# AMATH 562
# Advanced Stochastic Processes
# **Homework 2**

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Due: February 2, 2023

1. Consider a measurable space $(\Omega, \mathcal{F})$ with finite elementary event set $\Omega = \{1, \ldots, n\}$, the corresponding $\mathcal{F} = 2^\Omega$, and the Lebesgue-measure like counting measure $\nu_i = 1, 1 \le i \le n$. A stochastic Markov (chain) dynamics, $X_k$, has one step transitions in terms of a set of conditional probabilities $p^{(\boldsymbol{\nu})}(i,j) = \Pr\{X_{k+1} = j | X_k = i\}$. This assumption of a "counting measure" $\boldsymbol{\nu}$ is implicit in all Markov chain theory.

   (a) If a Markov chain with $p^{(\boldsymbol{\nu})}(i,j)$ has a unique invariant probability $\boldsymbol{\pi} = \{\pi_1, \ldots, \pi_n\}$ with all positive $\pi_k$, express the transition probability as the Radon-Nikodym derivative w.r.t. $\boldsymbol{\pi}$, denoted as $p^{(\boldsymbol{\pi})}(i,j)$.

   **Solution:** From the definition of the RND, we see that for the given discrete space the transition probability with respect to $\boldsymbol{\pi}$ is given by

$$p^{(\boldsymbol{\pi})}(i,j) = \frac{p^{(\boldsymbol{\nu})}(i,j)}{\pi_j}$$

   (b) Show that

$$\boldsymbol{\pi} P^{(\boldsymbol{\pi})} = \mathbf{1}$$

   and

$$P^{(\boldsymbol{\pi})} \boldsymbol{\pi}^T = \mathbf{1}^T$$

   where $P^{(\boldsymbol{\pi})}$ is the transition probability matrix w.r.t. $\boldsymbol{\pi}$, $\mathbf{1} = (1, \ldots, 1)$, and $\mathbf{1}^T$ is the column vector of 1's. Please explain these two equations.

   **Solution:** For the first equation we consider the $j$th entry of the product $\boldsymbol{\pi} P^{(\boldsymbol{\pi})}$.

$$(\boldsymbol{\pi} P^{(\boldsymbol{\pi})})_j = \sum_i \pi_i p^{(\boldsymbol{\pi})}(i,j) = \sum_i \frac{\pi_i}{\pi_j} p^{(\boldsymbol{\nu})}(i,j) = \frac{1}{\pi_j} \sum_i \pi_i p^{(\boldsymbol{\nu})}(i,j) = 1$$

where the last equality comes from the fact that $\boldsymbol{\pi}$ is the stationary distribution, which by definition means that $\boldsymbol{\pi} P^{(\boldsymbol{\nu})} = \boldsymbol{\pi}$. Hence,

$$\boldsymbol{\pi} P^{(\boldsymbol{\pi})} = \mathbf{1}$$

Furthermore, transposing this equation results in

$$P^{(\boldsymbol{\pi})} \boldsymbol{\pi}^T = \mathbf{1}^T$$

which was the second equation we wanted to show.

Both of these equations express the fact that the total probability of being in any state at the current time step should equal the total probability of being in any state at the next time step, as the chain transitions from one state to another.

(c) The *reversibility* of a Markov chain is introduced in §4.5 of MLN. What is the $P^{(\boldsymbol{\pi})}$ of a reversible Markov chain?

**Solution:** A Markov chain is reversible if its one-step transition matrix is invariant under time reversal. That is,

$$\pi_i p^{(\boldsymbol{\nu})}(i,j) = \pi_j p^{(\boldsymbol{\nu})}(j,i)$$

And since $p^{(\boldsymbol{\nu})}(i,j) = \pi_j p^{(\boldsymbol{\pi})}(i,j)$, this means that

$$\pi_i \pi_j p^{(\boldsymbol{\pi})}(i,j) = \pi_j \pi_i p^{(\boldsymbol{\pi})}(j,i) \Rightarrow p^{(\boldsymbol{\pi})}(i,j) = p^{(\boldsymbol{\pi})}(j,i)$$

Hence, for reversible Markov chains, $P^{(\boldsymbol{\pi})}$ is symmetric.

(d) In discrete time, a deterministic first-order "dynamics" in the $\Omega$ is defined by a one step map $S : \Omega \to \Omega$. Since a deterministic first-order dynamics is just a special case of a Markov dynamics, express the transition probability $p^{(\boldsymbol{\nu})}(i,j)$ corresponding to the map $S$.

**Solution:** For a deterministic discrete time system with a one step map $S : \Omega \to \Omega$, the transition probability matrix $P^{(\boldsymbol{\nu})}$ is a matrix whose rows contain all zero entries except for at one point, where the value is one. Specifically,

$$p^{(\boldsymbol{\nu})}(i,j) = \delta_j^{S(i)}$$

where $\delta_j^i$ is the Kronecker delta.

(e) Show the deterministic dynamics in (d) has an invariant probability $\boldsymbol{\pi} = (\frac{1}{n}, \ldots, \frac{1}{n})$ if and only if the map $S$ is one to one. Within the context of a deterministic $S$, discuss the notion of *irreducibility* defined in §4.3 of MLN.

**Solution:** Consider the system in (d) with deterministic dynamics defined by a one-step map $S$ and with an invariant probability $\boldsymbol{\pi}$. Then by the definition of $\boldsymbol{\pi}$ we must have

$$\boldsymbol{\pi} = \boldsymbol{\pi} P^{(\boldsymbol{\nu})} \Rightarrow \pi_i = \sum_i \pi_i p^{(\boldsymbol{\nu})}(i,j) = \sum_i \pi_i \delta_j^{S(i)} = \pi_{S(i)}$$

Hence, each $\pi_n$ is equal to the value of the next state in the sequence generated by $S$.

Let $i, j \in C_k \subseteq \Omega$ be two states which are each members of the same closed set of persistent states in the Markov decomposition $\Omega = T \cup C_1 \cup C_2 \cup \ldots$ with respect to the dynamics generated by the one-step map $S$. Then the above result implies $\pi_i = \pi_j$, since $\exists n : S^n(i) = j$ and $\exists m : S^m(j) = i$. Since all of the states in $C_k$ have the same probability, it follows that $\mathbb{P}\{X = i \in C_k\} = \mathbb{P}(X \in C_k)/|C_k|$. In particular, if $\Omega$ is irreducible then all of the $\pi_i$ must be the same value of $1/|\Omega|$, which means that

$$\boldsymbol{\pi} = (\frac{1}{n}, \ldots, \frac{1}{n})$$

Likewise, if $\boldsymbol{\pi} = (\frac{1}{n}, \ldots, \frac{1}{n})$, then $S$ must be one-to-one. As shown in (d), for a deterministic discrete-time system the values in the matrix $P^{(\boldsymbol{\nu})}$ are all zeros and ones. Hence, if $S$ mapped two different states to the same state the resulting matrix rows would not sum to one, violating conservation of probability.

2. Consider the continuous time Markov chain with generator

$$\mathbf{G} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \ldots \\ \mu & -\mu-\lambda & \lambda & 0 & 0 \\ \ldots \\ 0 & 2\mu & -2\mu-\lambda & \lambda & 0 \\ \ldots \\ 0 & 0 & 3\mu & -3\mu-\lambda & \lambda \\ \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ddots \end{pmatrix}$$

in which $\lambda, \mu > 0$.

(a) Find its invariant probability distribution $\boldsymbol{\pi}$.

**Solution:** The invariant probability distribution $\boldsymbol{\pi}$ must satisfy $\boldsymbol{\pi}\mathbf{G} = 0$. From the matrix $\mathbf{G}$ above, we can see that this means that

$$\begin{aligned} -\lambda\pi_0 + \mu\pi_1 &= 0 \\ \lambda\pi_0 - (\mu+\lambda)\pi_1 + 2\mu\pi_2 &= 0 \\ \lambda\pi_1 - (2\mu+\lambda)\pi_2 + 3\mu\pi_3 &= 0 \\ &\vdots \\ \lambda\pi_{n-1} - (n\mu+\lambda)\pi_n + (n+1)\mu\pi_{n+1} &= 0 \end{aligned}$$

Rearranging and re-indexing this equation allows us to express $\pi_n$ in terms of $\pi_{n-1}$ and $\pi_{n-2}$ as

$$\pi_n = \frac{(n-1)\mu + \lambda}{n\mu}\pi_{n-1} - \frac{\lambda}{n\mu}\pi_{n-2}$$

Furthermore, from the first equation we have $\pi_1 = \frac{\lambda}{\mu}\pi_0$, hence by recursively applying the above formula we can find all $\pi_n$ in terms of $\pi_0$. When we do this we find

$$\pi_2 = \frac{\lambda^2}{2\mu^2}\pi_1, \qquad \pi_3 = \frac{\lambda^3}{6\mu^3}\pi_1, \qquad \pi_4 = \frac{\lambda^4}{24\mu^4}\pi_1, \qquad \ldots, \qquad \pi_n = \frac{\lambda^n}{n!\mu^n}\pi_0$$

All that remains now is to find $\pi_0$ by requiring that $\sum_n \pi_n = 1$. This results in

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{n!\mu^n} = \pi_0 e^{\lambda/\mu}$$

$$\Rightarrow \pi_0 = e^{-\lambda/\mu}$$

Hence, the invariant probability distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$ is given by the Poisson distribution:

$$\pi_n = \frac{\lambda^n}{n!\mu^n}e^{-\lambda/\mu}$$

(b) Assume $X_0 = 0$. Using the matrix exponential symbol $(e^{\mathbf{G}t})_{ij}$, give the joint probability for the finite trajectory

$$\Pr\{X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n\},$$

where $0 < t_1 < t_2 < \cdots < t_n$.

**Solution:** Since $(X_n)$ is Markov, the probability of a given transition in a trajectory is independent of all the other transitions in the trajectory. Hence, we can express the joint probability of the trajectory as the product of the individual transition probabilities

$$\Pr\{X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n\} = \prod_{k=1}^{n} \Pr\{X_{t_k} = i_k | X_{t_{k-1}} = i_{k-1}\}$$

Furthermore, the time-independence of $\mathbf{G}$ implies that $(X_n)$ is homogeneous, and therefore its dynamics at each step in the trajectory are given by the transition matrices $(\mathbf{P}_\tau)_{\tau \geq 0}$, where $\tau$ is the duration between the time steps in the partition $\{0, t_1, \ldots, t_n\}$. Let $\tau_k = t_k - t_{k-1}$ represent the time difference between the $k$th $t_i$ and the $k-1$th $t_i$, then we may write

$$\Pr\{X_{t_k} = i_k | X_{t_{k-1}} = i_{k-1}\} = (\mathbf{P}_{\tau_k})_{i_{k-1}i_k}$$

Lastly, we can write these transition matrices in terms of the generator $\mathbf{G}$, and write the finite trajectory probability as

$$\Pr\{X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n\} = \prod_{k=1}^{n} (e^{\mathbf{G}\tau_k})_{i_{k-1}i_k}$$

(c) Introducing the probability generating function (see MLN §5.2)

$$G_X(s,t) = \mathbb{E}[s^{X_t}].$$

Show that $G_X(s,t)$ satisfies the following partial differential equation

$$\frac{\partial}{\partial t} G_X(s,t) = u\left(s, t, G_X, \frac{\partial G_X}{\partial s}, \frac{\partial^2 G_X}{\partial s^2}\right).$$

Give the explicit form for the function on the rhs.

**Solution:** Recall the definition of $G_X(s,t)$,

$$G_X(s,t) := \mathbb{E}[s^{X_t} | X_0 = i] = \sum_{j=0}^{\infty} s^j p_t(i,j)$$

from which it follows that

$$\frac{\partial G_X(s,t)}{\partial t} = \sum_{j=0}^{\infty} s^j \frac{\partial p_t(i,j)}{\partial t} \qquad \frac{\partial G_X(s,t)}{\partial s} = \sum_{j=0}^{\infty} j s^{j-1} p_t(i,j)$$

By the Kolmogorov forward equation we have

$$\frac{d}{dt} p_t(i,0) = -\lambda p_t(i,0) + \mu p_t(i,1)$$
$$\frac{d}{dt} p_t(i,j) = \lambda p_t(i,j-1) - (j\mu + \lambda) p_t(i,j) + (j+1)\mu p_t(i,j+1) \qquad j \geq 1$$

Multiplying the $j$th equation by $s^j$ and summing over $j$, we have

$$\sum_{j=0}^{\infty} s^j \frac{d}{dt} p_t(i,j) = \lambda \sum_{j=1}^{\infty} s^j p_t(i,j-1) - \sum_{j=0}^{\infty} s^j (j\mu + \lambda) p_t(i,j) + \sum_{j=0}^{\infty} s^j (j+1)\mu p_t(i,j+1)$$

Let us consider each of these four summations in turn, from left to right.

i.
$$\sum_{j=0}^{\infty} s^j \frac{d}{dt} p_t(i,j) = \frac{\partial G_X(s,t)}{\partial t}$$

ii.

$$\lambda\sum_{j=1}^{\infty} s^j p_t(i,j-1) = \lambda s\sum_{j=0}^{\infty} s^j p_t(i,j) = \lambda s G_X(s,t)$$

iii.

$$\sum_{j=0}^{\infty} s^j(j\mu+\lambda)p_t(i,j) = \sum_{j=0}^{\infty} js^j\mu p_t(i,j)+\lambda\sum_{j=0}^{\infty} s^j p_t(i,j) = \mu s\frac{\partial G_X(s,t)}{\partial s}+\lambda G_X(s,t)$$

iv.

$$\sum_{j=0}^{\infty} s^j(j+1)\mu p_t(i,j+1) = \mu\sum_{j=0} js^{j-1}p_t(i,j) = \mu\frac{\partial G_X(s,t)}{\partial s}$$

Plugging these in, we end up with the following PDE in terms of $G_X(s,t)$:

$$\frac{\partial}{\partial t}G_X = \lambda(s-1)G_X + \mu(1-s)\frac{\partial}{\partial s}G_X$$

(d) Show that the solution to the PDE in (c), with initial data $G_X(s,0)$, is

$$G_X(s,t) = G_X\left(1+(s-1)e^{-\mu t},0\right)\exp\left\{\frac{\lambda}{\mu}(s-1)\left(1-e^{-\mu t}\right)\right\}.$$

**Solution:** Let us assume solutions of the form $G_X(s,t) = g(s)f(t)$. Then our differential equation becomes

$$gf' = \lambda(s-1)gf + \mu(1-s)g'f$$

Dividing both sides by $gf$, we have

$$\frac{f'}{f} = \lambda(s-1) + \mu(1-s)\frac{g'}{g}$$

We see that the lhs is a function of only $t$, while the rhs is a function only of $s$. It follows that both sides must be constant.

$$\frac{f'}{f} = \lambda(s-1) + \mu(1-s)\frac{g'}{g} = C$$

We have reduced the system to a pair of ODEs for $f(t)$ and $g(s)$. We will begin with the $f(t)$ equation.

$$\frac{f'}{f} = C \Rightarrow f' = Cf \Rightarrow f(t) = f_0e^{Ct}$$

where $f_0$ is some constant.

Next, we have the equation for $g(s)$. We have

$$
\begin{aligned}
\lambda(s-1)+\mu(1-s)\frac{g'}{g} &= C \\
\Rightarrow \frac{g'}{g} &= \frac{1}{\mu}\left[\frac{C}{1-s}+\lambda\right] \\
\Rightarrow \int \frac{dg}{g} &= \frac{1}{\mu}\int\left[\frac{C}{1-s}+\lambda\right] ds \\
\Rightarrow \log\left(\frac{g}{g_0}\right) &= \frac{1}{\mu}\left[-C\log(1-s')+\lambda s'\right]\Big|_0^s \\
\Rightarrow g = g_0 \exp&\left\{\frac{1}{\mu}\left[\lambda s - C\log(1-s)\right]\right\}
\end{aligned}
$$

$$
\Rightarrow g = g_0\left(\frac{e^{\lambda s}}{(1-s)^C}\right)^{1/\mu}
$$

where $g_0$ is another constant. Now that we have solved for the $s$ and $t$ dependencies of $G_X$, we can write down the full solution $G_X(s,t) = g(s)f(t)$. Letting $G_0 = f_0 g_0$, we have

$$
G_X(s,t) = G_0 e^{Ct}\left(\frac{e^{\lambda s}}{(1-s)^C}\right)^{1/\mu}
$$

To show that this solution is equivalent to the one provided, we note that

$$
G(s,0) = G_0\left(\frac{e^{\lambda s}}{(1-s)^C}\right)^{1/\mu}
$$

$$
\Rightarrow G_X\left(1+(s-1)e^{-\mu t},0\right) = G_0\left(\frac{\exp\left\{\lambda(1+(s-1)e^{-\mu t})\right\}}{(1-s)^C e^{-\mu C t}}\right)^{1/\mu}
$$

$$
\begin{aligned}
\Rightarrow G_X\left(1+(s-1)e^{-\mu t},0\right)\exp\left\{\frac{\lambda}{\mu}(s-1)\left(1-e^{-\mu t}\right)\right\} &= G_0 e^{Ct}\left(\frac{e^{\lambda s}}{(1-s)^C}\right)^{1/\mu} \\
&= G_X(s,t)
\end{aligned}
$$

(e) Verify that the limit of $G_X(s,t)$ as $t\to\infty$ agrees with the $\boldsymbol{\pi}$ obtained in (a).
**Solution:** We have the following limit

$$
\begin{aligned}
\lim_{t\to\infty} G_X(s,t) &= \lim_{t\to\infty} G_X\left(1+(s-1)e^{-\mu t},0\right)\exp\left\{\frac{\lambda}{\mu}(s-1)\left(1-e^{-\mu t}\right)\right\} \\
&= G_X(1,0)\exp\left\{\frac{\lambda}{\mu}(s-1)\right\} \\
&= \exp\left\{\frac{\lambda}{\mu}(s-1)\right\}
\end{aligned}
$$

Where for the last inequality we used

$$G_X(s,t) = \sum_{j=0}^{\infty} s^j p_t(i,j) \Rightarrow G_X(1,0) = \sum_{j=0}^{\infty} p_0(i,j) = 1$$

Next, we can expand the exponential on the righthand side to find

$$\lim_{t\to\infty} G_X(s,t) = e^{-\lambda/\mu} \sum_{n=0}^{\infty} s^n \frac{\lambda^n}{n!\mu^n}$$

One of the properties of the probability generating function $G_X(s,t)$ is that the coefficient of the $\mathcal{O}(s^n)$ term in its power series expansion is the probability $\mathbb{P}(X_t = n)$, and so we would expect these coefficients to approach $\pi_k$ as $t \to \infty$. Indeed, this is exactly what we find.

3. Let the generator $\mathbf{G}$ of a three-state continuous time Markov chain $X_t$ be given by

$$\begin{pmatrix} -\alpha-\beta & \alpha & \beta \\ \beta & -\alpha-\beta & \alpha \\ \alpha & \beta & -\alpha-\beta \end{pmatrix} = \alpha \begin{pmatrix} -1-b & 1 & b \\ b & -1-b & 1 \\ 1 & b & -1-b \end{pmatrix},$$

in which $\alpha, \beta > 0$, $b = \beta/\alpha$. Note the $\mathbf{G}$ matrix is circulant, so its eigenvalues and eigenvectors have special forms which are readily obtained. Assuming that $X_0$ follows the invariant probability distribution $\boldsymbol{\pi}$; therefore $X_t^{(st)}$ is a stationary Markov chain. Let the function $y(X) = -1, 0, 1$ corresponding to the states $X = 1, 2, 3$.

(a) Compute $\mu = \mathbb{E}[y(X_t^{(st)})]$ and $\sigma^2 = \mathbb{V}[y(X_t^{(st)})]$.
**Solution:** It's simple to see that $\mathbf{1} = (1,1,1)$ is a left eigenvector of $\mathbf{G}$ with eigenvalue 0, hence the invariant distribution can be found by normalizing this row vector.

$$\boldsymbol{\pi} = (1/3, 1/3, 1/3)$$

Using this distribution we can calculate the mean to be:

$$\mu = \mathbb{E}[y(X_t^{(st)})] = \frac{1}{3}(y(1) + y(2) + y(3)) = 0$$

Similarly, the variance is given by

$$\sigma^2 = \mathbb{V}[y(X_t^{(st)})] = \mathbb{E}[y(X_t^{(st)})^2] - 0 = \frac{1}{3}(y^2(1) + y^2(2) + y^2(3)) = \frac{2}{3}$$

(b) For two random variables $V(\omega)$ and $W(\omega)$,

$$\mathbb{E}\left[(V - \mathbb{E}[V])(W - \mathbb{E}[W])\right]$$

is called covariance between $V$ and $W$. Find an analytical expression for the covariance function

$$g(\tau) = \mathbb{E}\left[\left(y(X_{t+\tau}^{(st)}) - \mu\right)\left(y(X_t^{(st)}) - \mu\right)\right].$$

**Solution:** In part (a) we found that $\mu = 0$. Using this along with the Markov property of $X_t$ we may write

$$g(\tau) = \mathbb{E}\left[\left(y(X_{t+\tau}^{(st)})\right)\left(y(X_t^{(st)})\right)\right] = \mathbb{E}\left[\left(y(X_\tau^{(st)})\right)\left(y(X_0^{(st)})\right)\right]$$

Next, by the definition of expectation we have

$$\begin{aligned}
\mathbb{E}\left[\left(y(X_\tau^{(st)})\right)\left(y(X_0^{(st)})\right)\right] &= \sum_{i,j=1}^{3} y(i)y(j)\mathbb{P}(X_0 = i, X_\tau = j) \\
&= \sum_{i,j=1}^{3} y(i)y(j)p_\tau(i,j)\pi_i \\
&= \frac{1}{3}\sum_{i,j=1}^{3} y(i)y(j)(e^{\mathbf{G}\tau})_{ij}
\end{aligned}$$

So, using the fact that $y(2) = 0$, we can write

$$g(\tau) = \frac{1}{3}\left(y(1)^2(e^{\mathbf{G}\tau})_{11} + y(1)y(3)\left[(e^{\mathbf{G}\tau})_{13} + (e^{\mathbf{G}\tau})_{31}\right] + y(3)^2(e^{\mathbf{G}\tau})_{33}\right)$$

$$= \frac{1}{3}\left((e^{\mathbf{G}\tau})_{11} - \left[(e^{\mathbf{G}\tau})_{13} + (e^{\mathbf{G}\tau})_{31}\right] + (e^{\mathbf{G}\tau})_{33}\right)$$

We turn now to computing the components of $e^{\mathbf{G}\tau}$. Since $\mathbf{G}$ is a $3\times 3$ circulant matrix, its eigenvectors and eigenvalues are given by

$$v_{j+1} = \frac{1}{\sqrt{3}}(1, \omega^j, \omega^{2j}) \qquad \lambda_{j+1} = \alpha(\omega^j - 1) + \beta(\omega^{2j} - 1) \qquad j = 0, 1, 2$$

where $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is the cubic root of unity. Note that $\omega^2 = \omega^*$. Using this, we may write $\mathbf{G}$ in the form $\mathbf{G} = \mathbf{U\Lambda U}^{-1}$ with

$$\mathbf{U} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^* & \omega \\ 1 & \omega & \omega^* \end{pmatrix} \qquad \mathbf{U}^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix} = \mathbf{U}^*$$

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha(\omega - 1) + \beta(\omega^* - 1) & 0 \\ 0 & 0 & \alpha(\omega^* - 1) + \beta(\omega - 1) \end{pmatrix}$$

Using this decomposition, we may write

$$(e^{\mathbf{G}\tau})_{ij} = \sum_k u_{ik} e^{\tau\lambda_k} u^*_{kj}$$

where $u_{ij} = (\mathbf{U})_{ij}$. With this expression and the above matrices, we may proceed with calculating the relevant components.

i. $(e^{\mathbf{G}\tau})_{11} = \sum_{k=1}^{3} u_{1k} e^{\tau\lambda_k} u^*_{k1} = (e^{\tau\lambda_1} + e^{\tau\lambda_2} + e^{\tau\lambda_3})/3$
ii. $(e^{\mathbf{G}\tau})_{13} = \sum_{k=1}^{3} u_{1k} e^{\tau\lambda_k} u^*_{k3} = (e^{\tau\lambda_1} + \omega^* e^{\tau\lambda_2} + \omega e^{\tau\lambda_3})/3$
iii. $(e^{\mathbf{G}\tau})_{31} = \sum_{k=1}^{3} u_{3k} e^{\tau\lambda_k} u^*_{k1} = (e^{\tau\lambda_1} + \omega e^{\tau\lambda_2} + \omega^* e^{\tau\lambda_3})/3$
iv. $(e^{\mathbf{G}\tau})_{33} = \sum_{k=1}^{3} u_{3k} e^{\tau\lambda_k} u^*_{k3} = (e^{\tau\lambda_1} + e^{\tau\lambda_2} + e^{\tau\lambda_3})/3$

Plugging in these components, and using $\omega + \omega^* = -1$, we find $g(\tau)$ to be

$$g(\tau) = \frac{1}{3}(e^{\tau\lambda_2} + e^{\tau\lambda_3})$$

Lastly, we plug in the values of $\lambda_2$ and $\lambda_3$ to find

$$g(\tau) = \frac{1}{3}(e^{\tau(\alpha(\omega-1)+\beta(\omega^*-1))} + e^{\tau(\alpha(\omega^*-1)+\beta(\omega-1))})$$

which can be simplified to the form

$$g(\tau) = \frac{2}{3}\cos\left(\frac{\sqrt{3}}{2}(\alpha - \beta)\tau\right) e^{-\frac{3}{2}\tau(\alpha+\beta)}$$

(c) Show that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T y(X_t^{(st)})dt = \mu$$

where the convergence is by $L^2$.

**Solution:** By the definition of $L^2$ convergence, we have

$$
\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{T}\int_0^T y(X_t^{(st)})dt-\mu\right)^2\right] &= \mathbb{E}\left[\left(\frac{1}{T}\int_0^T y(X_t^{(st)})dt\right)^2\right] \\
&= \frac{1}{T^2}\mathbb{E}\left[\left(\int_0^T y(X_t^{(st)})dt\right)^2\right] \\
&= \frac{1}{T^2}\mathbb{E}\left[\left(\int_0^T y(X_t^{(st)})dt\right)\left(\int_0^T y(X_\tau^{(st)})d\tau\right)\right] \\
&= \frac{1}{T^2}\mathbb{E}\left[\int_0^T\int_0^T y(X_t^{(st)})y(X_\tau^{(st)})d\tau dt\right] \\
&= \frac{2}{T^2}\int_0^T\int_0^t \mathbb{E}\left[y(X_t^{(st)})y(X_{t+\tau}^{(st)})\right]d\tau dt \\
&= \frac{2}{T^2}\int_0^T\int_0^t g(\tau)d\tau dt \\
&\leq \frac{2}{T^2}\int_0^T\int_0^t e^{-a\tau}d\tau dt
\end{aligned}
$$

Where in the final step we have bounded $g(\tau)$ by $e^{-a\tau}$, where $a=\frac{3}{2}(\alpha+\beta)$. So we have

$$
\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{T}\int_0^T y(X_t^{(st)})dt-\mu\right)^2\right] &\leq \frac{2}{aT^2}\int_0^T\left(1-e^{-at}\right)dt \\
&= \frac{2}{aT^2}\left(T+\frac{1}{a}e^{-aT}\right)
\end{aligned}
$$

We find that our expectation term is bounded by $\mathcal{O}(\frac{1}{T})$, which means that

$$
\lim_{T\to\infty}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T y(X_t^{(st)})dt-\mu\right)^2\right]=0
$$

and hence that

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T y(X_t^{(st)})dt=\mu
$$

by $L^2$ convergence.

(d) Use a computer and Monte Carlo simulation to verify that

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T y(X_{t+\tau}^{(st)})y(X_t^{(st)})dt-\mu^2
$$

agrees with $g(\tau)$ you obtained from (b)

**Solution:** See attached code. Monte Carlo integral estimate with N=10000 and T=10000 is -0.0084. This agrees with our above result

4. Let $W(t)$ be a standard Brownian motion. Introducing a function of the Brownian motion

$$\tilde{W}(s) = (1-s)W\left(\frac{s}{1-s}\right) \qquad\qquad 0 < s < 1$$

Compute its expected value, variance, and covariance function

$$\mathbb{C}\text{ov}[\tilde{W}(s_1), \tilde{W}(s_2)] \qquad\qquad 0 < s_1 < s_2 < 1$$

$\tilde{W}(s)$ is known as a *Brownian bridge.*

**Solution:** Let $t = \frac{s}{1-s}$, then

$$\tilde{W}(s) = \frac{1}{1+t}W(t)$$

With this substitution we have written $\tilde{W}(s)$ in terms of the standard Brownian motion $W(t)$, which we know is Gaussian distributed.

(a)

$$\mathbb{E}[\tilde{W}(s)] = \mathbb{E}\left[\frac{1}{1+t}W(t)\right] = \frac{1}{(1+t)\sqrt{2\pi t}}\int_{-\infty}^{\infty} xe^{-x^2/2t}dx = 0$$

Since the integrand is odd, we see immediately that $\mathbb{E}[\tilde{W}(s)] = 0$.

(b)

$$\begin{aligned}
\mathbb{V}[\tilde{W}(s)] &= \mathbb{V}\left[\frac{1}{1+t}W(t)\right] \\
&= \mathbb{E}\left[\left(\frac{1}{1+t}W(t)\right)^2\right] \\
&= \frac{1}{(1+t)^2}\mathbb{E}\left[W(t)^2\right] \\
&= \frac{t}{(1+t)^2} = s(1-s)
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{Co}\mathbb{V}\left[\tilde{W}(s_1),\tilde{W}(s_2)\right] &= \mathbb{E}\left[\left(\tilde{W}(s_1)-\mathbb{E}[\tilde{W}(s_1)]\right)\left(\tilde{W}(s_2)-\mathbb{E}[\tilde{W}(s_2)]\right)\right] \\
&= \mathbb{E}\left[\left(\tilde{W}(s_1)\right)\left(\tilde{W}(s_2)\right)\right] \\
&= \mathbb{E}\left[\left(\frac{W(t_1)}{1+t_1}\right)\left(\frac{W(t_2)}{1+t_2}\right)\right] \\
&= \frac{1}{(1+t_1)(1+t_2)}\mathbb{E}\left[W(t_1)W(t_2)\right] \\
&= \frac{\min(t_1,t_2)}{(1+t_1)(1+t_2)} = \frac{t_1}{(1+t_1)(1+t_2)} \\
&= s_1(1-s_2)
\end{aligned}$$

5. $W(t)$ is a standard Brownian motion. What is the characteristic function of $W(N_t)$ where $N_t$ is a Poisson process with intensity $\lambda$, and the brownian motion $W(t)$ is independent of the Poisson process $N_t$.

   **Solution:** We would like to calculate the characteristic function

$$\phi(\tau) = \mathbb{E}e^{i\tau W(N_t)}$$

This involves computing the following quantity

$$\phi(\tau) = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\frac{1}{\sqrt{2\pi n}}\int_{-\infty}^{\infty} e^{i\tau x}e^{-x^2/(2n)}dx$$

Let us begin by evaluating the integral.

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2n}+i\tau x}dx \\
&= e^{-\tau^2 n/2}\int_{-\infty}^{\infty} e^{-(\frac{x}{\sqrt{2n}}-i\tau\sqrt{\frac{n}{2}})^2}dx
\end{aligned}$$

Let $u = \frac{x}{\sqrt{2n}}$ and $v = -\tau\sqrt{\frac{n}{2}}$, then

$$I = \sqrt{2n}e^{-\tau^2 n/2}\int_{-\infty}^{\infty} e^{-(u+iv)^2}du$$

As a corollary, note that

$$
\begin{aligned}
\frac{d}{dv}\int_{-\infty}^{\infty} e^{-(u+iv)^2}du &= \int_{-\infty}^{\infty} \frac{d}{dv}e^{-(u+iv)^2}du \\
&= \int_{-\infty}^{\infty} -2i(u+iv)e^{-(u+iv)^2}du \\
&= i\int_{-\infty}^{\infty} \frac{d}{du}e^{-(u+iv)^2}du \\
&= ie^{-(u+iv)^2}\Big|_{-\infty}^{\infty} = 0
\end{aligned}
$$

It follows that the Gaussian integral is invariant under shifts in the complex plane. Hence we may conclude that

$$
I = \sqrt{2\pi n}e^{-\tau^2 n/2}
$$

Returning to our above sum, we have

$$
\begin{aligned}
\phi(\tau) &= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}e^{-\tau^2 n/2} = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t e^{-\tau^2/2})^n}{n!} \\
&= \exp\left\{\lambda t(e^{-\tau^2/2}-1)\right\}
\end{aligned}
$$