AMATH 567, Homework 6

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1. **Problem 1:**

   (a) Let $\hat{f}(s)$ and $\hat{g}(s)$ be the Laplace transforms of one-sided functions $f(t)$ and $g(t)$, respectively. Show that the inverse Laplace transform of $\hat{f}(s)\hat{g}(s)$ is

   $$\int_0^t f(t-\tau)g(\tau)d\tau$$

   Solution: We begin by writing

   $$\hat{f}(s)\hat{g}(s) = \left[\int_0^\infty f(t_1)e^{-st_1}dt_1\right]\left[\int_0^\infty g(t_2)e^{-st_2}dt_2\right] = \int_0^\infty \int_0^\infty f(t_1)g(t_2)e^{-s(t_1+t_2)}dt_1dt_2$$

   We make the substitution $\tau = t_2$ and $t = t_1 + \tau$, so that we can write this expression as

   $$\hat{f}(s)\hat{g}(s) = \int_0^\infty \int_\tau^\infty f(t-\tau)g(\tau)e^{-st}dtd\tau$$

   Exchanging the order of integration, we can express this as

   $$\hat{f}(s)\hat{g}(s) = \int_0^\infty e^{-st}\int_0^t f(t-\tau)g(\tau)d\tau dt$$

   But this we can recognize as the Laplace transform of the convolution of $f(t)$ and $g(t)$. Therefore, the inverse Laplace transform of this expression will result in the convolution expression

   $$\int_0^t f(t-\tau)g(\tau)d\tau.$$

   (b) Use the Laplace transform and the result in (a) to solve the following ordinary differential equation $\frac{d^2y}{dt^2} + 4y = f(t)$, subject to the initial conditions $y(0) = 0$, $\frac{dy}{dt}(0) = 0$

   Solution: We begin by taking the Laplace transform of the terms in our equation.

   $$\mathcal{L}[\frac{d^2y}{dt^2}] + 4\mathcal{L}[y] = \mathcal{L}[f] \Rightarrow s^2Y + 4Y = F \Rightarrow Y = \frac{F}{s^2+4}$$

Here we are using the notation $\mathcal{L}[y(t)] = Y(s)$ and $\mathcal{L}[f(t)] = F(s)$. We also used the properties of the Laplace transform applied to derivatives, together with the boundary conditions that $y(0) = y'(0) = 0$, so that these terms don't appear in the final result.

We now recall the well known result that $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$. Comparing with our above expression, we find that

$$Y = \frac{1}{2} F \mathcal{L}[\sin 2t]$$

The right-hand side is therefore the product of two Laplace transforms. We can now use the inverse Laplace transform and our result from part (a) so solve for $y(t)$ in terms of $f(t)$ as

$$y = \frac{1}{2}\mathcal{L}^{-1}[F\mathcal{L}[\sin 2t]] = \frac{1}{2}\int_0^t f(t-\tau)\sin(2\tau)d\tau$$

2. **Problem 2:** Solve the following Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

in the upper half plane $x \in (-\infty, \infty)$ $y \in [0, \infty)$, subject to the boundary conditions $\phi \to 0$ as $y \to \infty$; $\phi \to 0$ as $x \to \pm\infty$;

$$\phi(x, 0) = \frac{x}{x^2 + a^2}$$

Solution: We assume ϕ is integrable and Fourier transform to write

$$\Phi(\lambda, y) = \int_{-\infty}^{\infty} e^{i\lambda x}\phi(x, y)dx$$

Using this and the derivative properties of the Fourier transform, our equation is transformed to

$$-\lambda^2\Phi + \Phi_{yy} = 0 \Rightarrow \Phi_{yy} = \lambda^2\Phi$$

This is a second order linear differential equation which we can solve using standard techniques to find the general solution

$$\Phi(\lambda, y) = A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y}$$

In order to satisfy our boundary condition that $\phi \to 0$ for $y \to \infty$, we require that $A(\lambda) = 0$ for $\lambda > 0$, and $B(\lambda) = 0$ for $\lambda < 0$. Hence, we can rewrite this general equation as

$$\Phi(\lambda, y) = C(\lambda)e^{-|\lambda| y}$$

To solve for $C(\lambda) = \Phi(\lambda, 0)$ we use our other boundary condition for $\phi(x, 0)$. We have

$$C(\lambda) = \mathcal{F}[\phi(x,0)] = \int_{-\infty}^{\infty} e^{i\lambda x} \frac{x}{x^2 + a^2} dx$$

To evaluate this integral, we start by making the change of variables $\xi = \lambda x$, so that our integral becomes

$$\int_{-\infty}^{\infty} e^{i\xi} \frac{\xi}{\xi^2 + a^2\lambda^2} d\xi = \int_{-\infty}^{\infty} e^{i\xi} \frac{\xi}{(\xi + ia\lambda)(\xi - ia\lambda)} d\xi$$

Since the non-exponential term in our integral goes to zero as $|\xi| \to \infty$, we may proceed by Jordan's lemma and evaluate this integral using a semicircular contour in the upper half of the complex plane. This integral has two poles at $\xi = \pm ia\lambda$. Hence we must find the residue at $\xi = i|a||\lambda|$. Taking care to account for the sign of λ, we calculate this residue to be

$$\text{Res}(\xi = i|a||\lambda|) = \frac{1}{2} \operatorname{sgn}(\lambda) e^{-|\lambda||a|}$$

Therefore, by the residue theorem, we have

$$C(\lambda) = i\pi \operatorname{sgn}(\lambda) e^{-|\lambda||a|}$$

Plugging this back into our expression above for Φ, we have

$$\Phi(\lambda, y) = i\pi \operatorname{sgn}(\lambda) e^{-|\lambda|(y+|a|)}$$

Now we need only take the inverse Fourier transform to recover our full solution.

$$\phi(x,y) = \frac{i}{2} \int_{-\infty}^{\infty} \operatorname{sgn}(\lambda) e^{-i\lambda x} e^{-|\lambda|(y+|a|)} d\lambda$$

$$\Rightarrow \phi(x,y) = \frac{i}{2} \left[\int_0^{\infty} e^{-i\lambda x} e^{-\lambda(y+|a|)} d\lambda - \int_{-\infty}^{0} e^{-i\lambda x} e^{\lambda(y+|a|)} d\lambda \right]$$

For the second integral, for the negative values of λ, we can flip the integration bounds (and pick up a minus sign) and make the substitution $\lambda \to -\lambda$ (resulting in another minus sign which cancels out the first). With this our expression becomes

$$\phi(x,y) = \frac{i}{2} \int_0^{\infty} \left[e^{-i\lambda x} e^{-\lambda(y+|a|)} - e^{i\lambda} e^{-\lambda(y+|a|)} \right] d\lambda = \int_0^{\infty} e^{-\lambda(y+|a|)} \sin(\lambda x) d\lambda$$

If we let $s = y + |a|$, we see that this expression is the Laplace transform of $\sin(\lambda x)$. Using the the well known result for $\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$, we can therefore write down the solution of this integral as

$$\phi(x,y) = \frac{x}{x^2 + (y + |a|)^2}$$

3. **Problem 3:** Use the Fourier transform to solve the following wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $x \in (-\infty, \infty)$ and $t \in [0, \infty)$, subject to the initial condition $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = \delta(x)$ and the boundary conditions $u(x, t) \to 0$ as $x \to \pm\infty$.

Solution: We begin by Fourier transforming both sides of our equation with respect to x. Letting $\mathcal{F}[u(x, t)] = \mathcal{U}(\lambda, t)$, our equation becomes

$$\mathcal{U}_{tt} = -\lambda c^2 \mathcal{U}$$

This is a linear second order differential equation which we can easily solve to find the general solution

$$\mathcal{U}(\lambda, t) = A(\lambda) e^{ic\lambda t} + B(\lambda) e^{-ic\lambda t}$$

From our first boundary condition $u(x, 0) = 0$, we have

$$\mathcal{F}^{-1}[\mathcal{U}(\lambda, 0)] = \mathcal{F}^{-1}[A + B] = 0$$

Hence, we conclude that $A = -B$, we we can write our general solution as

$$\mathcal{U}(\lambda, t) = A(\lambda) \sin(c\lambda t)$$

Next, we use our second boundary condition $u_t(x, 0) = \delta(x)$. This gives us

$$\mathcal{U}_t(\lambda, 0) = \int_{-\infty}^{\infty} e^{i\lambda x} \delta(x) dx = 1$$

Comparing this with our general expression for $\mathcal{U}$, we see that

$$\mathcal{U}_t(\lambda, t) = c\lambda A \cos(c\lambda t) \Rightarrow \mathcal{U}_t(\lambda, 0) = c\lambda A = 1 \Rightarrow A = \frac{1}{c\lambda}$$

Hence, our final solution for $\mathcal{U}$ is

$$\mathcal{U}(\lambda, t) = \frac{\sin(c\lambda t)}{c\lambda}$$

All that is left to do is to take the inverse Fourier transform to recover our solution $u(x, t)$.

$$u(x, t) = \mathcal{F}^{-1}[\mathcal{U}(\lambda, t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \frac{\sin(c\lambda t)}{c\lambda} d\lambda$$

We will use the Leibniz integration rule to differentiate under the integral sign. Consider the parameterized integral

$$I(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\frac{\sin(c\lambda\tau)}{c\lambda}d\lambda$$

Clearly, $u(x,t) = I(t)$. Differentiating once with respect to τ, we find

$$I'(\tau) = \frac{1}{2\pi}\frac{d}{d\tau}\int_{-\infty}^{\infty} e^{-i\lambda x}\frac{\sin(c\lambda\tau)}{c\lambda}d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\cos(c\lambda\tau)d\lambda$$

This is the inverse Fourier transform of $\cos(c\lambda\tau)$ with respect to λ. By writing cos in complex exponential form, we can evaluate this expression in terms of delta functions.

$$\Rightarrow I'(\tau) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[e^{i\lambda(c\tau - x)} + e^{-i\lambda(c\tau + x)}\right]d\lambda = \frac{1}{2}(\delta(x - c\tau) + \delta(x + c\tau))$$

We can now integrate $I'(\tau)$ from $\tau = 0$ to $\tau = t$ to recover $u(x,t) = I(t)$

$$\Rightarrow u(x,t) = I(t) = \frac{1}{2}\int_0^t (\delta(x - c\tau) + \delta(x + ct\tau))d\tau$$

Looking at this integral, we see that it is zero when $t < \left|\frac{x}{c}\right|$, and it is $\frac{1}{2}$ when $t > \left|\frac{x}{c}\right|$. Hence, we can express $u(x,t)$ in terms of the Heaviside function $H(x) = 1$ if $x > 0$, $\frac{1}{2}$ at $x = 0$ and 0 for $x < 0$.

$$u(x,t) = \frac{1}{2}H\left(t - \left|\frac{x}{c}\right|\right)$$