Lecture 9

Example: The Drunken Sailor problem

PDE: $\frac{\partial}{\partial t} u = D \frac{\partial^2}{\partial x^2} u, \quad -\infty < x < \infty$

$t > 0$

BC: $u(x,t) \to 0$ as $x \to \pm\infty, \quad t > 0$

IC: $u(x,0) = \delta(x), \quad -\infty < x < \infty$

Fourier transform in x:

$$U(\omega, t) = \mathcal{F}[u(x,t)]$$

$$= \int_{-\infty}^{\infty} e^{i\omega x} u(x,t)\, dx$$

$$\mathcal{F}[u_t] = D\, \mathcal{F}[u_{xx}]$$

$$\mathcal{F}[u_t] = U_t$$

$$\mathcal{F}[u_{xx}] = \int_{\infty}^{\infty} e^{i\omega x} u_{xx}\, dx$$

$$= -\omega^2 U$$

after applying BC $u(x,t) \to 0$ as $x \to \pm\infty$

and assuming $u_x(x,t) \to 0$ as $x \to \pm\infty$.

$$\boxed{\frac{\partial}{\partial t}\widetilde{U} = -D\omega^2 \widetilde{U},}$$

an ODE !

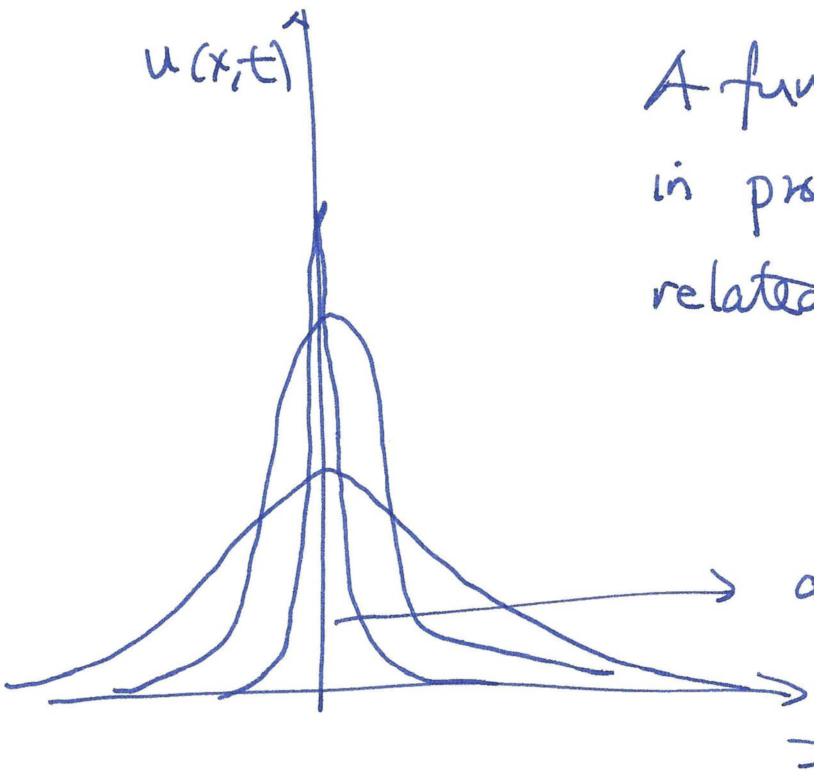
Solution: $\widetilde{U}(\omega,t) = \widetilde{U}(\omega,0)e^{-D\omega^2 t}$

$\widetilde{U}(\omega,0) = \mathcal{F}[u(x,0)] = \mathcal{F}[\delta(x)] = 1$

$u(x,t) = \mathcal{F}^{-1}[\widetilde{U}(\omega,t)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x - D\omega^2 t}\, d\omega$$

$$= \frac{1}{\sqrt{4\pi D t}} \exp\left\{-\frac{x^2}{4Dt}\right\}$$



A fundamental solution in probability, finance, related to random walk.

Area under the curve conserved.

$\int_{-\infty}^{\infty} u(x,t)\,dx = 1.$

3-D Heat equation

PDE: $\frac{\partial}{\partial t} u = D \nabla^2 u$, $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

$$-\infty < x_1 < \infty$$

$$-\infty < x_2 < \infty$$

$$-\infty < x_3 < \infty$$

BCs: $u(\vec{x}, t) \to 0$ as $x_1 \to \pm\infty$
$x_2 \to \pm\infty$
$x_3 \to \pm\infty$

IC: $u(\vec{x}, 0) = \delta^3(\vec{x}) \equiv \delta(x_1)\delta(x_2)\delta(x_3)$

Apply 3-D Fourier transform in the 3 space dimensions:

$$U(\vec{\lambda}, t) = \mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 [u(\vec{x}, t)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\vec{x}, t)\, e^{i(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)}\, dx_1 dx_2 dx_3$$

$$\boxed{\frac{\partial}{\partial t} U = -D\lambda^2 U} \qquad \lambda^2 \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

Solution of the ODE: $U(\vec{\lambda}, t) = U(\vec{\lambda}, 0)\, e^{-\lambda^2 D t}$

$$U(\vec{\lambda},0) = \mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 [u(\vec{x},0)] = \mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 [\delta^3(\vec{x})]$$

$$= \mathcal{F}_1[\delta(x_1)]\, \mathcal{F}_2[\delta(x_2)]\, \mathcal{F}_3[\delta(x_3)]$$

$$= 1 \cdot 1 \cdot 1 = 1$$

$$\boxed{U(\vec{\lambda},t) = \exp\{-\lambda^2 Dt\}}$$

$$= e^{-\lambda_1^2 Dt}\, e^{-\lambda_2^2 Dt}\, e^{-\lambda_3^2 Dt}$$

Inverse transform:

$$u(\vec{x},t) = \mathcal{F}_1^{-1} \mathcal{F}_2^{-2} \mathcal{F}_3^{-3} [U(\vec{\lambda},t)]$$

$$= \mathcal{F}_1^{-1}[e^{-\lambda_1^2 Dt}]\, \mathcal{F}_2^{-1}[e^{-\lambda_2^2 Dt}]\, \mathcal{F}_3^{-1}[\quad]$$

$$= \frac{1}{\sqrt{4\pi Dt}} \exp\left\{-\frac{x_1^2}{4Dt}\right\} \frac{1}{\sqrt{4\pi Dt}} \exp\left\{-\frac{x_2^2}{4Dt}\right\}$$

$$\cdot \frac{1}{\sqrt{4\pi Dt}} \exp\left\{-\frac{x_3^2}{4Dt}\right\}$$

$$= \frac{1}{(4\pi Dt)^{3/2}} \exp\left\{-\frac{(x_1^2+x_2^2+x_3^2)}{4Dt}\right\}$$

n - D Heat equation

$$u(\vec{x}, t) = \frac{1}{(4\pi D t)^{n/2}} \exp\left\{ -\frac{r^2}{4Dt} \right\}$$

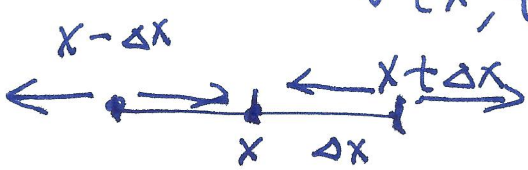
where $r^2 = |\vec{x}|^2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2$

Spreads radially from the origin.

Random walk:

Let $w(x,t)$ be the probability that a particle (or a drunken sailor) which starts at the origin at time $t=0$ is located at point x at a later time t.

It moves by Δx in Δt with equal likelihood of going forward or backward



$$w(x, t+\Delta t) = \tfrac{1}{2} w(x-\Delta x, t) + \tfrac{1}{2} w(x+\Delta x, t)$$

Initial condition: $w(0,0) = 1$

$$w(x,0) = 0, \quad x \neq 0$$

Consider the case of large number of steps, with x, t finite, but $\Delta x \to 0$, $\Delta t \to 0$

Taylor series expansion:

$$w(x, t+\Delta t) = w(x,t) + w_t(x,t)\,\Delta t + O(\Delta t^2)$$

$$w(x-\Delta x, t) = w(x,t) + w_x(x,t)\cdot(-\Delta x) + \frac{1}{2} w_{xx}(x,t)\,\Delta x^2 + \cdots$$

$$w(x+\Delta x, t) = w(x,t) + w_x(x,t)\cdot\Delta x + \frac{1}{2} w_{xx}(x,t)\,\Delta x^2 + \cdots$$

$$w_t(x,t)\,\Delta t + O(\Delta t^2) = \frac{1}{2} w_{xx}(x,t)\,(\Delta x^2) + O(\Delta x^3)$$

Divide by Δt

$$w_t(x,t) = \frac{(\Delta x)^2}{2\Delta t} w_{xx} + O\left(\frac{\Delta x^3}{\Delta t}\right) + \cdots$$

An interesting case

$$D = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{2\Delta t} \neq 0, \text{ finite}$$

For stock movements, D is determined by the "volatility" of that stock.

Let

$$u(x,t) \equiv \frac{W(x,t)}{\Delta x},$$

which is finite no matter how small Δx is

The probability of finding the particle between a and b is

$$\int_a^b u(x,t)\,dx \ ; \quad \int_{-\infty}^{\infty} u(x,t)\,dx = 1$$

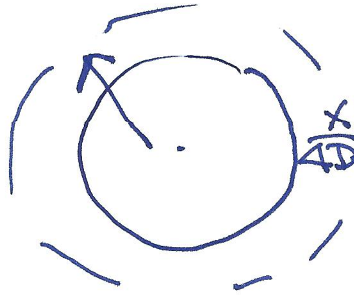
It is governed by a "diffusion" equation

$$u_t = D\,u_{xx}$$

Solution:

$$u(x,t) = \frac{1}{(4\pi D t)^{1/2}} \exp\left(-\frac{x^2}{4Dt}\right)$$

For $\frac{x^2}{4Dt}$ = a const, this contour moves a distance $x \sim \sqrt{4Dt}$ at time t.



$\frac{x^2}{4Dt} = 1$