Lecture 8

Wave equation in 3-D (8.7 of my book)

PDE :

$$u_{tt} = c^2 \nabla^2 u$$

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

BC : $u \to 0$ as $r^2 \equiv x_1^2 + x_2^2 + x_3^2 \to \infty$

IC : $u(\vec{x}, 0) = u_0(r)$

$u_t(\vec{x}, 0) = 0$

The initial u is assumed to have radial symmetry about the origin.

We apply Fourier transform to each space dimension by letting

$$U(\vec{\lambda}, t) = \mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 [u(\vec{x}, t)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\vec{x}, t)\, e^{i(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)}\, dx_1\, dx_2\, dx_3$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\vec{x}, t)\, e^{i\vec{\lambda}\cdot\vec{x}}\, d^3\vec{x}$$

$d^3\vec{x} = dx_1\, dx_2\, dx_3$, $\vec{x} = (x_1, x_2, x_3)$

$\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$

We take the triple Fourier transform of the PDE:

$$\frac{\partial^2}{\partial t^2} U = -c^2 \lambda^2 U$$

where $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$

$$U(\vec{\lambda}, t) = A(\vec{\lambda}) \cos c\lambda t + B(\vec{\lambda}) \sin c\lambda t$$

Applying the ICs, we find

$$B(\vec{\lambda}) = 0, \quad A(\vec{\lambda}) = U(\vec{\lambda}, 0),$$

where $U(\vec{\lambda}, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(r) e^{i\vec{\lambda}\cdot\vec{x}} d^3\vec{x}$

$$= \int_0^{\infty} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \; u_0(r) e^{i\lambda r \cos\theta}$$

in spherical coordinates. We have oriented the coordinates so that θ is the angle the vector $\vec{x}$ makes relative to a (fixed) vector $\vec{\lambda}$.

$$U(\vec{\lambda},0) = 2\pi \int_0^\infty r^2 dr\, u_0(r) \int_0^\pi d(-\cos\theta)\, e^{i\lambda r\cos\theta}$$

$$= 2\pi \int_0^\infty r^2 dr\, u_0(r)\, \frac{e^{i\lambda r\cos\theta}}{-i\lambda r}\Big|_0^\pi$$

$$= 4\pi \int_0^\infty u_0(r)\, \frac{\sin\lambda r}{\lambda}\, r\,dr$$

$$\equiv U_0(\lambda),$$

a function of the magnitude of $\vec{\lambda}$ only.

$$U(\vec{\lambda},t) = U_0(\lambda)\cos c\lambda t$$

Inverse Fourier transform:

$$u(\vec{x},t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_0(\lambda)\cos c\lambda t\, e^{-i\vec{\lambda}\cdot\vec{x}}\, d^3\vec{\lambda}$$

$$= \frac{1}{(2\pi)^3}\int_0^\infty \lambda\, d\lambda\, U_0(\lambda)\, \frac{\cos c\lambda t\, \sin\lambda r}{r}$$

Since

$$\sin\lambda r \cos c\lambda t = \frac{1}{2}\sin\lambda(r-ct) + \frac{1}{2}\sin\lambda(r+ct)$$

$$u(\vec{x},0) = u_0(r)$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_0(\lambda)\, e^{-i\vec{\lambda}\cdot\vec{x}}\, d^3\vec{\lambda}$$

$$= \frac{2}{(2\pi)^2}\int_0^{\infty} \lambda\, d\lambda\, U_0(\lambda)\, \frac{\sin\lambda r}{r}$$

we then have

$$r u(\vec{x},t) = \frac{1}{(2\pi)^2}\int_0^{\infty}\lambda\, d\lambda\, U_0(\lambda)\sin\lambda(r-ct) + \frac{1}{(2\pi)^2}\int_0^{\infty}\lambda\, d\lambda\, U_0(\lambda)\sin\lambda(r+ct)$$

$$= \frac{1}{2}(r-ct)\,u_0(r-ct) \quad \text{outgoing}$$

$$+ \frac{1}{2}(r+ct)\,u_0(r+ct) \quad \text{incoming.}$$