AMATH 562 Homework Assignment #5
[Due online via Canvas: Thursday 11:59pm, March 2, 2022]

1. Please give an integer-valued stochastic Lévy process $(\eta_t)_{t\geq 0}$,

$$\eta_t \in \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\},$$

which, furthermore, is a martingale.

2. We have discussed in the class that "independent, stationary increments", under certain conditions, gives rise to the Gaussian nature of Brownian motion due to the central limit theorem: The item 3 in MLN's Definition 7.2.1. Discuss why the Lévy process as defined in Definition 10.1.1, item 3 cannot be improved to a Gaussian distribution?

3. A probability distribution is *infinitely divisible* if it can be expressed as the probability distribution of the sum of an arbitrary number of independent and identically distributed (i.i.d.) random variables. It turns out, every infinitely divisible probability distribution corresponds to a Lévy process.

(a) Show that normal distribution on $\mathbb{R}$ is infinitely divisible.

(b) Show that Poisson distribution on $\mathbb{N}$ is infinitely divisible.

(c) The pdf of Cauchy distribution is

$$f_X(x) = \frac{\gamma}{\pi(\gamma^2 + x^2)}.$$

Show its characteristic function $\phi_X(u) := \mathbb{E}\left[e^{iuX}\right]$ is $e^{-\gamma|u|}$. What are its expected value and its variance?

(d) A real-valued random variable X is infinitely divisible if and only if its characteristic function $\phi_X(u)$ is of the form $e^{\psi(u)}$, with

$$\psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iuz} - 1 - iuz\mathbb{1}_{|z|<1}\right)\nu(\mathrm{d}z). \tag{1}$$

We see that when the $\nu(\mathrm{d}z) = 0$, the $\phi_X(u)$ corresponds to the normal distribution $\mathcal{N}(\mu, \sigma^2)$. Taking Eq. 1 as given, with $\mu = \sigma = 0$, find the $\nu(\mathrm{d}z)$ such that X has a Cauchy distribution.

4. Let P_t be a Poisson process with rate λ. Introducing "compensated Poisson process"

$$\tilde{P}_t = P_t - \mathbb{E}[P_t].$$

(a) Show that its characteristic function has the form

$$\mathbb{E}\left[e^{iu\tilde{P}_t}\right] = e^{t\psi(u)}.$$

(b) Give the expression for $\psi(u)$.

5. Professor Matt Lorig's notes, exercise 10.1.