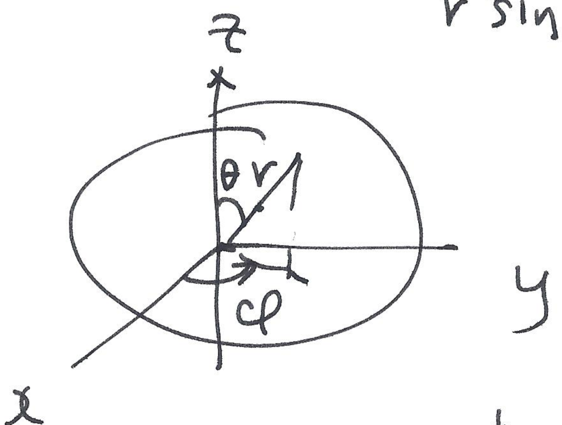
Helmholtz's eigenvalue problem in a sphere

$$\nabla^2 \phi = -\lambda^2 \phi$$

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\phi\right)$$

$$+ \frac{1}{r^3 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\phi$$



Separation of variables

$$\phi(r,\theta,\varphi) = R(r)\,Y(\theta,\varphi)$$

$$\frac{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}R\right)}{R} + \frac{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}Y\right)}{Y}$$

$$+ \frac{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}Y}{Y} = -\lambda^2$$

$$\frac{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y}{Y}$$

$$= -\lambda^2 r - \frac{\frac{d}{dr}\left(r^2\frac{d}{dr}R\right)}{R} = \text{const}$$

$$\equiv -\mu$$

$$R'' + \frac{2}{r}R' - \left(\frac{\mu}{r^2} - \lambda^2\right)R = 0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y + \mu Y = 0$$

<u>The spherical harmonics</u>:

$$Y(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$$

$$\frac{\sin\theta(\sin\theta\,\Theta')' + \mu\sin^2\theta\,\Theta}{\Theta}$$

$$= -\frac{\Phi''}{\Phi} = \text{const} \equiv \alpha^2$$

$$\Phi''(\varphi) + \alpha^2 \Phi(\varphi) = 0, \quad 0 \le \varphi \le 2\pi$$

subject to the periodic boundary condition

$$\Phi(\varphi + 2\pi) = \Phi(\varphi)$$

Solution : $\Phi(\varphi) = \Phi_m(\varphi)$

$$= A_m e^{im\varphi}, \quad m = 0, \pm 1, \pm 2, \cdots$$

$$\alpha = m = \alpha_m$$

Θ - equation becomes

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \Theta \right) + \left(\mu - \frac{m^2}{\sin^2\theta} \right) \Theta = 0$$

$$0 \le \theta \le \pi$$

change of variable :

$$x = \cos\theta, \quad dx = -\sin\theta\, d\theta, \quad \frac{d}{d\theta} = -\sin\theta \frac{d}{dx}$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$$

$$\frac{d}{dx} \left[(1 - x^2) \frac{d}{dx} \Theta \right] + \left(\mu - \frac{m^2}{1 - x^2} \right) \Theta = 0$$

$$-1 \le x \le 1$$

Associated Legendre equation.

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$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\Theta\right] + \left(\mu - \frac{m^2}{1-x^2}\right)\Theta = 0$$

$$-1 \le x \le 1$$

It has regular singular points at $x = \pm 1$.

Boundary conditions : $\Theta(\theta)$ bounded at $x = \pm 1$. (North pole : $\theta = 0$, $x = 1$
South pole : $\theta = \pi$, $x = -1$)

This happens only when

$$\mu = n(n+1) \equiv \mu_n \quad , n = 0, 1, 2, 3, \cdots$$

$$\mu_n = n(n+1), \quad m = -n, -n+1, \cdots, n-1, n$$

$$\Theta = \Theta_n = P_n^m(x),$$

The spherical harmonics are defined by

$$Y(\theta, \varphi) = Y_{nm}(\theta, \varphi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \cdot P_n^n(\cos\theta) e^{im\varphi}$$

$$-n \le m < n$$

The integral over the surface of a sphere

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \; Y_{nm}(\theta,\varphi)\, Y^*_{n'm'}(\theta,\varphi) \sin\theta$$

$$= \begin{cases} 0 & \text{if } n \neq n', \text{ or } m \neq m' \\ 1 & \text{if } n = n' \text{ and } m = m' \end{cases}$$

The spherical harmonics are orthogonal since $e^{im\varphi}$ and P_n^m are orthogonal.

The R- equation

$$r^2 R''(r) + 2r R'(r) + [\lambda^2 r^2 - n(n+1)] R(r) = 0$$

BCs: $R(0)$ bounded

$R(a) = 0$, a the radius of the sphere.

This equation is called the spherical Bessel equation.

Let $x = \lambda r$, $R(r) = r^{-1/2} y(x)$

$$x^2 y''(x) + x y'(x) + (x^2 - p^2) y(x) = 0$$

with $p = n + \frac{1}{2}$

The solution is $J_p(x)$ and $Y_p(x)$

The bounded solution is $J_p(x)$

The solution that is bounded at $r = 0$ is the spherical Bessel function of the first kind.

$$R(r) = j_n(x) \equiv \left(\frac{\pi}{2x}\right)^{1/2} J_{n+\frac{1}{2}}(x)$$

$$n = 0, 1, 2, 3, \ldots$$

To satisfy the boundary condition at $r=a$,

$$R(a) = 0$$

$$j_n(\lambda a) = \left(\frac{\pi}{2\lambda a}\right)^{1/2} J_{n+\frac{1}{2}}(\lambda a) = 0$$

This determines λ:

$$\lambda = \lambda_{nk} = \frac{z_{n+\frac{1}{2},k}}{a}, \quad k = 1, 2, 3, \ldots$$

where z_{pk} is the k^{th} zero of $J_p(x)$

$$j_n(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$$ (shown in chapter 11)

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\vdots$$