Lecture 18

<u>Third method</u>: Integral representation.

Write

$$u(x) = \frac{1}{2\pi}\int_\Gamma U(k)e^{-ikx}\,dk$$

$$q(x) = \frac{1}{2\pi}\int_\Gamma Q(k)\,e^{-ikx}\,dk$$

Γ unknown contour.

Substitute into the ODE

$$u_{xx} + k_0^2\, u = -\,q(x)/c^2$$

$$\int_\Gamma U\cdot(-k^2+k_0^2)\,e^{-ikx}\,dk = -\frac{1}{c^2}\int_\Gamma Q(k)\,e^{-ikx}\,dk$$

To satisfy the ODE we choose

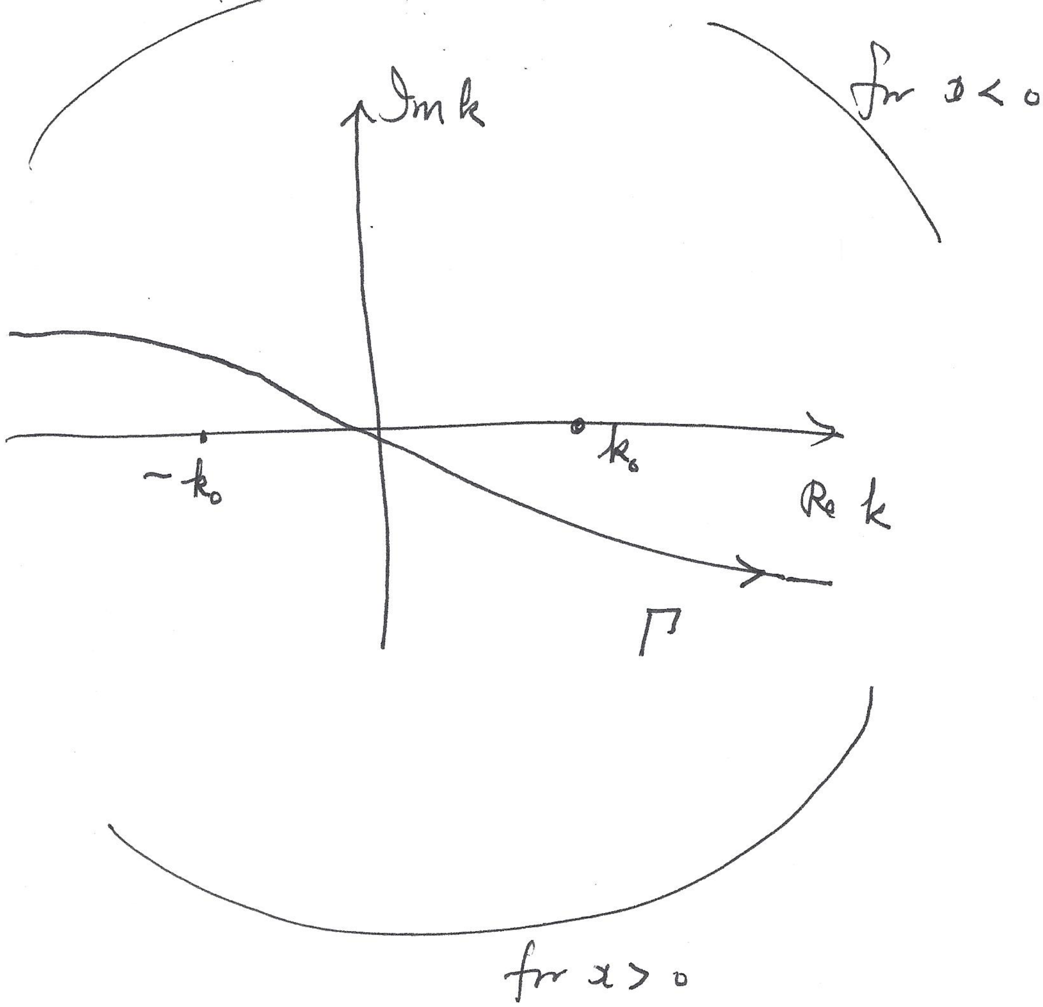
$$U(k) = \frac{Q(k)/c^2}{k^2-k_0^2}$$

$$u(x) = \frac{1}{2\pi c^2}\int_\Gamma \frac{Q(k)/c^2}{k^2-k_0^2}\,e^{-ikx}\,dk$$

Choose Γ to satisfy BC at $x\to\pm\infty$.

Poles at $k=\pm k_0$.

To get e^{ik_0x} type solution as $x\to\infty$, Γ must be above $k=-k_0$. To get e^{-ik_0x} type solution as $x\to-\infty$, Γ must be below $k=k_0$



By Jordan's Lemma: close below for $x > 0$ and close above for $x < 0$.

$$u(x) = \frac{1}{2\pi c^2} \int \frac{Q(k)}{k^2 - k_0^2} e^{-ikx} dk.$$

This explains the mysterious method often used by physicists of "indenting the contour" to avoid the singularity at the real axis.

Lecture 18

Approach 4 : Generalized Fourier Transform

(optional)

$$\frac{d^2}{dx^2} u + k_0^2 u = \frac{-q(x)}{c^2}$$

$u(x)$ not integrable. So we cannot use the standard Fourier transform for $-\infty < x < \infty$.

Similar to Laplace transform, we define two one-sided functions:

$$u_+(x) = \begin{cases} u(x), & x > 0 \\ 0 & x < 0 \end{cases}$$

$$u_-(x) = \begin{cases} 0, & x > 0 \\ u(x), & x < 0 \end{cases}$$

We define one-sided Fourier transform as

$$U_+(k) = \int_0^\infty u(x) e^{ikx} dx, \quad \operatorname{Im} k > 0$$

$$= \int_{-\infty}^\infty u_+(x) e^{ikx} dx$$

$$U_-(k) = \int_{-\infty}^{0} u(x)\, e^{ikx}\, dx \quad , \ \Im k < 0.$$

$$= \int_{-\infty}^{\infty} u_-(x)\, e^{ikx}\, dx$$

If $u \to O(e^{bx})$ as $x \to \infty$,

we need $\Im k > b$.

If $u \to O(e^{c|x|})$ as $x \to -\infty$

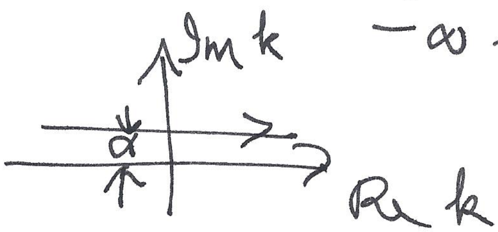
we need $\Im k < -c$.

Inverse transform:

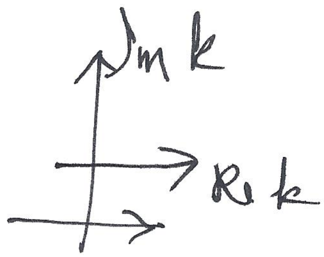
Since $U_+ = \int_{-\infty}^{\infty} u_+(x)\, e^{ikx}\, dx = \mathcal{F}[u_+]$

$u_+(x) = \mathcal{F}^{-1}[U_+]$

$$= \frac{1}{2\pi} \int_{-\infty + i\alpha}^{\infty + i\alpha} U_+(k)\, e^{-ikx}\, dk, \quad \alpha > b$$



$$u_-(x) = \mathcal{F}^{-1}[U_-]$$



$$= \frac{1}{2\pi}\int_{-\infty - i\beta}^{\infty - i\beta} U(k)\, e^{-ikx}\, dk \,, \quad \beta > c$$

$$u(x) = u_+(x) + u_-(x) \,.$$

Perform $\int_0^\infty dx\, e^{ikx}$ of the ODE:

$$\int_0^\infty u_{xx}\, e^{ikx}\, dx = -k^2 U_+(k) - u_x(0)$$
$$+ iku(0)$$

Let $Q_+(k) = \int_0^\infty q(x)\, e^{ikx}\, dx$

Then

$$U_+(k) = \frac{-Q_+(k)/c^2 + u_x(0) - iku(0)}{k_0^2 - k^2}$$

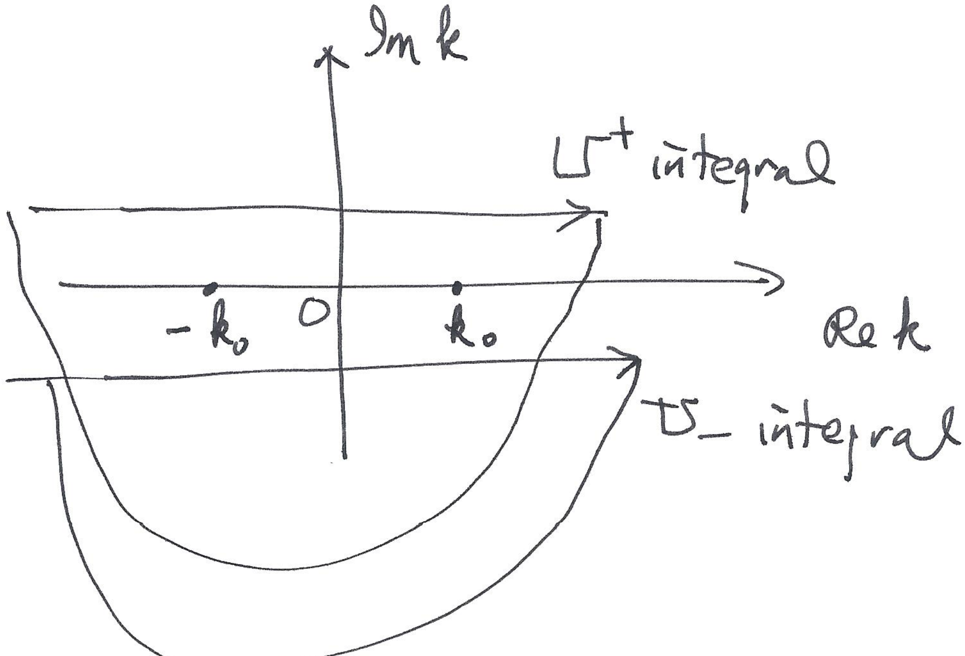
Similarly, perform $\int_{-\infty}^{0} e^{ikx}\,dx$ of the ODE

and let $Q_-(k) = \int_{-\infty}^{0} q(x)\, e^{ikx}\,dx$.

$$U_-(k) = \frac{-Q_-(k)/c^2 - u_x(0) + ik\,u(0)}{k_0^2 - k^2}$$

$$u(x) = \frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} U_+(k)\, e^{-ikx}\,dk$$

$$+ \frac{1}{2\pi}\int_{-\infty-i\beta}^{\infty-i\beta} U_-(k)\, e^{-ikx}\,dk$$



For $x > 0$, close below

U_- integral does not enclose any singularity

Only the U_+ integral contributes

$$U_+(k) = \frac{\frac{1}{c^2} Q_+(k) - u_x(0) + iku(0)}{k^2 - k_0^2}$$

$$u(x) = \frac{1}{2\pi c^2} \int \frac{Q_+(k)}{k^2 - k_0^2} e^{-ikx} \, dk$$

$$- u_x(0) \frac{1}{2\pi} \int \frac{e^{-ikx}}{k^2 - k_0^2} \, dk$$

$$+ u(0) \frac{i\cancel{k}}{2\pi} \int \frac{k e^{-ikx}}{k^2 - k_0^2} \, dk$$

$$= \frac{-1}{2\pi c^2} \int \frac{Q_+(k)}{k^2 - k_0^2} e^{-ikx} \, dk$$

$$+ u_x(0) \frac{1}{2\pi} \int \frac{e^{-ikx}}{k^2 - k_0^2} \, dk$$

$$- u(0) \frac{i}{2\pi} \int \frac{k e^{-ikx}}{k^2 - k_0^2} \, dk$$

$$\text{let } g(x) = \frac{1}{2\pi} \int \frac{e^{-ikx}}{k^2 - k_0^2}\, dk \qquad x > 0$$

$$= \frac{i}{2k_0} \left[e^{-ik_0 x} - e^{ik_0 x} \right]$$

$$h(x) = \frac{1}{2\pi} \int \frac{k\, e^{-ikx}}{k^2 - k_0^2}\, dk \quad , \quad x > 0$$

$$= \frac{i}{2} \left[e^{-ik_0 x} + e^{ik_0 x} \right]$$

$$u(x) = \frac{-i}{2k_0} \int_0^x dy\, \frac{1}{c^2} q(y) \left[e^{-ik_0(x-y)} - e^{ik_0(x-y)} \right]$$

$$+ u_x(0) \left(\frac{+i}{2k_0} \right) \left[e^{-ik_0 x} - e^{ik_0 x} \right]$$

$$+ u(0)\, \frac{1}{2} \left[e^{-ik_0 x} + e^{ik_0 x} \right]$$

$$= \frac{i}{2k_0 c^2} \int_0^x dy\, q(y) \left[e^{ik_0(x-y)} - e^{-ik_0(x-y)} \right]$$

$$+ \frac{i}{2k_0} \left[u_x(0) - ik_0 u(0) \right] e^{-ik_0 x}$$

$$- \frac{i}{2k_0} \left[u_x(0) + ik_0 u(0) \right] e^{ik_0 x}$$

$x > 0$:

$$u(x) = \left[A + \frac{i}{2k_0 c^2} \int_0^x q(y) e^{-ik_0 y} dy \right] e^{ik_0 x}$$

$$+ \left[B - \frac{i}{2k_0 c^2} \int_0^x q(y) e^{ik_0 y} dy \right] e^{-ik_0 x}$$

where

$$A = \frac{+i}{2k_0} \left[u_x(0) - ik_0 u(0) \right]$$

$$B = \frac{i}{2k_0} \left[u_x(0) + ik_0 u(0) \right]$$

$$A = \frac{i}{2k_0 c^2} \int_{-\infty}^{0} q(y) e^{-ik_0 y} dy$$

$$B = \frac{i}{2k_0 c^2} \int_0^{\infty} q(y) e^{ik_0 y} dy$$

$$u(x) = \frac{i}{2k_0 c^2} \left[\int_{-\infty}^{x} e^{-ik_0 y} q(y) dy \right] e^{ik_0 x}$$

$$+ \frac{i}{2k_0 c^2} \left[\int_x^{\infty} e^{ik_0 y} q(y) dy \right] e^{-ik_0 x}$$

$$= \frac{i}{2k_0 c^2} \int_{-\infty}^{\infty} q(y) e^{ik_0 |x-y|} dy \quad //$$

Similarly for $x < 0$