Lecture 24: Convergence Properties

In the last few lectures we focused our attention on implementation details of a kernel PDE solver that approximates the solution of the PDE

(P†)

$$\begin{cases} P(u^\dagger) = f & \text{in } \Omega \\ u^\dagger = 0 & \text{on } \partial\Omega \end{cases}$$

By solving the problem

(PM)

$$u^M = \begin{cases} \operatorname{argmin} \|u\|_K \\ \text{s.t. } P(u)(\underline{x}_i) = f(\underline{x}_i) & \underline{x}_i \in \Omega \\ \quad\quad u(\underline{x}_i) = 0 & \underline{x}_i \in \partial\Omega \end{cases}$$

A natural question arises as to how good of an approximation u^M is? whether $u^M \to u^\dagger$ & how fast? we will provide an answer to these questions in this lecture.

23.1 A simple proof of Convergence

We begin with a simple convergence result that is widely applicable but will not give us rates

Recall our assumption that P is a second order uniformly elliptic operator & assume $f \in H^s(\Omega)$ for $s > d/2$ suff large so that not only $f(\underline{x})$ is well-defined but also $u^t \in H^{s+2}(\Omega)$ & $P(u^t)(\underline{x})$ is well defined.

$\boxed{\text{Simply put, } u^t \text{ is a strong sol}^n.}$

Let us now write $X_M = \{\underline{x}_1, \dots, \underline{x}_M\} \subset \overline{\Omega}$ & suppose these pts are chosen so that $\lim_{M\to\infty} X_M$ is dense in $\overline{\Omega}$ that is, $\overline{\lim_{M\to\infty} X_M} = \overline{\Omega}$

Thm: Let $K : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ be a PDS kernel with RKHS $\mathcal{H}_K$. Suppose $u^t \in \mathcal{H}_K$ & $\mathcal{H}_K$ is compactly embedded in the Sobolev Space $H^t(\Omega)$ with $t > d/2 + 2$. Then $u^M \longrightarrow u^t$ pointwise in Ω as $M \to \infty$.

Proof: Since $u^t \in \mathcal{H}_K$ then $\|u^t\|_K < +\infty$. Additionally since u^M satisfies fewer constraints than u^t then we have that $\|u^M\|_K \leq \|u^t\|_K$. In otherwords, u^M is a bounded sequence in $\mathcal{H}_K$. Since we assumed $\mathcal{H}_K$ is compactly embedded in $H^t(\Omega)$,

$$\|u\|_{H^t(\Omega)} \leq C \|u\|_{\mathcal{H}_K} \quad \forall u \in \mathcal{H}_K$$

& $\mathcal{H}_K$ is a compact subset of $H^t(\Omega)$ & so bounded sequences in $\mathcal{H}_K$ have convergent subsequences in $H^t(\Omega)$!

Thus, u^n has convergent subsequences that converge to a solution of $P(u)=f$ in $H^t(\Omega)$. Since u^t is the unique solution of the PDE then all subsequences of u^M should converge to u^t, which in turn implies $u^M \longrightarrow u^t$ in $H^t(\Omega)$

Finally, since $t > d/2 + 2$, the Sobolev embedding thm implies that $H^t(\Omega) \subset C^2(\Omega)$ & so $u^M \to u^t$ pointwise. $\blacksquare$

- The theorem already indicates that K should somehow be adapted to the PDE to ensure that $u^t \in \mathcal{H}_K$. For eg. if $u^t \in H^1(\Omega)$ but we pick $K = RBF$ then $u^t \notin \mathcal{H}_K$ & the result is not applicable.
- One can also assume the X_n are a nested seq. of collocation pts. In this case $\|u^M\|_K$ is an increasing seq. & proof can be simplified.

• Also note that the uniqueness of u^+ is crucial. Otherwise, convergent subsequences of u^M will converge to different solutions of the PDE.

That is, the accumulation pts of u^M are sol^n's to the PDE.

23.2 Convergence Rates

While our convergence proof is simple & widely applicable, simply proving convergence is not enough in practice. It is natural to ask whether one could prove a rate as well?

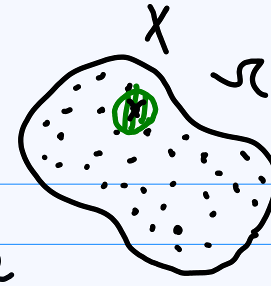
We will do so by making some additional assumptions & using what is called a sampling inequality.

Thm (Sobolev Sampling ineq.) Suppose $X \subset \Omega$ is a set of distinct pts & define the fill distance

$$h := \sup_{\underline{x}' \in \Omega} \inf_{\underline{x} \in X} \|\underline{x} - \underline{x}'\|$$

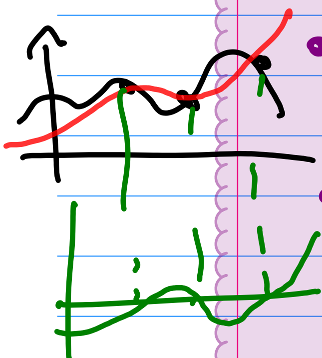
← size of largest hole!

Let $u \in H^s(\Omega)$ for $s > d/2$ s.t. $u(\underline{x}_i) = 0$ for all $\underline{x}_i \in X$. Then, there exist constants $C, h_0 > 0$ s.t. if $h < h_0$ then



$$\|u\|_{H^t(\Omega)} \le C h^{s-t} \|u\|_{H^s(\Omega)}$$

for $0 \le t \le s$.



- In particular, $\|u\|_{L^2(\Omega)} \le C h^s \|u\|_{H^s(\Omega)}$.
- If u^h interpolates $u \in H^s(\Omega)$ at the X's then $\|u - u^h\|_{H^t(\Omega)} \le C h^{s-t} \|u - u^h\|_{H^s(\Omega)}$

We will now use the above sampling ineq. to get a rate of convergence for our kernel collocation method.

<u>Thm</u>: Suppose $u^\dagger$ is the unique strong sol^n of the PDE, (P†) & u^h is the solution to (P$_h$), with a kernel K. Suppose the following conditions hold:

Can be dropped {(i) K is chosen so that $u \in \mathcal{H}_K$ satisfy $u = 0$ on $\partial\Omega$.

(ii) $u^t \in \mathcal{H}_K \subset H^{s+2}(\Omega)$ for $s > d/2$.

(iii) PDE is well-posed. there exist constants $C_1, C_2 > 0$ s.t. for a u_1 & $u_2 \in H^{s+2}(\Omega)$

$$\|P(u_1) - P(u_2)\|_{H^s(\Omega)} \leq C_1 \|u_1 - u_2\|_{H^{s+2}(\Omega)}$$

& that there exists $t < s$ s.t.

$$\|u_1 - u_2\|_{H_0^{t+2}(\Omega)} \leq C_2 \|P(u_1) - P(u_2)\|_{H^t(\Omega)}$$

Then there exists $C, h_0 > 0$ so that if $h < h_0$ then

$$\|u^M - u^t\|_{H_0^{t+2}(\Omega)} \leq C h^{s-t} \|u^t\|_K$$

<u>Proof</u>: By (iii) we have that

$$\|u^t - u^M\|_{H_0^{t+2}(\Omega)} \leq C_2 \|P(u^t) - P(u^M)\|_{H^t(\Omega)}$$

$$= C_2 \|f - P(u^M)\|_{H^t(\Omega)}$$

since u^n is the collocation sol^n, i.e.,
$P(u^M)(\underline{x}_i) = f(\underline{x}_i) = P(u^t)(\underline{x}_i)$ Then
$P(u^M)$ interpolates f. Thus, the Sampling
ineq gives. for $h < h_0$ that

$$\|f - P(u^M)\|_{H^t(\Omega)} \leq C h^{s-t} \underline{\|f - P(u^M)\|_{H^s(\Omega)}}$$

Using (iii) again we get

$$\|u^t - u^M\|_{H_0^{t+2}(\Omega)} \leq C h^{s-t} \underbrace{\|u^t - u^M\|}_{} {}_{H^{s+2}(\Omega)}$$

Now recall that $\|u^M\|_K \leq \|u^t\|_K$ & that
we assumed in (ii) that $H_K \subset H^{s+2}(\Omega)$ thus,

$$\|u^t - u^M\|_{H_0^{t+2}(\Omega)} \leq C h^{s-t} \left(\|u^t\|_{H^{s+2}(\Omega)} + \|u^M\|_{H^{s+2}(\Omega)} \right)$$

$$\leq 2C h^{s-t} \|u^t\|_K.$$

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