AMATH 562 Homework Assignment #4
[Due online via Canvas: Monday 11:59pm, February 20, 2023]

1. Professor Matt Lorig's notes, exercises 8.5

2. The Ornstein-Uhlenbeck process, defined by time-homogeneous linear SDE

$$\mathrm{d}X(t) = -\mu X(t)\mathrm{d}t + \sigma \mathrm{d}W(t),\ X(0) = x_0,$$

in which σ and $\mu > 0$ are two constants, has its Kolmogorov forward equation

$$\frac{\partial}{\partial t}\Gamma(x_0; t, x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\Gamma(x_0; t, x) + \frac{\partial}{\partial x}\Big(\mu x \Gamma(x_0; t, x)\Big), \tag{1}$$

with the initial condition $\Gamma(x_0; 0, x) = \delta(x - x_0)$.

(a) Show that the solution to the linear PDE (1) has a Gaussian form and find the solution.

(b) What is the limit of

$$\lim_{t\to\infty} \Gamma(x_0; t, x)?$$

(c) Find $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$.

(d) You note that $\mathbb{E}[X(t)]$ is the same as the solution to the ODE $\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu x$, which is obtained when $\sigma = 0$. Is this result true for a nonlinear SDE?

3. The time-independent solution to a Kolmogorov forward equation gives a stationary probability density function for the Ito process $\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t)$:

$$-\frac{\partial}{\partial x}\Big(\mu(x)f(x)\Big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x)f(x)\Big) = 0.$$

This is a linear, second-order ODE. We assume that both $\mu(x)$ and $\sigma(x)$ satisfy the conditions required to have a solution $f(x)$ on the entire $\mathbb{R}$. Find the expression for the general solution. There are two constants of integration, which should be determined according to appropriate probabilistic reasoning.

4. Professor Matt Lorig's notes, exercises 9.3

5. Consider a continuous-time $(n+1)$-state Markov process $X(t)$, $X \in \mathcal{S} = \{0, 1, 2, \cdots, n\}$, with transition rates

$$g(i, j) = \frac{1}{\mathrm{d}t}\mathbb{P}\big\{X(t + \mathrm{d}t) = j | X(t) = i\big\},\ j \neq i.$$

Let state 0 be an absorbing state, e.g., all $g(0, j) = 0$, $1 \le j \le n$. Let τ_k be a hitting time:

$$\tau_k := \inf \left\{ t \ge 0 : X(t) = 0, X(0) = k \right\}.$$

(a) Show that

$$\sum_{1 \le k \le n} g(j, k) \mathbb{E}[\tau_k] = -1.$$

(b) Derive a system of equations relating $\mathbb{E}[\tau_k^2]$ to $\mathbb{E}[\tau_j]$, $1 \le j, k \le n$.

(c) Now if both states 0 and n are absorbing, let u_k be the probability of $X(t)$, starting with $X(0) = k$, being absorbed into state 0 and $1 - u_k$ be the probability being absorbed into state n. Derive a system of equations for u_k.