Hubble Parameter as a function of time

$$\boxed{\dot{R}^2(t) \;=\; \frac{8\pi G}{3}\rho(t)R^2(t) \; -Kc^2}$$

Friedmann's First Equation

$$\boxed{H(t)^2\left[1-(\Omega_m+\Omega_{rad}+\Omega_\Lambda)\right] = -\frac{Kc^2}{R^2}}$$

$$\Omega(t) = \Omega_m(t) + \Omega_{rad}(t) + \Omega_\Lambda(t)$$

$$H(t)^2(1-\Omega(t))R^2 = -Kc^2 \; = H_o^2(1-\Omega_0)R_0^2$$

Density Parameters as a function of time

$$\Omega(t) = \Omega_m(t) + \Omega_{rad}(t) + \Omega_\Lambda(t)$$

$$\boxed{\Omega_m(t) = \Omega_{m0}(1+z)^3 \frac{H_o^2}{H(t)^2}}$$

Baryons, Dark Matter

$$\boxed{\Omega_{rad}(t) = \Omega_{rad0}(1+z)^4 \frac{H_0^2}{H(t)^2}}$$

Photons, Neutrinos

$$\boxed{\Omega_\Lambda(t) = \Omega_{\Lambda 0} \frac{H_0^2}{H(t)^2}}$$

Dark Energy

Hubble Parameter as a function of time

$$\boxed{H(t)^2 = H_o^2\left[\Omega_{m,o}(1+z)^3 + \Omega_{rad,o}(1+z)^4 + \Omega_{\Lambda,o} + (1-\Omega_o)(1+z)^2\right]}$$

Where $1 - \Omega_0 = \Omega_k$

Curvature Density Parameter

This equation describes the fractional rate of expansion of the universe as a function of time.

Where now every density parameter is defined in terms of their present day values.

Benchmark Cosmology:

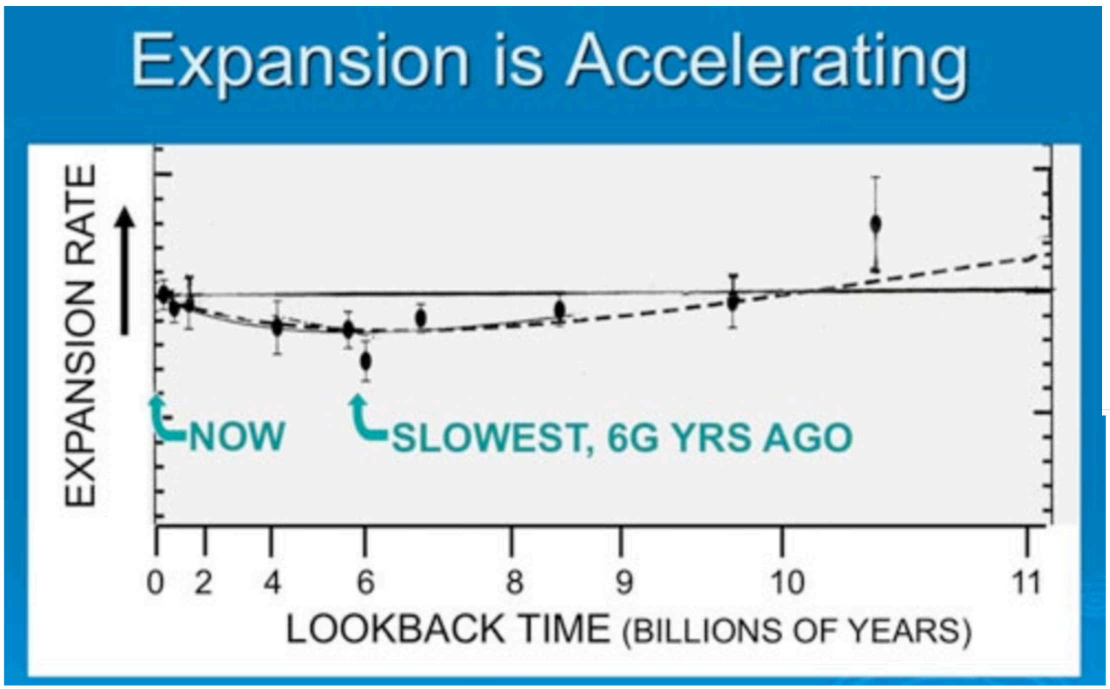
2015 Planck results (Table 4 column 2)

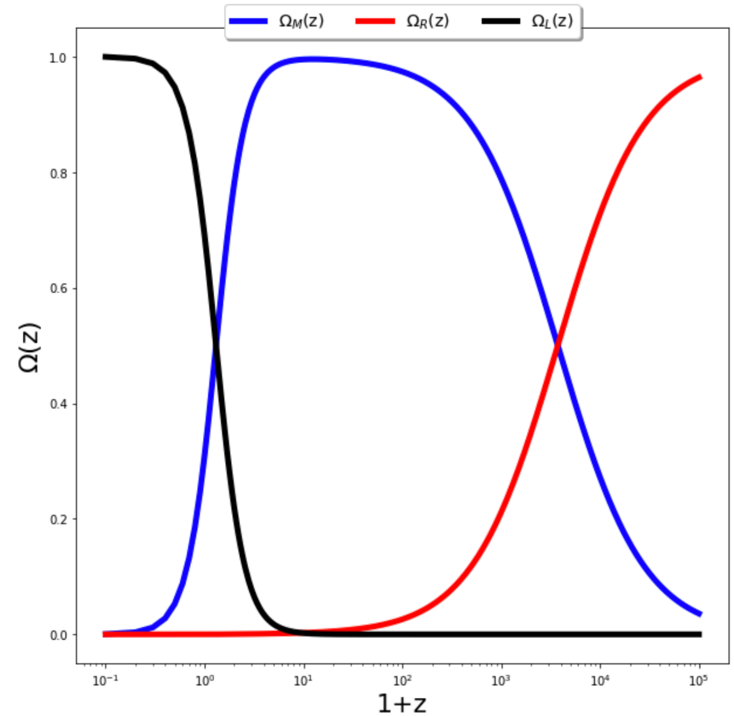
$\Omega_{m0} = 0.308 \pm 0.012$ $\Omega_{\Lambda 0} = 0.692 \pm 0.012$ $\Omega_{\mathrm{rad}0} = 8.24 \times 10^{-5}$ $H_o = 67.81 \pm 0.92$

In Class Lab 12 Discussion

Era of Dark Energy

Rate of Expansion (dR/dt) as a function of time





In Class Lab 12 Discussion

Matter-Radiation Equality

Setting $\Omega_m(t) = \Omega_{rad}(t)$

$$\Omega_{m0}(1+z)^3 \frac{H_0^2}{H(t)^2} = \Omega_{rad0}(1+z)^4 \frac{H_0^2}{H^2} \qquad (1+z) = \frac{\Omega_{m0}}{\Omega_{rad0}} = 0.308/8.24e-5 \sim 3700$$

(0.05 Myr after the big bang)

Era of Dark Energy

At some point the Cosmological Constant started to dominate (but it was pretty much negligible before then).

$$(1+z) = \frac{\Omega_\Lambda}{\Omega_m}^{1/3} = (0.698/0.308)^{1/3} = 1.3 \tag{2}$$

Look Back Time

Ryden Chapter 6

Because light travels at a finite speed, we see a younger cosmos as we look toward more distant galaxies at higher redshift.

If you observe a galaxy at redshift z, at what time (t_e) did those photons leave on their journey towards us?

$$\boxed{R(t) = \frac{1}{1+z} = \lambda_e/\lambda_{obs}}$$

$$1 + z = \frac{1}{R(t)}$$

$$\boxed{H(t) = \frac{\dot{R}}{R}}$$

$$\frac{dz}{dt} = -\frac{1}{R(t)^2}\frac{dR}{dt} = -\frac{1}{R(t)}H(z) = -(1+z)H(z)$$

$$-\int_{t_0}^{t_e} dt = t_0 - t_e = \boxed{\int_0^z \frac{1}{H(z)}\frac{dz'}{(1+z')} = t_L}$$

Look Back Time (Gyr ago) [inverse of Hubble Parameter)

Proper Distance:

Recall: $r = R(t)u$

Robertson Walker Metric

$$ds^2 = (cdt)^2 - R(t)^2\left[\left(\frac{du}{\sqrt{1-u^2K}}\right)^2 + (ud\theta)^2 + (usin\theta d\phi)^2\right]$$

Light rays travel paths with $ds^2 = 0$, called null geodesics. $c^2dt^2 = d\ell^2$

Light rays travel on radial paths, where $(\theta = 0,\ \phi = 0)$ $cdt = R(t)\dfrac{du}{\sqrt{1-Ku^2}}$

Proper Distance:

$$\boxed{c\int_{t_e}^{t_0}\frac{dt}{R(t)} = \int_{u_e}^{u_0}\frac{du}{\sqrt{1-ku^2}}}$$

Soln to RHS

$$= \begin{cases} |\kappa|^{-1/2}\sinh^{-1}\sqrt{|\kappa|}\,u & \text{if } \kappa < 0 \text{ (a negatively curved ‘hyperbolic’ universe)} \\ u & \text{if } \kappa = 0 \text{ (a spatially flat universe)} \\ |\kappa|^{-1/2}\sin^{-1}\sqrt{|\kappa}\,u & \text{if } \kappa > 0 \text{ (a positively curved ‘spherical’ universe)} \end{cases}$$

Comoving Radial Distance (K=0)

In a flat universe, K = 0, Moving at *c*, over a time interval Δt, light travels a proper distance:

$$cdt = R(t)\frac{du}{\sqrt{1-Ku^2}} = R(t)\Delta u$$

Comoving Radial Distance:

$$u = \int_{t_e}^{t_0} \frac{cdt}{R(t)}$$

Recall: $\frac{dz}{dt} = -(1+z)H(z)$

$$= \int_0^z \frac{c}{R(t)} \frac{dz'}{(1+z)H(z)}$$

$$R(t) = \frac{1}{1+z}$$

$$\boxed{D_C = c\int_0^{z_e} \frac{dz'}{H(z)}}$$

Proper Distance in terms of D_C

$$\boxed{D_C = c\int_0^{z_e} \frac{dz'}{H(z)}}$$

This is true regardless of the observer's redshift

Proper Distance at any z_o:
(Ruler Distance, K=0)

$$c\Delta t = R(t)u = R(t)D_c = \frac{D_c}{1+z_0}$$

Today, $z_0 = 0$ And $R(0) = 1$

$$c\Delta t = R(t_o)u = D_c$$

So the Proper Distance to an object from us TODAY IS the SAME as its Comoving Distance TODAY.
This is the **line of sight distance** to a galaxy at a given redshift at the present day.

I.e. This is the distance you would put into Hubble's Law

$$v = H(t_0)R(t_o)u = H(t_0)D_C$$

Horizon Distance: Size of the Observable Universe

- The Size of the observable universe is the proper distance traveled by a photon over the age of the universe.

Horizon Distance today = Comoving Radial Distance

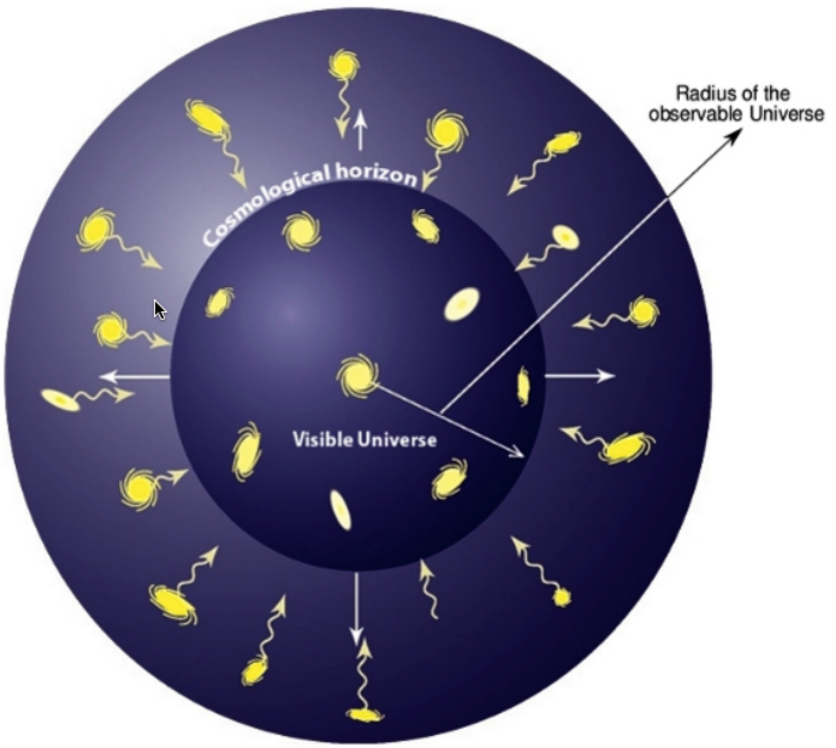
$$d_H = c\Delta t = R(t_o)u = D_c$$

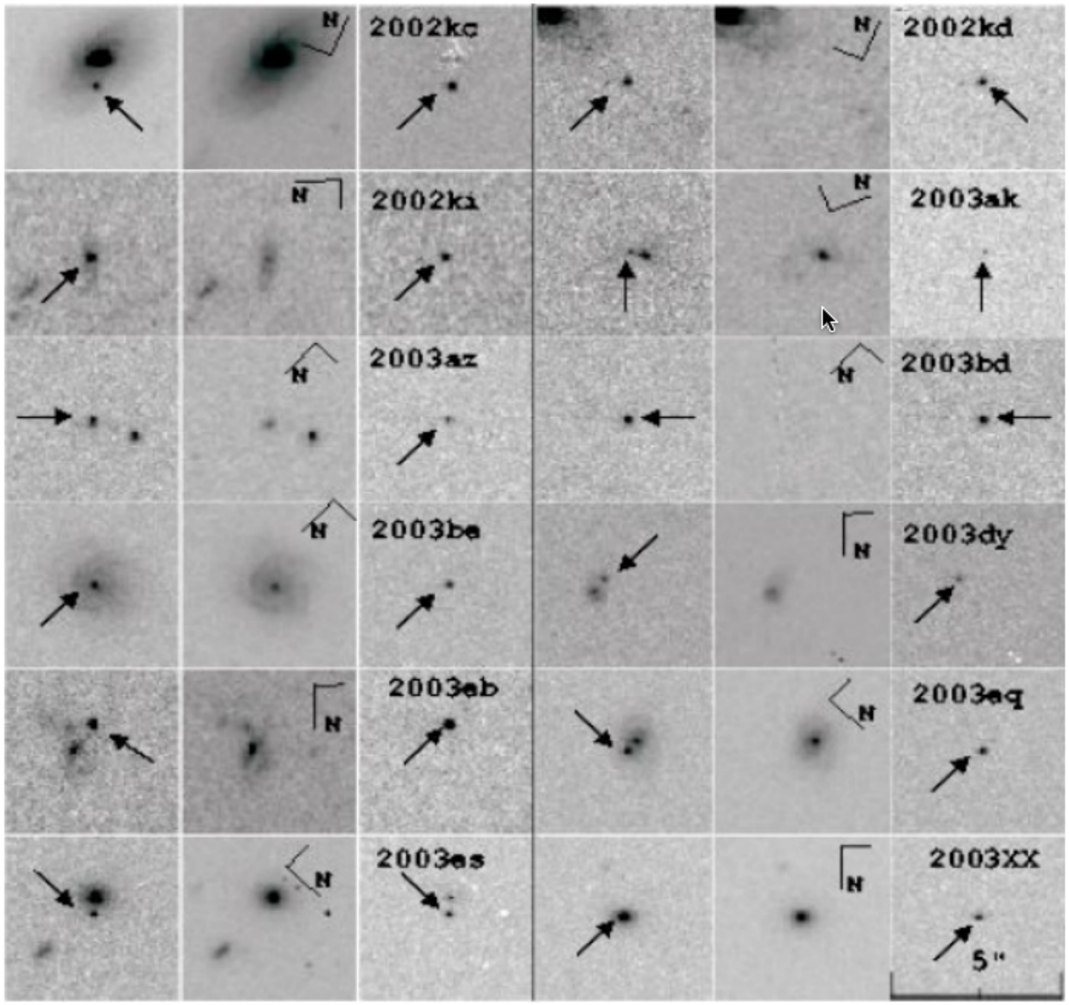
$$D_C = c\int_0^z \frac{dz'}{H(z)}$$ z is large

At an arbitrary redshift, however:

$$\boxed{d_H(t_{obs}) = c\Delta t = R(t_{obs})D_c = \frac{D_c}{1+z_{obs}}}$$

Proper Distance at an arbitrary observer distance





Supernovae in distant galaxies found by HST

How do we measure distances to standard candles in an expanding universe???

Luminosity Distance

$$F = \frac{L}{4\pi d^2}$$

Ryden Chapter 7

In an expanding universe, far away objects appear dimmer :

$$L_e = \frac{h\nu_e}{\Delta t_e} \qquad L_o = \frac{h\nu_o}{\Delta t_o}$$

Le is the intrinsic luminosity of the source

$$L_o = L_e \frac{\nu_o}{\nu_e}\frac{\Delta t_e}{\Delta t_o} = L_e R(t) R(t)$$

Energy per photon:

$$h\nu \propto 1/(1+\mathrm{z})$$

From redshift calculation

$$\frac{\Delta t_e}{R(t_e)} = \frac{\Delta t_0}{R(t_0)}$$

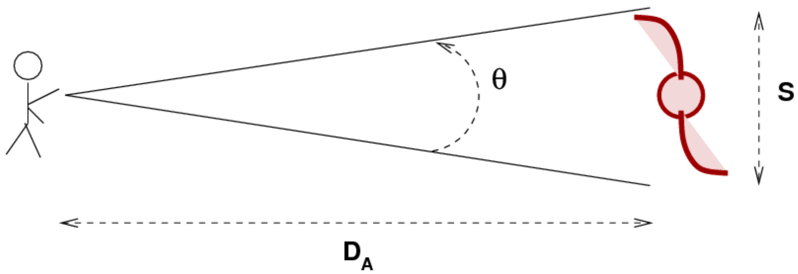
Changing to comoving coordinates:

$$F = \frac{L_\mathrm{e}}{4\pi \boxed{R^2 u^2} }\frac{1}{(1+z)}\frac{1}{(1+z)} \equiv \frac{L}{4\pi d_L^2}$$

$$\boxed{d_L = (1+z)R(t_0)u_e = c(1+z)\int_0^z \frac{dz'}{H(t)} = (1+z)D_C}$$

Luminosity Distance: how far an object of known luminosity L would have to be in Euclidean space so that we measure a total flux F.

Angular Diameter Distance



D_A is the distance to the source, such that it subtends the same angle it would have in Euclidean Space

$$D_A \equiv \frac{S}{\theta}$$

If K = 0

$$\theta\ [\mathrm{rad}] = \frac{S}{R(t_e)u_e} \equiv \frac{S}{D_A}$$

This is the Proper Distance at the time of emission – so that you see the correct angle

$$D_A = R(t_e)u_e = \frac{R(t_0)u_e}{1+z} = \frac{D_C}{(1+z)} = \frac{D_L}{(1+z)^2} = \frac{S}{\theta}$$

So, the size of a galaxy, or equivalently, the separation between two galaxies, that subtend angle Theta would be

$$S = \theta D_A = \frac{D_C}{1+z}\theta$$