Lecture 11

More general IC

$$\begin{cases} u_t = D u_{xx}, & -\infty < x < \infty \\ & t > 0 \\ u(x,t) \to 0 & \text{as } x \to \pm\infty \\ u(x,0) = \delta(x), & -\infty < x < \infty \end{cases}$$

we had the solution:

$$u(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left\{-\frac{x^2}{4Dt}\right\}$$

More general initial condition

$$\begin{cases} u_t = D u_{xx}, & -\infty < x < \infty \\ & t > 0 \\ u(x,t) \to 0 & \text{as } x \to \pm\infty \\ u(x,0) = f(x), & -\infty < x < \infty \end{cases}$$

Since $f(x) = \int_{-\infty}^{\infty} f(y)\,\delta(x-y)\,dy$

and the solution with initial condition $u(x,0) = \delta(x-y)$ is

$$\frac{1}{\sqrt{4\pi D t}} \exp\left\{-\frac{(x-y)^2}{4Dt}\right\}$$

Same as superposition

So

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi D t}} \exp\left\{-\frac{(x-y)^2}{4Dt}\right\} dy}$$

Heat equation in a semi-infinite domain

(chapter 6 of Bernard's notes)

PDE: $u_t = D u_{xx}, \quad x > 0, \quad t > 0$

BCs: $u(0,t) = 0, \quad u(x,t) \to 0$ as $x \to \infty$

IC: $u(x,0) = f(x), \quad x > 0.$

Could use Laplace transform in x, but there is a simpler way.

Let $g(x)$ be the odd extension of $f(x)$, i.e.

$$g(x) = \begin{cases} f(x) & \text{for } x > 0 \\ 0 & x = 0 \\ -f(-x) & \text{for } x < 0 \end{cases}$$

Use our previous results in infinite domain.

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi D t}} \exp\left\{-\frac{(x-y)^2}{4Dt}\right\} g(y)\, dy$$

$$u(x,t) = -\int_{-\infty}^{0} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-y)^2}{4Dt}} f(-y)\,dy$$

$$+\int_{0}^{\infty} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

$$= \int_{\infty}^{0} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x+y)^2}{4Dt}} f(y)\,dy$$

$$+\int_{0}^{\infty} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{4\pi Dt}} \left\{ e^{-\frac{(x-y)^2}{4Dt}} - e^{-\frac{(x+y)^2}{4Dt}} \right\}$$

$$f(y)\,dy$$

Greens formula

(16.2.2 of my book)

Note the vector identity

$$\vec{\nabla}\cdot(u\vec{\nabla}v) = u\nabla^2 v + \vec{\nabla}u\cdot\vec{\nabla}v$$

$$\vec{\nabla}\cdot(v\vec{\nabla}u) = v\nabla^2 u + \vec{\nabla}v\cdot\vec{\nabla}u$$

-2 ______

$$u\nabla^2 v - v\nabla^2 u = \vec{\nabla}\cdot(u\vec{\nabla}v - v\vec{\nabla}u)$$

Integrale :

$$\iiint_V [u\nabla^2 v - v\nabla^2 u]\,dV$$

$$= \iiint_V \vec{\nabla}\cdot(u\vec{\nabla}v - v\vec{\nabla}u)\,dV$$

$$= \iint_{\partial V} (u\vec{\nabla}\underset{\substack{\uparrow \\ \text{typo}}}{v} - v\vec{\nabla}u)\cdot\vec{n}\,dS$$

In 2-D,

$$\iint_A [u\nabla^2 \underset{\substack{\uparrow \\ \text{typo}}}{v} - v\nabla^2 u]\,dA$$

$$= \int_{\partial A} (u\vec{\nabla}\underset{\substack{\uparrow \\ \text{typo}}}{v} - v\vec{\nabla}u)\cdot\vec{n}\,ds$$

Greens function for the Poisson equation

It describes e.g. the gravitational potential due to a distribution of mass in space, or the electrostatic potential due to a given distribution of electric charges.

$$\nabla^2 u = f(\vec{x})$$

Let G satisfies

$$\nabla^2 G = \delta_3(\vec{x}-\vec{\xi}) \quad , \quad G = G(\vec{x},\vec{\xi})$$

subject to homogeneous BCs.

Greens formula:

$dV_{\vec{\xi}} = d^3\vec{\xi} = d\xi\, d\eta\, d\zeta$

$$\iiint_V (u\nabla^2 G - G\nabla^2 u)\, dV_{\vec{\xi}}$$

$$= \iint_{\partial V} (u\vec{\nabla}G - G\vec{\nabla}u)\cdot\vec{n}\, dS_{\vec{\xi}}$$

$= 0$ if u satisfies homogeneous BCs.

$$\iiint_V u \nabla^2 G \, dV_{\vec{\xi}} = \iiint_V u \, \delta_3(\vec{x}-\vec{\xi}) \, dV_{\vec{\xi}}$$

$$= u(\vec{x})$$

$$\iiint_V (G \nabla^2 u) \, dV_{\vec{\xi}} = \iiint_V G f(\vec{\xi}) \, dV_{\vec{\xi}}$$

So $$\boxed{u(\vec{x}) = \iiint_V G(\vec{x},\vec{\xi}) f(\vec{\xi}) \, dV_{\vec{\xi}}}$$

This is the same as the principle of superposition.

Greens function for Poisson equation

in 3-D in infinite domain.

PDE: $\nabla^2 G = \delta_3(\vec{x} - \vec{\xi})$, $G(\vec{x}, \vec{\xi})$

BC : $G(\vec{x}, \vec{\xi})$ bounded as $|\vec{x}| \to \infty$

$G(\vec{x}, \vec{\xi})$ represents the response observed at $\vec{x}$ to a point source located at $\vec{\xi}$

Let $\vec{r} = \vec{x} - \vec{\xi}$

be the distance measured from this single source.

In terms of $\vec{r}$, there is no reference in the angular direction. So seek a solution that depends on

$r = |\vec{r}|$ only.

We let $G(\vec{x}, \vec{\xi}) = G(r)$

In spherical coordinates

$$\nabla^2 G(\vec{x}, \vec{\xi}) = \nabla^2 G(r)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} G\right)$$

For $r \neq 0$, the delta function is zero:

$$\frac{d}{dr}\left(r^2 \frac{d}{dr} G\right) = 0, \quad r > 0$$

Integrate: $r^2 \frac{d}{dr} G = A$

$$\frac{d}{dr} G = A/r^2$$

Integrate again:

$$G(r) = -\frac{A}{r} + B. \quad r > 0$$

We cannot impose boundedness condition at $r=0$ to get rid of A. Solution not valid at $r=0$.

Use the equation that is valid at $r=0$:

$$\nabla^2 G = \delta_3(\vec{x}-\vec{\xi})$$

Integrate over a sphere of radius r.

$$\iiint_V \nabla^2 G\, dV = \iiint_V \delta(\vec{x}-\vec{\xi})\, dV = 1$$

LHS is, by the Divergence Th$^{\underline{m}}$:

$$\iiint_V \vec{\nabla}\cdot(\vec{\nabla} G)\, dV = \iint_{\partial V} \vec{\nabla} G \cdot \vec{n}\, ds$$

where $\vec{n} = \vec{r}/r$ is pointing radially.

$$\vec{\nabla} G \cdot \vec{n} = \frac{\partial}{\partial r} G$$

$$1 = \iint_{\partial V} \vec{\nabla} G \cdot \vec{n}\, ds = \int_0^{2\pi} d\varphi \int_0^{\pi} \left(\frac{\partial}{\partial r} G\right) r^2 \sin\theta\, d\theta$$

$$= 4\pi r^2 \frac{\partial}{\partial r} G$$

Thus $$\lim_{r\to 0^+} r^2 \frac{\partial}{\partial r} G = \frac{1}{4\pi}$$

$$\frac{\partial}{\partial r} G(r) = A/r^2$$

$$r^2 \frac{\partial}{\partial r} G = A = \frac{1}{4\pi}$$

Finally

$$\boxed{G(\vec{x}, \vec{\xi}\,) = -\frac{1}{4\pi r} + B}$$

B is arbitrary since a potential is determined only up to a constant.

We can set $B = 0$

$$\boxed{G(\vec{x}, \vec{\xi}\,) = -\frac{1}{4\pi} \frac{1}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}}}$$

Poisson equation in 2-D in infinite domain

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in polar coordinates.

Assume $G(\vec{x}, \vec{\xi})$ depend on r only

$$\vec{r} = \vec{x} - \vec{\xi}, \quad r = |\vec{r}|.$$

$$\nabla^2 G = \delta_2(\vec{x} - \vec{\xi})$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}G\right) = \delta_2(\vec{r})$$

For $r \neq 0$, $\quad \frac{d}{dr}\left(r\frac{d}{dr}G\right) = 0$

$$r\frac{d}{dr}G = A, \quad \frac{d}{dr}G = A/r$$

$$G(r) = A\ln r + B.$$

From

$$\iint_A \nabla^2 G \, dS = \iint_A \delta(\vec{x}-\vec{\xi}\,)\, dS = 1$$

$$\iint_A \nabla^2 G \, dS = \iint_A \vec{\nabla}\cdot(\vec{\nabla} G)\, dS$$

$$= \int_{\partial A} \vec{\nabla} G \cdot \vec{n}\, dl$$

$dl = r\,d\theta$, A is a circular disk

$$\int_{\partial A} \vec{\nabla} G \cdot \vec{n}\, dl = \int_0^{2\pi} \frac{d}{dr} G \cdot r\, d\theta$$

$$= 2\pi r \frac{d}{dr} G$$

Therefor $2\pi r \frac{d}{dr} G = 1$

$$\lim_{r \to 0^+} r \frac{d}{dr} G = \frac{1}{2\pi}$$

Since $G = A \ln r + B$, $\frac{d}{dr} G = A/r$

$A = \frac{1}{2\pi}$.

Setting the arbitrary constant to zero

$$\boxed{\begin{aligned} G(\vec{x},\vec{\xi}) &= \frac{1}{2\pi}\ln r \quad , \quad r>0 \\ &= \frac{1}{2\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right]^{1/2}. \end{aligned}}$$