AMATH 573 Homework 4

Lucas Cassin Cruz Burke

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Feel free to use maple or mathematica or other software. If you do, you can upload your worksheets on Canvas.

1. Consider the Modified Vector Derivative NLS equation

$$\boldsymbol{B}_t + (\|\boldsymbol{B}\|^2\boldsymbol{B})_x + \gamma(\boldsymbol{e}_1 \times \boldsymbol{B}_0)\,(\boldsymbol{e}_1 \cdot (\boldsymbol{B}_x \times \boldsymbol{B}_0)) + \boldsymbol{e}_1 \times \boldsymbol{B}_{xx} = 0.$$

This equation describes the transverse propagation of nonlinear Alfvén waves in magnetized plasmas. Here $\boldsymbol{B} = (0, u, v)$, $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{B}_0 = (0, B_0, 0)$, and γ is a constant. The boundary conditions are $\boldsymbol{B} \to \boldsymbol{B}_0$, $\boldsymbol{B}_x \to 0$ as $|x| \to \infty$. By looking for stationary solutions $\boldsymbol{B} = \boldsymbol{B}(x - Wt)$, one obtains a system of ordinary differential equations. Integrating once, one obtains a first-order system of differential equations for u and v.

a) Show that this system is Hamiltonian with canonical Poisson structure, by constructing its Hamiltonian $H(u, v)$.

Solution: By letting $\mathbf{B} = \mathbf{B}(x - Wt)$, we can evaluate the relevant terms in the Modified Vector Derivative NLS equation as follows:

(a) $\boldsymbol{B}_t = -W\mathbf{B}'$
(b) $(\|\boldsymbol{B}\|^2\boldsymbol{B})_x = [(u^2 + v^2)\boldsymbol{B}]_x = [(u^2 + v^2)\boldsymbol{B}]'$
(c) $(\boldsymbol{e}_1 \times \boldsymbol{B}_0)\,(\boldsymbol{e}_1 \cdot (\boldsymbol{B}_x \times \boldsymbol{B}_0)) = -v' B_0^2 \boldsymbol{e}_3$
(d) $\boldsymbol{e}_1 \times \boldsymbol{B}_{xx} = u''\boldsymbol{e}_3 - v''\boldsymbol{e}_2$

Plugging these terms in to our equation gives us two equations for the $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ coefficients.

$$\begin{aligned}
\boldsymbol{e}_2 :&\quad -Wu' + [(u^2 + v^2)u]' - v'' = 0\\
\boldsymbol{e}_3 :&\quad -Wv' + [(u^2 + v^2)v]' - v' B_0^2 \gamma + u'' = 0
\end{aligned}$$

We can integrate each of these equations once to find

$$-Wu + (u^2 + v^2)u - v' = C_1$$

$$-Wv + (u^2 + v^2)v - B_0^2\gamma v + u' = C_2$$

Since these equations hold for all x, we can solve for C_1 and C_2 by applying our boundary conditions as $|x| \to \infty$, where $u \to B_0$ and $v \to 0$. We find

$$C_1 = -WB_0 + B_0^3 \qquad\qquad C_2 = 0$$

Plugging these values in and rearranging our equations gives us

$$u' = -(u^2 + v^2)v + Wv + B_0^2\gamma v$$
$$v' = (u^2 + v^2)u - Wu + WB_0 - B_0^3$$

We would like to construct a Hamiltonian $H(u, v)$ such that $u' = -\frac{\partial H}{\partial v}$ and $v' = \frac{\partial H}{\partial u}$. From the first equation we have

$$\frac{\partial H}{\partial v} = (u^2+v^2)v - Wv - B_0^2\gamma v \Rightarrow H = \frac{1}{2}u^2v^2 + \frac{1}{4}v^4 - \frac{1}{2}Wv^2 - \frac{11}{2}B_0^2\gamma v^2 + C(u)$$

while for the second equation we have

$$\frac{\partial H}{\partial u} = (u^2+v^2)u - Wu + WB_0 - B_0^3 \to H = \frac{1}{4}u^4 + \frac{1}{2}v^2u^2 - \frac{1}{2}Wu^2 + WB_0u - B_0^3u + D(v)$$

Comparing the above two expressions, we see that

$$H = \frac{1}{4}(u^2 + v^2)^2 - \frac{1}{2}W(v^2 + u^2) + u(WB_0 - B_0^3) - \frac{1}{2}B_0^2\gamma v^2$$

Note that $C(u)$ and $D(v)$ may include constant terms which could show up in H. However, constant shifts in H have no impact on the dynamics of the system in question, and so we are free to set them to zero.

b) Find the value of the Hamiltonian such that the boundary conditions are satisfied. Then $H(u, v)$ equated to this constant value defines a curve in the (u, v)-plane on which the solution lives. In the equation of this curve, let $U = u/B_0$, $V = v/B_0$, and $W_0 = W/B_0^2$. Now there are only two parameters in the equation of the curve: W_0 and γ.

Solution: At the boundary as $|x| \to \infty$ we have $v \to 0$ and $u \to B_0$, so that

$$\lim_{|x|\to\infty} H = \frac{1}{4}B_0^4 - \frac{1}{2}WB_0^2 + WB_0^2 - B_0^4 = \frac{1}{2}WB_0^2 - \frac{3}{4}B_0^4$$

Hence, our boundary conditions at $x \to \pm\infty$ are satisfied when $H(u, v) = \frac{1}{2}WB_0^2 - \frac{3}{4}B_0^4 = H_\infty$. By setting our expression for $H(u, v)$ to this value, and making the variable substitutions $(u, v, W) \to (U, V, W_0)$, we find

$$H(U,V) = \frac{B_0^4}{4}(U^2+V^2)^2 - \frac{B_0^4}{2}W_0(U^2+V^2) + B_0^4 U(W_0-1) - \frac{B_0^4}{2}\gamma V^2 = \frac{B_0^4}{2}W_0 - \frac{3}{4}B_0^4$$

Dividing both sides by B_0^4 then gives us

$$\frac{1}{4}(U^2+V^2)^2 - \frac{1}{2}W_0(U^2+V^2) + U(W_0-1) - \frac{1}{2}\gamma V^2 = \frac{1}{2}W_0 - \frac{3}{4}$$

which does indeed correspond to a family of curves in the (U,V) plane parameterized by γ and W_0.

c) With $\gamma = 1/10$, plot the curve for $W_0 = 3$, $W_0 = 2$, $W_0 = 1.1$, $W_0 = 1$, $W_0 = 0.95$, $W_0 = 0.9$. All of these curves have a singular point at $(1,0)$. This point is an equilibrium point for the Hamiltonian system, corresponding to the constant solution which satisfies the boundary condition. The curves beginning and ending at this equilibrium point correspond to soliton solutions of the Modified Vector Derivative NLS equation. How many soliton solutions are there for the different velocity values you considered? Draw a qualitatively correct picture of the solitons for all these cases.

Solution:

(a) For $W_0 = 3$ we have a single soliton solution, up to time reversal, which moves along a cardioid in the (U,V) plane. For this soliton we see that the angle of $\mathbf{B}$ from its equilibrium state completes a full rotation.

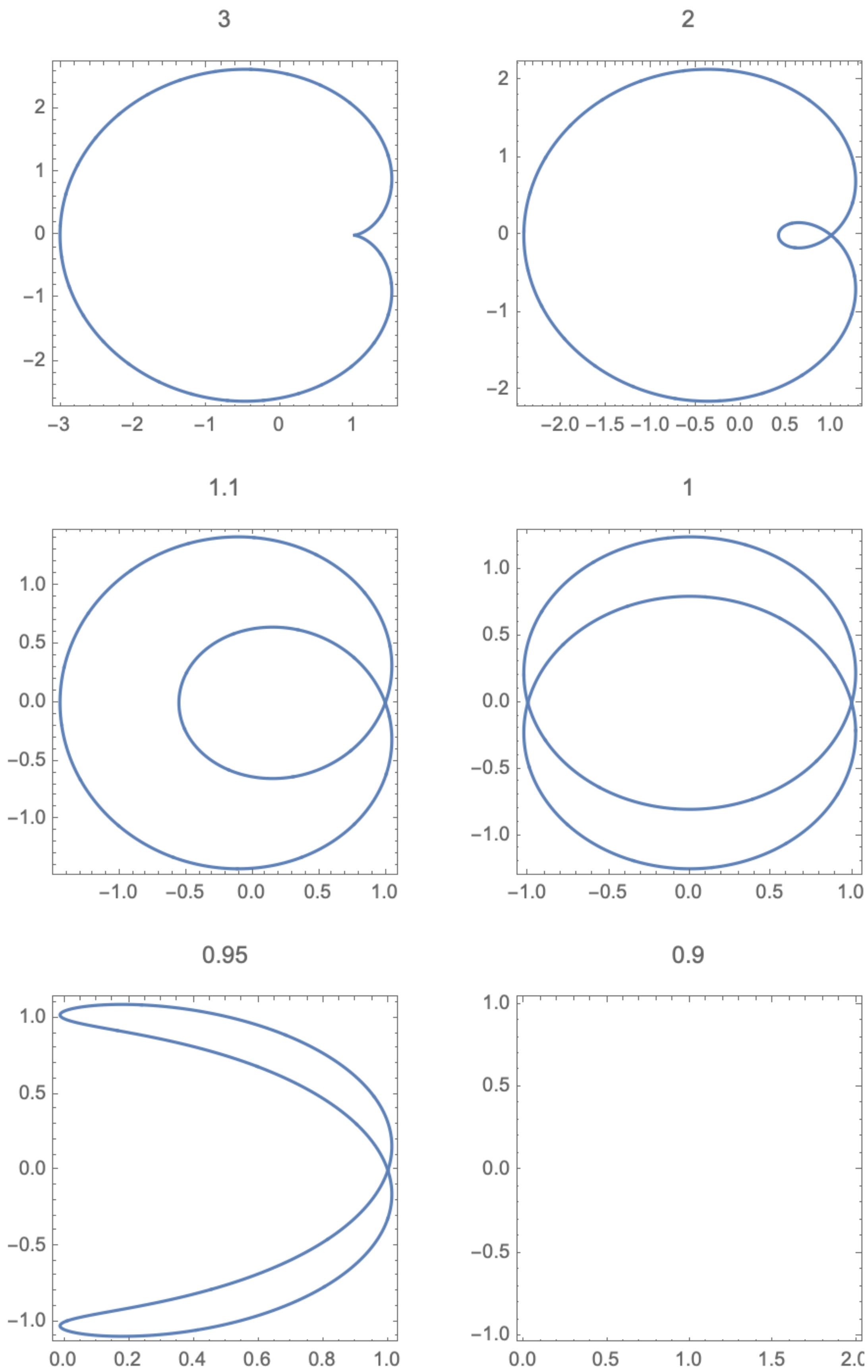
(b) For $W_0 = 2$ we have two solitons up to time reversal. The first soliton solution is nearly identical to the cartioid solution we had for $W_0 = 3$. However, the second solution stays much closer to the equilibrium solution, and the angle of $\mathbf{B}$ never makes a full rotation. We note that as W_0 decreases, so too does the amplitude of $\mathbf{B}$.

(c) For $W_0 = 1.1$ we have two soliton solutions. One of them is a cartioid, and the other is similar to the one for $W_0 = 2$, except this time it loops around the origin, meaning that $\mathbf{B}$ makes a full revolution around the $\boldsymbol{e}_1$ axis.

(d) For $W_0 = 1$ we see two solutions which appear elliptical. Up till now the two soliton solutions we have encountered have been symmetric about the V axis. We now see solutions which favor positive or negative values of V.

(e) For $W_0 = 0.95$ we have again two solitons. The anti-symmetric phenomena encountered in the previous W_0 value is even more extreme, and V never changes sign through the duration of the soliton travels.

(f) There is no solution for $W_0 = 0.9$, so sayeth Mathematica.


3
2
1.1
1
0.95
0.9
−3 −2 −1 0 1
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−2.0 −1.5 −1.0 −0.5 0.0 0.5 1.0
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0.0 0.5 1.0 1.5 2.0

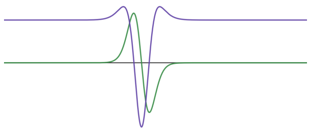

Figure 1: Drawing of the soliton solution for $W_0 = 3$.

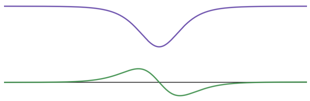

Figure 2: Drawing of one of the soliton solutions for $W_0 = 2$.

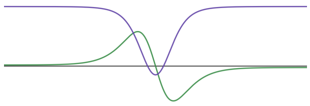

Figure 3: Drawing of one of the soliton solutions for $W_0 = 1.1$.

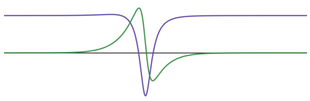

Figure 4: Drawing of one of the soliton solutions for $W_0 = 1$.

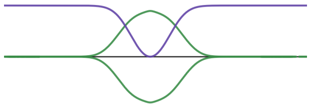

Figure 5: Drawing of both soliton solutions for $W_0 = 0.95$.

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2. Show that the canonical Poisson bracket

$$\{f,g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right)$$

satisfies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

Solution: To make my life easier, we will employ the Einstein summation convention and introduce the notation $f_i = \frac{\partial f}{\partial q_i}$ and $f^i = \frac{\partial f}{\partial p_i}$, so that the canonical Poisson bracket can be written as $\{f,g\} = f_i g^i - f^i g_i$.

To find $\{\{f,g\},h\}$ we will first calculate the partial derivatives of $\{f,g\}$.

(a) $\{f,g\}_j = (f_i g^i - f^i g_i)_j = f_{ij} g^i + f_i g^i_j - f^i_j g_i - f^i g_{ij}$

(b) $\{f,g\}^j = (f_i g^i - f^i g_i)^j = f^j_i g^i + f_i g^{ij} - f^{ij} g_i - f^i g^j_i$

With these expressions, we find that $\{\{f,g\},h\}$ can be written as

$$\{\{f,g\},h\} = \{f,g\}_j h^j - \{f,g\}^j h_j = (f_{ij}g^i + f_i g^i_j - f^i_j g_i - f^i g_{ij})h^j - (f^j_i g^i + f_i g^{ij} - f^{ij} g_i - f^i g^j_i)h_j$$

The Jacobi identity means that cyclically permuting $(f\ g\ h) \to (h\ f\ g) \to (g\ h\ f)$ should result in terms which cancel out all of the above terms. Let us first focus on the terms where either f or g has two lower indices. We have

$$f_{ij}g^i h^j - f^i g_{ij} h^j + h_{ij} f^i g^j - h^i f_{ij} g^j + g_{ij} h^i f^j - g^i h_{ij} f^j = 0$$

We see that each term in this expression cancels out the third term to the right (or left) of it. It follows from the symmetry of the equation that any term where f or g has two upper indices will similarly cancel out when cyclically permuted. We are left with terms in which either f or g have both an upper and lower index. Writing out these terms and their permutations explicitly, we have

$$\begin{aligned} & f_i g^i_j h^j - f^i_j g_i h^j - f^j_i g^i h_j + f^i g^j_i h_j \\ + & h_i f^i_j g^j - h^i_j f_i g^j - h^j_i f^i g_j + h^i f^j_i g_j \\ + & g_i h^i_j f^j - g^i_j h_i f^j - g^j_i h^i f_j + g^i h^j_i f_j = 0 \end{aligned}$$

We see that the positive terms of each permutation cancel out the negative terms of the previous permutation, and so their sum must be zero.

It follows that the canonical Poisson bracket does indeed satisfy the Jacobi identity.

3. Show that the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

is Hamiltonian with canonical Poisson structure and Hamiltonian

$$H = \int \left(\frac{1}{2}p^2 + \frac{1}{2}q_x^2 + 1 - \cos(q)\right) dx,$$

where $q = u$, and $p = u_t$.

Solution: The Sine-Gordon equation is Hamiltonian with canonical Poisson structure if its dynamics are given by $\dot{q} = \frac{\delta H}{\delta p}$ and $\dot{p} = -\frac{\delta H}{\delta q}$. We can verify this explicitly by calculating the variational derivatives of H with respect to p and q. We have

$$\begin{aligned}
\frac{\delta H}{\delta p} &= \frac{\partial H}{\partial p} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p_x}\right) = p = q_t \\
\frac{\delta H}{\delta q} &= \frac{\partial H}{\partial q} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial q_x}\right) = \sin q - q_{xx} = -p_t
\end{aligned}$$

Hence, the Sine-Gordon equation is indeed Hamiltonian with canonical Poisson structure and the above Hamiltonian.

4. Check explicitly that the conserved quantities $F_{-1} = \int u dx$, $F_0 = \int \frac{1}{2}u^2 dx$, $F_1 = \int \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right) dx$, $F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2\right) dx$ are mutually in involution with respect to the Poisson bracket defined by the Poisson structure given by ∂_x.

Solution: We begin by calculating the variational derivatives of each of these conserved quantities. We have

$$\begin{aligned}
\frac{\delta F_{-1}}{\delta u} &= \frac{\partial f_{-1}}{\partial u} = 1 \\
\frac{\delta F_0}{\delta u} &= \frac{\partial f_0}{\partial u} = u \\
\frac{\delta F_1}{\delta u} &= \frac{\partial f_1}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f_1}{\partial u_x} = \frac{1}{2}u^2 + u_{xx} \\
\frac{\delta F_2}{\delta u} &= \frac{\partial f_2}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f_2}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f_2}{\partial u_{xx}} = \frac{1}{6}u^3 + \frac{1}{2}u_x^2 + uu_{xx} + \frac{3}{5}u_{4x}
\end{aligned}$$

Note that since $\frac{\delta F_{-1}}{\delta u}$, F_{-1} is a Casimir function and is trivially in involution with the other quantities.

For the remaining conserved quantities, we can calculate their Poisson brackets now.

(a) For $\{F_1, F_0\}$ we have:

$$\{F_1, F_0\} = \int (\frac{1}{2}u^2 + u_{xx})u_x dx = \int \left(\frac{1}{2}u^2 u_x + u_{xx}u_x\right) dx = \frac{1}{6}u^3 + \frac{1}{2}u_x^2\Big|_{-\infty}^{\infty} = 0$$

So $\{F_1, F_0\} = -\{F_0, F_1\} = 0$, hence F_0 and F_1 are in involution.

(b) For $\{F_2, F_0\}$ we have:

$$\begin{aligned}\{F_2, F_0\} &= \int \left(\frac{1}{6}u^3 + \frac{1}{2}u_x^2 + uu_{xx} + \frac{3}{5}u_{4x}\right) u_x dx \\ &= \int \left(\frac{1}{6}u_x u^3 + \frac{1}{2}u_x^3 + uu_x u_{xx} + \frac{3}{5}u_x u_{4x}\right) dx\end{aligned}$$

We note that $u_x u^3 = \frac{1}{2}(u^4)_x$, and so this term will integrate to zero due to the boundary conditions. Likewise, by applying our boundary conditions at infinity we can use integration by parts to write $\int uu_x u_{xx} dx = -\int \frac{1}{2}u_x^3 dx$, hence the second and third terms cancel each other out. Lastly, using integration by parts twice we find $\int u_x u_{4x} dx = -\int uu_{5x} dx = \int u_{6x} dx$, and so the final term will also evaluate to zero. Thus we find that

$$\{F_2, F_0\} = -\{F_0, F_2\} = 0$$

And so F_0 and F_2 are in involution.

(c) Lastly, for $\{F_2, F_1\}$ we have:

$$\begin{aligned}\{F_2, F_1\} &= \int \left(\frac{1}{6}u^3 + \frac{1}{2}u_x^2 + uu_{xx} + \frac{3}{5}u_{4x}\right)(uu_x + u_{3x})dx \\ &= \int \left(\frac{1}{6}u^4 u_x + \frac{1}{2}uu_x^3 + u^2 u_x u_{xx} + \frac{3}{5}uu_x u_{4x}\right. \\ &\quad \left. + \frac{1}{6}u^3 u_{3x} + \frac{1}{2}u_x^2 u_{3x} + uu_{xx}u_{3x} + \frac{3}{5}u_{3x}u_{4x}\right) dx\end{aligned}$$

We note that $u^4 u_x = (u^5)_x$, and so this term can be eliminated on account of our boundary conditions. We also see that $u_{3x}u_{4x} = (\frac{1}{2}u_{3x}^2)_x$, so this term can likewise be eliminated. This brings us down to

$$\int \left(\frac{1}{2}uu_x^3 + u^2 u_x u_{xx} + \frac{3}{5}uu_x u_{4x} + \frac{1}{6}u^3 u_{3x} + \frac{1}{2}u_x^2 u_{3x} + uu_{xx}u_{3x}\right) dx$$

Next we note that $\frac{1}{2}uu_x^3 + u^2 u_x u_{xx} + \frac{1}{6}u^3 u_{3x} = (\frac{1}{4}u^2 u_x^2 + \frac{1}{6}u^3 u_{xx})_x$, so these terms can be eliminated, which leaves us with

$$\int \left(\frac{3}{5}uu_x u_{4x} + \frac{1}{2}u_x^2 u_{3x} + uu_{xx}u_{3x}\right) dx$$

We now note that

$$\begin{aligned}
(uu_{xx}^2)_x &= u_x u_{xx}^2 + 2uu_{xx}u_{3x} \\
(uu_x u_{3x})_x &= u_x^2 u_{3x} + uu_{xx}u_{3x} + uu_x u_{4x} \\
(u_x^2 u_{xx})_x &= 2u_x u_{xx}^2 + u_x^2 u_{3x}
\end{aligned}$$

We find that for a general linear expression of these terms we have

$$(Auu_{xx}^2+Buu_xu_{3x}+Cu_x^2u_{xx})_x = (A+2C)u_xu_{xx}^2+(2A+B)uu_{xx}u_{3x}+(B+C)u_x^2u_{3x}+Buu_xu_{4x}$$

Comparing with our remaining terms, we see that

$$A = -2C \qquad 2A + B = 1 \qquad B + C = \frac{1}{2} \qquad B = \frac{3}{5}$$

We see that $A = \frac{1}{5}$, $B = \frac{3}{5}$, and $C = \frac{-1}{10}$. Hence,

$$(\frac{1}{5}uu_{xx}^2 + \frac{3}{5}uu_xu_{3x} - \frac{1}{10}u_x^2u_{xx})_x = \frac{3}{5}uu_xu_{4x} + \frac{1}{2}u_x^2u_{3x} + uu_{xx}u_{3x}$$

And so it follows that

$$\{F_2, F_1\} = -\{F_1, F_2\} = 0$$

and F_1 and F_2 are in involution.

We have shown that F_{-1}, F_0, F_1, and F_2 are all mutually in involution with respect to the Poisson bracket defined by the Poisson structure $B = \partial_x$.

5. Find the fourth conserved quantity for the KdV equation $u_t = uu_x + u_{xxx}$, *i.e.*, the conserved quantity which contains $\frac{1}{24}\int u^4 dx$.

Solution: Only quantities with even weight can be conserved in KdV, so for the fourth conserved quantity F_2 we seek terms with weight $[f] = 8$. The possible non-x-derivative terms which are independent with respect to differentiation are u^4, uu_x^2, and u_{xx}^2, so we begin with

$$f = a_1u^4 + a_2uu_x^2 + a_3u_{xx}^2$$

For f to be conserved, there must be a conservation law of the form $f_t + g_x = 0$, which implies that the variational derivative $\frac{\delta F_2}{\delta u} = 0$. We will use this fact to solve for a_1, a_2, and a_3. See Mathematica notebook for detailed derivation.

Our final result for the variational derivative comes out to be

$$\begin{aligned}
\frac{\delta F_2}{\delta u} &= 12a_1(u_x^3 + 3uu_xu_{xx}) \\
&+ a_2(u_x^3 + 3uu_xu_{xx} - 6u_{xx}u_{3x} - 3u_xu_{4x}) \\
&- 5a_3(2u_{xx}u_{3x} + u_xu_{4x})
\end{aligned}$$

Comparing terms, we find that $\frac{\delta F_2}{\delta u} = 0$ when $a_1 = -\frac{1}{12}a_2$ and $a_2 = -\frac{5}{3}a_3$. If we let $a_3 = \frac{3}{10}$, then we have

$$F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2 \right) dx$$

6. **Recursion operator** For a Bi-Hamiltonian system with two Poisson structures given by B_0, B_1, one defines a recursion operator $R = B_1 B_0^{-1}$, which takes one element of the hierarchy of equations to the next element. For the KdV equation with $B_0 = \partial_x$ and $B_1 = \partial_{xxx} + \frac{1}{3}(u\partial_x + \partial_x u)$, we get $B_0^{-1} = \partial_x^{-1}$, integration with respect to x. Write down the recursion operator. Apply it to u_x (the zero-th KdV flow) to obtain the first KdV flow. Now apply it to $uu_x + u_{xxx}$ to get (up to rescaling of t_2) the second KdV equation. What is the third KdV equation?

Solution: By the definition of the recursion operator, we have

$$R = (\partial_{xxx} + \frac{1}{3}(u\partial_x + \partial_x u))\partial_x^{-1} = \partial_{xx} + \frac{1}{3}(u + \partial_x u \partial_x^{-1})$$

Applying this operator to the zero-th order KdV flow u_x gives us

$$Ru_x = (\partial_{xx} + \frac{1}{3}(u + \partial_x u \partial_x^{-1}))u_x = u_{xxx} + \frac{1}{3}(uu_x + \partial_x u^2) = u_{xxx} + \frac{1}{3}uu_x + \frac{2}{3}uu_x$$
$$\Rightarrow Ru_x = u_{xxx} + uu_x$$

Which is the first KdV flow, as expected. Applying the recursion operator again to this result, we find

$$\begin{aligned}
R(uu_x + u_{xxx}) &= (\partial_{xx} + \frac{1}{3}(u + \partial_x u \partial_x^{-1}))(uu_x + u_{xxx}) \\
&= \partial_{xx}(uu_x) + \frac{1}{3}u^2u_x + \frac{1}{3}\partial x u \partial_x^{-1}(uu_x) + u_{5x} + \frac{1}{3}uu_{3x} + \frac{1}{3}\partial_x(uu_{xx}) \\
&= 3u_xu_{xx} + uu_{3x} + \frac{1}{3}u^2u_x + \frac{1}{2}u^2u_x + u_{5x} + \frac{1}{3}uu_{3x} + \frac{1}{3}u_xu_{xx} + \frac{1}{3}uu_{3x} \\
&= \frac{10}{3}u_xu_{xx} + \frac{5}{3}uu_{3x} + \frac{5}{6}u^2u_x + u_{5x}
\end{aligned}$$

Comparing this with the second KdV equation $u_{t_2} = \frac{1}{2}u^2u_x + 2u_xu_{xx} + uu_{3x} + \frac{3}{5}u_{5x}$, we see that the expression we found is $\frac{5}{3}u_{t_2}$. Since rescaling t_2 enables us to freely scale the right-hand side of our equation by a constant coefficient, it follows that these two equations are equivalent up to rescaling of t_2.

Continuing this process, we may find the third equation by the same method

$$R(\frac{1}{2}u^2u_x + 2u_xu_{xx} + uu_{3x} + \frac{3}{5}u_{5x}) = (\partial_{xx} + \frac{1}{3}u + \frac{1}{3}\partial_x u \partial_x^{-1})(\frac{1}{2}u^2u_x + 2u_xu_{xx} + uu_{3x} + \frac{3}{5}u_{5x})$$

For the second derivative operator ∂_{xx} we find

(a) $(u^2u_x)_{xx} = (2uu_x^2 + u^2u_{xx})_x = 2u_x^3 + 6uu_xu_{xx} + u^2u_{3x}$

(b) $(u_xu_{xx})_{xx} = (u_{xx}^2 + u_xu_{3x})_x = 3u_{xx}u_{3x} + u_xu_{4x}$

(c) $(uu_{3x})_{xx} = (u_xu_{3x} + uu_{4x})_x = u_{xx}u_{3x} + 2u_xu_{4x} + uu_{5x}$

(d) $(u_{5x})_{xx} = u_{7x}$

Meanwhile, for the antiderivative operator $\partial_x u \partial_x^{-1}$ we find

(a) $\partial_x u \partial_x^{-1}(u^2u_x) = \frac{1}{3}(u^4)_x = \frac{4}{3}u^3u_x$

(b) $\partial_x u \partial_x^{-1}(u_xu_{xx}) = \frac{1}{2}(uu_x^2)_x = \frac{1}{2}u_x^3 + uu_xu_{xx}$

(c) $\partial_x u \partial_x^{-1}(uu_{3x}) = -\partial_x u \partial_x^{-1}(u_xu_{xx}) = -\frac{1}{2}(uu_x^2)_x = -\frac{1}{2}u_x^3 - uu_xu_{xx}$

(d) $\partial_x u \partial_x^{-1}(u_{5x}) = (uu_{4x})_x = u_xu_{4x} + uu_{5x}$

Note that for (c) above we used integration by parts and the boundary conditions $u_x, u_{3x} \to 0$ at $x \to \pm\infty$ to evaluate the antiderivative, hence the minus sign.

The effect of the u operator is straightforward, so we can now plug in our terms to find

$$
\begin{aligned}
Ru_{t_2} = {} & u_x^3 + 3uu_xu_{xx} + \frac{1}{2}u^2u_{3x} + 6u_{xx}u_{3x} + 2u_xu_{4x} + u_{xx}u_{3x} + 2u_xu_{4x} + uu_{5x} + \frac{3}{5}u_{7x} \\
& + \frac{1}{6}u^3u_x + \frac{2}{3}uu_xu_{xx} + \frac{1}{3}u^2u_{3x} + \frac{1}{5}uu_{5x} \\
& + \frac{2}{9}u^3u_x + \frac{1}{3}u_x^2 + \frac{2}{3}uu_xu_{xx} - \frac{1}{6}u_x^3 - \frac{1}{3}uu_xu_{xx} + \frac{1}{5}uu_{5x}
\end{aligned}
$$

$$
\Rightarrow u_{t_3} = Ru_{t_2} = \frac{1}{3}u_x^2 + \frac{5}{6}u_x^3 + 4uu_xu_{xx} + \frac{5}{6}u^2u_{3x} + 7u_{xx}u_{3x} + u_x(\frac{7}{18}u^3 + 4u_{4x}) + \frac{7}{5}uu_{5x} + \frac{3}{5}u_{7x}
$$

7. Consider the function $U(x) = 2\partial_x^2 \ln\left(1 + e^{kx+\alpha}\right)$. Show that for a suitable k, $U(x)$ is a solution of the first member of the stationary KdV hierarchy (as you've already seen, it is the one-soliton solution):

$$
6uu_x + u_{xxx} + c_0u_x = 0.
$$

(Note: it may be easier to define c_0 in terms of k, instead of the other way around)

Having accomplished this, let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots)}$. Determine the dependence of α on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations:

$$
\begin{aligned}
u_{t_1} &= 6uu_x + u_{xxx}, \\
u_{t_2} &= 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x}, \\
u_{t_3} &= 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u_{xx}u_{xxx} + 70u^2u_{xxx} + 42u_xu_{xxxx} + 14uu_{5x} + u_{7x}.
\end{aligned}
$$

Based on this, write down a guess for the one-soliton solution that solves the entire KdV hierarchy.

Solution: Using Mathematica, we find that $U(x)$ can be simplified to the standard sech^2 form as

$$\frac{1}{2}k^2\text{sech}^2\left(\frac{1}{2}(\alpha+kx)\right)$$

Plugging this in to the first member of the stationary KdV hierarchy, we find

$$\begin{aligned} 6UU_x + U_{xxx} + c_0 U_x = k\left(c_0+k^2\right)\text{sech}\left(\frac{1}{2}(\alpha+kx)\right) = 0 \\ \Rightarrow c_0 = -k^2 \end{aligned}$$

We now let $u(x,t_1,t_2,t_3,\ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots)}$. Plugging this into each of the above equations, we find that the first, second and third KdV equations can be reduced to

$$\begin{aligned} k\left(k^3-\alpha_{t_1}\right)\text{sech}\left(\frac{1}{2}(\alpha+kx)\right) &= 0 \Rightarrow \alpha_{t_1} = k^3 \\ k\left(k^5-\alpha_{t_2}\right)\text{sech}\left(\frac{1}{2}(\alpha+kx)\right) &= 0 \Rightarrow \alpha_{t_2} = k^5 \\ k\left(k^7-\alpha_{t_3}\right)\text{sech}\left(\frac{1}{2}(\alpha+kx)\right) &= 0 \Rightarrow \alpha_{t_1} = k^7 \end{aligned}$$

Based off of this pattern, we might guess the following one-soliton solution for the entire KdV hierarchy:

$$\begin{aligned} u(x,t_1,t_2,t_3,\ldots) &= \frac{1}{2}k^2\text{sech}^2\left(\frac{1}{2}(\alpha(t_1,t_2,t_3,\ldots)+kx)\right) \\ \alpha(t_1,t_2,t_3,\ldots) &= t_1k^3+t_2k^5+t_3k^7+\cdots = \sum_{n=1}^{\infty} t_n k^{2n+1} \end{aligned}$$

8. **Warning: maple/mathematica-intensive.** Consider the function

$$U(x) = 2\partial_x^2 \ln\left(1+e^{k_1x+\alpha}+e^{k2_x+\beta}+\left(\frac{k_1-k_2}{k_1+k_2}\right)^2 e^{k_1x+k_2x+\alpha+\beta}\right).$$

Show that for a suitable k_1, k_2, $U(x)$ is a solution of the second member of the stationary KdV hierarchy:

$$30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x+u_{xxx}) + c_0u_x = 0.$$

(Note: it may be easier to define c_1, c_0 in terms of k_1 and k_2 instead of the other way around)

Having accomplished this, let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots),\beta=\beta(t_1,t_2,t_3,\ldots)}$. Determine the dependence of α and β on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations, given above.

Based on this, write down a guess for the two-soliton solution of the entire KdV hierarchy.

Solution: —