Lecture 6: Sobolev Spaces as RKHSs

Recall that we introduced RKHS's as, Hilbert spaces of functions where pointwise evaluation is a bdd lin. func.

$f \in H \quad \delta_x(f) = \langle K_x, f \rangle$

& used this simple fact to show that the Riesz rep. of δ_x in $\mathcal{H}$ defines a PDS kernel

$$K(x,y) = \langle k_x, k_y \rangle$$

& the kernel satisfies the reproducing property

$f(x) = \langle K(x,\cdot), f \rangle$

So, the main thing we need to check is whether $\delta_x \in H^*$ for some Hilbert space of functions We will introduce an example called Sobolev spaces. These are extremly useful in PDEs, ML, Statistics, ...

Note: I will introduce a very narrow version of Sobolev spaces. For more see:

- Ch5 of "Partial Differential Equations" by L.C. Evans
- Ch4 of "Partial Diff. Eqs I: Basic Theory" by Taylor
- "Sobolev Spaces" by Adams & Fournier

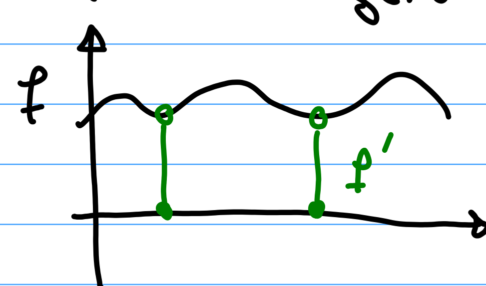
Let $\Omega \subseteq \mathbb{R}^d$ & recall the space $L^2(\Omega)$

$$L^2(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \|f\|_{L^2(\Omega)} < +\infty \right\}$$

where
$$\|f\|_{L^2(\Omega)} := \left(\int_\Omega |f(x)|^2 \, dx \right)^2 \qquad (*)$$

The above integral is defined with respect to (wrt) the Lebesgue measure. This leads to some important & subtle theoretical issues since given f & f' s.t. $f = f'$ almost everywhere (a.e) i.e, f & f' only differ on a set of measure zero.

Then $\|f\|_{L^2(\Omega)} = \|f'\|_{L^2(\Omega)}$



So, To make $L^2(\Omega)$ into a Banach space we need to Think of f & f' as the same thing!

There are called equivalence classes of functions!

(2)

Hence, we take $L^2(\Omega)$ as the space of equivalence classes of functions s.t $\|\cdot\|_{L^2(\Omega)}$ is bdd. $L^2(\Omega)$ is also a Hilbert space

$$\langle f, g\rangle_{L^2(\Omega)} := \int_\Omega f(x)g(x)\,dx, \qquad f,g \in L^2(\Omega)$$

But, Pointwise eval. is not bdd. on $L^2(\Omega)$!

eg. $f(x) = \dfrac{1}{|x|^\alpha}$ is in $L^2(-1,1)$ for $\alpha < 1/2$

The idea now is to look at functions that are not only in $L^2(\Omega)$ but also, some of their derivatives live in $L^2(\Omega)$.

Given a multi index $\underline{\alpha} \in \mathbb{N}^d$

$$\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$$

write $$\partial^{\underline{\alpha}} f := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f$$

Then we define the Sobolev Space $H^s(\Omega)$ for an integer $k \in \mathbb{N}$ as

$$H^s(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \partial^{\underline{\alpha}} f \in L^2(\Omega),\ \forall \underline{\alpha} \text{ s.t. } \|\underline{\alpha}\|_1 \leq s \right\}$$

Convention $H^0(\Omega) \equiv L^2(\Omega)$.

In words, $H^k(\Omega)$ is the space of (eq. classes) of functions such that all of the s-order partial derivatives belong to $L^2(\Omega)$.

$H^s(\Omega)$ can be equipped with a norm

$$\|f\|_{H^s(\Omega)} := \left(\sum_{|\alpha| \leq s} \int_\Omega |\partial^\alpha f(x)|^2 \, dx \right)^{1/2}$$

which arises from an inner product (inherited from L^2)

$$\langle f, g \rangle_{H^s(\Omega)} := \sum_{|\alpha| \leq s} \int_\Omega \partial^\alpha f(x) \, \partial^\alpha g(x) \, dx$$

It is indeed a Hilbert space!

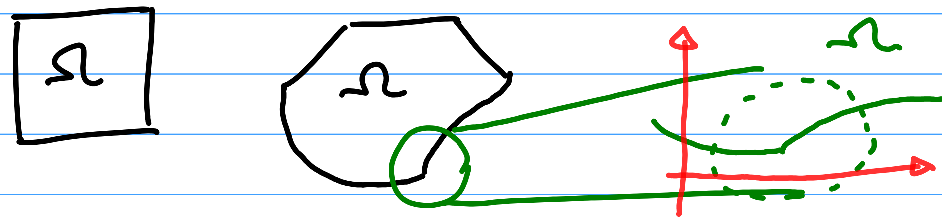
Now it turns out that if s is sufficiently large the $H^s(\Omega)$ is an RKHS thanks to the celebrated Sobolev Embedding theorem.

<u>Thm</u> (Sobolev Embedding) Suppose Ω is a Lipschitz domain. If $s > d/2 + \ell$ then $H^s(\Omega)$ is continuously embedded in $C^\ell(\Omega)$. That is, $H^s(\Omega) \subset C^\ell(\Omega)$ & there exists a constant $C > 0$ s.t.

$$\|f\|_{C^\ell(\Omega)} \leq C \|f\|_{H^s(\Omega)}$$

In words, if $s > d/2 + \ell$ & Ω has a nice boundary. Then elements of $H^s(\Omega)$ have a representative that is in $C^\ell(\Omega)$.

- Lipschitz domain Ω is a domain s.t. its bdry looks like the graph of a Lipschitz func



- what sort of bdry is not included?
  fractals, sharp slits, etc. (see Adams & Fournier)

- In almost all practical applications we can get away with Lipschitz bdry or simply a smooth bdry.

Now, the embedding theorem tells us that $H^s(\Omega) \subset C(\Omega)$ if $s > d/2$ & so, δ_x is a bdd lin. func on $H^s(\Omega)$! ie, $\delta_x \in (H^s(\Omega))^*$

We also knew that $H^s(\Omega)$ is a Hilbert space.

(5) So, from last lecture we infer that $H^s(\Omega)$ is an RKHS!

The difficulty here is that finding the Kernel of $H^s(\Omega)$, in general, is not easy!

$$\langle \delta_x, f \rangle = \langle K_x, f \rangle_{H^s(\Omega)}$$

Note: there is actually a way to show explicitly that the kernel coincides with the Green's function of elliptic operators of the form

$$\left(-\Delta + \tau^2 I\right)^s$$

for $\tau, s > 0$ But we won't go there as it requires a lot of background in PDEs

The good news is that there is an alternative, more convenient way of building RKHSs by explicitly prescribing the Kernel & we will discuss this in the next lecture.