Lecture 2: Review of functional analysis continued.

Last time we introduced Banach spaces as complete normed spaces. We will now continue discussing Banach spaces & introduce their duals.

2.1 Linear operators

Def^n: Let X & Y be vector spaces. A map $T: X \longmapsto Y$ is a linear operator if

$$T(x+x') = T(x) + T(x') \quad \forall x, x' \in X$$
$$T(\alpha x) = \alpha \cdot T(x).$$

Common notation is to write Tx instead of $T(x)$ whenever T is linear, in line with matrix notation.

Analogously to linear algebra we also define the domain $\mathrm{dom}(T)$, the range $\mathrm{range}(T)$ & the null-space $\mathrm{null}(T)$.

eg. Identity map $\quad Id(x) = x$

Differentiation $\quad T(x)(t) = x'(t), \quad x \in C([a,b])$

Integration $\quad T(x)(t) = \int_a^t x(\tau)\, d\tau$

Matrices $\quad Tx = y \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m$

Def^n Let $T : X \longrightarrow Y$ be linear. We say T is **bounded** if there is a real number $C > 0$ s.t.

$$\|Tx\|_Y \leq C \|x\|_X$$

It is natural to think about the smallest value of $c > 0$,

$$x \neq 0 \qquad \frac{\|Tx\|_Y}{\|x\|_X} \leq C$$

so c should equal the sup of the LHS. Turns out this quantity defines a norm on T

$$\boxed{\|T\|_{X \to Y}} = \|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Tx\|_Y$$

Notation

Note we immediately have the inequality
$\|Tx\|_Y \leq \|T\| \cdot \|x\|_X$ (Think about linear algebra)

Indeed, it follows that $\|T\|$ satisfies the four axioms of a norm. In fact, the space $L(X, Y)$ of bounded linear maps from $X \longrightarrow Y$ is a Banach space!

Linear operators between normed spaces have a remarkable property where boundedness is equivalent to continuity.

Thm: Let X, Y be normed spaces & suppose $T: X \to Y$ is linear, then

(a) T is continuous iff T is bounded

(b) If T is continuous at a single point then it is continuous.

Recall, $T: X \to Y$ is cont. at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|Tx_0 - Tx\|_Y < \varepsilon$, $\forall x \in X$ satisfying $\|x - x_0\|_X < \delta$.

Linear operators are in many ways analogous to matrices. For example, the notion of the inverse of a linear operator can be defined similarly to matrices.

(Scribe recall invertible matrices)

Thm: Suppose X, Y are vector spaces & let $T: Dom(T) \longmapsto Y$ be a linear operator with $Dom(T) \subset X$ & $Range(T) \subset Y$. Then.

(i) The inverse $T^{-1}: Rang(T) \to Dom(T)$ exists iff $Tx = 0 \implies x = 0$

(ii) If T^{-1} exists, it is a linear op.

(iii) If $\dim(\mathrm{Dom}(T)) = n < \infty$ & T^{-1} exists, then $\dim(\mathrm{Rang}(T)) = \dim(\mathrm{Dom}(T))$.

It further follows that if $T : X \to Y$ & $S : Z \to X$ are invertible linear maps then $(TS)^{-1} = S^{-1}T^{-1}$

2.2 Dual spaces & pairings

When working with Banach spaces, a particular class of bdd lin. operators turn out to be very useful.

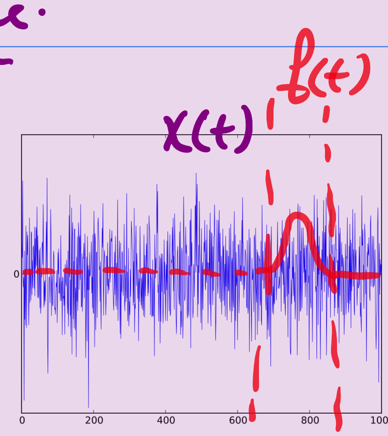
Def^n: A bdd & linear operator $\phi : X \to \mathbb{R}$ is called a bdd linear functional on X, ie, $\exists\ c \geq 0$ s.t. $|\phi(x)| \leq c\|x\|\ \forall x \in X$.

Similar to the case of bdd lin. op. we can define a norm on ϕ:

$$\|\phi\| := \sup_{x \neq 0} \frac{|\phi(x)|}{\|x\|} \equiv \sup_{\|x\|=1} |\phi(x)|$$

Intuition for bdd lin. func.

Imagine we are given a noisy/uncertain signal x. In the real world, we can never observe x perfectly. what we can do is take measurents of x!



The simplest possible class of such measurmnts would be a bdd lin. func. $\phi(x)$. eg.

$$\phi(x) = \int_{\mathbb{R}} x(t)\, f(t)\, dt$$

Then working with bdd lin. func. is very natural. They also apper naturally in the analysis of Banach spaces as we intuitively expect that sufficient "measurements" of a signal x will tell us a lot of information about x itself!

eg: • Dot product $x \in \mathbb{R}^n$, fixed $y \in \mathbb{R}^n$ then $x^T y$ is a bdd lin. func.

• Given $f \in C([a,b])$ the integral

$$\phi(x) = \int_a^b x(t)\, f(t)\, dt$$

is a bdd lin. func.

• Pointwise evaluation is a bdd lin. func. on $C([a,b])$

$$\delta_{t_0}(x) = x(t_0)$$

$$|x(t_0)| \leq \sup_{t\in[a,b]} |x(t)| = \|x\|_\infty$$

in fact $\|\delta_{t_0}\| = 1$ (bdd)

$$\delta_{t_0}(x+x') = (x+x')(t_0) = x(t_0) + x'(t_0) = \delta_{t_0}(x) + \delta_{t_0}(x')$$ (lin)

• The norm $\|\cdot\| : X \to \mathbb{R}$ is a functional (it maps to $\mathbb{R}$) but it is not linear.

Defn Given a Banach space X, its dual space (Topological dual) is the space of all bdd lin. func. on X. We denote this space with the notation X^*.

$$X^* := \{\phi : X \to \mathbb{R} \mid \phi \text{ is bdd lin.}\}$$

We already defined a norm on X^*, called the dual norm, $\|\phi\|_* := \sup_{\|x\|=1} |\phi(x)|$

so it is natural to ask what sort of structure does X^* have? in particular is it Banach?

__Thm__: The dual space X^* of a normed space X is a Banach space (whether or not X is!)

In fact, this is a special case of the fact that the space $B(X,Y)$ of bdd lin. op. from a normed space X to a Banach space Y is a Banach space! (see Thm. 2.10-2 of Kreyszig).

(see Kreyszig for proofs)

eg. • The dual of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself.

• The dual of $\ell^1 := \{ \underline{x} \in \mathbb{R}^\infty \mid \sum_{j=1}^{\infty} |x_j| < +\infty \}$ is $\ell^\infty := \{ \underline{x} \in \mathbb{R}^\infty \mid \sup_j |x_j| < +\infty \}$

• The dual of $L^p([a,b])$ is $L^q([a,b])$ where $1/p + 1/q = 1$.

You will often hear/read the terminology duality Pairing often denoted as $[\cdot,\cdot]: X^* \times X \longrightarrow \mathbb{R}$, as a bilinear mapping (lin. in each arg) from the product of X^* & X to $\mathbb{R}$. This is nothing but new notation for $[\phi, x] = \phi(x)$

However, it has important implications for us later on since these pairings can be defined in different ways! For ex. through an intermediate space.
(look up Gelfand Triples or Rigged Hilbert spaces).