AMATH 569 Homework 3

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Problem 1. Solve using Fourier transform in x and Laplace transform in t:

$$u_t - Du_{xx} = \delta(x-\xi)\delta(t-\tau) \qquad \begin{cases} -\infty < x < \infty, & t > 0 \\ -\infty < \xi < \infty, & \tau > 0 \end{cases}$$

where $u(x,t) \to 0$ as $x \to \pm\infty$ and $u(x,0) = 0$.

Solution. We begin by Fourier transforming in x. Let

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} e^{ikx} u(x,t) dx$$

Then, taking the Fourier transform of both sides of the PDE gives us the following ODE:

$$\hat{u}_t + Dk^2 \hat{u} = e^{ik\xi}\delta(t-\tau)$$

We now take the Laplace transform in t. Let $U(x,s)$ denote the Laplace transform of $u(x,t)$, and similarly let $\hat{U}(k,s)$ denote the Laplace transform of $\hat{u}(k,t)$. Then taking the Laplace transform of both sides of the above ODE results in

$$s\hat{U} + Dk^2\hat{U} = e^{ik\xi}e^{-s\tau}$$

Rearranging, we have

$$\hat{U}(k,s) = \frac{\exp\{ik\xi - s\tau\}}{s + Dk^2}$$

Taking the inverse Laplace transform, we have

$$\mathcal{L}^{-1}\{\hat{U}\}(k,t) = \hat{u}(k,t) = \theta(t-\tau)e^{ik\xi}e^{-Dk^2(t-\tau)}$$

where $\theta(x)$ is the Heaviside step function. We now take the inverse Fourier transform to find u. We have

$$\begin{aligned}
u(x,t) = \mathcal{F}^{-1}\{\hat{u}\}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta(t-\tau)e^{-ikx}e^{ik\xi}e^{-Dk^2(t-\tau)}dk \\
&= \frac{\theta(t-\tau)}{2\pi}\int_{-\infty}^{\infty} \exp\left[-D(t-\tau)\left(k + \frac{i(x-\xi)}{2D(t-\tau)}\right)^2 - \frac{(x-\xi)^2}{4D(t-\tau)}\right] dk \\
&= \frac{\theta(t-\tau)}{2\pi}\exp\left(-\frac{(x-\xi)^2}{4D(t-\tau)}\right)\int_{-\infty}^{\infty} \exp\left[-D(t-\tau)\left(k + \frac{i(x-\xi)}{2D(t-\tau)}\right)^2\right] dk \\
&= \frac{\theta(t-\tau)}{2\pi}\exp\left(-\frac{(x-\xi)^2}{4D(t-\tau)}\right)\cdot I
\end{aligned}$$

Where we define

$$I = \int_{-\infty}^{\infty} \exp\left[-D(t-\tau)\left(k + \frac{i(x-\xi)}{2D(t-\tau)}\right)^2\right] dk$$

Substitute $\gamma = k + \frac{i(x-\xi)}{2D(t-\tau)}$, then

$$I = \int_{-\infty}^{\infty} \exp\left[-D(t-\tau)\gamma^2\right] d\gamma$$

Now substitute $\phi = \frac{\gamma}{\sqrt{D(t-\tau)}}$, then

$$I = \sqrt{D(t-\tau)} \int_{-\infty}^{\infty} \exp\left[-\phi^2\right] d\phi = \sqrt{\pi D(t-\tau)}$$

Plugging this back into our solution gives us

$$u(x,t) = \begin{cases} 0 & \text{for } t < \tau \\ \frac{1}{2}\sqrt{\frac{D(t-\tau)}{\pi}} \exp\left[-\frac{(x-\xi)^2}{4D(t-\tau)}\right] & \text{for } t > \tau \end{cases}$$

Problem 2. Solve the same problem without using a Laplace transform in t. Figure out the matching condition for your ODE across $t = \tau$.

Solution. We return now to the original ODE which we derived in Problem 1.

$$\hat{u}_t + Dk^2\hat{u} = e^{ik\xi}\delta(t-\tau)$$

For $t \neq \tau$, this equation is of the form

$$\hat{u}_t + Dk^2\hat{u} = 0$$

This is a homogeneous ODE with the solution

$$\hat{u} = A\exp\left[-Dk^2(t-\tau)\right]$$

In order to satisfy our initial condition $u(x,0) = 0$ we must have $A = 0$ for $t < \tau$. We can determine the coefficient for $t > \tau$ by integrating across a small duration of time in which the impulse occurs.

$$\begin{aligned} \Delta\hat{u} &= \lim_{\epsilon\to 0}\int_{\tau-\epsilon}^{\tau+\epsilon} \left[\hat{u}_t + Dk^2\hat{u}\right] dt \\ &= \lim_{\epsilon\to 0}\int_{\tau-\epsilon}^{\tau+\epsilon} e^{ik\xi}\delta(t-\tau)dt \\ &= e^{ik\xi} \end{aligned}$$

Hence, for $t > \tau$ we have $A = e^{ik\xi}$. With this, we can write the ODE solution $\hat{u}$ as

$$\hat{u} = e^{ik\xi}\exp\left[-Dk^2(t-\tau)\right]$$

Lastly, we take the inverse Fourier transform to find our PDE solution. We have

$$u(x,t) = \mathcal{F}^{-1}\left\{\hat{u}\right\}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta(t-\tau)e^{-ikx}e^{ik\xi}\exp\left[-Dk^2(t-\tau)\right] dk$$

We recognize this as the same integral which we evaluated for $u(x,t)$ in Problem 1. This shows that these two approaches yield the same result.