AMATH 569 Homework 1

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Problem 1. Solve the PDE:

$$u_t + uu_x = 0 \qquad -\infty < x < \infty \qquad t > 0$$

subject to the initial condition

$$u(x,0) = u_0(x) = \begin{cases} -1 & -\infty < x \le -a \\ \frac{x}{a} & -a < x < a \\ 1 & a \le x < \infty \end{cases} \qquad a > 0$$

Solution. Since this is a first order quasi-linear PDE we can solve it using the method of characteristics. We seek characteristic curves $x(t)$ along which $u(x(t),t)$ is constant.

$$\frac{d}{dt}u(x(t),t) = \dot{u} = u_t + \dot{x}u_x = 0$$

For the given PDE we have

$$\dot{x} = \frac{dx}{dt} = u = u_0(\xi)$$

along the characterstic curve that starts at $x = \xi$. Integrating with respect to t, we therefore find the equation for the characterstic curve to be

$$x(t) = \xi + tu_0(\xi) = \begin{cases} \xi - t & -\infty < \xi \le -a \\ \xi\left(1 + \frac{t}{a}\right) & -a < \xi < a \\ \xi + t & a \le x < \infty \end{cases}$$

We note that the characterstic curves fan out and do not intersect. We now can solve for $\xi(x,t)$.

$$\xi(x,t) = \begin{cases} x + t & -\infty < x \le -a \\ \frac{x}{1+t/a} & -a < x < a \\ x - t & a \le x < \infty \end{cases}$$

Hence, our solution is given by

$$u(x,t) = u_0(\xi(x,t)) = \begin{cases} -1 & -\infty < \xi(x,t) \le -a \\ \frac{\xi(x,t)}{a} & -a < \xi(x,t) < a \\ 1 & a \le \xi(x,t) < \infty \end{cases}$$

Problem 2. Consider the initial value problem in infinite domain:

$$u_t + uu_x = 0 \qquad\qquad u(x,0) = u_0(x) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

(a) Find where and when a shock first forms.

(b) Solve the problem and sketch or plot the solution before when a shock first forms.

(c) Find the shock speed using the Rankine-Hugoniot condition.

(d) Solve the problem and sketch or plot the solution after the shock has formed.

Solution.

(a) A shock wave will form at the point in the (x,t) plane where two characterstic curves intersect, resulting in a discontinuity in u. That is, the point at which $|u_x| \to \infty$ or $|u_t| \to \infty$. By the chain rule, we have

$$u_x = u_0'(\xi)\xi_x \qquad\qquad u_t = u_0'(\xi)\xi_t$$

We also note that $|u_0'(x)| < \infty$ for all x, hence the shock must form at some time $t > 0$. Let us now find the form for the derivatives ξ_x and ξ_t. Just as in Problem 1, we have the characteristic curves

$$x = \xi + tu_0(\xi)$$

which we can use to implicitely solve for ξ_x and ξ_t. Differentiating with respect to x gives us

$$1 = \xi_x + tu_0'(\xi)\xi_x \qquad \Rightarrow \qquad \xi_x = \frac{1}{1+tu_0'(\xi)}$$

and differentiating with respect to t gives us

$$0 = \xi_t + u_0(\xi) + tu_0'(\xi)\xi_t \qquad \Rightarrow \qquad \xi_t = \frac{-u_0(\xi)}{1+tu_0'(\xi)}$$

We see that both $|\xi_x|$ and $|\xi_t|$ blow up at the time

$$t^* = \frac{-1}{u_0'(\xi^*)}$$

where ξ^* corresponds to characteristics for which $u_0'(\xi^*) < 0$ and $|u_0'(\xi^*)|$ is maximal. From the definition of $u_0(x)$ we have $u_0(\xi) < 0$ only for $\xi \in (0,1)$, in which case we have $u_0'(\xi) = -1$. Hence we find the critical time to be

$$t^* = 1$$

To find where the shock occurs in the x-axis we again use the formula for the characterstic curves

$$x = \xi + tu_0(\xi)$$

and sustituting $t \to t^* = 1$ and $\xi \to \xi^* \in (0, 1)$ gives us

$$x^* = \xi^* + 1 - \xi^* = 1$$

Hence, we have found that the shock first forms at the point $(x^*, t^*) = (1, 1)$.

(b) We return again to the formula we derived in Problem 1 for the characterstic curves $x(t)$. We will consider times prior to the shock formation $t < t^* = 1$, during which no characterstic curves intersect and therefore the curves can be inverted to solve for $\xi(x, t)$. We have

$$x(\xi, t) = \xi + tu_0(\xi) = \begin{cases} \xi + t & \xi \leq 0 \\ \xi + t - \xi t & 0 < \xi < 1 \\ \xi & \xi \geq 1 \end{cases}$$

We can use a contour diagram to visualize the characteristic curves $x(\xi, t)$, as shown in Figure 1.

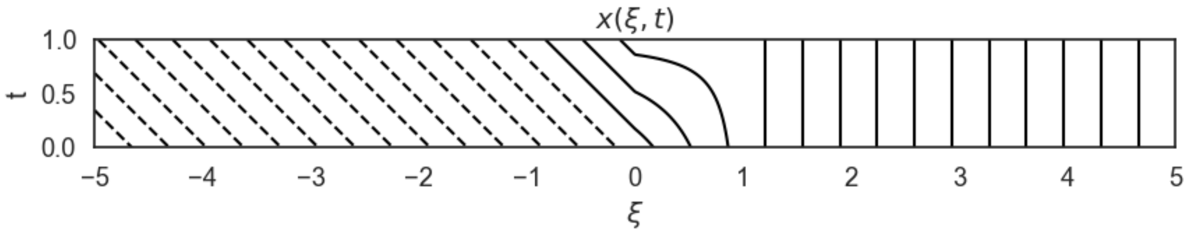

Figure 1: Plot of $x(\xi, t)$

We now invert $x(\xi, t)$ to find $\xi(x, t)$. Since we are restricting ourselves to the pre-shock domain $t < 1$, it is straightforward to invert the above expression to give us

$$\xi(x, t) = \begin{cases} x - t & x \leq t \\ \frac{x-t}{1-t} & t < x < 1 \\ x & x \geq 1 \end{cases}$$

Using this expression, we may write our solution as

$$u(x, t) = u_0(\xi(x, t)) = \begin{cases} 1 & \xi(x, t) \leq 0 \\ 1 - \xi(x, t) & 0 < \xi(x, t) < 1 \\ 0 & \xi(x, t) \geq 1 \end{cases}$$

which we have plotted in Figure 2.

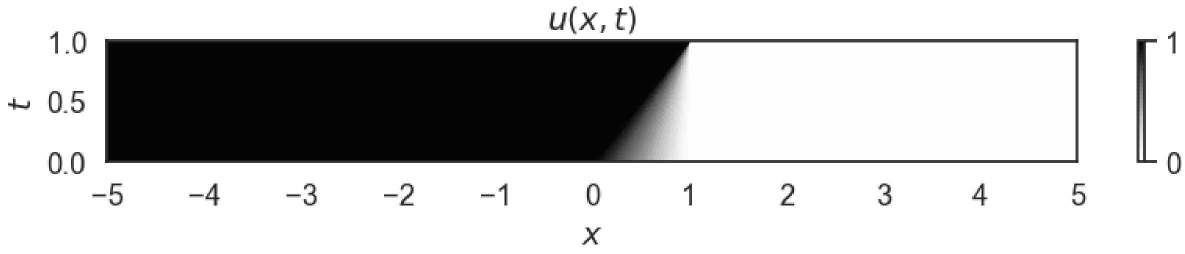

Figure 2: Plot of $u(x, t)$ for $t < 1$

(c) In part (a) we determined the location of shock formation. To find the speed of the shock we can use the Rankine-Hugoniot condition. We first note that we can write our PDE in the form

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\frac{1}{2}u^2 = 0$$

which we recognize to be the conservation law form $f_t + g_x = 0$, where u is a conserved density and $\frac{1}{2}u^2$ is its flux. We may write this conservation law in integral form over a small region $[x_1, x_2] = [\mathcal{X} - \epsilon, \mathcal{X} + \epsilon]$ which contains the location of the shock $\mathcal{X}(t)$. Let u_1 and u_2 denote the value of $u(x, t)$ on either side of the shock.

$$\frac{d}{dt}\int_{x_1}^{x_2} udx = -\frac{1}{2}(u_2^2 - u_1^2)$$

Now, splitting this integral up at $\mathcal{X}(t)$, we have

$$\begin{aligned} \frac{d}{dt}\int_{x_1}^{\mathcal{X}} udx + \frac{d}{dt}\int_{\mathcal{X}}^{x_2} udx &= -\frac{1}{2}(u_2^2 - u_1^2) \\ \dot{\mathcal{X}}u_1 - \dot{\mathcal{X}}u_2 &= -\frac{1}{2}(u_2^2 - u_1^2) \end{aligned}$$

and finally we have the shock speed

$$\dot{\mathcal{X}} = \frac{1}{2}\frac{u_2^2 - u_1^2}{u_2 - u_1} = \frac{1}{2}(u_1 + u_2)$$

For our PDE, we have $u_1 = 1$ and $u_2 = 0$, hence we have a shock speed of

$$\dot{\mathcal{X}} = \frac{1}{2}$$

(d) In part (b) we found the solution for $t < t^* = 1$. For $t > 1$ the solution has a shock discontinuity which moves at constant speed $\dot{x}(t) = \frac{1}{2}$ beginning at the point $(1, 1)$. All points to the left of the shock have value 1, and all values to the right of the shock have value 0, as the characteristic curves on either side meet at the shock. See Figure 3.

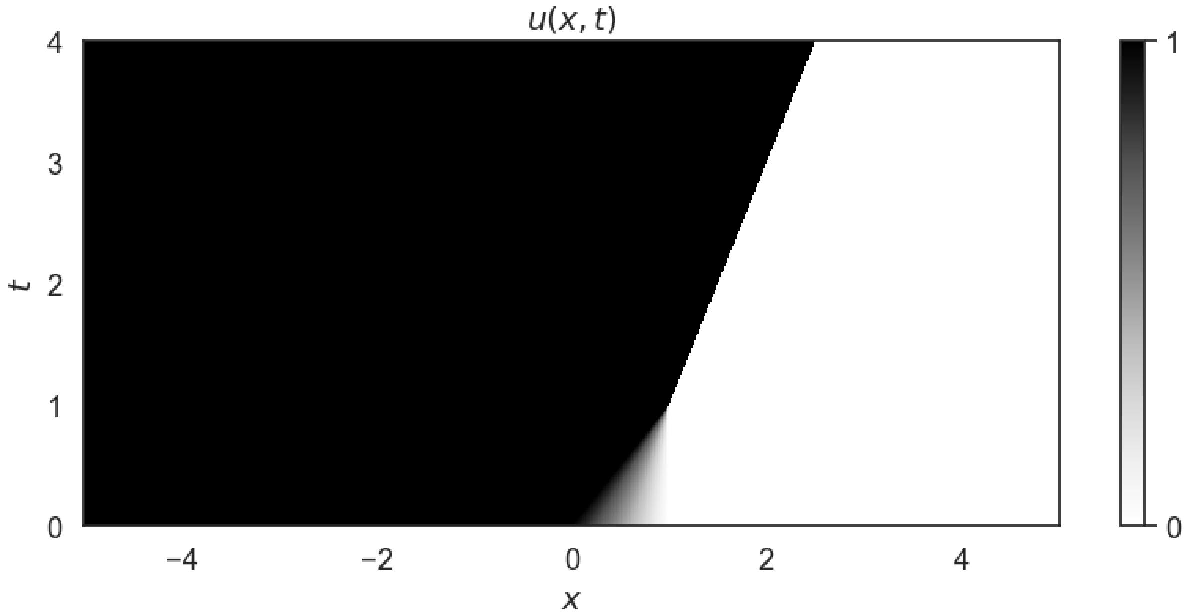

Figure 3: Full plot of $u(x,t)$

Problem 3. Solve the PDE:

$$u_t + uu_x = u \qquad -\infty < x < \infty \qquad t > 0$$

subject to the initial condition

$$u(x,0) = u_0(x) = 2x \qquad 0 \le x \le 2$$

Where in the x-t plane is the solution valid?

Solution. This is Burgers equation with dissipation. As above, its characteristics are given by

$$\dot{x} = u$$

However due to the dissipation term this PDE is no longer homogeneous, and hence the solution along this curve must satisfy

$$\dot{u} = u \qquad \Rightarrow \qquad u = u_0(\xi)\exp(t)$$

Combining this with the first equation, we have

$$\dot{x} = u_0(\xi)\exp(t) \qquad \Rightarrow \qquad x = u_0(\xi)\exp(t) - u_0(\xi) + \xi$$

We now use our initial condition.

$$x = 2\xi\exp(t) - 2\xi + \xi = 2\xi\exp(t) - \xi$$

Rearranging this, we solve for $\xi(x,t)$

$$\xi(x,t) = \frac{x}{2\exp(t) - 1}$$

Our solution is therefore given by

$$u(x,t) = u_0(\xi(x,t))\exp(t) = \frac{2x\exp(t)}{2\exp(t) - 1}$$

Note that as our initial condition was specified only for $x \in [0,2]$ it follows that our result is valid only for $\xi(x,t) \in [0,2]$. This corresponds to a region bounded by $x = 0$ and $x(t) = 4\exp(t) - 2$, inside of which our solution is valid. Hence, our final solution is given by:

$$u(x,t) = \frac{2x\exp(t)}{2\exp(t) - 1} \qquad 0 \le x \le 4\exp(t) - 2 \qquad t > 0$$

See Figure 4 for a plot of this solution in its domain.

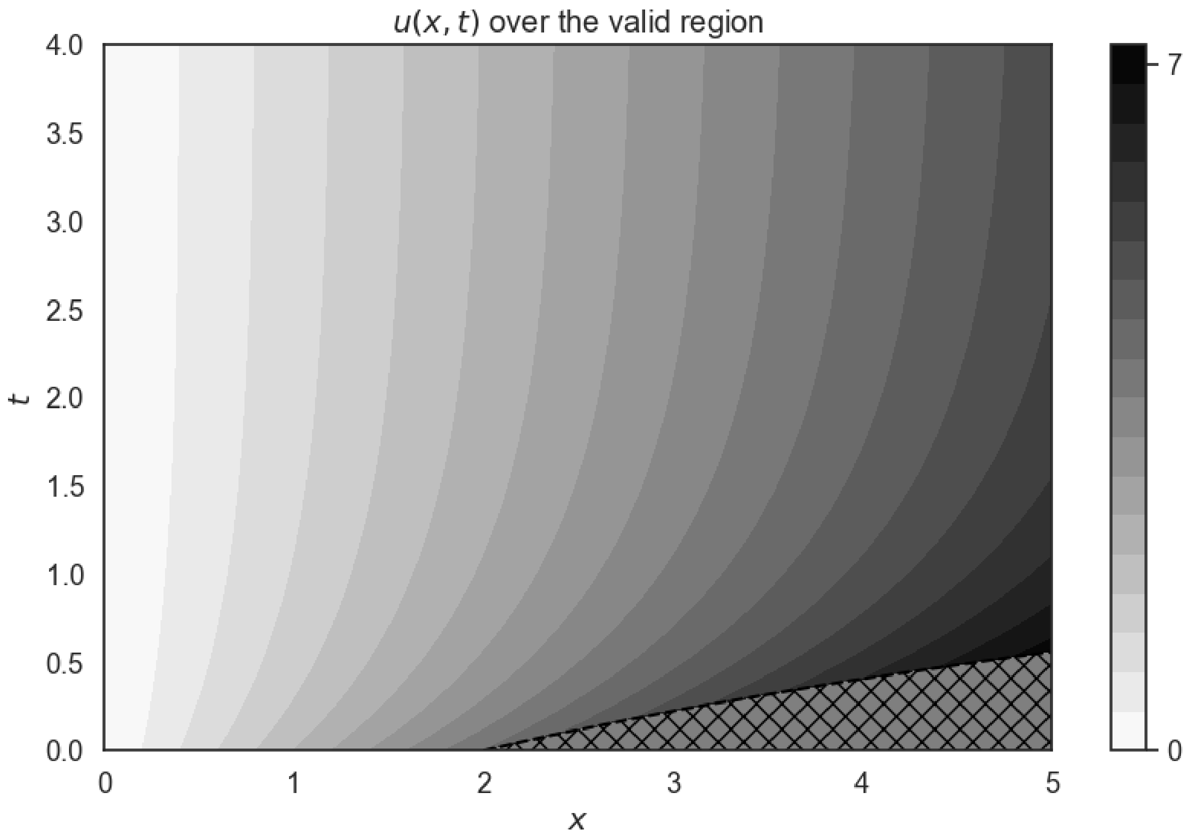

Figure 4: Plot of $u(x,t)$ for $0 \le x \le 4\exp(t) - 2$ and $t > 0$.