AMATH 563 Homework 1

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Problem 1. Prove that $C([a,b])$ equipped with the $L^2([a,b])$ norm is not a Banach space.

Solution. To show that $C([a,b])$ equipped with the $L^2([a,b])$ norm is not a Banach space it is sufficient to show that $C([a,b])$ is not complete. That is, there exists a Cauchy sequence of functions $(f_i \in C([a,b]))_{i=1}^{\infty}$ which converges to a limit function $f \notin C([a,b])$.

Consider the following discontinuous step function defined on the $[a,b]$ interval.

$$f(x) = \begin{cases} 0, & x \in [a, \frac{a+b}{2}) \\ 1, & x \in [\frac{a+b}{2}, b] \end{cases}$$

A standard result from Fourier theory is that step functions over finite intervals can be constructed using a Fourier series. In particular, $f(x)$ can be can be constructed as the limit as $i \to \infty$ of the sequence of Fourier partial sums defined by

$$f_i(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{i} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\,\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right)$$

As finite sums of analytic functions, it is clear that $f_i \in C([a,b])$ for all $i \in \mathbb{N}$. Furthermore the convergence properties of the Fourier series imply that the sequence of partial sums converges to $f(x)$. Since the sequence converges, it is a Cauchy sequence. However, the limit function $f(x)$ is discontinuous, and so it does not belong to $C([a,b])$.

We have found a Cauchy sequence $f_i \in C([a,b])$ which converges in the $L^2([a,b])$ norm to a discontinuous function $f \notin C([a,b])$. This shows that $C([a,b])$ equipped with the $L^2([a,b])$ norm is not a Banach space.

Problem 2. If $(X_1, ||\cdot||_1)$ and $(X_2, ||\cdot||_2)$ are normed spaces, show that the (Cartesian) product space $X = X_1 \times X_2$ becomes a normed space with the norm $||x|| = \max(||x_1||_1, ||x_2||_2)$ where $x \in X$ is defined as the tuple $x = (x_1, x_2)$ with addition and scalar multiplication operations $(x_1, x_2) + (y_1, y_2) = (x_1 + x_2, y_1 + y_2)$ and $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.

Solution. To begin, we note that the product space X equipped with binary addition and scalar multiplication operations as defined forms a vector space. To show that $(X, ||\cdot||)$ is a normed space we must show that $||\cdot||$ satisfies the following norm axioms:

1. $||x|| \geq 0$: We have $||x|| = \max(||x_1||_1, ||x_2||_2)$. Since $||\cdot||_1$ and $||\cdot||_2$ are norms it follows by definition that $||x_1||_1, ||x_2||_2 \geq 0$ for all x_1, x_2. Hence $\max(||x_1||_1, ||x_2||_2) \geq 0$, so $||x|| \geq 0$.

2. $||x|| = 0 \Leftrightarrow x = 0$: Since $||x_1||_1, ||x_2||_2 \geq 0$ we have that $||x|| = \max(||x_1||_1, ||x_2||_2) = 0$ only if $||x_1||_1 = ||x_2||_2 = 0$, and since $||\cdot||_1$ and $||\cdot||_2$ are both norms this implies that $x_1 = x_2 = 0$, and hence that $x = 0$. Conversely, if $x = 0 = (0,0)$ we have $||x|| = \max(||0||_1, ||0||_2) = \max(0,0) = 0$. This proves that $||x|| = 0 \Leftrightarrow x = 0$.

3. $\forall \alpha \in \mathbb{R} : ||\alpha x|| = |\alpha| \cdot ||x||$: We have $||\alpha x|| = ||\alpha(x_1, x_2)|| = ||(\alpha x_1, \alpha x_2)|| = \max(||\alpha x_1||_1, ||\alpha x_2||_2) = \max(|\alpha| \cdot ||x_1||_1, |\alpha| \cdot ||x_2||_2) = |\alpha| \max(||x_1||_1, ||x_2||_2) = |\alpha| \cdot ||x||$.

4. The triangle inequality, $||x + x'|| \leq ||x|| + ||x'||$: We have $||x + x'|| = ||(x_1 + x_1', x_2 + x_2')|| = \max(||x_1 + x_1'||_1, ||x_2 + x_2'||_2)$. Since $||\cdot||_1$ and $||\cdot||_2$ are norms, we can use the triangle inequality for each of them, resulting in $\max(||x_1 + x_1'||_1, ||x_2 + x_2'||_2) \leq \max(||x_1||_1 + ||x_1'||_1, ||x_2||_2 + ||x_2'||_2)$. Hence $||x + x'|| \leq ||x|| + ||x'||$ and so $||\cdot||$ satisfies the triangle inequality.

Since the norm $||\cdot||$ satisfies the four norm axioms it follows that $(X, ||\cdot||)$ is a normed space.

Problem 3. Show that the product (composition) of two linear operators, if it exists, is a linear operator.

Solution. Let $f : X \to Y$ and $g : Y \to Z$ be linear operators. Now consider the composition $g \circ f : X \to Z$. Let $x_1, x_2 \in X$, then we have

$$g\circ f(x_1+x_2) = g(f(x_1+x_2)) = g(f(x_1)+f(x_2)) = g(f(x_1))+g(f(x_2)) = g\circ f(x_1)+g\circ f(x_2)$$

Hence $g \circ f$ satisfies the additive property. Additionally, we have for $x \in X$

$$g \circ f(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha g \circ f(x)$$

Hence $g \circ f$ satisfies homogeneity of scalar multiplication. Since it satisfies these properties by definition $g \circ f$ is a linear operator. Hence, the product of any two linear operators, if it exists, is a linear operator.

Problem 4. Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$. Show that the Range$(T) = Y$ if and only if T^{-1} exists.

Solution. We will begin by showing that Range$(T) = Y \implies \exists T^{-1}$, and then we will prove the reverse implication. First, we note that if Range$(T) = Y$ then by definition T is surjective. Now fix some $y \in Y$ and let $x_1, x_2 \in X$ be two vectors such that $T(x_1) = T(x_2) = y$. Subtracting $T(x_2)$ and applying the linearity property of T gives us

$$T(x_1) - T(x_2) = T(x_1 - x_2) = 0$$

Hence $x_1 - x_2$ lies in the nullspace of the operator T. We recall that for finite dimensional linear maps the rank-nullity theorem states that

$$\begin{aligned}\text{Rank}(T) + \text{Nullity}(T) &= \dim X = n \\ \Rightarrow n + \text{Nullity}(T) &= n \\ \Rightarrow \text{Nullity}(T) &= 0\end{aligned}$$

From this is follows that $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$. Hence T is also injective. Since T is both injective and surjective it is a bijection, and there exists an inverse linear operator T^{-1}.

We will now show the reverse implication, $\exists T^{-1} \implies \text{Range}(T) = Y$. In this case we have that T is a bijection by definition, from which it follows that T is surjective, and hence that $\text{Range}(T) = Y$.

Hence, we have

$$\text{Range}(T) = Y \Leftrightarrow \exists T^{-1}$$

Problem 5. Let T be a bounded linear operator from a normed space X onto a normed space Y. Show that if there is a positive constant b such that $||Tx|| \geq b||x||$ for all $x \in X$ then T^{-1} exists and is bounded.

Solution. We will begin by showing that T^{-1} exists, that is, T is a bijection. First, we note from the problem statement that T is surjective by definition. Next, we note that $||Tx|| \geq b||x||$ implies that the null space of T is trivial. Now let $x_1, x_2 \in X$ such that $Tx_1 = Tx_2$, then by the linearity of T we have $T(x_1 - x_2) = 0$. Since the null space of T is trivial we must have $x_1 = x_2$, and so T is injective. Since T is both injective and surjective, it is a bijection and hence T^{-1} exists.

We will now show that T^{-1} satisfying the problem statement is necessarily bounded. Let $Tx = y$, then from the problem statement we have $||Tx|| = ||y|| \geq b||x||$. Then, using $x = T^{-1}y$, we have $||y|| \geq b||T^{-1}y||$ and hence

$$||T^{-1}y|| \leq \frac{1}{b}||y||$$

This inequality holds for all $y \in Y$, which means that T^{-1} is bounded.

Problem 6. Consider the functional $f(x) = \max_{t\in[a,b]} x(t)$ on $C([a,b])$ equipped with the sup norm. Is this functional linear? Is it bounded?

Solution. f is not linear, but it is bounded.

To show this, we will begin with the conditions for linearity. $f(x)$ satisfies the additivity property. That is, for $x, y \in C([a,b])$,

$$f(x+y) = \max_{t\in[a,b]}[x(t) + y(t)] = \max_{t\in[a,b]} x(t) + \max_{t\in[a,b]} y(t) = f(x) + f(y)$$

However, f is not invariant under scalar multiplication. Instead, we have

$$f(\alpha x) = \max_{t\in[a,b]} \alpha x(t) = |\alpha| \max_{t\in[a,b]} \text{sgn}(\alpha)x(t) = |\alpha| f(\text{sgn}(\alpha)x) \neq \alpha f(x)$$

It follows that f is not a linear functional.

We continue now to the question of boundedness. Using the sup norm we have $||x|| = \sup_{t\in[a,b]} |x(t)|$. We note that

$$f(x) = \max_{t\in[a,b]} x(t) \leq \sup_{t\in[a,b]} |x(t)| = ||x||$$

from which it follows that

$$||f(x)|| \leq ||x||$$

and so f is bounded.

Problem 7. Let X be a Banach space and denote its dual as X^*. Show that $||\varphi|| : \varphi \mapsto \sup_{||x||=1} |\varphi(x)|$ is a norm on X^*.

Solution. We recall the axioms which define the vector norm:

1. $||\varphi|| \geq 0$: We have $||\varphi|| = \sup_{||x||=1} |\varphi(x)|$, and so since $|\varphi(x)| \geq 0$ it follows that $||\varphi|| \geq 0$.
2. $||\varphi|| = 0 \Leftrightarrow \varphi(x) = 0$: We have

$$||\varphi|| = \sup_{||x||=1} |\varphi(x)| = 0 \implies \varphi(x) = 0.$$

We also have

$$\varphi(x) = 0 \implies \sup_{||x||=1} |\varphi(x) = ||\varphi|| = 0.$$

Hence we have $||\varphi|| = 0 \Leftrightarrow \varphi(x) = 0$.

3. $||\alpha\varphi|| = |\alpha| \cdot ||\varphi||$: We have

$$||\alpha\varphi|| = \sup_{||x||=1} |\alpha\varphi(x)| = |\alpha| \cdot \sup_{||x||=1} |\varphi(x)| = |\alpha| \cdot ||\varphi||$$

4. Triangle inequality $||x + x'|| \leq ||x|| + ||x'||$: We have

$$\begin{aligned} ||\varphi + \phi|| = \sup_{||x||=1} |\varphi(x) + \phi(x)| &\leq \sup_{||x||=1} \Big[|\varphi(x)| + |\phi(x)|\Big] \\ &\leq \sup_{||x||=1} |\varphi(x)| + \sup_{||x||=1} |\phi(x)| \\ &\leq ||\varphi|| + ||\phi|| \end{aligned}$$

Hence $|| \cdot ||$ satisfies the triangle inequality.

Since the given functional satisfies all of the above properties, it is indeed a norm on X^*.

Problem 8. Prove the Schwartz inequality on inner product spaces: $|\langle x, y\rangle| \leq ||x|| \cdot ||y||$ for all $x, y \in X$, where equality holds if and only if x, y are linearly dependent.

Solution. Consider any two vectors $x, y \in X$. We define the projection

$$y_{||} = \frac{\langle x, y\rangle}{||x||^2} \cdot x = \alpha x \qquad\qquad y_\perp = y - y_{||}$$

Using this, we can write the inner product as

$$\langle x, y\rangle = \langle x, y_{||} + y_\perp\rangle = \langle x, y_{||}\rangle + \langle x, y_\perp\rangle = \langle x, y_{||}\rangle = \alpha\langle x, x\rangle$$

from which it follows that

$$|\langle x, y\rangle| = |\alpha| \cdot ||x||^2 = |\alpha| \cdot ||x|| \cdot ||x|| = ||x|| \cdot ||y_{||}||$$

Now, in the case where x, y are linearly dependent we have $y = y_{||}$ and hence $|\langle x, y\rangle| = ||x|| \cdot ||y_{||}|| = ||x|| \cdot ||y||$. If x and y are not linearly dependent then $y_{\perp} \neq 0$, and therefore $||y|| = ||y_{||} + y_{\perp}|| > ||y_{||}||$, where the inequality is guaranteed by the orthogonality of $y_{||}$ and $y_{\perp}$. From this it follows that

$$|\langle x, y\rangle| = ||x|| \cdot ||y_{||}|| \leq ||x|| \cdot ||y||$$

which gives us the Schwartz inequality

$$\forall x, y \in X : |\langle x, y\rangle| \leq ||x|| \cdot ||y||$$

where the equality holds if and only if x, y are linearly dependent.