AMATH 569 Homework Assignment #5, Spring 2023
Assigned: May 17, 2023
Due, May 24, 2023

1. For the 1-dimensional heat equation for conduction in a copper rod:

$$\frac{\partial}{\partial t}u = \alpha^2 \frac{\partial^2}{\partial x^2}u, \;\; 0 < x < L,\, t > 0.$$
$$u(x,t) = 0 \text{ at } x = 0 \text{ and } x = L.$$
$$u(x,0) = f(x),\; 0 < x < L.$$

(a) solve using separation of variables. Show that the time-dependence for the n^{th} standing mode is
$\exp(-n^2(t/t_e))$, where $t_e \equiv (L/\pi\alpha)^2$, about 1 hour for a 2m-copper rod .

(b) For $t > t_e$, find the mode that dominates the solution, and thus write down the approximate solution (in time and in space). Describe in words how an initial condition, which may not look like a sine wave, becomes the sine shape with the largest wavelength fitting in between the boundaries.

2. Sound waves in a box satisfies

PDE: $\frac{\partial^2}{\partial t^2}u - c^2\nabla^2 u = 0, \quad \text{in } V$

BC: $u = 0 \text{ on } \partial V.$

Use separation of variables to solve this problem for the following configurations and find the quantized frequency of oscillation ω, where ω appears in the time dependence of the solution in the form of a sine or cosine of ωt.
(a) V is a one-dimensional box: $0<x<L$.
(b) V is a two-dimensional box: $0<x<L;\ 0<y<L$.
(c) V is a three-dimensional box: $0<x<L;\ 0<y<L;\ 0<z<L$.

3. (a) Solve:
$$\frac{d^2}{dx^2}u + (k_0^2 + i\varepsilon k_0 / c)u = -\delta(x-y)/c^2, \quad -\infty < x < \infty,\ \varepsilon > 0,\ y \text{ finite.}$$
subject to the boundary condition $u \to 0$ as $x \to \pm\infty$. Consider separately $x < y$ and $x > y$, and matching across $x = y$. Do not use Fourier transform.

(b) Solve the above equation for the case of $\varepsilon = 0$, subject to the Sommerfeld radiation condition. Show that the solution from (a) reduce to this solution as $\varepsilon \to 0$.