AMATH 573, Problem Set 3

Lucas Cassin Cruz Burke

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1. **The KdV equation for ion-acoustic waves in plasmas.** Ion-acoustic waves are low-frequency electrostatic waves in a plasma consisting of electrons and ions. We consider the case with a single ion species.

   Consider the following system of one-dimensional equations

$$
\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) &= 0 \\
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} &= -\frac{e}{m}\frac{\partial \phi}{\partial z} \\
\frac{\partial^2 \phi}{\partial z^2} &= \frac{e}{\varepsilon_0}\left[N_0 \exp\left(\frac{e\phi}{\kappa T_e}\right) - n\right]
\end{aligned}
$$

   Here n denotes the ion density, v is the ion velocity, e is the electron charge, m is the mass of an ion, ϕ is the electrostatic potential, ε_0 is the vacuum permittivity, N_0 is the equilibrium density of the ions, κ is Boltzmann's constant, and T_e is the electron temperature.

   (a) Verify that $c_s = \sqrt{\frac{\kappa T_e}{m}}$, $\lambda_{De} = \sqrt{\frac{\varepsilon_0 \kappa T_e}{N_0 e^2}}$, and $\omega_{pi} = \sqrt{\frac{N_0 e^2}{\varepsilon_0 m}}$ have dimensions of velocity, length and frequency, respectively. These quantities are known as the ion acoustic speed, the Debye wavelength for the electrons, and the ion plasma frequency.

   Solution: The relevant constants used in the definitions of c_s, λ_{De}, and ω_{pi} have the units:

   i. κ: (Length)^2(Mass)(Time)$^{-2}$(Temperature)$^{-1}$
   ii. T_e: (Temperature)
   iii. m: (Mass)
   iv. ε_0: (Charge)^2(Time)^2(Mass)$^{-1}$(Length)$^{-3}$
   v. N_0: (Length)$^{-3}$
   vi. e: (Charge)

   Therefore, the units of c_s are $[(\text{Length})^2(\text{Time})^{-2})]^{1/2} = (\text{Length})(\text{Time})^{-1}$, which is speed; the units of λ_{De} are $[(\text{Length})^2]^{1/2} = (\text{Length})$; and the units of ω_{pi} are $[(\text{Time})^{-2}]^{1/2} = (\text{Time})^{-1}$, which is frequency.

(b) Nondimensionalize the above system, using

$$n = N_0 n^*, \quad v = c_s v^*, \quad z = \lambda_{De} z^*, \quad t = \frac{t^*}{\omega_{pi}}, \quad \phi = \frac{\kappa T_e}{e}\phi^*.$$

Solution: Making the given substitutions transforms our system of equations into

$$\begin{aligned}
N_0\omega_{pi}\frac{\partial n^*}{\partial t^*} + \frac{N_0 c_s}{\lambda_{De}}\frac{\partial}{\partial z^*}(n^* v^*) &= 0 \\
c_s\omega_{pi}\frac{\partial v^*}{\partial t^*} + \frac{c_s^2}{\lambda_{De}} v^* \frac{\partial v^*}{\partial z^*} &= -\frac{e}{m}\frac{\kappa T_e}{e\lambda_{De}}\frac{\partial \phi^*}{\partial z^*} \\
\frac{\kappa T_e}{e^2_{De}}\frac{\partial^2 \phi^*}{\partial z^{*2}} &= \frac{e}{\varepsilon_0}\left[N_0 \exp\left(\frac{e}{\kappa T_e}\frac{\kappa T_e}{e}\phi^*\right) - N_0 n^*\right]
\end{aligned}$$

Canceling terms and dividing out common factors gives us

$$\begin{aligned}
\omega_{pi}\frac{\partial n^*}{\partial t^*} + \frac{c_s}{\lambda_{De}}\frac{\partial}{\partial z^*}(n^* v^*) &= 0 \\
c_s\omega_{pi}\frac{\partial v^*}{\partial t^*} + \frac{c_s^2}{\lambda_{De}} v^* \frac{\partial v^*}{\partial z^*} &= -\frac{\kappa T_e}{m\lambda_{De}}\frac{\partial \phi^*}{\partial z^*} \\
\frac{\kappa T_e}{e\lambda_{De}^2}\frac{\partial^2 \phi^*}{\partial z^{*2}} &= \frac{N_0 e}{\varepsilon_0}\left[e^{\phi^*} - n^*\right]
\end{aligned}$$

From the definitions of c_s, λ_{De}, and ω_{pi} given in (a) we see that $\frac{c_s}{\lambda De} = \omega_{pi}$, that $c_s\omega_{pi} = \frac{c_s^2}{\lambda_{De}} = \frac{\kappa T_e}{m\lambda_{De}}$, and that $\frac{\kappa T_e}{e\lambda_{De}^2} = \frac{N_0 e}{\varepsilon_0}$. Dividing each of our equations by each of these respectively gives us the nondimensionalized system of equations

$$\begin{aligned}
\frac{\partial n^*}{\partial t^*} + \frac{\partial}{\partial z^*}(n^* v^*) &= 0 \\
\frac{\partial v^*}{\partial t^*} + v^* \frac{\partial v^*}{\partial z^*} &= -\frac{\partial \phi^*}{\partial z^*} \\
\frac{\partial^2 \phi^*}{\partial z^{*2}} &= e^{\phi^*} - n^*
\end{aligned}$$

(c) You have obtained the system

$$\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) &= 0 \\
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} &= -\frac{\partial \phi}{\partial z} \\
\frac{\partial^2 \phi}{\partial z^2} &= e^{\phi} - n
\end{aligned}$$

for the dimensionless variables. Note that we have dropped the $*$'s, to ease the notation. Find the linear dispersion relation for this system, linearized around the trivial solution $n = 1$, $v = 0$, and $\phi = 0$.

Solution: We let $n = 1 + \varepsilon\eta$, $v = \varepsilon\nu$, and $\phi = \varepsilon\varphi$, where we assume $\varepsilon << 1$. Plugging this in to our nondimensionalized system of equations and considering only $\mathcal{O}(\varepsilon)$ terms gives us the linearized system of equations

$$\eta_t + \nu_z = 0, \quad \nu_t = -\varphi_z, \quad \varphi_{zz} = \varphi - \eta$$

We look for plane wave solutions $\eta(z,t) = A_1 e^{ik_1 z - i\omega_1 t}$, $\nu(z,t) = A_2 e^{ik_2 z - i\omega_2 t}$, $\varphi(z,t) = A_3 e^{ik_3 z - i\omega_3 t}$. Plugging these into the first of our linearized equations gives us

$$-i\omega_1\eta + ik_2\nu = 0 \Rightarrow \nu = \frac{\omega_1}{k_2}\eta$$

Since ν is a scalar multiple of η, it follows that $k_1 = k_2 = k$ and $\omega_1 = \omega_2 = \omega$, so

$$\nu = \frac{\omega}{k}\eta = c_p\eta$$

where $c_p = \frac{\omega}{k}$. The second linearized equation gives us

$$-i\omega\nu = -ik_3\varphi \Rightarrow \varphi = \frac{\omega}{k_3}\nu$$

again we see that φ is a scalar multiple of ν, and therefore $k_3 = k$ and $\omega_3 = \omega$, so that

$$\varphi = \frac{\omega}{k}\nu = c_p\nu$$

The third linearized equation gives us

$$-k^2\varphi = \varphi - \eta$$

Combining this with the results from the first two linearized equation gives us the dispersion relation

$$-k^2 - 1 + \frac{k^2}{\omega^2} = 0 \Rightarrow \omega^2 = \frac{k^2}{k^2+1}$$

(d) Rewrite the system using the "stretched variables"

$$\xi = \epsilon^{1/2}(z - t), \quad \tau = \epsilon^{3/2}t$$

Given that we are looking for low-frequency waves, explain how these variables are inspired by the dispersion relation.

Solution: Using the chain rule, we find that $\partial_t = -\epsilon^{1/2}\partial_\xi + \epsilon^{3/2}\partial_\tau$ and $\partial_z = \epsilon^{1/2}\partial_\xi$. Plugging our stretched variables into our linearized system of equations therefore gives us

i. $n_t + (nv)_z = 0 \Rightarrow -\epsilon^{1/2}n_\xi + \epsilon^{3/2}n_\tau + \epsilon^{1/2}(nv)_\xi = 0 \Rightarrow -n_\xi + (nv)_\xi + \epsilon n_\tau = 0$

ii. $v_t + vv_z = -\phi_z \Rightarrow -\epsilon^{1/2}v_\xi + \epsilon^{3/2}v_\tau + \epsilon^{1/2}vv_\xi = -\epsilon^{1/2}\phi_\xi \Rightarrow -v_\xi + vv_\xi + \epsilon v_\tau = -\phi_\xi$

iii. $\phi_{zz} = e^\phi - n \Rightarrow \epsilon\phi_{\xi\xi} = e^\phi - n$

So we can rewrite our system in terms of ξ and τ as

$$n_\xi - (nv)_\xi = \epsilon n_\tau \tag{1}$$

$$(1-v)v_\xi - \phi_\xi = \epsilon v_\tau \tag{2}$$

$$e^\phi - n = \epsilon\phi_{\xi\xi} \tag{3}$$

We see that for the $\mathcal{O}(1)$ terms only ξ derivatives show up.

We are looking for low frequency waves, which means $\omega^2 << 1$, and from the dispersion relation we see that this happens only when $k^2 << 1$. This suggests that we should Taylor expand the dispersion relation and consider small values of $k = \sqrt{\epsilon} << 1$. Doing so gives us $\omega = \frac{k}{\sqrt{k^2+1}} = \sqrt{\frac{\epsilon}{\epsilon+1}} = \epsilon^{1/2} - \frac{1}{2}\epsilon^{3/2} + \dots$.

Our solutions are of the form $e^{i(kz-\omega(k)t)}$. If we plug in this expression for the dispersion relation, we find that the argument of the solution becomes

$$kz - \omega(k)t \Rightarrow \epsilon^{1/2}z - \epsilon^{1/2}t + \frac{1}{2}\epsilon^{3/2}t + \dots = \epsilon^{1/2}(z - t + \frac{1}{2}\epsilon t + \dots)$$

Hence our solutions become

$$e^{i\epsilon^{1/2}(z\mp t)\pm\frac{i}{2}\epsilon^{3/2}t+\dots}$$

This is a traveling wave moving to the right (left) which is being modulated on a time scale of $\epsilon^{3/2}t$ by a left (right) moving wave. Hence we recognize the stretched variables ξ and τ as the arguments of the two most dominant phenomena for low frequency waves.

(e) Expand the dependent variables as

$$n = 1 + \epsilon n_1 + \epsilon^2 n_2 + \dots$$
$$v = \epsilon v_1 + \epsilon^2 v_2 + \dots$$
$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \dots$$

Using that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$, find a governing equation which determines how ϕ_1 depends on ξ and τ.

Solution: Plugging these expanded dependent variables into the equations derived in (d) and considering only $\mathcal{O}(\epsilon^2)$ terms, we find

i. $n_\xi - (nv)_\xi = \epsilon n_\tau \Rightarrow \epsilon n_{1\xi} + \epsilon^2 n_{2\xi} + \epsilon^2 (v_1 n_1)_\xi = \epsilon^2 n_{1\tau}$
$\Rightarrow \epsilon(n_{1\xi} - v_{1\xi}) + \epsilon^2(n_{2\xi} - v_{2\xi} - (v_1 n_1)_\xi) = \epsilon^2 n_{1\tau}$

ii. $(1-v)v_\xi - \phi_\xi = \epsilon v_\tau \Rightarrow (1 - \epsilon v_1 - \epsilon^2 v_2)(\epsilon v_{1\xi} + \epsilon^2 v_{2\xi}) - \epsilon \phi_{1\xi} - \epsilon^2 \phi_{2\xi} = \epsilon^2 v_1 \tau$
$\Rightarrow \epsilon(v_{1\xi} - \phi_{1\xi}) + \epsilon^2(v_{2\xi} - v_1 v_{1\xi} - \phi_{2\xi}) = \epsilon^2 v_{1\tau}$

iii. $e^\phi - n = \epsilon \phi_{\xi\xi} \Rightarrow 1 + \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^2\phi_1^2 - 1 - \epsilon n_1 - \epsilon^2 n_2 = \epsilon^2 \phi_{1\xi\xi}$
$\Rightarrow \epsilon(\phi_1 - n_1) + \epsilon^2(\phi_2 + \phi_1^2 - n_2) = \epsilon^2 \phi_{1\xi\xi}$

When we group by order of ϵ we get six equations for $n_1, n_2, v_1, v_2, \phi_1$, and ϕ_2.

i. $n_{1\xi} = v_{1\xi}$
ii. $v_{1\xi} = \phi_{1\xi}$
iii. $\phi_1 = n_1$
iv. $n_{2\xi} - v_{2\xi} - (v_1 n_1)_\xi = n_{1\tau}$
v. $v_{2\xi} - v_1 v_{1\xi} - \phi_{2\xi} = v_{1\tau}$
vi. $\phi_2 + \phi_1^2 - n_2 = \phi_{1\xi\xi}$

Integrating (ii) with respect to ξ gives us $v_1 = \phi_1 + C$. Since $\phi_1, v_1 \to 0$ as $\xi \to \pm\infty$, we deduce that $C = 0$ and therefore $v_1 = \phi_1$. Likewise, integrating (i) and using the fact that $n_1, v_1 \to 0$ as $\xi \to \pm\infty$ we find that $n_1 = v_1 = \phi_1$. Using these results from the first three equations and applying them to equation (v) we find

$$\phi_{2\xi} = v_{2\xi} - \phi_1\phi_{1\xi} - \phi_{1\tau}$$

which we can plug in to (vi) to find

$$v_{2\xi} - \phi_1\phi_{1\xi} - \phi_{1\tau} + 2\phi_1\phi_{1\xi} - n_{2\xi} = \phi_{1\xi\xi\xi} \Rightarrow n_{2\xi} - v_{2\xi} = \phi_1\phi_{1\xi} - \phi_{1\tau} - \phi_{1\xi\xi\xi}$$

Lastly, rearranging (iv) and plugging in this expression for $n_{2\xi} - v_{2\xi}$ leads us to

$$\phi_1\phi_{1\tau} - \phi_{1\tau} - \phi_{1\xi\xi\xi} - (\phi_1^2)_\xi = \phi_{1\tau}$$

And finally, rearranging, we have

$$2\phi_{1\tau} = \phi_1\phi_{1\xi} - \phi_{1\xi\xi\xi} - (\phi_1^2)_\xi$$

2. **Obtaining the KdV equation from the NLS equation.** We have shown that the NLS equation may be used to describe the slow modulation of periodic wave trains of the KdV equation. In this problem we show that the KdV equation describes the dynamics of long-wave solutions of the NLS equation.

Consider the defocusing NLS equation

$$ia_t = -a_{xx} + |a|^2 a.$$

(a) Let

$$a(x,t) = e^{i\int V dx}\rho^{1/2}.$$

Derive a system of equations for the phase function $V(x,t)$ and for the amplitude function $\rho(x,t)$, by substituting this form of $a(x,t)$ in the NLS equation, dividing out the exponential, and separating real and imaginary parts. Write your equations in the form $\rho_t = \ldots$, and $V_t = \ldots$. Due to their similarity with the equations of hydrodynamics, this new form of the NLS equation is referred to as its *hydrodynamic form*.

Solution: We begin by calculating the relevant quantities in terms of ρ and V.

i. $a_t = \frac{\partial}{\partial t}\left(e^{i\int V dx}\rho^{1/2}\right) = \left(i\int \dot{V} dx + \frac{\dot{\rho}}{2\rho}\right) a$

ii. $a_{xx} = \frac{\partial}{\partial x} a_x = \frac{\partial}{\partial x}\left[\left(iV + \frac{\rho'}{2\rho}\right)a\right] = \left(iV + \frac{\rho'}{2\rho}\right)' a + \left(iV + \frac{\rho'}{2\rho}\right) a'$
$= \left(iV' + \left(\frac{\rho'}{2\rho}\right)'\right) a + \left(iV + \frac{\rho'}{2\rho}\right)^2 a = \left[\frac{\rho''}{2\rho} - \frac{\rho'^2}{4\rho^2} - V^2 + i\left(V' + V\frac{\rho'}{\rho}\right)\right] a$

iii. $|a|^2 a = |\rho^{1/2}|^2 a = \rho a$

Plugging these in to our equation and dividing out a gives us

$$i\left(i\int \dot{V} dx + \frac{\dot{\rho}}{2\rho}\right) = -\left[\frac{\rho''}{2\rho} - \frac{\rho'^2}{4\rho^2} - V^2 + i\left(V' + V\frac{\rho'}{\rho}\right)\right] + \rho$$

$$\Rightarrow -\int \dot{V} dx + i\frac{\dot{\rho}}{2\rho} = -\frac{\rho''}{2\rho} + \frac{\rho'^2}{4\rho^2} + V^2 - i\left(V' + V\frac{\rho'}{\rho}\right) + \rho$$

The imaginary part of this equation gives us

$$\frac{\dot{\rho}}{2\rho} = -V' - V\frac{\rho'}{\rho} \Rightarrow \dot{\rho} = -2V'\rho - 2V\rho' \Rightarrow \dot{\rho} = -2(V\rho)'$$

and the real part of the equation gives

$$-\int \dot{V} dx = -\frac{\rho''}{2\rho} + \frac{\rho'^2}{4\rho^2} + V^2 + \rho \Rightarrow \dot{V} = \left(\frac{\rho''}{2\rho} - \frac{\rho'^2}{4\rho^2} - V^2 - \rho\right)'$$

Summarizing, we have found the following expressions for V_t and ρ_t

$$V_t = \left(\frac{\rho_{xx}}{2\rho} - \frac{\rho_x^2}{4\rho^2} - V^2 - \rho\right)_x$$

$$\rho_t = -2(V\rho)_x$$

(b) Find the linear dispersion relation for the hydrodynamic form of the defocusing NLS equation, linearized around the trivial solution $V = 0$, $\rho = 1$. In other words, we are examining perturbations of the so-called Stokes wave solution of the NLS equation, which is given by a signal of constant amplitude.

Solution: We will let $V(x,t) = \epsilon v(x,t)$ and $\rho(x,t) = 1 + \epsilon p(x,t)$, where $\epsilon << 1$. Plugging these in to the two equations we found in part (a) gives us, for the first equation,

$$\epsilon \dot{v} = \left(\frac{\epsilon p''}{2(1+\epsilon p)} - \frac{\epsilon^2 p'^2}{4(1+\epsilon p)^2} - \epsilon^2 v^2 - (1+\epsilon p) \right)'$$

In the limit as $\epsilon \to 0$ this becomes

$$\epsilon \dot{v} = \left(\frac{\epsilon p''}{2} - \frac{\epsilon^2 p'^2}{4} - \epsilon^2 v^2 - (1+\epsilon p) \right)' = \frac{1}{2}\epsilon p''' - \frac{1}{2}\epsilon^2 p' p'' - 2\epsilon^2 v v' - \epsilon p'$$

Keeping only $\mathcal{O}(\epsilon)$ terms, we end up with the linearized equation

$$\dot{v} = \frac{1}{2} p''' - p'$$

For the second equation, we have

$$\epsilon \dot{p} = -2(\epsilon v(1+\epsilon p))' = -2\epsilon v' - 2\epsilon^2 (vp)'$$

Again keeping only $\mathcal{O}(\epsilon)$ terms, we are left with

$$\dot{p} = -2v'$$

We will now find the linear dispersion relation by considering traveling wave solutions $v = A_1 e^{ik_1 x - i\omega_1 t}$ and $p = A_2 e^{ik_2 x - i\omega_2 t}$. Plugging these ansatz' into the first of our linearized equations gives us

$$-i\omega_1 v = -i\frac{k_2^3}{2} p - ik_2 p \Rightarrow v = \frac{k_2}{\omega_1}\left(\frac{k_2^2}{2} + 1 \right) p$$

Since v is a scalar multiple of p, it follows that $k_1 = k_2 = k$ and $\omega_1 = \omega_2 = \omega$, so we can write

$$v = \frac{k}{\omega}\left(\frac{k^2}{2} + 1 \right) p$$

Plugging our wave solution expressions into the second linearized equation gives us

$$-i\omega p = -2ikv \Rightarrow p = 2\frac{k}{\omega} v$$

We can plug this back into the expression derived above from the first equation to find

$$v = 2\frac{k^2}{\omega^2}\left(\frac{k^2}{2}+1\right)v \Rightarrow 1 = 2\frac{k^2}{\omega^2}\left(\frac{k^2}{2}+1\right) \Rightarrow \omega^2 = k^2(k^2+2)$$

(c) Rewrite the system using the "stretched variables"

$$\xi = \epsilon(x-\beta t), \quad \tau = \epsilon^3 t$$

Given that we are looking for long waves, explain how these variables are inspired by the dispersion relation. What should the value of β be?

Solution: By the chain rule we have $\partial t = -\epsilon\beta\partial_\xi + \epsilon^3\partial_\tau$ and $\partial_x = \epsilon\partial_\xi$. We can therefore rewrite our equations as

$$V_t = \left(\frac{\rho_{xx}}{2\rho} - \frac{\rho_x^2}{4\rho^2} - V^2 - \rho\right)_x \Rightarrow -\epsilon\beta V_\xi + \epsilon^3 V_\tau = \epsilon\left(\epsilon^2\frac{\rho_{\xi\xi}}{2\rho} - \epsilon^2\frac{\rho_\xi^2}{4\rho^2} - V^2 - \rho\right)_\xi$$

$$\rho_t = -2(V\rho)_x \Rightarrow -\epsilon\beta\rho_\xi + \epsilon^3\rho_\tau = -2\epsilon(V\rho)_\xi$$

Dividing out ϵ gives us the equations

$$\begin{aligned}-\beta V_\xi + \epsilon^2 V_\tau &= \frac{\epsilon^2}{2}\left(\frac{\rho_{\xi\xi}}{\rho} - \frac{\rho_\xi^2}{2\rho^2}\right)_\xi - 2VV_\xi - \rho_\xi \\ -\beta\rho_\xi + \epsilon^2\rho_\tau &= -2(V\rho)_\xi\end{aligned}$$

Lastly, grouping by orders of ϵ, we are left with

$$\begin{aligned}2VV_\xi - \beta V_\xi + \rho_\xi &= \epsilon^2\left[\frac{1}{2}\left(\frac{\rho_{\xi\xi}}{\rho} - \frac{\rho_\xi^2}{2\rho^2}\right)_\xi - V_\tau\right] \\ 2(V\rho)_\xi - \beta\rho_\xi &= \epsilon^2\rho_\tau\end{aligned}$$

We are looking for long waves, i.e. $k << 1$. For small $k = \epsilon << 1$ we can Taylor expand the dispersion relation as $\omega = \epsilon\sqrt{\epsilon^2+2} = \sqrt{2}\epsilon + \frac{1}{2\sqrt{2}}\epsilon^3 + \mathcal{O}(k^5)$. Then our solutions will be of the form

$$e^{i(\epsilon x - \sqrt{2}\epsilon t) - \frac{i}{2\sqrt{2}}\epsilon^3 t + \ldots}$$

As before, we see that the dominant phenomena is a rightward traveling wave solution which is being modulated on a timescale of $\tau = \epsilon^3 t$ by a leftward traveling, and we see that β should be $\sqrt{2}$. Our stretched variables ξ and τ are hence appear as the dominant arguments of our solutions for long waves.

(d) Expand the dependent variables as

$$V = \epsilon^2 V_1 + \epsilon^4 V_2 + \dots$$
$$\rho = 1 + \epsilon^2 \rho_1 + \epsilon^4 \rho_2 + \dots$$

Using that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$, find a governing equation which determines how V_1 depends on ξ and τ. This equation should be equivalent to the KdV equation.

Solution: We have the two equations

$$2VV_\xi - \beta V_\xi + \rho_\xi = \frac{\epsilon^2}{2}\left(\frac{\rho_{\xi\xi}}{\rho} - \frac{\rho_\xi^2}{2\rho^2}\right)_\xi - \epsilon^2 V_\tau \tag{1}$$
$$2(V\rho)_\xi - \beta\rho_\xi = \epsilon^2 \rho_\tau \tag{2}$$

Starting with (1), we use the Taylor expansions $\frac{1}{1+x} = 1 - x + x^2 + \mathcal{O}(x^3)$ and $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 + \mathcal{O}(x^3)$ to write

$$\frac{1}{\rho} = \frac{1}{1+\epsilon^2\rho_1+\epsilon^4\rho_2} = 1 - \epsilon^2\rho_1 - \epsilon^4\rho_2 + \epsilon^4\rho_1^2 + \mathcal{O}(\epsilon^5)$$

and
$$\frac{1}{\rho^2} = \frac{1}{(1+\epsilon^2\rho_1+\epsilon^4\rho_2)^2} = 1 - 2\epsilon^2\rho_1 - 2\epsilon^4\rho_2 + 3\epsilon^4\rho_1^2 + \mathcal{O}(\epsilon^5)$$

Plugging in our expanded V and ρ variables and using these Taylor expansions for the denominators in the right-hand side then gives us, to $\mathcal{O}(\epsilon^4)$,

$$2\epsilon^4 V_1V_{1\xi} - \beta\epsilon^2 V_{1\xi} - \beta\epsilon^4 V_{2\xi} + \epsilon^2\rho_{1\xi} + \epsilon^4\rho_{2\xi} = \frac{\epsilon^4}{2}\rho_{1\xi\xi\xi} - \epsilon^4 V_{1\tau}$$

and rearranging this gives us

$$\epsilon^2[\rho_{1\xi} - \beta V_{1\xi}] = \epsilon^4\left[\frac{1}{2}\rho_{1\xi\xi\xi} - V_{1\tau} - 2V_1V_{1\xi} + \beta V_{2\xi} - \rho_{2\xi}\right]$$

Looking now at (2), we can plug in our expanded variables to get, to order $O(\epsilon^4)$,

$$2(\epsilon^2 V_1 + \epsilon^4 V_2 + \epsilon^4 V_1\rho_1)_\xi - \beta(\epsilon^2\rho_{1\xi} + \epsilon^4\rho_{2\xi}) = \epsilon^4\rho_{1\tau}$$

which we can rearrange to find

$$\epsilon^2[2v_{1\xi} - \beta\rho_{1\xi}] = \epsilon^4[\rho_{1\tau} - 2v_{2\xi} - 2(v_1\rho_1)_\xi + \beta\rho_{2\xi}]$$

And so, grouping by order of ϵ, we have four equations in terms of V_1, V_2, ρ_1, and ρ_2:

i. $\rho_{1\xi} = \beta V_{1\xi}$
ii. $2V_{1\xi} = \beta \rho_{1\xi}$
iii. $V_{1\tau} + 2V_1 V_{1\xi} - \beta V_{2\xi} = \frac{1}{2}\rho_{1\xi\xi\xi} - \rho_{2\xi}$
iv. $2V_{2\xi} + 2(V_1\rho_1)_\xi = \beta\rho_{2\xi} + \rho_{1\tau}$

Integrating (i) we find $\int \rho_{1\xi} d\xi = \beta \int V_{1\xi} d\xi \Rightarrow \rho_1 = \beta V_1 + C$. Since we know that $\rho_1, V_1 \to 0$ as $\xi \to \pm\infty$, it follows that $C = 0$, and therefore $\rho_1 = \beta V_1$. Applying this result to (iii) gives us

$$V_{1\tau} + 2V_1 V_{1\xi} - \beta V_{2\xi} = \frac{1}{2} V_{1\xi\xi\xi} - \rho_{2\xi} \Rightarrow \beta V_{2\xi} - \rho_{2\xi} = V_{1\tau} + 2V_1 V_{1\xi} - \frac{1}{2} V_{1\xi\xi\xi}$$

We now use our earlier result from part (C) that $\beta = \sqrt{2}$ and multiply both sides by β to write this in the form

$$2V_{2\xi} - \beta\rho_{2\xi} = \beta V_{1\tau} + 2\beta V_1 V_{1\xi} - \frac{\beta}{2} V_{1\xi\xi\xi}$$

Lastly, using our earlier result that $\rho_1 = V_1$, we can rearrange (iv) to

$$2V_{2\xi} - \beta\rho_{2\xi} = V_{1\tau} - 4V_1 V_{1\xi}$$

Now we use our result from (iii) to rewrite the left-hand side to

$$\beta V_{1\tau} + 2\beta V_1 V_{1\xi} - \frac{\beta}{2} V_{1\xi\xi\xi}$$

Lastly, we rearrange to get

$$(1 - \beta) V_{1\tau} = (2\beta + 4) V_1 V_{1\xi} - \frac{\beta}{2} V_{1\xi\xi\xi}$$

Which we can recognize as the KdV equation $u_\tau = Auu_\xi + Bu_{\xi\xi\xi}$, where $u = V_1$, $A = \frac{2\sqrt{2}+4}{1-\sqrt{2}}$, and $B = \frac{-1}{\sqrt{2}(1-\sqrt{2})}$.

3. Consider the previous problem, but with the focusing NLS equation

$$ia_t = -a_{xx} - |a|^2 a.$$

The method presented in the previous problem does not allow one to describe the dynamics of long-wave solutions of the focusing NLS equation using the KdV equation. How does this show up in the calculations?

Solution: If we were to perform the same substitution as done in problem (2) we would find that we would end up with a $-\rho$ term in the real part of the equation rather than a $+\rho$ term. The result of this sign change is that the dispersion relationship ends up being $\omega = \pm k\sqrt{k^2 - 2}$, as opposed to $\omega = \pm k\sqrt{k^2 + 2}$. This means that if we are interested in long-wave solutions where $k << 1$ we end up with imaginary values for ω, which causes the solutions to blow up at $|x| \to \pm\infty$.

4. The mKdV equation considered in the text is known as the *focusing* mKdV equation, because of the behavior of its soliton solutions. This behavior is similar to that of the focusing NLS equation. In this problem, we study the *defocusing* mKdV equation

$$4u_t = -6u^2 u_x + u_{xxx}.$$

You have already seen that you can scale the coefficients of this equation to your favorite values, except for the ratio of the signs of the two terms on the right-hand side.

(a) Examine, using the potential energy method and phase plane analysis, the traveling-wave solutions.

Solution: We make the traveling wave ansatz $u(x,t) = U(x - vt) = U(z)$, and plug this into the defocusing mKdV equation to get

$$-4vU' = -6U^2U' + U'''$$

We can integrate this equation, then multiply both sides by U' and integrate again, as follows

$$-4vU' = -6U^2U' + U''' \Rightarrow -4vU = -2U^3 + U'' + \alpha$$
$$\Rightarrow -4vUU' = -2U^3U' + U''U' + \alpha U' \Rightarrow -2vu^2 = -\frac{1}{2}U^4 + \frac{1}{2}U'^2 + \alpha U - \beta$$

Which we can rearrange to write

$$\frac{1}{2}U'^2 + V(U; v, \alpha) = \beta$$

where

$$V(U; v, \alpha) = 2vU^2 - \frac{1}{2}U^4 + \alpha U$$

We can interpret this equation as a conservation of energy equation of a particle of unit mass moving under the influence of a conservative force. U' is interpreted as a momentum, $V(U)$ is an energy potential, and β is the total energy of the system.

$V(U)$ has three terms:

i. $-\frac{1}{2}U^4$ is fixed by the original equation and dominates for large U. The effect of the defocusing term is that $V \to -\infty$ as $U \to \pm\infty$. It follows that there are real solutions which are unbounded.

ii. $2vU^2$ is proportional to the velocity v of the traveling wave solution and is quadratic in U, and will therefore dominate the $-U^4/4$ term for small values of U. This term allows for the possibility of periodic and solitary wave solutions.

iii. αU is proportional to the integration constant α. This is the only odd term in the potential V and it allows for non-symmetric potentials. In particular, these allow for the possibility of dynamics in which U reaches some maximal (or minimal) point, changes direction, and then approaches but never reaches some minimal (or maximal) point.

For the phase plane analysis, we consider the dynamics of

$$U'' = -\frac{\partial V}{\partial U}(U; v, \alpha) = -4vU + U^3 - \alpha$$

.

Letting $u_1 = U$ and $u_2 = U'$, we can write this as a first-order system

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -4vu_1 + u_1^3 - \alpha \end{aligned}$$

Equilibrium solutions occur when $\partial V/\partial U = 0$, and the number of these solutions is determined by the discriminant, which is

$$\Delta = 256v^3 - 27a^2$$

i. When $\Delta = 0 \Rightarrow 256v^3 = 27a^2$ we have one equilibrium point and solutions are real but unbounded.

ii. When $\Delta > 0 \Rightarrow 256v^3 > 27a^2$ we have three real equilibrium solutions, including solutions which are bounded between some U_{min} and U_{max} where $V(U_{min}) = V(U_{max}) = \beta$. If either U_{min} or U_{max} is an equilibrium point then the solution is a solitary wave which will asymptotically approach the point but take an infinite amount of time to reach it. Otherwise the solution will oscillate between U_{min} and U_{max}.

iii. When $\Delta < 0$ we have one real equilibrium point and solutions are unbounded.

(b) If you have found any homoclinic or heteroclinic connections, find the explicit form of the profiles corresponding to these connections.

Solution: A heteroclinic connection occurs when $a = 0$ and $v > 0$ between the equilibrium points at $U = \pm\sqrt{2v}$. For simplicity, we will allow $v = 1$ so that $\beta_s = 2$. Any solution which has energy $\beta_s = 2$ in the range between $U = \pm\sqrt{s}$ will therefore asymptotically approach one of these equilibrium points.

We have

$$U' = \pm\sqrt{2(\beta - V(U))} \Rightarrow U' = \pm\sqrt{(4 - 4U^2 + U^4)} = \pm(U^2 - 2)$$

which we can solve using separation of variables

$$\Rightarrow \int_{U_0}^{U} \frac{dU}{U^2 - 2} = \pm z \Rightarrow \frac{1}{\sqrt{2}} \tanh^{-1}\left(\frac{U}{\sqrt{2}}\right)\Bigg|_{U_0}^{U} = \pm z$$

$$\Rightarrow \tanh^{-1}\left(\frac{U}{\sqrt{2}}\right) = \pm\sqrt{2}z - \tanh^{-1}\left(\frac{U_0}{\sqrt{2}}\right)$$

And our final answer is

$$\Rightarrow U(z) = \pm\sqrt{2}\tanh\left(\sqrt{2}z \mp A_0\right)$$

where $A_0 = \tanh^{-1}\left(\frac{U_0}{\sqrt{2}}\right)$. Note that U(z) asymptotically approaches the equilibrium point at either $\pm\sqrt{2}$.

5. Consider the so-called Derivative NLS equation (DNLS)

$$b_t + \alpha\left(b|b|^2\right)_x - ib_{xx} = 0.$$

This equation arises in the description of quasi-parallel waves in space plasmas. Here $b(x,t)$ is a complex-valued function.

(a) Using a polar decomposition

$$b(x,t) = B(x,t)e^{i\theta(x,t)},$$

with B and θ real-valued functions, and separating real and imaginary parts (after dividing by the exponential), show that you obtain the system

$$B_t + 3\alpha B^2 B_x + \frac{1}{B}(B^2\theta_x)_x = 0,$$
$$\theta_t + \alpha B^2\theta_x + \theta_x^2 - \frac{1}{B}B_{xx} = 0.$$

Solution: We will begin by calculating the relevant partial derivatives for $b(x,t) = B(x,t)e^{i\theta(x,t)}$.

i. $b_t = B_t e^{i\theta} + i\theta_t B e^{i\theta}$
ii. $(b|b|^2)_x = \left(B^3 e^{i\theta}\right)_x = 3B_x B^2 e^{i\theta} + i\theta_x B^3 e^{i\theta}$
iii. $b_{xx} = \left[B_x e^{i\theta} + i\theta_x B e^{i\theta}\right]_x = B_{xx}e^{i\theta} + 2i\theta_x B_x e^{i\theta} + i\theta_{xx}Be^{i\theta} - \theta_x^2 Be^{i\theta}$

Plugging these expressions into the DNLS equation and multiplying by $e^{i\theta}$ gives us

$$B_t + i\theta_t B + 3\alpha B^2 B_x + i\alpha\theta_x B^3 - iB_{xx} + 2\theta_x B_x + \theta_{xx}B + i\theta_x^2 B = 0$$

The real and imaginary parts of this equation give us, respectively,

$$B_t + 3\alpha B^2 B_x + 2\theta_x B_x + \theta_{xx}B = 0$$
$$\theta_t B + \alpha\theta_x B^3 - B_{xx} + \theta_x^2 B = 0$$

Diving both of these by B, we see that

$$B_t + 3\alpha B^2 B_x + \frac{1}{B}(B^2\theta_x)_x = 0$$
$$\theta_t + \alpha\theta_x B^2 + \theta_x^2 - \frac{1}{B}B_{xx} = 0$$

(b) Assuming a traveling-wave envelope, $B(x,t) = R(z)$, with $z = x - vt$ and constant v, show that $\theta(x,t) = \Phi(z) - \Omega t$, with constant Ω, is consistent with these equations. You can show (but you don't have to) that assuming a traveling-wave amplitude results in only this possibility for $\theta(x,t)$. At this point, we have reduced the problem of finding solutions with traveling envelope to that of finding two one-variable functions $R(z)$ and $\Phi(x)$. The problem also depends on two parameters v (envelope speed) and Ω (frequency like).

Solution: We will begin by calculating the partial derivatives $B_t = -vR'$, $B_x = R'$, $\theta_t = -v\Phi' - \Omega$, and $\theta_x = \Phi'$. Plugging these in, we find the first equation gives us

$$-vR' + 3\alpha R^2 R' + 2\Phi' R' + \Phi'' R = 0$$

and the second equation gives us

$$-v\Phi' - \Omega + \alpha\Phi' R^2 + \Phi'^2 - \frac{R''}{R} = 0$$

which depends only on z.

(c) Substituting these ansatz in the first equation of the above system, show that

$$\Phi' = \frac{C + vs - 3s^2}{2s},$$

where C is a constant and $s = \alpha R^2/2$.

Solution: Plugging these ansatz into the first equation and rearranging gives us

$$\frac{1}{R}(R^2\Phi')' = vR' - 3\alpha R^2 R' \Rightarrow (R^2\Phi')' = vR'R - 3\alpha R^3 R'$$
$$\Rightarrow R^2\Phi' = \frac{1}{2}vR^2 - \frac{3\alpha}{4}R^4 + C \Rightarrow \Phi' = \frac{1}{2}v - \frac{3\alpha}{4}R^2 + \frac{C}{R^2}$$

We now make the substitution $s = \alpha R^2/2$ to write

$$\Phi' = \frac{C + vs - 3s^2}{2s}$$

(d) Lastly, by substituting your results in the second equation of the system, show that $s(z)$ satisfies

$$\frac{1}{2}s'^2 + V(s) = E,$$

the equation for the motion of a particle with potential $V(s)$. Find the expression for $V(s)$ and for E.

Solution: We have

$$-v\Phi' - \Omega + \alpha\Phi' R^2 + \Phi'^2 - \frac{R''}{R} = 0$$

Where $\Phi' = \frac{1}{2s}(C + vs - 3s^2)$ and $s = \frac{\alpha}{2}R^2$. Plugging this expression for Φ' into our left-hand side and using $\alpha = 2s/R^2$ gives us

$$\begin{aligned}\frac{1}{2s}(C + vs - 3s^2)(-v + 2s) + \frac{1}{4s^2}(C + vs - 3s^2)^2 &= \frac{R''}{R} + \Omega\\ \Rightarrow \frac{C^2}{4s^2} - \frac{C}{2} - \frac{v^2}{4} + sv - \frac{3s^2}{4} &= \frac{R''}{R} + \Omega\end{aligned}$$

We now multiply both sides by s', and recalling that $s' = 2\alpha RR'$, we have

$$\frac{C^2 s'}{4s^2} - \frac{Cs'}{2} - \frac{v^2 s'}{4} + ss'v - \frac{3s^2 s'}{4} = 2\alpha R' R'' + \Omega s'$$

We can integrate this equation with respect to z to give

$$-\frac{C^2}{4s} - \frac{C}{2}s - \frac{1}{4}s(s - v)^2 = \alpha R'^2 + \Omega s - E$$

Where we have written our integration constant as $-E$.
Lastly, we can use $R' = \frac{s'}{\alpha R}$ and $R = \sqrt{\frac{2s}{\alpha}}$ to write $R' = \frac{s'}{\sqrt{2\alpha s}}$, so that our equation becomes

$$-\frac{C^2}{4s} - \frac{C}{2}s - \frac{1}{4}s(s - v)^2 = \frac{s'^2}{2s} + \Omega s - E$$

Multiplying both sides by s and rearranging allows us to write this in the form

$$\frac{1}{2}s'^2 + V(s) = E$$

where

$$V(s) = \left(\Omega + \frac{C}{2}\right)s + \frac{C^2}{4s} + \frac{1}{4}s(s - v)^2$$

You can (don't do this; it would take a lot of work; there's a lot of cases[1]) use this observation to classify the traveling-envelope solutions of the DNLS, in the same vein that we did at the beginning of this chapter for KdV.

6. Consider example 5.2 in the notes. Check that $y = x^2/t$ and $t^{1/2}q$ are both scaling invariant. Find the ordinary differential equation satisfied by $G(y)$, for similarity solutions of the form $q(x,t) = t^{-1/2}G(y)$. Show that this results in the same similarity solutions as in the example.

Solution: We will first check that $y = x^2/t$ and $t^{1/2}q$ are both scaling invariant. We know that the NLS equation has the scaling symmetry $x = x^*/a$, $t = t^*/a^2$, and $q = aq^*$. Applying this scaling transformation on y results in $y = \frac{x^{*2}}{a^2}\frac{a^2}{t^*} = x^{*2}/t$ and applying it to $t^{1/2}q$ gives back $\frac{t^{*1/2}}{a}aq^* = t^{*1/2}q^*$, which confirms that both quantities are scaling invariant.

We will now find the ODE satisfied by $G(y)$ for similarity solutions of the form $q(x,t) = t^{-1/2}G(y)$. To start, we calculate the derivatives $y_x = 2t^{-1/2}y^{1/2}$ and $y_t = -yt^{-1}$, which we will use to calculate the following quantities:

(a) $q_t = (t^{-1/2}G)_t = -\frac{1}{2}t^{-3/2}G + t^{-1/2}G'y_t = -\frac{1}{2}t^{-3/2}G - yt^{-3/2}G'$
(b) $q_{xx} = (t^{-1/2}G)_{xx} = (2t^{-1}y^{1/2}G')_x = 2t^{-3/2}G' + 4t^{-3/2}yG''$
(c) $|q|^2q = |t^{-1/2}G|^2t^{-1/2}G = t^{-3/2}|G|^2G$

Plugging these expressions back into the NLS equations gives us

$$-\frac{i}{2}t^{-3/2}G - iyt^{-3/2}G' = -2t^{-3/2}G' - 4t^{-3/2}yG'' + \sigma t^{-3/2}|G|^2G$$
$$\Rightarrow -\frac{i}{2}G - iyG' = -2G' - 4yG'' + \sigma|G|^2G$$

We can show that these are the same similarity solutions as those we got in example 5.2. In the example we had $z = xt^{-1/2} = y^{1/2}$. By the chain rule we have

$$\frac{d}{dy} = \frac{1}{2z}\frac{d}{dz}, \quad \text{and} \quad \frac{d^2}{dy^2} = -\frac{1}{4z^3}\frac{d}{dz} + \frac{1}{4z^2}\frac{d^2}{dz^2}$$

Using these and substituting in z^2 for y gives us

$$-\frac{i}{2}G - iz^2\left(\frac{1}{2z}\right)G' = -2\left(\frac{1}{2z}\right)G' - 4z^2\left(\frac{-1}{4z^3}G' + \frac{1}{4z^2}G''\right) + \sigma|G|G$$
$$\Rightarrow G'' - \frac{i}{2}G - \frac{iz}{2}G' = -G'' + \sigma|G|^2G$$
$$\Rightarrow \left(G' - \frac{i}{2}(zG)\right)' = \sigma|G|^2G$$

Which is indeed the same ODE we got in the example.

[1]Two semicolons so close together? Really? And a footnote?

7. One way to write the **Toda Lattice** is

$$\frac{da_n}{dt} = a_n(b_{n+1} - b_n),$$
$$\frac{db_n}{dt} = 2(a_n^2 - a_{n-1}^2),$$

where a_n, b_n, $n \in \mathbb{Z}$, are functions of t.

(a) Find a scaling symmetry of this form of the Toda lattice, *i.e.*, let[2] $a_n = \alpha A_n$, $b_n = \beta B_n$, $t = \gamma\tau$, and determine relations between α, β and γ so that the equations for the Toda lattice in the (A_n, B_n, t) variables are identical to those using the (a_n, b_n, τ) variables.

Solution: Making the variable substitution $a_n = \alpha A_n$, $b_n = \beta B_n$, and $t = \gamma\tau$, and applying the chain rule $\frac{d}{dt} = \frac{1}{\gamma}\frac{d}{d\tau}$ transforms our equations into

$$\frac{1}{\gamma}\frac{dA_n}{d\tau} = \beta A_n(B_{n+1} - B_n),$$
$$\frac{\beta}{\gamma}\frac{dB_n}{d\tau} = 2\alpha^2(A_n^2 - A_{n-1}^2),$$

We see that in order for this scaling to leave the first equation invariant, we must have $\beta = \frac{1}{\gamma}$, while for the second equation to be invariant we require that $2\alpha^2 = \frac{\beta}{\gamma} = \beta^2$. Hence we have the scaling symmetry

$$a_n = \pm\alpha A_n, \; b_n = \alpha B_n, \; \tau = \alpha t$$

(b) Using this scaling symmetry, find a two-parameter family of similarity solutions of the Toda lattice. If necessary, find relations among the parameters that guarantee the solutions you found are real for all n and for $t > 0$.

Solution: We note that $a_n t$ and $b_n t$ are each scale invariant, so we look for similarity solutions for which $a_n(t) = t^{-1}x_n$ and $b_n(t) = t^{-1}y_n$, where x_n and y_n are constants.

Plugging these ansatz solutions into the first equation gives us

$$-t^{-2}x_n = t^{-2}x_n(y_{n+1} - y_n) \Rightarrow y_{n+1} = y_n - 1 \Rightarrow y_n = y_0 - n$$

From the second equation we have

$$-t^{-2}y_n = 2t^{-2}(x_n^2 - x_{n-1}^2) \Rightarrow x_n^2 = x_{n-1}^2 - \frac{1}{2}y_n \Rightarrow x_{n+1}^2 = x_n^2 - \frac{1}{2}y_{n+1}$$

[2]In principle, we could let α, β and γ depend on n. If you did this, you quickly discover that a scaling symmetry only exists if they do not

Applying the result of the first equation to the second equation gives

$$x_n^2 = x_{n-1}^2 - \frac{1}{2}(y_0 - n - 1)$$

Therefore to ensure that all x_n are real, we require that $x_n^2 > \frac{1}{2}(y_0 - n)$. Note that this means that $x_n \to \infty$ as $n \to -\infty$.

The Toda Lattice was introduced originally by Toda in 1967 in the form

$$\begin{aligned} \frac{dq_n}{dt} &= p_n, \\ \frac{dp_n}{dt} &= e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}, \end{aligned}$$

where q_n, p_n, $n \in \mathbb{Z}$, are functions of t. It is clear that this form does not lend itself to a scaling symmetry: the quantities q_n show up as arguments of the exponential function, and they cannot be scaled. This can be remedied by returning to the physical setting of the derivation, where a constant would multiply these exponents. This constant, being a dimensional quantity, scales in its own way under a scaling transformation.

8. Consider the equation

$$u_t = 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x},$$

which we will encounter more in later chapters, due to its relation to the KdV equation. Show that it has a scaling symmetry.

When we look for the scaling symmetry of the KdV equation, we have two equations for three unknowns: we have three quantities (x, t, u) to scale, and after normalizing one coefficient to 1, two remaining terms that need to remain invariant. Thus it is no surprise that we find a one-parameter family of scaling symmetries. The above equation has two more terms, and it should be clear that some "luck" is needed in order for there to be a scaling symmetry.

Solution: We have $x^* = \alpha x$, $t^* = \beta t$ and $u = \gamma u^*$. By chain rule, we have $\partial_x = \alpha_{x^*}$ and $\partial_t = \beta \partial_{t^*}$. Substituting these scaled variables we find

$$\beta\gamma u_t = \gamma^3 \alpha 30u^2 u_x + \gamma^2 \alpha^3 20u_x u_{xx} = \gamma^2 \alpha^3 10uu_{xxx} + \gamma\alpha^5 u_{5x}$$

Hence we see that we require $\gamma^2 \frac{\alpha}{\beta} = \gamma \frac{\alpha^3}{\beta} = \gamma \frac{\alpha^3}{\beta} = \frac{\alpha^5}{\beta} = 1$, and so our scaling symmetry is $x^* = \alpha x$, $t^* = \alpha^5 t$ and $u = \alpha^2 u^*$.

9. Consider a Modified KdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0.$$

(a) Find its scaling symmetry.

Solution: We have $x^* = \alpha x$, $t^* = \beta t$ and $u = \gamma u^*$. By chain rule, we have $\partial_x = \alpha_{x^*}$ and $\partial_t = \beta \partial_{t^*}$. Substituting these scaled variables we find

$$\beta\gamma u_t - \gamma^3 \alpha 6 u^2 u_x + \gamma \alpha^3 u_{xxx} = 0$$

and so we find our scaling symmetry to be $x^* = \alpha x$, $t^* = \alpha^3 t$ and $u = \pm \alpha u^*$

(b) Using the scaling symmetry, write down an ansatz for any similarity solutions of the equation.

Solution: $y = x^3/t$ and ux^{-1} are scaling invariant, therefore we can ansatz $u(x,t) = xF(y)$.

(c) Show that your ansatz is compatible with $u = (3t)^{-1/3} w(z)$, with $z = x/(3t)^{1/3}$.

Solution: Comparing with above, we see that $y = 3z^3$. Since they are both scale invariant, the solutions must be compatible with one another.

(d) Use the above form of u to find an ordinary differential equation for $w(z)$. This equation will be of third order. It can be integrated once (do this) to obtain a second-order equation. The second-order equation you obtain this way is known as the second of the Painlevé equations. We will see more about these later.

Solution: We begin by finding the relevant partial derivatives. We begin with $z_t = -x(3t)^{-4/3} = -z(3t)^{-1}$ and $z_x = (3t)^{-1/3}$, which we use to find

i. $u_t = [(3t)^{-1/3} w]_x = -(3t)^{-4/3} (zw)'$
ii. $u_x = [(3t)^{-1/3} w]_x = (3t)^{-2/3} w'$
iii. $u_{xxx} = [(3t)^{-1/3} w]_{xxx} = (3t)^{-4/3} w'''$

Plugging these in to the mKdV equation gives us

$$-(3t)^{-4/3}(zw)' - 6(3t)^{-2/3} w^2 (3t)^{-2/3} w' + (3t)^{-4/3} w''' = 0$$
$$\Rightarrow -(zw)' - 6w^2 w' + w''' = 0$$

Integrating this equation gives us

$$-zw - 2w^3 + w'' = C$$