AMATH 568
Advanced Differential Equations
Homework 6

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1. Consider the singular equation:

$$\epsilon y'' + (1+x)^2 y' + y = 0$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$.

(a) Obtain the leading order uniform solution using the WKB method.

Solution: We begin by making the WKB ansatz

$$y(x) = e^{S(x)/\delta},$$

where $S(x) = S_0(x) + \delta S_1(x) + \dots$ and $0 < \delta \ll 1$. We can calculate the relevant derivatives to $\mathcal{O}(1)$ as

$$\begin{aligned}
y' &= \frac{S'}{\delta} y = \left(\frac{1}{\delta} S_0' + S_1'\right) y \\
y'' &= \left(\frac{S''}{\delta} + \frac{S'^2}{\delta^2}\right) y = \left(\frac{1}{\delta}(S_0'' + \delta S_1'') + \frac{1}{\delta^2}({S_0'}^2 + 2\delta S_0' S_1')\right) y \\
&= \left(\frac{1}{\delta^2}{S_0'}^2 + \frac{1}{\delta}(S_0'' + 2S_0'S_1') + S_1''\right) y
\end{aligned}$$

Plugging our ansatz into the above singular equation results in the following condition for $y(x) \neq 0$:

$$\begin{aligned}
\epsilon\left(\frac{S''}{\delta} + \frac{S'^2}{\delta^2}\right) + (1+x)^2 \frac{S'}{\delta} + 1 = 0 \\
\Rightarrow \epsilon\left(\frac{1}{\delta^2}{S_0'}^2 + \frac{1}{\delta}(S_0'' + 2S_0'S_1') + S_1''\right) + (1+x)^2\left(\frac{1}{\delta}S_0' + S_1'\right) + 1 = 0
\end{aligned}$$

To leading order, we have the terms

$$\frac{\epsilon}{\delta^2}{S_0'}^2 + (1+x)^2 \frac{1}{\delta} S_0' = 0$$

Hence, to establish a dominant balance we require that

$$\frac{\epsilon}{\delta^2} \sim \frac{1}{\delta},$$

which we can satisfy by setting $\delta = \epsilon$.
With this, our equation for $S(x)$ becomes, to leading order $\mathcal{O}(1/\epsilon)$,

$$S_0'^2 + (1+x)^2 S_0' = 0$$

This is trivially satisfied when $S_0' = 0 \Rightarrow S_0 = C$. In this case, our governing equation becomes, to $\mathcal{O}(1)$,

$$\begin{aligned}
(1+x)^2 S_1' + 1 &= 0 \\
S_1' &= \frac{-1}{(1+x)^2} \\
\Rightarrow S_1 &= \frac{1}{1+x}
\end{aligned}$$

and we find the WKB solution

$$y(x) = C_1 \exp\left[\frac{1}{1+x}\right]$$

If $S_0' \neq 0$, we have

$$S_0' = -(1+x)^2 \Rightarrow S_0 = -\frac{1}{3}(1+x)^3 + C$$

In this case we have, to $\mathcal{O}(1)$,

$$\begin{aligned}
S_0'' + 2S_0'S_1' + (1+x)^2 S_1' &= 0 \\
\Rightarrow -2(1+x) - 2(1+x)^2 S_1' + (1+x)^2 S_1' &= 0 \\
\Rightarrow -2(1+x) - (1+x)^2 S_1' &= 0 \\
\Rightarrow S_1' = \frac{-2}{1+x} \Rightarrow S_1 &= \log\left[\frac{1}{(1+x)^2}\right]
\end{aligned}$$

Hence we find the WKB solution

$$y(x) = \frac{C_2}{(1+x)^2} \exp\left[-\frac{1}{3\epsilon}(1+x)^3\right]$$

We can now combine these two solutions to find

$$y(x) = C_1 \exp\left[\frac{1}{1+x}\right] + \frac{C_2}{(1+x)^2} \exp\left[-\frac{1}{3\epsilon}(1+x)^3\right]$$

Our boundary conditions require that $y(0) = 1$, which gives us

$$y(0) = C_1 e + C_2 e^{-1/3\epsilon} = 1 \qquad \Rightarrow \qquad C_2 = e^{1/3\epsilon}(1 - C_1 e)$$

And our second boundary condition requires that $y(1) = 1$, which gives us

$$\begin{aligned}
y(1) = C_1 e^{1/2} + \frac{C_2}{4} e^{-8/3\epsilon} &= 1 \\
C_1 e^{1/2} + \frac{1}{4}(1 - C_1 e) e^{-7/3\epsilon} &= 1 \\
C_1 \left(e^{1/2} - \frac{1}{4} e^{1-\frac{7}{3\epsilon}}\right) &= 1 - \frac{1}{4} e^{-7/3\epsilon} \\
C_1 &= \left(1 - \frac{1}{4} e^{-7/3\epsilon}\right)\left(e^{1/2} - \frac{1}{4} e^{1-\frac{7}{3\epsilon}}\right)^{-1}
\end{aligned}$$

All together, our final WKB uniform solution is given to leading order by

$$y(x) = A(\epsilon) \exp\left[\frac{1}{1+x}\right] + \frac{e^{\frac{1}{3\epsilon}}(1 - eA(\epsilon))}{(1+x)^2} \exp\left[-\frac{1}{3\epsilon}(1+x)^3\right],$$

where

$$A(\epsilon) = \left(1 - \frac{1}{4} e^{-7/3\epsilon}\right)\left(e^{1/2} - \frac{1}{4} e^{1-\frac{7}{3\epsilon}}\right)^{-1}$$

(b) Plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution:

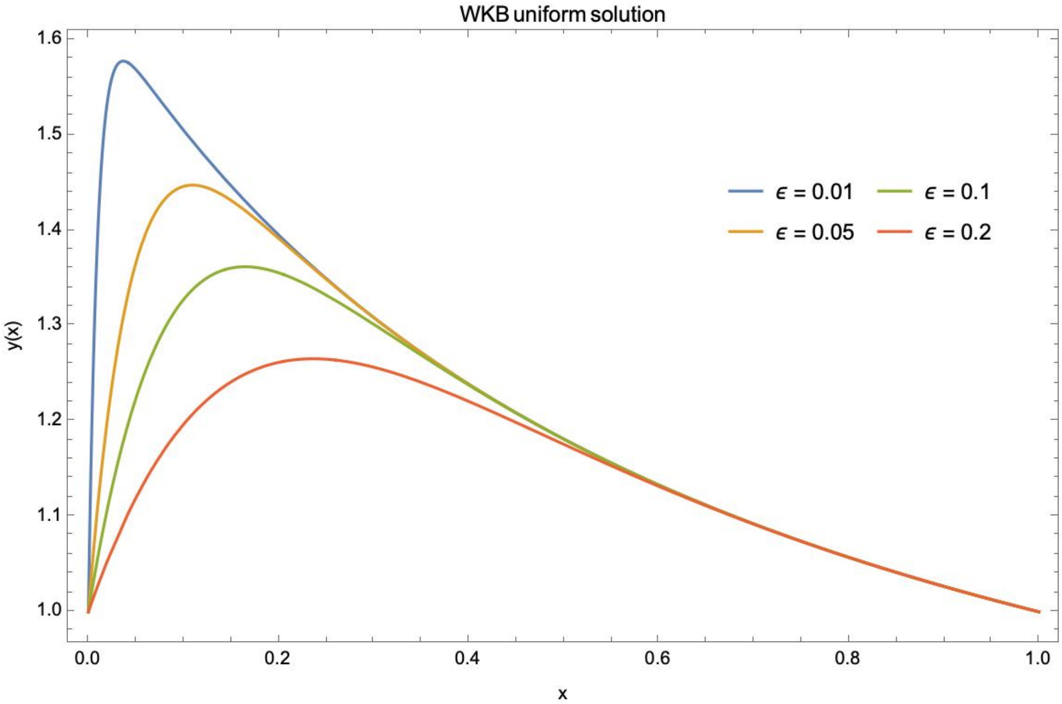

Figure 1: Leading order uniform solution for various values of ϵ.