AMATH 569 Homework 2

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Problem 1. Consider the wave equation:

$$c^2 u_{xx} - u_{tt} = 0,$$

which is a special case of the general quasi-linear equation:

$$au_{xx} + 2bu_{xy} + cu_{yy} = f$$

with $y = t$.
Find the slope of each of the two characteristics:

$$\frac{dy}{dx} = -z_1 \qquad \text{along } \alpha = \text{const. charactersitic}$$
$$\frac{dy}{dx} = -z_2 \qquad \text{along } \beta = \text{const. charactersitic}$$

Find the expression in terms of x and t for α and β, so that the wave equation simplifies to

$$u_{\alpha\beta} = 0$$

Solution. In order to express the wave equation in the canonical form $u_{\alpha\beta} = 0$ we need α and β to satisfy

$$a\varphi_x^2 + 2b\varphi_x\varphi_t + c\varphi_t^2 = 0 \qquad \Rightarrow \qquad a\left(\frac{\varphi_x}{\varphi_t}\right)^2 + 2b\frac{\varphi_x}{\varphi_t} + c = 0$$

Plugging in $a = c^2$ and $c = -1$, we find the relevant quadratic to be

$$c^2 z^2 - 1 = 0$$

where $z = \varphi_x/\varphi_t$. This has roots at $z_1 = 1/c$ and $z_2 = -1/c$. Since the roots are real and distinct the equation is hyperbolic. Now let

$$\alpha_x = z_1\alpha_t = \alpha_t/c \qquad\qquad \beta_x = z_2\beta_t = -\beta_t/c$$

These are first order differential equations whose solutions give the new coordinates α and β as functions of the old ones x and t. The solutions are:

$$\alpha = x + ct \qquad \text{and} \qquad \beta = x - ct$$

Therefore, the characteristic slopes are $\frac{dy}{dx} = -z_1 = -1/c$ along $\alpha =$ const. and $\frac{dy}{dx} = -z_2 = 1/c$ along $\beta =$ const.

In these new coordinates (α, β), the wave equation simplifies to $u_{\alpha\beta} = 0$.

Problem 2. Use the Fourier transform method to solve the 2D Laplace equation in the upper plane for the bounded solution

$$\begin{aligned} \nabla^2 u &= 0, & y \in \mathbb{R}^+ \text{ and } x \in \mathbb{R} \\ u(x,0) &= f(x), & x \in \mathbb{R} \end{aligned}$$

Assume $f(x)$ is of compact support and $u(x,y) \to 0$ as $|x| \to \infty$.

Solution. We begin by Fourier transforming $u(x,y)$.

$$\hat{u}(k,y) = \mathcal{F}[u] = \int_{-\infty}^{\infty} e^{ikx} u(x,y) dx$$

wherefore

$$u(x,y) = \mathcal{F}^{-1}[\hat{u}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{u}(k,y) dk$$

Then, using the well-known properties of the Fourier transform under the assumption that $u(x,y) \to 0$ as $|x| \to \infty$, we have

$$\begin{aligned} \mathcal{F}\big[\nabla^2 u\big] = \mathcal{F}\left[\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right) u\right] &= \hat{u}_{yy} - k^2 \hat{u} = 0 \\ &\Rightarrow \hat{u}_{yy} = k^2 \hat{u} \end{aligned}$$

We have reduced our second order PDE to a second order ODE, whose well-known solutions are given by

$$\hat{u}(k,y) = c_1(k) e^{ky} + c_2(k) e^{-ky}$$

Since we seek a bounded solution we discard the first term to give us

$$\hat{u}(k,y) = c(k) e^{-ky}$$

We now apply our boundary condition at $y = 0$. We have

$$\begin{aligned} u(x,0) &= f(x) \\ \Rightarrow \mathcal{F}[u(x,0)] &= \mathcal{F}[f(x)] \\ \hat{u}(k,0) = c(k) &= \mathcal{F}[f(x)] = F(k) \end{aligned}$$

Having found this expression for $c(k)$, we may write the Fourier transform $\hat{u}$ as

$$\hat{u}(k, y) = F(k)e^{-ky}$$

Hence our solution is given by the inverse Fourier transform of this,

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ikx}e^{-ky}dk$$

Problem 3. Solve the following problem in two ways:

$$\begin{array}{ccc} u_t = u_{xx} & t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^+ & \\ u(x, 0) = 0 & u(x, t) \text{ bounded as } x \to \infty & \forall t > 0 : u(0, t) = T_0 \end{array}$$

1. by the method of similarity transformation. Look for the value of α such that the PDE reduces to an ODE in η, $\eta = x/t^\alpha$.

   Solution. We have $\eta = x/t^\alpha$. Therefore by the chain rule our differential operators can be written in terms of η as

$$\frac{\partial}{\partial x} \mapsto \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = t^{-\alpha}\frac{\partial}{\partial \eta} \qquad \frac{\partial}{\partial t} \mapsto \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -\alpha x t^{-\alpha-1}\frac{\partial}{\partial \eta}$$

   Now let $v(\eta(x, t)) = u(x, t)$. Under our similarity transformation, our PDE becomes

$$u_t - u_{xx} = 0 \qquad \Rightarrow \qquad -\alpha x t^{-\alpha-1}v_\eta - t^{-2\alpha}v_{\eta\eta} = 0$$

   We now multiply both sides by xt^α to get

$$-\alpha x^2 t^{-1}v_\eta - xt^{-\alpha}v_{\eta\eta} = 0$$

   From here we can see that for $\alpha = 1/2$ we have $\eta = x/\sqrt{t}$ and our equation can be reduced to an ODE in η.

$$\begin{aligned} -\frac{1}{2}\frac{x^2}{t}v_\eta - \frac{x}{\sqrt{t}}v_{\eta\eta} &= 0 \\ \Rightarrow -\frac{1}{2}\eta^2 v_\eta - \eta v_{\eta\eta} &= 0 \end{aligned}$$

   To solve this ODE we start by dividing both sides by η which is valid since $x, t \neq 0$ by our problem statement. We have a second order ODE in η which is first order and separable in terms of v_η.

$$v_{\eta\eta} = -\frac{1}{2}\eta v_\eta \qquad \Rightarrow \qquad v_\eta = c_1 \exp\left\{-\frac{\eta^2}{4}\right\}$$

We now integrate once more to find our solution $v(\eta)$ to be

$$v = c_1 \operatorname{erf}\left(\frac{\eta}{2}\right) + c_2 = c_1 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + c_2$$

where $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ is the Gaussian error function. Having found the form of our solution, we revert back to $u(x,t)$.

$$u(x,t) = c_1 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + c_2$$

From our problem statement we have $u(0,t) = T_0$ for all $t > 0$, and hence

$$u(0,t) = c_1 \operatorname{erf}(0) + c_2 = c_2 = T_0$$

We also have that $u(x,0) = 0$, which implies that $v(\eta) \to 0$ as $t \to 0$, and so

$$\lim_{t \to 0^+} u(x,t) = \lim_{t \to 0^+} c_1 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + T_0 = c_1 + T_0 = 0$$

From which it follows that $c_1 = -T_0$. Putting these together, we find our full solution to be

$$u(x,t) = T_0 \left(1 - \operatorname{erf}\left\{\frac{x}{2\sqrt{t}}\right\}\right)$$

2. by an integral transform in t, in this case a Laplace transform.

   Solution. Let $\hat{u}(x,s) = \mathcal{L}[u(x,t)]$ be the Laplace transform of the solution $u(x,t)$. Taking the Laplace transform of both sides of the PDE, we have

$$\begin{aligned} \mathcal{L}[u_t] &= \mathcal{L}[u_{xx}] \\ s\hat{u}(x,s) - u(x,0) &= \hat{u}_{xx}(x,s) \\ \Rightarrow s\hat{u} &= \hat{u}_{xx} \end{aligned}$$

   We have reduced our equation to a second order ODE in the x variable, whose solution is given by

$$\hat{u}(x,s) = c_1 \exp\left\{\sqrt{s}x\right\} + c_2 \exp\left\{-\sqrt{s}x\right\}$$

   Since $u(x,t)$ is bounded as $x \to \infty$, we can discard the unbounded term, leaving us with

$$\hat{u}(x,s) = c_2 \exp\left\{-\sqrt{s}x\right\}$$

To find c_2, we use the boundary condition $u(0,t) = T_0$ for all $t > 0$, which after Laplace transform gives $\hat{u}(0,s) = c_2 = T_0/s$. Hence our solution in the Laplace domain is given by

$$\hat{u}(x,s) = \frac{T_0}{s} \exp\left\{-\sqrt{s}x\right\}$$

We apply the inverse Laplace transform and find the solution to be

$$u(x,t) = T_0 \left(1 - \operatorname{erf}\left\{\frac{x}{2\sqrt{t}}\right\}\right)$$

which agrees with the result we obtained using the similarity transformation method.