Electron in a box:

$$\lambda^2 = 2\mu E/\hbar^2$$

$$\nabla^2 \phi = -\lambda^2 \phi$$

BC: $\phi = 0$ at $x=0$ and $x=L$

$\phi = 0$ at $y=0$ and $y=L$

$\phi = 0$ at $z=0$ and $z=L$

$$\phi(x,y,z) = X(x)Y(y)Z(z)$$

$$X''YZ + XY''Z + XYZ'' = -\lambda^2 XYZ$$

$$\frac{X''}{X(x)} = -\frac{Y''(y)}{Y(y)} - \frac{Z''(z)}{Z(z)} - \lambda^2 \equiv -a^2$$

LHS is a function of x only, while the RHS is a function of y and z only. So each must be equal to a constant.

$$X''(x) = -a^2 X(x)$$

$$X(0) = 0, \quad X(L) = 0$$

$$X(x) = X_n(x) = \sin \frac{n\pi x}{L}$$

$$a = a_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

$$\frac{Y''(y)}{Y(y)} = -\frac{Z''(z)}{Z(z)} + a_n^2 - \lambda^2$$

$$= \text{const} \equiv -b^2$$

$$Y''(y) = -b^2 Y$$

$$Y(0) = 0, \quad Y(L) = 0$$

$$Y(y) = Y_m(y) = \sin \frac{m\pi x}{L}$$

$$b = b_m = \frac{m\pi}{L}, \quad m = 1, 2, 3, \ldots$$

$$\frac{Z''(z)}{Z(z)} = -c^2 \quad , \quad c^2 \equiv \lambda^2 - (a_n^2 + b_m^2)$$

$$Z(0) = 0 \, , \quad Z(L) = 0$$

$$Z(z) = Z_l(z) = \sin \frac{l\pi z}{L}$$

$$c = c_l = \frac{l\pi}{L} \, , \quad l = 1, 2, 3, \cdots$$

$$\lambda^2 = \lambda_{nml}^2 = a_n^2 + b_m^2 + c_l^2$$

$$= \pi^2 (n^2 + m^2 + l^2) / L^2$$

$$n = 1, 2, 3, \cdots , \quad m = 1, 2, 3, \cdots , \quad l = 1, 2, 3, \cdots$$

$$\phi(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} T_{nml}(0) \, e^{-i(E_{nml}/\hbar)t}$$

$$\sin \frac{n\pi x}{L} \sin \frac{m\pi y}{L} \sin \frac{l\pi z}{L}$$

$$E = E_{nml} = \frac{\hbar^2 \pi^2}{2\mu L^2} (n^2 + m^2 + l^2)$$

quantized energy !

Sound wave in a rectangular cavity

$$\frac{\partial^2}{\partial t^2} u = c^2 \nabla^2 u$$

$$u(\vec{x}, t) = T(t)\,\phi(\vec{x})$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{\nabla^2 \phi}{\phi} = -\lambda^2$$

Solve for the time dependence

$$T''(t) = -c^2 \lambda^2 T$$

$$T(t) = A \sin(c\lambda t) + B \cos(c\lambda t)$$

The frequency of the oscillation

$$\omega \equiv c\lambda$$

depends on the eigenvalue λ of the Helmholtz equation.

$$\nabla^2 \phi = -\lambda^2 \phi$$

$\phi = 0$ at $x = 0,\ x = L_1$

$y = 0,\ y = L_2$

$z = 0,\ z = L_3$

Eigenfunction

$$\phi_{nml}(\vec{x}) = \sin\frac{n\pi x}{L_1} \cdot \sin\frac{m\pi y}{L_2} \cdot \sin\frac{l\pi z}{L_3}$$

Eigenvalue:

$$\lambda = \lambda_{nml} = \pi\left[\left(\frac{n}{L_1}\right)^2 + \left(\frac{m}{L_2}\right)^2 + \left(\frac{l}{L_3}\right)^2\right]^{1/2}$$

$n = 1, 2, 3, \ldots$

$m = 1, 2, 3, \ldots$

$l = 1, 2, 3, \ldots$

Frequency of the sound made by the cavity:

$$\omega = c\lambda_{nml} \equiv \omega_{nml}$$

$$= c\pi\left[\left(\frac{n}{L_1}\right)^2 + \left(\frac{m}{L_2}\right)^2 + \left(\frac{l}{L_3}\right)^2\right]^{1/2}$$

One-dimensional oscillators, such as the violin string, has "harmonic" frequencies

$$\omega_n = n\,\omega_1, \quad \omega_1 \equiv \frac{c\pi}{L_1}, \quad n = 1, 2, 3, \ldots$$

The higher frequencies are integer multiples of the fundamental frequency ω_1.
Human ears find the superposition of harmonic frequencies pleasing.
On the other hand, sounds from 2-D oscillators, such as drums, are not pleasing to the ear because their sounds are a superposition of incommensurable frequencies

$$\omega_{nm} = c\pi \left[\left(\frac{n}{L_1}\right)^2 + \left(\frac{m}{L_2}\right)^2 \right]^{1/2}.$$

Helmholtz eigenvalue problem

in a cylinder

$$\nabla^2 \phi = -\lambda^2 \phi$$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\phi + \frac{\partial^2}{\partial z^2}\phi$$

separation of variables:

$$\phi(r,\theta,z) = R(r)\,\Theta(\theta)\,Z(z)$$

$$\frac{Z''(z)}{Z(z)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{\frac{1}{r}(rR'(r))'}{R} = -\lambda^2$$

$$\frac{Z''(z)}{Z(z)} = -b^2, \quad Z(0)=0, \quad Z(L)=0$$

$$Z(z) = Z_l(z) = \sin\frac{l\pi z}{L}, \quad l = 1,2,3,\ldots$$

$$b = b_l = \frac{l\pi}{L}.$$

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\frac{r(rR'(r))'}{R} - (\lambda^2 - b^2) r^2$$

$$= \text{const} = -m^2,$$

$$\Theta(\theta) = e^{im\theta}, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots$$

satisfies the BC that ϕ is 2π-periodic in θ.

$$r\frac{d}{dr}\left(r\frac{d}{dr}R\right) + [(\lambda^2 - b^2) r^2 - m^2] R = 0$$

BCs: $R(0)$ bounded

$R(a) = 0$, a the radius of the cylinder.

This equation can be put into the form of Bessel's equation by letting:

$$x = r\sqrt{\lambda^2 - b^2},$$

$$y(x) = R(r)$$

$$\boxed{x^2\frac{d}{dx^2}y + x\frac{d}{dx}y + (x^2 - m^2)y = 0}$$

Bessel's equation.

The Bessel's equation can be written in the standard Sturm-Liouville form:

$$\frac{d}{dx}\left(x \frac{d}{dx} y\right) + \left[(\lambda^2 - b^2) r - \frac{m^2}{r} \right] y = 0$$

The solution to the Bessel's equation is the ~~pre~~ Bessel's function

$$y(x) = A J_m(x) + B Y_m(x)$$

$J_m(x)$ is the Bessel's function of the first kind, and has the series expansion:

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+m)!} \left(\frac{x}{2}\right)^{2k+m} .$$

$Y_m(x)$ is the Bessel's function of the second kind. It has a $\ln(\frac{x}{2})$ singularity at $x=0$ and blows up there.

$y(0)$ bounded implies $B=0$.

$$R(r) = A\, J_m\left(\sqrt{\lambda^2 - b_l^2}\; r\right)$$

$y(a) = 0$ implies

$$J_m\left(\sqrt{\lambda^2 - b^2}\; a\right) = 0$$

Since the Bessel function J_m is similar to cosine

$\left(J_m(z) \simeq \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{1}{2}m\pi - \frac{1}{4}\pi\right)\right.$

for large $|z|$),

it has infinite number of zeros for each m. These zeros are tabulated.

$$0 < z_{m1} < z_{m2} < z_{m3} < \cdots .$$

Therefore $y(a)=0$ yields

$$\sqrt{\lambda^2 - b^2} = \frac{z_{mn}}{a}$$

$$\lambda^2 = \lambda^2_{nml} = \frac{z_{mn}^2}{a^2} + \frac{l^2\pi^2}{L^2},$$

$l = 1, 2, 3, \ldots, \quad n = 1, 2, 3, \ldots$
$m = 0, \pm 1, \pm 2, \ldots$

The eigenfunction is

$$\phi_{mnl}(r,\theta,z) = J_m(z_{mn} r/a)\, e^{im\theta} \sin\frac{l\pi z}{L}.$$

For sound waves in a circular cylinder, the frequency of the oscillation is given by

$$\omega = \omega_{nml} = c\lambda_{nml} = c\left[\frac{z_{mn}^2}{a^2} + \frac{l^2\pi^2}{L^2}\right]^{1/2}$$

For a long cylinder, L large, almost harmonic

$$\omega_{nml} \simeq c z_{mn}/a$$