Lecture 10

(1)

Laplace transform of $f(t)$ for $0<t<\infty$

$$\mathcal{L}[f(t)] \equiv \int_0^\infty f(t)\, e^{-st}\, dt \equiv \tilde{f}(s).$$

Functions that are zero for $t<0$ are called one-sided functions.

For one-sided $f(t)$, the Laplace transform is the same as the Fourier transform if we replace t by x and is by ω.

$$F(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^\infty f(x)\, e^{i\omega x}\, dx$$

$$= \int_0^\infty f(x)\, e^{-sx}\, dx \qquad i\omega = -s$$

$$= \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \tilde{f}(s)$$

$$\boxed{F(\underbrace{is}_{\omega}) = \tilde{f}(s)}$$

There is no "Laplace Integral Formula".
Theory of Laplace transform comes from Fourier transform.

Let
$$f(t) = \mathcal{L}^{-1}[\tilde{f}(s)]$$

From Fourier Integral formula

$$\begin{aligned} f(t) &= \mathcal{F}^{-1}[F(\omega)] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \tilde{f}(s)\, e^{st}\, ds \\ &= \mathcal{L}^{-1}[\tilde{f}(s)] \end{aligned}$$

$$\boxed{\begin{aligned} &-i\omega \\ &= s \end{aligned}}$$

The inverse transform involves integration in the complex plane.

The problem with <u>Fourier Integral Formula</u> is the requirement that

$f(x)$ be integrable in $-\infty < x < \infty$, which is very restrictive.

For example, e^{-x}, e^{x}, 1, x, x^2 etc are not integrable in $-\infty < x < \infty$.

We can relax this requirement in Laplace transform.

Suppose $f(t)$ is not integrable because $f(t) \to \infty$ as $t \to \infty$. Or even because $f(t) \not\to 0$ as $t \to \infty$.

Suppose for some positive real α,

$$g(t) \equiv f(t) e^{-\alpha t} \to 0 \text{ as } t \to \infty$$

is integrable.

$$\tilde{g}(s) \equiv \mathcal{L}[g(t)] = \int_0^\infty \underline{f(t)\, e^{-\alpha t}}\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-(\alpha+s)t}\, dt$$

still let

$$\tilde{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

Then $\tilde{g}(s) = \tilde{f}(\alpha + s)$

the inverse of $\tilde{g}(s)$ is

$$g(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \tilde{g}(s)\, e^{st}\, ds$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \tilde{f}(\alpha+s)\, e^{st}\, ds$$

Let $s' = s + \alpha$

$$f(t)\, e^{-\alpha t} = g(t) = \frac{1}{2\pi i} \int_{-i\infty}^{\infty} \tilde{f}(\alpha+s)\, e^{st}\, ds$$

$$= \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \tilde{f}(s')\, e^{-\alpha t}\, e^{s't}\, ds'$$

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \hat{f}(s)\, e^{st}\, ds}$$

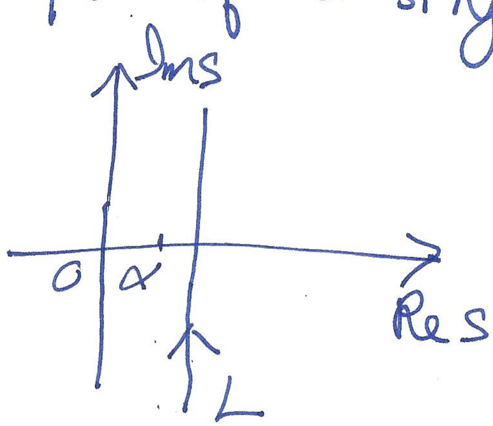
Thus, as long as $f(t)\, e^{-\alpha t}$ is integrable in $0 < t < \infty$, we have

$$\hat{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

valid for $\mathrm{Re}\, s \geq \alpha$

$$f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \hat{f}(s)\, e^{st}\, ds$$

In practice we do the integration in the s-complex plane along a vertical path to the right of all singularities of $\hat{f}(s)$.



(8.6 of my book)

Solving the Drunken Sailor problem using Laplace transform

PDE: $u_t = D u_{xx}$, $-\infty < x < \infty$

$0 < t < \infty$

BC: $u(x,t) \to 0$ as $x \to \pm\infty$

IC: $u(x,0) = \delta(x)$, $-\infty < x < \infty$

Laplace transform in t (more difficult)

Let $\tilde{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt$

$= \mathcal{L}[u]$

$\mathcal{L}[\frac{\partial}{\partial t} u] = D\,\tilde{u}_{xx}$

$\mathcal{L}\{\frac{\partial}{\partial t} u\} = \int_0^\infty \frac{\partial}{\partial t} u\, e^{-st}\,dt = u\,e^{-st}\Big|_0^\infty$

$+ s\int_0^\infty u\,e^{-st}\,dt$, $\operatorname{Re} s > 0$

$= s\tilde{u} - u(x,0)$

$$\boxed{\tilde{u}_{xx} - \frac{s}{D}\tilde{u} = -\frac{1}{D}\delta(x)}$$

$$\boxed{\tilde{u}_{xx} - \frac{s}{D}\tilde{u} = -\frac{1}{D}\delta(x)}$$

Second-order ODE, nonhomogeneous.

Since $\delta(x) = 0$ for $x \neq 0$, we have

$x > 0$: $\tilde{u}_{xx} - \frac{s}{D}\tilde{u} = 0$, want $\tilde{u} \to 0$ as $x \to \infty$

$x < 0$: $\tilde{u}_{xx} - \frac{s}{D}\tilde{u} = 0$, want $\tilde{u} \to 0$ as $x \to -\infty$

$x>0$: $\tilde{u}(x,s) = \begin{cases} A(s)\exp\{-(s/D)^{1/2}x\} & \text{if } \operatorname{Re} s^{1/2} > 0 \\ A'(s)\exp\{+(s/D)^{1/2}x\} & \text{if } \operatorname{Re} s^{1/2} < 0 \end{cases}$

Pick one of them, say $\operatorname{Re} s^{1/2} > 0$.

Keep this condition consistent throughout.

$x>0$: $\tilde{u}(x,s) = A(s)\exp\{-(s/D)^{1/2}x\}$

$x<0$: $\tilde{u}(x,s) = B(s)\exp\{+(s/D)^{1/2}x\}$

Match two solutions across $x=0$:

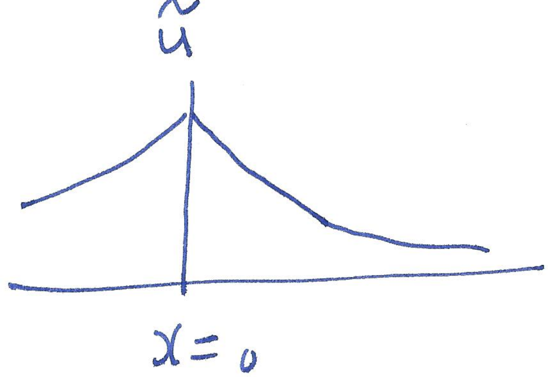
$$\int_{0^-}^{0^+} (\tilde{u}_{xx} - \frac{s}{D}\tilde{u})\,dx = \int_{0^-}^{0^+} -\frac{1}{D}\delta(x)\,dx$$

$$= -\frac{1}{D}$$

$\int_{0^-}^{0^+} \tilde{u}\,dx = 0$ if $\tilde{u}$ is finite

$\int_{0^-}^{0^+} \tilde{u}_{xx}\,dx = \tilde{u}_x \Big|_{0^-}^{0^+}$

So $\quad \tilde{u}_x \Big|_{0^-}^{0^+} = -\frac{1}{D}$.



$\tilde{u}$ is continuous at $x=0$
otherwise $\tilde{u}_x$ is infinite.

If $\tilde{u}$ is continuous at $x=0$

$A(s) = B(s)$

$\tilde{u}_x$ at 0^+ is $(-s/D)^{1/2} A$

$\tilde{u}_x$ at 0^- is $(s/D)^{1/2} B$

$$2 (s/D)^{1/2} A = -\frac{1}{D}$$

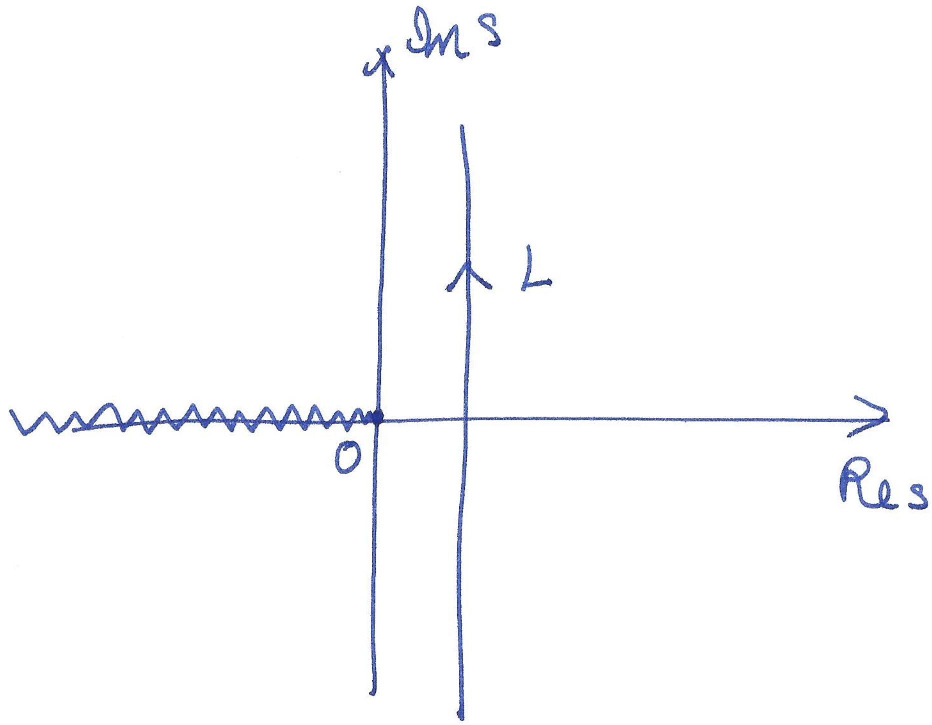
$$A = -\frac{1}{2D(s/D)^{1/2}}$$

$$\boxed{\tilde{u}(x,s) = \frac{1}{2(Ds)^{1/2}} \exp\{-(\tfrac{s}{D})^{1/2} |x|\}}$$

Inverse Laplace transform (optional)

$$u(x,t) = \frac{1}{2\pi i} \int_L \frac{ds}{2(Ds)^{1/2}} \exp\{st - (s/D)^{1/2}|x|\}$$

where L is the vertical line in the complex plane to the right of all singularities. There is a branch point at $s = 0$.

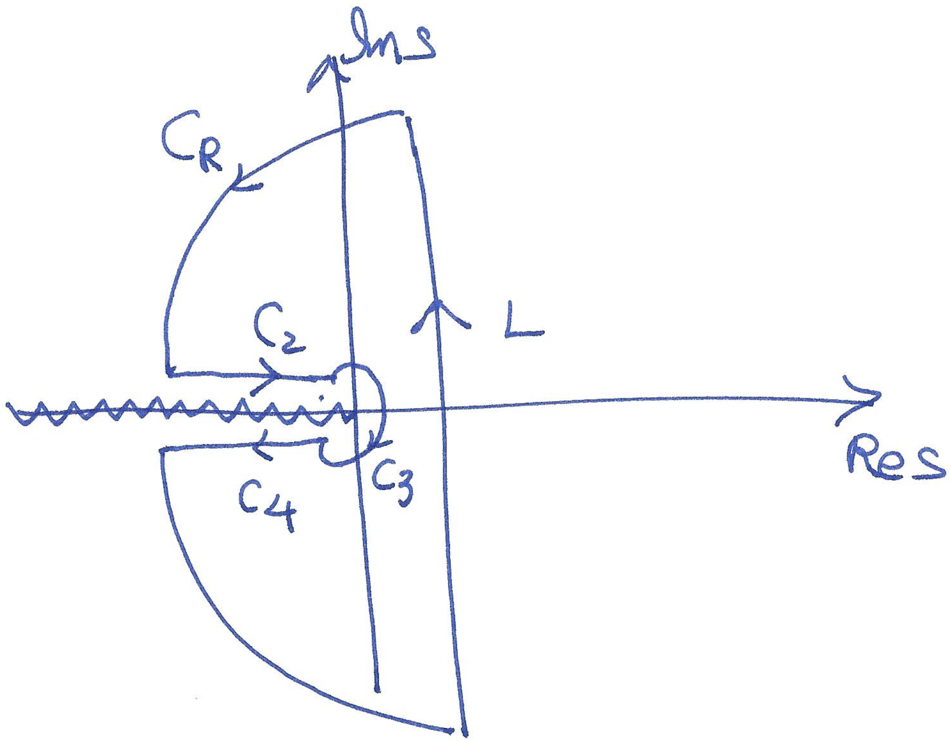


The branch with $\text{Re}\, s^{1/2} > 0$ is defined by

$$s = re^{i\theta}, \quad s^{1/2} = r^{1/2} e^{i\theta/2}$$

$$\text{Re}\, s^{1/2} = r^{1/2} \cos \theta/2 > 0 \Rightarrow -\pi < \theta < \pi$$

$\Rightarrow$ branch cut across the negative real axis



Consider the closed contour C defined by

$$C = L + C_R + C_2 + C_3 + C_4$$

There is no singularity inside C, so

$$\oint_C \; ds = 0.$$

$\int_{C_R} \; ds \to 0$ by Jordan's lemma for $t > 0$ and the radius $\to \infty$

$\int_{C_3} \; ds \to 0$ as the radius $\to 0$.

On the upper horizontal line C_2,

$$s = re^{i\pi^-}$$

$$\frac{1}{2\pi i}\int_{C_2} = \frac{1}{2\pi i}\int_{\infty}^{0} e^{-rt - i(r/D)^{1/2}|x|} \frac{e^{i\pi}\,dr}{2i(Dr)^{1/2}}$$

$$= -\frac{1}{4\pi D^{1/2}}\int_0^{\infty} \frac{e^{-rt - i(r/D)^{1/2}|x|}}{r^{1/2}}\,dr$$

On the lower horizontal line C_4,

$$s = re^{-i\pi^+}$$

$$\frac{1}{2\pi i}\int_{C_4} = \frac{1}{2\pi i}\int_0^{\infty} e^{-rt + i(r/D)^{1/2}|x|} \frac{e^{-i\pi}\,dr}{-2i(Dr)^{1/2}}$$

$$= -\frac{1}{4\pi D^{1/2}}\int_0^{\infty} \frac{e^{-rt + i(r/D)^{1/2}|x|}}{r^{1/2}}\,dr$$

$$u(x,t) = \frac{1}{2\pi i}\int_L = \frac{1}{2\pi i}\left\{ \oint_C - \int_{C_R} - \int_{C_2} - \int_{C_3} - \int_{C_4} \right\}$$

$$= -\frac{1}{2\pi i}\left\{ \int_{C_2} + \int_{C_4} \right\}$$

$$= \frac{1}{4\pi D^{1/2}} \int_0^\infty \frac{dr}{r^{1/2}} e^{-rt} \left\{ e^{i(r/D)^{1/2}|x|} + e^{-i(r/D)^{1/2}|x|} \right\}$$

Let $y = r^{1/2}$:

$$u(x,t) = \frac{1}{4\pi D^{1/2}} \int_0^\infty 2\,dy\, e^{-y^2 t} \left\{ e^{i(|x|/D^{1/2})y} + e^{-i(|x|/D^{1/2})y} \right\}$$

$$= \frac{1}{2\pi D^{1/2}} \int_{-\infty}^\infty e^{-ty^2 + i\frac{|x|}{D^{1/2}}y}\, dy$$

$$= \frac{1}{(4\pi D t)^{1/2}} \exp\left\{ -\frac{x^2}{4Dt} \right\}$$

Same as previously found using Fourier T.