Approach 2: Add a damping term εu_t, keep the original boundary condition of $u(x,t) \to 0$ as $x \to \pm\infty$. Later take $\varepsilon \to 0$.

$$u_{tt} + \varepsilon u_t - c^2 u_{xx} = q(x) e^{-i\omega_0 t}$$

$\varepsilon \to 0$

When $\varepsilon > 0$, no matter how small, the travelling waves are damped. At large distances from the source (at finite x), the waves that get there have travelled long distances and are the waves that were generated a long time ago. Any damping, no matter how small, will have a significant finite effect on the wave amplitude.

Still let $u(x,t) = u(x) e^{-i\omega_0 t}$

$$\frac{d^2}{dx^2} u + (k_0^2 + i\varepsilon k_0/c) u = -\frac{q(x)}{c^2}$$

$u(x) \to 0$ as $x \to \pm\infty$.

Since $u(x)$ is now integrable, we can define a Fourier transform in x:

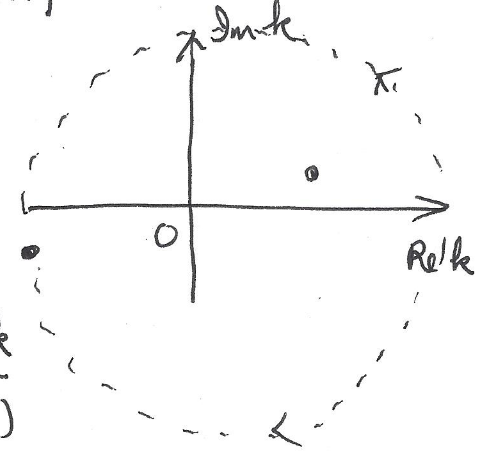
$$U(k) = \mathcal{F}[u(x)] = \int_{-\infty}^{\infty} e^{ikx} u(x)\, dx. \quad Q(k) = \mathcal{F}[q(x)].$$

Applying the boundary condition $u(x) \to 0$ as $x \to \pm\infty$ and assuming $u_x(x) \to 0$ as $x \to \pm\infty$ (to be verified later),

$$[-k^2 + (k_0^2 + i\varepsilon k_0/c)]\, U(k) = -\frac{1}{c^2} Q(k)$$

$$U(k) = \frac{Q(k)/c^2}{[k^2 - (k_0^2 + i\varepsilon k_0/c)]}$$

Let us do the problem first with $q(x) = \delta(x-y)$. Afterwards we can use the solution to reconstruct the solution for a general (localized) $q(x)$. $Q(k) = e^{iky}$



The inverse Fourier transform is:

$$u_0(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k)\, e^{-ikx}\, dk$$

$$= \frac{1}{2\pi c^2}\int_{-\infty}^{\infty} \frac{e^{-ik(x-y)}}{k^2 - (k_0^2 + i\varepsilon k_0/c)}\, dk$$

This integral is well defined for $\varepsilon \neq 0$. Without ε there would have been two singularities, $k = \pm k_0$, along the path of integration. Then the integral would have been <u>undefined</u>. For small ε, the singularity now is located at

$$k = \pm k_0 \sqrt{1 + i\varepsilon/\omega_0} \simeq \pm k_0 (1 + \tfrac{1}{2} i\varepsilon/\omega_0)$$

The real integral can be converted into a complex contour integral by closing in the upper half plane if $x - y < 0$ (using Jordan's Lemma), and in the lower half plane if $x - y > 0$. The Residue Thm then yields

$$u_0(x,y) = \frac{i}{2k_0 c^2}\, \frac{\exp[\, i k_0 |x-y| \sqrt{1 + i\varepsilon/\omega_0}\,]}{\sqrt{1 + i\varepsilon/\omega_0}}$$

as $\varepsilon \to 0$ $\quad \rightarrow \dfrac{i}{2k_0 c^2} e^{ik_0|x-y|}$

The full solution is obtained by superposition

$$u(x) = \int_{-\infty}^{\infty} q(y)\, u_0(x,y)\, dy = \frac{i}{2k_0 c^2}\int_{-\infty}^{\infty} q(y)\, e^{ik_0|x-y|}\, dy.$$

One could alternatively do the problem

$$u_{xx} + (k_0^2 + i\varepsilon k_0/c)u = -\frac{q(x)}{c^2}$$

using variation of paramaters and impose only the boundary condition $u \to 0$ as $x \to \pm\infty$. The method will automatically pick the Sommerfeld's radiating waves. There is no need to impose a separate radiation boundary condition.

Let us examine $u_0(x,y)$ in more detail.

$$u_0(x,y,\varepsilon) = \frac{i}{2k_0c^2}\frac{\exp[ik_0|x-y|\sqrt{1+i\varepsilon/\omega_0}\,]}{\sqrt{1+i\varepsilon/\omega_0}}$$

for small ε $\quad \simeq \frac{i}{2k_0c^2}\frac{1}{(1+\frac{1}{2}i\varepsilon/\omega_0)}\exp[ik_0|x-y| - \frac{1}{2}\frac{\varepsilon}{c}|x-y|]$

For fixed $\varepsilon > 0$, no matter how small

$$u_0(x,y,\varepsilon) \to 0 \quad \text{as } |x-y| \to \infty$$

(which is the same as $|x| \to \infty$, since y is finite and fixed)

It automatically got rid of the solution

$$\exp[-ik|x-y|\sqrt{1+i\varepsilon/\omega_0}\,]$$

which grows with $|x|$.

Details for approach 2

$$u_{tt} + \varepsilon u_t - c^2 u_{xx} = q(x) e^{-i\omega_0 t}$$

$$\varepsilon \to 0^+$$

How do we know adding εu_t provides damping?

$$u_{tt} + \varepsilon u_t \sim 0 \quad \rightarrow \quad u_t \sim e^{-\varepsilon t}$$

$$\cancel{\frac{\partial}{\partial t}\left(u_t e^{\varepsilon t}\right)}$$

still keep the BC $u(x,t) \to 0$ as $x \to \pm\infty$

Let $u(x,t) = u(x) e^{-i\omega_0 t}$, $\omega_0 = k_0 c$

$$\frac{d^2}{dx^2} u + (k_0^2 + i\varepsilon k_0/c) u = -\frac{q(x)}{c^2}$$

$$u(x) \to 0 \text{ as } x \to \pm\infty$$

$u(x)$ is integrable; use Fourier transform.

Let $U(k) = \mathcal{F}[u(x)] = \int_{-\infty}^{\infty} e^{ikx} u(x)\,dx$

$Q(k) = \mathcal{F}[q(x)]$

$$\mathcal{F}\left[\frac{d^2}{dx^2}u + (k_0^2 + i\varepsilon k_0/c)u = -\mathcal{F}\left[\frac{q(x)}{c^2}\right]\right.$$

$$[-k^2 + (k_0^2 + i\varepsilon k_0/c]\,U(k) = -\frac{1}{c^2}Q(k)$$

after applying the bc $u \to 0$ as $x \to \pm\infty$

and assuming $u_x \to 0$ as $x \to \pm\infty$

$$U(k) = \frac{\frac{1}{c^2}Q(k)}{[k^2 - (k_0^2 + i\varepsilon k_0/c)]}$$

Do $q(x) = \delta(x-y)$ first. Denote $U(k)$ as $U_0(k)$

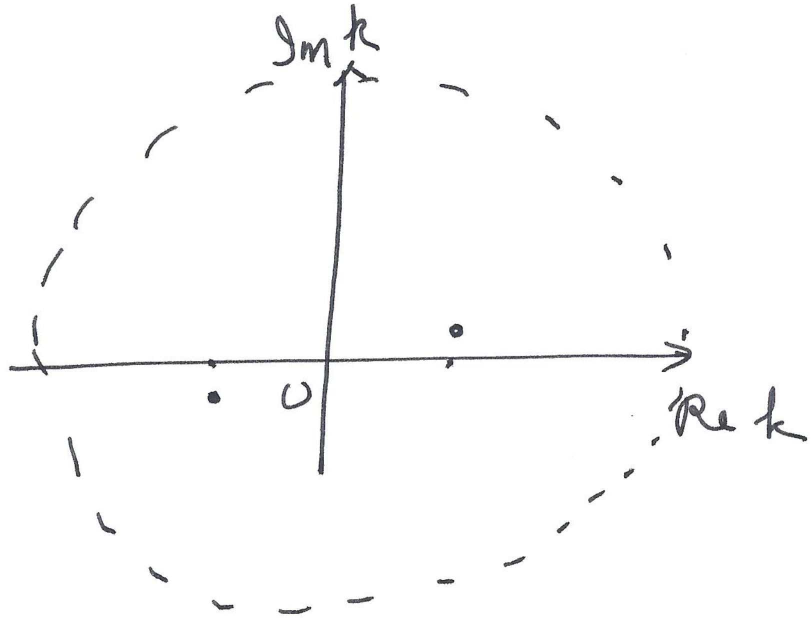
$$Q(k) = \mathcal{F}[q(x)] = \mathcal{F}[\delta(x-y)] = e^{iky}$$

$$U_0(k) = \frac{\frac{1}{c^2}e^{iky}}{[k^2 - (k_0^2 + i\varepsilon k_0/c)]}$$

Let

$$u_0(x,y) = \mathcal{F}^{-1}[\,U_0(k)\,]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ik(x-y)}\,dk}{k^2 - (k_0^2 + i\varepsilon k_0/c)}$$

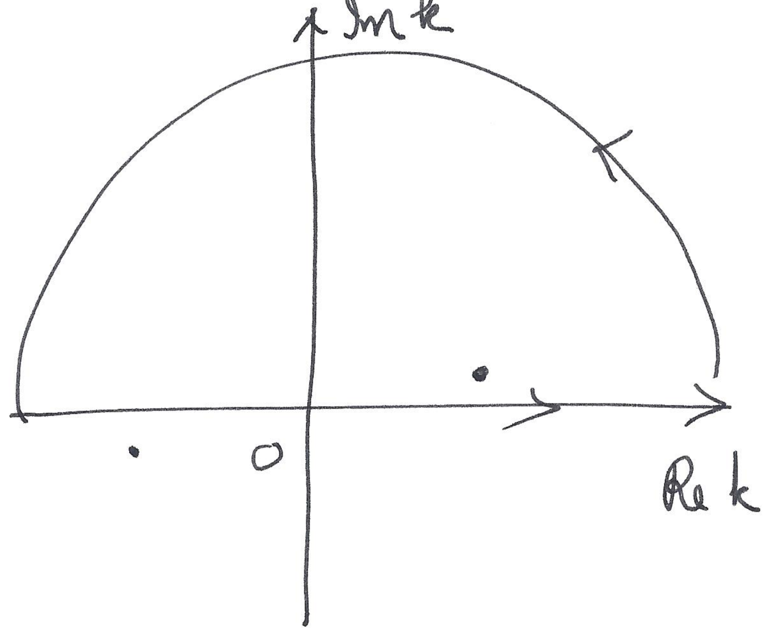


Integral well-defined. No singularity on the path of integration.

For small ε, the singularities are located at

$$k = \pm k_0\sqrt{1 + i\varepsilon/\omega_0}$$

$$\simeq \pm k_0\left(1 + \tfrac{1}{2}\, i\varepsilon/\omega_0\right)$$



If $x-y<0$, close in the upper half plane by Jordan's lemma

$$u_0(x,y) = \frac{2\pi i}{2\pi c^2} \underset{\text{in the UHP}}{\text{Res of}} \frac{e^{-ik(x-y)}}{k^2-(k_0^2+i\varepsilon k_0/c)}$$

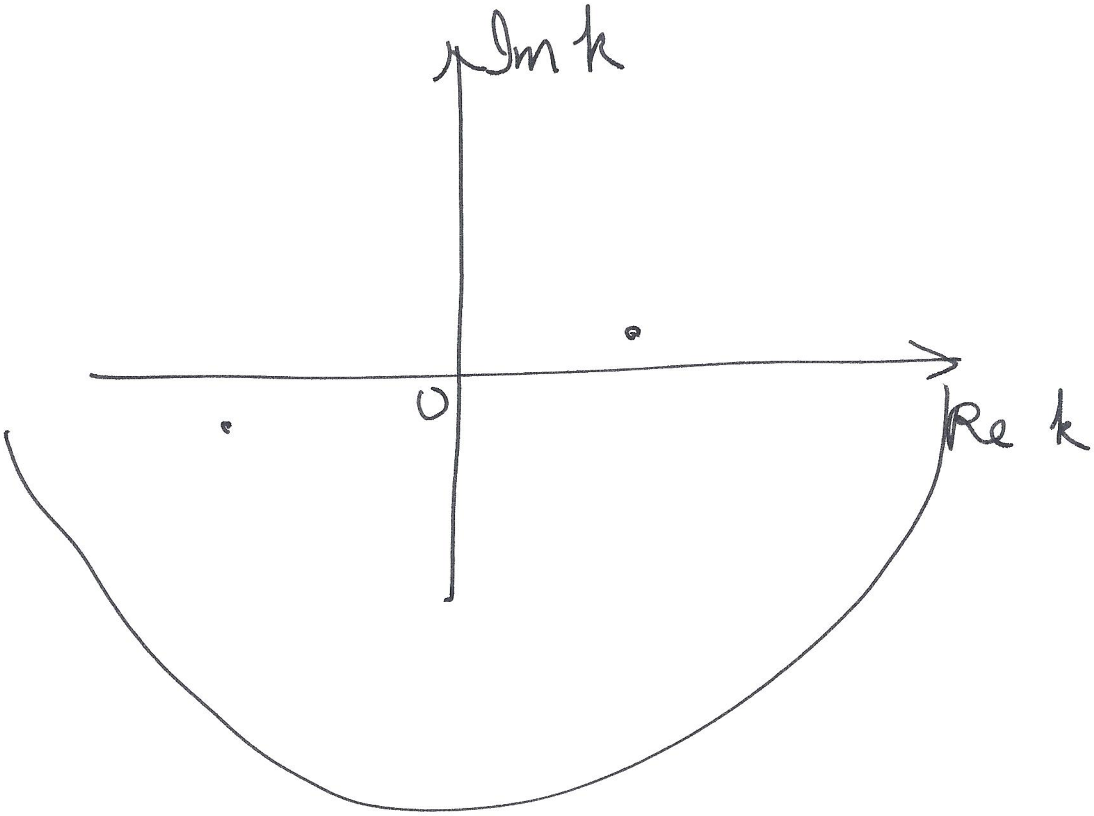
that is, at

$$k = k_0\sqrt{1+i\varepsilon/\omega_0}$$

$$u_0(x,y) = \frac{i}{c^2} \frac{e^{-ik_0(x-y)\sqrt{1+i\varepsilon/\omega_0}}}{2k_0\sqrt{1+i\varepsilon/\omega_0}}$$

$$= \frac{i}{2k_0c^2} \frac{\exp\{ik_0|x-y|\sqrt{1+i\varepsilon/\omega_0}\}}{\sqrt{1+i\varepsilon/\omega_0}}$$

If $x - y > 0$, close in the lower half plane by Jordan's lemma:



$$u_0(x,y) = \frac{1}{2\pi c^2} \int$$

$$= -\frac{2\pi i}{2\pi c^2} \text{ Res of } \frac{e^{-ik(x-y)}}{k^2 - (k_0^2 + i\varepsilon k_0/c}$$

$$\text{at } k = -k_0\sqrt{1+i\varepsilon/\omega_0}$$

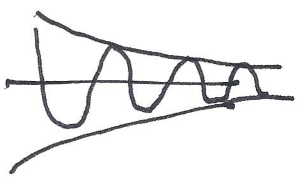
$$= \frac{i}{2k_0 c^2} \frac{\exp\{ik|x-y|\sqrt{1+i\varepsilon/\omega_0}\}}{\sqrt{1+i\varepsilon/\omega_0}}$$

Examine $u_0(x,y)$ in more detail:

$\varepsilon \neq 0$

$$u_0(x,y) = \frac{i}{2k_0c^2}\,\frac{\exp\left[ik|x-y|\sqrt{1+i\varepsilon/\omega_0}\,\right]}{\sqrt{1+i\varepsilon/\omega_0}}$$

$$\simeq \frac{i}{2k_0c^2}\,\frac{1}{(1+\frac{1}{2}i\varepsilon/\omega_0)}$$



$$\times \exp\left\{ik_0|x-y| - \frac{1}{2}\frac{\varepsilon}{c}|x-y|\right\}$$

for small ε

For fixed $\varepsilon > 0$, no matter how small

$$u_0(x,y) \to 0 \quad \text{as } |x-y| \to \infty$$

$(y \text{ fixed}, x \to \pm\infty)$

It satisfies the bc $u \to 0$ as $x \to \pm\infty$.

As $\varepsilon = 0$, $\quad u_0(x,y) \to \exp\{ik_0|x-y|\}$

$$\to \begin{cases} e^{ik_0x} & \text{as } x \to \infty \\ e^{-ik_0x} & \text{as } x \to -\infty \end{cases}$$

Now take the limit $\varepsilon \to 0^+$

$$u_0(x,y) = \frac{i}{2k_0 c^2} e^{ik_0|x-y|}$$

Supperposition for a general $q(x)$:

$$u(x) = \int_{-\infty}^{\infty} q(y)\, u_0(x,y)\, dy$$

$$= \frac{i}{2k_0 c^2} \int_{-\infty}^{\infty} q(y)\, e^{ik_0|x-y|}\, dy$$