AMATH 562
Advanced Stochastic Processes
Homework 4

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1. **Exercise 8.5** Let $X = (X_t)_{0 \le t \le T}$ be an OU process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$dX_t = K(\theta - X_t)dt + \sigma dW_t$$

Where $W = (W_t)_{0 \le t \le T}$ is a Brownian motion under probability measure $\mathbb{P}$. Then we can define a new probability measure $\tilde{\mathbb{P}}$ such that the process $\tilde{W} = (\tilde{W}_t)_{0 \le t \le T}$ is a Brownian motion under $\tilde{\mathbb{P}}$. Then the OU process $X = (X_t)_{0 \le t \le T}$ on the new probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ will be

$$dX_t = K(\theta^* - X_t)dt + \sigma d\tilde{W}_t.$$

Find the Radon-Nikodym derivative $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$.

Solution: Since $\tilde{W}$ is a Brownian motion under $\tilde{\mathbb{P}}$, there must exist some $\mathbb{F}$-adapted Θ_t such that $d\tilde{W}_t = \Theta_t dt + dW_t$. Then we may write dX_t as

$$\begin{aligned} dX_t &= K(\theta^* - X_t)dt + \sigma d\tilde{W}_t \\ &= K(\theta^* - X_t)dt + \sigma(\Theta_t dt + dW_t) \\ &= (K\theta^* - KX_t + \sigma\Theta_t)dt + \sigma dW_t \end{aligned}$$

The drift term here must be equal to the drift term in the original definition of dX_t, which means that

$$K\theta dt = (K\theta^* + \sigma\Theta_t)dt \qquad \Rightarrow \qquad \Theta_t = \frac{K}{\sigma}(\theta - \theta^*)$$

Then by Girsanov's theorem we may define a Radon-Nikodym derivative $Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ given by

$$
\begin{aligned}
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= \exp\left(-\int_0^T \frac{1}{2}\Theta_t^2 dt - \int_0^T \Theta_t dW_t\right) \\
&= \exp\left(-\frac{K^2}{2\sigma^2}(\theta-\theta^*)^2 \int_0^T dt - \frac{K}{\sigma}(\theta-\theta^*) \int_0^T dW_t\right) \\
&= \exp\left(-\frac{K^2}{2\sigma^2}(\theta-\theta^*)^2 T - \frac{K}{\sigma}(\theta-\theta^*) W_T\right)
\end{aligned}
$$

2. The Ornstein-Uhlenbeck process, defined by the time-homogeneous linear SDE

$$
dX(t) = -\mu X(t)dt + \sigma dW(t) \qquad\qquad X(0) = x_0
$$

in which $\sigma, \mu > 0$ are two constants, has its Kolmogorov forward equation

$$
\frac{\partial}{\partial t}\Gamma(x_0;t,x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\Gamma(x_0;t,x) + \frac{\partial}{\partial x}\left(\mu x \Gamma(x_0;t,x)\right), \tag{1}
$$

with the initial condition $\Gamma(x_0;0,x) = \delta(x-x_0)$.

(a) Show that the solution to the linear PDE (1) has a Gaussian form and find the solution.

Solution: For some functions $g(t)$, $f(t)$ such that $g(t) > 0$ and $f(0) = x_0$, define

$$
\Gamma(x_0;t,x) = \frac{1}{\sqrt{2\pi g(t)}} \exp\left\{-\frac{(x-f(t))^2}{2g(t)}\right\}
$$

Using Mathematica to evaluate the forward equation for this function, we find that for Γ to satisfy (1) f and g must satisfy

$$
\begin{aligned}
0 = x^2\left(\sigma^2 - g' - 2\mu g\right) &+ x\left(-2gf' + 2fg\mu + g' - \sigma^2\right) \\
&+ 2\mu g^2 + f^2(\sigma^2 - g') + g(g' - \sigma^2) + 2ff'g
\end{aligned}
$$

Since this must hold for all x, the coefficients of each power of x must each individually cancel to zero. Therefore, for $\mathcal{O}(x^2)$ we have the following first order linear ODE for g.

$$
\sigma^2 - g' - 2\mu g = 0 \qquad \Rightarrow \qquad g = \frac{1}{2\mu}[\sigma^2 - e^{-2\mu t}]
$$

Plugging this expression in for $g(t)$ gives us a new expression for Γ.

$$\Gamma(x_0; t, x) = \sqrt{\frac{\mu}{\pi(\sigma^2 - e^{-2\mu t})}} \exp\left(-\frac{\mu(x - f(t))^2}{\sigma^2 - e^{-2\mu t}}\right)$$

When we plug this new expression in to (1), we find that its solvability requires that f satisfy the following linear first order ODE.

$$f'(t) + \mu f(t) \qquad \Rightarrow \qquad f(t) = x_0 e^{-\mu t}$$

Here we have used the initial condition $\Gamma(x_0; 0, x) = \delta(x - x_0)$. Having found both $f(t)$ and $g(t)$, we may write the full Gaussian solution to the OU process Kolmogorov forward equation as

$$\Gamma(x_0; t, x) = \sqrt{\frac{\mu}{\pi(\sigma^2 - e^{-2\mu t})}} \exp\left(-\frac{\mu\left(x - x_0 e^{-\mu t}\right)^2}{\sigma^2 - e^{-2\mu t}}\right)$$

(b) What is the limit of

$$\lim_{t\to\infty} \Gamma(x_0; t, x)?$$

Solution: As $t \to \infty$, the time-dependent exponential terms go to zero, and Γ approaches $\mathcal{N}(0, \sigma^2/2\mu)$.

$$\lim_{t\to\infty} \Gamma(x_0; t, x) = \sqrt{\frac{\mu}{\pi\sigma^2}} \exp\left(-\frac{\mu x^2}{\sigma^2}\right)$$

(c) Find $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$.

Solution: Since y is Gaussian distributed, we can read its expected value and variance directly from its equation. From part (a) we had

$$\Gamma(x_0; t, x) = \sqrt{\frac{\mu}{\pi(\sigma^2 - e^{-2\mu t})}} \exp\left(-\frac{\mu\left(x - x_0 e^{-\mu t}\right)^2}{\sigma^2 - e^{-2\mu t}}\right)$$

Hence we have

$$\mathbb{E}[X(t)] = x_0 e^{-\mu t} \qquad \text{and} \qquad \mathbb{V}[X(t)] = \frac{\sigma^2 - e^{-2\mu t}}{2\mu}$$

(d) Note that $\mathbb{E}[X(t)]$ is the same as the solution to the ODE $\frac{dx}{dt} = -\mu x$, which is obtained when $\sigma = 0$. Is this result true for a nonlinear SDE?

Solution: No this would in general not hold for a nonlinear SDE. To see this $\hat{x}(t) = \mathbb{E}[X(t)]$. If we write the linear SDE in integral form and take expectation, we have

$$\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}\left[x_0 - \mu \int_0^t X(s)ds + \sigma \int_0^t dW(s)\right] \\
\Rightarrow \hat{x}(t) &= x_0 - \mu \int_0^t \hat{x}(s)ds
\end{aligned}$$

We see that $\hat{x}(t)$ satisfies the ODE $d\hat{x}/dt = -\mu\hat{x}$. Now consider a new process $Y(t)$ which satisfies nonlinear SDE defined by

$$Y(t) = y_0 - \frac{1}{2}\mu \int_0^t Y^2(s)ds + \sigma \int_0^t dW(s)$$

Repeating the same steps as above results in

$$\begin{aligned}
\mathbb{E}[Y(t)] &= \mathbb{E}\left[y_0 - \frac{\mu}{2}\int_0^t Y^2(s)ds + \sigma\int_0^t dW(s)\right] \\
\Rightarrow \hat{y}(t) &= y_0 - \frac{\mu}{2}\int_0^t \mathbb{E}[Y^2(s)]ds \\
&\neq y_0 - \frac{\mu}{2}\int_0^t \hat{y}^2(s)ds
\end{aligned}$$

3. The time-independent solution to a Kolmogorov forward equation gives a stationary probability density function for the Itô process $dX_t = \mu(X_t)dt + \sigma(X_t)dW(t)$:

$$-\frac{\partial}{\partial x}\Big(\mu(x)f(x)\Big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x)f(x)\Big) = 0.$$

This is a linear, second-order ODE. We assume that both $\mu(x)$ and $\sigma(x)$ satisfy the conditions required to have a solution $f(x)$ on the entire $\mathbb{R}$. Find the expression for the general solution. There are two constants of integration, which should be determined according to appropriate probabilistic reasoning.

Solution: Since both left-hand side terms are derivatives, we can integrate once, picking up an integration constant A.

$$-\mu(x)f(x) + \frac{1}{2}(\sigma^2(x)f(x))_x = A$$

Now let $g(x) = \frac{1}{2}\sigma^2(x)f(x)$, so that we may write

$$-\frac{2\mu}{\sigma^2}g + g_x = A$$

We now introduce the integrating factor $\phi = \phi(x)$, satisfying the following

$$\phi_x = \frac{-2\mu}{\sigma^2}\phi \qquad \Rightarrow \qquad \phi(x) = \exp\left\{\int_{-\infty}^x \frac{-2\mu(\xi)}{\sigma(\xi)^2}d\xi\right\}$$

Multiplying both sides of our equation by ϕ gives us

$$
\begin{aligned}
A\phi &= -\frac{2\mu}{\sigma^2} g\phi + g_x\phi \\
&= g\phi_x + g_x\phi = (g\phi)_x
\end{aligned}
$$

$$\Rightarrow g(x) = \frac{1}{\phi(x)}\left(A\int_{-\infty}^{x}\phi(\xi)d\xi + B\right)$$

$$\Rightarrow f(x) = \frac{1}{\sigma^2(x)\phi(x)}\left(A\int_{-\infty}^{x}\phi(\xi)d\xi + B\right)$$

For integrability, we require that $f(x) \to 0$ as $|x| \to \infty$. Additionally, since $f(x)$ is a density, we require that $\int_{\mathbb{R}} f(x)dx = 1$. A simple way to achieve this is to let $A = 0$ and set

$$B = \left[\int_{-\infty}^{\infty} \frac{dx}{\sigma^2(x)\phi(x)}\right]^{-1}$$

Then, provided that $\sigma^2(x)$ is unbounded as $|x| \to \infty$ and grows sufficiently quickly, $f(x)$ will satisfy the required integrability conditions.

4. **Exercise 9.3** For $i = 1, 2, \ldots, n$, let $X^{(i)}$ satisfy

$$dX_t^{(i)} = -\frac{b}{2}X_t^{(i)}dt + \frac{1}{2}\sigma dW_t^{(i)},$$

where the $(W^{(i)})_{i=1}^{n}$ are independent Brownian motions. Define

$$R_t := \sum_{i=1}^{n}(X_t^{(i)})^2, \qquad B_t := \sum_{i=1}^{n}\int_0^t \frac{1}{\sqrt{R_s}}X_s^{(i)}dW_s^{(i)}.$$

Show that B is a Brownian motion. Derive an SDE for R that involves only dt and dB_t terms (i.e., no $dW_t^{(i)}$ terms should appear).

Solution: To show that B is a Brownian motion, we first note that $B_0 = 0$. Additionally, B_t is a finite sum of stochastic integrals, and hence is both martingale and continuous. Lastly, since B_t is a sum of Itô processes we may calculate the quadratic variation as

$$
\begin{aligned}
d[B,B]_t &= \sum_{i=1}^{n} d\left(\int_0^t \frac{X_s^{(i)}}{\sqrt{R_s}} dW_s^{(i)}\right) \\
&= \sum_{i=1}^{n} \frac{\left(X_t^{(i)}\right)^2}{R_t} dt \\
&= \frac{1}{R_t} \sum_{i=1}^{n} (X_t^{(i)})^2 dt \\
&= dt
\end{aligned}
$$

Hence, by the Lévy characterization of Brownian motion, B must be a Brownian motion.

To derive an SDE for R, we first note that since $X^{(i)}$ is an Itô process with diffusion term given by $\sigma_t^{ij} = \frac{\sigma}{2}\delta_{ij}$, where δ_{ij} is the Kronecker delta. Hence its quadratic variation can be calculated using Itô's formula to be

$$
\begin{aligned}
d[X^{(i)}, X^{(j)}]_t &= \sum_{k=1}^{n} \sigma_t^{ik}\sigma_t^{jk} dt \\
&= \frac{\sigma^2}{4} \sum_{k=1}^{n} \delta_{ik}\delta_{jk} dt \\
&= \frac{\sigma^2}{4} \delta_{ij} dt
\end{aligned}
$$

Using this result, we can use Itô's formula to calculate dR_t as follows

$$
\begin{aligned}
dR_t &= \sum_{i=1}^{n} \frac{\partial R_t}{\partial x_i} dX_t^{(i)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 R_t}{\partial x_i \partial x_j} d[X^{(i)}, X^{(j)}]_t \\
&= \sum_{i=1}^{n} \frac{\partial R_t}{\partial x_i} dX_t^{(i)} + \frac{\sigma^2}{8} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 R_t}{\partial x_i \partial x_j} \delta_{ij} dt \\
&= \sum_{i=1}^{n} \frac{\partial R_t}{\partial x_i} dX_t^{(i)} + \frac{\sigma^2}{8} \sum_{i=1}^{n} \frac{\partial^2 R_t}{\partial x_i^2} dt
\end{aligned}
$$

Using the provided definition of R_t to calculate the derivative terms, and writing out $X_t^{(i)}$ explicitely, results in

$$
\begin{aligned}
dR_t &= 2\sum_{i=1}^{n} X_t^{(i)} dX_t^{(i)} + \frac{\sigma^2}{4}\sum_{i=1}^{n} dt \\
&= \sum_{i=1}^{n}\left[\left(\frac{\sigma^2}{4} - b\left(X_t^{(i)}\right)^2\right) dt + \sigma X_t^{(i)} dW_t^{(i)}\right] \\
&= \left(\frac{n\sigma^2}{4} - bR_t\right) dt + \sigma \sum_{i=1}^{n} X_t^{(i)} dW_t^{(i)}
\end{aligned}
$$

Lastly, we can use $dB_t^{(i)} = \frac{X_t^{(i)}}{\sqrt{R_t}} dW_t^{(i)} \Rightarrow dW_t^{(i)} = \frac{\sqrt{R_t}}{X_t^{(i)}} dB_t^{(i)}$ to write

$$
dR_t = \left(\frac{n\sigma^2}{4} - bR_t\right) dt + \sigma\sqrt{R_t} dB_t
$$

This is a SDE for R_t in terms of t and B_t. Note that we can also divide both sides by $2\sqrt{R_t}$ to find a SDE for $\sqrt{R_t}$ that has a constant diffusion term $\sigma/2$.

5. Consider a continuous-time $(n+1)$-state Markov process $X(t)$, $X \in \mathcal{S} = \{0, 1, 2, \ldots, n\}$, with transition rates

$$
g(i,j) = \frac{1}{dt}\mathbb{P}\{X(t+dt) = j | X(t) = i\}, \qquad j \neq i
$$

Let state 0 be an absorbing state. E.g., all $g(0, j) = 0$ for $1 \leq j \leq n$. Let τ_k be a hitting time:

$$
\tau_k := \inf\{t \geq 0 : X(t) = 0, X(0) = k\}.
$$

(a) Show that

$$
\sum_{1 \leq k \leq n} g(j,k)\mathbb{E}[\tau_k] = -1
$$

Solution: Note that since $\mathbf{G} = (g(i,j))_{i,j}$ is a generator, its entries $g(i,j)$ must satisfy

$$
g(i,i) \leq 0 \qquad \forall j \neq i : g(i,j) \geq 0 \qquad \sum_{j} g(i,j) = 0
$$

from which it follows that

$$
g(i,i) = -\sum_{j \neq i} g(i,j)
$$

Next, we note that since 0 is absorbing, $\tau_0 = 0$, and so we may write

$$\sum_{1\le k\le n} g(j,k)\mathbb{E}[\tau_k] = \sum_{k\in\mathcal{S}} g(j,k)\mathbb{E}[\tau_k]$$

Using these results, we can write our quantity of interest as

$$\begin{aligned}
\sum_{1\le k\le n} g(j,k)\mathbb{E}[\tau_k] &= g(j,j)\mathbb{E}[\tau_j] + \sum_{k\neq j} g(j,k)\mathbb{E}[\tau_k] \\
&= -\mathbb{E}[\tau_j]\sum_{k\neq j} g(j,k) + \sum_{k\neq j} g(j,k)\mathbb{E}[\tau_k] \\
&= \sum_{k\neq j} g(j,k)\left(\mathbb{E}[\tau_k] - \mathbb{E}[\tau_j]\right) \\
&= \sum_{k} g(j,k)\mathbb{E}[\tau_k - \tau_j]
\end{aligned}$$

The last equality comes from linearity of expectation and from recognizing that for $k = j$ the summand will cancel to zero. Next, multiplying by by dt gives us

$$\begin{aligned}
\sum_{k} g(j,k)\mathbb{E}[\tau_k - \tau_j]dt &= \sum_{k} \mathbb{E}[\tau_k - \tau_j]\mathbb{P}\left[X(t+dt) = k | X(t) = j\right] \\
&= \mathbb{E}[\mathbb{E}[\tau_{X(t+dt)} - \tau_{X(t)}] | X(t) = j] \\
&= \mathbb{E}[\tau_{X(t+dt)} - \tau_{X(t)} | X(t) = j] \\
&= \mathbb{E}[d\tau | X(t) = j] \\
&= -dt
\end{aligned}$$

We see that this quantity is equivalent to the expected infinitesimal change in the hitting time τ following the passage of time dt. Regardless of starting state, in the expected hitting time must decrease in proportion with dt. Hence, dividing by dt, we find the equality which we wanted to show.

$$\sum_{1\le k\le n} g(j,k)\mathbb{E}[\tau_k] = -1$$

(b) Derive a system of equations relating $\mathbb{E}[\tau_k^2]$ to $\mathbb{E}[\tau_j]$, for $1 \le j, k \le n$.

Solution: Let us define $\mathbf{u}(\lambda) \in \mathbb{R}^{n-1}$ by $[\mathbf{u}(\lambda)]_k = u_k(\lambda)$ where $u_k(\lambda)$ is the Laplace transform of τ_k, defined by

$$u_k(\lambda) = \mathbb{E}\left[e^{-\lambda\tau} | X(0) = k\right] = \mathbb{E}[e^{-\lambda\tau_k}] \qquad\qquad k \ge 1$$

We note that $u_k'(0) = -\mathbb{E}[\tau_k]$ and that $u_k''(0) = \mathbb{E}[\tau_k^2]$. Furthermore, from Lorig Corollary 9.4.2. we know that $\mathbf{u}(\lambda)$ satisfies

$$(\mathbf{G}-\lambda)\mathbf{u}=0 \qquad \Rightarrow \qquad \sum_k g(j,k)u_k(\lambda)=\lambda u_j(\lambda)$$

Differentiating this equation twice with respect to λ, we find

$$\begin{aligned}
\frac{d^2}{d\lambda^2}\sum_k g(j,k)u_k(\lambda) &= \frac{d^2}{d\lambda^2}\lambda u_j(\lambda)\\
\Rightarrow \frac{d}{d\lambda}\sum_k g(j,k)u_k'(\lambda) &= \frac{d}{d\lambda}(u_j(\lambda)+\lambda u_j'(\lambda))\\
\Rightarrow \sum_k g(j,k)u_k''(\lambda) &= 2u_j'(\lambda)+\lambda u_j''(\lambda)
\end{aligned}$$

Lastly, we evaluate this expression at $\lambda=0$ and use the relationships above to write

$$\mathbb{E}[\tau_j]=-\frac{1}{2}\sum_k g(j,k)\mathbb{E}[\tau_k^2]$$

(c) Now if both states 0 and n are absorbing, let u_k be the probability of $X(t)$, starting with $X(0)=k$, being absorbed into state 0 and $1-u_k$ be the probability of it being absorbed into state n. Derive a system of equations for u_k.

Solution: Let $p_t(i,j)=\mathbb{P}[X_{s+t}=j|X_t=i]$, then it is clear that

$$u_k=\lim_{t\to\infty}p_t(k,0)=\lim_{t\to\infty}\left(e^{t\mathbf{G}}\right)_{k,0}$$