Waveguides: Numerical Homework 7

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All code used to produce the figures in this write-up can be found in this Github repository.

1 Deriving the equations for TE waves (exercises 1 and 2)

We can begin with the equations:
(i) $\tilde{E}(x, y, z, t) = \tilde{E_0} e^{i(kz-\omega t)}$
(ii) $\tilde{B}(x, y, z, t) = \tilde{B_0} e^{i(kz-\omega t)}$
Following Griffiths, we can solve Maxwell's equations to obtain the following equations. All of these equations are complex, but, like Griffiths, I'm going to be lazy with the complex notation since LaTeXis being weird about it for me.

$$E_x = \frac{i}{(\omega/c)^2 - k^2}(k\frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y}) \tag{1}$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}(k\frac{E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x}) \tag{2}$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2}(k\frac{\partial B_z}{\partial y} - \omega \frac{\partial E_z}{\partial x}) \tag{3}$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x}) \tag{4}$$

Starting from:

$$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2]E_z = 0 \tag{5}$$

$$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2]B_z = 0 \tag{6}$$

In the case of a TE wave, $E_z = 0$, so, using equation 6, we can solve for B_z. To solve the differential equation, we set $B_z = X(x)Y(y)$. This transforms equation 6 into:

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + [(\omega/c)^2 - k^2]XY = 0 \tag{7}$$

Diving by XY:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + [(\omega/c)^2 - k^2]XY = 0 \tag{8}$$

It must be true that the x-dependent and y-dependent terms are constants. So, $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2$ and $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$. This is a differential equation that has the solution:

$$X = A\cos k_x x + B\sin k_x x \tag{9}$$

$$Y = C\cos k_y y + D\sin k_y y \tag{10}$$

In order to figure out the coefficients A, B, C, and D, we can apply the following boundary conditions:

$$\begin{cases} \frac{\partial B_z}{\partial x} = 0 & \text{at x = 0 and x = a} \\ \frac{\partial B_z}{\partial y} = 0 & \text{at y = 0 and y =b} \end{cases}$$

This is, of course, in the case of a rectangular waveguide with height and width a and b. In order to simplify the calculations, we can apply each boundary condition separately (e.g. apply the y boundary condition to the y equation only). This is possible because the X equation depends only on x and the Y equation depends only on y.
We can begin with the X equation:
When $x = 0$:

$$\frac{\partial X}{\partial x} = -Ak_x \sin k_x x + Bk_x \cos k_x x \tag{11}$$

$$0 = -Ak_x \sin(0) + Bk_x \cos(0) \tag{12}$$

For this to be true, B must be 0. So we can rewrite our X equation as:

$$X = A \cos k_x x \tag{13}$$

We apply our second boundary condition, $x = a$:

$$\frac{\partial X}{\partial x} = 0 = -Ak_x \sin k_x a \tag{14}$$

For this to be true, $\sin k_x a = 0$. If, like B, A were 0, then we would never have any magnetic field or electric field traveling along the waveguide, which cannot be the case. So, the other term, $\sin k_x a$ must be 0.

$$k_x a = \pi \tag{15}$$

$$k_x = \pi m/a \tag{16}$$

Here, m is an integer $\epsilon[0, 1, 2, ...]$.
We can repeat this procedure for the Y equation:
When $y = 0$:

$$\frac{\partial B_z}{\partial y} = -Ck_y \sin k_y y + Dk_y \cos k_y y \tag{17}$$

$$0 = -Ck_y \sin(0) + Dk_y \cos(0) \tag{18}$$

So, D = 0 for this to be true.

$$Y = C \cos k_y y \tag{19}$$

We apply our second boundary condition, $y = b$:

$$\frac{\partial B_z}{\partial y} = 0 = -Ck_y \sin k_y b \tag{20}$$

Thus, similar to in the X case, $\sin k_y b = 0$. Therefore,

$$k_y b = \pi \tag{21}$$

$$k_y = \pi n/b \tag{22}$$

Again, n is an integer $\epsilon[0, 1, 2, ...]$. Thus, $B_z = X(x)Y(y)$ and

$$X(x) = A \cos\left(\frac{\pi m x}{a}\right) \tag{23}$$

$$Y(y) = C \cos\left(\frac{\pi n y}{b}\right) \tag{24}$$

Therefore, $B_z = A\cos(\frac{\pi m x}{a})C\cos(\frac{\pi n y}{b})$ where $AC = B_0$.

$$B_z = B_0 \cos(\frac{\pi m x}{a})\cos(\frac{\pi n y}{b}) \tag{25}$$

Although, as mentioned earlier, we've been lazy about complex notation. So, via equation (ii), B_z can properly be written as: $B_z(x,y,z,t) = B_0\cos(\frac{\pi m x}{a})\cos(\frac{\pi n y}{b})e^{i(kz-\omega t)} = B_0\cos(\frac{\pi m x}{a})\cos(\frac{\pi n y}{b})[(\cos kz - \omega t) + i\sin(kz-\omega t)]$ Using this expression for B_z, we can find the other components of the electromagnetic wave. Of course, $E_z = 0$, but we can find E_x, E_y, B_x, and B_y. We do this by applying equations 1-4.

1.1 Deriving the components generally

Starting with E_x:
Since $E_z = 0$ we can just take the partial $\frac{\partial B_z}{\partial y}$:

$$\frac{\partial B_z}{\partial y} = -B_0\frac{\pi n}{b}\cos(\frac{\pi m x}{a})\sin\frac{\pi n y}{b})[\cos(kz-\omega t) + i\sin(kz-\omega t)] \tag{26}$$

Therefore,

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\omega(-B_0\frac{\pi n}{b}\cos(\frac{\pi m x}{a})\sin\frac{\pi n y}{b})[\cos(kz-\omega t) + i\sin(kz-\omega t)]) \tag{27}$$

Since $i \cdot i = -1$, $\cos(kz-\omega t)$ will become the imaginary part while $\sin(kz-\omega t)$ will become real and gain a negative coefficient. Therefore, by taking only the real part, we simplify E_x to the following:

$$E_x = \frac{\omega}{(\omega/c)^2 - k^2}B_0\frac{\pi n}{b}\cos(\frac{\pi m x}{a})\sin\frac{\pi n y}{b})\sin(kz-\omega t) \tag{28}$$

Next, we can move on to E_y:
Here, again, we are not concerned with the partial of E_z, but we need to compute $\frac{\partial B_z}{\partial x}$.

$$\frac{\partial B_z}{\partial x} = -B_0\frac{\pi m}{a}\sin(\frac{\pi m x}{a})\cos(\frac{\pi n y}{b})[\cos(kz-\omega t) + i\sin(kz-\omega t)] \tag{29}$$

Thus,

$$E_y = \frac{-i\omega}{(\omega/c)^2 - k^2}(-B_0)\frac{\pi m}{a}\sin(\frac{\pi m x}{a})\cos(\frac{\pi n y}{b})[\cos(kz-\omega t) + i\sin(kz-\omega t)] \tag{30}$$

Again, when we expand this out, cos becomes the imaginary part and sin because real and negative. We have an additional negative sign present in E_y, this makes the final equation negative rather than positive like E_x.

$$E_y = \frac{-\omega}{(\omega/c)^2 - k^2}(-B_0)\frac{\pi m}{a}\sin(\frac{\pi m x}{a})\cos(\frac{\pi n y}{b})\sin(kz-\omega t) \tag{31}$$

Next, we look at B_x:
We can reuse $\frac{\partial B_z}{\partial x}$. This gives us the following equation:

$$\frac{-ik}{(\omega/c)^2 - k^2}B_0\frac{\pi m}{a}\sin(\frac{\pi m x}{a})\cos(\frac{\pi n y}{b})[\cos(kz-\omega t) + i\sin(kz-\omega t)] \tag{32}$$

Which, when we take only the real part, becomes:

$$B_x = \frac{k}{(\omega/c)^2 - k^2}B_0\frac{\pi m}{a}\sin(\frac{\pi m x}{a})\cos(\frac{\pi n y}{b})\sin(kz-\omega t) \tag{33}$$

Finally, we can address B_y:
Now, we can reuse $\frac{\partial B_z}{\partial y} = -B_0\frac{\pi n}{b}\cos(\frac{\pi m x}{a})\sin\frac{\pi n y}{b})[\cos(kz-\omega t) + i\sin(kz-\omega t)]$. This gives,

$$B_y = \frac{ik}{(\omega/c)^2 - k^2}(-B_0)\frac{\pi n}{b}\cos(\frac{\pi m x}{a})\sin\frac{\pi n y}{b})[\cos(kz-\omega t) + i\sin(kz-\omega t)] \tag{34}$$

We can simplify this by expanding the equation and taking only the real part:

$$B_y = \frac{k}{(\omega/c)^2 - k^2} B_0 \frac{\pi n}{b} \cos\left(\frac{\pi m x}{a}\right) \sin\frac{\pi n y}{b}) \sin(kz - \omega t) \tag{35}$$

Of course, we know $E_z = 0$ since this is a TE wave and we've already found

$$B_z = B_0 \cos\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{b}\right)[(\cos kz - \omega t) + i \sin(kz - \omega t)](36)$$

So, we can take the real part of B_z to find:

$$B_z = B_0 \cos\left(\frac{\pi m x}{a}\right) \cos\left(\frac{\pi n y}{b}\right) \cos(kz - \omega t)(37)$$

1.2 Exercise 1

We can take these equations and simplify them to the case were $m = 1$ and $n = 0$.

$$\begin{cases} E_z = 0 \\ B_z = B_0 \cos\left(\frac{\pi m x}{a}\right) \cos(kz - \omega t) \\ E_x = 0 \\ E_y = \frac{-\omega}{(\omega/c)^2 - k^2}(-B_0)\frac{\pi m}{a} \sin\left(\frac{\pi m x}{a}\right) \sin(kz - \omega t) \\ B_x = \frac{k}{(\omega/c)^2 - k^2} B_0 \frac{\pi m}{a} \sin\left(\frac{\pi m x}{a}\right) \sin(kz - \omega t) \\ B_y = 0 \end{cases}$$

We can simplify this further by noting that $k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}$ becomes $k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2]}$. Therefore k^2 can be written as $(\omega/c)^2 - \pi^2[(m/a)^2]$. Thus, we can take $(\omega/c)^2 - k^2$ to cancel out $(\omega/c)^2$ and leave $\pi^2[(m/a)^2$. We can plug this into the equations above and discover that factors of π, m, and a will cancel. This gives:

$$\begin{cases} E_z = 0 \\ B_z = B_0 \cos\left(\frac{\pi m x}{a}\right) \cos(kz - \omega t) \\ E_x = 0 \\ E_y = \frac{-\omega B_0 a}{\pi} \sin\left(\frac{\pi m x}{a}\right) \sin(kz - \omega t) \\ B_x = \frac{ka}{\pi} B_0 \sin\left(\frac{\pi m x}{a}\right) \sin(kz - \omega t) \\ B_y = 0 \end{cases}$$

1.3 Exercise 2

Now, for exercise 2, we can take our general expressions again and make them a bit nicer. Our general expressions are:

$$\begin{cases} E_z = 0 \\ B_z = B_0 \cos\left(\frac{\pi m x}{a}\right) \cos(kz - \omega t) \\ E_x = \frac{\omega}{(\omega/c)^2 - k^2} B_0 \frac{\pi n}{b} \cos\left(\frac{\pi m x}{a}\right) \sin\frac{\pi n y}{b}) \sin(kz - \omega t) \\ E_y = \frac{-\omega}{(\omega/c)^2 - k^2}(-B_0)\frac{\pi m}{a} \sin\left(\frac{\pi m x}{a}\right) \sin(kz - \omega t) \\ B_x = \frac{k}{(\omega/c)^2 - k^2} B_0 \frac{\pi m}{a} \sin\left(\frac{\pi m x}{a}\right) \sin(kz - \omega t) \\ B_y = \frac{k}{(\omega/c)^2 - k^2} B_0 \frac{\pi n}{b} \cos\left(\frac{\pi m x}{a}\right) \sin\frac{\pi n y}{b}) \sin(kz - \omega t) \end{cases}$$

Now, we can't just get rid of n, but, we can still use our equation for k^2 to make the formula a bit cleaner. We can still see that $(\omega/c)^2 - k^2$ will get rid of $(\omega/c)^2$, but this time, it will leave $\pi^2[(m/a)^2 + (n/b)^2]$ in the denominator of the coefficient. However, we can throw in another equation:

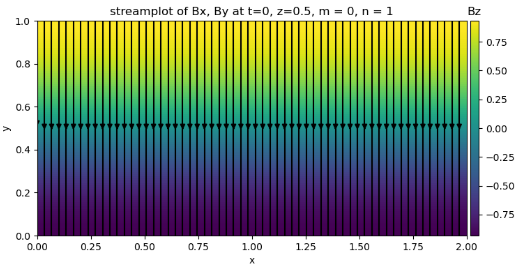
$$\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2} \tag{38}$$

ω_{mn} is the cutoff frequency. For any value of ω below this, the wave will not propagate through the waveguide. We can use this to rewrite $(\omega/c)^2 - k^2$ as $\frac{\omega_{mn}^2}{c^2}$. Therefore, we can plug this simplification into the equations above to retrieve the following generalized results for a TE wave:

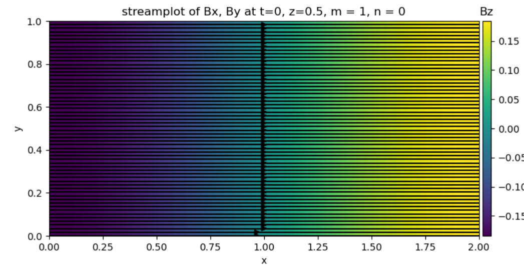
$$\begin{cases} E_z = 0 \\ B_z = B_0 \cos\left(\frac{\pi m x}{a}\right) \cos\left(kz - \omega t\right) \\ E_x = \frac{\omega}{\frac{\omega_{mn}^2}{c^2}} B_0 \frac{\pi n}{b} \cos\left(\frac{\pi m x}{a}\right) \sin\frac{\pi n y}{b}) \sin\left(kz - \omega t\right) \\ E_y = \frac{-\omega}{\frac{\omega_{mn}^2}{c^2}} (-B_0) \frac{\pi m}{a} \sin\left(\frac{\pi m x}{a}\right) \sin\left(kz - \omega t\right) \\ B_x = \frac{k}{\frac{\omega_{mn}^2}{c^2}} B_0 \frac{\pi m}{a} \sin\left(\frac{\pi m x}{a}\right) \sin\left(kz - \omega t\right) \\ B_y = \frac{k}{\frac{\omega_{mn}^2}{c^2}} B_0 \frac{\pi n}{b} \cos\left(\frac{\pi m x}{a}\right) \sin\frac{\pi n y}{b}) \sin\left(kz - \omega t\right) \end{cases}$$

2 Plotting TE waves for various m and n values (Exercise 3)

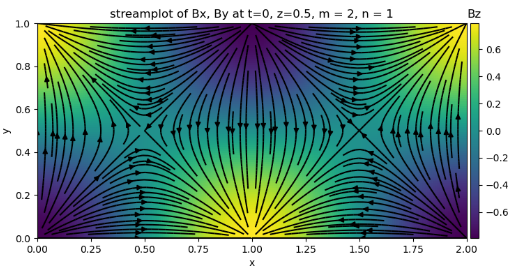
For each of the following plots, a grid size of 100×600 was chosen. x and y are plotting against each other and each pixel represents a position (x, y) within the waveguide. x ranges from 0 to a and y ranges from 0 to b. z is held constant at 0.5. The angular frequency, ω, of the wave is set to be $1.5w_{mn}$, allowing the wave to always propagate through the waveguide. For each pixel, the associated B_z value is plotted. A stream plot describing the movement of B_z is overlaid. In order to discover the relationship between m and n and the propagation of waves within the waveguide, we can change only m and observe how the propagation changes and then change only n and observe the changes.



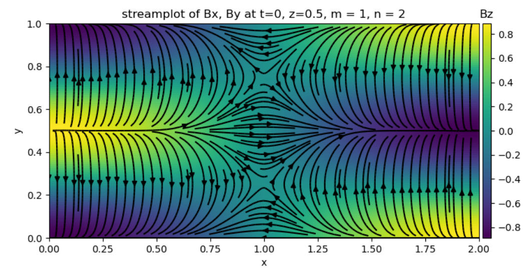
(a) m = 0, n =1



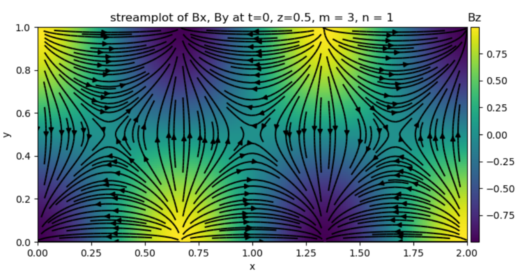
(b) m = 1, n = 0



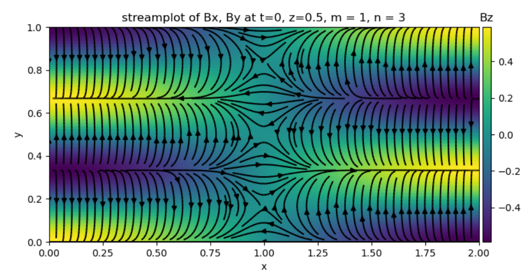
(c) m = 2, n = 1



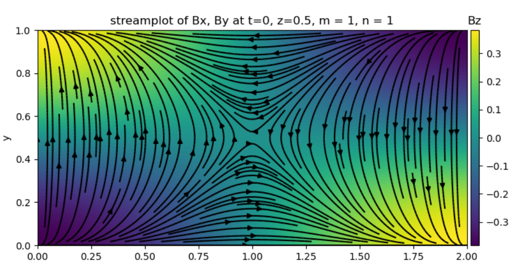
(d) m = 1, n = 2



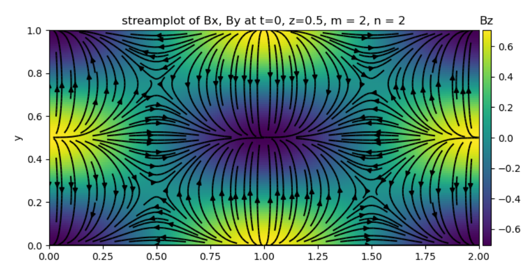
(e) m = 3, n= 1



(f) m = 1, n = 3



(g) m = 1, n= 1



(h) m = 2, n = 2

As seen above, when n is held at 1 and m is changed, the number of nodes in the horizontal direction increases. For instance, in case (c), when m is two and n is one, we have two nodes along the horizontal and one node along the vertical. By contrast, in case (d), m is one and n is two. In this case, there are two notes along the vertical and one node along the horizontal. In the case where m is zero and n is one, we see one node that runs horizontally. When m and n are equal as in cases (g) and (h), we see the same pattern where the number of nodes horizontally corresponds to m and the number of nodes vertically corresponds to n.

3 Deriving the equations for a TM wave (exercise 4)

The general equations for a TM wave can be derived similarly to the equations for a TE wave, except with $B_z = 0$. In this case, we solve Equation [5] for E_z.
We write: $E_z = X(x)Y(y)$, which leads to Equations [7] and [8], the same as a TE wave. We then solve these differential equations and retrieve Equations 9 and 10. However, now, the boundary conditions are the following:

$$E_x = \begin{cases} 0 & \text{at x = 0 and x = a} \\ 0 & \text{at y = 0 and y = b} \end{cases}$$

Unlike in the TE_{mn} case, we don't have to take derivatives of equations 9 and 10 in order to enforce the boundary conditions. But, again, equation 9 which represents the X equation is independent of y and equation 10, which represents the Y equation is independent of x, so the boundary conditions in the x and y directions can be enforced separately.
Starting with the x-direction:
When $x = 0$:

$$0 = A\sin(0) + B\cos(0) \tag{39}$$

In order for this to be true, $B = 0$. Therefore, the X equation becomes:

$$X = A\sin k_x x \tag{40}$$

When $x = a$:

$$0 = A\sin k_x a \tag{41}$$

Here, $\sin k_x a = 0$, so $k_x a = \pi$. So,

$$k_x = \frac{\pi m}{a} \tag{42}$$

Thus,

$$X = A\sin(\frac{\pi m}{a}x) \tag{43}$$

The y-direction:
When $y = 0$:

$$0 = C\sin(0) + D\cos(0) \tag{44}$$

For this to be true, $D = 0$. So, the Y equation is:

$$Y = C\sin k_y y \tag{45}$$

We can enforce the secondary boundary condition $y = b$:

$$0 = C\sin k_y b \tag{46}$$

For this to be true, $\sin k_y b$ must be 0. For the wave to **not** be 0 for all values of y, $C \neq 0$. So, $k_y b = \pi$

$$k_y = \frac{\pi n}{b} \tag{47}$$

Thus,

$$Y = C\sin(\frac{\pi n}{b}y) \tag{48}$$

Now that the boundary conditions have been enforced in both the x and y directions, we can find E_z using our knowledge that $E_z = X(x)Y(y)$.

$$E_z = A\sin(\frac{\pi m}{a}x)C\sin(\frac{\pi n}{b}y) \tag{49}$$

We can set $AC = E_0$ since A and C are constants:

$$E_z = E_0 \sin(\frac{\pi m}{a} x) \sin(\frac{\pi n}{b} y) \tag{50}$$

However, as before, we neglected proper notation for complex components, so E_z should really have a tilde overhead and the full expression for E_z should be written as:

$$E_z = E_0 \sin(\frac{\pi m}{a} x) \sin(\frac{\pi n}{b} y) e^{i(kz-\omega t)} \tag{51}$$

We get this from equation (i).
This can be rewritten as:

$$E_z = E_0 \sin(\frac{\pi m}{a} x) \sin(\frac{\pi n}{b} y)[\cos(kz - \omega t) + i \sin(kz - \omega t)] \tag{52}$$

Like for the TE waves, we can use equations 1-4 to find the components.
Starting with E_x:
We can find:

$$\frac{\partial E_z}{\partial x} = \frac{\pi m}{a} E_0 \cos(\frac{\pi m}{a} x) \sin(\frac{\pi m}{b} y[\cos(kz - \omega t) + i \sin(kz - \omega t)] \tag{53}$$

In section 1.3 (exercise 2), we showed that $(\omega/c)^2 - k = \frac{\omega_{mn}}{c^2}$. Thus, we can write:

$$E_x = \frac{ik}{\omega_{mn}^2/c^2} \frac{\pi m}{a} E_0 \cos(\frac{\pi m}{a} x) \sin(\frac{\pi m}{b} y[\cos(kz - \omega t) + i \sin(kz - \omega t)] \tag{54}$$

By expanding this and taking only the real part, we get:

$$E_x = \frac{-k}{\omega_{mn}^2/c^2} \frac{\pi m}{a} E_0 \cos(\frac{\pi m}{a} x) \sin(\frac{\pi m}{b} y \sin(kz - \omega t) \tag{55}$$

Then, we move to E_y:
We find:

$$\frac{\partial E_z}{\partial y} = \frac{n\pi}{b} E_0 \sin(\frac{\pi n}{a} x) \cos(\frac{n\pi}{b} y)[\cos(kz - \omega t) + i \sin(kz - \omega t)] \tag{56}$$

Therefore,

$$E_y = \frac{ik}{\omega_{mn}^2/c^2} \frac{n\pi}{b} E_0 \sin(\frac{\pi n}{a} x) \cos(\frac{n\pi}{b} y)[\cos(kz - \omega t) + i \sin(kz - \omega t)] \tag{57}$$

After expanding this out again and taking the real part, we get:

$$E_y = \frac{-k}{\omega_{mn}^2/c^2} \frac{n\pi}{b} E_0 \sin(\frac{\pi n}{a} x) \cos(\frac{n\pi}{b} y) \sin(kz - \omega t) \tag{58}$$

Now, we move onto the x and y equations of B:
For B_x, we can reuse our expression for $\frac{\partial E_z}{\partial y}$.

$$B_x = \frac{-i}{\omega_{mn}^2/c^2} \frac{\omega}{c^2} \frac{n\pi}{b} E_0 \sin(\frac{m\pi}{a} x) \cos(\frac{n\pi}{b} y)[\cos(kz - \omega t) + i \sin(kz - \omega t)] \tag{59}$$

Since the c^2s cancel:

$$B_x = \frac{-i\omega}{\omega_{mn}^2} \frac{n\pi}{b} E_0 \sin(\frac{m\pi}{a} x) \cos(\frac{n\pi}{b} y)[\cos(kz - \omega t) + i \sin(kz - \omega t)] \tag{60}$$

We can expand this and take the real part:

$$B_x = \frac{\omega}{\omega_{mn}^2} \frac{n\pi}{b} E_0 \sin(\frac{m\pi}{a} x) \cos(\frac{n\pi}{b} y) \sin(kz - \omega t) \tag{61}$$

Finally, we can find B_y:
We can, again, reuse our expression for $\frac{\partial E_z}{\partial x}$

$$B_y = \frac{i}{\omega_{mn}^2/c^2}\frac{\omega}{c^2}\frac{\pi m}{a}E_0 \cos{(\frac{\pi m}{a}x)}\sin{(\frac{\pi m}{b}y)}[\cos{(kz-\omega t)}+i\sin{(kz-\omega t)}] \tag{62}$$

Again, we can cancel our c^2s, giving:

$$B_y = \frac{i\omega}{\omega_{mn}^2}\frac{\pi m}{a}E_0 \cos{(\frac{\pi m}{a}x)}\sin{(\frac{\pi m}{b}y)}[\cos{(kz-\omega t)}+i\sin{(kz-\omega t)}] \tag{63}$$

Expanding this and taking the real part gives:

$$B_y = \frac{-\omega}{\omega_{mn}^2}\frac{\pi m}{a}E_0 \cos{(\frac{\pi m}{a}x)}\sin{(\frac{\pi m}{b}y)}\sin{(kz-\omega t)} \tag{64}$$

Finally, we can take the real part of our E_z equation to get the real form:

$$E_z = E_0 \sin{(\frac{\pi m}{a}x)}\sin{(\frac{\pi n}{b}y)}\cos{(kz-\omega t)} \tag{65}$$

To summarize, we can retrieve the following generalized equations for TM_{mn} waves:

$$\begin{cases} E_z = E_0 \sin{(\frac{\pi m}{a}x)}\sin{(\frac{\pi n}{b}y)}\cos{(kz-\omega t)} \\ B_z = 0 \\ E_x = \frac{-k}{\omega_{mn}^2/c^2}\frac{\pi m}{a}E_0\cos{(\frac{\pi m}{a}x)}\sin{(\frac{\pi m}{b}y\sin{(kz-\omega t)}} \\ E_y = \frac{-k}{\omega_{mn}^2/c^2}\frac{n\pi}{b}E_0\sin{(\frac{\pi n}{a}x)}\cos{(\frac{n\pi}{b}y)}\sin{(kz-\omega t)} \\ B_x = \frac{\omega}{\omega_{mn}^2}\frac{n\pi}{b}E_0\sin{(\frac{m\pi}{a}x)}\cos{(\frac{n\pi}{b}y)}\sin{(kz-\omega t)} \\ B_y = \frac{-\omega}{\omega_{mn}^2}\frac{\pi m}{a}E_0\cos{(\frac{\pi m}{a}x)}\sin{(\frac{\pi m}{b}y)}\sin{(kz-\omega t)} \end{cases}$$

3.1 Characteristics of TM waves

Now, we can find the lowest mode, the cutoff frequencies, the wave and group velocities, and the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency.
Since, $B_z = 0$, if $E_z = 0$, there would be no wave propagation. In the TE case, since B_z is written with cosines, m or n could be 0 and a wave could still propagate. However, in the TM case, since E_z is expressed in terms of sines, if either m or n is 0, then there is no wave propagation. So, the lowest mode is TM_{11}.
The cutoff frequency is the same for TM waves as it is for TE waves: $\omega_{mn} = c\pi\sqrt{(m/a)^2+(n/b)^2}$.
The wave velocity can be written as:

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}} \tag{66}$$

The group velocity can be written as:

$$v_g = c\sqrt{1-(\omega_{mn}/\omega)^2} \tag{67}$$

We can find the frequency of the lowest TM cutoff frequency to the lowest TE cutoff frequency:

$$\omega_{01}^{TE} = c\pi\sqrt{(1/a)^2+(1/b)^2} \tag{68}$$

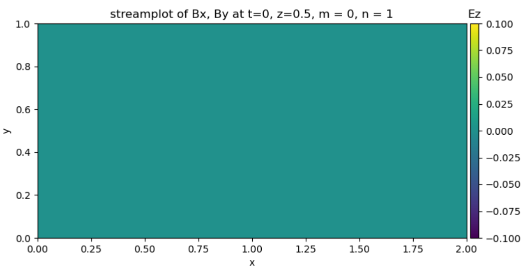
$$\omega_{11}^{TM} = c\pi\sqrt{(1/a)^2} = c\pi/a \tag{69}$$

Therefore,

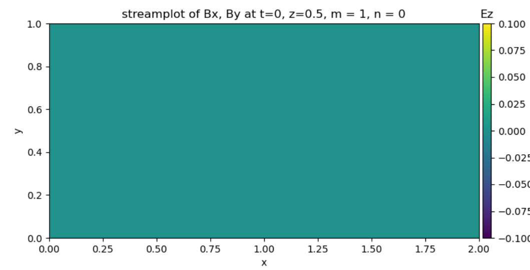
$$\frac{\omega_{11}^{TE}}{\omega_{10}^{TM}} = \frac{c\pi\sqrt{(1/a)^2+(1/b)^2}}{c\pi\sqrt{(1/a)^2} = c\pi/a} \tag{70}$$

$$\frac{\omega_{11}^{TE}}{\omega_{10}^{TM}} = \sqrt{a^2[(1/a)^2+(1/b)^2]} = \sqrt{1+(a/b)^2} \tag{71}$$

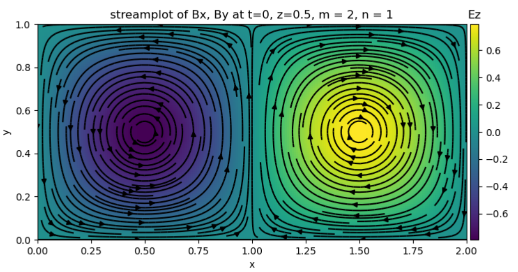
Now, we can repeat the same procedure that was used to visualize the TE waves to visualize the TM waves for various m and n values.
We define the general equations for a TM_{mn} wave and then define a grid. We use a grid size of 200 in the x-direction and 600 in the z-direction. Then, we plot E_z using a `pcolormesh` plot. We add a streamplot on top that shows the direction of B_x and B_y.



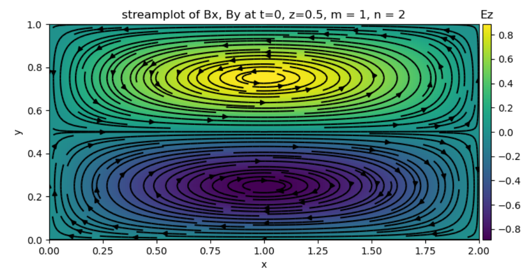
(i) m = 0, n =1



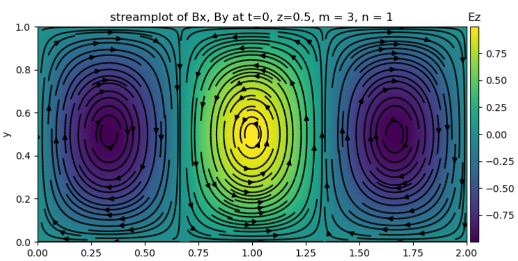
(j) m = 1, n = 0



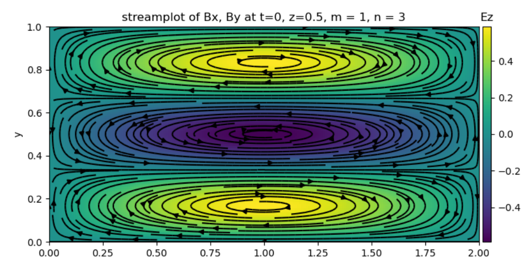
(k) m = 2, n = 1



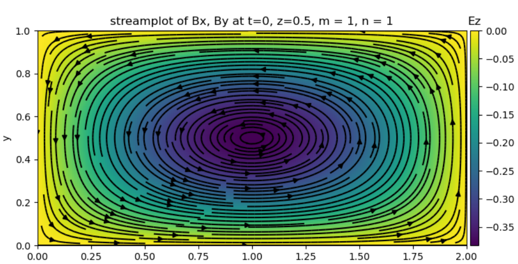
(l) m = 1, n = 2



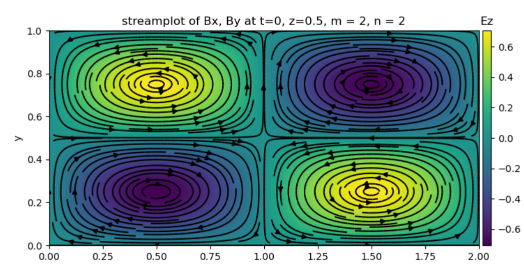
(m) m = 3, n= 1



(n) m = 1, n = 3



(o) m = 1, n= 1



(p) m = 2, n = 2

While with the TE waves the m and n modes describe the nodes of the wave, in the TM case, the anti-nodes are described by the m and n modes. As seen above, when m increases, another anti-node is added in the horizontal direction. When n increases, an anti-node is added in the vertical direction. Therefore, when n is two, we have two nodes in the vertical direction (cases l and p). When m is two, we have two nodes in the

horizontal direction (cases k and p).
A movie describing this wave's movement over time is listed in the repository as `TM11xz.mp4`. This movie is made by varying t from 0 to 6 over 60 frames and plotting each point along the 200 by 600 grid.

4 TEM wave in a coaxial transmission line

The equations for the wave can be taken from Griffiths:

$$\begin{cases} E(r,\phi,z,t) = \frac{A\cos(kz-\omega t)}{r}\hat{r} \\ B(r,\phi,z,t) = \frac{A\cos(kz-\omega t)}{cr}\hat{\phi} \end{cases}$$

Utilizing a polar plot, we repeated the same procedure with the `pcolormesh` function in order to plot the electric field equation. Then, we overlaid a quiver plot showing the direction of the electric field and magnetic field. A snapshot of the coaxial cable at $z = 0.5$, $\omega = 1.5$, and $t = 0$ can be seen below:

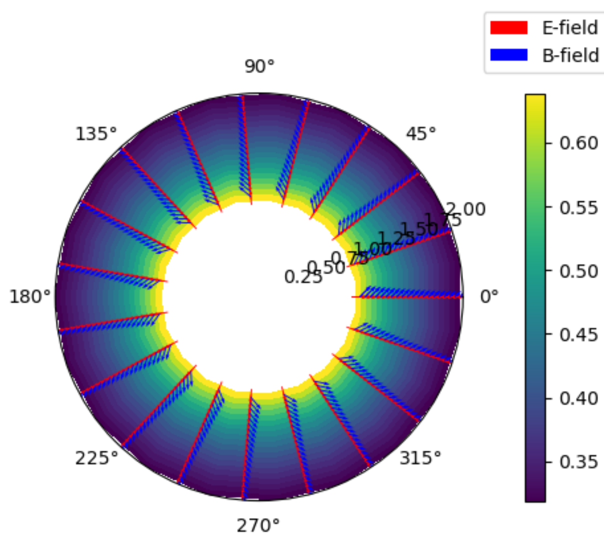


Figure 1: TEM wave in a coaxial cable at $z = 0.5$, $\omega = 1.5$, and $t = 0$

As seen in the figure, the E and B field magnitudes are larger at smaller radii and decrease following $1/r$. The E field points outward radially and the B field points in the ϕ direction. The movie showing the evolution of the wave over time is called `coax.mp4`.The colormesh plot was removed to better show the direction of the electric and magnetic fields.