Eigenfunction expansion

Example:

$$[(1-x^2)y']' + \mu y = 1$$

$$-1 < x < 1$$

μ given.

BC: $y(x)$ bounded at $x = \pm 1$

Express $y(x)$ in an eigenfunction expansion:
You can choose your basis function.

$$y(x) = \sum_{n=0}^{\infty} a_n P_n(x), \quad -1 < x < 1$$

$$[(1-x^2)y']' = \sum_{n=0}^{\infty} a_n \frac{d}{dx}[(1-x^2)\frac{d}{dx}P_n]$$

$$= \sum_{n=0}^{\infty} -a_n n(n+1) P_n(x)$$

since $\frac{d}{dx}[(1-x^2)\frac{d}{dx}P_n] + n(n+1)P_n = 0$

The boundary conditions on $y(x)$ is automatically satisfied since $P_n(x)$ satisfies the same BC.

$$\sum_{n=0}^{\infty} [\mu - n(n+1)] a_n P_n(x) = 1$$

Expand the forcing term in the same way

$$1 = \sum_{n=0}^{\infty} f_n P_n(x)$$

$$f_0 = 1, \quad f_n = 0, \quad n = 1, 2, 3, \cdots$$

because $P_0(x) = 1$.

$$(\mu - 0] a_0 P_0(x) = f_0 P_0(x)$$

$$(\mu - 1 \cdot 2) a_1 P_1(x) = 0$$

$$(\mu - 2 \cdot 3) a_2 P_2(x) = 0$$

$$\vdots$$

$$a_0 = \frac{1}{\mu}$$

$$a_n = 0 \quad n = 1, 2, 3, \cdots$$

if μ is not an integer

$$y(x) = \frac{1}{\mu}$$

If μ is $m(m+1)$, $m = 1, 2, 3, \ldots$

then there is a homogeneous solution

$$b_n P_m(x)$$

that should be added to the forced solution

$$y(x) = \frac{1}{\mu} + b_m P_m(x).$$

b_m is an arbitrary ~~condit~~ constant.

The special case of $\mu = 0$:

$$[(1-x^2)y']' = 1$$

Integrate $(1-x^2)y' = x + B$

$$y' = \frac{x}{1-x^2} + \frac{B}{1-x^2}$$

Integrate again:

$$y(x) = \ln[(1-x^2)^{-1/2}]$$

$$+ B \ln\left[\frac{(1+x)^{1/2}}{(1-x)^{1/2}}\right] + C$$

Cannot satisfy BC at both $x = \pm 1$.

This solution can be rule out.

A PDE example :

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx} + \underbrace{\sin(3\pi x)}_{f(x,t)}$$

$$0 < x < 1, \; t > 0$$

BCs: $u(0,t) = 0, \; u(1,t) = 0$

IC : $u(x,0) = \sin \pi x, \; 0 < x < 1.$

step 1: Find the eigenfunctions of the homogeneous PDE through separation of variables :

Drop the RHS :

$$u_t = \alpha^2 u_{xx}$$

Let $u(x,t) = T(t) X(x)$

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = \text{const} = -\lambda^2$$

$$X(x) = X_n(x) = \sin \lambda_n x, \quad X(0) = 0, \; X(1) = 0$$

$$\lambda = \lambda_n = n\pi, \quad n = 1, 2, 3, \ldots$$

Do not solve for $T(x)$ yet.

step 2 :

Eigenfunction expansion of the solution

$$u(x,t) = \sum_n T_n(t) X_n(x)$$

$$f(x,t) = \sum_n f_n(t) X_n(x)$$

For this example, $f_3 = 1$, $f_n = 0$, $n \neq 3$

step 3 :

Substitute into PDE :

$$\sum_n [T_n'(t) + \alpha^2 \lambda_n^2 T_n(t) - f_n(t)] X_n(x) = 0$$

Because $X_n(x)$'s are orthogonal,

$$[T_n'(t) + \alpha^2 \lambda_n^2 T_n(t) - f_n(t)] = 0$$

$$T_n(t) = T_n(0) e^{-\alpha^2 \lambda_n^2 t}$$

$$+ \int_0^t f_n(\tau) e^{-\alpha^2 \lambda_n^2 (t-\tau)} d\tau .$$

For this example $f_n = 0$ if $n \neq 3$.

$n \neq 3$:

$$T_n(t) = T_n(0)\, e^{-\alpha^2 n^2 \pi^2 t}.$$

$n = 3$:

$$T_3(t) = T_3(0)\, e^{-9\pi^2\alpha^2 t} + \frac{1}{(3\pi\alpha)^2}\left[1 - e^{-9\pi^2\alpha^2 t}\right]$$

To satisfy the initial condition, we expand:

$$f = \sin \pi x = \sum_{n=1}^{\infty} T_n(0) \sin n\pi x, \quad 0 < x < 1$$

$T_n(0) = 0$, except $T_n(0) = 1$ for $n = 1$.

$$T_1(t) = e^{-\alpha^2\pi^2 t}$$

$$T_2(t) = 0$$

$$T_3(t) = \frac{1}{(3\pi\alpha)^2}\left[1 - e^{-(3\pi\alpha)^2 t}\right]$$

$$T_4(t) = 0 \quad \cdots$$

$$u(x,t) = e^{-(\alpha\pi)^2 t} \sin \pi x + \frac{1}{(3\pi\alpha)^2} \times \left[1 - e^{-(3\pi\alpha)^2 t}\right] \sin 3\pi x.$$