AMATH 562
Advanced Stochastic Processes
Homework 3

Lucas Cassin Cruz Burke

Due: February 13, 2023

1. $W(t)$ is a standard Brownian motion.

   (a) Let $c > 0$ be a constant. Show that the process defined by $B(t) = cW(t/c^2)$ is a standard Brownian motion.

   Solution: We see that $B(t)$ is a scaled Brownian motion with respect to a slow-time parameter $\tau = t/c^2$. In particular we can show that it fits all of the criteria for Brownian motion according to Lévy's characterization. B has continuous sample paths and, since W is martingale with respect to a filtration $\mathbb{F}$, it follows that B must also be martingale with respect to the same filtration. We can also see that $B(0) = cW(0) = 0$. Lastly, letting $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, t]$, we have:

$$
\begin{aligned}
[B, B]_t &= \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |B(t_{j+1}) - B(t_j)|^2 \\
&= \lim_{||\Pi|| \to 0} c^2 \sum_{j=0}^{n-1} |W(t_{j+1}/c^2) - W(t_j/c^2)|^2 \\
&= c^2 [W, W]_{t/c^2} \\
&= t
\end{aligned}
$$

   Thus, by the Lévy characterization of Brownian motion, $B(t)$ must be a standard Brownian motion.

   (b) For $t = n = 0, 1, \ldots$, show that $W^2(n) - n$ is a discrete time martingale.

   Solution: Define $M(n) = W^2(n) - n$ and let $\mathbb{F} = \{\mathcal{F}_n\}$ be a filtration of the Brownian motion W, i.e. $\forall n : W(n) \in \mathcal{F}_n$. Since $M = M(n, W)$ is a deterministic function of W and n, it follows that M is also $\mathbb{F}$-adapted. To determine whether M is a discrete-time martingale with respect to the filtration $\mathbb{F}$, we consider the expectation

$$
\begin{aligned}
\mathbb{E}[M(n+m)|\mathcal{F}_n] &= \mathbb{E}[W^2(n+m) - n - m|\mathcal{F}_n] \\
&= \mathbb{E}[W^2(n+m)|\mathcal{F}_n] - n - m
\end{aligned}
$$

We may use the property of Brownian motion $\forall n, m : W(n+m) - W(n) \sim \mathcal{N}(0, m)$ to calculate the expectation term. We have

$$\begin{aligned}\mathbb{E}[W^2(n+m)|\mathcal{F}_n] &= \mathbb{E}[(W(n+m) - W(n) + W(n))^2|\mathcal{F}_n] \\ &= \mathbb{E}[(W(n+m) - W(n))^2 + 2W(n)(W(n+m) - W(n)) + W^2(n)|\mathcal{F}_n] \\ &= m + W^2(n)\end{aligned}$$

Then, plugging this into the above, we have

$$\begin{aligned}\mathbb{E}[M(n+m)|\mathcal{F}_n] &= \mathbb{E}[W^2(n+m)|\mathcal{F}_n] - n - m \\ &= W^2(n) - n \\ &= M(n)\end{aligned}$$

Hence, we conclude that $M(n) = W^2(n) - n$ is a discrete time martingale.

2. The nth variation of a function f, over the interval $[0, T]$, is defined as

$$V_T(n, f) := \lim_{||\Pi|| \to 0} \sum_{j=0}^{m-1} |f(t_{j+1}) - f(t_j)|^n$$

in which $\Pi = \{0 = t_0, t_1, \ldots, t_m = T\}$ is a *partition* of $[0, T]$, and

$$||\Pi|| = \max_{0 \le j \le m-1} (t_{j+1} - t_j).$$

Show that $V_T(1, W) = \infty$ and $V_T(3, W) = 0$, where W is a Brownian motion. **Solution:** Beginning with the first variation, we have

$$V_T(1, W) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|$$

Let us consider consider the sampled first variation

$$F_\Pi := \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|$$

Using the fact that $W(t_{j+1}) - W(t_j) \sim \mathcal{N}(0, t_{j+1} - t_j)$, we can calculate the expectation of this quantity to be

$$\begin{aligned}\mathbb{E}F_\Pi &= \sum_{j=0}^{m-1} \mathbb{E}|W(t_{j+1}) - W(t_j)| \\ &= \sum_{j=0}^{m-1} \sqrt{t_{j+1} - t_j} = \sum_{j=0}^{m-1} \frac{t_{j+1} - t_j}{\sqrt{t_{j+1} - t_j}} \\ &\ge \frac{1}{\sqrt{||\Pi||}} \sum_{j=0}^{m-1} t_{j+1} - t_j = \frac{T}{\sqrt{||\Pi||}}\end{aligned}$$

Thus, as $||\Pi|| \to 0$ we have $\mathbb{E}F_\Pi \to \infty$, which proves that

$$V_T(1, W) = \lim_{||\Pi|| \to 0} F_\Pi = \infty \qquad \text{(a.s.)}$$

Next, let us consider the cubic variation

$$V_T(3, W) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^3$$

We start by considering the sampled cubic variation

$$C_\Pi := \sum_{j=0}^{m-1} |W(t_{j+1}) - W(t_j)|^3$$

We again use the fact that $W(t_{j+1}) - W(t_j) \sim \mathcal{N}(0, t_{j+1} - t_j)$ to calculate the expectation of this quantity.

$$\begin{aligned}
\mathbb{E}C_\Pi &= \sum_{j=0}^{m-1} \mathbb{E}|W(t_{j+1}) - W(t_j)|^3 \\
&= \sum_{j=0}^{m-1} (t_{j+1} - t_j)^{3/2} \\
&\leq \sqrt{||\Pi||} \sum_{j=0}^{m-1} t_{j+1} - t_j = T\sqrt{||\Pi||}
\end{aligned}$$

Thus, as $||\Pi|| \to 0$ we have $\mathbb{E}C_\Pi \to 0$, which proves that

$$V_T(3, W) = \lim_{||\Pi|| \to 0} C_\Pi = 0 \qquad \text{(a.s.)}$$

3. (a) Show that the transition probability density function for standard Brownian motion $W(t)$:

$$\frac{1}{dx} \Pr\{x < W(t+s) \leq x + dx |\, W(s) = y\} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} = f(x; t|y)$$

in which $t, s > 0$.

Solution: By the definition of Brownian motion, we have

$$W(s+t) - W(s) \sim \mathcal{N}(0, t) \qquad \Rightarrow \qquad W(s+t) \sim \mathcal{N}(W(s), t)$$

This means that, for a fixed $W(s) = y$, the probability density function of $W(s+t)$ is given by

$$\Pr\{x < W(s+t) \le x + dx | W(s) = y\} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dx = f(x; t|y) dx$$

Dividing all terms by dx gives us the transition probability density function

$$\frac{1}{dx} \Pr\{x < W(t+s) \le x + dx|\, W(s) = y\} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} = f(x; t|y)$$

(b) Show that $f(x; t|y)$ satisfies the following two linear partial differential equations:

$$\frac{\partial f(x;t|y)}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 f(x;t|y)}{\partial x^2}\right) \quad \text{and} \quad \frac{\partial f(x;t|y)}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 f(x;t|y)}{\partial y^2}\right)$$

Solution: This can be shown by direct differentiation. We have

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} = -\frac{t - (x-y)^2}{2t^2\sqrt{2\pi t}} e^{-(x-y)^2/(2t)}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2}{\partial x^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} = \frac{\partial}{\partial x} \frac{-(x-y)}{t\sqrt{2\pi t}} e^{-(x-y)^2/(2t)} \\
&= -\frac{t - (x-y)^2}{t^2\sqrt{2\pi t}} e^{-(x-y)^2/(2t)} \\
&= 2\frac{\partial f}{\partial t}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial^2}{\partial y^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} = \frac{\partial}{\partial y} \frac{x-y}{t\sqrt{2\pi t}} e^{-(x-y)^2/(2t)} \\
&= -\frac{t - (x-y)^2}{t^2\sqrt{2\pi t}} e^{-(x-y)^2/(2t)} \\
&= 2\frac{\partial f}{\partial t}
\end{aligned}$$

Hence, we conclude that

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}$$

4. **Exercise 8.1**: Compute $d(W_t^4)$. Write W_T^4 as an integral with respect to W plus an integral with respect to t. Use this representation of W_T^4 to show that $\mathbb{E}W_T^4 = 3T^2$. Compute $\mathbb{E}W_T^6$ using the same technique.

Solution: We can use the differential form of Itô's formula in one dimension. Recall that for $f \in C^2(\mathbb{R})$ we have

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X,X]_t$$

Hence, for $f(x) = x^4$ we have

$$df(W_t) = dW_t^4 = 4W_t^3 dW_t + 6W_t^2 dt$$

where we have used $d[W,W]_t = dt$. Using this, we may write W_T^4 as

$$W_T^4 = 4\int_0^T W_t^3 dW_t + 6\int_0^T W_t^2 dt$$

and we can calculate the expectation $\mathbb{E}W_T^4$ as

$$\begin{aligned}
\mathbb{E}W_T^4 &= 4\int_0^T \mathbb{E}W_t^3 dW_t + 6\int_0^T \mathbb{E}W_t^2 dt \\
&= 6\int_0^T t dt \\
&= 3T^2
\end{aligned}$$

Let us now use this same technique to compute $\mathbb{E}W_T^6$. Let $g(x) = x^6$, then by the Itô formula in one dimension, we have

$$dg(W_t) = dW_t^6 = 6W_t^5 dW_t + 30W_t^4 dt$$

$$\Rightarrow W_T^6 = 6\int_0^T W_t^5 dW_t + 30\int_0^T W_t^4 dt$$

We can now calculate the expectation $\mathbb{E}W_T^6$. Using our prior result for $\mathbb{E}W_T^4$, we have

$$\begin{aligned}
\mathbb{E}W_T^6 &= 6\int_0^T \mathbb{E}W_t^5 dW_t + 30\int_0^T \mathbb{E}W_t^4 dt \\
&= 30\int_0^T (3t^2) dt \\
&= 30T^3
\end{aligned}$$

5. **Exercise 8.2**: Find an explicit expression for Y_T where

$$dY_t = rdt + \alpha Y_t dW_t$$

Solution: Let $Z_t := \exp(-\alpha W_t + \frac{1}{2}\alpha^2 t)$. Then by Itô's formula we have

$$\begin{aligned} dZ_t &= -\alpha Z_t dW_t + \frac{1}{2}\alpha^2 Z_t dt + \frac{1}{2}\alpha^2 Z_t d[W,W]_t \\ &= -\alpha Z_t dW_t + \alpha^2 Z_t dt \end{aligned}$$

Using this, we may compute $d(Y_t Z_t)$ as

$$\begin{aligned} d(Y_t Z_t) &= Z_t dY_t + Y_t dZ_t + dY_t dZ_t \\ &= Z_t r dt + \alpha Y_t Z_t dW_t - \alpha Y_t Z_t dW_t + \alpha^2 Y_t Z_t dt - \alpha^2 Y_t Z_t dt \\ &= r Z_t dt \end{aligned}$$

Integrating both sides, we find

$$\begin{aligned} \int_0^T d(Y_t Z_t) = Y_T Z_T - Y_0 Z_0 = r \int_0^T Z_t dt \\ \Rightarrow Y_T = \frac{1}{Z_T}\left(r \int_0^T Z_t dt + Y_0 Z_0 \right) \end{aligned}$$

Lastly, we plug back in the explicit expression for Z_t to find

$$Y_T = \left(r \int_0^T e^{-\alpha W_t + \frac{1}{2}\alpha^2 t} dt + Y_0 \right) e^{\alpha W_T - \frac{1}{2}\alpha^2 T}$$

6. **Exercise 8.3**: Suppose X, Δ and Π are given by

$$dX_t = \sigma X_t dW_t, \qquad \Delta_t = \frac{\partial f}{\partial x}(t, X_t), \qquad \Pi_t = X_t \Delta_t$$

where f is some smooth function. Show that if f satisfies

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) f(t,x) = 0$$

for all (t,x), then Π is a martingale with respect to a filtration $\mathcal{F}_t$ for all W.

Solution: We begin by using Itô's formula to compute $d\Delta_t$. Using subscripts to denote partial derivatives of $f(t, X_t)$, we have

$$
\begin{aligned}
d\Delta_t = d(f_x) &= f_{xt}dt + f_{xx}dX_t + \frac{1}{2}f_{xxx}dX_t^2 \\
&= f_{xt}dt + \sigma X_t f_{xx}dW_t + \frac{1}{2}\sigma^2 X_t^2 f_{xxx}dt \\
&= (f_{xt} + \frac{1}{2}\sigma^2 X_t^2 f_{xxx})dt + \sigma X_t f_{xx}dW_t
\end{aligned}
$$

Using this, we may compute $d\Pi_t$ as

$$
\begin{aligned}
d\Pi_t &= X_t d\Delta_t + \Delta_t dX_t + d[X,\Delta]_t \\
&= X_t(f_{xt} + \frac{1}{2}\sigma^2 X_t^2 f_{xxx})dt + \sigma X_t^2 f_{xx}dW_t + \sigma X_t f_x dW_t + \sigma^2 X_t^2 f_{xx}dt \\
&= X_t(f_{xt} + \frac{1}{2}\sigma^2 X_t^2 f_{xxx} + \sigma^2 X_t f_{xx})dt + \sigma X_t(X_t f_{xx} + f_x)dW_t
\end{aligned}
$$

From this we see that Π is an Itô process with a drift term given by

$$
\Theta_t = X_t(f_{xt} + \frac{1}{2}\sigma^2 X_t^2 f_{xxx} + \sigma^2 X_t f_{xx})
$$

However, since $f(t, X_t)$ satisfies

$$
\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) f(t,x) = 0
$$

it follows that

$$
\begin{aligned}
f_t &= -\frac{1}{2}\sigma^2 x^2 f_{xx} \\
\Rightarrow f_{xt} &= -\frac{1}{2}\sigma^2 \left(2x f_{xx} + x^2 f_{xxx}\right)
\end{aligned}
$$

Evaluating this expression at $x = X_t$ and substituting it into our drift term results in

$$
\Theta_t = 0
$$

It follows that Π is an Itô process with zero drift. Consequently, Π must be a martingale with respect to any filtration $\mathbb{F}$ of the brownian motion W.

7. **Exercise 8.4**: Suppose X is given by

$$
dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.
$$

For any smooth function f define

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mu(s, X_s) \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(s, X_s) \frac{\partial^2}{\partial x^2} \right) f(s, X_s) ds.$$

Show that M^f is a martingale with respect to a filtration $\mathcal{F}_t$ for W.

Solution: We note that we can write the given expression for M_t^f as

$$M_t^f = \int_0^t df - \int_0^t \left(f_s + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} \right) ds$$

Additionally, using Itô's formula we may write df as

$$\begin{aligned} df &= f_x dX_t + f_t dt + \frac{1}{2} f_{xx} dX_t^2 \\ &= f_x(\mu dt + \sigma dW_t) + f_t dt + \frac{1}{2}\sigma^2 f_{xx} dt \\ &= \left(f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x dW_t \end{aligned}$$

Hence, we have

$$\begin{aligned} M_t^f &= \int_0^t \sigma f_x dW_s + \int_0^t \left(f_s + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} \right) ds - \int_0^t \left(f_s + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} \right) ds \\ &= \int_0^t \sigma(s, X_s) f_x(s, X_s) dW_s \end{aligned}$$

Since M_t^f is an Itô process with zero drift, it follows that it must be Martingale with respect to any filtration $\mathbb{F}$ for the Brownian motion W.