AMATH 567, Homework 4

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1. **Problem 1:** Using residue calculus, calculate

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x} dx$$

Solution: We substitute the complex variable z for x and consider the contour integral

$$I_C = \oint_C \frac{e^{iz}}{\sinh z} dz$$

where C is a positively oriented semicircular contour in the upper half of the complex plane, in the limit as $R \to \infty$. We note that $\frac{1}{\sinh z} \to 0$ as $|z| \to \infty$, and therefore by Jordan's lemma, $I = \Im(I_C)$. The challenge now is to evaluate this contour integral using the residue theorem.

We begin by identifying the poles of the integrand. The denominator of the integrand is $\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right)$, and so we have a simple pole whenever $e^z = e^{-z}$, which occurs whenever $z = i\pi n$ for $n \in Z$. Since C surrounds only the upper half of the complex plane, only the poles at $z = 0, i\pi, 2i\pi, \ldots$ will contribute to the integral.

Lastly, we note that the simple pole at $z = 0$ lies on the contour C, and therefore this is an improper integral, and so we will evaluate the principle value PI_C. We recall that simple poles on the contour of a principle value integral contribute half the normal residue as a pole inside the contour, and we will make use of this fact later when evaluating the integral.

With that, we can now evaluate the residue of I_C at a simple pole $z_0 = i\pi n$. Since the numerator of our integrand is analytic, we can make use of the formula for calculating simple poles of functions of the form $f(z) = P(z)/Q(z)$, where P is analytic, which is $Res\{f(z); z_0\} = P(z_0)/Q'(z_0)$. Through this formula, we see that

$$Res(i\pi n) = \frac{e^{-\pi n}}{\cosh(i\pi n)} = \frac{e^{-\pi n}}{\cos(n\pi)} = \left(-e^{-\pi}\right)^n$$

We also see that, for the pole at $z = 0$, $Res(0) = \frac{e^0}{\cosh(0)} = 1$. Hence, by the residue theorem

$$I_C = \pi i Res(0) + 2\pi i \sum_{n=1}^{\infty} Res(i\pi n) = \pi i - 2\pi i e^{-\pi} \sum_{n=0}^{\infty} \left(-e^{-\pi}\right)^n$$

Note that the righthand sum is a geometric series of the form $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$. In this case, $r = -e^{-\pi} \Rightarrow |r| = e^{-\pi} < 0$, so this sum converges to $\frac{1}{1+e^{-\pi}}$. Plugging this result in to the above equation gives us

$$I_C = \pi i - 2\pi i \frac{e^{-\pi}}{1 + e^{-\pi}} = i\pi \tanh \frac{\pi}{2}$$

And so our final answer is

$$I = \Im(I_C) = \pi \tanh \frac{\pi}{2}$$

2. **Problem 2:** Using residue calculus, calculate

$$I = \int_{-\infty}^{\infty} \frac{1 + \cos(x)}{(x - \pi)^2} dx$$

Solution: We will begin by making the substitution $x' = x - \pi$, so that our integral becomes

$$I = \int_{-\infty}^{\infty} \frac{1 + \cos(x' + \pi)}{x'^2} dx' = \int_{-\infty}^{\infty} \frac{1 - \cos(x')}{x'^2} dx'$$

Next we will use Jordan's lemma and consider the contour integral

$$I_C = \oint_C \frac{1 - e^{iz}}{z^2} dz$$

Where C is a positively oriented semicircular contour in the upper half of the complex plane. Note that $I = \Re(I_C)$. It is clear that I_C has a pole at $z = 0$, which lies on the contour C. By expanding the numerator of the integrand into the infinite series $1 - e^{iz} = -iz + \frac{z^2}{2} + \ldots$, we see that the integrand is of the form $-\frac{i}{z} + \frac{1}{2} + \ldots$. Hence $z = 0$ is a simple pole, and I_C is an improper integral whose principle value we can evaluate.

The residue at $z = 0$ can be evaluated using L'Hôpital's rule

$$Res(0) = \lim_{z \to 0} \frac{1 - e^{iz}}{z} = \lim_{z \to 0} -ie^{iz} = -i$$

Therefore, using the residue theorem formula for the principle value of an improper integral, we find that

$$I_C = \pi i(-i) = \pi$$

And since $I = \Re(I_C)$, we have

$$I = \pi$$

3. **Problem 3:** Evaluate the following integral using residue calculus

$$I = \int_0^\infty \frac{x^a}{a + 2x\cos b + x^2} dx$$

where $-1 < a < 1$, $a \neq 0$, and $-\pi < b < \pi$, $b \neq 0$. Justify all key steps. Do not use the general formula for this integral.

Solution: Due the the x^a in the numerator, the integrand is multi-valued for complex valued arguments. We can verify this using an AB test, where we select a point A on the unit circle with $\arg A = 0$ and a point B on the unit circle with $\arg B \to 2\pi$. We find that $f(B) = e^{2\pi a} f(A)$. Therefore our integrand is in general multi-valued. To resolve this, we restrict ourselves to a branch where $0 \leq \theta < 2\pi$ in which the function is analytic. In doing so we create a branch cut on the positive real axis over which the function is discontinuous, however we are now able to use residue calculus on our chosen branch.

We begin by considering the circular contour C with radius $R \to \infty$ which encompasses the complex plane and has a branch cut along the positive real axis. In particular we can decompose C into four components $C = C_0 + C_R + C_1 + C_2$. C_0 is a straight contour along the positive real axis from $(0, R]$; C_R is a circular arc of radius R with argument $\theta \in [0, 2\pi)$; C_1 is a straight contour along the positive real axis from $[R, 0)$; and C_2 is a small circular contour of radius ρ with argument $\theta \in (2\pi, 0]$. (See diagram in Prof. Tung's Complex Analysis Lecture 14 notes, page 3). We note that $I = I_{C_0}$.

We will begin by evaluating the contour integral

$$I_C = \oint_C \frac{z^a}{a + 2z\cos b + z^2} dz$$

where $\arg z \in [0, 2\pi)$.

The integrand has simple poles when $1 + 2z\cos b + z^2 = 0$. We can rewrite this expression as $1 + 2z\cos b + z^2 = (z + \cos b)^2 + \sin^2(b) = (z + \cos b + i\sin b)(z + \cos b - i\sin b) = (z + e^{ib})(z + e^{-ib}) = 0$. From this it is clear that this expression has roots, and therefore our integrand has (simple) poles, at $z = -e^{\pm ib}$. Since the problem description tells us that $b \neq 0$, neither of these points lie on the real axis, and therefore they must lie within the contour C. We will now calculate each of their respective residues.

(a)

$$Res(-ie^{ib}) = \lim_{z \to -e^{ib}} \frac{z^a}{z + e^{-ib}} = \frac{e^{ia(b+\pi)}}{-e^{ib} + e^{-ib}} = \frac{ie^{ia(b+\pi)}}{2\sin b}$$

Where we have used $-1 = e^{i\pi}$.

(b)

$$Res(-e^{-ib}) = \lim_{z\to -e^{-ib}} \frac{z^a}{z+e^{ib}} = \frac{e^{-ia(b-\pi)}}{-e^{-ib}+e^{ib}} = \frac{-ie^{-ia(b-\pi)}}{2\sin b}$$

Where we have again used $-1 = e^{i\pi}$. (Note that, since we have restricted ourselves to the particular branch with $\theta \in [0, 2\pi)$, we cannot use $-1 = e^{-i\pi}$).

Therefore, by the residue theorem,

$$I_C = 2\pi i \left[\frac{ie^{ia\pi}}{2\sin b}\left(e^{iab} - e^{-iab}\right)\right] = -2\pi i e^{ia\pi}\frac{\sin ab}{\sin b}$$

Next, we will analyze the contour integrals along each of the sub-contours. We will begin with the fixed-radius circular contour C_R. We have

$$|I_{C_R}| \le \oint_{C_R} |zf(z)|dz = \oint_{C_R} \lim_{R\to\infty} \frac{R^{a+1}}{|a+2z\cos b + z^2|}dz$$

We note that in the denominator of the right-hand side that the z^2 term will dominate both a ($|a| < 1$) and $2z\cos b$ ($|2z\cos b| < |2z|$) in the limit as $R \to \infty$, and hence

$$|I_{C_R}| \le \oint_{C_R} \lim_{R\to\infty} \frac{R^{a+1}}{R^2}dz$$

Since $|a| < 1$ it follows that $a+1 < 2$, and so $\lim_{R\to\infty} \frac{R^{a+1}}{R^2} = 0$, and hence

$$|I_{C_R}| \le 0 \Rightarrow I_{C_R} = 0$$

Next we will consider the contour integral along the small circular contour C_2. Here we have $z = \rho e^{i\theta}$ where $\rho \to 0^+$. Therefore

$$I_{C_2} = \int_{2\pi}^{0} \lim_{\rho\to 0^+} \frac{\rho^a e^{ia\theta}}{a + 2\rho e^{i\theta}\cos b + \rho^2 e^{2i\theta}} i\rho e^{i\theta}d\theta = -i\int_0^{2\pi} \lim_{\rho\to 0^+} \frac{\rho^{a+1}e^{i(a+1)\theta}}{a+2\rho e^{i\theta}\cos b + \rho^2 e^{2i\theta}}d\theta$$

We note that as $\rho \to 0^+$ the denominator will approach a, as it is the only term that does not include a factor of ρ. Taking the magnitude of the integrand in the limit as $\rho \to 0^+$, we see that

$$|I_{C_2}| \le \int_0^{2\pi} \lim_{\rho\to 0^+} \frac{\rho^{a+1}}{a}d\theta = 0 \Rightarrow I_{C_2} = 0$$

Next we will consider the contour integral over C_1, the straight line running from R to 0, with . We have

$$I_{C_1} = \int_R^0 \lim_{\theta\to 2\pi} \frac{r^a e^{ia\theta}}{a + 2re^{i\theta}\cos b + r^2 e^{2i\theta}} e^{i\theta}dr = -e^{2\pi ai}\int_0^R \frac{r^a}{a+2r\cos b + r^2}dr$$

We note that in the limit as $R \to \infty$, $I_{C_1} \to -e^{2\pi a i} I$, where I is the integral we are trying to evaluate. Therefore, combining this with our earlier observation that $I_{C_0} = I$ and that $I_{C_R} = I_{C_2} = 0$, we have

$$I_C = \left(1 - e^{2\pi a i}\right) I = -2\pi i e^{ia\pi} \frac{\sin ab}{\sin b}$$

Therefore, solving for I gives us the answer

$$I = \frac{\sin ab}{\sin b \sin \pi a}$$