AMATH 573
Solitons and nonlinear waves
Homework 6

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1. Show that $\bar{N}(x,k)$ is analytic in the open lower-half plane, Im$k < 0$, by showing $\bar{N}(x,k)$ and $\partial\bar{N}/\partial k$ are bounded there. What are the conditions you need for this to be true?

   Solution: We begin by writing the fundamental equation for $\bar{N}$ as an integral equation.

$$\begin{aligned} \bar{N}_{xx} - 2ik\bar{N}_x &= -u\bar{N} \\ \Rightarrow e^{-2ikx}\bar{N}_{xx} - 2ike^{-2ikx}\bar{N}_x &= -ue^{-2ikx}\bar{N} \\ \Rightarrow \left(e^{-2ikx}\bar{N}_x\right)_x &= -ue^{-2ikx}\bar{N} \\ \Rightarrow -e^{-2ikx}\bar{N}_x &= -\int_x^\infty u(z)e^{-2ikz}\bar{N}(z,k)dz \end{aligned}$$

$$\Rightarrow \bar{N}_x = \int_x^\infty e^{2ik(x-z)}u(z)\bar{N}(z,k)dz$$

   Note that when integrating we have assumed that $\lim_{x\to\infty}\bar{N}(x,k) = 1$ is a uniform limit, so that the derivatives of $\bar{N}$ go to zero as $|x| \to \infty$. This allowed us to eliminate the additional term in the left hand side for $z \to \infty$.

   Continuing, we integrate once more to get an integral equation for $\bar{N}$.

$$\begin{aligned} \int_x^\infty \bar{N}_y(y,k)dy &= \int_x^\infty dy \int_x^\infty dz e^{2ik(y-z)}u(z)\bar{N}(z,k) \\ \Rightarrow 1 - \bar{N}(x,k) &= \int_x^\infty \int_x^\infty e^{2ik(y-z)}u(z)\bar{N}(z,k)dzdy \\ \Rightarrow \bar{N}(x,k) &= 1 - \int_x^\infty \int_x^\infty e^{2ik(y-z)}u(z)\bar{N}(z,k)dzdy \end{aligned}$$

Since $u(z)$ and $\bar{N}(z,k)$ do not depend on y, we can swap the order of integration, pull these terms, and explicitly evaluate the y integral. Taking care to set our bounds of integration appropriately, we have

$$\bar{N}(x,k) = 1 - \int_x^\infty u(z)\bar{N}(z,k)\int_x^z e^{2ik(y-z)}dydz$$
$$\Rightarrow \bar{N}(x,k) = 1 - \int_x^\infty u(z)\bar{N}(z,k)\left[\frac{e^{2ik(y-z)}}{2ik}\right]_x^z dz$$
$$\Rightarrow \bar{N}(x,k) = 1 - \int_x^\infty u(z)\bar{N}(z,k)\frac{1-e^{2ik(x-z)}}{2ik}dz$$

To establish bounds on $\bar{N}(x,k)$, we first solve this integral equation by iteration by letting

$$\bar{N}(x,k) = 1 + \sum_{j=1}^{\infty}\bar{N}_j(x,k)$$

where

$$\bar{N}_j(x,k) = -\int_x^\infty u(z)\bar{N}_{j-1}(z,k)\frac{1-e^{2ik(x-z)}}{2ik}dz$$

is a recursion formula that defines $\bar{N}_j$ in terms of $\bar{N}_{j-1}$, with $\bar{N}_0 \equiv 1$.

We also note that, for $\Im(k) < 0$ and $x \leq 0$,

$$\left|\frac{1-e^{2ikx}}{2ik}\right| \leq \frac{1}{|k|}$$

Using this bound, we have, for $\Im(k) < 0$,

$$\left|\bar{N}_1(x,k)\right| = \left|-\int_x^\infty u(z)\bar{N}_0(z,k)\frac{1-e^{2ik(x-z)}}{2ik}dz\right|$$
$$\leq \int_x^\infty |u(z)|\left|\frac{1-e^{2ik(x-z)}}{2ik}\right|dz$$
$$\leq \frac{1}{|k|}\int_x^\infty |u(z)|dz = \frac{U(x)}{|k|}$$

where $U(x) = \int_x^\infty |u(z)|dz$, provided this integral is defined.

We now note that

$$|\bar{N}_j| \leq \frac{U^m(x)}{m!|k|^m} \implies |\bar{N}_{j+1}| \leq \frac{U^{m+1}(x)}{(m+1)!|k|^{m+1}}$$

This can be proven using our recursive definition above. We have

$$
\begin{aligned}
\bar{N}_{j+1} &= -\int_x^\infty u(z)\bar{N}_j(z,k)\frac{1-e^{2ik(y-z)}}{2ik}dz \\
\Rightarrow |\bar{N}_{j+1}| &\le \frac{1}{m!|k|^{m+1}}\int_x^\infty |u(z)|U^m(z)dz = \frac{U^{m+1}(x)}{(m+1)!|k|^{m+1}}
\end{aligned}
$$

Since this bound applies for $\bar{N}_1$, it follows by induction that

$$
|\bar{N}_j(x,k)| \le \frac{U^j(x)}{j!|k|^j}
$$

for $j = 1, 2, \ldots$. With these results, we can determine an upper bound for $\bar{N}(x,k)$ in the lower half of the complex plane. We have

$$
|\bar{N}(x,k)| \le \sum_{j=0}^\infty |\bar{N}_j(x,k)| \le \sum_{j=0}^\infty \frac{U^j(x)}{j!|k|^j} = e^{U(x)/|k|}
$$

Therefore, it follows that if $\int_{-\infty}^\infty |u(x)|dx < \infty$ then $|\bar{N}(x,k)|$ is bounded by $e^{U(-\infty)/|k|}$.

Having established that $|\bar{N}(x,k)|$ is bounded in the lower half plane, we turn our attention now to its derivative $\partial\bar{N}/\partial k$. Differentiating our integral equation above, we find

$$
\begin{aligned}
\frac{\partial \bar{N}}{\partial k} &= -\int_x^\infty u(z)\left[\frac{\partial \bar{N}}{\partial k}\left(\frac{1-e^{2ik(x-z)}}{2ik}\right) - \bar{N}\frac{x-z}{k}e^{2ik(x-z)} - \bar{N}\left(\frac{1-e^{2ik(x-z)}}{2ik^2}\right)\right]dz \\
&= F(x,k) - \int_x^\infty u(z)\frac{\partial \bar{N}}{\partial k}(z,k)\frac{1-e^{2ik(x-z)}}{2ik}dz
\end{aligned}
$$

Where we have defined the function

$$
F(x,k) = \int_x^\infty u(z)\bar{N}(z,k)\left[\frac{x-z}{k}e^{2ik(x-z)} + \frac{1-e^{2ik(x-z)}}{2ik^2}\right]dz
$$

We note that this equation is nearly identical to the equation we found for $\bar{N}$, except for the nonhomogeneous term $F(x,k)$. Hence, if we can find a uniform (in x) bound $F_0(k)$ for $F(x,k)$, it would follow that $\left|\frac{\partial \bar{N}}{\partial k}\right|$ is also bounded.

We now let $k = k_R - ik_I$, with $k_I > 0$, and write

$$
\begin{aligned}
|F(x,k)| &\le \int_x^\infty |u(z)||\bar{N}(z,k)|\frac{(z-x)e^{-2k_I(z-x)}}{|k|}dz + \int_x^\infty |u(z)||\bar{N}(z,k)|\frac{1-e^{-2k_I(z-x)}}{2|k|^2}dz \\
&\le \frac{1}{|k|}\int_x^\infty (z-x)|u(z)||\bar{N}(z,k)|dz + \frac{1}{|k|^2}\int_x^\infty |u(z)||\bar{N}(z,k)|dz \\
&\le \frac{e^{U(-\infty)/|k|}}{|k|}\int_x^\infty (z-x)|u(z)|dz + \frac{e^{U(-\infty)/|k|}}{|k|^2}\int_x^\infty |u(z)|dz \\
&= \frac{e^{U(-\infty)/|k|}}{|k|}V(x) + \frac{e^{U(-\infty)/|k|}}{|k|^2}U(x) \\
&\le \frac{e^{U(-\infty)/|k|}}{|k|}V(-\infty) + \frac{e^{U(-\infty)/|k|}}{|k|^2}U(-\infty)
\end{aligned}
$$

where $V(x) = \int_x^\infty (z-x)|u(z)|dz$.
It follows that $\partial\bar{N}(x,k)/\partial k$ is bounded in the lower half plane provided that

$$
\int_{-\infty}^\infty |u(z)| < \infty \qquad \text{and} \qquad \lim_{x\to-\infty}\int_x^\infty (z-x)|u(z)|dz < \infty
$$

If the above conditions are satisfied and both $\bar{N}$ and $\partial\bar{N}/\partial k$ are bounded in the lower half plane, then it follows that $\bar{N}$ must be analytic in that region.

2. Recall the second member of the stationary KdV hierarchy from HW4:

$$
30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x + u_{xxx}) + c_0u_x = 0.
$$

Integrating once, it can be rewritten as

$$
(10u^3 + 10u_x^2 + 10uu_{xx} - 5u_x^2 + u_{xxxx}) + c_1(3u^2 + u_{xx}) + c_0u + c_{-1} = 0. \tag{1}
$$

This is an ordinary differential equation for u as a function of x. You already know that the two soliton is a solution of this. You know this equation can be written as

$$
\frac{\delta T_2}{\delta u} = 0 \iff \frac{\delta}{\delta u}\left(F_2 + c_1F_1 + c_0F_0 + c_{-1}F_{-1}\right) = 0, \tag{2}
$$

where F_k, $k = -1, 0, \ldots$ are the conserved quantities of the KdV equation. These conserved quantities are in involution,

$$
\{F_j, F_k\} = 0 \quad \Rightarrow \{T_j, F_k\} = 0,
$$

for $j, k = -1, 0, \ldots$. From HW4, you know that this implies (using $j = 2$)

$$
\frac{\delta T_2}{\delta u}\frac{d}{dx}\frac{\delta F_k}{\delta u} = \frac{dH_k}{dx}, \quad k = 0, 1. \tag{3}
$$

For some functions H_0 and H_1. Thus, H_0 and H_1 are conserved quantities of (1)[1].

(a) Find H_0 and H_1 explicitly.

Solution: We will find H_0 first. We have the 0-th conserved quantity $F_0 = \frac{1}{2}\int u^2 dx$, and we would look to find H_0 satisfying

$$\frac{\delta T_2}{\delta u}\frac{d}{dx}\frac{\delta F_0}{\delta u} = \frac{dH_0}{dx}$$

To begin, we calculate the variational derivative of F_0 as

$$\frac{\delta F_0}{\delta u} = \frac{\partial}{\partial u}\frac{1}{2}u^2 = u$$

and so

$$\frac{d}{dx}\frac{\delta F_0}{\delta u} = u_x$$

Using this and (1), we have

$$\begin{aligned}\frac{\delta T_2}{\delta u}\frac{d}{dx}\frac{\delta F_0}{\delta u} &= 10u^3u_x + 5u_x^3 + 10uu_xu_{xx} + u_xu_{4x} + 3c_1u^2u_x + c_1u_xu_{xx} + c_0uu_x + c_{-1}u_x \\ &= \left(\frac{5}{2}u^4 + 5uu_x^2 + \frac{1}{2}c_1u_x^2 + \frac{1}{2}c_0u^2 + c_{-1}u + u_xu_{3x} - \frac{1}{2}u_{xx}^2 + c_1u^3\right)_x\end{aligned}$$

where we have used integration by parts and the condition that u and its derivatives go to zero as $|x| \to \infty$ to evaluate the antiderivative of u_xu_{4x}. Therefore, by equation (3) we have

$$\left(\frac{5}{2}u^4 + 5uu_x^2 + \frac{1}{2}c_1u_x^2 + \frac{1}{2}c_0u^2 + c_{-1}u + u_xu_{3x} - \frac{1}{2}u_{xx}^2 + c_1u^3 + c_2\right)_x = H_{0,x}$$

Which we can integrate to solve for H_0:

$$H_0 = \frac{5}{2}u^4 + 5uu_x^2 + \frac{1}{2}c_1u_x^2 + \frac{1}{2}c_0u^2 + c_{-1}u + u_xu_{3x} - \frac{1}{2}u_{xx}^2 + c_1u^3 + c_2$$

For H_1 we have $F_1 = \int\left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right)dx$, the KdV Hamiltonian. We again begin by calculating the variational derivative of this conserved quantity.

$$\begin{aligned}\frac{\delta F_1}{\delta u} &= \frac{\partial}{\partial u}\left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right) - \frac{\partial}{\partial x}\frac{\partial}{\partial u_x}\left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right) \\ &= -\frac{1}{2}u^2 - u_{xx}\end{aligned}$$

$$\Rightarrow \frac{d}{dx}\frac{\delta F_1}{\delta u} = -uu_x - u_{xxx}$$

[1] Why did we not include $k = -1$? **Sol:** Because $\delta F_{-1}/\delta u = 1$, hence it is a Casimir and (3) is trivial, leading to $H_{-1} = C$.

And so we can calculate (3) for $k = 1$ to be

$$-H_{1,x} = 10u^4u_x + 5uu_x^3 + 10u^2u_xu_{xx} + uu_xu_{4x} + 3c_1u^3u_x + c_1uu_xu_{xx} + c_0u^2u_x + c_{-1}uu_x$$
$$+10u^3u_{xxx} + 5u_x^2u_{xxx} + 10uu_{xx}u_{xxx} + u_{xxx}u_{4x} + 3c_1u^2u_{xxx} + c_1u_{xx}u_{xxx} + c_0uu_{xxx} + c_{-1}u_{xxx}$$

$$\Rightarrow -H_{1,x} = [2u^5 + \frac{1}{2}u_{xxx}^2 + c_1(\frac{3}{4}u^4 + \frac{1}{2}u_{xx}^2) + c_0(\frac{1}{3}u^3 + \frac{1}{2}u_x^2) + c_{-1}(u_{xx} + \frac{1}{2}u^2)]_x$$
$$+ 5uu_x^3 + 10u^2u_xu_{xx} + uu_xu_{4x} + 10u^3u_{xxx} + 5u_x^2u_{xxx} + 10uu_{xx}u_{xxx}$$
$$+ c_1(uu_xu_{xx} + 3u^2u_{xxx})$$

Integrating this expression, we find

$$-H_1 = 2u^5 + \frac{1}{2}u_{xxx}^2 + c_1(\frac{3}{4}u^4 + \frac{1}{2}u_{xx}^2) + c_0(\frac{1}{3}u^3 + \frac{1}{2}u_x^2) + c_{-1}(u_{xx} + \frac{1}{2}u^2)$$
$$+ 5\int_{-\infty}^{\infty}\left(5uu_x^3 - \frac{5}{2}u_xu_{xx}^2 + \frac{1}{2}c_1u^2u_{xxx}\right)dx$$

(b) Check explicitly, by taking an x derivative, that H_0 and H_1 are conserved along solutions of (1).

Solution: From our earlier results in part (a) we found

$$H_{0,x} = 10u^3u_x + 5u_x^3 + 10uu_xu_{xx} + u_xu_{4x} + 3c_1u^2u_x + c_1u_xu_{xx} + c_0uu_x + c_{-1}u_x$$

This is simply (1) multiplied by u_x. Therefore, along any solution of (1) H_0 must be conserved.
Likewise, for H_1 we have

$$-H_{1,x} = 10u^4u_x + 5uu_x^3 + 10u^2u_xu_{xx} + uu_xu_{4x} + 3c_1u^3u_x + c_1uu_xu_{xx} + c_0u^2u_x + c_{-1}uu_x$$
$$+10u^3u_{xxx} + 5u_x^2u_{xxx} + 10uu_{xx}u_{xxx} + u_{xxx}u_{4x} + 3c_1u^2u_{xxx} + c_1u_{xx}u_{xxx} + c_0uu_{xxx} + c_{-1}u_{xxx}$$

As seen in part (a), this is simply (1) multiplied by $-uu_x - u_{xxx}$. Hence, along any solution to (1) H_1 is conserved as well.

Reinterpreting what you just did: $\delta F_0/\delta u$ and $\delta F_1/\delta u$ are integrating factors for (1): factors with which to multiply the equation so it can be integrated. In other words, (1) has two different ways in which it can be made *exact*!

Since (2) shows that (1) is a Lagrangian equation, we expect it to be Hamiltonian. Novikov & Bogoyavlenski (1974) showed this is indeed the case. In fact, the Hamiltonian system has canonical Poisson structure. Usually, one uses the Legendre transformation to go from a Lagrangian to a Hamiltonian system. Here the more general Ostrogradski transformation has to be used. In addition, using the ideas outlined above, they showed that the resulting Hamiltonian system is completely integrable, as it has enough conserved quantities. Indeed, (1) is 4-th order, thus it will be a Hamiltonian system with q_1, q_2 and p_1, p_2. We found two conserved quantities for this system, which is the required number, following the Liouville-Arnol'd theorem.

3. **The Ostrovsky equation** is used to model weakly nonlinear long waves in a rotating frame. It is given by

$$(\eta_t + \eta\eta_x + \eta_{xxx})_x = \gamma\eta,$$

with $\gamma \neq 0$. In what follows, we assume that as $|x| \to \infty$, η and its derivatives approach zero as fast as we need them to.

- Show that $\int_{-\infty}^{\infty} \eta dx = 0$. In other words, not only is $\int_{-\infty}^{\infty} \eta dx$ conserved, but its value is fixed at zero.

  Solution: Let $\phi(x) = \int_{-\infty}^{x} \eta(z)dz$. Then

$$\int_{-\infty}^{x} (\eta_t + \eta\eta_z + \eta_{zzz})_z dz = \gamma\phi(x) \Rightarrow \eta_t + \eta\eta_z + \eta_{zzz}|_{-\infty}^{x} = \gamma\phi(x)$$

  Since η and its derivatives go to zero as $|x| \to \infty$, this is

$$\eta_t + \eta\eta_x + \eta_{xxx} = \gamma\phi(x)$$

  Taking the limit as $x \to \infty$ all of the terms on the lefthand side vanish, and hence

$$\lim_{x\to\infty} \gamma\phi(x) = \gamma \int_{-\infty}^{\infty} \eta dx = 0 \Rightarrow \int_{-\infty}^{\infty} \eta dx = 0$$

  And so $\int_{-\infty}^{\infty} \eta dx$ is a conserved quantity whose value is fixed at zero.

- Using this result, show that $\int_{-\infty}^{\infty} \eta^2 dx$ is a conserved quantity. Do this by rewriting the equation in evolution form, with an indefinite integral on the right-hand side.

  Solution: We begin by taking the indefinite integral of the Ostrovsky equation to write it in the form

$$\eta_t + \eta\eta_x + \eta_{xxx} = \gamma \int \eta dx$$

  Moving all x derivative terms to the righthand side and multiplying both sides by 2η gives

$$2\eta\eta_t = 2\eta\gamma \int \eta dx - 2\eta^2\eta_x - 2\eta\eta_{xxx}$$

Letting $f = \eta^2$ be the density of the quantity $\int_{-\infty}^{\infty} \eta^2 dx$, we can write this as

$$f_t = 2\eta\gamma \int \eta dx - 2\eta^2\eta_x - 2\eta\eta_{xxx}$$

Each of the terms in the righthand side are either a derivative, or they can be integrated via integration by parts due to the boundary condition that $|x| \to \infty$. Hence we can rewrite this equation in the form

$$f_t = \left(\gamma \left[\int \eta dx\right]^2 - \frac{2}{3}\eta^3 - 2\eta\eta_{xx} + \eta_x^2\right)_x$$

This is a conservation law. It follows that $\int_{-\infty}^{\infty} \eta^2 dx$ is a conserved quantity.

- Use the definition of the variational derivative to verify that the Ostrovsky equation is Hamiltonian with Poisson operator ∂_x and Hamiltonian

$$H = \frac{1}{2}\int_{-\infty}^{\infty} \left(\eta_x^2 - \frac{1}{3}\eta^3 - \gamma\phi^2\right) dx,$$

where $\phi_x = \eta$.

Solution: Since our Hamiltonian in this case depends on the integral of η, rather than just the derivatives of η, we need to generalize the standard equation for the variational derivative.

We recall that the Euler-Lagrange equations were derived by considering the effect of a small variation of a path on an action functional. To that end, consider a functional $S[\eta] = \int_{-\infty}^{\infty} L(\phi, \eta, \eta_x, \ldots, \eta^{(n)})dx$. By the definition of the variational derivative, we have

$$\frac{dS}{d\epsilon}[\eta(x) + \epsilon y(x)]\bigg|_{\epsilon=0} = \lim_{\epsilon\to 0} \frac{S[\eta + \epsilon y] - S[\eta]}{\epsilon}$$

When we Taylor expand $S[\eta + \epsilon y]$ at order $\mathcal{O}(\epsilon)$ we find the expression

$$\int_{-\infty}^{\infty} \left(\frac{\partial L}{\partial \phi} Y + \frac{\partial L}{\partial \eta} y + \frac{\partial L}{\partial \eta'} y' + \cdots + \frac{\partial L}{\partial \eta^{(n)}} y^{(n)}\right) dx$$

and using our boundary conditions and integration by parts we are able to derive the generalized Euler-Lagrange equations

$$\frac{\delta S}{\delta \eta} = -\int \frac{\partial L}{\partial \phi} dx + \frac{\partial L}{\partial \eta} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \eta_x} + \cdots + (-1)^n \frac{\partial^n}{\partial x^n}\frac{\partial L}{\partial \eta^{(n)}}$$

Using this, we can calculate the variational derivative $\delta H/\delta\eta$ as follows

$$\begin{aligned}
\frac{\delta H}{\delta \eta} &= -\int \frac{\partial \mathcal{H}}{\partial \phi} dx + \frac{\partial \mathcal{H}}{\partial \eta} - \frac{\partial}{\partial x}\frac{\partial \mathcal{H}}{\partial \eta_x} \\
&= \int \gamma \phi dx - \frac{1}{2}\eta^2 - \eta_{xx}
\end{aligned}$$

And hence the dynamics of η are given by

$$\Rightarrow \eta_t = \{\eta, H\} = \frac{\partial}{\partial x}\frac{\delta H}{\delta \eta} = \frac{\partial}{\partial x}\left[\int \gamma \phi dx - \frac{1}{2}\eta^2 - \eta_{xx}\right]$$

$$\Rightarrow \eta_t = \gamma \phi - \eta \eta_x - \eta_{xxx}$$

$$\Rightarrow \eta_t + \eta \eta_x + \eta_{xxx} = \gamma \phi$$

We recognize this as the Ostrovsky equation in evolution form as derived earlier in this problem. Hence we have shown that the Ostrovsky equation is Hamiltonian with the given Poisson operator and Hamiltonian.

4. Using the Painlevé test, discuss the integrability of

$$u_t = u^p u_x + u_{xxx}.$$

Solution: We make the ansatz

$$u(x,t) = \frac{1}{(x-x_0)^q}\sum_{k=0}^{\infty} \alpha_k(t)(x-x_0)^k$$

where $q \in \mathbb{Z}$. Using this we calculate

$$\begin{aligned}
u_t &\sim \alpha_0'(x-x_0)^{-q} \\
u_x &\sim -q\alpha_0(x-x_0)^{-q-1} \\
u_{xx} &\sim q(q+1)\alpha_0(x-x_0)^{-q-2} \\
u_{xxx} &\sim -q(q+1)(q+2)\alpha_0(x-x_0)^{-q-3}
\end{aligned}$$

Plugging this into our equation and considering only the most singular terms, we have

$$\alpha_0' X^{-q} = -q\alpha_0^{p+1} X^{-q(p+1)-1} - q(q+1)(q+2)\alpha_0 X^{-q-3}$$

Where $X = x - x_0$. We have three possibilities:

(a) $\mathbf{-q(p+1)-1}$ **is dominant**: For a solution to exist we would require that

$$-q\alpha_0^{p+1} = 0$$

meaning that either $q = 0$ or $\alpha_0 = 0$. If $\alpha_0 = 0$ then the above expression for $u(x,t)$ can be reindexed and expressed as a new Laurent series with $q \to q-1$, so we can ignore this option. This leaves us with $q = 0$. However, if $q = 0$ then this term dissappears on account of the $-q$ coefficient. We conclude that there are no solutions for which $-q(p+1)-1$ is dominant.

(b) $\mathbf{-q-3}$ **is dominant**: For a solution to exist we require

$$q(q+1)(q+2) = 0$$

Which gives us the possibilities $\alpha_0 = 0$, $q = 0$, $q = -1$, and $q = -2$. We discard the $\alpha_0 = 0$ possibility for the same reason as in (a), and the remaining three options result in this term dissappearing, so we again conclude that there are no solutions for which $-q-3$ is dominant.

(c) $\mathbf{-q(p+1)-1}$ **and** $\mathbf{-q-3}$ **are equally dominant**: For this to be true means that

$$-q(p+1) - 1 = -q - 3 \Rightarrow pq = 2$$

If p and q are both integers, this gives us four possible options $p = \pm 1$ and $q = \pm 2$ or $p = \pm 2$ and $q = \pm 1$. We can eliminate the negative options, since if $q = -1$ or $q = -2$ then the X^{-q-3} term disappears, and the two terms cannot be balanced.

Hence, we conclude via the Painlevé test that the only values of p which make this equation integrable are $p = 1$ (KdV) and $p = 2$ (mKdV).