central charge c when they satisfy eqs. (H.2), (H.6) and (H.7).

The twisted small $\mathcal{N}=4$ SCA with central charge c can be constructed from the untwisted one in the same way as we constructed the twisted $\mathcal{N}=2$ SCA from eq. (H.2), i.e., by shifting the stress-tensor as in eq. (H.4). With respect to the twisted stress-tensor, the conformal weight of J^{++} becomes zero, that of J^{--} becomes 2, the conformal weight of G^{+} and $\widetilde{G}^{+}$ gets shifted to 1 and that of G^{-} and $\widetilde{G}^{-}$ gets shifted to 2. Consequently, we say that the twisted generators $\{J, J^{\pm\pm}, G^{\pm}, \widetilde{G}^{\pm}, T\}$ form a twisted small $\mathcal{N}=4$ SCA with central charge c when they obey eqs. (H.5), (H.6) and

$$J^{\pm\pm}(y)G^{\mp}(z) \sim \mp\frac{\widetilde{G}^{\pm}(z)}{(y-z)}, \tag{H.8a}$$

$$J^{\pm\pm}(y)\widetilde{G}^{\mp}(z) \sim \pm\frac{G^{\pm}(z)}{(y-z)}, \tag{H.8b}$$

$$G^{+}(y)\widetilde{G}^{+}(z) \sim \frac{2J^{++}(z)}{(y-z)^2} + \frac{\partial J^{++}(z)}{(y-z)}, \tag{H.8c}$$

$$\widetilde{G}^{-}(y)G^{-}(z) \sim \frac{2J^{--}(z)}{(y-z)^2} + \frac{\partial J^{--}(z)}{(y-z)}, \tag{H.8d}$$

$$\widetilde{G}^{+}(y)\widetilde{G}^{-}(z) \sim \frac{\frac{c}{3}}{(y-z)^3} + \frac{J(z)}{(y-z)^2} + \frac{T(z)}{(y-z)}, \tag{H.8e}$$

$$T(y)J^{++}(z) \sim \frac{\partial J^{++}(z)}{(y-z)}, \tag{H.8f}$$

$$T(y)J^{--}(z) \sim \frac{2J^{--}(z)}{(y-z)^2} + \frac{\partial J^{--}(z)}{(y-z)}, \tag{H.8g}$$

$$T(y)\widetilde{G}^{+}(z) \sim \frac{\widetilde{G}^{+}(z)}{(y-z)^2} + \frac{\partial\widetilde{G}^{+}(z)}{(y-z)}\,, \tag{H.8h}$$

$$T(y)\widetilde{G}^{-}(z) \sim \frac{2\widetilde{G}^{-}(z)}{(y-z)^2} + \frac{\partial\widetilde{G}^{-}(z)}{(y-z)}\,. \tag{H.8i}$$

With respect to the $\mathfrak{su}(2)$ symmetry, T transforms as a singlet and G^{+} (G^{-}) transforms as an upper (lower) component of an $\mathfrak{su}(2)$ doublet whose lower (upper) component is $\widetilde{G}^{-}(\widetilde{G}^{+})$. This $\mathfrak{su}(2)$ rotates the different choices of the $\mathrm{U}(1)$ current J into one another and computations are equivalent no matter what choice of this $\mathrm{U}(1)$ one picks [18].

In addition, there is another SU(2) symmetry (that we refer to as SU(2)$_{outer}$) of the $\mathcal{N}=4$ SCA which acts by outer automorphisms. To see that, consider the following linear combinations of the fermionic generators[2]

$$\mathbf{G}^{+} = u_1^{*}G^{+} - u_2^{*}\widetilde{G}^{+}\,, \tag{H.9a}$$

$$\mathbf{G}^{-} = u_1 G^{-} - u_2\widetilde{G}^{-}\,, \tag{H.9b}$$

$$\widetilde{\mathbf{G}}^{+} = u_1\widetilde{G}^{+} + u_2 G^{+}\,, \tag{H.9c}$$

$$\widetilde{\mathbf{G}}^{-} = u_1^{*}\widetilde{G}^{-} + u_2^{*}G^{-}\,, \tag{H.9d}$$

by demanding that $\mathbf{G}^{\pm}$ and $\widetilde{\mathbf{G}}^{\pm}$ satisfy the same algebra as $G^{\pm}$ and $\widetilde{G}^{\pm}$ we get the relation $|u_1|^2+|u_2|^2=1$, i.e., u_1 and u_2 are elements of SU(2)$_{outer}$. This symmetry that rotates the supercurrents parametrizes the different embeddings of the $\mathcal{N}=2$ SCA into the $\mathcal{N}=4$ SCA and, in general, is not a symmetry of the theory [18] [46].

Lastly, we should mention the important fact that a small $\mathcal{N}=4$ SCA can be constructed from any $c=6$ $\mathcal{N}=2$ SCA by defining the $SU(2)$ currents to be J, $J^{++}=-e^{\int J}$ and $J^{--}=e^{-\int J}$. The condition $c=6$ is necessary in order for J^{++} and J^{--} to have conformal weight 1 when the algebra is not twisted. As an example, the RNS superstring has a description as a $c=6$ $\mathcal{N}=2$ string and, therefore, can also be described as an $\mathcal{N}=4$ topological string [18].

[2] Note that here they obey the hermiticity properties $(\mathbf{G}^{\pm})^{*}=\widetilde{\mathbf{G}}^{\mp}$ and $(\widetilde{\mathbf{G}}^{\pm})^{*}=\widetilde{\mathbf{G}}^{\mp}$.

Appendix I

Another basis for the bosonic currents

I.1 Choosing a $\mathrm{U}(1)$ direction

In order to label the physical states, it is convenient to single out an $\mathrm{U}(1) \in \mathrm{SL}(2,\mathbb{R})$ and an $\mathrm{U}(1) \in \mathrm{SU}(2)$ direction [80].

We build the $\mathrm{SL}(2,\mathbb{R})_k$ generators in a standard basis from the currents $K_{\underline{a}}$ by defining

$$J_\pm = -\frac{i}{2}(K_1 \pm iK_2)\,, \qquad\qquad J_3 = -\frac{i}{2}K_0\,, \tag{I.1}$$

which satisfy the current algebra

$$J_3(y)J_3(z) \sim -\frac{k}{2}(y-z)^{-2}\,, \tag{I.2a}$$

$$J_3(y)J_\pm(z) \sim \pm(y-z)^{-1}J_\pm\,, \tag{I.2b}$$

$$J_+(y)J_-(z) \sim k(y-z)^{-2} - 2(y-z)^{-1}J_3\,. \tag{I.2c}$$

If desired, one can do the same for the $\mathrm{SU}(2)_k$ part. We define the linear combinations[1]

$$K_{3'} = -\frac{i}{2}K_5\,, \qquad\qquad K_{\pm'} = -\frac{i}{2}(K_3 \pm iK_4)\,, \tag{I.3a}$$

which satisfy

$$K_{3'}(y)K_{3'}(z) \sim \frac{k}{2}(y-z)^{-2}\,, \tag{I.3b}$$

$$K_{3'}(y)K_{\pm'}(z) \sim \pm(y-z)^{-1}K_{\pm'}\,, \tag{I.3c}$$

$$K_{+'}(y)K_{-'}(z) \sim k(y-z)^{-2} + 2(y-z)^{-1}K_{3'}\,. \tag{I.3d}$$

[1]The choice to single out the "five" direction in $K_{3'} = K_5$ comes because σ_5 is block diagonal in our conventions, see eqs. (B.2)

I.2 $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ quantum numbers

One can label the vertex operator $\mathcal{V}$ in (4.43) by the $\mathrm{SL}(2,\mathbb{R})$ quantum numbers $\{j,m\}$ and $\mathrm{SU}(2)$ quantum numbers $\{j',m'\}$. As before, we will consider zero amount of spectral flow in this section. Since all the physical degrees of freedom are contained in the superfield V from eq. (4.50a), we focus on describing its components in what follows. Moreover, the superfield $V_0 \subset \mathcal{V}$ decouples from amplitude computations presented in this work.

Accordingly, if $\mathcal{V} \supset V$ has quantum numbers $\{j,m,j',m'\}$, we write

$$V = V(\vec{j},\vec{m})\,, \tag{I.4}$$

where

$$\vec{j} = \{j,j'\}\,, \qquad \vec{m} = \{m,m'\}\,, \tag{I.5}$$

with $j = j'+1$, the half-BPS condition. The vector $\vec{j}_i$ labels the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ spin of the representation and $\vec{m}$ characterize the state in the given representation. As a consequence, under the zero-modes of the diagonal currents defined in Section I.1, we then have

$$\vec{\nabla}_3 V = \vec{m} V\,, \qquad \vec{\nabla}_3 = \{\nabla_3, \nabla_{3'}\}\,. \tag{I.6}$$

Consequently, the wavefunctions $\{\chi_{\alpha 2}, a^{\underline{a}}, \psi^{\alpha 2}\}$ in (4.50a) can be written as

$$\chi_{\alpha 2} = V^{\mathrm{SL}(2,\mathbb{R})\times \mathrm{SU}(2)}_{\vec{j},\vec{m}_\alpha} = V^{\mathrm{SL}(2,\mathbb{R})}_{j,m_\alpha} V^{\mathrm{SU}(2)}_{j',m'_\alpha}\,, \tag{I.7a}$$

$$a^{\underline{a}} = V^{\mathrm{SL}(2,\mathbb{R})\times \mathrm{SU}(2)}_{\vec{j},\vec{m}^{\underline{a}}} = V^{\mathrm{SL}(2,\mathbb{R})}_{j,m^{\underline{a}}} V^{\mathrm{SU}(2)}_{j',m'^{\underline{a}}}\,, \tag{I.7b}$$

$$\psi^{\alpha 2} = V^{\mathrm{SL}(2,\mathbb{R})\times \mathrm{SU}(2)}_{\vec{j},\vec{m}^\alpha} = V^{\mathrm{SL}(2,\mathbb{R})}_{j,m^\alpha} V^{\mathrm{SU}(2)}_{j',m'^\alpha}\,, \tag{I.7c}$$

where $V^{\mathrm{SL}(2,\mathbb{R})}_{j_i,m_i}$ and $V^{\mathrm{SU}(2)}_{j'_i,m'_i}$ are $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ current algebra primaries, respectively. From the properties

$$[\nabla_{\underline{a}}, \theta^\alpha] = -f_{\underline{a}\,\beta 1}{}^{\alpha 1}\theta^\beta\,, \tag{I.8a}$$

$$[\nabla_{\underline{a}}, (\theta\sigma_{\underline{b}}\theta)] = f_{\underline{a}\underline{b}}{}^{\underline{c}}(\theta\sigma_{\underline{c}}\theta)\,, \tag{I.8b}$$

$$[\nabla_{\underline{a}}, (\theta^3)_\alpha] = f_{\underline{a}\,\alpha 1}{}^{\beta 1}(\theta^3)_\beta\,, \tag{I.8c}$$

one finds that

$$\vec{m}^{3} = \{m, m'\}\,, \qquad \vec{m}^{\pm} = \{m \mp 1, m'\}\,, \tag{I.9a}$$

$$\vec{m}^{3'} = \{m, m'\}\,, \qquad \vec{m}^{\pm'} = \{m, m' \mp 1\}\,, \tag{I.9b}$$

and $\vec{m}_\alpha = \{m_\alpha, m'_\alpha\}$ with

$$\vec{m}_1 = \{m + \tfrac{1}{2}, m' - \tfrac{1}{2}\}\,, \qquad \vec{m}_2 = \{m - \tfrac{1}{2}, m' + \tfrac{1}{2}\}\,, \tag{I.10a}$$

$$\vec{m}_3 = \{m - \tfrac{1}{2}, m' - \tfrac{1}{2}\}\,, \qquad \vec{m}_4 = \{m + \tfrac{1}{2}, m' + \tfrac{1}{2}\}\,, \tag{I.10b}$$

and, finally, $\vec{m}^\alpha = \{m^\alpha, m'^\alpha\}$ with

$$\vec{m}^1 = \{m - \tfrac{1}{2}, m' + \tfrac{1}{2}\}\,, \qquad \vec{m}^2 = \{m + \tfrac{1}{2}, m' - \tfrac{1}{2}\}\,, \tag{I.11a}$$

$$\vec{m}^3 = \{m + \tfrac{1}{2}, m' + \tfrac{1}{2}\}\,, \qquad \vec{m}^4 = \{m - \tfrac{1}{2}, m' - \tfrac{1}{2}\}\,. \tag{I.11b}$$

Appendix J

Gauge invariance in the hybrid description

Let us analyze the consistency of the hybrid vertices (4.80) appearing in the three-point amplitude for the states in the NS-sector. We take $V = \frac{i}{2}(\theta\sigma_{\underline{a}}\theta)a^{\underline{a}}$, so that eqs. (4.80) become

$$\mathcal{V} = e^{\rho+i\sigma}\frac{i}{2}(\theta\sigma_{\underline{a}}\theta)a^{\underline{a}}\,, \tag{J.1a}$$

$$\widetilde{G}_0^+\mathcal{V} = e^{2\rho+i\sigma}J_C^{++}\frac{i}{2}(\theta\sigma_{\underline{a}}\theta)a^{\underline{a}}\,, \tag{J.1b}$$

$$G_0^+\mathcal{V} = -\frac{1}{2k}\frac{1}{\sqrt{2}}e^{i\sigma}\left(K_{\underline{a}}a^{\underline{a}} + iS_{\alpha 1}(\sigma^{\underline{ab}}\theta)^\alpha D_{\underline{a}}a_{\underline{b}} + i\sqrt{2}S_{\alpha 1}(\widehat{\delta}\sigma_{\underline{a}}\theta)^\alpha a^{\underline{a}}\right). \tag{J.1c}$$

Up to a constant, the integrated vertex operator is then given by

$$\int G_{-1}^-G_0^+\mathcal{V} = \int \left(K_{\underline{a}}a^{\underline{a}} + iS_{\alpha 1}(\sigma^{\underline{ab}}\theta)^\alpha D_{\underline{a}}a_{\underline{b}} + i\sqrt{2}S_{\alpha 1}(\widehat{\delta}\sigma_{\underline{a}}\theta)^\alpha a^{\underline{a}}\right). \tag{J.2}$$

We now check that the integrand of (J.2) is gauge invariant up to a total derivative. From eq. (4.47), one finds that the gauge transformation for $a_{\underline{a}}$ is

$$\delta a_{\underline{a}} = \nabla_{\underline{a}}\lambda\,, \tag{J.3}$$

for some $\lambda = \lambda(g)$ where $g \in \mathrm{PSU}(1,1|2)$ and $\nabla_{\alpha 1}\lambda = 0$. Therefore, under (J.3), the integrated vertex operator (J.2) transforms as

$$\begin{aligned} &K^{\underline{a}}\nabla_{\underline{a}}\lambda - S_{\alpha 1}\theta^\beta f_{\underline{a}\beta 1}{}^{\alpha 1}\nabla^{\underline{a}}\lambda \\ &= K^{\underline{a}}\nabla_{\underline{a}}\lambda + S^{\alpha j}\nabla_{\alpha j}\lambda \\ &= \partial\lambda\,, \end{aligned} \tag{J.4}$$

as we wanted to show. In arriving to the second line above we used that

$$\nabla_{\alpha 1}\lambda = 0 \qquad \Rightarrow \qquad S^{\alpha 2}\nabla_{\alpha 2}\lambda = -S_{\alpha 1}\theta^{\beta} f_{\underline{a}\,\beta 1}{}^{\alpha 1}\nabla^{\underline{a}}\lambda \, . \tag{J.5}$$

As a result, we conclude that the integrated vertex operator (J.2) is gauge invariant up to a total derivative in the group manifold of $\mathrm{PSU}(1,1|2)$.

Appendix K

$d = 6$ $\mathcal{N} = 1$ super-Yang-Mills

In this section, we closely follow the $d = 10$ $\mathcal{N} = 1$ super-Yang-Mills description presented in ref. [106, Appendix B].

To describe $d = 6$ super-Yang-Mills in $\mathcal{N} = 1$ superspace, we define the super-covariant derivatives

$$\mathfrak{D}_{\underline{a}} = \partial_{\underline{a}} + A_{\underline{a}}\,, \tag{K.1a}$$

$$\mathfrak{D}_{\alpha j} = \nabla_{\alpha j} + A_{\alpha j}\,, \tag{K.1b}$$

where $\nabla_{\alpha j} = \frac{\partial}{\partial\theta^{\alpha j}} - \frac{i}{2}\epsilon_{jk}\theta^{\beta k}\sigma^{\underline{a}}_{\alpha\beta}\partial_{\underline{a}}$ with $\{\nabla_{\alpha j}, \nabla_{\beta k}\} = -i\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}\partial_{\underline{a}}$. Then, the field-strengths are

$$F_{\alpha j\beta k} = \{\mathfrak{D}_{\alpha j}, \mathfrak{D}_{\beta k}\} + i\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}\mathfrak{D}_{\underline{a}}\,, \tag{K.2a}$$

$$F_{\alpha j\underline{a}} = [\mathfrak{D}_{\alpha j}, \mathfrak{D}_{\underline{a}}]\,, \tag{K.2b}$$

$$F_{\underline{ab}} = [\mathfrak{D}_{\underline{a}}, \mathfrak{D}_{\underline{b}}]\,, \tag{K.2c}$$

which are invariant under the gauge transformations

$$\delta A_{\alpha j} = \nabla_{\alpha j}\Lambda\,, \qquad \delta A_{\underline{a}} = \partial_{\underline{a}}\Lambda\,, \tag{K.3}$$

for any Λ.

Explicitly, the superspace field-strength constraint $F_{\alpha j\beta k} = 0$ reads [43]

$$\nabla_{\alpha j}A_{\beta k} + \nabla_{\beta k}A_{\alpha j} + \{A_{\alpha j}, A_{\beta k}\} + i\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}A_{\underline{a}} = 0\,. \tag{K.4}$$

Multiplying the above equation by $(\sigma^{\underline{abc}})^{\alpha\beta}$ and using that $(\sigma^{\underline{abc}})^{\alpha\beta}\sigma^{\underline{d}}_{\alpha\beta} = 0$, we obtain

$$(\sigma^{\underline{abc}})^{\alpha\beta}(\nabla_{\alpha j}A_{\beta k} + \nabla_{\beta k}A_{\alpha j} + \{A_{\alpha j}, A_{\beta k}\}) = 0\,. \tag{K.5}$$

The converse also follows.

From the Bianchi identity

$$[\{\mathfrak{D}_{\alpha j},\mathfrak{D}_{\beta k}\},\mathfrak{D}_{\gamma l}]+[\{\mathfrak{D}_{\gamma l},\mathfrak{D}_{\alpha j}\},\mathfrak{D}_{\beta k}]+[\{\mathfrak{D}_{\beta k},\mathfrak{D}_{\gamma l}\},\mathfrak{D}_{\alpha j}]=0\,, \tag{K.6}$$

we have

$$i\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}[\mathfrak{D}_{\underline{a}},\mathfrak{D}_{\gamma l}]+i\epsilon_{lj}\sigma^{\underline{a}}_{\gamma\alpha}[\mathfrak{D}_{\underline{a}},\mathfrak{D}_{\beta k}]+i\epsilon_{kl}\sigma^{\underline{a}}_{\beta\gamma}[\mathfrak{D}_{\underline{a}},\mathfrak{D}_{\alpha j}]=0\,, \tag{K.7}$$

which is satisfied if $F_{\alpha j\underline{a}}=-i\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}W^{\beta k}$ by using the Schouten identity [6, Appendix A]. Therefore, $F_{\alpha j\underline{a}}=[\mathfrak{D}_{\alpha j},\mathfrak{D}_{\underline{a}}]=-i\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}W^{\beta k}$ gives

$$\partial_{\underline{a}}A_{\alpha j}-\mathfrak{D}_{\alpha j}A_{\underline{a}}-i\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}W^{\beta k}=0\,. \tag{K.8}$$

The Bianchi identity

$$[\{\mathfrak{D}_{\alpha j},\mathfrak{D}_{\beta k}\},\mathfrak{D}_{\underline{a}}]+\{[\mathfrak{D}_{\underline{a}},\mathfrak{D}_{\alpha j}],\mathfrak{D}_{\beta k}\}-\{[\mathfrak{D}_{\beta k},\mathfrak{D}_{\underline{a}}],\mathfrak{D}_{\alpha j}\}=0\,, \tag{K.9}$$

gives

$$\epsilon_{jk}\sigma^{\underline{b}}_{\alpha\beta}F_{\underline{ab}}+\epsilon_{jl}\sigma_{\underline{a}\alpha\gamma}\mathfrak{D}_{\beta k}W^{\gamma l}+\epsilon_{kl}\sigma_{\underline{a}\beta\gamma}\mathfrak{D}_{\alpha j}W^{\gamma l}=0\,. \tag{K.10}$$

Multiplying eq. (K.10) by $\sigma^{\underline{a}\alpha\beta}$, we obtain

$$\epsilon_{jl}\mathfrak{D}_{\alpha k}W^{\alpha l}-\epsilon_{kl}\mathfrak{D}_{\alpha j}W^{\alpha l}=0\,, \tag{K.11}$$

which imply $\mathfrak{D}_{\alpha j}W^{\alpha j}=0$. Contracting (K.10) with $\sigma^{\underline{a}\beta\sigma}$ and $\sigma^{\underline{a}\alpha\sigma}$, we get

$$-i\epsilon_{jk}(\sigma^{\underline{ab}})^{\sigma}{}_{\alpha}F_{\underline{ab}}-4\epsilon_{lk}\mathfrak{D}_{\alpha j}W^{\sigma l}+\epsilon_{lk}\delta^{\sigma}_{\alpha}\mathfrak{D}_{\beta j}W^{\beta l}+\epsilon_{jk}\mathfrak{D}_{\alpha l}W^{\sigma l}=0\,, \tag{K.12a}$$

$$i\epsilon_{jk}(\sigma^{\underline{ab}})^{\sigma}{}_{\beta}F_{\underline{ab}}-4\epsilon_{lk}\mathfrak{D}_{\beta j}W^{\sigma l}+\epsilon_{lk}\delta^{\sigma}_{\beta}\mathfrak{D}_{\alpha j}W^{\alpha l}+3\epsilon_{jk}\mathfrak{D}_{\beta l}W^{\sigma l}=0\,. \tag{K.12b}$$

From (K.12b) $-\,3\times$ (K.12a), it follows that

$$2i\epsilon_{jk}(\sigma^{\underline{ab}})^{\beta}{}_{\alpha}F_{\underline{ab}}-4\mathfrak{D}_{\alpha j}W^{\beta}_{k}+\delta^{\beta}_{\alpha}\mathfrak{D}_{\gamma j}W^{\gamma}_{k}=0\,, \tag{K.13}$$

where $W^{\alpha}_{j}=\epsilon_{jk}W^{\alpha k}$. Consequently,

$$i\epsilon_{jk}(\sigma^{\underline{ab}})^{\beta}{}_{\alpha}F_{\underline{ab}}-\mathfrak{D}_{\alpha[j}W^{\beta}_{k]}=0\,, \tag{K.14a}$$

$$4\mathfrak{D}_{\alpha(j}W^{\beta}_{k)} - \delta^{\beta}_{\alpha}\mathfrak{D}_{\gamma(j}W^{\gamma}_{k)} = 0\,. \tag{K.14b}$$

Furthermore, using the equation of motion of $d = 6$ super-Yang-Mills [43], i.e., $\mathfrak{D}_{\alpha(j}W^{\alpha}_{k)} = 0$, we then have

$$\mathfrak{D}_{\alpha j}W^{\beta}_{k} - \frac{i}{2}\epsilon_{jk}(\sigma^{\underline{ab}})^{\beta}{}_{\alpha}F_{\underline{ab}} = 0\,, \tag{K.15}$$

Now, consider the Bianchi identity

$$[[\mathfrak{D}_{\underline{a}}, \mathfrak{D}_{\underline{b}}], \mathfrak{D}_{\alpha j}] + [[\mathfrak{D}_{\alpha j}, \mathfrak{D}_{\underline{a}}], \mathfrak{D}_{\underline{b}}] + [[\mathfrak{D}_{\underline{b}}, \mathfrak{D}_{\alpha j}], \mathfrak{D}_{\underline{a}}] = 0\,, \tag{K.16}$$

that implies

$$\mathfrak{D}_{\alpha j}F_{\underline{ab}} = i\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}\mathfrak{D}_{\underline{b}}W^{\beta k} - i\epsilon_{jk}\sigma_{\underline{b}\alpha\beta}\mathfrak{D}_{\underline{a}}W^{\beta k}\,. \tag{K.17}$$

Finally, acting with $\mathfrak{D}_{\gamma l}$ in (K.15), symmetrizing in the indices $\{\alpha j, \gamma l\}$, then using (K.17) and multiplying by δ^{γ}_{β}, we end up with

$$\sigma^{\underline{a}}_{\alpha\beta}\mathfrak{D}_{\underline{a}}W^{\beta j} = 0\,. \tag{K.18}$$

In summary, the equations describing $d = 6$ super-Yang-Mills obtained in this section are

$$\nabla_{\alpha j}A_{\beta k} + \nabla_{\beta k}A_{\alpha j} + \{A_{\alpha j}, A_{\beta k}\} + i\epsilon_{jk}\sigma^{\underline{a}}_{\alpha\beta}A_{\underline{a}} = 0\,, \tag{K.19a}$$
$$\partial_{\underline{a}}A_{\alpha j} - \mathfrak{D}_{\alpha j}A_{\underline{a}} - i\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}W^{\beta k} = 0\,, \tag{K.19b}$$
$$\mathfrak{D}_{\alpha j}W^{\beta}_{k} - \frac{i}{2}\epsilon_{jk}(\sigma^{\underline{ab}})^{\beta}{}_{\alpha}F_{\underline{ab}} = 0\,, \tag{K.19c}$$
$$\sigma^{\underline{a}}_{\alpha\beta}\mathfrak{D}_{\underline{a}}W^{\beta j} = 0\,, \tag{K.19d}$$

which were shown to follow from the equation of motion $\mathfrak{D}_{\alpha(j}W^{\alpha}_{k)} = 0$ and the superspace constraint $F_{\alpha j\beta k} = 0$.

Note also that the superfields $\{A_{\underline{a}}, W^{\alpha j}, F_{\underline{ab}}\}$ can be written as

$$A_{\underline{a}} = -\frac{i}{4}\epsilon^{jk}\sigma^{\alpha\beta}_{\underline{a}}(\nabla_{\alpha j}A_{\beta k} + \nabla_{\beta k}A_{\alpha j} + \{A_{\alpha j}, A_{\beta k}\})\,, \tag{K.20a}$$
$$W^{\alpha j} = \frac{i}{3}\epsilon^{jk}\sigma^{\underline{a}\alpha\beta}(\partial_{\underline{a}}A_{\beta k} - \mathfrak{D}_{\beta k}A_{\underline{a}})\,, \tag{K.20b}$$
$$F_{\underline{ab}} = \mathfrak{D}_{\underline{a}}A_{\underline{b}} - \mathfrak{D}_{\underline{b}}A_{\underline{a}}\,. \tag{K.20c}$$

The θ expansion of the $d = 6$ SYM superfields is given by

$$A_{\alpha j} = -\frac{i}{2}\epsilon_{jk} a_{\alpha\beta}\theta^{\beta k} + \frac{1}{3}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{jk}\epsilon_{lm}\theta^{\beta k}\psi^{\gamma l}\theta^{\delta m} + \dots\,, \tag{K.21a}$$

$$A_{\underline{a}} = a_{\underline{a}} + i\epsilon_{jk}\sigma_{\underline{a}\alpha\beta}\psi^{\alpha j}\theta^{\beta k} + \dots\,, \tag{K.21b}$$

$$W^{\alpha j} = \psi^{\alpha j} - \frac{i}{2}(\sigma^{\underline{ab}})^{\alpha}{}_{\beta}\theta^{\beta j} f_{\underline{ab}} + \dots\,, \tag{K.21c}$$

$$F_{\underline{ab}} = f_{\underline{ab}} + \dots\,, \tag{K.21d}$$

where $a_{\underline{a}}$ is the gluon, $\psi^{\alpha j}$ the gluino and $f_{\underline{ab}} = \partial_{\underline{a}} a_{\underline{b}} - \partial_{\underline{b}} a_{\underline{a}}$ the gluon field-strength. Note further that the first component of $A_{\alpha j}$ can be gauged away.

Appendix L

$\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$

The Lie superalgebra $\mathfrak{g}$ of $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ contains 12 bosonic and 16 fermionic generators $T_{\underline{A}}$ where $\underline{A} = \{[\underline{ab}], \alpha j, \underline{a}, \widehat{\alpha} j\}$. The index $\underline{a}$ ranges from $\{0 \text{ to } 5\}$, the $\mathrm{SU}(4)$ indices α and $\widehat{\alpha}$ range from $\{1 \text{ to } 4\}$, $j = \{1,2\}$ and $[\underline{ab}] = \{[ab], [a'b']\}$ with $a = \{0,1,2\}$ and $a' = \{3,4,5\}$.

Beyond that, the Lie superalgebra $\mathfrak{g}$ has a $\mathbb{Z}_4$-automorphism [91],[1] which means that it can be decomposed as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3\,, \tag{L.1}$$

where

$$T_{[\underline{ab}]} \in \mathfrak{g}_0\,, \qquad T_{\alpha j} \in \mathfrak{g}_1\,, \qquad T_{\underline{a}} \in \mathfrak{g}_2\,, \qquad T_{\widehat{\beta} k} \in \mathfrak{g}_3\,, \tag{L.2}$$

and, in turn, we have that

$$[\mathfrak{g}_r, \mathfrak{g}_s\} = \mathfrak{g}_{r+s} \pmod 4\,. \tag{L.3}$$

Note that this property is manifest in the structure constants (4.113). The supertrace over the generators must also be $\mathbb{Z}_4$-invariant, so that

$$\mathrm{sTr}(\mathfrak{g}_r \mathfrak{g}_s) = 0 \quad \text{unless} \quad r + s = 0 \pmod 4\,, \tag{L.4}$$

where we are denoting the supertrace over the Lie superalgebra by $\mathrm{sTr}(\ldots)$.

The structure constants (4.113) of the $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ Lie superalge-

[1] The supergroup properties presented in this section also hold for the super-coset descriptions of $\mathrm{AdS}_2 \times \mathrm{S}^2$ and $\mathrm{AdS}_5 \times \mathrm{S}^5$ backgrounds as well [91].

bra satisfy the super-Jacobi identities

$$(-)^{|\underline{A}||\underline{C}|}[T_{\underline{A}},[T_{\underline{B}},T_{\underline{C}}\}\}+(-)^{|\underline{A}||\underline{B}|}[T_{\underline{B}},[T_{\underline{C}},T_{\underline{A}}\}\}+(-)^{|\underline{C}||\underline{B}|}[T_{\underline{C}},[T_{\underline{A}},T_{\underline{B}}\}\}$$
$$=-\left((-)^{|\underline{A}||\underline{C}|}f_{\underline{BC}}{}^{\underline{D}}f_{\underline{AD}}{}^{\underline{E}}+(-)^{|\underline{A}||\underline{B}|}f_{\underline{CA}}{}^{\underline{D}}f_{\underline{BD}}{}^{\underline{E}}+(-)^{|\underline{C}||\underline{B}|}f_{\underline{AB}}{}^{\underline{D}}f_{\underline{CD}}{}^{\underline{E}}\right)T_{\underline{E}}=0\,, \tag{L.5}$$

where $|\underline{A}| = 0$ if it corresponds to a bosonic and $|\underline{A}| = 1$ if it corresponds to a fermionic indice.

The supertrace can be used to further relate the structure constants of the supergroup with the help of the following identity

$$\mathrm{sTr}\left([T_{\underline{A}},T_{\underline{B}}\}T_{\underline{C}}\right)=\mathrm{sTr}\left(T_{\underline{A}}[T_{\underline{B}},T_{\underline{C}}\}\right)\quad\Rightarrow\quad f_{\underline{AB}}{}^{\underline{D}}\eta_{\underline{DC}}=\eta_{\underline{AD}}f_{\underline{BC}}{}^{\underline{D}}\,, \tag{L.6}$$

where we defined the $\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)$ metric

$$\mathrm{sTr}(T_{\underline{A}}T_{\underline{B}})=\eta_{\underline{AB}}\,. \tag{L.7}$$

In our conventions, some important properties of the metric are

$$\eta^{\underline{AB}}\eta_{\underline{BC}}=\delta^{\underline{A}}_{\underline{C}}\,, \tag{L.8a}$$
$$\eta_{\underline{AB}}=(-)^{|\underline{A}||\underline{B}|}\eta_{BA}\,, \tag{L.8b}$$
$$X^{\underline{A}}=\eta^{\underline{AB}}X_{\underline{B}}\,, \tag{L.8c}$$
$$X_{\underline{A}}=\eta_{\underline{AB}}X^{\underline{B}}\,, \tag{L.8d}$$
$$f_{\underline{A}}{}^{\underline{BC}}=\eta^{\underline{BD}}f_{\underline{AD}}{}^{\underline{C}}\,, \tag{L.8e}$$

for X an element of the Lie superalgebra. With the help of $\eta_{\underline{AB}}$, one can define the structure constants with all indices down $f_{\underline{ABC}}$. Under permutation of the indices, they satisfy

$$f_{\underline{ABC}}=\eta_{\underline{CD}}f_{\underline{AB}}{}^{\underline{D}}\,, \tag{L.9a}$$
$$f_{\underline{ABC}}=-(-)^{|\underline{A}||\underline{B}|}f_{\underline{BAC}}\,, \tag{L.9b}$$
$$f_{\underline{ABC}}=-(-)^{|\underline{A}||\underline{C}|}f_{\underline{CBA}}\,, \tag{L.9c}$$
$$f_{\underline{ABC}}=-(-)^{|\underline{A}||\underline{B}|+|\underline{A}||\underline{C}|+|\underline{B}||\underline{C}|}f_{\underline{ACB}}\,. \tag{L.9d}$$

Explicitly, the non-vanishing components of the $\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)$ metric are

$$\eta_{\underline{ab}} = \{\eta_{ab}\,, \eta_{a'b'}\} = \{\mathrm{diag}(-1,1,1), \mathrm{diag}(1,1,1)\}\,, \tag{L.10a}$$

$$\eta_{[\underline{ab}][\underline{cd}]} = \left\{\frac{1}{2}\eta_{a[c}\eta_{d]b}\,, -\frac{1}{2}\eta_{a'[c'}\eta_{d']b'}\right\}, \tag{L.10b}$$

$$\eta_{\alpha j\,\widehat{\beta}k} = \widehat{\delta}_{\alpha\widehat{\beta}}\epsilon_{jk}\,. \tag{L.10c}$$

Note that $\widehat{\delta}_{\alpha\widehat{\beta}} = 2\sqrt{2}(\sigma^{012})_{\alpha\widehat{\beta}}$, $\widehat{\delta}_{\alpha\widehat{\beta}} = \widehat{\delta}_{\widehat{\beta}\alpha}$ and the inverse components of the metric are defined according to

$$\widehat{\delta}_{\alpha\widehat{\beta}}\widehat{\delta}^{\widehat{\beta}\gamma} = \delta_{\alpha}^{\gamma}\,, \qquad \widehat{\delta}_{\widehat{\alpha}\beta}\widehat{\delta}^{\beta\widehat{\gamma}} = \delta_{\widehat{\alpha}}^{\widehat{\gamma}}\,, \tag{L.11a}$$

$$\eta^{\underline{ab}}\eta_{\underline{bc}} = \delta_{\underline{c}}^{\underline{a}}\,, \qquad \eta^{[\underline{ab}][\underline{ef}]}\eta_{[\underline{ef}][\underline{cd}]} = \frac{1}{2}\delta_{\underline{c}}^{[\underline{a}}\delta_{\underline{d}}^{\underline{b}]}\,. \tag{L.11b}$$

Furthermore, the sigma-matrices obey the relations

$$\sigma^{a}_{\alpha\beta} = \widehat{\delta}_{\alpha\widehat{\alpha}}\widehat{\delta}_{\beta\widehat{\beta}}\sigma^{a\widehat{\alpha}\widehat{\beta}}\,, \qquad \sigma^{a\alpha\beta} = \widehat{\delta}^{\alpha\widehat{\alpha}}\widehat{\delta}^{\beta\widehat{\beta}}\sigma^{a}_{\widehat{\alpha}\widehat{\beta}}\,, \tag{L.12a}$$

$$\sigma^{a'}_{\alpha\beta} = -\widehat{\delta}_{\widehat{\alpha}\alpha}\widehat{\delta}_{\widehat{\beta}\beta}\sigma^{a'\widehat{\alpha}\widehat{\beta}}\,, \qquad \sigma^{a'\alpha\beta} = -\widehat{\delta}^{\alpha\widehat{\alpha}}\widehat{\delta}^{\beta\widehat{\beta}}\sigma^{a'}_{\widehat{\alpha}\widehat{\beta}}\,. \tag{L.12b}$$

Appendix M

$d = 6\ \mathcal{N} = 2$ superfields

In terms of the bi-spinor superfield $A_{\alpha j\,\widehat{\beta}k}$, the linearized $d = 6\ \mathcal{N} = 2$ supergravity connections and field-strengths appearing in the massless integrated vertex operator of the Type IIB superstring (4.144) are

$$A_{\underline{a}\,\widehat{\gamma}l} = -\frac{i}{2}\epsilon^{jk}\sigma_{\underline{a}}^{\alpha\beta}D_{\alpha j}A_{\beta k\,\widehat{\gamma}l}\,, \tag{M.1a}$$

$$A_{\underline{a}\,\beta k} = \frac{i}{2}\epsilon^{jl}\sigma_{\underline{a}}^{\widehat{\alpha}\widehat{\gamma}}D_{\widehat{\alpha}j}A_{\beta k\,\widehat{\gamma}l}\,, \tag{M.1b}$$

$$E_{\widehat{\beta}k}{}^{\gamma l} = \frac{i}{3}\epsilon^{lj}\sigma^{\underline{a}\gamma\alpha}\left(D_{\alpha j}A_{\underline{a}\,\widehat{\beta}k} - \partial_{\underline{a}}A_{\alpha j\,\widehat{\beta}k}\right), \tag{M.1c}$$

$$E_{\beta k}{}^{\widehat{\gamma}l} = \frac{i}{3}\epsilon^{lj}\sigma^{\underline{a}\widehat{\gamma}\widehat{\alpha}}\left(D_{\widehat{\alpha}j}A_{\underline{a}\,\beta k} + \partial_{\underline{a}}A_{\beta k\,\widehat{\alpha}j}\right), \tag{M.1d}$$

$$\begin{aligned} A_{\underline{ab}} &= -\frac{i}{2}\epsilon^{jk}\sigma_{\underline{b}}^{\alpha\beta}D_{\alpha j}A_{\underline{a}\,\beta k} \\ &= -\frac{i}{2}\epsilon^{jk}\sigma_{\underline{a}}^{\widehat{\alpha}\widehat{\beta}}D_{\widehat{\alpha}j}A_{\underline{b}\,\widehat{\beta}k}\,, \end{aligned} \tag{M.1e}$$

$$\begin{aligned} E_{\underline{a}}{}^{\beta k} &= \frac{i}{3}\epsilon^{kj}\sigma^{\underline{b}\beta\alpha}\left(\partial_{\underline{b}}A_{\underline{a}\,\alpha j} - D_{\alpha j}A_{\underline{ab}}\right) \\ &= -\frac{i}{2}\epsilon^{jl}\sigma_{\underline{a}}^{\widehat{\alpha}\widehat{\gamma}}D_{\widehat{\alpha}j}E_{\widehat{\gamma}l}{}^{\beta k}\,, \end{aligned} \tag{M.1f}$$

$$\begin{aligned} E_{\underline{b}}{}^{\widehat{\beta}k} &= \frac{i}{3}\epsilon^{kj}\sigma^{\underline{a}\widehat{\beta}\widehat{\alpha}}\left(\partial_{\underline{a}}A_{\underline{b}\,\widehat{\alpha}j} - D_{\widehat{\alpha}j}A_{\underline{ab}}\right) \\ &= -\frac{i}{2}\epsilon^{jl}\sigma_{\underline{b}}^{\alpha\gamma}D_{\gamma l}E_{\alpha j}{}^{\widehat{\beta}k}\,, \end{aligned} \tag{M.1g}$$

$$\begin{aligned} F^{\beta k\,\widehat{\gamma}l} &= -\frac{i}{3}\epsilon^{lj}\sigma^{\underline{a}\widehat{\gamma}\widehat{\alpha}}\left(D_{\widehat{\alpha}j}E_{\underline{a}}{}^{\beta k} - \partial_{\underline{a}}E_{\widehat{\alpha}j}{}^{\beta k}\right) \\ &= \frac{i}{3}\epsilon^{kj}\sigma^{\underline{a}\beta\alpha}\left(D_{\alpha j}E_{\underline{a}}{}^{\widehat{\gamma}l} - \partial_{\underline{a}}E_{\alpha j}{}^{\widehat{\gamma}l}\right), \end{aligned} \tag{M.1h}$$

$$\Omega_{\underline{abc}} = \frac{i}{2}(\sigma_{\underline{bc}})^{\alpha}{}_{\beta}D_{\alpha j}E_{\underline{a}}{}^{\beta j}\,, \tag{M.1i}$$

$$\widehat{\Omega}_{\underline{abc}} = \frac{i}{2}(\sigma_{\underline{bc}})^{\widehat{\alpha}}{}_{\widehat{\beta}}D_{\widehat{\alpha}j}E_{\underline{a}}{}^{\widehat{\beta}j}\,. \tag{M.1j}$$

Let us add to the integrated vertex operator (4.144) the remaining terms not containing the chiral bosons

$$
\begin{aligned}
W_{\text{SG}} = {} & \overline{\partial}\widehat{\theta}^{\widehat{\beta}k}\partial\theta^{\alpha j}A_{\alpha j\,\widehat{\beta}k} + \partial\theta^{\alpha j}\overline{\Pi}^{\underline{a}}A_{\underline{a}\,\alpha j} + \overline{\partial}\widehat{\theta}^{\widehat{\beta}k}\Pi^{\underline{a}}A_{\underline{a}\,\widehat{\beta}k} + \Pi^{\underline{b}}\overline{\Pi}^{\underline{a}}A_{\underline{ab}} + d_{\alpha j}\overline{\partial}\widehat{\theta}^{\widehat{\beta}k}E_{\widehat{\beta}k}{}^{\alpha j} \\
& + d_{\alpha j}\overline{\Pi}^{\underline{a}}E_{\underline{a}}{}^{\alpha j} + \widehat{d}_{\widehat{\beta}k}\partial\theta^{\alpha j}E_{\alpha j}{}^{\widehat{\beta}k} + \widehat{d}_{\widehat{\beta}k}\Pi^{\underline{a}}E_{\underline{a}}{}^{\widehat{\beta}k} + d_{\alpha j}\widehat{d}_{\widehat{\beta}k}F^{\alpha j\,\widehat{\beta}k} \\
& - \frac{i}{2}N_{\underline{ab}}\left(\overline{\partial}\widehat{\theta}^{\widehat{\beta}k}\Omega_{\widehat{\beta}k}{}^{\underline{ab}} + \overline{\Pi}^{\underline{c}}\Omega_{\underline{c}}{}^{\underline{ab}}\right) - \frac{i}{2}\widehat{N}_{\underline{ab}}\left(\partial\theta^{\alpha j}\widehat{\Omega}_{\alpha j}{}^{\underline{ab}} + \Pi^{\underline{c}}\widehat{\Omega}_{\underline{c}}{}^{\underline{ab}}\right) \\
& - \frac{i}{2}N_{\underline{ab}}\widehat{d}_{\widehat{\beta}k}C^{\widehat{\beta}k\,\underline{ab}} - \frac{i}{2}\widehat{N}_{\underline{ab}}d_{\alpha j}\widehat{C}^{\alpha j\,\underline{ab}} - \frac{1}{4}R^{\underline{abcd}}N_{\underline{cd}}\widehat{N}_{\underline{ab}} + (\dots)\,,
\end{aligned} \tag{M.2}
$$

where in $(\dots)$ we gathered all terms proportional to the $\{\rho,\sigma\}$-ghosts and

$$
\{\Omega_{\widehat{\beta}k}{}^{\underline{ab}}, \widehat{\Omega}_{\alpha j}{}^{\underline{ab}}, C^{\widehat{\beta}k\,\underline{ab}}, \widehat{C}^{\alpha j\,\underline{ab}}, R^{\underline{abcd}}\}\,, \tag{M.3}
$$

are superfields functions of the zero-modes of $\{x^{\underline{a}}, \theta^{\alpha j}, \widehat{\theta}^{\widehat{\alpha}j}\}$. From BRST invariance of the integrated vertex, one then obtains that the additional superfields in (M.2) are related to those in (4.144) by the following equations

$$
\begin{aligned}
D_{\alpha j}E_{\widehat{\gamma}l}{}^{\beta k} - \frac{i}{2}\delta_j^k(\sigma_{\underline{ab}})^{\beta}{}_{\alpha}\Omega_{\widehat{\gamma}l}{}^{\underline{ab}} &= 0\,, && \text{(M.4a)}\\
D_{\widehat{\alpha}j}E_{\gamma l}{}^{\widehat{\beta}k} - \frac{i}{2}\delta_j^k(\sigma_{\underline{ab}})^{\widehat{\beta}}{}_{\widehat{\alpha}}\Omega_{\gamma l}{}^{\underline{ab}} &= 0\,, && \text{(M.4b)}\\
D_{\alpha j}F^{\beta k\,\widehat{\gamma}l} - \frac{i}{2}\delta_j^k(\sigma_{\underline{ab}})^{\beta}{}_{\alpha}C^{\widehat{\gamma}l\,\underline{ab}} &= 0\,, && \text{(M.4c)}\\
D_{\widehat{\alpha}j}F^{\beta k\,\widehat{\gamma}l} + \frac{i}{2}\delta_j^l(\sigma_{\underline{ab}})^{\widehat{\gamma}}{}_{\widehat{\alpha}}\widehat{C}^{\beta k\,\underline{ab}} &= 0\,, && \text{(M.4d)}\\
D_{\alpha j}\widehat{C}^{\beta k\,\underline{ab}} + \frac{i}{2}\delta_j^k(\sigma_{\underline{cd}})^{\beta}{}_{\alpha}R^{\underline{abcd}} &= 0\,, && \text{(M.4e)}\\
D_{\widehat{\alpha}j}C^{\widehat{\beta}k\,\underline{ab}} + \frac{i}{2}\delta_j^k(\sigma_{\underline{cd}})^{\widehat{\beta}}{}_{\widehat{\alpha}}R^{\underline{abcd}} &= 0\,, && \text{(M.4f)}\\
(\sigma_{\underline{abc}})^{\alpha\gamma}D_{\alpha j}C^{\widehat{\beta}k\,\delta}{}_{\gamma} &= 0\,, && \text{(M.4g)}\\
(\sigma_{\underline{abc}})^{\widehat{\alpha}\widehat{\gamma}}\widehat{D}_{\widehat{\alpha}j}\widehat{C}^{\beta k\,\widehat{\delta}}{}_{\widehat{\gamma}} &= 0\,, && \text{(M.4h)}
\end{aligned}
$$

where $C^{\widehat{\beta}k\,\delta}{}_{\gamma} = (\sigma_{\underline{ab}})^{\delta}{}_{\gamma}C^{\widehat{\beta}k\,\underline{ab}}$ and $\widehat{C}^{\beta k\,\widehat{\delta}}{}_{\widehat{\gamma}} = (\sigma_{\underline{ab}})^{\widehat{\delta}}{}_{\widehat{\gamma}}\widehat{C}^{\beta k\,\underline{ab}}$.

We also have that

$$
\begin{aligned}
D_{\widehat{\alpha}j}\Omega_{\widehat{\beta}k}{}^{\underline{ab}} + D_{\widehat{\beta}k}\Omega_{\widehat{\alpha}j}{}^{\underline{ab}} + i\epsilon_{jk}\sigma^{\underline{c}}_{\widehat{\alpha}\widehat{\beta}}\Omega_{\underline{c}}{}^{\underline{ab}} &= 0\,, && \text{(M.5a)}\\
D_{\alpha j}\widehat{\Omega}_{\beta k}{}^{\underline{ab}} + D_{\beta k}\widehat{\Omega}_{\alpha j}{}^{\underline{ab}} + i\epsilon_{jk}\sigma^{\underline{c}}_{\alpha\beta}\widehat{\Omega}_{\underline{c}}{}^{\underline{ab}} &= 0\,, && \text{(M.5b)}\\
D_{\widehat{\alpha}j}\Omega_{\underline{c}}{}^{\underline{ab}} - \partial_{\underline{c}}\Omega_{\widehat{\alpha}j}{}^{\underline{ab}} + i\epsilon_{jk}\sigma_{\underline{c}\widehat{\alpha}\widehat{\beta}}C^{\widehat{\beta}k\,\underline{ab}} &= 0\,, && \text{(M.5c)}
\end{aligned}
$$

$$D_{\alpha j}\widehat{\Omega}_{\underline{c}}{}^{\underline{ab}} - \partial_{\underline{c}}\widehat{\Omega}_{\alpha j}{}^{\underline{ab}} + i\epsilon_{jk}\sigma_{\underline{c}\alpha\beta}\widehat{C}^{\beta k\,\underline{ab}} = 0\,, \tag{M.5d}$$

and

$$(\sigma^{\underline{abc}})^{\alpha\delta}D_{\alpha j}\Omega_{\widehat{\beta}k\,\delta}{}^{\gamma} = 0\,, \tag{M.6a}$$
$$(\sigma^{\underline{abc}})^{\widehat{\alpha}\widehat{\delta}}D_{\widehat{\alpha}j}\widehat{\Omega}_{\beta k\,\widehat{\delta}}{}^{\widehat{\gamma}} = 0\,, \tag{M.6b}$$
$$(\sigma^{\underline{abc}})^{\alpha\delta}D_{\alpha j}\Omega_{\underline{c}\,\delta}{}^{\gamma} = 0\,, \tag{M.6c}$$
$$(\sigma^{\underline{abc}})^{\widehat{\alpha}\widehat{\delta}}D_{\widehat{\alpha}j}\widehat{\Omega}_{\underline{c}\,\widehat{\delta}}{}^{\widehat{\gamma}} = 0\,, \tag{M.6d}$$
$$(\sigma^{\underline{efg}})^{\alpha\delta}(\sigma_{\underline{ab}})^{\gamma}{}_{\delta}D_{\alpha j}R^{\underline{abcd}} = 0\,, \tag{M.6e}$$
$$(\sigma^{\underline{efg}})^{\widehat{\alpha}\widehat{\delta}}(\sigma_{\underline{ab}})^{\widehat{\gamma}}{}_{\widehat{\delta}}D_{\widehat{\alpha}j}R^{\underline{abcd}} = 0\,, \tag{M.6f}$$

where $\Omega_{\widehat{\beta}k\,\delta}{}^{\gamma} = \Omega_{\widehat{\beta}k}{}^{\underline{ab}}(\sigma_{\underline{ab}})_{\delta}{}^{\gamma}$, $\widehat{\Omega}_{\beta k\,\widehat{\delta}}{}^{\widehat{\gamma}} = \widehat{\Omega}_{\widehat{\beta}k}{}^{\underline{ab}}(\sigma_{\underline{ab}})_{\widehat{\delta}}{}^{\widehat{\gamma}}$, $\Omega_{\underline{c}\,\delta}{}^{\gamma} = \Omega_{\underline{c}}{}^{\underline{ab}}(\sigma_{\underline{ab}})_{\delta}{}^{\gamma}$ and $\widehat{\Omega}_{\underline{c}\,\widehat{\delta}}{}^{\widehat{\gamma}} = \Omega_{\underline{c}}{}^{\underline{ab}}(\sigma_{\underline{ab}})_{\widehat{\delta}}{}^{\widehat{\gamma}}$.

Furthermore, from eqs. (M.4), we can write

$$D_{\widehat{\alpha}j}D_{\delta l}F^{\gamma l\,\widehat{\beta}k} - \frac{1}{2}\delta_j^k(\sigma_{\underline{ab}})^{\gamma}{}_{\delta}(\sigma_{\underline{cd}})^{\widehat{\beta}}{}_{\widehat{\alpha}}R^{\underline{abcd}} = 0\,, \tag{M.7a}$$
$$D_{\alpha j}D_{\widehat{\delta}l}F^{\beta k\,\widehat{\gamma}l} + \frac{1}{2}\delta_j^k(\sigma_{\underline{ab}})^{\widehat{\gamma}}{}_{\widehat{\delta}}(\sigma_{\underline{cd}})^{\beta}{}_{\alpha}R^{\underline{abcd}} = 0\,, \tag{M.7b}$$

and

$$(\sigma_{\underline{abc}})^{\alpha\gamma}D_{\alpha j}D_{\gamma l}F^{\delta l\,\widehat{\beta}k} = 0\,, \tag{M.8a}$$
$$(\sigma_{\underline{abc}})^{\widehat{\alpha}\widehat{\gamma}}D_{\widehat{\alpha}j}D_{\widehat{\gamma}l}F^{\beta k\,\widehat{\delta}l} = 0\,. \tag{M.8b}$$

Appendix N

Background field expansion

After plugging (4.158) into $\mathcal{H}_{NS}$ defined by (4.127a) and only keeping terms quadratic in the fluctuations, using the Maurer-Cartan eqs. (4.119) and $\nabla^2 X^A = X^B R_B{}^A$, the three-dimensional integral over $\mathcal{H}_{NS}$ can be written as a two-dimensional integral over the one-forms $J^{\underline{A}}$, which is given by

$$\begin{aligned}
&-\frac{i}{f^2}\int_{\mathcal{B}} \mathcal{H}_{NS} \\
&\quad = -i\int\Big[-\frac{1}{2}J^{\underline{c}}X^{\underline{b}}\nabla X^{\underline{a}}H_{\underline{abc}} + \frac{1}{2}J^{\underline{a}}\Big(-X^{\widehat{\beta}k}\nabla X^{\alpha j} + \nabla X^{\widehat{\beta}k}X^{\alpha j}\Big)H_{\alpha j\,\widehat{\beta}k\,\underline{a}} \\
&\quad -\frac{1}{2}\Big(J^{\widehat{\beta}k}X^{\underline{a}}\nabla X^{\alpha j} + J^{\alpha j}X^{\underline{a}}\nabla X^{\widehat{\beta}k}\Big)H_{\alpha j\,\widehat{\beta}k\,\underline{a}} - \frac{3}{2}\Big(J^{\widehat{\beta}k}\nabla X^{\underline{a}}X^{\alpha j} \\
&\quad + J^{\alpha j}\nabla X^{\underline{a}}X^{\widehat{\beta}k}\Big)H_{\alpha j\,\widehat{\beta}k\,\underline{a}} + \frac{1}{4}\Big(J^{\gamma l}J^{\delta m}if_{\delta m\,\gamma l}{}^{\underline{a}} - J^{\widehat{\gamma}l}J^{\widehat{\delta}m}if_{\widehat{\delta}m\,\widehat{\gamma}l}{}^{\underline{a}}\Big)X^{\widehat{\beta}k}X^{\alpha j}H_{\alpha j\,\widehat{\beta}k\,\underline{a}} \\
&\quad + \frac{1}{2}\Big(J^{\gamma l}J^{\alpha j}if_{\gamma l\,\underline{b}}{}^{\widehat{\beta}k} + J^{\widehat{\gamma}l}J^{\widehat{\beta}k}if_{\widehat{\gamma}l\,\underline{b}}{}^{\alpha j}\Big)X^{\underline{b}}X^{\underline{a}}H_{\alpha j\,\widehat{\beta}k\,\underline{a}} + \Big(J^{\underline{b}}J^{\gamma l}X^{\underline{a}}X^{\alpha j}if_{\gamma l\,\underline{a}}{}^{\widehat{\beta}k} \\
&\quad + J^{\underline{b}}J^{\widehat{\gamma}l}X^{\underline{a}}X^{\widehat{\beta}k}if_{\widehat{\gamma}l\,\underline{a}}{}^{\alpha j}\Big)H_{\alpha j\,\widehat{\beta}k\,\underline{a}}\Big]\,. \qquad \text{(N.1)}
\end{aligned}$$

Substituting (4.158) into (4.126), the terms independent of $\mathcal{H}_{\mathrm{NS}}$ and quadratic in the fluctuations X^A are given by

$$\begin{aligned}
&\frac{1}{f^2}\int d^2z\ \mathrm{sTr}\Big[\frac{1}{2}J^2\overline{J}^2 + \overline{J}^1J^3 - \frac{1}{4}(2-\tfrac{f_{RR}}{f})\Big(\overline{J}^1J^3 - J^1\overline{J}^3\Big) + w\overline{\nabla}\lambda + \widehat{w}\nabla\widehat{\lambda} - N\widehat{N}\Big] \\
&\quad = \int d^2z\ \mathrm{sTr}\Big[\frac{1}{2}\nabla X^2\overline{\nabla}X^2 + \overline{\nabla}X^1\nabla X^3 + \frac{1}{4}\tfrac{f_{RR}}{f}\Big(\nabla X^1\overline{\nabla}X^3 - \overline{\nabla}X^1\nabla X^3\Big) \\
&\quad + \frac{1}{4}(3-\tfrac{f_{RR}}{f})J^2[X^1,\overline{\nabla}X^1] + \frac{1}{4}(-1+\tfrac{f_{RR}}{f})J^2[X^3,\overline{\nabla}X^3] \\
&\quad + \frac{1}{4}(-1+\tfrac{f_{RR}}{f})\overline{J}^2[X^1,\nabla X^1] + \frac{1}{4}(3-\tfrac{f_{RR}}{f})\overline{J}^2[X^3,\nabla X^3] + \frac{1}{2}J^2[[\overline{J}^2,X^2],X^2] \\
&\quad + \frac{1}{4}(2-\tfrac{f_{RR}}{f})J^2[[\overline{J}^2,X^1],X^3] + \frac{1}{4}(-2+\tfrac{f_{RR}}{f})J^2[[\overline{J}^2,X^3],X^1] \\
&\quad + \frac{1}{8}(4-\tfrac{f_{RR}}{f})J^1[X^1,\overline{\nabla}X^2] + \frac{1}{8}(8-3\tfrac{f_{RR}}{f})J^1[X^2,\overline{\nabla}X^1]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{8}\big(4-\tfrac{f_{RR}}{f}\big)\bar{J}^3[X^3,\nabla X^2]+\frac{1}{8}\big(8-3\tfrac{f_{RR}}{f}\big)\bar{J}^3[X^2,\nabla X^3]\\
&+\frac{1}{2}\big(-2+\tfrac{f_{RR}}{f}\big)J^1[[\bar{J}^3,X^2],X^2]+\frac{1}{4}\tfrac{f_{RR}}{f}J^1[[\bar{J}^3,X^1],X^3]\\
&+\frac{1}{4}\big(-2+\tfrac{f_{RR}}{f}\big)J^1[[\bar{J}^3,X^3],X^1]+\frac{1}{8}\tfrac{f_{RR}}{f}\bar{J}^1[X^1,\nabla X^2]\\
&+\frac{1}{8}\big(3\tfrac{f_{RR}}{f}-4\big)\bar{J}^1[X^2,\nabla X^1]+\frac{1}{8}\tfrac{f_{RR}}{f}J^3[X^3,\overline{\nabla}X^2]+\frac{1}{8}\big(3\tfrac{f_{RR}}{f}-4\big)J^3[X^2,\overline{\nabla}X^3]\\
&+\frac{1}{2}\big(2-\tfrac{f_{RR}}{f}\big)\bar{J}^1[[J^2,X^2],X^2]+\frac{1}{4}\big(4-\tfrac{f_{RR}}{f}\big)\bar{J}^1[[J^3,X^1],X^3]\\
&+\frac{1}{4}\big(2-\tfrac{f_{RR}}{f}\big)\bar{J}^1[[J^3,X^3],X^1]+\frac{1}{2}N\Big([\overline{\nabla}X^1,X^3]+[\overline{\nabla}X^3,X^1]+[\overline{\nabla}X^2,X^2]\Big)\\
&+\frac{1}{2}\widehat{N}\big([\nabla X^1,X^3]+[\nabla X^3,X^1]+[\nabla X^2,X^2]\Big)\\
&+\big(\text{terms involving }\{X^1X^2,X^2X^3,X^1X^1,X^3X^3\}\text{ and no cov. derivatives}\big)\Big].
\end{aligned}
\tag{N.2}
$$

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