AMATH 568
Advanced Differential Equations
Homework 7

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1. Consider the Optical Parametric Oscillator as given in Lecture 23 of the notes (pages 99-102).

   (a) Assuming slow time $\tau = \epsilon^2 t$ and slow space $\xi = \epsilon x$, derive the Fisher-Kolmogorov equation for the slow equation of the instability (the expression after Eq. (518)).

   Solution: The Optical Parametric Oscillator equation is given by

$$U_t = \frac{i}{2}U_{xx} + VU^* - (1 + i\Delta_1)U$$
$$V_t = \frac{i}{2}\rho V_{xx} - U^2 - (\alpha + i\Delta_2)V + S$$

The stable uniform steady-state response of the OPO is given by

$$U = 0$$
$$V = \frac{S}{\alpha + i\Delta_2}$$

It can be shown via linear stability analysis that this solution becomes unstable once the external pumping amplitude $|S|$ becomes greater than $|S_c|$, where S_c is the critical pumping strength

$$S_c = (\alpha + i\Delta_2)(1 + i\Delta_1).$$

We consider an OPO system with an external pumping term S near S_c, which we asymptotically expand as

$$S = S_c + \epsilon^2 C + \mathcal{O}(\epsilon^3)$$

where C is a constant and $0 < \epsilon \ll 1$. Next, we expand about the steady-state solution by letting

$$
\begin{aligned}
U &= \epsilon u(\tau,\xi) + \mathcal{O}(\epsilon^3) \\
V &= \frac{S_c + \epsilon^2 C}{\alpha + i\Delta_2} + \epsilon^2 v(\tau,\xi) + \mathcal{O}(\epsilon^3) \\
&= (1+i\Delta_1)(1+\epsilon^2 C/S_c) + \epsilon^2 v(\tau,\xi) + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

where we have assumed the slow time $\tau = \epsilon^2 t$ and slow space $\xi = \epsilon x$. Plugging these in to the OPO equation and applying the chain rule $\partial_x \to \epsilon\partial_\xi$ and $\partial_t \to \epsilon^2\partial_\tau$ results in the equations

$$
\begin{aligned}
\epsilon^2 u_\tau &= \frac{i}{2}\epsilon^2 u_{\xi\xi} + u^* \left(\frac{S_c + \epsilon^2 C}{\alpha + i\Delta_2} + \epsilon^2 v\right) - (1+i\Delta_1)u \\
\epsilon^2 v_\tau &= \frac{i}{2}\rho\epsilon^2 v_{\xi\xi} - u^2 - (\alpha + i\Delta_2)v
\end{aligned}
$$

which we can manipulate to the form

$$
(1+i\Delta_1)(u^* - u) = \epsilon^2\left(\frac{i}{2}u_{\xi\xi} - u_\tau + vu^* + \frac{C}{\alpha+i\Delta_2}u^*\right) \tag{1}
$$

$$
(\alpha + i\Delta_2)v = -u^2 + \epsilon^2\left(\frac{i}{2}\rho v_{\xi\xi} - v_\tau\right) \tag{2}
$$

To leading order, (1) gives us

$$
\begin{aligned}
(1+i\Delta_1)(u^* - u) &= \mathcal{O}(\epsilon^2) \\
\Rightarrow \Im(u) &\sim \epsilon^2
\end{aligned}
$$

Hence we conclude that u is real to order $\mathcal{O}(1)$. Next, to leading order, (2) gives us

$$
\begin{aligned}
(\alpha + i\Delta_2)v &= -u^2 \\
\Rightarrow v &= \frac{-u^2}{\alpha + i\Delta_2}
\end{aligned}
$$

Using this we may calculate vu^* to $\mathcal{O}(1)$ as

$$
vu^* = \frac{-u^2 u^*}{\alpha + i\Delta_2} = \frac{-|u|^2 u}{\alpha + i\Delta_2} + \mathcal{O}(\epsilon^2)
$$

Using these expressions for u^* and vu^* we can write the right hand forcing of (1) as

$$R = \epsilon^2 \left(\frac{i}{2} u_{\xi\xi} - u_\tau - \frac{u^3}{\alpha + i\Delta_2} + \frac{C}{\alpha + i\Delta_2} u \right) + \mathcal{O}(\epsilon^4)$$

Note that since the null space of the adjoint of the leading $\mathcal{O}(1)$ governing equation is the space of all real functions, by the Fredholm Alternative theorem R must be purely imaginary. Hence, at $\mathcal{O}(\epsilon^2)$ we have the governing equation

$$u_\tau = \frac{i}{2} u_{\xi\xi} + \frac{Cu - u^3}{\alpha + i\Delta_2} \tag{3}$$

We now perform the following coordinate transformations. Define the scaled function φ by $u = \sqrt{\alpha + i\Delta_2}\varphi$, along with the scaled slow space ζ defined by $\zeta = \sqrt{\frac{2}{i}}\xi$. Additionally, define $\gamma = C/(\alpha + i\Delta_2)$. Then substituting $u \to \varphi$ and $\xi \to \zeta$ into the (2) gives us the **Fisher-Kolmogorov equation** for the slow equation of the instability.

$$\varphi_\tau = \varphi_{\zeta\zeta} + \gamma\varphi - \varphi^3$$