AMATH 567, Homework 7

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December 7, 2022

1. **Problem 1:**

(a) Construct the bilinear transformation

$$w(z) = \frac{az+b}{cz+d}$$

that maps the region between the two circles $|z - \frac{1}{4}| = \frac{1}{4}$ and $|z - \frac{1}{2}| = \frac{1}{2}$ into an infinite strip bounded by the vertical lines $u = \Re\{w\} = 0$ and $u = \Re\{w\} = 1$. To avoid ambiguity, suppose that the outer circle is mapped to $u = 1$.

Solution: We can construct this transformation by considering certain specific points z and finding coefficients a, b, c and d which map them where we would like for them to end up.

To begin, we would like for the point $z = 0$ to be mapped to ∞, so we let $d = 0$. Next, since our domain is symmetric about the imaginary axis, we want values which lie on the real axis to remain on the real axis. In particular, we would like that $w(\frac{1}{2}) = 0$ and $w(1) = 1$. This first condition gives us

$$w(\frac{1}{2}) = \frac{a/2+b}{c/2} = 0 \Rightarrow a/2 + b = 0 \Rightarrow b = -a/2$$

Using this, the second condition gives us

$$\frac{a - a/2}{c} = 1 \Rightarrow c = \frac{a}{2}$$

Plugging these in, we find

$$w(z) = \frac{az - a/2}{az/2} \Rightarrow w(z) = \frac{2z-1}{z} = 2 - \frac{1}{z}$$

(b) Upon finding the appropriate transformation w, carefully show that the image of the inner circle under w is the vertical line $u = 0$, and similarly for the outer circle.

Solution: We can parameterize the inner circle using $C_1(\theta) = \frac{1}{4}(1 - e^{i\theta})$ for $\theta \in (0, 2\pi)$, and see what happens to the image of C_1 under w. We have

$$w(C_1(\theta)) = \frac{(1 - e^{i\theta})/2 - 1}{(1 - e^{i\theta})/4} = \frac{-2e^{i\theta} - 2}{1 - e^{i\theta}} = -2\frac{e^{i\theta/2} + e^{-i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = -2\frac{2\cos(\theta/2)}{-2i\sin(\theta/2)}$$

$$\Rightarrow w(C_1(\theta)) = \frac{-2i}{\tan(\theta/2)}$$

We see that the real part of the image of the inner circle is always zero, while the imaginary part ranges between $\pm\infty$. Hence, the inner circle is in fact mapped to the vertical line $u = 0$.

Following a similar process for the outer circle, we use the parameterization $C_2(\theta) = \frac{1}{2}(1 - e^{i\theta})$, and again apply w. We find

$$w(C_2(\theta)) = \frac{(1 - e^{i\theta}) - 1}{(1 - e^{i\theta})/2} = \frac{-2e^{i\theta}}{1 - e^{i\theta}} = \frac{-2e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = \frac{e^{i\theta/2}}{i\sin(\theta/2)}$$

$$\Rightarrow w(C_2(\theta)) = \frac{-i}{\tan(\theta/2)} + 1$$

Here we see that the real part of the image of the outer circle is always 1, while the imaginary part again ranges between $\pm\infty$. So the image of the outer circle is the vertical line in the complex plane which passes through $u = 1$.

2. **Problem 2:** Use the result of Problem 1 to find the steady state temperature $T(x, y)$ in the region bounded by the two circles, where the inner circle is maintained at $T = 0$°C and the outer circle at $T = 100$°C. Assume T satisfies the two-dimensional Laplace equation.

   Solution: We can solve this problem by first solving the Laplace equation for the transformed region $\{w : \Re(w) \in [0, 1]\}$, where the vertical line passing through $w = 0$ is maintained at 0°C and the vertical line passing through $w = 1$ is maintained at 100°C, and then use the inverse of the transformation given in Problem 1 to recover the solution to the problem we are interested in.

   To begin, we note that $f(w) = 100w$ is an analytic function, and therefore both its real and imaginary parts are solutions to the Laplace equation. Furthermore, the real part of this equation satisfies our boundary conditions, since $\Re\{f(0 + iy)\} = 0$ and $\Re\{f(1 + iy)\} = 100$.

   Now that we have found a solution in the transformed space, we can use the transformation from Problem 1 to write the solution for the region bounded by the two circles to find

$$T(x, y) = \Re\{100(2 - \frac{1}{z})\} = 100\left(2 - \Re\left\{\frac{1}{x + iy}\right\}\right)$$

$$\Rightarrow T(x, y) = 200 - \frac{100x}{x^2 + y^2}$$