Some theories on Laplace and Poisson

~~Review:~~ Divergence Th^{m}

$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iint_{\partial V} \vec{F}\cdot\vec{n}\,dS$$

Let $\vec{F} = v\vec{\nabla}u$

$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}v\cdot\vec{\nabla}u + v\nabla^2 u$$

$$\boxed{\iiint_V [\vec{\nabla}v\cdot\vec{\nabla}u]\,dV + \iiint_V v\nabla^2 u\,dV = \iint_{\partial V} v\vec{\nabla}u\cdot\vec{n}\,dS}$$

For Laplace equation : $\nabla^2 u = 0$

Let $v = u$

$$\boxed{\iiint_V |\vec{\nabla}u|^2\,dV = \iint_{\partial V} u\vec{\nabla}u\cdot\vec{n}\,dS}$$ Energy Th^{m}

The total "kinetic energy" in the interior of V $\frac{1}{2}\iiint_V |\vec{\nabla}u|^2\,dV$, is related to $u\vec{\nabla}u$ on the boundary.

If $u=0$ on ∂V, RHS $=0$

LHS $\Rightarrow$ $\vec{\nabla} u$ must vanish everywhere in the interior

$\Rightarrow \quad u \equiv 0$ in V.

If $\vec{\nabla} u \cdot \vec{n} = 0$ on ∂V

$u =$ constant in V.

<u>Gauss Integral theorem</u>

Let $\vec{F} = v \vec{\nabla} u$, $v=1$

Then for Laplace equation $\nabla^2 u = 0$

$$\boxed{\iint_{\partial V} \vec{\nabla} u \cdot \vec{n}\, dS = 0}$$

from the Divergence Thm

Uniqueness Theorem for Laplace and Poisson equations

Laplace : $\nabla^2 u = 0$

Poisson : $-\nabla^2 u = f$

Let ϕ_1 and ϕ_2 be the solution of

$$-\nabla^2 \phi_1 = f, \quad -\nabla^2 \phi_2 = f \quad \text{in } V$$

(i) Dirichlet boundary conditions :

ϕ_1, ϕ_2 specified to be the same on ∂V

Let $u \equiv \phi_1 - \phi_2$

then
$$\nabla^2 u = 0 \quad \text{in } V$$
$$u = 0 \quad \text{on } \partial V$$

From the previous page

$$u \equiv 0 \quad \text{in } V$$

i.e. ϕ_1 and ϕ_2 are the same. Unique.

(ii) Neumann boundary condition:

$\vec{\nabla}\phi_1 \cdot \vec{n}$, $\vec{\nabla}\phi_2 \cdot \vec{n}$ specified to be the same on ∂V

Let $u \equiv \phi_1 - \phi_2$

then $\nabla^2 u = 0$ in V

$\vec{\nabla} u \cdot \vec{n} = 0$ on ∂V

Then $\vec{\nabla} u \equiv 0$ in V

$u =$ constant in V

ϕ_1 and ϕ_2 can differ by at most a constant.

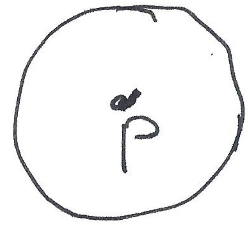
(iii) Mixed boundary condition:

ϕ prescribed on part of ∂V and $\vec{\nabla}\phi \cdot \vec{n}$ on the remainder. Then

$u \equiv \phi_1 - \phi_2 = 0$ in V

Solution to the Laplace or Poisson equation is <u>unique</u>

Mean-value Th$^{\underline{m}}$



The value of a solution of the Laplace equation at any point P is the average of values it takes on any sphere surrounding that point.

Let $P(x, y, z)$ be a fixed point inside V and $\nabla^2 u = 0$ in V.

Greens formula

$$\iiint_V [u \nabla^2 v - v \nabla^2 u] \, dV = \iint_{\partial V} (u \vec{\nabla} v - v \vec{\nabla} u) \cdot \vec{n} \, ds.$$

Let $Q(\xi, \eta, \zeta)$ be the dummy variable of integration. So inside the integral

$$\nabla_Q^2 u = 0.$$

Let $v = -\frac{1}{4\pi} \frac{1}{r_{PQ}}$

where $r_{PQ}^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$

So $\nabla_Q^2 v = \delta_3(\vec{\xi} - \vec{x})$ ~~(see later)~~

LHS of Greens formula

$$\iiint_V [\, u(Q)\nabla_Q^2 v - v(Q)\nabla_Q^2 u \,] dV_Q$$

$$= \iiint_V u(Q)\,\delta_3(\vec{\xi} - \vec{x})\, dV_Q$$

$$= \not{g} \quad u(P)$$

RHS of Greens formula, with Q on a sphere of radius R:

$$\iint_{\partial V:\ r_{PQ}=R} (u(Q)\vec{\nabla}_Q v - v(Q)\vec{\nabla}_Q u)\cdot \vec{n}\, dS_Q$$

1st term $= \displaystyle\iint_{r_{PQ}=R} u(Q)\frac{\partial}{\partial \rho}\left(-\frac{1}{4\pi\rho}\right)\Big|_{\rho=R} dS_Q$

$$= \frac{1}{4\pi R^2}\iint_{r_{PQ}=R} u\, dS_Q$$

For the 2nd term : $-\iint_{r_{PQ}=R} v \vec{\nabla} u \cdot \vec{n} \, dS_Q$

$v = -\frac{1}{4\pi R}$ when $r_{PQ} = R$

Then the 2nd term $= \frac{1}{4\pi R} \iint_{\partial V} \vec{\nabla} u \cdot \vec{n} \, dS_Q$

$= 0$ by the Gauss Integral Th$^{\underline{m}}$.

So

$$\boxed{u(p) = \frac{1}{4\pi R^2} \iint_{r_{PQ}=R} u \, dS_Q}$$

Maximum - Minimum Theorem

The maximum and minimum value of a solution to the Laplace equation, if it exists, must occur on the boundary.

pf: Suppose a maximum occurs in the interior point P. Surround that point by a small sphere of radius R. The average value on the sphere must be less than the value at P because by supposition P is a maximum. But this contradicts the mean-value thm.

Similarly for a minimum at P.

Continuity with respect to boundary data

$$-\nabla^2\phi_1 = f, \quad -\nabla^2\phi_2 = f \quad \text{in } V$$

$$\phi_1 = h \text{ on } \partial V, \quad \phi_2 = h + \Delta h \text{ on } \partial V$$

Let $u \equiv \phi_2 - \phi_1$

$$\nabla^2 u = 0 \text{ in } V$$

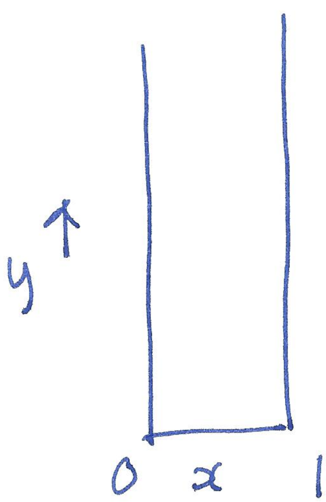
$$u = \Delta h \text{ on } \partial V$$

$$\min_{\text{on } \partial V} \Delta h \le \underset{\text{in } V}{u} \le \max_{\text{on } \partial V} \Delta h.$$ (see max-min principle later)

Thus a small variation in boundary data produces a small variation in the solution in the interior.

→ The Dirichlet problem is <u>well-posed</u>

A famous ill-posed problem due to Hadamard.



$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0 \quad \text{in } V$$

$$V: \quad 0 < x < 1$$
$$\quad\;\; 0 < y < \infty$$

$$u(x, 0) = h(x)$$

$$\frac{\partial}{\partial y} u(x, 0) = 0$$

$$u(0, y) = 0, \quad u(1, y) = 0$$

$$h(x) = e^{-\sqrt{n}} \sin(4n+1)\pi x$$

Solution:

$$u(x, y) = u_n(x, y) = e^{-\sqrt{n}} \cosh(4n+1)\pi y \, \sin(4n+1)\pi x$$

unique, exists.

Pick a point in the interior:

$$u_n(\tfrac{1}{2}, y) = e^{-\sqrt{n}} \cosh (4n+1)\pi y$$

$$\to \infty \text{ as } n \to \infty$$

for any fixed $y > 0$.

But $h(x) \to 0$ as $n \to \infty$

A small change in the boundary data h from 0 to 0^+ changes solution u from 0 to ∞.

An ill-posed problem.

Problem: Both u and $\frac{\partial}{\partial n} u$ specified on the same arc Γ in the (x,y) plane.

This is appropriate for the wave equation

$$\frac{\partial^2}{\partial x^2} u - \frac{\partial^2}{\partial y^2} u = 0$$

but is ill-posed for the Laplace equation.

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$