AMATH 573
Solitons and nonlinear waves
Homework 5
Lucas Cassin Cruz Burke

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1. Show that

$$X = \begin{pmatrix} -i\zeta & q \\ \pm q^* & i\zeta \end{pmatrix}, T = \begin{pmatrix} -i\zeta^2 \mp \frac{i}{2}|q|^2 & q\zeta + \frac{i}{2}q_x \\ \pm q^*\zeta \mp \frac{i}{2}q_x^* & i\zeta^2 \pm \frac{i}{2}|q|^2 \end{pmatrix}$$

are Lax Pairs for the Nonlinear Schrödinger equations

$$iq_t = -\frac{1}{2}q_{xx} \pm |q|^2 q.$$

Here the top (bottom) signs of one matrix correspond to the top (bottom) signs of the other. In other words, show that the X, T with the top (bottom) sign are a Lax pair for the Nonlinear Schrödinger equation with the top (bottom) sign.

Solution: To show that (X, T) as defined above are Lax pairs of the Nonlinear Schrödinger equation, we must show that the compatibility condition

$$T_x - X_t = [X, T] = XT - TX$$

implies the given NLSE. We begin this endeavor by calculating the partial derivatives T_x and X_t.

We have

$$X_t = \begin{pmatrix} 0 & q_t \\ \pm q_t^* & 0 \end{pmatrix}$$

$$T_x = \begin{pmatrix} \mp\frac{i}{2}(|q|^2)_x & q_x\zeta + \frac{i}{2}q_{xx} \\ \pm q_x^*\zeta \pm \frac{i}{2}q_{xx}^* & \pm\frac{i}{2}(|q|^2)_x \end{pmatrix}$$

And so

$$T_x - X_t = \begin{pmatrix} \mp\frac{i}{2}(|q|^2)_x & q_x\zeta + \frac{i}{2}q_{xx} - q_t \\ \pm q_x^*\zeta \mp \frac{i}{2}q_{xx}^* \mp q_t^* & \pm\frac{i}{2}(|q|^2)_x \end{pmatrix}$$

For the commutator $[X,T] = XT - TX$ we have

$$XT = \begin{pmatrix} -i\zeta(-i\zeta^2 \mp \frac{i}{2}q^*q) + q(\pm q^*\zeta \mp \frac{i}{2}q_x^*) & -i\zeta(q\zeta + \frac{i}{2}q_x) + q(i\zeta^2 \pm \frac{i}{2}q^*q) \\ \pm q^*(-i\zeta^2 \mp \frac{i}{2}q^*q) + i\zeta(\pm q^*\zeta \mp \frac{i}{2}q_x^*) & \pm q^*(q\zeta + \frac{i}{2}q_x) + i\zeta(i\zeta^2 \pm \frac{i}{2}q^*q) \end{pmatrix}$$

$$= \begin{pmatrix} -\zeta^3 \pm \frac{1}{2}\zeta|q|^2 \mp \frac{i}{2}q_x^*q & \frac{1}{2}\zeta q_x \pm \frac{i}{2}|q|^2q \\ -\frac{i}{2}|q|^2q^* \pm \frac{1}{2}\zeta q_x^* & \pm\frac{1}{2}|q|^2\zeta \pm \frac{i}{2}q^*q_x - \zeta^3 \end{pmatrix}$$

$$TX = \begin{pmatrix} -i\zeta(-i\zeta^2 \mp \frac{i}{2}|q|^2) \pm q^*(q\zeta + \frac{i}{2}q_x) & q(-i\zeta^2 \mp \frac{i}{2}|q|^2) + i\zeta(q\zeta + \frac{i}{2}q_x) \\ -i\zeta(\pm q^*\zeta \mp \frac{i}{2}q_x^*) \pm q^*(i\zeta^2 \pm \frac{i}{2}|q|^2) & q(\pm q^*\zeta \mp \frac{i}{2}q_x^*) + i\zeta(i\zeta^2 \pm \frac{i}{2}|q|^2) \end{pmatrix}$$

$$= \begin{pmatrix} -\zeta^3 \pm \frac{1}{2}\zeta|q|^2 \pm \frac{i}{2}q^*q_x & \mp\frac{i}{2}|q|^2q - \frac{1}{2}\zeta q_x \\ \mp\frac{1}{2}q_x^*\zeta + \frac{i}{2}|q|^2q^* & \pm\frac{1}{2}|q|^2\zeta \mp \frac{i}{2}q_x^*q - \zeta^3 \end{pmatrix}$$

Taking their difference, we find the commutator to be

$$[X,T] = XT - TX = \begin{pmatrix} \mp\frac{i}{2}(|q|^2)_x & \zeta q_x \pm i|q|^2q \\ -i|q|^2q^* \pm q_x^*\zeta & \pm\frac{i}{2}(|q|^2)_x \end{pmatrix}$$

Therefore, our continuity condition for X and T becomes

$$\begin{pmatrix} \mp\frac{i}{2}(|q|^2)_x & q_x\zeta + \frac{i}{2}q_{xx} - q_t \\ \pm q_x^*\zeta \mp \frac{i}{2}q_{xx}^* \mp q_t^* & \pm\frac{i}{2}(|q|^2)_x \end{pmatrix} = \begin{pmatrix} \mp\frac{i}{2}(|q|^2)_x & \zeta q_x \pm i|q|^2q \\ -i|q|^2q^* \pm q_x^*\zeta & \pm\frac{i}{2}(|q|^2)_x \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & \frac{i}{2}q_{xx} \mp i|q|^2q \\ \mp\frac{i}{2}q_{xx}^* + i|q|^2q^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & q_t \\ \pm q_t^* & 0 \end{pmatrix}$$

This matrix equation gives us four scalar equations, one for each matrix component. The entries on the main diagonal are trivially satisfied for any q, however the skew diagonal entries impose conditions which $q(x,t)$ must satisfy for the relation to hold.

The top right equation, when multiplied by i, is equivalent to the Nonlinear Schrödinger equations

$$iq_t = -\frac{1}{2}q_{xx} \pm |q|^2q$$

and the bottom left equation, multiplied by $-i$ is equivalent to the the complex conjugate of the NLSEs

$$-iq_t^* = -\frac{1}{2}q_{xx}^* \pm |q|^2q^*$$

Since the Nonlinear Schrödinger equation is the compatibility condition of X and T, and therefore (X,T) is indeed a Lax pair of the NLSE.

2. Let $\psi_n = \psi_n(t)$, $n \in \mathbb{Z}$. Consider the difference equation

$$\psi_{n+1} = X_n \psi_n,$$

and the differential equation

$$\frac{\partial \psi_n}{\partial t} = T_n \psi_n.$$

What is the compatibility condition of these two equations? Using this result, show that

$$X_n = \begin{pmatrix} z & q_n \\ q_n^* & 1/z \end{pmatrix}, T_n = \begin{pmatrix} iq_n q_{n-1}^* - \frac{i}{2}\left(1/z - z\right)^2 & \frac{i}{z} q_{n-1} - izq_n \\ -izq_{n-1}^* + \frac{i}{z} q_n^* & -iq_n^* q_{n-1} + \frac{i}{2}\left(1/z - z\right)^2 \end{pmatrix}$$

is a Lax Pair for the semi-discrete equation

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - |q_n|^2 (q_{n+1} + q_{n-1})$$

Note that this is a discretization of the NLS equation. It is known as the Ablowitz-Ladik lattice. It is an integrable discretization of NLS. For numerical purposes, it is far superior in many ways to the "standard" discretization of NLS:

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - 2|q_n|^2 q_n.$$

Solution: Differentiating the difference equation with respect to t, we find

$$\dot{\psi}_{n+1} = (X_n \psi_n)_t = \dot{X}_n \psi_n + X_n \dot{\psi}_n = (\dot{X}_n + X_n T_n)\psi_n$$

But by the second equation

$$\dot{\psi}_{n+1} = T_{n+1}\psi_{n+1} = T_{n+1} X_n \psi_n$$

From this follows the compatibility condition

$$T_{n+1} X_n = \dot{X}_n + X_n T_n \Rightarrow T_{n+1} X_n - X_n T_n = \dot{X}_n$$

We can verify explicitly that the X_n and T_n given are a Lax pair for the Ablowitz-Ladik lattice.

$$T_{n+1} X_n = \begin{pmatrix} -\frac{i}{2} z(1/z - z)^2 + \frac{i}{z}|q_n|^2 & iq_{n+1}|q_n|^2 - \frac{i}{2} q_n (1/z - z)^2 + \frac{i}{z^2} q_n - iq_{n-} \\ -iz^2 q_n^* + iq_{n+1}^* - iq_{n+1}^* |q_n|^2 + \frac{i}{2} q_n^* (1/z - z)^2 & -iz|q_n|^2 + \frac{i}{2z}(1/z - z)^2 \end{pmatrix}$$

$$X_n T_n = \begin{pmatrix} -\frac{i}{2} z(1/z - z)^2 + \frac{i}{z}|q_n|^2 & iq_{n-1} - iz^2 q_n - i|q_n|^2 q_{n-1} + \frac{i}{2} q_n (1/z - z)^2 \\ i|q_n|^2 q_{n-1}^* - \frac{i}{2} q_n^* (1/z - z)^2 - iq_{n-1}^* + \frac{i}{z^2} q_n^* & -iz|q_n|^2 + \frac{i}{2z}(1/z - z)^2 \end{pmatrix}$$

And so we find

$$T_{n+1}X_n - X_nT_n = i\begin{pmatrix} 0 & (|q_n|^2-1)(q_{n-1}+q_{n+1})+2q_n \\ (1-|q_n|^2)(q_{n-1}^*+q_{n+1}^*)-2q_n^* & 0 \end{pmatrix}$$

Hence our compatibility condition becomes

$$T_{n+1}X_n - X_nT_n = \dot{X}_n$$

$$\Rightarrow i\begin{pmatrix} 0 & (|q_n|^2-1)(q_{n-1}+q_{n+1})+2q_n \\ (1-|q_n|^2)(q_{n-1}^*+q_{n+1}^*)-2q_n^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dot{q}_n \\ \dot{q}_n^* & 0 \end{pmatrix}$$

We are left with two equations which $q(x,t)$ must satisfy in order for the flows of X and T to commute. The upper right hand equation, when multiplied by $-i$, is equivalent to

$$-i\dot{q}_n = q_{n-1} - 2q_n + q_{n+1} - |q_n|^2(q_{n+1}+q_{n-1})$$

Which is the Ablowitz-Ladik equation given above. Additionally, the bottom left hand equation, multiplied by i, gives us

$$i\dot{q}_n^* = q_{n-1}^* - 2q_n^* + q_{n+1}^* - |q_n|^2(q_{n+1}^*+q_{n-1}^*)$$

which is the complex conjugate of the Ablowitz-Ladik equation.

Since the Ablowitz-Ladik equation is the compatibility condition for X_n and T_n, it follows that (X_n, T_n) is a Lax pair of the equation.

3. For the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ with initial condition $u(x,0) = 0$ for $x \in (-\infty, -L) \cup (L, \infty)$, and $u(x,0) = d$ for $x \in (-L, L)$, with L and d both positive, consider the forward scattering problem.
   - Find $a(k)$, for all time t.
   - Knowing that the number of solitons emanating from the initial condition is the number of zeros of $a(k)$ on the positive imaginary axis (*i.e.*, $k = i\kappa$, with $\kappa > 0$), discuss how many solitons correspond to the given initial condition, depending on the value of $2L^2d$. You might want to use Maple, Mathematica or Matlab for this.
   - What happens for $d < 0$?
   - In the limit $L \to 0$, but $2dL = \alpha$, $u(x,0) \to \alpha\delta(x)$. What happens to $a(k)$ when you take this limit? Discuss.

Solution: We have $u(x,0) = d[H(-L) - H(L)]$, so forward scattering requires constructing the scattering data for $\psi_{xx} + [d(H(-L) - H(L)) + k^2]\psi = 0$. Since we are interested in finding the expansion coefficient $a(k)$, we only need to find ϕ and φ in some region, since $a(k) = \frac{W(\phi,\varphi)}{2ik}$. Furthermore, since $u(x,0)$ is invariant under the transformation $x \to -x$, it follows that the eigenfunctions ϕ and φ are equivalent up to a sign change in x.

$$\varphi(x) = \phi(-x)$$

Hence, once we have found ϕ it is straightforward to calculate $a(k)$.

For $x < -L$ we have $\psi_{xx} + k^2\psi = 0$, and so we have, for all $x < -L$, $\phi = e^{-ikx}$. Meanwhile, in the region $x \in (-L, L)$ we have $\psi_{xx} + (d + k^2)\psi = 0$. Here the general solution is

$$\psi(x) = Ae^{i\sqrt{d+k^2}x} + Be^{-i\sqrt{d+k^2}x} = Ae^{i\xi x} + Be^{-\xi x}$$

Where $\xi = \sqrt{d + k^2}$. Hence, in the region $x \in (-L, L)$ we have

$$\phi(x) = c_1e^{i\xi x} + c_2e^{-\xi x}$$

We require that our solutions are continuous at $-L$, so that

$$e^{ikL} = c_1e^{-i\xi L} + c_2e^{i\xi L} \Rightarrow c_2 = e^{i(k-\xi)L} - c_1e^{-2i\xi L}$$

To solve for c_1, we integrate the equation over the small region $[-L-\epsilon, -L+\epsilon]$, where $\epsilon \to 0$. Doing so gives us

$$\int_{-L-\epsilon}^{-L+\epsilon} \psi_{xx}dx + d\int_{-L-\epsilon}^{-L+\epsilon} H(-L)\psi dx + k^2\int_{-L-\epsilon}^{-L+\epsilon} \psi dx = 0$$

The third term must go to zero as $\epsilon \to 0$ by the continuity of ψ. The second term must likewise go to zero if ψ is bounded. This leaves us with

$$\int_{-L-\epsilon}^{-L+\epsilon} \psi_{xx}dx = \psi_x|_{-L-\epsilon}^{-L+\epsilon} = \psi_x(-L+\epsilon) - \psi_x(-L-\epsilon) = 0$$

So the first derivative of ϕ is also continuous. Applying this to ϕ at $-L$, we find

$$\begin{aligned}
-ke^{ikL} &= i\xi c_1e^{-i\xi L} - i\xi c_2e^{i\xi L}\\
&= i\xi c_1e^{-i\xi L} - i\xi\left(e^{i(k-\xi)L} - c_1e^{-2i\xi L}\right)e^{i\xi L} = 2i\xi c_1e^{-i\xi L} - i\xi e^{ikL}\\
&\Rightarrow c_1 = \frac{\xi - k}{2\xi}e^{i(k+\xi)L} \qquad\qquad c_2 = \frac{\xi + k}{2\xi}e^{i(k-\xi)L}
\end{aligned}$$

Hence, in the region $(-L, L)$,

$$\phi = \frac{\xi - k}{2\xi}e^{i(k+\xi)L + i\xi x} + \frac{\xi + k}{2\xi}e^{i(k-\xi)L - i\xi x}$$

Now that we have defined ϕ in this region, by the symmetry transformation $x \to -x$ we also have φ defined in this region, as

$$\varphi = \frac{\xi - k}{2\xi} e^{i(k+\xi)L - i\xi x} + \frac{\xi + k}{2\xi} e^{i(k-\xi)L + i\xi x}$$

We can now calculate the Wronskian $W(\phi, \varphi)$. We have

$$\begin{aligned} W(\phi,\varphi) &= \left(\frac{\xi-k}{2\xi}\right)^2 e^{2i(k+\xi)L} W(e^{i\xi x}, e^{-i\xi x}) + \left(\frac{\xi+k}{2\xi}\right)^2 e^{2i(k-\xi)L} W(e^{i\xi x}, e^{i\xi x}) \\ &= -2i\xi \left(\frac{\xi-k}{2\xi}\right)^2 e^{2i(k+\xi)L} + 2i\xi \left(\frac{\xi+k}{2\xi}\right)^2 e^{2i(k-\xi)L} \\ &= \frac{e^{2ikL}}{\xi}[2ik\xi\cos(2\xi L) + (k^2+\xi^2)\sin(2\xi L)] \end{aligned}$$

And we calculate $a(k) = \frac{W(\phi,\varphi)}{2ik}$ as

$$\begin{aligned} a(k) &= \frac{e^{2ikL}}{2\xi k}[2k\xi\cos(2\xi L) - i(k^2+\xi^2)\sin(2\xi L)] \\ &= \frac{e^{2ikL}}{2\sqrt{d+k^2}k}[2k\sqrt{d+k^2}\cos(2\sqrt{d+k^2}L) - i(d+2k^2)\sin(2\sqrt{d+k^2}L)] \end{aligned}$$

To find the number of soliton solutions associated with our initial condition, we let $k = i\kappa$ and look for zeros of $a(\kappa)$ where $\kappa > 0$. Substituting in this new variable, we have

$$a(\kappa) = \frac{e^{-2\kappa L}}{2i\sqrt{d-\kappa^2}\kappa}[2i\kappa\sqrt{d-\kappa^2}\cos(2\sqrt{d-\kappa^2}L) - i(d-2\kappa^2)\sin(2\sqrt{d-\kappa^2}L)]$$

From the above, we see that $a(\kappa) = 0$ when

$$2i\kappa\sqrt{d-\kappa^2}\cos(2\sqrt{d-\kappa^2}L) = i(d-2\kappa^2)\sin(2\sqrt{d-\kappa^2}L)$$

$$\Rightarrow \frac{2\kappa\sqrt{d-\kappa^2}}{d-2\kappa^2} = \tan(2\sqrt{d-\kappa^2}L)$$

We note immediately the existence of a trivial solution when $d = \kappa^2$.

Letting $\gamma = 2L^2 d$, we can write $2\sqrt{\kappa^2 - d}L = \sqrt{2(1-\kappa^2/d)\gamma} = \theta$, so that the right hand side becomes $\tan(\sqrt{4(1-\kappa^2/d)L^2 d})$. For real κ and positive d, the real values of θ range from 0 to $\sqrt{2\gamma}$. Since the left hand side is continuous in κ, there are

$$\lfloor 2\sqrt{2\gamma}/(3\pi)\rfloor - 1$$

solutions.

If $d < 0$, then θ is imaginary, and hence the equation becomes

$$\frac{2\kappa\sqrt{|d|+\kappa^2}}{d-2\kappa^2} = \tanh(2\sqrt{|d|+\kappa^2}L)$$

In the limit as $u(x,0) \to \alpha\delta(x)$, we would expect $a(\kappa)$ to approach $(\alpha + 2\kappa)/(2\kappa)$. Looking at the expression derived above for $a(\kappa)$ and substituting $d \to \alpha/2L$, we find that the cosine term reduces to one, leaving us with

$$\lim_{u(x,0)\to\alpha\delta(x)} a(\kappa) = 1 - \frac{\sqrt{d}\sin(2\sqrt{d}L}{2k}$$

Taylor expanding the sin then gives us

$$\lim_{u(x,0)\to\alpha\delta(x)} a(\kappa) = 1 - \frac{dL}{\kappa} = 1 - \frac{\alpha}{2\kappa} = \frac{\alpha + 2\kappa}{2\kappa}$$

So in the limit as $u(x,0) \to \alpha\delta(x)$ we do indeed recover the correct $a(k)$.

A little about Bäcklund transformations. We have occasionally name-dropped Bäcklund transformations: transformations from one nonlinear equation to another, providing ways to link the solutions of the equations. For instance, the Cole-Hopf transformation is a Bäcklund transformation linking the Burgers equation to the heat equation. Similarly, the Miura transformation links the KdV and mKdV equations. Below you'll work with two more Bäcklund transformations.

5. **The Liouville equation**. Consider the horribly nonlinear[1] PDE

$$u_{xy} = e^u,$$

known as Liouville's equation. Consider the transformation

$$v_x = -u_x + \sqrt{2}e^{(u-v)/2},$$
$$v_y = u_y - \sqrt{2}e^{(u+v)/2},$$

where $u(x,y)$ satisfies Liouville's equation above.

(a) Find an equation satisfied by $v(x,y)$: $v_{xy} = \ldots$. Your right-hand side cannot have any u's. Those should all be eliminated.

Solution: Direct differentiation of the first equation with respect to y gives us

$$v_{xy} = -u_{xy} + \frac{1}{\sqrt{2}}(u_y - v_y)e^{(u-v)/2}$$

Using Liouville's equation and the above expression for v_y, this becomes

$$v_{xy} = -e^u + \frac{1}{\sqrt{2}}(\sqrt{2}e^{(u+v)/2})e^{(u-v)/2} = -e^u + e^u$$

Hence, $v(x,y)$ must satisfy

[1] Technical term.

$$v_{xy} = 0$$

(b) Write down the general solution for $v(x, y)$ from the equation you obtained.

Solution: We can find a general solution for $v(x, y)$ by direct integration.

$$\int v_{xy} dx = \int 0 dx \Rightarrow v_y = g(y)$$

$$\int v_y dy = \int g(y) dy \Rightarrow v(x, t) = F(x) + G(y)$$

Therefore, the general solution for $v(x, y)$ is

$$v(x, y) = F(x) + G(y)$$

(c) Use this solution for v in your Bäcklund transformation and solve for u, obtaining the general solution of the Liouville equation!

Solution: Using this general solution for $v(x, y)$ in the above Bäcklund transformation yields us

$$v_x = F' = -u_x + \sqrt{2} e^{(u-F-G)/2}$$
$$v_y = G' = u_y - \sqrt{2} e^{(u+F+G)/2}$$

6. **The sine-Gordon equation.** Consider the sine-Gordon equation

$$u_{xt} = \sin u,$$

also horribly nonlinear.

(a) Show that the transformation

$$v_x = u_x + 2 \sin \frac{u+v}{2},$$
$$v_t = -u_t - 2 \sin \frac{u-v}{2},$$

is an *auto-Bäcklund transformation* for the sine-Gordon equation. In other words, v satisfies the same equation as u.

Solution: Differentiating the first equation with respect to t yields

$$v_{xt} = u_{xt} + (u_t + v_t) \cos \frac{u+v}{2}$$

From the second equation we see that $u_t + v_t = -2\sin\frac{u-v}{2}$. Using this, along with the original sine-Gordon equation, allows this expression to be rewritten as

$$v_{xt} = \sin u - 2\sin\frac{u-v}{2}\cos\frac{u+v}{2}$$

We now apply the trigonometric identity $\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$ and find that

$$v_{xt} = \sin u - \sin u + \sin v$$
$$\Rightarrow v_{xt} = \sin v$$

This means that v does indeed satisfy the sine-Gordon equation, and so the transformation is in fact an auto-Bäcklund transformation.

(b) Let $u(x,t)$ be the simplest[2] solution of the sine-Gordon equation. With this $u(x,y)$ solve the auto-Bäcklund transformation for $v(x,t)$, to find a more complicated solution of the sine-Gordon equation. Congratulations! You just found the one-soliton solution of the sine-Gordon equation.

Solution: The simplest solution to the sine-Gordon equation is the trivial solution

$$u(x,t) = 0$$

Using this solution and applying the auto-Bäcklund transformation supplied above, we find

$$v_x = 2\sin\frac{v}{2}$$
$$v_t = -2\sin\frac{-v}{2} = 2\sin\frac{v}{2} = v_x$$

Since $v_x = v_t$, v must be a traveling wave solution of the form $v(x,t) = f(x+t) = f(z)$. Hence, our problem becomes solving the ODE $f' = 2\sin\frac{f}{2}$. This is a first order autonomous differential equation which can be solved using separation of variables.

$$\frac{df}{dz} = 2\sin(f/2)$$

$$\Rightarrow \frac{1}{2}\int\frac{1}{\sin(f/2)}df = \int dz$$
$$\Rightarrow \int\frac{1}{\sin(f/2)}df = 2z + C$$

To integrate the left-hand side, we let $u = \cos(f/2)$, so that the integral becomes

$$\int\frac{1}{\sin(f/2)}df = -2\int\frac{1}{1-u^2}du = -2\int\frac{1}{(1+u)(1-u)}du$$

[2] I mean it.

$$= -\int \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du = -(\log(1+u) - \log(1-u)) + C$$

$$\Rightarrow \int \frac{1}{\sin \frac{v}{2}} dv = \log \left(\frac{1-\cos \frac{v}{2}}{1+\cos \frac{v}{2}} \right) = \log(\tan^2(f/2))$$

Hence, we have

$$\log(\tan^2(f/2)) = 2z + C \Rightarrow f = 2\arctan(Ae^z)$$

Plugging back in our original variables, we find the one soliton solution

$$v(x,t) = 2\arctan(Ae^{x+t})$$