AMATH 569 Homework 6

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Problem 1. Consider the sound waves generated by

$$\psi_{tt} = c^2\nabla^2\psi, \qquad \nabla^2 = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2$$

in a circular cylinder of radius a and length L, where $\psi = 0$ at $r = a$ and $\psi = 0$ at $z = 0, L$.

Assume that the sound produced in this tube is symmetric, i.e. no θ dependence. Find the lowest three frequencies. Take $c = 300\text{m/s}$, $a = 1\text{cm}$, $L = 0.5\text{m}$.

Solution. Since our domain is symmetric, we can drop the θ component of the Laplacian and write

$$\nabla^2 = \frac{1}{r}\partial_r(r\partial_r) + \partial_z^2$$

Our equation is then

$$\psi_{tt} = c^2\left[\frac{1}{r}\partial_r(r\psi_r) + \psi_{zz}\right]$$

We assume a separable solution of the form $\psi(r, z, t) = R(r)Z(z)T(t)$. Plugging this in and diving by $c^2\psi$ gives us the equation

$$\frac{T''}{c^2T} = \frac{(rR')'}{rR} + \frac{Z''}{Z} = -k^2$$

where $-k^2$ is a separation constant. For the T dependence, we have

$$T'' = -c^2k^2T \quad \Rightarrow \quad T = A\cos(ckt) + B\sin(ckt)$$

For R and Z, we note that the LHS is the sum of two constants, and so we separate our equation once more into the equations

$$(rR')' = rR'' + R' = -\alpha^2 rR \quad \text{and} \quad Z'' = -\beta^2 Z$$

where $\alpha^2 + \beta^2 = k^2$. For Z we have solutions $Z(z) = C\cos(\beta z) + D\sin(\beta z)$, while for the R function, we can multiply by r to get the order 0 Bessel equation

$$r^2R'' + rR' + \alpha^2r^2R = 0$$

whose solution is given by the Bessel function of the first kind $R(r) = J_0(\alpha r)$.

We can apply our boundary conditions for r and z to find acceptable values for α and β. We need $Z(0) = Z(l) = 0$, hence we must have $Z = A\sin(\frac{n\pi z}{L})$ for $n \in \{1, 2, \dots\}$. We also need $R(a) = 0$, hence we must have $R(r) = J_0(\frac{j_{0,m}r}{a})$, where $j_{0,m}$, $m \in \{1, 2, \dots\}$ is the mth root of J_0.

By our separation process, we must have $\alpha^2 + \beta^2 = k^2$, hence our k values must satisfy

$$k^2 = \frac{n^2\pi^2}{L^2} + \frac{j_{0,m}^2}{a^2} = [4\pi^2 n^2 + 10^6 j_{0,m}^2][\frac{1}{\mathrm{m}^2}]$$

We see that energy jump to the second lowest radial mode is much greater than the corresponding jump for the lengthwise mode. Hence our lowest k values will correspond to $m = 1$ and $n = 1, 2, 3, 4$. The corresponding k values are then given by

$$\begin{aligned}
k_1 &= \sqrt{4\pi^2 + 10^6 j_{0,1}^2}[\frac{1}{\mathrm{m}}] \approx 2404.80[1/\mathrm{m}] \\
k_2 &= \sqrt{16\pi^2 + 10^6 j_{0,1}^2}[\frac{1}{\mathrm{m}}] \approx 2404.83[1/\mathrm{m}] \\
k_3 &= \sqrt{36\pi^2 + 10^6 j_{0,1}^2}[\frac{1}{\mathrm{m}}] \approx 2404.87[1/\mathrm{m}] \\
k_4 &= \sqrt{64\pi^2 + 10^6 j_{0,1}^2}[\frac{1}{\mathrm{m}}] \approx 2404.93[1/\mathrm{m}]
\end{aligned}$$

Hence our lowest frequences, given by $\frac{ck_n}{2\pi} = \frac{300[\mathrm{m/s}]\times k_n}{2\pi}[1/\mathrm{s}]$, are

$$f_1 = 114820 \text{ Hz} \qquad f_2 = 114822 \text{ Hz} \qquad f_3 = 114824 \text{ Hz} \qquad f_4 = 114827 \text{ Hz}$$

Problem 2. Consider the wave function ψ for an electron of mass μ in a sphere surrounded by an infinite potential at a radius a from the nucleus, which just means that $\psi = 0$ at $r = a$.

$$i\hbar\psi_t = -\frac{\hbar^2}{2\mu}\nabla^2\psi$$

Find the energy levels for the symmetric case, where ψ does not depend on θ and ϕ. Your answer should be exact and in terms of parameters given.

Solution. We assume a separable solution $\psi(r, t) = R(t)T(t)$. Then, using the spherical Laplacian, we have

$$i\hbar RT_t = -\frac{\hbar^2}{2\mu r}(rR)_{rr}T \qquad \Rightarrow \qquad i\hbar\frac{T'}{T} = -\frac{\hbar^2}{2\mu}\frac{(rR)''}{rR} = E$$

where E is a separation constant. For the t dependence we have

$$T' = \frac{-iE}{\hbar}T \qquad \Rightarrow \qquad T(t) = Ae^{-iEt/\hbar}$$

For the r dependence we have

$$rR'' + 2R' + \frac{2\mu E}{\hbar^2}rR = 0$$

Multiplying by r gives us the first order spherical Bessel equation

$$r^2R'' + 2rR' + \frac{2\mu E}{\hbar^2}r^2R = 0$$

whose solution is given by the spherical Bessel function

$$R(r) = Bj_0\left(\frac{\sqrt{2\mu E}}{\hbar}r\right) + Cy_0\left(\frac{\sqrt{2\mu E}}{\hbar}r\right)$$

where j_0 and y_0 are the spherical Bessel functions of the first and second kind, respectively. Since we require that R is finite at $r = 0$, we must have $C = 0$. Furthermore, the boundary condition $\psi(a) = 0$ implies that $R(a) = 0$, hence we have

$$R(a) = Bj_0\left(\frac{\sqrt{2\mu E}}{\hbar}a\right) = 0 \qquad \Rightarrow \qquad \frac{\sqrt{2\mu E}}{\hbar}a = \lambda_n$$

where λ_n is the n root of the spherical Bessel function j_0. Hence, our energy levels are given by

$$E_n = \frac{\hbar^2\lambda_n^2}{2\mu a^2} \qquad n \in \{1, 2, \dots\}$$

where λ_n are the zeros of the spherical Bessel function.

Problem 3. Consider the Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}y(x)\right] + n(n+1)y(x) = 0, \quad -1 \leq x \leq 1,$$

with the condition that $y(\pm 1)$ are bounded. The solutions are the Legendre polynomials $P_n(x)$, which are given by the Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

For example, $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

Compute the first four coefficients in the Legendre expansion

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \qquad \text{where } a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx \quad \text{for } f(x) = \begin{cases} 0 & \text{for } -1 < x < 0 \\ x & \text{for } 0 < x < 1 \end{cases}$$

Plot the approximation of the sum consisting of one, two, three and four terms along with the original function $f(x)$.

Solution. Let's begin by writing out the first four Legendre polynomials explicitely. We have

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= \frac{1}{2}\frac{d}{dx}(x^2-1) = x \\
P_2(x) &= \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1) \\
P_3(x) &= \frac{1}{48}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{2}(5x^3-3x)
\end{aligned}$$

Using these, we can compute the coefficients a_0, a_1, a_2, a_3 using the above formula for $f(x)$. Starting with a_0, we have

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)P_0(x)dx = \frac{1}{2}\int_{0}^{1} xdx = \frac{1}{4}x^2\Big|_0^1 = \frac{1}{4}$$
$$a_1 = \frac{3}{2}\int_{-1}^{1} f(x)P_1(x) = \frac{3}{2}\int_{0}^{1} x^2dx = \frac{3}{6}x^3\Big|_0^1 = \frac{1}{2}$$
$$a_2 = \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)dx = \frac{5}{4}\int_{0}^{1} (3x^3 - x)dx = \frac{5}{4}\left(\frac{3}{4}x^4 - \frac{1}{2}x^2\right)\Big|_0^1 = \frac{5}{16}$$
$$a_3 = \frac{7}{2}\int_{-1}^{1} f(x)P_2(x)dx = \frac{7}{4}\int_{0}^{1} (5x^4 - 3x^2)dx = \frac{7}{4}\left(x^5 - x^3\right)\Big|_0^1 = 0$$

Hence our fourth order Legendre expansion approximation for $f(x)$ is given by

$$f(x) \approx \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) + 0 \cdot P_3(x) = \frac{1}{4} + \frac{x}{2} + \frac{5}{32}(3x^2 - 1)$$

The $f(x)$ function along with the first four Legendre approximations are plotted below.

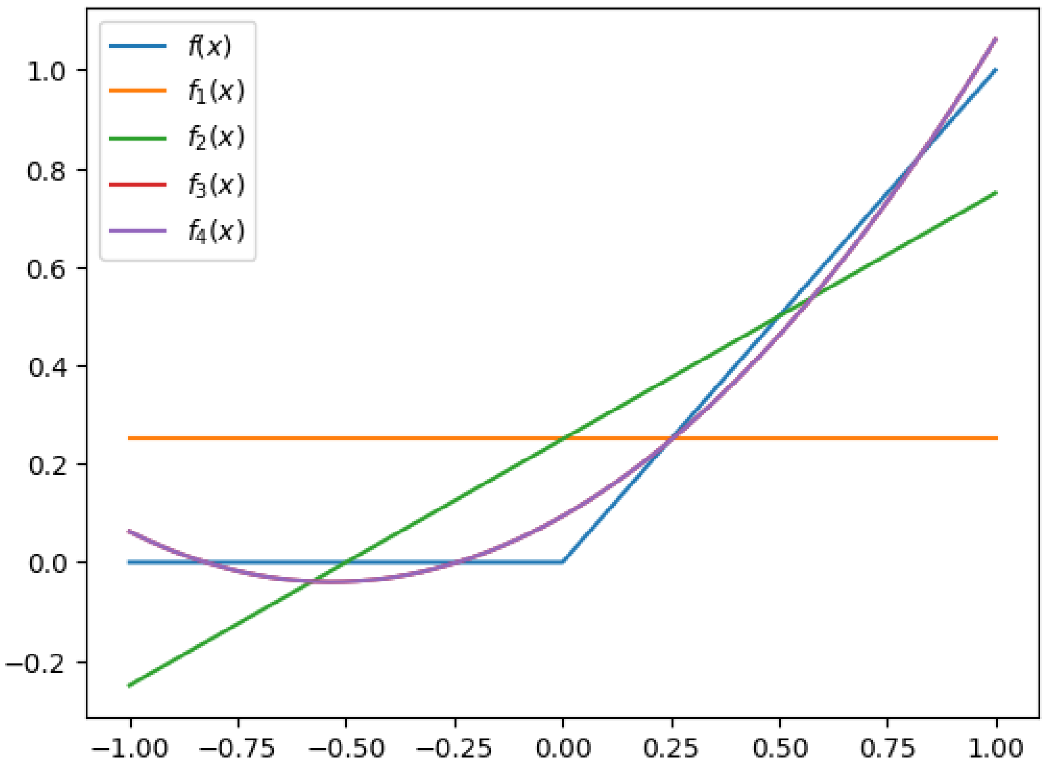

Figure 1: The function $f(x)$ along with its first four Legendre approximations.