AMATH 568
Advanced Differential Equations
Homework 1

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Due: January 11, 2023

1. Determine the eigenvalues and eigenvectors (real solutions), (b) sketch the behavior and classify the behavior.

   (a) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

   Solution: We seek eigenvalues λ satisfying

   $$\det \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5 = 0$$

   Expanding the polynomial expression gives us the quadratic equation

   $$\lambda^2 + 1 = 0$$

   Which has the solutions $\lambda = \pm i$. We will now find the corresponding eigenvectors $\mathbf{v}_{\pm i}$.

   i. $\mathbf{v}_i$: We have

   $$\begin{aligned} 2v_1 - 5v_2 &= iv_1 \\ v_1 - 2v_2 &= iv_2 \end{aligned}$$

   From the second equation we see that $v_1 = (i+2)v_2$. Plugging this into the first equation gives us

   $$2(i+2)v_2 - 5v_2 = i(i+2)v_2 \Rightarrow v_2 \cdot 0 = 0$$

   Hence, we find the eigenvector

   $$\mathbf{v}_i = \begin{pmatrix} i+2 \\ 1 \end{pmatrix}$$

ii. $\mathbf{v}_{-i}$: We have

$$\begin{aligned} 2v_1 - 5v_2 &= -iv_1 \\ v_1 - 2v_2 &= -iv_2 \end{aligned}$$

Again we see from the second equation that $v_1 = (2-i)v_2$, and so the first equation becomes

$$2(2-i)v_2 - 5v_2 = -i(2-i)v_2 \Rightarrow v_2 \cdot 0 = 0$$

Hence, we find the eigenvector

$$\mathbf{v}_{-i} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

Since the eigenvalues are purely imaginary, the solutions are completely oscillatory with elliptic trajectories.

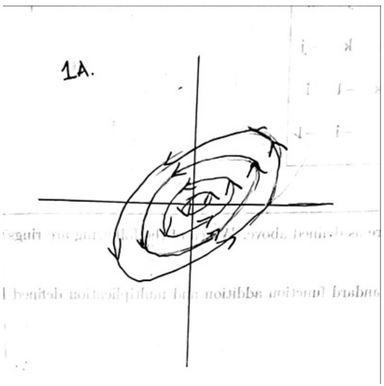


Figure 1: Sketch of 1A behaviour.

(b) $\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix} \mathbf{x}$

Solution: We seek eigenvalues λ satisfying

$$\det \begin{vmatrix} -1-\lambda & -1 \\ 0 & -0.25-\lambda \end{vmatrix} = (-1-\lambda)(-0.25-\lambda) = 0$$

Which has solutions $\lambda = -1$ and $\lambda = -1/4$. We will now find the corresponding eigenvectors $\mathbf{v}_{-1}$ and $\mathbf{v}_{-0.25}$.

i. $\mathbf{v}_{-1}$: We have

$$\begin{aligned} -v_1 - v_2 &= -v_1 \\ -0.25v_2 &= -v_2 \end{aligned}$$

From the second equation we see that $v_2 = 0$, and hence

$$v_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

ii. $\mathbf{v}_{-0.25}$: We have

$$-v_1 - v_2 = -0.25v_1$$
$$-0.25v_2 = -0.25v_2$$

From the first equation we see that $v_2 = -0.75v_1$, which results in the eigenvector

$$\mathbf{v}_{-0.25} = \begin{pmatrix} 1 \\ -0.75 \end{pmatrix}$$

Since the eigenvalues are real, unequal, and negative, the sytems behavior is that of a nodal sink, and all trajectories go to zero.

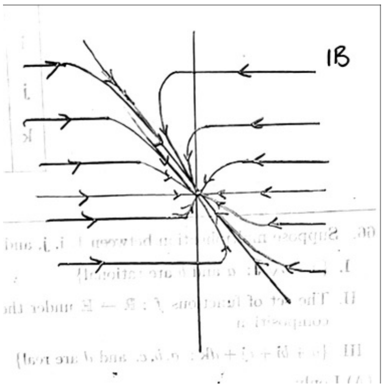


Figure 2: Sketch of 1B behaviour.

(c) $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

Solution: We seek eigenvalues λ satisfying

$$\det \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 4 = 0$$

This polynomial can be expanded as

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$$

Which has the double root $\lambda = 1$. The behaviour of this system will depend on whether there exist two independent eigenvectors which share the eigenvalue $\lambda = 1$.

We have

$$3v_1 - 4v_2 = v_1$$
$$v_1 - v_2 = v_2$$

From either equation we can see that $v_1 = 2v_2$. Therefore any eigenvector must be along

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Since there is only one linearly independent eigenvector, we must create a generalized eigenvector of A. That is, a vector $\mathbf{u}$ such that $(A - I)\mathbf{u} = \mathbf{v}$. In particular, we are looking to solve the system of equations

$$\begin{aligned} 2u_1 - 4u_2 &= 2 \\ u_1 - 2u_2 &= 1 \end{aligned}$$

We can see from either equation that $u_1 = 2u_2 + 1$, and so $\mathbf{u}$ is of the form

$$\mathbf{u} = \begin{pmatrix} 2a + 1 \\ a \end{pmatrix}$$

Where $a \in \mathbf{R}$. One may choose $a = 0$ for simplicity, or $a = -2/5$ so that $\mathbf{v}$ and $\mathbf{u}$ are orthogonal, for instance.

Having generated a generalized eigenvector $\mathbf{u}$, we may write a general solution to the system $\mathbf{x}' = A\mathbf{x}$ as

$$\mathbf{x} = c_1 \mathbf{v} e^{\lambda t} + c_2 \left[\mathbf{v} t e^{\lambda t} + \mathbf{u} e^{\lambda t} \right]$$

and the behavior of this system is that of an improper node.

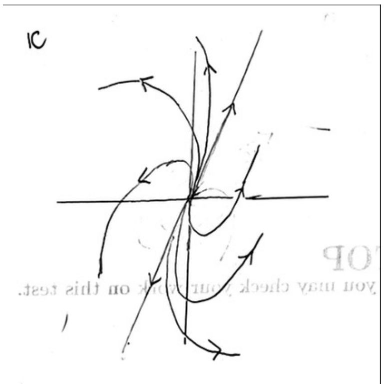


Figure 3: Sketch of 1C behaviour.

(d) $\mathbf{x}' = \begin{pmatrix} 2 & -5/2 \\ 9/5 & -1 \end{pmatrix} \mathbf{x}$

Solution: We seek eigenvalues λ satisfying

$$\det \begin{vmatrix} 2 - \lambda & -5/2 \\ 9/5 & -1 - \lambda \end{vmatrix} = (2 - \lambda)(-2 - \lambda) + 9/2 = 0$$

This polynomial can be expanded as

$$\lambda^2 - \lambda + 5/2 = 0$$

which has solutions $\lambda_{\pm} = (1 \pm 3i)/2$.

i. $\mathbf{v}_+$: We have

$$2v_1 - \frac{5}{2}v_2 = \frac{1}{2}(1+3i)v_1$$
$$\frac{9}{5}v_1 - v_2 = \frac{1}{2}(1+3i)v_2$$

From either equation we can see that $v_1 = \frac{5}{6}(1+i)v_2$. Hence we find the eigenvector

$$\mathbf{v}_+ = \begin{pmatrix} 5+5i \\ 6 \end{pmatrix}$$

ii. $\mathbf{v}_-$: We have

$$2v_1 - \frac{5}{2}v_2 = \frac{1}{2}(1-3i)v_1$$
$$\frac{9}{5}v_1 - v_2 = \frac{1}{2}(1-3i)v_2$$

From either equation we can see that $v_1 = \frac{5}{6}(1-i)v_2$, and so we find the eigenvector

$$v_- = \begin{pmatrix} 5-5i \\ 6 \end{pmatrix}$$

Since the eigenvalues are complex valued with positive real part, the behaviour of this system is that of an outward spiral.

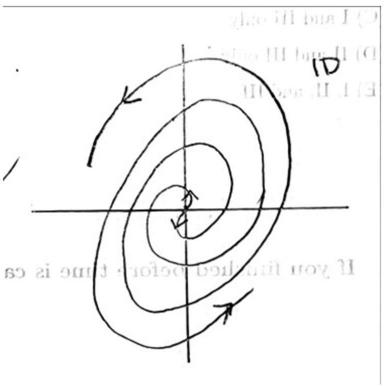


Figure 4: Sketch of 1D behaviour.

(e) $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

Solution: We seek eigenvalues λ satisfying

$$\det \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 3 = 0$$

which can be expanded into the form

$$\lambda^2 - 1 = 0$$

Which has the solutions $\lambda_\pm = \pm 1$.

i. $\mathbf{v}_+$: We have

$$2v_1 - v_2 = v_1$$
$$3v_1 - 2v_2 = v_2$$

We see from either equation that $v_1 = v_2$, and so we have the eigenvector

$$\mathbf{v}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

ii. $\mathbf{v}_-$: We have

$$2v_1 - v_2 = -v_1$$
$$3v_1 - 2v_2 = -v_2$$

We see from either equation that $3v_1 = v_2$, and so we get the eigenvector

$$\mathbf{v}_- = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Since the eigenvalues are real with opposite sign, the behavior of this system is that of a saddle, wherein vectors along $\mathbf{v}_+$ grow while those along $\mathbf{v}_-$ decay.

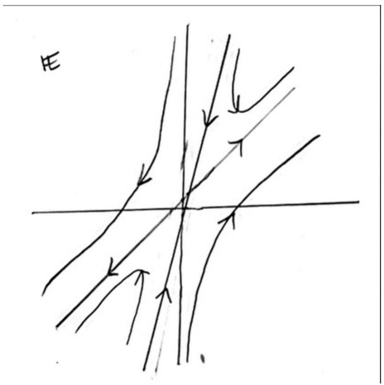


Figure 5: Sketch of 1E behaviour.

(f) $\mathbf{x}' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \mathbf{x}$

Solution: We seek eigenvalues λ satisfying

$$\det \begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 3 = 0$$

which can be expanded into the form

$$\lambda^2 - 4 = 0$$

which has solutions $\lambda_\pm = \pm 2$.

i. $\mathbf{v}_+$: We have

$$\begin{aligned} v_1 + \sqrt{3}v_2 &= 2v_1 \\ \sqrt{3}v_1 - v_2 &= 2v_2 \end{aligned}$$

From either equation we see that $v_1 = \sqrt{3}v_2$, and so we have the eigenvector

$$\mathbf{v}_+ = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

ii. $\mathbf{v}_-$: We have

$$\begin{aligned} v_1 + \sqrt{3}v_2 &= -2v_1 \\ \sqrt{3}v_1 - 1v_2 &= -2v_2 \end{aligned}$$

From either equation we can see that $-\sqrt{3}v_1 = v_2$, and so we have the eigenvector

$$\mathbf{v}_- = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

Since the eigenvalues are real with opposite sign, the behavior of this system is that of a saddle, wherein vectors along $\mathbf{v}_+$ grow while those along $\mathbf{v}_-$ decay.

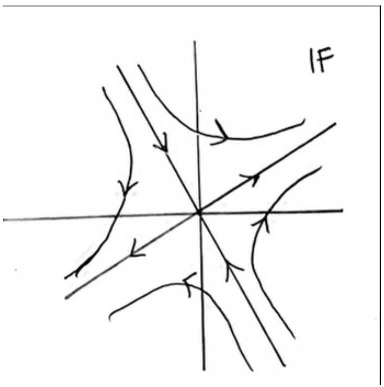


Figure 6: Sketch of 1F behaviour.

(g) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

Solution: We seek eigenvalues λ satisfying

$$\det \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = 0$$

which can be expanded into the form

$$\lambda^2 - \lambda - 2 = 0$$

which has the solutions $\lambda = -1$ and $\lambda = 2$.

i. $\mathbf{v}_{-1}$: We have

$$\begin{aligned}3v_1 - 2v_2 &= -v_1\\ 2v_1 - 2v_2 &= -v_2\end{aligned}$$

From either equation we see that $2v_1 = v_2$, and so we have the eigenvector

$$\mathbf{v}_{-1} = \begin{pmatrix}1\\2\end{pmatrix}$$

ii. $\mathbf{v}_2$: We have

$$\begin{aligned}3v_1 - 2v_2 &= 2v_1\\ 2v_1 - 2v_2 &= 2v_2\end{aligned}$$

From either equation we see that $v_1 = 2v_2$, and so we find the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix}2\\1\end{pmatrix}$$

Since the eigenvalues are real valued and have opposite sign, the behavior of this system is that of a saddle point.

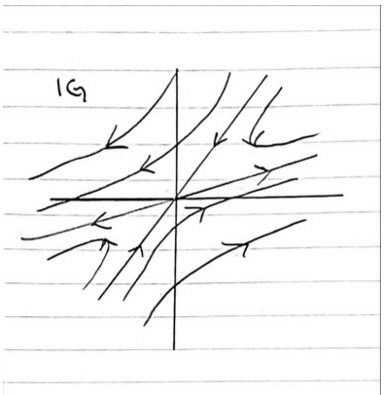


Figure 7: Sketch of 1G behaviour.

2. Consider $x' = -(x-y)(1-x-y)$ and $y' = x(2+y)$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.

Solution: We seek equilibrium solutions for which $x' = y' = 0$. From the first equation we have $x' = 0$ when $x = y$ or when $x + y = 1$, and from the second equation we have $y' = 0$ when $x = 0$ or when $y = -2$. Therefore we have equilibrium points at $(0,0), (0,1), (-2,-2),$ and $(3,-2)$. We will consider the behaviour of the system near each of these points.

(a) $(0,0)$: We let $x = 0 + \tilde{x}$ and $y = 0 + \tilde{y}$, where $\tilde{x}, \tilde{y} << 1$. Then the system of equations becomes

$$\begin{aligned}&\tilde{x}' = -(\tilde{x} - \tilde{y})(1 - \tilde{x} - \tilde{y}) = -\tilde{x} + \tilde{y} + \mathcal{O}(\tilde{x}^2, \tilde{y}^2)\\ &\tilde{y}' = \tilde{x}(2 + \tilde{y}) = 2\tilde{x} + \mathcal{O}(\tilde{x}\tilde{y})\end{aligned}$$

This gives us the linearized system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$.

To solve this system, we let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. That is, we seek eigenvalues λ satisfying

$$\det \begin{vmatrix} -1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = (-1-\lambda)(-\lambda) - 2 = 0$$

This equation can be expanded and written in the form

$$\lambda^2 + \lambda - 2 = 0$$

which has solutions $\lambda = -2$ and $\lambda = 1$. Since these eigenvalues are real and have opposite sign we conclude that $(0, 0)$ is a saddle point. Decay occurs along eigenvector $\mathbf{v}_{-2}$ and growth occurs along $\mathbf{v}_1$

(b) $(0, 1)$: We let $x = \tilde{x}$ and $y = 1 + \tilde{y}$. Then the system of equations becomes

$$\begin{aligned} \tilde{x}' &= -(\tilde{x} - \tilde{y} - 1)(1 - \tilde{x} - \tilde{y} - 1) = -\tilde{x} - \tilde{y} + \mathcal{O}(\tilde{x}^2, \tilde{y}^2) \\ \tilde{y}' &= \tilde{x}(2 + 1 + \tilde{y}) = 3\tilde{x} + \mathcal{O}(\tilde{x}\tilde{y}) \end{aligned}$$

This results in the linearized system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix}$.

To solve this system, we again let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. This time we seek eigenvalues λ satisfying

$$\det \begin{vmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} = (-1-\lambda)(-\lambda) + 3 = 0$$

which can be expanded into the form

$$\lambda^2 + \lambda + 3 = 0$$

with solutions $\lambda = (-1 \pm i\sqrt{11})/2$. Since these are complex eigenvalues with negative real part, we conclude that there is a stable equilibrium point at $(0, 1)$ where nearby values spiral inward.

(c) $(-2, -2)$: We now let $x = -2 + \tilde{x}$ and $y = -2 + \tilde{y}$. Then the system of equations becomes

$$\begin{aligned} \tilde{x}' &= -(-2 + \tilde{x} + 2 - \tilde{y})(1 + 2 - \tilde{x} + 2 - \tilde{y}) = -5\tilde{x} + 5\tilde{y} + \mathcal{O}(\tilde{x}^2, \tilde{y}^2) \\ \tilde{y}' &= (-2 + \tilde{x})(2 - 2 + \tilde{y}) = -2\tilde{y} + \mathcal{O}(\tilde{x}\tilde{y}) \end{aligned}$$

Resulting in the linearized system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix}$.

To solve this system, we again let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. We seek eigenvalues λ satisfying

$$\det \begin{vmatrix} -5-\lambda & 5 \\ 0 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) = 0$$

This characteristic equation has solutions $\lambda = -5$ and $\lambda = -2$. Since these eigenvalues are real, unequal, and negative, we conclude that the equilibrium point at $(-2, -2)$ is a nodal sink.

(d) $(3, -2)$: We now let $x = 3 + \tilde{x}$ and $y = -2 + \tilde{y}$. Then the system of equations becomes

$$\begin{aligned}\tilde{x}' &= -(3 + \tilde{x} + 2 - \tilde{y})(1 - 3 - \tilde{x} + 2 - \tilde{y}) = 5\tilde{x} + 5\tilde{y} + \mathcal{O}(\tilde{x}^2, \tilde{y}^2) \\ \tilde{y}' &= (3 + \tilde{x})(2 - 2 + \tilde{y}) = 3\tilde{y} + \mathcal{O}(\tilde{x}\tilde{y})\end{aligned}$$

Resulting in the linearized system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix}$.

To solve this system, we again let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. We therefore seek eigenvalues λ satisfying

$$\det \begin{vmatrix} 5 - \lambda & 5 \\ 0 & 3 - \lambda \end{vmatrix} = (5 - \lambda)(3 - \lambda) = 0$$

Which has solutions $\lambda = 5$ and $\lambda = 3$. Since these eigenvalues are real, unequal, and positive, we conclude that the equilibrium point at $(3, -2)$ is an unstable nodal source.

From the above linearization analyses, we would expect to see a saddle point at $(0, 0)$, an inward spiral at $(0, 1)$, a nodal sink at $(-2, -2)$, and a nodal source at $(3, -2)$. When we plot the solution curves in Mathematica (Fig. 8) this is exactly the behavior we find.

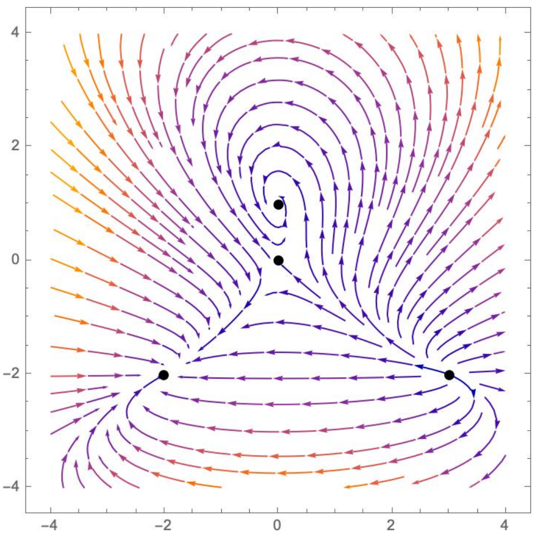


Figure 8: Problem 2 system dynamics.

3. Consider $x' = x - y^2$ and $y' = y - x^2$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.

Solution: We again seek equilibrium solutions for which $x' = y' = 0$. From the first equation we see that $x' = 0$ when $x = y^2$, and from the second equation we get that $y' = 0$ when $y = x^2$. The two equilibrium points which satisfy this are $(0, 0)$, $(1, 1)$. We will consider the behaviour of the system near each of these points.

(a) $(0,0)$: We let $x = 0 + \tilde{x}$ and $y = 0 + \tilde{y}$, where $\tilde{x}, \tilde{y} << 1$, so that the system of equations becomes

$$\begin{aligned}\tilde{x}' &= \tilde{x} - \tilde{y}^2 = \tilde{x} + \mathcal{O}(\tilde{y}^2)\\ \tilde{y}' &= \tilde{y} - \tilde{x}^2 = \tilde{y} + \mathcal{O}(\tilde{x}^2)\end{aligned}$$

This results in the linearized system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

To solve this system, we let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. Since this matrix is already diagonal, the eigenvalues and eigenvectors are immediately apparent. We have the double eigenvalue $\lambda = 1$, and our eigenvectors are simply $\mathbf{e}_1$ and $\mathbf{e}_2$. Since our eigenvalues are linearly independent, we conclude that the equilibrium point at $(0,0)$ is an unstable proper node.

(b) $(1,1)$: We now let $x = 1 + \tilde{x}$ and $y = 1 + \tilde{y}$, so that the system of equations becomes

$$\begin{aligned}\tilde{x}' &= 1 + \tilde{x} - (1 + \tilde{y})^2 = \tilde{x} - 2\tilde{y} + \mathcal{O}(\tilde{y}^2)\\ \tilde{y}' &= 1 + \tilde{y} - (1 + \tilde{x})^2 = \tilde{y} - 2\tilde{x} + \mathcal{O}(\tilde{x}^2)\end{aligned}$$

This results in the linearized system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

To solve this system, we let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. We therefore seek eigenvalue λ for which

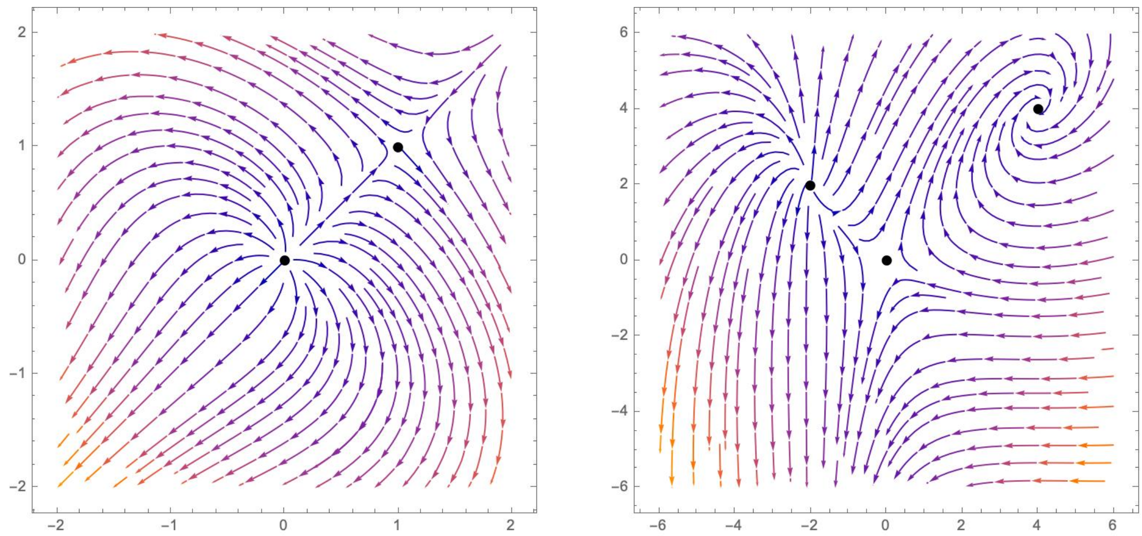
$$\det \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$$

This characteristic equation can be written in the form

$$\lambda^2 - 2x - 3 = 0$$

and has solutions $\lambda = -1$ and $\lambda = 3$. Since these eigenvalues are real with opposite sign, we conclude that the $(1,1)$ equilibrium point is a saddle point.

In summary, from the above analysis we would expect to find an outward directed proper node at $(0,0)$ and a saddle point at $(1,1)$. Plotting the solution curves in Mathematica (Fig. 9a) confirms this analysis.



(a) Problem 3 system dynamics.

(b) Problem 4 system dynamics.

Figure 9: System dynamics of Problem 3 and 4.

4. Consider $x' = (2+x)(y-x)$ and $y' = (4-x)(y+x)$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.

Solution: We once again seek equilibrium points for which $x' = y' = 0$. From the first equation we see that $x' = 0$ when $x = -2$ or when $y = x$. From the second equation we get that $y' = 0$ when $x = 4$ and when $y = -x$. Hence, we have solutions at $(-2, 2)$, $(4, 4)$, and $(0, 0)$. We will consider the dynamics near each of these points.

(a) $(0, 0)$: We let $x = \tilde{x}$ and $y = \tilde{y}$, where $\tilde{x}, \tilde{y} << 1$, so that the system becomes

$$\tilde{x}' = (2+\tilde{x})(\tilde{y}-\tilde{x}) = 2\tilde{y} + \mathcal{O}(\tilde{x}\tilde{y}, \tilde{x}^2)$$
$$\tilde{y}' = (4-\tilde{y})(\tilde{y}+\tilde{x}) = 4\tilde{x} + \mathcal{O}(\tilde{x}\tilde{y}, \tilde{y}^2)$$

This linearized system is of the form $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$.

To solve this system, we let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. Hence, we seek eigenvalues λ for which

$$\det \begin{vmatrix} -\lambda & 2 \\ 4 & -\lambda \end{vmatrix} = \lambda^2 - 8 = 0$$

This characteristic equation has the solutions $\lambda = \pm\sqrt{8}$. Since these eigenvalues are real and have opposite signs, we conclude that the equilibrium point at $(0, 0)$ is a saddle point.

(b) $(-2, 2)$: We now let $x = -2 + \tilde{x}$ and $y = 2 + \tilde{y}$, so that the system becomes

$$\tilde{x}' = (2 - 2 + \tilde{x})(2 + \tilde{y} + 2 - \tilde{x}) = 4\tilde{x} + \mathcal{O}(\tilde{x}\tilde{y}, \tilde{x}^2)$$
$$\tilde{y}' = (4 + 2 - \tilde{x})(2 + \tilde{y} - 2 + \tilde{x}) = 6\tilde{x} + 6\tilde{y} + \mathcal{O}(\tilde{x}\tilde{y}, \tilde{x}^2)$$

This results in a linearized system of the form $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix}$

To solve this system, we again let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. Hence, we seek eigenvalues λ for which

$$\det \begin{vmatrix} 4-\lambda & 0 \\ 6 & 6-\lambda \end{vmatrix} = (4-\lambda)(6-\lambda) = 0$$

We see right away that the eigenvalues for this system must be $\lambda_1 = 4$ and $\lambda_2 = 6$. Since these are real, unequal, and positive, we conclude that the equilibrium point at $(-2, 2)$ is an unstable nodal source.

(c) $(4, 4)$: We now consider $x = 4 + \tilde{x}$ and $y = 4 + \tilde{y}$, so that our system becomes

$$\begin{aligned} \tilde{x}' &= (2 + 4 + \tilde{x})(4 + \tilde{y} - 4 - \tilde{x}) = -6\tilde{x} + 6\tilde{y} + \mathcal{O}(\tilde{x}\tilde{y}, \tilde{x}^2) \\ \tilde{y}' &= (4 - 4 - \tilde{x})(4 + \tilde{y} + 4 + \tilde{x}) = -8\tilde{x} + \mathcal{O}(\tilde{x}\tilde{y}, \tilde{x}^2) \end{aligned}$$

This results in the linear system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix}$

To solve this system, we again let $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and solve the eigenvalue problem $(A - \lambda I)\mathbf{x} = 0$. Hence, we seek eigenvalues λ for which

$$\det \begin{vmatrix} -6-\lambda & 6 \\ -8 & -\lambda \end{vmatrix} = (-6-\lambda)(-\lambda) + 48 = 0$$

This equation can be expanded into the form

$$\lambda^2 + 6\lambda + 48 = 0$$

and has solutions $\lambda = -3 \pm i\sqrt{39}$. Since these roots are complex with negative real part, we conclude that the equilibrium point at $(4, 4)$ is a stable inward-directed spiral.

In summary, the above analysis suggests that we should expect to find a saddle point at $(0, 0)$, an outward-directed nodal source at $(-2, 2)$, and an inward-directed spiral at $(4, 4)$. Plotting the solution curves in Mathematica (Fig. 9b) affirms these results.