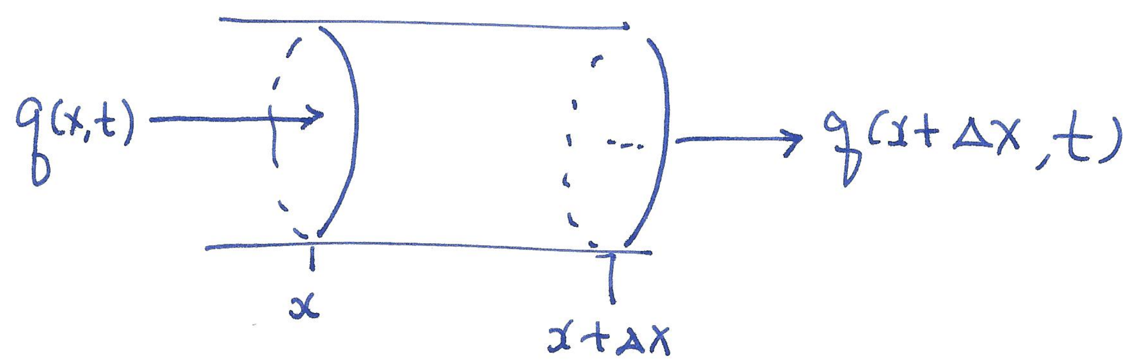
Lecture 1

(Chapter 2 of Deconinck's Notes)

First order PDEs.

Origin: Usually from conservation laws.



Consider the section Δx and cross-sectional area A:

Amount of "stuff" in this section:

$$u A \Delta x$$

where u is the concentration per unit volume

Time rate of increase in "stuff" in the section

= flux in at x − flux out at $x+\Delta x$

$= A q(x,t) - A q(x+\Delta x, t)$

where q is flux per unit area.

$$\frac{\partial}{\partial t} A \Delta x\, u(x,t) = A q(x,t) - A q(x+\Delta x, t)$$

$$\frac{\partial}{\partial t} u(x,t) = - \frac{q(x+\Delta x, t) - q(x,t)}{\Delta x}$$

Take the limit $\Delta x \to 0$

$$\boxed{\frac{\partial}{\partial t} u(x,t) = -\frac{\partial}{\partial x} q}$$

Different equation will result depending on the form of q.

e.g. Brownian motion, diffusion process: downgradient transport

$q \propto -\frac{\partial}{\partial x} u$, write

$q = -\alpha^2 \frac{\partial}{\partial x} u$

$\Rightarrow$ Diffusion equation:

$$\boxed{\frac{\partial}{\partial t} u = \frac{\partial}{\partial x}\left(\alpha^2 \frac{\partial}{\partial x} u\right)}.$$

Example: Advection of tracers:

$$q(x,t) = c\,u(x,t)$$

$$\boxed{\frac{\partial}{\partial t}u + c\frac{\partial}{\partial x}u = 0}$$

Another example: If the quantity being advected is momentum, and the velocity of advection is momentum/mass, then it is nonlinear advection:

$$q = \frac{1}{2}u^2$$

$$\boxed{\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0}$$

More generally,

$$\boxed{\frac{\partial}{\partial t}u + c(u)\frac{\partial}{\partial x}u = 0}$$

where $c(u) \equiv \dfrac{dq}{du}$.

Method of characteristics for solving first order pdes, linear or nonlinear.

Along curve C : $x = x(t)$ defined by

$$\frac{dx}{dt} = c(u)$$

$$\frac{d}{dt}u = \frac{\partial}{\partial t}u + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{\partial}{\partial t}u + c(u)\frac{\partial}{\partial x}u$$

"Lagrangian view"

$u = u(t, x(t))$.

So $\frac{\partial}{\partial t}u + c(u)\frac{\partial}{\partial x}u = 0$

is the same as

$$\boxed{\frac{du}{dt} = 0 \text{ along } \frac{dx}{dt} = c(u)}.$$

A few examples

(1) c = a constant

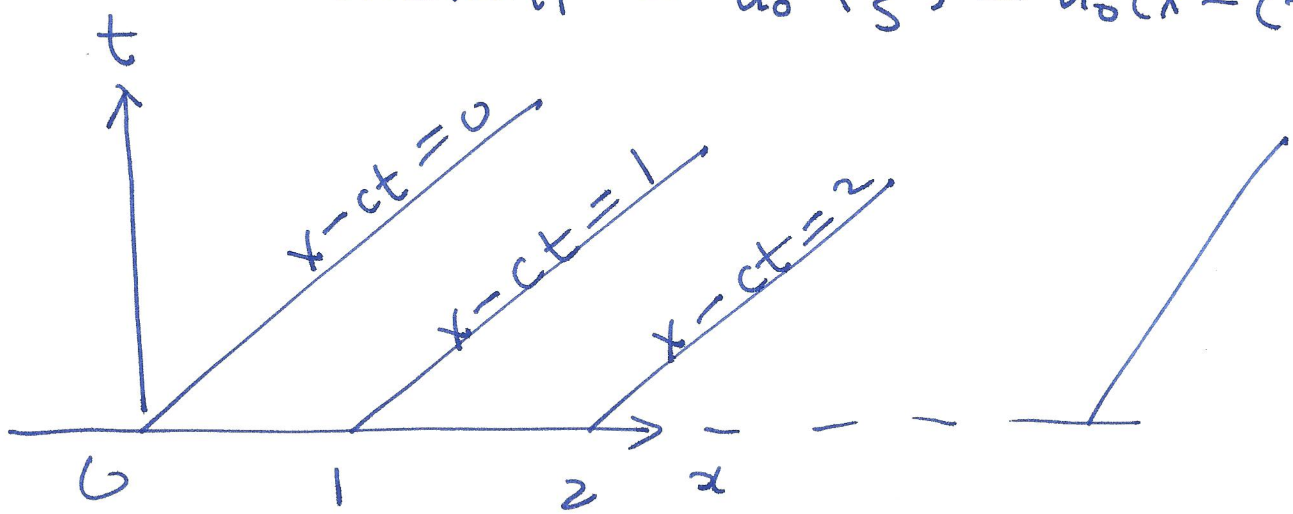
$$\frac{\partial u}{\partial t} + c\frac{\partial}{\partial x} u = 0$$

$$x(0) = \xi$$

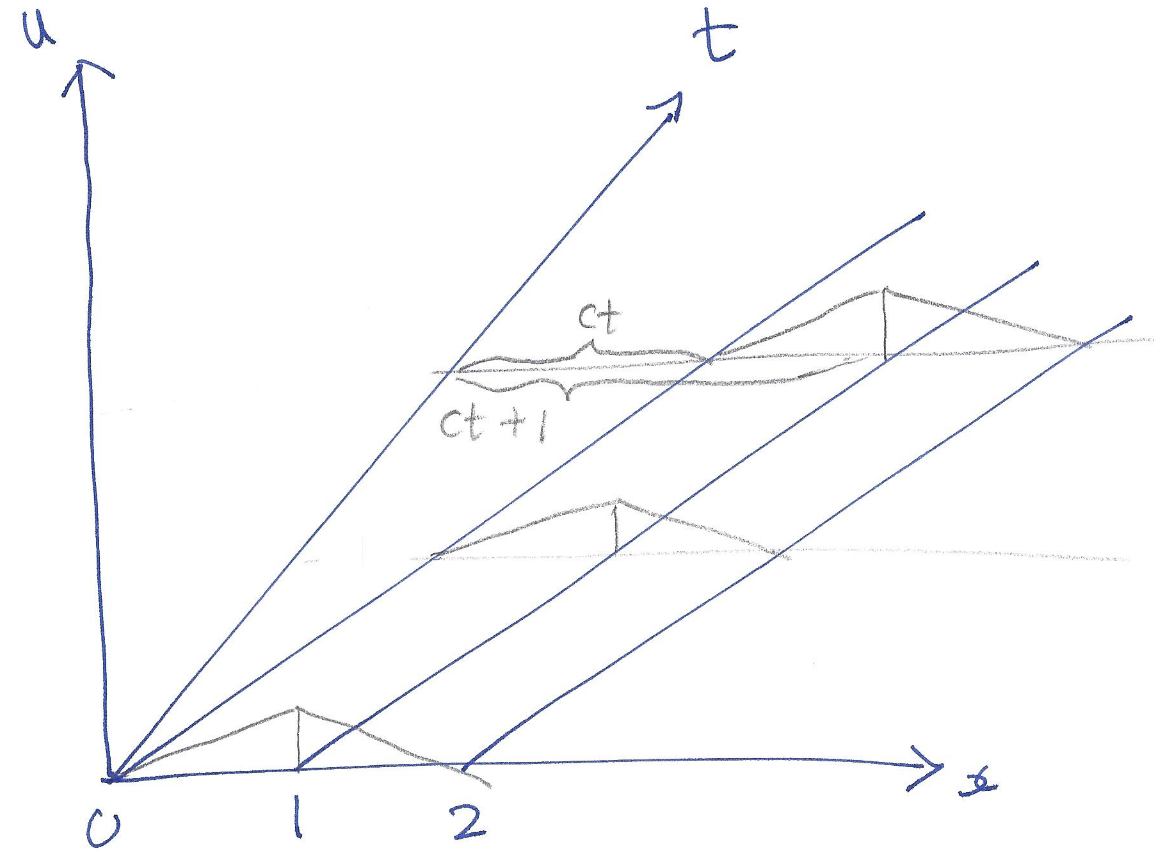
$$\frac{du}{dt} = 0 \quad \text{along } \frac{dx}{dt} = c, \text{ or } x = \xi + ct$$

$\frac{du}{dt} = 0$ along C means

$$u(x,t) = \text{constant} = u_0(\xi) = u_0(x - ct)$$



$\xi = x(0) = 0, 1, 2, \ldots$ etc.


u
t
ct
ct + 1
0
1
2
x

(2) <u>Wave breaking</u>

$c(u) = u \quad : \quad \frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = 0$

$$\frac{du}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = u$$

But since u is a constant, the slope is a constant,

$$\frac{dx}{dt} = u = u_0(\xi)$$

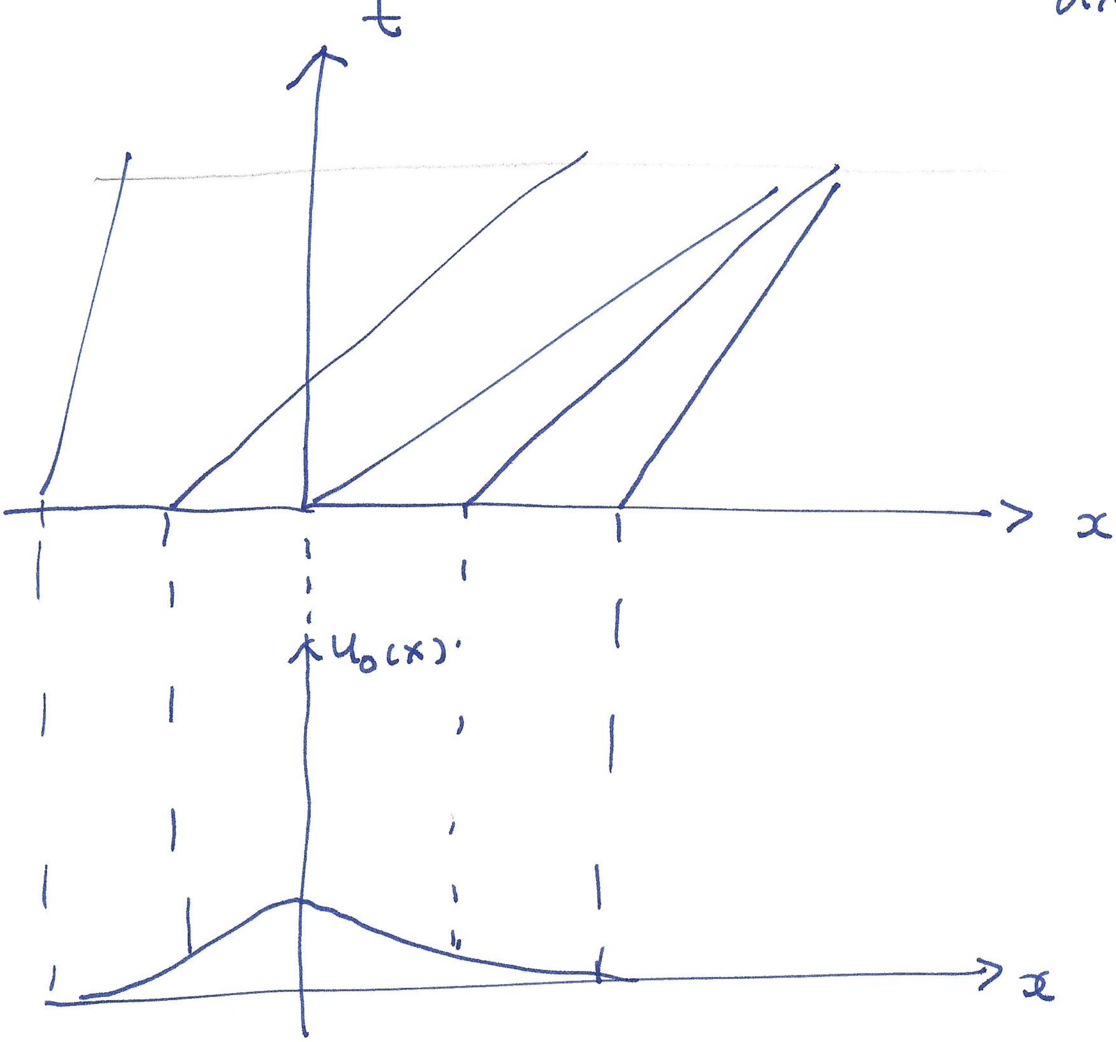
Therefore $\quad x = \xi + t\, u_0(\xi)$

$$\xi = x - u_0(\xi) t$$

$$u(x,t) = u_0(\xi) = u_0(x - u_0(\xi) t)$$

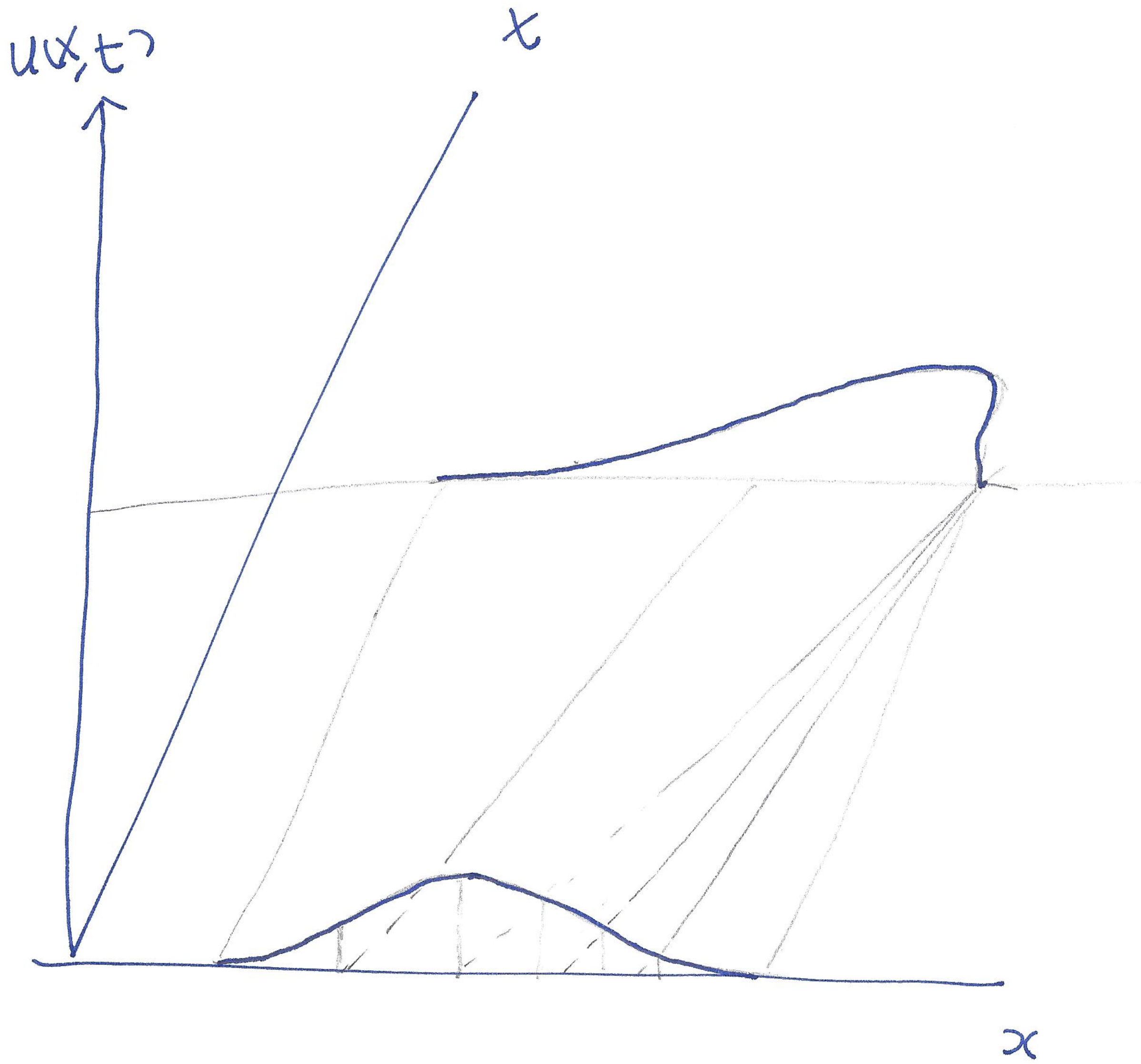
slope

$$\frac{dt}{dx} = \frac{1}{u}$$



slope:

$$\frac{dt}{dx} = \frac{1}{u}$$



wave steepens at the leading edge
Eventually forms a "shock".