Lecture 7

(Chapter 3 of Bernard's Notes
chapters 7-8 of my book)

Integral transform methods for solving PDEs

Fourier transform of $f(x)$, $-\infty < x < \infty$

$$F(\omega) \equiv \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx = \mathcal{F}[f(x)]$$

Fourier Integral formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x')\, e^{i\omega x'}\, dx' \right] e^{-i\omega x}\, d\omega$$

subject to the integrability of $f(x)$; $f(x)$ piecewise continuous.

Thus

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega$$

$$= \mathcal{F}^{-1}[F(\omega)]$$

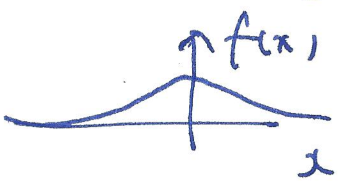
is the formula for <u>inverse transform</u>.

Wave equation in infinite domain:

PDE : $\frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u, \quad -\infty < x < \infty, \quad t > 0$

BCs : $u(x,t) \to 0$ as $x \to \pm\infty$

ICs : $u(x,0) = f(x)$, $f(x)$ of compact support.



$\frac{\partial}{\partial t} u(x,0) = 0, \quad -\infty < x < \infty$

[The second IC being 0 is a special case; can also be solved if it is a given function]

Let $U(\omega,t) = \mathcal{F}[u(x,t)]$

$\mathcal{F}[u_{tt}] = c^2 \mathcal{F}[u_{xx}]$

$\mathcal{F}[u_{tt}] = \frac{\partial^2}{\partial t^2} \mathcal{F}[u] = U_{tt}$

(can we interchange the order of integration and derivative?)

$$\mathcal{F}[u_{xx}] = \int_{-\infty}^{\infty} u_{xx} e^{i\omega x} dx$$

$$= u_x e^{i\omega x}\Big|_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} u_x e^{i\omega x} dx$$

$$= \underbrace{\cancel{u_x e^{i\omega x}\Big|_{-\infty}^{\infty}}}_{\text{assumed}}^{0} - i\omega \cancel{u e^{i\omega x}\Big|_{-\infty}^{\infty}}^{0 \text{ from BC}}$$

$$+ (-i\omega)^2 \int_{-\infty}^{\infty} u e^{i\omega x} dx$$

$$= -\omega^2 \mathcal{F}[u] = -\omega^2 U$$

The assumption of $u_x \to 0$ as $x \to \pm\infty$ will have to be verified a posteri.

Got rid of x as independent variable with derivatives in the PDE. Ended up with an ODE:

$$\boxed{\frac{\partial^2}{\partial t^2} U + c^2\omega^2 U = 0} \quad U(\omega, t)$$

Subject to ICs: $U(\omega, 0) = \mathcal{F}[u(x,0)] = F(\omega)$

$U_t(\omega, 0) = 0$.

$$\frac{\partial^2}{\partial t^2} U + c^2 \omega^2 U = 0$$

solution: $U(\omega, t) = A(\omega) \sin(c\omega t) + B(\omega) \cos(c\omega t)$

$U_t(\omega, 0) = 0$ implies $A = 0$

$U(\omega, 0) = F(\omega)$ implies $B = F(\omega)$

$$\boxed{U(\omega, t) = F(\omega) \cos(c\omega t)}$$

$$u(x,t) = \mathcal{F}^{-1}[U(\omega, t)]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cos(c\omega t) e^{-i\omega x} d\omega$$

$$\cos(c\omega t) = \frac{1}{2}\left(e^{ic\omega t} + e^{-ic\omega t}\right)$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2} F(\omega) e^{-i\omega(x-ct)} d\omega$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2} F(\omega) e^{-i\omega(x+ct)} d\omega$$

$$\boxed{u(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)}$$

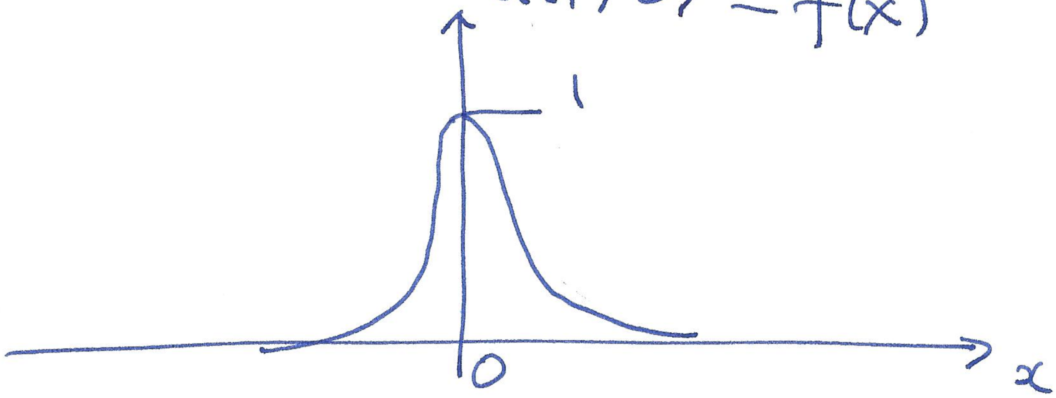
where $f(x)$ is the initial condition $u(x,0)$.

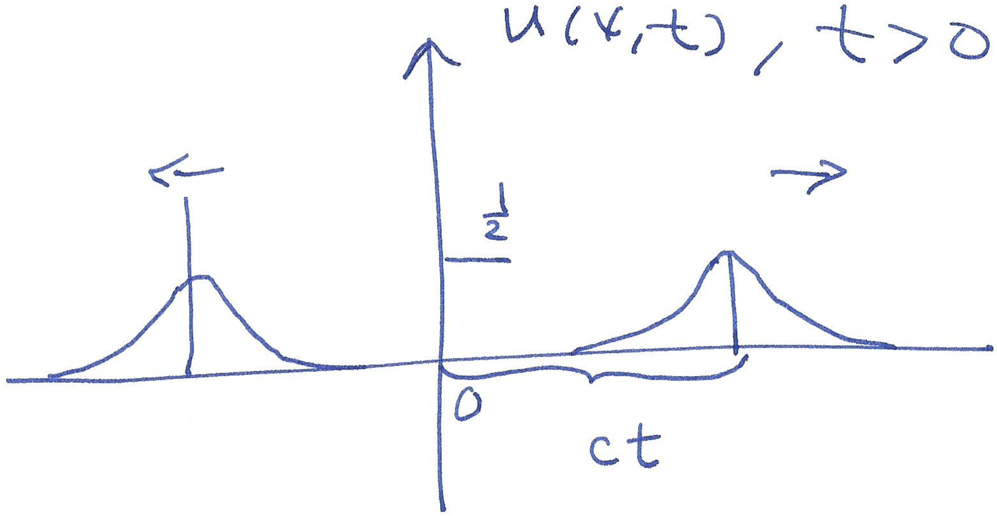
This is because

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

$$f(x-ct) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x-ct)} d\omega$$

and $f(x+ct) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x+ct)} d\omega$





An initial shape is split into two, each with half the amplitude but unchanged shape, one goes to the left and one goes to the right, with speed c.

[A posteri verification: if $f(x)$ does not have amplitude at $\pm\infty$, for finite t, it will not reach $\pm\infty$; so $u(x,t) \to 0$. $u_x(x,t) \to 0$ as $x \to \underline{\pm\infty}$.]

Solve

PDE: $\frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u$, $-\infty < x < \infty$, $t > 0$

BCs: $u \to 0$ as $x \to \pm\infty$

ICs: $u(x,0) = 0$, $\frac{\partial}{\partial t} u(x,0) = g(x)$, compact support.

$-\infty < x < \infty$

Take Fourier transform:

$$U(\omega,t) = \mathcal{F}[u(x,t)] = \int_{-\infty}^{\infty} u(x,t) e^{i\omega x} dx$$

$$\frac{\partial^2}{\partial t^2} U + c^2\omega^2 U = 0$$

subject to ICs: $U(\omega,0) = 0$

$$U_t(\omega,0) = G(\omega) \equiv \mathcal{F}[g(x)]$$

Solution: $U(\omega,t) = A(\omega)\sin(c\omega t) + B(\omega)\cos(c\omega t)$

$U(\omega,0) = 0 \Rightarrow B = 0$

$U_t(\omega,0) = c\omega A(\omega) = G(\omega)$

$$\boxed{U(\omega,t) = \frac{G(\omega)}{c\omega}\sin(c\omega t)}$$

First find the inverse transform of

$$\mathcal{U}_t(\omega,t) = G(\omega)\cos(c\omega t)$$

Since $\cos(c\omega t) = \frac{1}{2}\left(e^{ic\omega t} + e^{-i\omega t}\right)$

$$u_t(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2} G(\omega)\, e^{-i\omega(x-ct)}\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2} G(\omega)\, e^{-i\omega(x+ct)}\, d\omega$$

$$u_t(x,t) = \frac{1}{2} g(x-ct) + \frac{1}{2} g(x+ct)$$

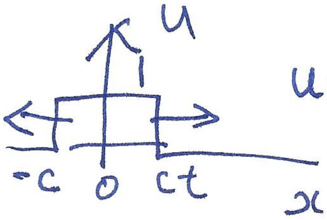
Integrate w.r.t. t (since $u(x,0)=0$):

$$u(x,t) = \int_0^t \frac{1}{2} g(x-ct')\,dt' + \int_0^t \frac{1}{2} g(x+ct')\,dt'$$

$$= -\frac{1}{2c}\int_x^{x-ct} g(\xi)\,d\xi + \frac{1}{2c}\int_x^{x+ct} g(\eta)\,d\eta$$

$$\boxed{u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi}$$

Example: $g(x) = \delta(x)$, $u(x,t) = \begin{cases} \frac{1}{2c} H(x+ct) \\ -\frac{1}{2c} H(x-ct) \end{cases}$



$$u(x,t) = \begin{cases} 1/2c & \text{for } -ct < x < ct \\ 0 & \text{elsewhere.} \end{cases}$$

Solve the same problems using the method of characteristics:

Let $\xi = x - ct$, $\eta = x + ct$

PDE: $\frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u$

BCs: $u \to 0$ as $x \to \pm\infty$

ICs: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

$$\frac{\partial u}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial u}{\partial \eta} \qquad \Bigg| \quad \frac{\partial \xi}{\partial t} = -c, \ \frac{\partial \xi}{\partial x} = 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta} \qquad \Bigg| \quad \frac{\partial \eta}{\partial t} = c, \ \frac{\partial \eta}{\partial x} = 1$$

$$u_t = -c u_\xi + c u_\eta$$

$$u_x = u_\xi + u_\eta$$

$$u_{tt} = c^2 u_{\xi\xi} + c^2 u_{\eta\eta} - 2c^2 u_{\xi\eta}$$

$$u_{xx} = u_{\xi\xi} + u_{\eta\eta} - c u_{\eta\xi} + c u_{\xi\eta}$$

$$u_{tt} - c^2 u_{xx} = -2c^2 u_{\xi\eta}$$

PDE becomes: $u_{\xi\eta} = 0$

solve $u_{\xi\eta} = 0$

by integration:

First integrate w.r.t. η:

$$u_\xi = A(\xi), \quad \text{where } A \text{ is a "constant" of integration}$$

Integrate w.r.t. ξ

$$u = \int^{\xi} A(\xi')\,d\xi' + B(\eta),$$

where B is a "constant" of integration.

solution is then of the form of

$$u(\xi,\eta) = F(\xi) + G(\eta),$$

(same as the form assumed by D'Alembert)

$$u(x,t) = F(x-ct) + G(x+ct)$$

Use ICs to find F and G.

$$u(x,0) = f(x) \Rightarrow F(x) + G(x) = f(x)$$

$$u_t(x,0) = g(x) \Rightarrow -cF'(x) + cG'(x) = g(x)$$

$$-F'(x) + G'(x) = \frac{1}{c} g(x)$$

Integrate :

$$-F(x) + G(x) = \frac{1}{c}\int^{x} g(x')\,dx' + K$$ constant

Add to : $F(x) + G(x) = f(x)$

$$2G(x) = f(x) + \frac{1}{c}\int^{x} g(x')\,dx' + K$$

subtract:

$$2F(x) = f(x) - \frac{1}{c}\int^{x} g(x')\,dx' - K$$

Solution :

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$\boxed{u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')\,dx'}$$