AMATH 569 Homework Assignment #6, Spring 2023
Assigned: May 24, 2023
Due, May 31, 2023

1. Consider the sound waves governed by

$$\frac{\partial^2}{\partial t^2}\psi = c^2\nabla^2\psi, \quad \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

in a circular cylinder of radius a and length L.

$\psi = 0$ at $r = a$; $\psi = 0$ at $z = 0, L$.

Assume that the sound produced in this tube is symmetric, i.e. no θ dependence. Find the lowest three frequencies. Take $c = 300 m/s$, $a = 1cm$, $L = 0.5m$.

2. Consider the wave function ψ for an electron of mass μ in a sphere surrounded by an infinite potential at a radius a from the nucleus, which just mean that $\psi = 0$ at $r = a$.

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2\mu}\nabla^2\psi.$$

Find the energy levels for the symmetric case, where ψ does not depend on θ and ϕ. Your answer should be exact and in terms of parameters given.

3. Consider the Legendre's equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}y(x)\right] + n(n+1)y(x) = 0, \quad -1 \le x \le 1,$$

with the condition that $y(\pm 1)$ are bounded. The solutions are the Legendre polynomials, $P_n(x)$, which are given by the Rodrigue's formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

For example $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

Compute the first four coefficients in the Legendre expansion (similar to Fourier sine or cosine series expansion):

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \quad \text{where } a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

$$\text{for } f(x) = \begin{cases} 0 \text{ for } -1 < x < 0 \\ x \text{ for } \quad 0 < x < 1 \end{cases}$$

Plot the approximation of the sum consisting of one, two, three and four terms along with the original function $f(x)$.