Lecture 6

Chapter 2 of Bernard's Notes

Quasi-linear second-order PDEs:

$$a(x,y)\,u_{xx} + 2b(x,y)\,u_{xy} + c(x,y)\,u_{yy} = f(x,y,u,u_x,u_y)$$

where the two independent variables are denoted by x, y. (can be generalized to higher dimensions)

Can it be transformed into the canonical form $u_{\alpha\beta} = F$?

Use the transformation:

$$\alpha = \phi(x,y), \quad \beta = \psi(x,y)$$

assume to be locally invertible, so require

$$\det \begin{bmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{bmatrix} \neq 0.$$

$$u_x = u_\alpha \alpha_x + u_\beta \beta_x$$

$$u_y = u_\alpha \alpha_y + u_\beta \beta_y$$

$$u_{xx} = u_\alpha \alpha_{xx} + u_\beta \beta_{xx} + u_{\alpha\alpha} \alpha_x^2 + 2 u_{\alpha\beta} \alpha_x \beta_x + u_{\beta\beta} \beta_x^2$$

$$u_{yy} = u_\alpha \alpha_{yy} + u_\beta \beta_{yy} + u_{\alpha\alpha} \alpha_y^2 + 2 u_{\alpha\beta} \alpha_y \beta_y + u_{\beta\beta} \beta_y^2$$

$$u_{xy} = u_\alpha \alpha_{xy} + u_\beta \beta_{xy} + u_{\alpha\alpha} \alpha_x \alpha_y + u_{\beta\beta} \beta_x \beta_y + u_{\alpha\beta} (\alpha_x \beta_y + \alpha_y \beta_x)$$

$$a u_{xx} + 2b u_{xy} + c u_{xx} = f$$

becomes

$$A(\alpha,\beta) u_{\alpha\alpha} + 2B(\alpha,\beta) u_{\alpha\beta} + C(\alpha,\beta) u_{\beta\beta} = F(\alpha, \beta, u, u_\alpha, u_\beta)$$

where

$$A = a\alpha_x^2 + 2b \alpha_x \alpha_y + c\alpha_y^2$$

$$C = a\beta_x^2 + 2b \beta_x \beta_y + c\beta_y^2$$

$$B = a \alpha_x \beta_x + b(\alpha_x \beta_y + \alpha_y \beta_x) + c \alpha_y \beta_y$$

Because A and C are of similar form, if α and β can be chosen to satisfy

$$a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 = 0$$

(i.e. two roots of the same eq.)

then $\quad A = 0$ and $C = 0$

The transformed equation will be of the canonical form:

$$u_{\alpha\beta} = \frac{F(\alpha, \beta, u, u_\alpha, u_\beta)}{2B(\alpha, \beta)}$$

Divide by φ_y^2 & get

$$a\left(\frac{\varphi_x}{\varphi_y}\right)^2 + 2b\left(\frac{\varphi_x}{\varphi_y}\right) + c = 0$$

Let $z \equiv \varphi_x/\varphi_y$, then

$$a(z - z_1)(z - z_2) = 0$$

z_1 and z_2 are the roots to the quadratic equation.

$$\boxed{az^2 + 2bz + c = 0}$$

Case 1: $b^2 - ac > 0$, "Hyperbolic" PDE

In this case both roots are real

$$a z^2 + 2bz + c = 0$$

$$\Rightarrow z_{1,2} = -b \pm \sqrt{b^2 - ac}$$

Assign one root to each (one to α and one to β)

$$\alpha_x = z_1 \alpha_y \,, \quad \beta_x = z_2 \beta_y$$

$\alpha(x,y) = \text{const}$ is one family of characteristics

$\beta(x,y) = \text{const}$ is a second family of characteristics.

The slope of the characteristics in the x-y plane: $\left(\frac{dy}{dx}\right)_{\alpha=\text{const}} = -z_1$

$$\left(\frac{dy}{dx}\right)_{\beta=\text{const}} = -z_2$$

[Details: $\alpha(x,y) = \text{const}$, $\frac{d\alpha}{dx} = 0 = \alpha_x + \frac{dy}{dx}\alpha_y$

$$\left(\frac{dy}{dx}\right)_{\alpha=\text{const}} = -\frac{\alpha_x}{\alpha_y} = -z_1$$]

In the part of the x-y plane where z_1 and z_2 are real and distinct, we have

$$\boxed{2B(\alpha,\beta)\, u_{\alpha\beta} = F}$$

All Hyperbolic eq$\underline{s}$ can be transformed into this form.

$$\det\begin{bmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{bmatrix} = 0 \Rightarrow \det\begin{bmatrix} z_1\alpha_y & \alpha_y \\ z_2\beta_y & \beta_y \end{bmatrix} \neq 0$$

$$\Rightarrow \alpha_y \beta_y (z_1 - z_2) \neq 0$$

$$\Rightarrow \alpha_y \beta_y \neq 0$$

$$B = a\,\alpha_x \beta_x + b(\alpha_x \beta_y + \alpha_y \beta_x) + c\,\alpha_y \beta_y$$

$$= 2\alpha_y \beta_y \frac{ac - b^2}{a} \neq 0$$

Case 2: $b^2 - ac \equiv 0$, $z_1 = z_2$ real

Parabolic case. Only one characteristic.

$$\alpha_x = z_1 \alpha_y = -\frac{b}{a} \alpha_y$$

$$A = 0$$

β can be anything that is linearly independent of α.

$$B = a \alpha_x \beta_x + b(\alpha_x \beta_y + \alpha_y \beta_x)$$
$$+ c \alpha_y \beta_y$$
$$= -b\alpha_y \beta_x + b(-\frac{b}{a}\alpha_y \beta_y + \alpha_y \beta_x)$$
$$+ c \alpha_y \beta_y$$
$$= -\frac{b^2}{a} \alpha_y \beta_y + c \alpha_y \beta_y$$
$$= \alpha_y \beta_y \frac{ac - b^2}{a} = 0$$

Only $C \neq 0$. Cannot be transformed into $u_{\alpha\beta} = F$, but can be put into

$$\boxed{u_{\beta\beta} = F(\alpha, \beta, u, u_\alpha, u_\beta) / C(\alpha, \beta)}$$

Case 3 : $b^2 - ac < 0$. Elliptic

$$z_2 = z_1^* \quad \text{complex}$$

Because the two roots are distinct, similar to Case 1 :

$$u_{\alpha\beta} = G(\alpha, \beta, u, u_\alpha, u_\beta)$$

However
$$\alpha_x = z_1 \alpha_y$$
$$\beta_x = z_2 \beta_y$$

leads to complex characteristics

$$\alpha = \xi + i\eta \quad , \xi, \eta \text{ real}$$
$$\beta = \xi - i\eta$$

$$\xi = (\alpha + \beta)/2$$
$$\eta = (\alpha - \beta)/2i$$

$$u_{\alpha\beta} = (u_\xi \xi_\alpha + u_\eta \eta_\alpha)_\beta$$
$$= \tfrac{1}{2}(u_\xi - i u_\eta)_\beta$$
$$= \tfrac{1}{2}(u_{\xi\xi}\xi_\beta + u_{\xi\eta}\eta_\beta - i u_{\eta\xi}\xi_\beta - i u_{\eta\eta}\eta_\beta)$$
$$= \tfrac{1}{4}(u_{\xi\xi} + u_{\eta\eta})$$

$$\boxed{u_{\xi\xi} + u_{\eta\eta} = 4G(\alpha, \beta, u, u_\alpha, u_\beta)}$$