Lecture 16. A Radiation problem

We will solve this problem 5 different ways to demonstrate the connection among different methods. Also to illustrate the connection between integrability, existence of Fourier transform, singularity along the path of integration in the inverse transform and causality

Consider the wave equation with a persistent source with a given frequency:

PDE: $\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = Q(x,t), \quad -\infty < x < \infty$, $t > 0$

$Q(x,t) = q(x) e^{-i\omega_0 t}$, ω_0 a given frequency

$q(x)$ is of compact support

BCs: $u(x,t) \to 0$ as $x \to \pm\infty$, $t > 0$

ICs: $u(x,0) = 0, \quad u_t(x,0) = 0$

(source switched on at $t > 0$. Actually $Q(x,t) = H(t) \cdot q(x) e^{-i\omega_0 t}$)

A well-posed problem, solution exists satisfying the PDE, BCs and ICs, as will be shown.

Persistent source : If the source has been switched on long time ago and has persisted for a long time, physically one may assume that the solution would take on the frequency of forcing, with perhaps some phase shift.

So instead of solving the initial-value problem, one could look for a solution of the form:

$$u(x,t) = u(x)\, e^{-i\omega_0 t}.$$

Substituting this assumed form into the original PDE yields:

$$\frac{d^2}{dx^2}u + k_0^2 u = -q(x)/c^2, \text{ an ODE}$$

$$u(x) \to 0 \text{ as } x \to \pm\infty \qquad k_0 \equiv \omega_0/c$$

Causality is lost with this "simplification"!

This is a commonly adopted simplification in many applications.

solve the ODE using Fourier transform in x

$$\mathcal{F}[\frac{d^2}{dx^2}u] + k_0^2\,\mathcal{F}[u] = -\frac{1}{c^2}\mathcal{F}[q]$$

where $\mathcal{F}[u] = \int_{-\infty}^{\infty} u(x)\, e^{ikx}\, dx$, $= U(k)$

$\mathcal{F}[q] = Q(k)$

Since $q(x)$ is integrable, $Q(k)$ exists.

We need to <u>assume</u> the integrability of the unknown solution in order to proceed with this method.

$$\mathcal{F}[\frac{d^2}{dx^2}u] = \int_{-\infty}^{\infty} \frac{d^2}{dx^2}u \cdot e^{ikx}\, dx$$

$$= \frac{d}{dx}u\, e^{ikx}\Big|_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty}\frac{d}{dx}u\, e^{ikx}\, dx$$

$$= \frac{d}{dx}u \cdot e^{ikx}\Big|_{-\infty}^{\infty} - ik\, u\, e^{ikx}\Big|_{-\infty}^{\infty} + (ik)^2\int_{-\infty}^{\infty} u e^{ikx}\, dx$$

$$= -k^2 U$$

<u>assuming</u> $\frac{d}{dx}u \to 0$ as $x \to \pm\infty$
(to be verified a posteri.)

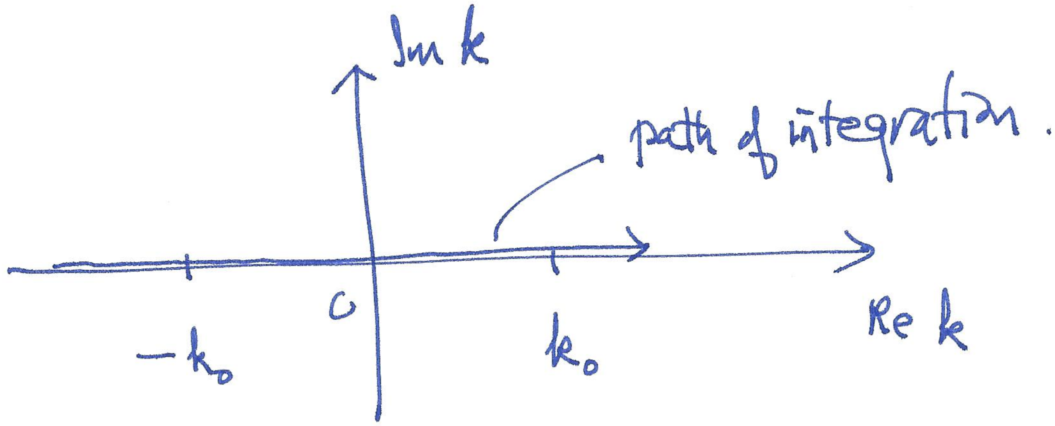
$u \to 0$ as $x \to \pm\infty$.

$$[-k^2 + k_0^2]\,\tilde{U}(k) = -\frac{1}{c^2}\,Q(k)$$

$$\tilde{U}(k) = \frac{\frac{1}{c^2}\,Q(k)}{[k^2 - k_0^2]}\,.$$

Inverse Fourier transform:

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(k)\, e^{-ikx}\, dk$$



There are two singularities along the path of integration. Integral undefined. Should we take the Principal value?

We can actually solve the ODE without using Fourier transform.

The solution consists of two homogeneous solutions plus a particular solution:

$$u(x) = u_1(x) + u_2(x) + u_p(x)$$

where $u_1(x) = A e^{ik_0 x}$, $u_2(x) = B e^{-ik_0 x}$ are the homogeneous solutions. The particular solution can be found using variation of parameters

$$u_p(x) = u_1(x) \int^x \frac{-u_2(y)\cdot\left(-\frac{g(y)}{c^2}\right) dy}{W(u_1(y), u_2(y))}$$

$$+ \; u_2(x) \int^x \frac{u_1(y)\cdot\left(-\frac{g(y)}{c^2}\right) dy}{W(u_1(y), u_2(y))}$$

$$= \frac{i}{2k_0 c^2} \int_0^x e^{ik_0(x-y)} g(y)\, dy$$

$$- \frac{i}{2k_0 c^2} \int_0^x e^{-ik_0(x-y)} g(y)\, dy$$

Thus

$$u(x) = \left[A + \frac{i}{2k_0c^2} \int_0^x e^{-ik_0 y} q(y)\,dy \right] e^{ik_0 x}$$

$$+ \left[B - \frac{i}{2k_0c^2} \int_0^x e^{ik_0 y} q(y)\,dy \right] e^{-ik_0 x}$$

The solution extends throughout the x domain even though the forcing $q(x)$ is localized. This explains why we ran into trouble with the Fourier transform; $u(x)$ does not have a Fourier transform. (We should have used Generalized Fourier transform to do this problem). Singularity along the real axis (the path of integration in the inverse Fourier transform) is a consequence of this non-integrability. Also, the original boundary condition: $u(x) \to 0$ as $x \to \pm\infty$ cannot be satisfied. The source has acted for an infinitely long time, and so the wave would have reached all space. How to determine A and B?

Approach 1: Sommerfeld's radiation condition

The physicist Arnold Sommerfeld suggested that one should impose a radiation condition, that waves should be outgoing as $x \to \pm\infty$. This is in lieu of causality.

That is, the wave should be left going as $x \to -\infty$, and right going as $x \to \infty$ in this simple case. (In more complicated physical problem it is the group velocity's direction that should be specified.)

We want $u(x) \propto e^{-ik_0 x}$ as $x \to -\infty$

so that $u(x,t) \propto e^{-ik_0(x+ct)}$.

And $u(x) \propto e^{ik_0 x}$ as $x \to +\infty$

so that $u(x,t) \propto e^{ik_0(x-ct)}$.

Thus, rewriting the solution

$$u(x) = \left[A + \frac{i}{2k_0c^2} \int_\infty^x e^{-ik_0y} q(y)\,dy \right] e^{ik_0x}$$

$$+ \left[B - \frac{i}{2k_0c^2} \int_{-\infty}^x e^{ik_0y} q(y)\,dy \right] e^{-ik_0x}$$

[The lower limits are changed to facilitate taking $x \to \infty$ or $x \to -\infty$, by changing A and B, which are arbitrary anyway.]

We want $[\quad]e^{-ik_0x}$ to vanish as $x \to \infty$

$$\lim_{x\to\infty} u(x) = \left[A + \frac{i}{2k_0c^2} \int_\infty^\infty e^{-ik_0y} q(y)\,dy \right] e^{ik_0x}$$

$$+ \left[B - \frac{i}{2k_0c^2} \int_{-\infty}^\infty e^{ik_0y} q(y)\,dy \right] e^{-ik_0x}$$

$$= A e^{ik_0x} \quad \text{if} \quad B = \frac{i}{2k_0c^2} \int_{-\infty}^\infty e^{ik_0y} q(y)\,dy.$$

As $x \to -\infty$, we want $[\quad]e^{ik_0x}$ to vanish, yielding

$$\lim_{x\to-\infty} u(x) = \left[A + \frac{i}{2k_0c^2} \int_\infty^{-\infty} e^{-ik_0y} q(y)\,dy \right] e^{ik_0x}$$

$$+ B e^{-ik_0x},$$

yielding $A = \frac{i}{2k_0c^2} \int_{-\infty}^\infty e^{-ik_0y} q(y)\,dy] \cancel{e^{ik}}$.

$$u(x) = \frac{i}{2k_0c^2}\left[\int_{-\infty}^{\infty} - \int_x^{\infty} e^{-ik_0 y} q(y)\,dy\right] e^{ik_0 x}$$

$$+ \frac{i}{2k_0c^2}\left[\int_{-\infty}^{\infty} - \int_{-\infty}^{x} e^{ik_0 y} q(y)\,dy\right] e^{-ik_0 x}$$

$$= \frac{i}{2k_0c^2}\left[\int_{-\infty}^{x} e^{ik_0(x-y)} q(y)\,dy \right. , \quad x > y$$

$$\left. + \int_x^{\infty} e^{-ik_0(x-y)} q(y)\,dy \right] , \quad y > x$$

$$= \frac{i}{2k_0c^2}\left[\int_{-\infty}^{x} e^{ik_0(x-y)} q(y)\,dy \right. , \quad x > y$$

$$\left. + \int_x^{\infty} e^{ik_0|x-y|} q(y)\,dy \right]$$

$$= \frac{i}{2k_0c^2}\int_{-\infty}^{\infty} q(y)\, e^{ik_0|x-y|}\,dy$$