AMATH 562 Final Exam
[Due online via Canvas: Thursday 11:59pm, March 16, 2023]

You will work alone on the exam. You may use everything that is on the course CANVAS. You may use Mathematica, or any other computational tool you find helpful. But you may not use the internet, or discuss the exam with others.

1. [40pt] W_t is a standard Brownian motion.

(a) Find the probability density of W_t^2.

(b) Evaluate the expectation:

$$\mathbb{E}\left[\left(\int_0^T W_t^2 \mathrm{d}W_t\right)^2\right].$$

(c) Show that $W_t^3 - 3tW_t$ is a martingale.

(d) Use Ito's formula to write the following stochastic process

$$X_t = e^{W_t} + t + 2$$

into the standard form

$$\mathrm{d}X_t = \mu(t,\omega)\mathrm{d}t + \sigma(t,\omega)\mathrm{d}W_t.$$

2. [40pt] The concept of *change of measure* in terms of a Radon-Nikodym derivative can be summarized as in the following diagram:

$$\begin{tikzcd}
(\Omega,\mathcal{F},\mathbb{P}) \arrow[r, "X(\omega)"] \arrow[d, "\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega)"'] & f_X(x) \arrow[d] \\
(\Omega,\mathcal{F},\tilde{\mathbb{P}}) \arrow[r, "X(\omega)"] & \tilde{f}_X(x)
\end{tikzcd}$$

(a) Assuming that in the diagram, both probability density functions $f_X(x)$ and $\tilde{f}_X(x)$ for a random variable $X(\omega)$ are given. Find the RND $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega)$ in terms of the $X(\omega)$.

(b) In the diagram below, $X : \Omega \to \mathbb{R}$ is a random variable with a smooth probability density function. A smooth function $g(x) : \mathbb{R} \to \mathbb{R}$ represents the RND

$$g\big(X(\omega)\big) = \frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega).$$

Let us consider a random variable $Y(\omega) = h^{-1}\big(X(\omega)\big)$, or $X(\omega) = h\big(Y(\omega)\big)$, where $h(x) : \mathbb{R} \to \mathbb{R}$ is a monotonic and smooth function on $\mathbb{R}$ and h^{-1} is the inverse function. If the random variable $Y(\omega)$ under the new measure $\tilde{\mathbb{P}}$ has a probability density function

$$\tilde{f}_Y(x) = f_X(x),$$

find the function $h(y)$.

$$\begin{tikzcd}
\big(\Omega, \mathcal{F}, \mathbb{P}\big) \arrow[r, "X(\omega)"] \arrow[d, "\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega) = g[X(\omega)]"'] & f_X(x) \arrow[d] \\
\big(\Omega, \mathcal{F}, \tilde{\mathbb{P}}\big) \arrow[r, "X(\omega)"] \arrow[ur, "Y(\omega)"] & \tilde{f}_X(x)
\end{tikzcd}$$

(c) Now consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $X(\omega) = \big(X_1, X_2, \cdots, X_n)(\omega)$ is a n-dimensional random variables, whose sccessive differences $X_j - X_{j-1}$ are all conditionally, normally distributed independent random variables:

$$X_{j+1} - X_j \sim \mathcal{N}\Big(\mu_{j+1}(X_j), \sigma^2_{j+1}(X_j)\Big).$$

Find the change of measures $Z(\omega) = \frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega)$ such that under the new measure $\tilde{\mathbb{P}}$,

$$X_{j+1} - X_j \sim \mathcal{N}\big(0, \sigma^2_{j+1}(X_j)\Big).$$

(d) What is the conditional expectation

$$\mathbb{E}\big[Z | X_1, \cdots, X_k\big]$$

for $k < n$?

3. [20pt] Let $(X, Y)(t)$ be an Ito process in $\mathbb{R}^2$, as the solution to the SDE

$$\left\{ \begin{array}{rcl} \mathrm{d}X(t) &=& \mu\big(t, X, Y\big)\mathrm{d}t + \sigma^2\big(t, X, Y\big)\mathrm{d}W(t), \\ \mathrm{d}Y(t) &=& \theta\big(t, X, Y\big)\mathrm{d}t, \end{array} \right.$$

in whch μ, σ, and θ are all continuous functions. Find the first and second variations of $Y(t)$.

4. [20pt] Consider SDE

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t.$$

(a) Show that

$$v(x,t) := \mathbb{E}\Big[\delta(x - X_t)\,\Big|\,X_0 = y\Big],$$

where $\delta(t)$ is the Dirac-δ function, satisfies the partial differential equation

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\sigma^2(x)}{2}v(x,t)\right) - \dfrac{\partial}{\partial x}\Big(\mu(x)v(x,t)\Big), \\ v(x,0) = \delta(x-y). \end{cases}$$

(b) Show that

$$u(x,t) = \mathbb{E}\Big[\varphi(X_t)\Big|X_0 = x\Big]$$

satisfies the partial differential equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\sigma^2(x)}{2}\dfrac{\partial^2 u(x,t)}{\partial x^2} + \mu(x)\dfrac{\partial u(x,t)}{\partial x}, \\ u(x,0) = \varphi(x). \end{cases}$$

5. [50pt] We denote Kolmogorov's backward and forward operators

$$\begin{aligned} \mathscr{L}_x[u] &= \frac{\sigma^2(x)}{2}\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + \mu(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}, \\ \mathscr{L}_x^*[f] &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\sigma^2(x)}{2}f(x,t)\right) - \frac{\mathrm{d}}{\mathrm{d}x}\Big(\mu(x)f(x,t)\Big). \end{aligned}$$

(a) Show that $\mathscr{L}_x$ has an alternative expression:

$$\mathscr{L}_x[u] = \frac{\sigma^2(x)s(x)}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(s^{-1}(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right), \tag{1}$$

where $s(x)$ is known as the scale density:

$$s(x) = \exp\left\{-\int\frac{2\mu(x)}{\sigma^2(x)}\mathrm{d}x\right\}.$$

(b) Give the corresponding expression, as in (1), for $\mathscr{L}_x^*$.

(c) Consider the linear partial differential equation (PDE)

$$\begin{cases} \dfrac{\partial f(x,t)}{\partial t} = \mathscr{L}_x^*\big[f(x,t)\big] - \gamma(t,x)f(x,t), \\ f(x,0) = \psi(x), \end{cases} \tag{2}$$

where the operator $\mathscr{L}_x^*$ is defined above. Express the solution to PDE (2) in terms of the Ito process X_t that satisfies

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t.$$

6. [30pt] Consider a two-dimensional SDE

$$\begin{aligned}\mathrm{d}X_1(t) &= \mu_1(X_1, X_2)\mathrm{d}t + \sigma\mathrm{d}W_1(t),\\ \mathrm{d}X_2(t) &= \mu_2(X_1, X_2)\mathrm{d}t + \sigma\mathrm{d}W_2(t),\end{aligned}$$

where

$$\big(\mu_1, \mu_2\big)(\mathbf{x}) = -\nabla U(\mathbf{x}) \text{ and } \mathbf{x} = (x_1, x_2),$$

and $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions.

(a) Give the generator $\mathcal{A}$ and its $L^2\big(\mathbb{R}^2, \mathrm{d}\mathbf{x}\big)$ adjoint $\mathcal{A}^*$. They are also known as Kolmogorov's backward and forward operators for the time-homogeneous Ito diffusion:

$$\mathcal{A}[u] = \frac{\sigma^2}{2}\nabla^2 u + \cdots,\ \mathcal{A}^*[f] = \frac{\sigma^2}{2}\nabla^2 f + \cdots.$$

(b) Show that under a proper choice of the weight $\rho(\mathbf{x}) > 0$ for the inner product between any $f, g \in L^2$

$$\langle f, g\rangle_\rho = \int_{\mathbb{R}^2} \rho(\mathbf{x}) f(\mathbf{x}) g(\mathbf{x})\mathrm{d}\mathbf{x},$$

$\mathcal{A}$ is self-adjoint, i.e.,

$$\langle f, \mathcal{A}g\rangle_\rho = \langle \mathcal{A}f, g\rangle_\rho.$$

You can assume that both $f(\mathbf{x})$ and $g(\mathbf{x})$, and their partial derivatives, go to 0 sufficiently fast as $|\mathbf{x}| \to \infty$.

(c) Similarly, with an alternative choice of the weight for the inner product, $\mathcal{A}^*$ is also self-adjoint:

$$\langle f, \mathcal{A}^* g\rangle_\rho = \langle \mathcal{A}^* f, g\rangle_\rho.$$

(Hint: This is the 2-d generalization of the material in MLN, Sec. 9.5)

7. [30pt] Let $X_t \in \mathbb{R}^2$ be a Lévy process defined by

$$X_t = \int_0^t \sigma(t)\mathrm{d}W(t) + \int_{\mathbb{R}^2} zN(t, \mathrm{d}z),$$

in which the second term is a compound Poisson process with scalar Poisson random measure $N(t, z, \omega)$ that is independent from the $W(t, \omega)$; $\sigma, N \in \mathbb{R}^1$ and $W, z \in \mathbb{R}^2$. If the Lévy measure

$$\nu(\mathrm{d}z) = \mathbb{E}\big[N(1, \mathrm{d}z)\big] = \frac{\lambda}{2\pi\eta^2} \exp\left(-\frac{z_1^2 + z_2^2}{2\eta^2}\right) \mathrm{d}z,$$

where $\mathrm{d}z = \mathrm{d}z_1 \mathrm{d}z_2$, and both λ and η are real positive numbers. Find the characteristic function for X_t.