AMATH 573, Problem Set 1

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1. Problem 1.2: The KdV equation

$$u_t = uu_x + u_{xxx}$$

is often written with different coefficients. By using a scaling transformation on all variables (dependent and independent), show that the choice of the coefficients is irrelevant: by choosing a suitable scaling, we can use any coefficients we please. Can you say the same for the modified KdV (mKdV) equation

$$u_t = u^2 u_x + u_{xxx}?$$

Solution:

A given KdV equation with arbitrary coordinates can be written in the form

$$u_t + auu_x + bu_{xxx} = 0$$

with $a, b \in R \setminus \{0\}$.

Let $x' = \alpha x$, $t' = \beta t$, and $v(x'(x), t'(t)) = u(x,t)/\gamma$, where $\alpha, \beta, \gamma \in R \setminus \{0\}$. Making these coordinate transformations and applying the chain rule results in the expression

$$\beta\gamma v_{t'} + \alpha\gamma^2 avv_{x'} + \alpha^3\gamma b v_{x'x'x'} = 0$$

We divide out $\beta\gamma$ to write this equation in the form $v_{t'} + a'vv_{x'} + b'v_{x'x'x'}$, and find

$$a' = \frac{\alpha\gamma}{\beta} a$$

and

$$b' = \frac{\alpha^3}{\beta} b$$

It is always possible to find scaling factors α, β, γ to transform the KdV equation coefficients from $(a, b) \to (a', b')$.

Let $A(\alpha, \beta, \gamma) = \frac{a'}{a}$ and $B(\alpha, \beta, \gamma) = \frac{b'}{b}$ be our coefficient scaling factors. We let $\alpha = 1$ and solve for γ and β in terms of A and B to find $\beta = B^{-1}$ and $\gamma = B/A$. This means that in order to make the coefficient transformation $(a, b) \to (a', b')$ it is sufficient to make the scaling transformations

$$t' = \frac{b}{b'} t$$

and

$$v(x', t') = \frac{ab'}{a'b} u(x, t)$$

For the modified KdV equation

$$u_t + au^2 u_x + bu_{xxx} = 0$$

performing the same scaling transformations as above yields the result

$$\beta\gamma v_{t'} + \alpha\gamma^3 avv_{x'} + \alpha^3\gamma bv_{x'x'x'} = 0$$

and dividing out $\beta\gamma$ gives us

$$a' = \frac{\alpha\gamma^2}{\beta} a$$

and

$$b' = \frac{\alpha^3}{\beta} b$$

We see that for the modified KdV equation, the inclusion of a u^2 term ends up giving us an extra factor of γ in the new a' coefficient, so a' is now proportional with γ^2, rather than γ. The result of this is that a'/a and b'/b must always have the same sign, either both positive or both negative. This is equivalent to being able to scale either coefficient by an arbitrary positive amount.

2. Problem 1.4: (Use a symbolic computing software for this problem.) Consider the KdV equation $u_t + uu_x + u_{xxx} = 0$. Show that

$$u = 12\partial_x^2 \ln\left(1 + e^{k_1 x - k_1^3 t + \alpha}\right)$$

is a one-soliton solution of the equation (i.e., rewrite it in sech^2 form). Now check that

$$u = 12\partial_x^2 \ln\left(1 + e^{k_1 x - k_1^3 t + \alpha} + e^{k_2 x - k_2^3 t + \beta} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x - k_1^3 t + \alpha + k_2 x - k_2^3 t + \beta}\right)$$

is also a solution of the equation. It is a two-soliton solution of the equation, as we will verify later. By changing t, we can see how the two solitons interact. With $\alpha = 0$ and $\beta = 1$, examine the following 3 regions of parameter space:

(a) $\frac{k_1}{k_2} > \sqrt{3}$

(b) $\sqrt{3} > \frac{k_1}{k_2} > \sqrt{(3 + \sqrt{5})/2}$

(c) $\frac{k_1}{k_2} < \sqrt{(3 + \sqrt{5})/2}$.

Discuss the different types of collisions. Here "examine" and "discuss" are supposed to be interpreted in an experimental sense: play around with this solution and observe what happens. The results you observe are the topic of the second part of Lax's seminal paper.

Solution:

See plot: https://www.desmos.com/calculator/0i40ispbv4

u can indeed be rewritten in sech^2 form, and is a one-soliton solution to the KdV equation. (See attached code, 1.4.1).

$$u(x,t) = 3k_1^2 \,\mathrm{sech}^2\left[\frac{1}{2}k_1 x - k_1^3 t + \alpha\right]$$

The second equation above can be verified as as solution to the KdV equation as well. (See attached code, 1.4.2).

Let $\gamma = \frac{k_1}{k_2}$. We observe that for for large $\gamma > \sqrt{3}$ the solution begins to resemble a linear superposition of single soliton solutions. The larger, faster wave overtakes the smaller, slower wave and essentially passes right through it. Conversely for small $1 < \gamma < \sqrt{(3 + \sqrt{5}/2}$ the two solitons begin to interact more like particles than waves. It looks like the first wave "bumps" into the second wave and pushes it forward, losing momentum in the process. In the intermediate region between these two values we see a mixture of both "passing through" and "bouncing off" type interactions.

As γ approaches 1, the starting distance between the two waves approaches infinity. For this reason we cannot have a two soliton solution where both solitons share the same k value.

3. Problem 1.5: **The Cole-Hopf transformation.** Show that every non-zero solution of the heat equation $\theta_t = \nu\theta_{xx}$ gives rise to a solution of the dissipative Burgers equation $u_t + uu_x = \nu u_{xx}$, through the mapping $u = -2\nu\theta_x/\theta$.

   Solution:

   Letting $u = -2\nu\theta_x/\theta$ and plugging this expression into the dissipative Burgers' equation gives us

   $$\frac{2\nu}{\theta^2}[\nu\theta\theta_{xxx} - \theta\theta_{xt} + \theta_t\theta_x - \nu\theta_x\theta_x x] = 0 \tag{1}$$

   Differentiating the heat equation with respect to x gives us

   $$\theta_{xt} = \nu\theta_{xxx} \tag{2}$$

   Substituting (2) into (1) results in all terms cancelling each other out, thereby proving that u is a solution to the dissipative Burgers' equation.

4. Problem 1.6: From the previous problem, you know that every solution of the heat equation $\theta_t = \nu\theta_{xx}$ gives rise to a solution of the dissipative Burger's equation $u_t + uu_x = \nu u_{xx}$, through the mapping $u = -2\nu\theta_x/\theta$.

   (a) Check that $\theta = 1 + \alpha e^{-kx+\nu k^2 t}$ is a solution of the heat equation. What solution of Burgers' equation does it correspond to? Describe this solution qualitatively (velocity, amplitude, steepness, etc) in terms of its parameters.

   (b) Check that $\theta = 1 + \alpha e^{-k_1 x+\nu k_1^2 t} + \beta e^{-k_2 x+\nu k_2^2 t}$ is a solution to the heat equation. What solution of burgers' equation does it correspond to? Describe the dynamics of this solution, i.e., how does it change in time?

Solution:

See plot: https://www.desmos.com/calculator/sqjtc3eyjo

(a) Plugging θ into the heat equation and simplifying confirms that θ is a solution to the heat equation. If we let $\psi = \alpha e^{-kx+\nu k^2 t}$ then the heat equation becomes $\theta_t - \nu\theta_{xx} = \nu k^2\psi - \nu k^2\psi = 0$.

θ corresponds to the Burgers' equation solution

$$u = \frac{2\alpha\nu k}{\alpha + e^{k(x-\nu kt)}}$$

This equation is of the form $f(x - ct)$, where $c = \nu k$, which means it is also a solution to the one-dimensional wave equation $u_{tt} = \nu^2 k^2 u_{xx}$. In particular, this is a wave solution which is moving to the right with velocity νk. Since the only effect of t is to shift $u(x, 0)$ (to the right when $\nu k > 0$ and left when $\nu k < 0$), we can simplify our analysis by focusing on $u(x, 0)$.

$$u(x, 0) = \frac{2\alpha\nu k}{\alpha + e^{kx}}$$

We see that for large values of x the exponential term in the denominator dominates and the solution goes to zero, while for very negative values of x the exponential goes to zero, and the solution approaches $2\nu k$. Hence, away from the origin α has nearly no effect on the solution. However, since $u(0, 0) = \frac{2\alpha\nu}{\alpha+1}$, the α parameter controls the value of u at $x = 0$, which must lie in the open interval $(0, 2\nu k)$. If α is zero then $u(x, t) = 0$, and if it is negative then $u(x, t)$ is undefined at $x = \ln(|\alpha|)/k$.

The νk term in the numerator means that increases in either ν or k will increase the amplitude of u, as well as its speed. However, since k is also present in the exponential terms as a decay factor, increasing k will also cause u to drop off faster, and the width of the intermediate region between the high and low values will shrink in proportion with the change in k. Combining these two effects means that increasing k results in u needing to fall further and over a shorter distance, and so the steepness of u increases quadratically with k. Furthermore, since k appears in an exponential, each higher order derivative includes an additional factor of k and is proportional to k^{n+1}, where n is the order of the derivative. Flipping the sign of k is equivalent to flipping the graph of u over both the x and y axes, and also equivalent to shifting u downwards by $2\nu k$.

Increasing ν will scale the height and velocity of the function as well, but it will not change the width of the intermediate region of u. Since it only scales u vertically, the effect of ν on the steepness of u is linear, and each successive derivative retains this linear relationship, in contrast with the increasing polynomial dependency seen with k.

If we were to fix $\nu k = 1$ and allow k to approach infinity while ν approaches 0, we would find that u would approach a step function which is 0 for positive values and 2 for negative values.

(b) Plugging θ into the heat equation and simplifying confirms that θ is a solution to the heat equation. (See attached code, 1.6.1). θ corresponds to the Burgers' equation solution:

$$\frac{2\nu\left(\alpha k_1 e^{k_1^2\nu t+k_2 x}+\beta k_2 e^{k_2^2\nu t+k_1 x}\right)}{\alpha e^{k_1^2\nu t+k_2 x}+\beta e^{k_2^2\nu t+k_1 x}+e^{(k_1+k_2)x}}$$

When $k_1/k_2 > 0$ and $k_1 \neq k_2$ there is always a rightward moving faster and steeper wave which overtakes the slower and smoother wave and "swallows" it. The waves do not add together as would be expected of linear waves, instead the larger, faster wave simply overtakes the smaller one and continues on unchanged. Clearly, this is a darker, more Darwinian wave mechanics.

When $k_1/k_2 < 0$ and $k_1 \neq k_2$ there is one leftward moving negatively valued wave and one rightward moving positively valued wave. When these collide they push back against each other. The larger and faster wave pushes ahead, but it is slowed down by the resistance of the smaller wave. If $k_1 = -k_2$ then the waves will collide at the origin and remain motionless; they will not pass through each other as would a linear wave.

A consequence of these dynamics is that $k_1 = k_2$ produces the same dynamics as $k_2 = 0$. Two waves moving in the same direction at the same speed is indistinguishable from one wave.