AMATH 568
Advanced Differential Equations
Homework 8

Lucas Cassin Cruz Burke

Due: March 8, 2023

1. Consider the inverted pendulum dynamics:

$$y'' + (\delta + \epsilon \cos \omega t) \sin y = 0$$

(a) Perform a Floquet analysis (computationally) of the pendulum with continuous forcing $\cos \omega t$.

Solution: We can use Python's solve-ivp function to directly solve this equation for different values of δ, ϵ, and ω. For each triple $(\delta, \epsilon, \omega)$ we solve the system twice on the interval $t \in [0, \frac{2\pi}{\omega}]$ to find $y_1(t)$ and $y_2(t)$ with initial conditions

$$\begin{aligned} y_1(0) = y_2'(0) = 1 \\ y_2(0) = y_1'(0) = 0 \end{aligned}$$

We can then use these to define the Floquet discriminant Γ as a function of the parameters δ, ϵ, and ω.

$$\Gamma(\delta, \epsilon, \omega) = y_1(T) + y_2'(T)$$

The stability of the periodic solutions for $(\delta, \epsilon, \omega)$ depends on the condition

$$|\Gamma(\delta, \epsilon, \omega)| < 2$$

Having defined this function in Python, we can directly compute Γ for different values of δ, ϵ and ω to determine the dependence of the stability on these parameters.

(b) Evaluate for what values of δ, ϵ, and ω, the pendulum is stabilized.

Solution: Using the function defined in (a) for numerically computing the Floquet discriminant we can explore the parameter space and see how stability is impacted by ϵ, δ, and ω.

Figures 1, 2 and 3 show the dependence of the Floquet analysis on the underlying parameters. Figure 1 shows the real value of the Γ, Figure 2 shows the absolute value $|\Gamma|$ For small values of ω, and Figure 3 shows which values satisfy the stability condition $|\Gamma| > 2$.

We observe from these pictures the existence of three behavioral regimes depending on the relative size of ω to δ and ϵ.. For small values of $\omega \ll \delta, \epsilon$ we observe the clearest shift in stabilization behavior depending on the relative magnitudes of ϵ and δ. For $\delta > \epsilon$ we observe a quasi-periodic structure in the Floquet discriminant Γ and the pendulum is stabilized for most parameter values. Conversely, for $\epsilon > \delta$ the stabilization structure appears chaotic for $\omega \ll 1$, or at least, the structure cannot be resolved efficiently using this method. However as ω grows to $\omega \sim \delta, \epsilon$ we observe new phenomena in the $\epsilon > \delta$ region including pattern formation (see $\omega = 0.02$) and eventual coherence (see $\omega = 0.2$) into a near-uniformly divergent solution.

Our observations in the $\omega \gg \delta, \epsilon$ regime agree with the analytical results which predict the pendulum stabilizing for $\omega \to \infty$. Indeed, we see that $\Gamma \to 2$ for all ϵ, δ, implying neutral stability.

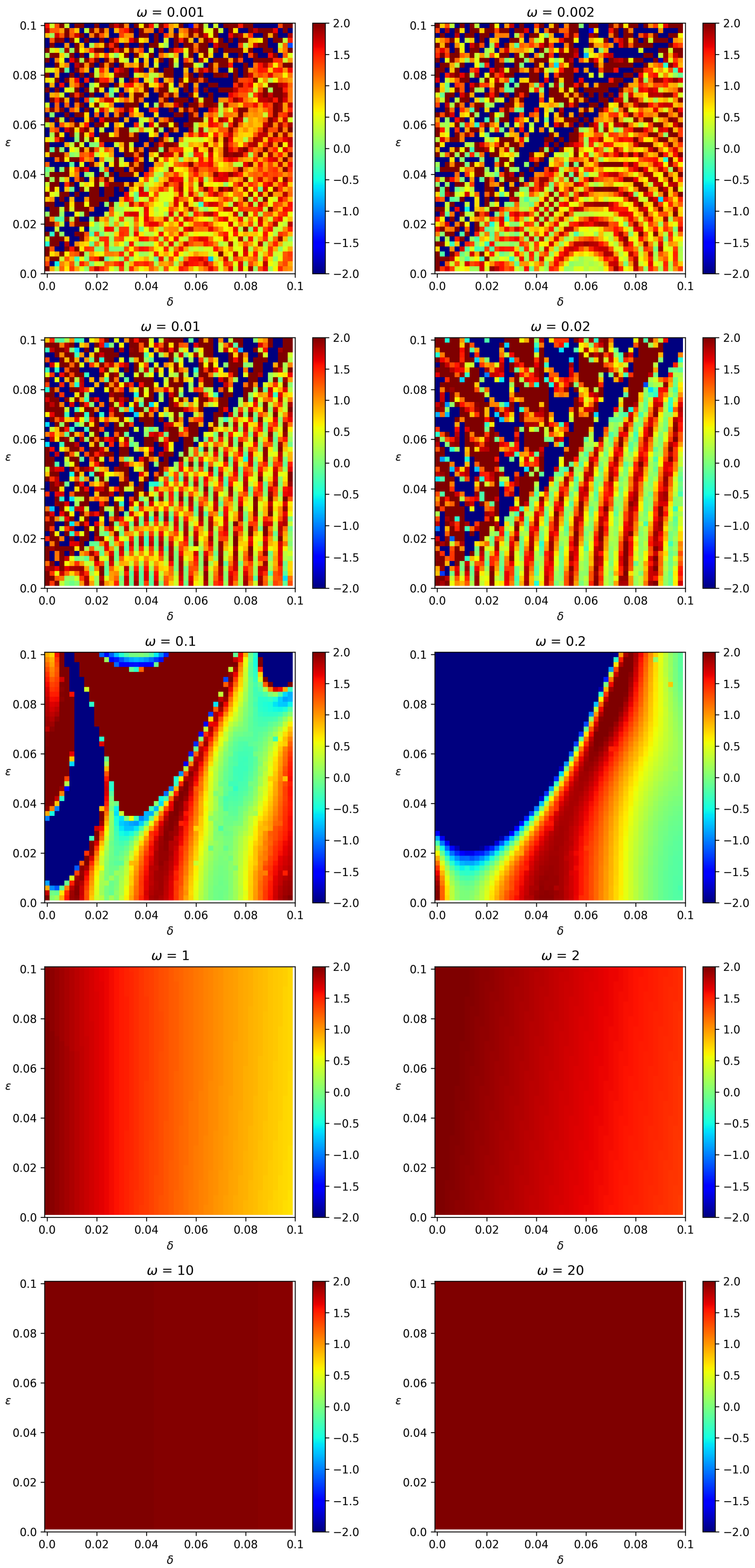

Figure 1: Value of Floquet discriminant $|\Gamma(\delta, \epsilon, \omega)|$ for various parameter values.

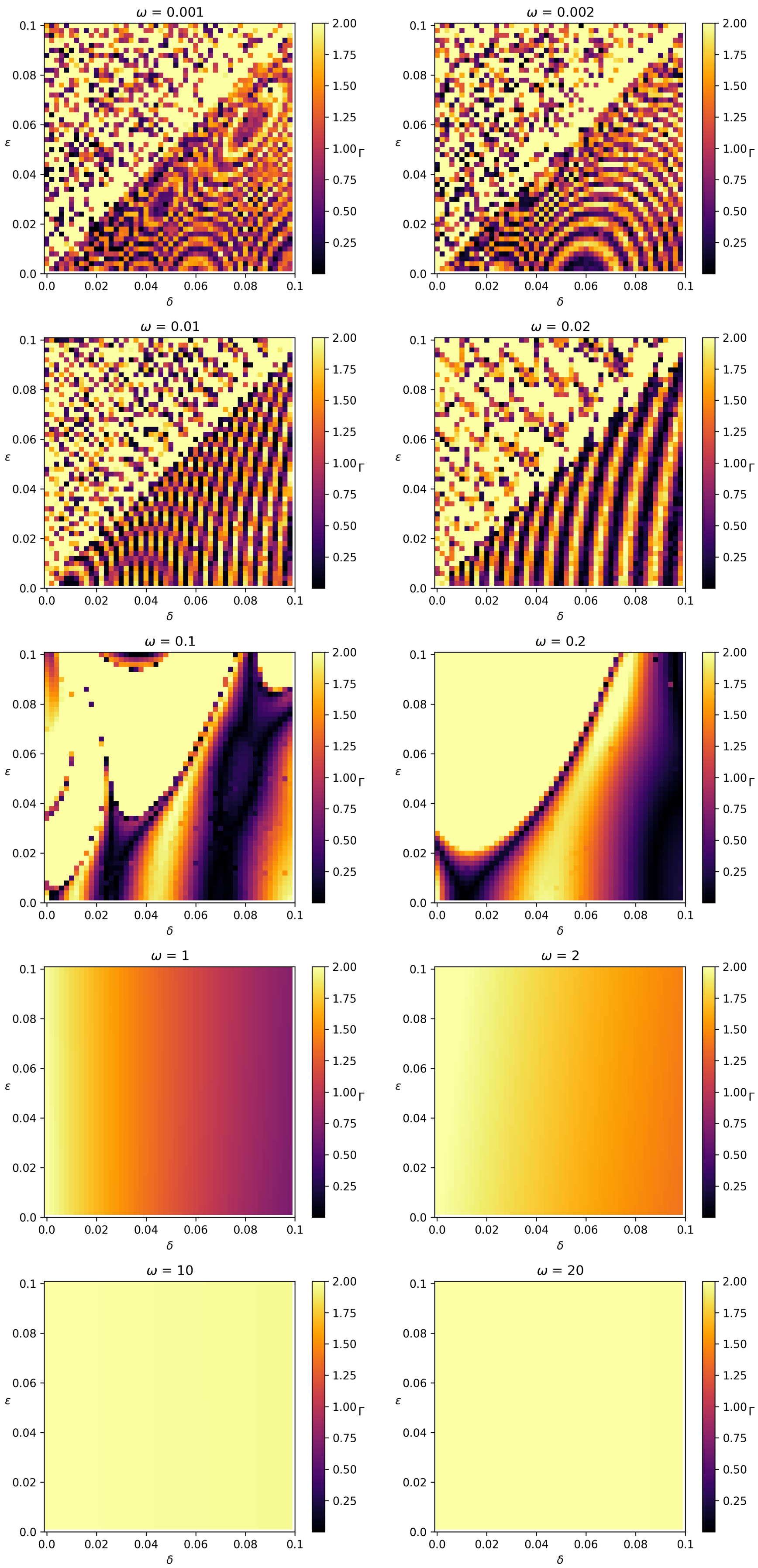

Figure 2: Absolute value of Floquet discriminant $|\Gamma(\delta, \epsilon, \omega)|$ for various parameter values.

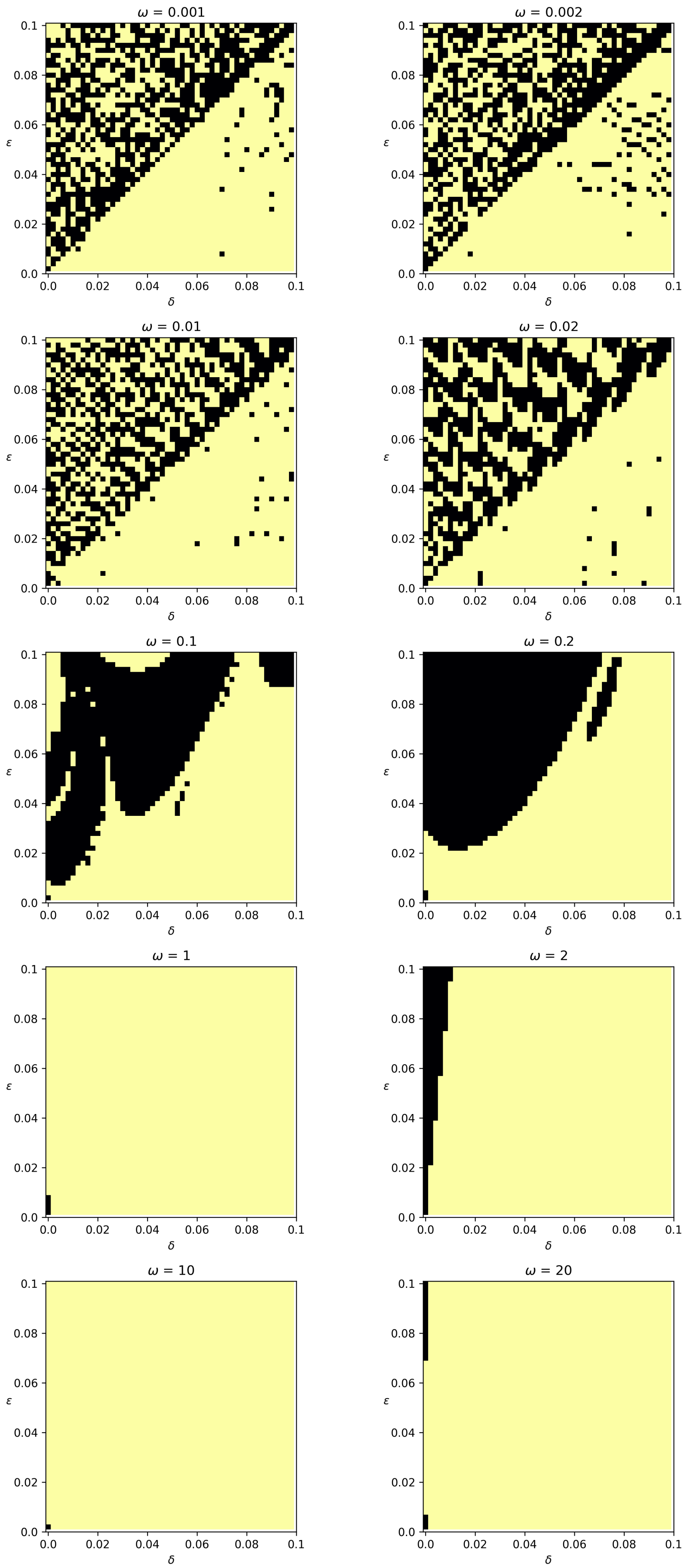

Figure 3: Stability of periodic solutions for parameters $(\delta, \epsilon, \omega)$. Yellow pixels correspond to stabilized pendulum.