Approach 5 : solving the full initial-value, boundary value problem.

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = Q(x,t), \qquad t > 0$$

$$Q(x,t) = q(x)\, e^{-i\omega_0 t} \qquad -\infty < x < \infty$$

$$u(x,0) = 0$$

$$u_t(x,0) = 0$$

$$u(x,t) \to 0 \text{ as } x \to \pm\infty . \quad \text{fixed } t .$$

Use Laplace transform in t. Assume $u(x,t)$ is one-sided.

We find it more convenient to let

$$s = -i\omega$$

$$\tilde{u}(x,s) = \mathcal{L}[u(x,t)] \equiv U(x,\omega) = \int_0^\infty u(x,t)\, e^{i\omega t}\, dt$$

$$\Big[\ \mathcal{L}[u(x,t)] = \int_0^\infty u(x,t)\, e^{-st}\, dt = \tilde{u}(x,s)$$

$$= \int_0^\infty u(x,t)\, e^{i\omega t}\, dt$$

$$\equiv U(x,\omega) \qquad \Big]$$

$$\mathcal{L}\left[\frac{\partial^2}{\partial t^2} u\right] = s^2 \tilde{u} = -\omega^2 U$$

$$\mathcal{L}[Q(x,t)] = \mathcal{L}[q(x) e^{-i\omega_0 t}]$$

$$= q(x) \cdot \int_0^\infty e^{-i\omega_0 t - st} dt$$

$$= q(x) \cdot \int_0^\infty e^{i(\omega - \omega_0)t} dt$$

$$= \frac{1}{i(\omega-\omega_0)} e^{i(\omega-\omega_0)t} \Big|_0^\infty$$

$$= -\frac{1}{i(\omega-\omega_0)}$$ provided that $\operatorname{Im} \omega > 0$ $(\operatorname{Re} s > 0)$

The PDE becomes an ODE

$$\boxed{\frac{d^2}{dx^2} U + k^2 U = \frac{q(x)}{i c^2 (\omega - \omega_0)}}$$

$-\infty < x < \infty$ $\qquad k \equiv \omega / c$

This ODE in x can be solved either with Fourier transform in x, or with variation of parameters.

Here we use the superposition principle and consider first

$$g(x) = \delta(x-y)$$

and then superpose.

$$\frac{d^2}{dx^2}\mho + k^2\mho = \frac{\delta(x-y)}{ic^3(k-k_0)}$$

For $x \neq y$, $\delta(x-y) = 0$

$\omega \equiv kc$

$$\frac{d^2}{dx^2}\mho + k^2\mho = 0$$

$$\mho(x,\omega) = \begin{cases} A(\omega)\, e^{ik(x-y)} & \text{for } x > y \\ B(\omega)\, e^{-ik(x-y)} & \text{for } x < y \end{cases}$$

These satisfy $\mho \to 0$ as $x \to \pm\infty$

since $\Im\omega > 0$.

No need for radiation condition.

Matching at $x=y$:

$\bar{U}$ continuous implies $A=B$

The second matching condition is obtained by integrating the 2nd order ODE once:

$$\frac{d}{dx}\bar{U}\Big|_{x=y^-}^{x=y^+} + k^2\int_{y^-}^{y^+}\bar{U}\,dx = \int_{y^-}^{y^+}\frac{\delta(x-y)\,dx}{ic^3(k-k_0)}$$

$$\int_{y^-}^{y^+}\delta(x-y)\,dx = 1$$

$\int_{y^-}^{y^+}\bar{U}\,dx \to 0$ as $y^+ - y^- \to 0$

unless $\bar{U}$ is a delta function; then $\frac{d}{dx}\bar{U}$ is $\delta'(x-y)$ and is inconsistent with the RHS.

$$\frac{d}{dx}\bar{U}\Big|_{x=y^-}^{x=y^+} = \frac{1}{ic^3(k-k_0)}$$

$$\frac{d}{dx} U \Big|_{x=y+} = A ik \, e^{ik(x-y)} \Big|_{x=y+}$$

$$= ikA$$

$$\frac{d}{dx} U \Big|_{x=y-} = -ikB \, e^{-ik(x-y)} \Big|_{x=y-}$$

$$= -ikB$$

$$2ikA = \frac{1}{ic^3(k-k_0)}$$

$$A = \frac{-1}{2c^3 k(k-k_0)}$$

The solution is

$$U(x,\omega) = \frac{-1}{2c^3 k(k-k_0)} e^{ik|x-y|}$$

Inverse Laplace transform:

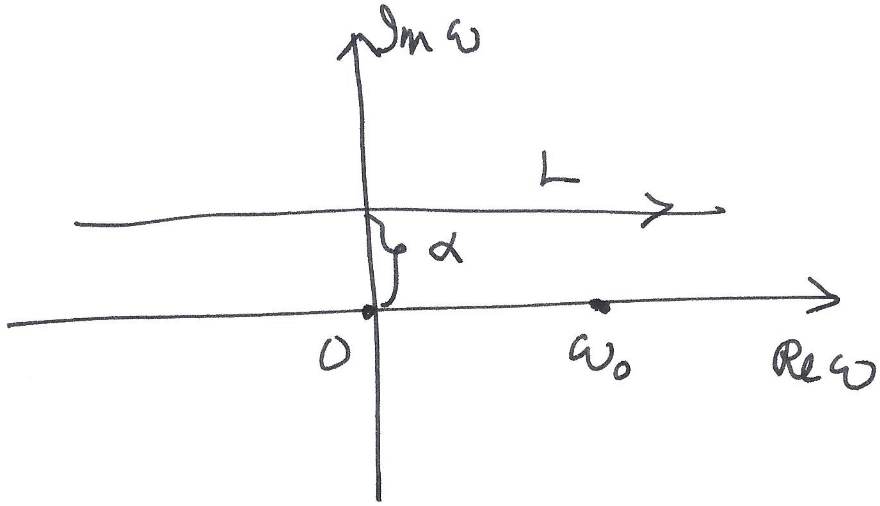
$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty+\alpha}^{i\infty+\alpha} \tilde{u}(x,s)\, e^{st}\, ds$$

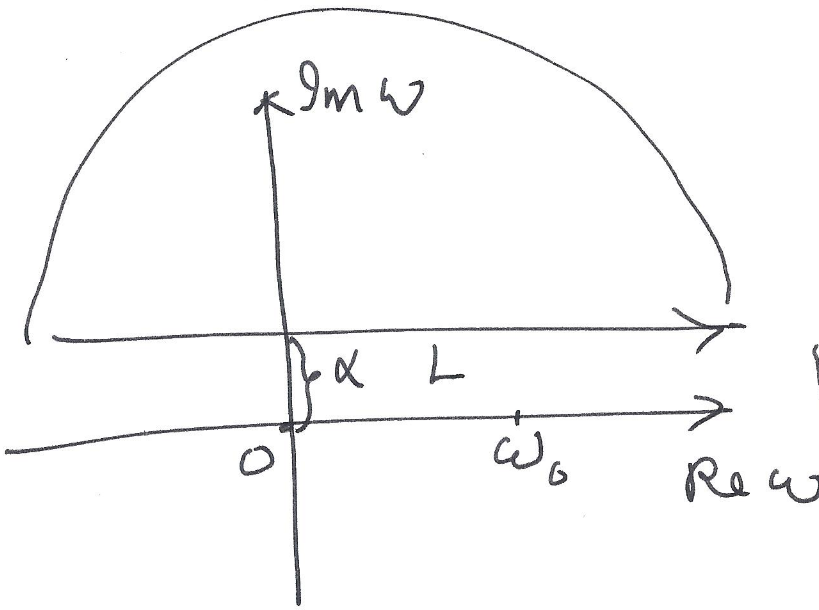
$(\alpha > 0)$

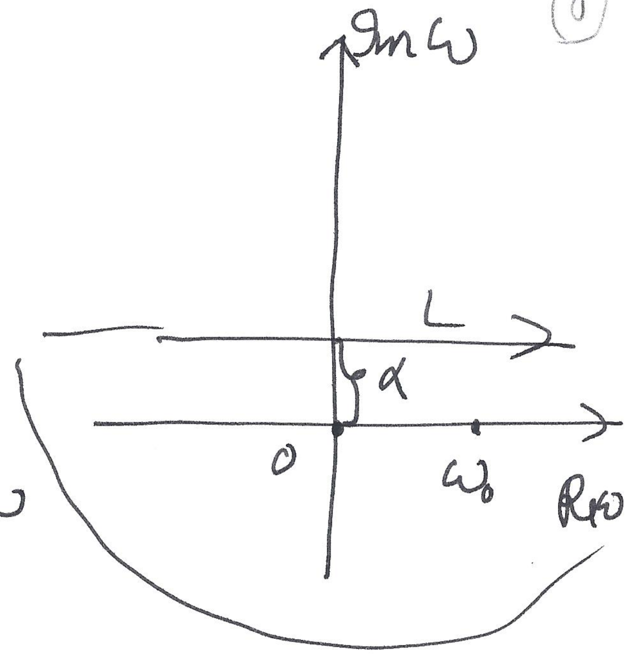
$$= \frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} \bar{U}(x,\omega)\, e^{-i\omega t}\, d\omega$$

$$= -\frac{1}{4\pi c}\int_{-\infty+i\alpha}^{\infty+i\alpha} \frac{1}{\omega(\omega-\omega_0)}$$

$$\times \exp\left\{-i\omega\left(t - |x-y|/c\right)\right\} d\omega$$







For $t < |x-y|/c$, close in the upper half plane by Jordan's Lemma

$$u(x,t) = \int d\omega = 0$$

because there is no singularity within the closed contour.

For $t > |x-y|/c$, close in the lower half plane; encloses two singularities

$$u(x,t) = \frac{i}{2c} \sum \left\{ \text{Res of } \frac{1 \exp\{-i\omega(t-|x-y|/c)\}}{\omega \cdot (\omega-\omega_0)} \right\}$$

at $\omega = 0$ and $\omega = \omega_0$

$$u(x,t) = \frac{i}{2c}\left\{ \text{Res of } \frac{1}{\omega(\omega-\omega_0)} e^{-i\omega(t-|x-y|/c)} \text{ at } \omega=0 \text{ and } \omega=\omega_0 \right\}$$

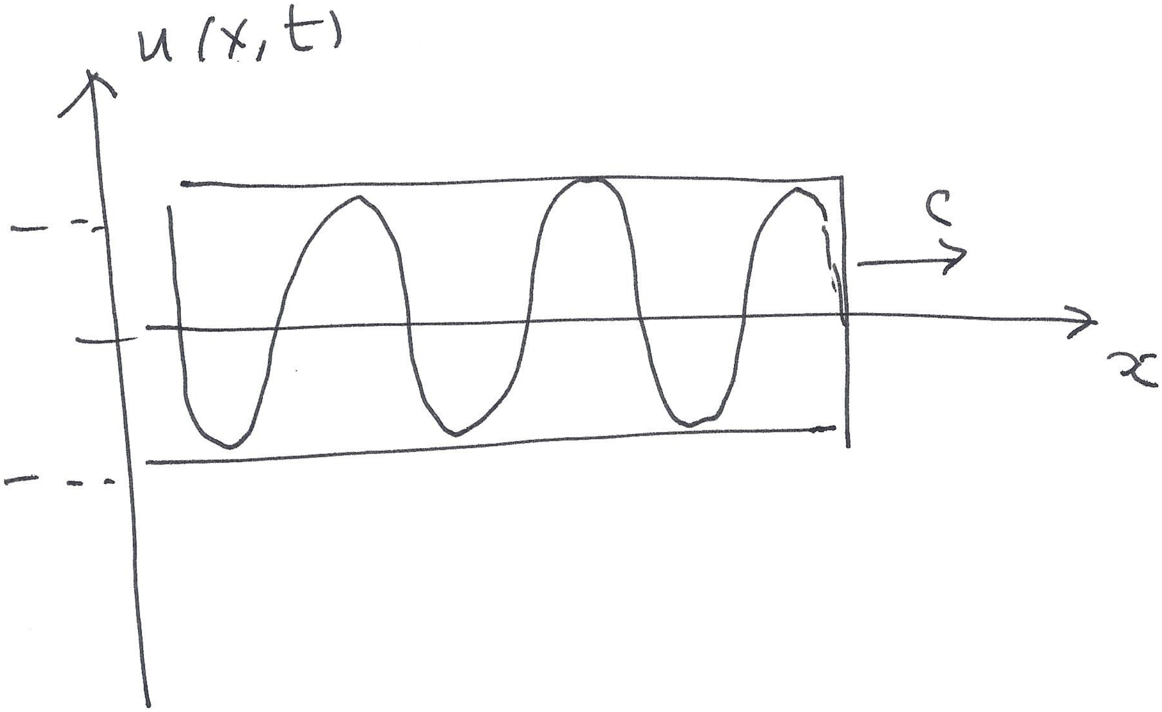
$$= \frac{i}{2c}\left\{ -\frac{1}{\omega_0} + \frac{1}{\omega_0} e^{-i\omega_0(t-|x-y|/c)} \right\}$$

Therefore:

$$\boxed{u(x,t) = \frac{i}{2c\omega_0}\left\{ e^{-i\omega_0(t-|x-y|/c)} - 1 \right\} \times H(t-|x-y|/c).}$$

The Heaviside function is there to express the fact that

$$u(x,t) = 0 \text{ for } t < |x-y|/c.$$



There is a wavefront travelling with speed c, ahead of which there is no signal (since $H(t-|x-y|/c) = 0$ for $|x-y| > ct$).

Behind the wavefront there is a radiating wave, satisfying Sommerfeld's radiation condition.

The -1 is the solution to the homogeneous PDE, needed to satisfy the zero initial condition.

For a fixed finite t,

$$u(x,t) \to 0 \quad \text{as } x \to \pm\infty$$

satisfying the zero boundary conditions.

The previous solutions can be reconciled as the limit of this solution with $t \to \infty$ first with x fixed.

For general $q(x)$, superposition yields

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} dy\, q(y)\, H(t - |x-y|/c) \frac{i}{2c\omega_0} \left\{ e^{-i\omega_0 (t - |x-y|/c)} - 1 \right\}}$$