Ridge, Suppose $n=2$, $p=2$,

$$x_{11} = x_{12}$$
$$x_{21} = x_{21}$$

$y_1 + y_2 = 0$, $x_{11} + x_{21} = 0$, $x_{12} + x_{22} = 0$

So, $\hat{\beta}_0 = 0$

(a) ridge optimization

$x_{11} = x_{12} = x_1$
$x_{21} = x_{22} = x_2$

minimizing $$(y_1 - \hat{\beta}_1 x_1 - \hat{\beta}_2 x_1)^2 + (y_2 - \hat{\beta}_1 x_2 - \hat{\beta}_2 x_2)^2 + \lambda(\hat{\beta}_1^2 + \hat{\beta}_2^2)$$

(b) if we'd like to minimize above problem, the need to keep

take first derivative wrt β_1 and β_2

wrt $\hat{\beta}_1$

$$2(y_1 - \hat{\beta}_1 x_1 - \hat{\beta}_2 x_1)\cdot(-x_1) + 2(y_2 - \hat{\beta}_1 x_2 - \hat{\beta}_2 x_2)\cdot(-x_2) + 2\lambda\hat{\beta}_1$$

$$= -2y_1x_1 + 2\hat{\beta}_1 x_1^2 + 2\hat{\beta}_2 x_1^2 - 2y_2x_2 + 2\hat{\beta}_1 x_2^2 + 2\hat{\beta}_2 x_2^2$$

$$+ 2\lambda\beta_1 \quad = 0$$

$$= -x_1y_1 + \hat{\beta}_1(x_1^2 + x_2^2 + \lambda) - y_2x_2$$

$$+ \hat{\beta}_2(x_1^2 + x_2^2) \quad = 0$$

$$\Rightarrow \quad \hat{\beta}_1(x_1^2 + x_2^2 + \lambda) + \hat{\beta}_2(x_1^2 + x_2^2) = x_2y_2 + x_1y_1 \cdots ①$$

wrt $\hat{\beta}_2$

$$\Rightarrow \quad \hat{\beta}_2(x_1^2 + x_2^2 + \lambda) + \hat{\beta}_1(x_1^2 + x_2^2) = x_2y_2 + x_1y_1 \cdots ②$$

① - ②

$$(\hat{\beta}_1 - \hat{\beta}_2)(x_1^2 + x_2^2 + \lambda) + (\hat{\beta}_2 - \hat{\beta}_1)(x_1^2 + x_2^2) = 0$$

Since $x_1^2 + x_2^2 \geq 0$

then $\hat{\beta}_1 - \hat{\beta}_2 = 0$ and $\hat{\beta}_2 - \hat{\beta} = 0$

then $\hat{\beta}_1 = \hat{\beta}_2$

(c) write out lasso optimation

minimizing $(y_1 - \hat{\beta}_1 x_1 - \hat{\beta}_2 x_1)^2$

$+ (y_2 - \hat{\beta}_1 x_2 - \hat{\beta}_2 x_2)^2$

$+ \lambda(|\hat{\beta}_1| + |\hat{\beta}_2|)$

(d) Assum true $\hat{\beta}_i = 5$ and $\hat{\beta}_1 + \hat{\beta}_2 = 5$.

then $\lambda(|\hat{\beta}_1| + |\hat{\beta}_2|)$

$$\begin{cases} \tilde{\beta}_1 = 0 & \tilde{\beta}_2 = 5 \\ \tilde{\beta}_1 = 1 & \tilde{\beta}_2 = 4 \\ \tilde{\beta}_1 = 2 & \tilde{\beta}_2 = 3 \end{cases}$$

$\Rightarrow$ indifferent for lasso problem.

So there're multiple solution