# Sturm-Liouville Theory

S-L equation

$$\frac{d}{dx}\left[ p(x) \frac{d}{dx} y \right] + \left[ \lambda r(x) - q(x) \right] y = 0$$

($\lambda$ is the eigenvalue, instead of $\lambda^2$ used previously)

Regular S-L problem:

- The domain $[a, b]$ is a closed finite interval.
- $p(x)$, $p'(x)$ and $r(x)$ are continuous in $[a,b]$.
- $p(x) \neq 0$, $r(x) \neq 0$ in $[a,b]$

  WLOG we assume $p(x) > 0$, $r(x) > 0$

  (the latter may involve changing the sign of $\lambda$).
- The boundary conditions:

$$c_1 y(a) + c_2 y'(a) = 0$$
$$d_1 y(b) + d_2 y'(b) = 0$$

## Singular S-L problem

Either on an infinite interval or on a finite interval but with at least one of the regularity properties not satisfied. Typically $p(x)$ goes to zero at one of the end points.

The type of singular S-L problem that we will consider here still have

$p(x)$, $p'(x)$ and $r(x)$ continuous

in the open interval $a < x < b$.

$p(x) > 0$, $r(x) > 0$ in $a < x < b$.

but $p(x)$ or $r(x)$ may vanish at $x = a$ or $x = b$. Then the boundary condition should be changed to boundedness of the solution at the singular point.

Examples of S-L system :

(a) Fourier :

$$y''(x) + \lambda y(x) = 0, \quad 0 \le y \le L$$

$$y(0) = 0, \quad y(L) = 0$$

(b) Legendre :

$$[(1-x^2)y']' + \mu y = 0, \quad -1 < x < 1$$

$y(x)$ bounded at $x = 1$, $x = -1$.

$p(x) = 1 - x^2$, $r(x) = 1$

$p(x) = 0$ at $x = \pm 1$, the boundary end points.

(c) Bessel :

$$[xy']' + [\lambda x - \frac{p^2}{x}]y = 0, \quad 0 < x < b$$

$p(x) = x$, $r(x) = x$, $q(x) = p^2/x$

(d) spherical Bessel:

$$(x^2y')' + [\lambda x^2 - n(n+1)]y = 0$$

$$0 < x < a$$

$p(x) = x^2, \quad p(0) = 0$

$r(x) = r^2, \quad r(0) = 0$

$q(x) = n(n+1)$

(e) Chebyshev:

$$(1-x^2)y'' - xy' + \lambda y = 0, \quad -1 < x < 1$$

This can be put into the S-L form by dividing by $(1-x^2)^{1/2}$

$$(1-x^2)^{1/2} y'' - \frac{x}{(1-x^2)^{1/2}} y' + \lambda \frac{1}{(1-x^2)^{1/2}} y = 0$$

$$[(1-x^2)^{1/2} y']' + \frac{\lambda}{(1-x^2)^{1/2}} y = 0$$

$p(x) = (1-x^2)^{1/2}, \quad r(x) = \frac{1}{(1-x^2)^{1/2}}$

$q(x) = 0$

$y(x)$ bounded at $x = \pm 1$.

# Orthogonality Th$^{\underline{m}}$

The eigenfunctions of the S-L system to different eigenvalues are orthogonal to each other with respect to the weight $r(x)$.

Pf: Consider two pairs of eigenfunctions and eigenvalues:

$$(\phi_k, \lambda_k), \quad (\phi_j, \lambda_j)$$

which satisfy:

$$(p\phi_k')' + [\lambda_k r - q]\phi_k = 0$$

$$(p\phi_j')' + [\lambda_j r - q]\phi_j = 0$$

multiply the first by $\phi_j$ and the second by $\phi_k$:

$$\phi_j (p\phi_k')' + \lambda_k r \phi_k \phi_j - q \phi_k \phi_j = 0$$

$$\phi_k (p\phi_j')' + \lambda_j r \phi_k \phi_j - q \phi_k \phi_j = 0$$

Subtract:

$$\phi_j (p\phi_k')' - \phi_k (p\phi_j')' = (\lambda_j - \lambda_k) r \phi_k \phi_j$$

The LHS can be written as

$$\frac{d}{dx}[\phi_j (p\phi_k') - \phi_k (p\phi_j')]$$

Integrate both sides:

$$\boxed{[\phi_j (p\phi_k') - \phi_k (p\phi_j')]\Big|_a^b = (\lambda_j - \lambda_k)\int_a^b r(x)\phi_j(x)\phi_k(x)\,dx}$$

Applying the regular boundary conditions, the LHS = 0.

For the singular S-L system, if $p(x) = 0$ at the boundary (one or both boundaries), the boundedness condition also makes the boundary terms vanish. Thus

$$(\lambda_j - \lambda_k) \int_a^b r(x)\, \phi_j(x)\, \phi_k(x)\, dx = 0$$

Hence:

$$\boxed{\int_a^b r(x)\, \phi_j(x)\, \phi_k(x)\, dx = \begin{cases} 0 & \text{if } \lambda_j \neq \lambda_k \\ \text{positive constant} & \text{if } \lambda_j = \lambda_k \end{cases}}$$

The positive constant is $\int_a^b r(x)\, \phi_j^2(x)\, dx$

Sometimes the eigenfunctions are normalized so that $\int_a^b r(x)\, \phi_j^2\, dx = 1$.

# Uniqueness of eigenfunctions

There is only one eigenfunction corresponding to an eigenvalue

Pf: Assume this is not true and so there are two different eigenfunctions $\phi_1(x)$ and $\phi_2(x)$ corresponding to the same $\lambda$; then (note the typos)

$$(p\phi_1')' + (\lambda r - q)\phi_1 = 0$$
$$(p\phi_2')' + (\lambda r - q)\phi_2 = 0$$

multiply the first by $\phi_2$ and the second by $\phi_1$, and then subtract:

$$0 = \phi_2(p\phi_1')' - \phi_1(p\phi_2')'$$

which is, $\frac{d}{dx}[p(\phi_2\phi_1' - \phi_1\phi_2')]$,

or $p(\phi_2\phi_1' - \phi_1\phi_2') = \text{constant}$

The constant is zero if evaluated at one of the boundaries with regular boundary conditions. It is also zero in the singular case where $p(x)=0$ at a boundary point. In the interior of the domain where $p(x) \neq 0$, we have

$$\phi_1 \frac{d}{dx}\phi_2 - \phi_2 \frac{d}{dx}\phi_1 = 0,$$

which is the same as

$$\frac{d}{dx}(\phi_2/\phi_1) = 0$$

Or $\quad \phi_2(x) = c\,\phi_1(x)$

So $\phi_1(x)$ and $\phi_2(x)$ are the same eigenfunction.

Note that this result fails for periodic boundary conditions.

All eigenvalues are real and positive:

$$[py']' + [\lambda r - q]y = 0$$

$$[py^{*\prime}]' + [\lambda^* r - q]y^* = 0$$

Then, as before

$$(\lambda - \lambda^*) \int_a^b r\,\phi\phi^* dx = 0$$

Since $\int_a^b r\,\phi\phi^* dx = \int_a^b r\,|\phi|^2 dx \neq 0$

we must have

$$\lambda = \lambda^*$$

so $\lambda$ is real.

Also, all eigenvalues are positive if $q(x) \geq 0$.

multiply the first equation by $y^*$

$$0 = \int_a^b y^* [py']' dx + \lambda \int_a^b r|y|^2 dx - \int_a^b q|y|^2 dx$$

$$0 = y^*[py']\Big|_a^b - \int_a^b p|y'|^2 dx + \lambda \int_a^b r|y|^2 dx - \int_a^b q|y|^2 dx$$

$$\lambda = \int_a^b \{p|y'|^2 + q|y|^2\} dx \Big/ \int_a^b r|y|^2 dx$$

So $\lambda > 0$ if $q > 0$.

For $q = 0$, the possibility of $\lambda = 0$ is allowed if $y'(x) \equiv 0$.