AMATH 567, Problem Set 1

Lucas Cassin Cruz Burke

October 12, 2022

1. Problem 1: Express each of the following in polar exponential form.

   (a) $-i$

   (b) $1+i$

   (c) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

   Solution:
   We can rewrite a general complex number of the form $a+bi$ to the form $re^{i\theta} = r(\cos\theta + i\sin\theta)$ by first calculating the magnitude $r = |z| = \sqrt{a^2+b^2}$ and then calculating the angle $\theta = \arcsin\frac{b}{r} = \arccos\frac{a}{r}$. Applying this technique, we find the following results:

   (a) $-i = e^{-i\pi/2}$

   (b) $1+i = \sqrt{2}e^{i\pi/4}$

   (c) $\frac{1}{2}+\frac{\sqrt{3}}{2}i = e^{i\pi/3}$

2. Problem 2: Express each of the following in the form of $a + bi$, where a and b are real.

   (a) $e^{2+i\pi/2}$

   (b) $\frac{1}{1+i}$

   (c) $(1+i)^3$

   (d) $|3+4i|$

   (e) $\cos(i\pi/4 + c)$, where c is real.

   Solution:

   (a) $e^{2+i\pi/2} = e^2 e^{i\pi/2} = e^2 i$

   (b) $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$

   (c) $(1+i)^3 = (\sqrt{2}e^{i\pi/4})^3 = 2^{3/2}e^{i3\pi/4} = 2^{3/2}(\cos 3\pi/4 + i\sin 3\pi/4) = 2^{3/2}(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = -2 + 2i$

   (d) $|3+4i| = \sqrt{(3+4i)(3-4i)} = \sqrt{9+16} = \sqrt{25} = 5$

   (e) $\cos(i\pi/4{+}c) = \frac{1}{2}\left(e^{i(i\pi/4+c)} + e^{-i(i\pi/4+c)}\right) = \frac{1}{2}\left(e^{-\pi/4+ic} + e^{\pi/4-ic}\right) = \frac{1}{2}\left(e^{-\pi/4} + e^{\pi/4}\right)\cos c + \frac{i}{2}\left(e^{-\pi/4} - e^{\pi/4}\right)\sin c$

3. Problem 3: Solve for the roots of the following equations.

   (a) $z^3 = 4$

   (b) $z^4 = -1$

   Solution:

   A general equation $z^n = w$ can be solved by writing z and w in polar form $z = |z|e^{i\theta}$ and $w = |w|e^{i\gamma}$. Then the equation becomes $|z|^n e^{in\theta} = |w|e^{i\gamma}$, which leads us to $|z| = |w|^{1/n}$ and $n\theta \pmod{2\pi} = \gamma$. Our solutions are therefore $|w|^{1/n}e^{i\gamma/n}$ multiplied by all the nth roots of unity.

   (a) $z \in 2^{2/3}\left\{1, e^{i2\pi/3}, e^{-i2\pi/3}\right\}$

   (b) $z \in \left\{e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{-i\frac{3\pi}{4}}, e^{-i\frac{\pi}{4}}\right\}$

4. Problem 4: Establish the following results.

   (a) $(z+w)* = z * + w*$

   (b) $\Re(z) \leq |z|$

   (c) $|wz * + w * z| \leq 2|wz|$

   (d) $|z_1 z_2| = |z_1||z_2|$

   Solution:

   (a) Write z and w in Cartesian form as $z = z_x + iz_y$ and $w = w_x + iw_y$ where $z_x, z_y, w_x, w_y \in \mathbf{R}$, then

   $$(z+w)^* = (z_x+w_x+i(z_y+w_y))^* = z_x+w_x-i(z_y+w_y) = (z_x-iz_y)+(w_x-iw_y) = z^*+w^*$$

   .

   (b) Let $z = \Re(z) + i\Im(z)$. By definition, $|z|^2 = \Re(z)^2 + \Im(z)^2$. Since $\Im(z)$ is real, $\Im(z)^2 \geq 0$, and therefore

   $$\Re(z)^2 \leq |z|^2$$

   Since both sides of this equation are real and positive definite, we can take the square root of either side to show that

   $$\Re(z) \leq |z|$$

   (c) We write w and z in polar form as $w = |w|e^{i\phi}$ and $z = |z|e^{i\varphi}$. Then, by Euler's formula

   $$wz^* + w^*z = |w||z|\left(e^{i(\phi-\varphi)} + e^{-i(\phi-\varphi)}\right) = 2|w||z|\cos(\phi-\varphi)$$

   In (d) we prove that $|w||z| = |wz|$. Taking the absolute value of the above and applying this fact leads us to

   $$|wz^* + w^*z| = 2|w||z||\cos(\phi-\varphi)| = 2|wz||\cos(\phi-\varphi)| \leq 2|wz|$$

   (d) Consider z_1 and z_2 in polar form $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_1e^{i\theta_2}$. We see that $|z_1| = r_1$ and $|z_2| = r_2$. We have

   $$|z_1z_2| = |r_1r_2e^{i(\theta_1+\theta_2)}| = r_1r_2 = |z_1||z_2|$$