Last Lecture:

Comoving Units $r = R(t)u$

Dimensionless Scale Factor & Redshift

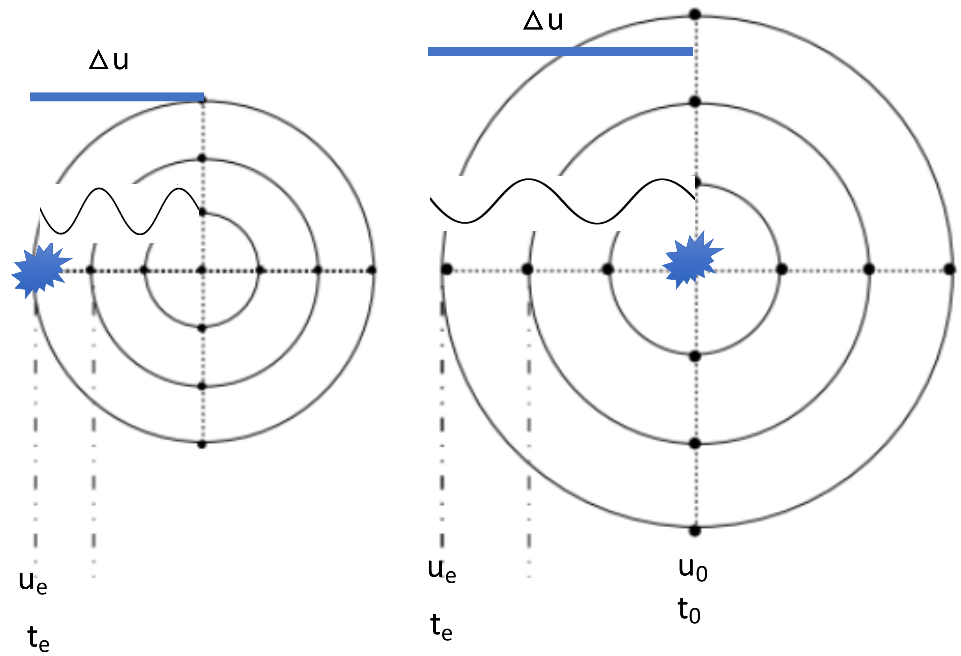
$$\boxed{R(t) = \frac{1}{1+z} = \lambda_e/\lambda_{obs}}$$

Hubble Law

$$\boxed{v_r = \dot{R}u = \frac{\dot{R}}{R}r}$$

Hubble Parameter

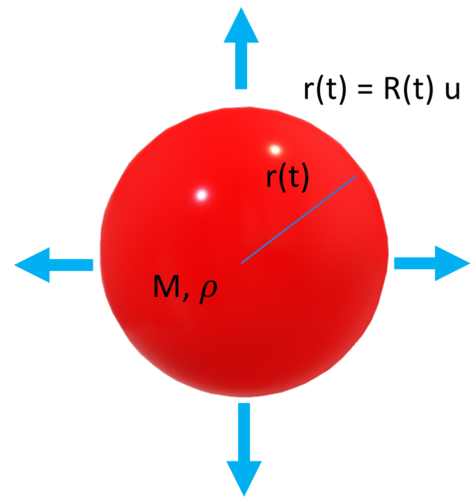
$$\boxed{H(t) = \frac{\dot{R}}{R}}$$



Ho = 100 *h* km/s/Mpc *h* = 0.7 at present day (z=0)

Ryden Chapters 2,3, (some) 4

Last Lecture



$$\frac{d^2r}{dt^2} = -\frac{GM(<r)}{r^2} = -\frac{4\pi G}{3}\rho(t)r$$

Changing to comoving units, r = R(t)u and integrating

First Friedmann Equation

$$\boxed{\dot{R}^2(t) = \frac{8\pi G}{3}\rho(t)R^2(t) - Kc^2}$$

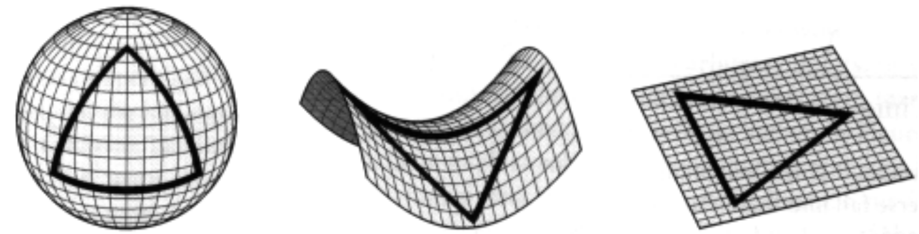
Modifying so that Density refers to *Energy Density*, ($E= mc^2$) : mass + radiation + dark energy

$$\boxed{\dot{R}^2(t) = \frac{8\pi G}{3c^2}\epsilon(t)R^2(t) - Kc^2}$$

Critical Density

The minimum density required to maintain a flat universe (K=0)

$$\dot{R}^2(t) = \frac{8\pi G}{3}\rho(t)R^2(t) - Kc^2$$



Set K=0 and rearrange the Friedmann equation for density (above), and substitute H(t)

$$\boxed{\frac{\epsilon(t)}{c^2} = \rho_{\rm crit}(t) \equiv \frac{3}{8\pi G}(\frac{\dot{R}}{R})^2 = \frac{3H^2(t)}{8\pi G}}$$

$\rho > \rho_{\rm crit}$ K > 0 Closed

$\rho < \rho_{\rm crit}$ K < 0 Open

H_0 = 100 *h* km/s/Mpc   *h* ~ 0.7 $\rho_{\rm crit}(t_o) = 3.3 \times 10^{11} h^2 \ M_\odot \ {\rm Mpc}^{-3}$

R_{200} = Edge of Dark Matter Halo = 200 x critical density

Density Parameter

We can express the abundance of various mass-energy components relative to the critical density as a "density parameter"

$$\boxed{\Omega(t) \equiv \rho(t)/\rho_{\rm crit}(t)}$$

We typically write Ω_0 to denote the value today.

Total Mass Density $\rho(t)$ = Mass + Radiation + Cosmological Constant (Dark Energy)

$$(\rho_m + \rho_{rad} + \rho_\Lambda)$$

Dividing by the critical density, yields the Total Density Parameter:

$$\boxed{\Omega(t) = \Omega_m(t) + \Omega_{rad}(t) + \Omega_\Lambda(t)}$$

$\Omega = 1$, K = 0 Flat

$\Omega > 0$, K > 0 Closed

$\Omega < 0$ K < 0 Open

The Cosmological Constant: Λ

Ryden Chapter 4, (some) 5, 6

Einstein originally concluded that the universe was static (before Hubble and expansion). For the universe to be static then acceleration must vanish everywhere.

$$a = -\Delta\Phi \quad = 0$$

Poissonʼs Equation: $\nabla^2\Phi = 4\pi G\rho \qquad \text{so } \rho = 0$

BUT clearly the density canʼt be 0. So Einstein introduced a “fudge” term to Newtonian physics:

$$\nabla^2\Phi + \Lambda = 4\pi G\rho$$

In Einsteinʼs Field equations, the correction has a factor of 8, not 4.

$$\nabla^2\Phi = 0 \text{ if } \boxed{\Lambda = 4\pi G\rho}$$

This is called the cosmological constant

$$\boxed{\rho_\Lambda = \frac{\Lambda}{8\pi G}}$$

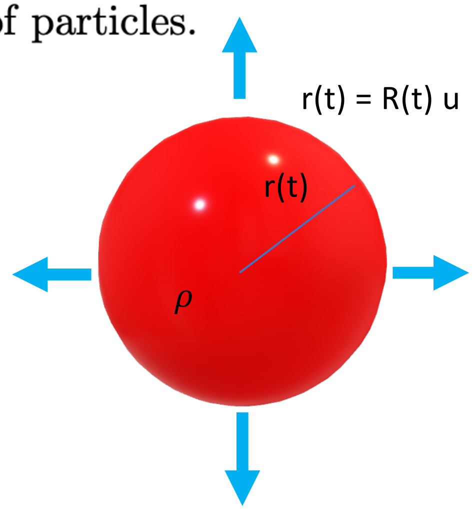
After Hubble showed that the universe was expanding, Einstein abandoned the term. But it turns out we need it to explain that the universeʼs expansion is *accelerating*

Fluid Equation: Evolution of Energy Density

First Law of Thermodynamics: $dE = -PdV + dQ$

The expansion of a homogeneous universe is *adiabatic*, as there is no place for "heat" to come from, and no "friction" to convert energy of bulk motion into random motions of particles.

$$dE + P\,dV = 0 \qquad \boxed{\frac{dE}{dt} + P\frac{dV}{dt} = 0}$$



1. dV/dt: Consider a spherical volume of r(t) = R(t)u $\quad V = 4/3\pi(Ru)^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi u^3 3R^2\frac{dR}{dt} = V3\frac{\dot{R}}{R}$$

2. dE/dt: The total energy is the energy density times the volume.

$$\frac{dE}{dt} = \frac{d(\epsilon V)}{dt} = \frac{d\epsilon}{dt}V + \epsilon\frac{dV}{dt} = \frac{d\epsilon}{dt}V + \epsilon V3\frac{\dot{R}}{R}$$

Fluid Equation: Evolution of Energy Density

$$\boxed{\frac{dE}{dt} + P\frac{dV}{dt} = 0}$$

$$\frac{d\epsilon}{dt}V + \epsilon V 3\frac{\dot{R}}{R} + PV3\frac{\dot{R}}{R} = 0$$

$$V(\frac{d\epsilon}{dt} + 3\frac{\dot{R}}{R}(\epsilon + P)) = 0$$

Or in terms of mass density $E = mc^2$

Fluid Equation: $$\boxed{\frac{d\epsilon}{dt} = -3\frac{\dot{R}}{R}(\epsilon + P)}$$ $$\boxed{\frac{d\rho}{dt} = -3\frac{\dot{R}}{R}(\rho + \frac{P}{c^2})}$$

The fluid equation describes evolution of density (in all forms) in an expanding universe

Equation of State

$$P = w\rho c^2 = w\epsilon$$

$$\boxed{\frac{d\rho}{dt} = -3\frac{\dot{R}}{R}(\rho + \frac{P}{c^2})}$$

The fluid equation then becomes: $$\frac{d\rho}{dt} = -3\frac{\dot{R}}{R}(\rho + \frac{w\rho c^2}{c^2}) = -3\frac{\dot{R}}{R}\rho(1+w)$$

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{R}}{R}$$

$$\frac{\epsilon}{\epsilon_0} = \boxed{\frac{\rho}{\rho_0} = \frac{R}{R_0}^{-3(1+w)}}$$

Relationship between density and scale factor

Recall, $R_0 = 1$ and

$$R = \frac{1}{1+z}$$

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\rho}{\rho_0} = (1+z)^{3(1+w)}}$$

Relationship between density and redshift

So, if you know the relevant equation of state for matter, radiation and dark energy, you can describe how the density of the universe evolves with redshift.

Eqn of state: $P = w\rho c^2 = w\epsilon$

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\rho}{\rho_0} = (1+z)^{3(1+w)}} \boxed{= \frac{R}{R_0}^{-3(1+w)}}$$

A) Non-Relativistic Particles
(Baryons, Matter)

$$\frac{P}{\epsilon} = \frac{mv_{th}^2}{mc^2} << 1 \qquad w = 0$$

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\rho}{\rho_0} = R^{-3} = (1+z)^3}$$

As you might expect, density evolves like the expansion of the volume.

If we know the density today, we can determine the density at any z

B) Relativistic Particles
(Photons, Radiation)

$$P = \frac{\epsilon}{3} \qquad w = 1/3$$

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\rho}{\rho_0} = R^{-4} = (1+z)^4}$$

Since λ∝R, the energy (h/λ) will decrease accordingly (1/R). Including the volume scaling R^-3 (number density of photons), we get ∝ R−4

C) Dark Energy

$$\boxed{\rho_\Lambda = \frac{\Lambda}{8\pi G}}$$

Consider a universe where the energy density was dominated by the cosmological constant (dark energy)

The energy density associated with the cosmological constant is → constant!

No redshift evolution!

From the fluid equation

$$\boxed{\frac{d\rho}{dt} = -3\frac{\dot{R}}{R}(\rho + \frac{P}{c^2})} = 0 \qquad P_\Lambda = -\rho_\Lambda c^2 \qquad P_\Lambda = -\epsilon_\Lambda \qquad w = -1$$

NEGATIVE pressure enables acceleration!

For a fluid to have negative pressure it must have **tension** — it takes work to expand the fluid, instead of taking work to compress it.

Expanding a **volume increases the energy of such a fluid** (since energy density is constant).

Why is the expansion of the universe accelerating?

To understand the rate of the change of the scale factor in terms of (mass) density and pressure.

Start with the

First Friedmann Equation

$$\boxed{\dot{R}^2(t) = \frac{8\pi G}{3c^2}\epsilon(t)R^2(t) - Kc^2}$$

Take the time derivative :

$$\frac{d}{dt}\dot{R}^2 = \frac{d}{dt}\left(\frac{8\pi G}{3c^2}\epsilon(t)R^2(t) - Kc^2\right)$$

$$2\dot{R}\ddot{R} = \frac{8\pi G}{3c^2}(\dot{\epsilon}R^2 + 2\epsilon R\dot{R})$$

Divide by $2\dot{R}R$

$$\frac{\ddot{R}}{R} = \frac{4\pi G}{3c^2}(\dot{\epsilon}\frac{R}{\dot{R}} + 2\epsilon)$$

Subbing in for $\dot{\epsilon}$ from fluid equation

$$\frac{\ddot{R}}{R} = \frac{4\pi G}{3c^2}(-3\frac{\dot{R}}{R}(\epsilon + P)\frac{R}{\dot{R}} + 2\epsilon)$$

Second Friedmann Equation

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3c^2}(\epsilon + 3P)}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3\frac{P}{c^2})$$

Why is the expansion of the universe accelerating?

Consider a universe where the energy density was dominated by the cosmological constant (dark energy)

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3c^2}(\epsilon + 3P)}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3c^2}(\epsilon_\Lambda - 3\epsilon_\Lambda) = +\frac{8\pi G}{3}\epsilon_\Lambda = +\frac{\Lambda}{3}$$

$$P_\Lambda = -\epsilon_\Lambda \qquad w = -1$$

This yields positive acceleration.

This is the universe we are currently experiencing, implying that dark energy must dominate the energy budget of the universe at the present day

Scale Factor as a function of time

How do we determine whether matter, radiation or dark energy is dominating the energy budget and therefore controlling the rate of expansion?

First Friedmann Equation

$$\boxed{\dot{R}^2(t) = \frac{8\pi G}{3}\rho(t)R^2(t) - Kc^2}$$

$$\left[\frac{\dot{R}^2}{R^2} - \frac{8\pi G}{3}(\rho_m + \rho_{rad} + \rho_\Lambda)\right]R^2 = -Kc^2$$

$$H(t) = \frac{\dot{R}}{R}$$

$$H(t)^2\left[1 - \frac{8\pi G}{3H(t)^2}(\rho_m + \rho_{rad} + \rho_\Lambda)\right] = -\frac{Kc^2}{R^2}$$

$$\rho_{\text{crit}}(t) = \frac{3H^2(t)}{8\pi G}$$

$$\boxed{H(t)^2\left[1 - (\Omega_m + \Omega_{rad} + \Omega_\Lambda)\right] = -\frac{Kc^2}{R^2}}$$

$$\Omega(t) \equiv \rho(t)/\rho_{\text{crit}}(t)$$

Density Parameters as a function of Time

A) Matter Density Parameter

Time evolution of matter density

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\rho}{\rho_0} = R^{-3} = (1+z)^3}$$

$$\Omega_m = \frac{\rho_m}{\rho_c}$$

$$= \frac{8\pi G}{3H(t)^2}\rho_m(t) = \frac{8\pi G}{3H(t)^2}\rho_{m0}(1+z)^3$$

$$\Omega_{m0} = \frac{8\pi G}{3H_o^2}\rho_{m0}$$

$$\boxed{\Omega_m(t) = \Omega_{m0}(1+z)^3\frac{H_o^2}{H(t)^2}}$$

Density Parameters as a function of Time

B) Radiation Density Parameter

Time evolution of radiation density

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\rho}{\rho_0} = R^{-4} = (1+z)^4}$$

$$\Omega_{rad}(t) = \frac{\rho_{rad}(t)}{\rho_c(t)}$$

$$= \frac{8\pi G}{3H(t)^2}\rho_{rad}(t) = \frac{8\pi G}{3H(t)^2}\rho_{rad0}(1+z)^4$$

$$\boxed{\Omega_{rad}(t) = \Omega_{rad0}(1+z)^4 \frac{H_0^2}{H(t)^2}}$$

Density Parameters as a function of Time

C) Dark Energy Density Parameter

$$\Omega_\Lambda(t) = \frac{\rho_\Lambda(t)}{\rho_c} \qquad \rho_\Lambda = \frac{\Lambda}{8\pi G} = \rho_{\Lambda 0}$$

No time evolution!

$$\boxed{\Omega_\Lambda(t) = \Omega_{\Lambda 0}\frac{H_0^2}{H(t)^2}}$$

Hubble Parameter as a function of time

$$\boxed{H(t)^2\left[1-(\Omega_m+\Omega_{rad}+\Omega_\Lambda)\right]=-\frac{Kc^2}{R^2}}$$

$$\Omega(t)=\Omega_m(t)+\Omega_{rad}(t)+\Omega_\Lambda(t)$$

$$H(t)^2(1-\Omega(t))R^2=-Kc^2$$

$$H(t)^2R(t)^2\left(1-\Omega_{m0}(1+z)^3\frac{H_0^2}{H(t)^2}-\Omega_{rad0}(1+z)^4\frac{H_0^2}{H(t)^2}-\Omega_{\Lambda 0}\frac{H_0}{H(t)^2}\right)=-Kc^2$$

$$=H_o^2(1-\Omega_0)R_0^2$$

$$\boxed{R(t)=\frac{1}{1+z}}$$

Dividing through by $H_0^2/H(t)^2$

$$\frac{H(t)^2}{H_0^2}-\Omega_{m0}(1+z)^3-\Omega_{rad0}(1+z)^4-\Omega_{\Lambda 0}=(1-\Omega_0)(1+z)^2$$

$$\boxed{H(t)^2=H_o^2\left[\Omega_{m,o}(1+z)^3+\Omega_{rad,o}(1+z)^4+\Omega_{\Lambda,o}+(1-\Omega_o)(1+z)^2\right]}$$

Hubble Parameter as a function of time

$$\boxed{H(t)^2 = H_o^2\left[\Omega_{m,o}(1+z)^3 + \Omega_{rad,o}(1+z)^4 + \Omega_{\Lambda,o} + (1-\Omega_o)(1+z)^2\right]}$$

Where $1 - \Omega_0 = \Omega_k$

Curvature Density Parameter

This equation describes the fractional rate of expansion of the universe as a function of time.

Where now every density parameter is defined in terms of their present day values.

Benchmark Cosmology:

2015 Planck results (Table 4 column 2)

$$\Omega_{m0} = 0.308 \pm 0.012 \qquad \Omega_{\Lambda 0} = 0.692 \pm 0.012 \qquad \Omega_{\rm rad0} = 8.24 \times 10^{-5} \qquad H_o = 67.81 \pm 0.92$$