AMATH 569 Homework Assignment #4 Spring 2023

Assigned: May 3, 2023

Due: May 10, 2023

1. Green's function of the 1-D heat equation in a semi-infinite domain, $G(x,t;\xi,\tau)$, is defined by:

$$\left(\frac{\partial}{\partial t}-D\frac{\partial^2}{\partial x^2}\right)G=\delta(x-\xi)\delta(t-\tau),\quad 0<x,\xi<\infty,\quad t>0,\tau>0.$$

subject to zero initial condition: $G=0$ at $t=0$.

The boundary condition is either (a): $G=0$ at $x=0$ and $x\to\infty$, or

(b): $\frac{\partial}{\partial x}G=0$ at $x=0$ and $x\to\infty$.

The solution in a semi-infinite domain can be constructed from the solution in the infinite domain by adding or subtracting another source located at $x=-\xi$, so that the contributions cancel at $x=0$ for (a), or the contributions are symmetric about $x=0$. Find the Green's function defined above for boundary condition (a). Then repeat the problem for boundary condition (b).

2. Find the Greens function for the wave equation in two-dimensions governed by

$$\frac{\partial^2}{\partial t^2}G-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)G=\delta(t)\delta(x)\delta(y).$$

$G\to 0$ as $r\to\infty$, where $r^2=x^2+y^2$.

$G\equiv 0$ for $t<0$.

The solution is

$$G=\frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}},\text{ where } H \text{ is the Heaviside function.}$$

(a) Derive this solution using Fourier transform in x and y.

Hint: In the inverse transform, use polar coordinates to get

$G=\frac{1}{2\pi}\int_0^\infty J_0(kr)\sin kt\,dk$. Then use integral tables.

(b) Derive this solution using Laplace transform in t.

Hint: First show that the Laplace transform of G is

$\tilde{G}=\frac{1}{2\pi}K_0(sr)$, where K is modified Bessel function of the second kind. Then use Laplace transform tables.