AMATH 567, Homework 5

Lucas Cassin Cruz Burke

November 9, 2022

1. **Problem 1:** Evaluate the integrals

$$\frac{1}{2\pi i}\oint_C f(z)dz$$

where C is the unit circle centered at the origin with $f(z)$ given below by

(a) enclosing the singular points inside C, and

(b) enclosing the singular points outside C by including the point at infinity $t = \frac{1}{z}$ as $z \to \infty$.

Show that you obtain the same result in both cases.

Solution:

By the residue theorem, $\frac{1}{2\pi i}\oint_C f(z)dz = \sum_k \mathrm{Res}(f, a_k)$, so this is equivalent to summing the residues for the following functions.

Including the point at infinity z_∞ in our domain is equivalent to working within the one point compactification of the complex plane, i.e. the Riemann sphere. Within this context, (a) is equivalent to enclosing the lower hemisphere of the Riemann sphere, and (b) is equivalent to enclosing its upper half. Furthermore, making the change of variable $z \to \frac{1}{z}$ has the effect of flipping the Riemann sphere about the z axis so that the upper point at z_∞ is mapped to the lower point at 0, and vice versa.

More explicitly, to enclose the the region outside of a contour C for an integral $I = \oint_C f(z)dz$ we can flip the orientation of C so that $I \to -I$ and make the change of variable $z = \frac{1}{\xi} \to dz = -\frac{1}{\xi^2}d\xi$ so that our integral becomes $I = -\oint_C f(\frac{1}{\xi})(-\frac{1}{\xi^2})d\xi = \oint_C \frac{f(1/\xi)}{\xi^2}d\xi$. In the following problems we will show that this integral is equivalent to the original integral.

(a) $f(z) = \frac{z^2+1}{z^2-a^2}$, $a^2 < 1$

i. We can factor the denominator to write $I = \frac{z^2+1}{(z+a)(z-a)}$.
The integrand has simple poles at $z = \pm a$. Since $a^2 < 1$, both of these points lie inside the unit circle. We can calculate $\mathrm{Res}(a)$ as $\lim_{z\to a} \frac{z^2+1}{(z+a)} = \frac{a^2+1}{2a}$. Likewise, we calculate $\mathrm{Res}(-a) = \lim_{z\to -a} \frac{z^2+1}{(z-a)} = \frac{a^2+1}{-2a}$. Since $\mathrm{Res}(a) + \mathrm{Res}(-a) = 0$, our final answer is $I = 0$.

ii. Using the method described above, we now consider the integral $I == \oint_C \frac{f(1/\xi)}{\xi^2} d\xi$, which is equivalent to our original integral after making the change of variable $z = \frac{1}{\xi}$ so that we can include the point at infinity $z_\infty = \lim_{\xi \to 0} \frac{1}{\xi}$.
Our integral has become

$$I = \oint_C \frac{\frac{1}{\xi^2} + 1}{\frac{1}{\xi^2} - a^2} \frac{d\xi}{\xi^2} = \oint_C \frac{1 + \xi^2}{\xi^2(1 - a^2\xi^2)} d\xi$$

The denominator of this integrand has zeros at $\xi = 0$ and at $\xi = \frac{1}{a}$. Since $a^2 < 1$, it follows that $\xi = \frac{1}{a} > 1$ and therefore lies outside of our contour. We now consider the point $\xi = 0$ which corresponds to the point at infinity, z_∞.
We will now evaluate the residue of this integral at $\xi = 0$. Since this is an order two pole, we have

$$\text{Res}(\xi = 0) = \lim_{\xi \to 0} \frac{d}{d\xi} \left[\frac{1}{1 - a^2\xi^2} + \frac{\xi^2}{(1 - a^2\xi^2)} \right] = \lim_{\xi \to 0} \left[\frac{2a^2\xi}{(1 - a^2\xi^2)^2} + \frac{2\xi}{(1 - a^2\xi^2)^2} \right] = 0$$

Which is the same result as the one we found in part (i).

(b) $f(z) = \frac{z^2+1}{z^3}$

i. We have an order three pole at $z = 0$, so we can calculate the residue as

$$\text{Res}(z = 0) = \lim_{z \to 0} \frac{1}{2} \frac{d^2}{dz^2}(z^2 + 1) = 1$$

ii. After making our our variable transformation from z to $\xi = \frac{1}{z}$, our integral becomes

$$I = \oint_C \left(\frac{1}{\xi} + \xi \right) d\xi$$

We have a simple pole at $\xi = 0$, and we can calculate the residue here to be

$$\text{Res}(\xi = 0) = \lim_{\xi \to 0} 1 + \xi^2 = 1$$

Which is the same result we found in part (i).

(c) $f(z) = z^2 e^{-1/z}$

i. We see that we have an essential singularity at $z = 0$. Expanding the exponential, we can write $f(z)$ as the Laurent series

$$f(z) = z^2 \left(1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{6z^3} + \dots \right) = z^2 - z + \frac{1}{2} - \frac{1}{6z} + \dots$$

From the above series expansion, we see that the Laurent coefficient $a_{-1} = -\frac{1}{6}$, and therefore the residue at $z = 0$ is $-\frac{1}{6}$.

ii. After making the variable transformation from z to $\xi = 1/z$, our integral becomes

$$I = \oint_C \frac{1}{\xi^2} e^{-\xi} \frac{d\xi}{\xi^2} = \oint_C \frac{1}{\xi^4} e^{-\xi} d\xi$$

This integral has a fourth order pole at $\xi = 0$. We can evaluate the residue at this point using

$$\text{Res}(\xi = 0) = \lim_{\xi \to 0} \frac{1}{6} \frac{d^3}{d\xi^3} e^{-\xi} = \lim \xi \to 0 \frac{1}{6} - e^{-\xi} = -\frac{1}{6}$$

Which is the same as the residue we found in part (i).

2. **Problem 2:** Find the Fourier transform of $f(t)$ where $f(t) = 1$ for $-a < t < a$ and $f(t) = 0$ otherwise.

Then, do the inverse transform using techniques of contour integration, e.g. Jordan's lemma, principle values, etc.

Solution: We begin by calculating the Fourier transform

$$F(x) = \mathcal{F}(f)(x) = \int_{-\infty}^{\infty} f(t) e^{ixt} dt = \int_{-a}^{a} e^{ixt} dt = \left. \frac{e^{ixt}}{ix} \right|_{-a}^{a} = \frac{e^{ixa} - e^{-ixa}}{ix} = \frac{2\sin(ax)}{x}$$

Next we calculate the inverse Fourier transform

$$\mathcal{F}^{-1}(F)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin(ax)}{x} e^{-ixt} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(ax)}{x} e^{-ixt} dx$$

Writing the sin function in exponential form, our problem becomes evaluating the integral

$$I = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(ax)}{x} e^{-ixt} dx = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{x} \left(e^{ix(a-t)} - e^{-ix(a+t)} \right) dx$$

Let $I_1 = \int_{-\infty}^{\infty} \frac{1}{x} e^{ix(a-t)} dx$ and $I_2 = \int_{-\infty}^{\infty} \frac{1}{x} e^{-ix(a+t)} dx$ so that $I = \frac{1}{2\pi i}(I_1 - I_2)$.

We will begin by evaluating $I_1 = \int_{-\infty}^{\infty} \frac{1}{x} e^{ix(a-t)} dx$. We would like to use Jordan's lemma to evaluate this with a contour integral, but to do this we need to ensure that $(a - t)$ is positive. If $a > t$ we may proceed forward, but if $a < t$ then we can make the variable change $x \to -x$ so that $I_1 \to \int_{\infty}^{-\infty} \frac{1}{x} e^{ix(t-a)} dx = -\int_{-\infty}^{\infty} \frac{1}{x} e^{ix(t-a)} dx = -I_1$.

To generalize, we can use the sgn function, where $\text{sgn}(x) = 1$ when $x > 0$, $\text{sgn}(x) = -1$ when $x < 0$, and $\text{sgn}(x) = 0$ when $x = 0$. Using this, we can write

$$I_1 = \text{sgn}(a - t) \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx$$

where $m := |a - t| \geq 0$. We can now use Jordan's lemma, as we see that $\frac{1}{z} \to 0$ as $|z| \to \infty$, and consider the contour integral

$$I_C = \oint_C \frac{e^{imz}}{z} dz$$

where C is a semicircular contour in the upper half of the complex plane. We see that I_C has a simple pole on the contour C at $z = 0$, so we must instead consider the principle value integral $\mathcal{P}I_C$. By the residue theorem, we have

$$\mathcal{P}I_C = i\pi \operatorname{Res}(0)$$

and Res(0) is easily evaluated as $\operatorname{Res}(0) = \lim_{z\to 0} e^{imz} = 1$. Hence, we have

$$\mathcal{P}I_C = i\pi \Rightarrow I_1 = \operatorname{sgn}(a - t)i\pi$$

We now turn our attention to $I_2 = \int_{-\infty}^{\infty} \frac{1}{x} e^{-ix(a+t)} dx$. For notational purposes we make the change of variable $x \to -x$ to write $I_2 = -\int_{-\infty}^{\infty} \frac{1}{x} e^{ix(a+t)} dx$. To use Jordan's lemma, we again require that $(a + t) > 0$. If $t < -a$, then we are free to make the change of variables $x \to -x$ once again so that $I_2 \to -I_2$, and to write the general case we again make use of the sgn function and write

$$I_2 = -\operatorname{sgn}(a + t) \int_{-\infty}^{\infty} \frac{e^{imx}}{x}$$

where $m := |a + t| \geq 0$. We can now use Jordan's lemma and consider the contour integral

$$I_C = \oint_C \frac{e^{imz}}{z} dz$$

But this is the same integral whose principle value we evaluated above. Therefore we know that $I_2 = -\operatorname{sgn}(a + t)i\pi$.

Having evaluated both I_1 and I_2, we can compute the value of our original integral I to be

$$I = \frac{1}{2\pi i}(I_1 - I_2) = \frac{1}{2}(\operatorname{sgn}(a - t) + \operatorname{sgn}(a + t))$$

We see that $\operatorname{sgn}(a - t) + \operatorname{sgn}(a + t) = 2$ for $-a < t < a$ and 0 for $|t| > a$, so it follows that $\mathcal{F}(F)(t) = 1$ for $-a < t < a$ and $\mathcal{F}(F)(t) = 0$ otherwise. So we have indeed recovered our original function $f(t)$.

3. **Problem 3:** Consider the function

$$f(z) = \ln(z^2 - 1)$$

made single-valued by restricting the angles in the following ways, with $z_1 = z - 1 = r_1 e^{i\theta_1}$ and $z_2 = z + 1 = r_2 e^{i\theta_2}$. Find where the branch cuts are for each case by locating where the function is discontinuous. Use AB tests and show your results.

Solution: We can write

$$f(z) = \ln(z^2-1) = \ln((z+1)(z-1)) = \log(z_1 z_2) = \log(r_1 r_2) + i(\theta_1 + \theta_2)$$

Below, we will perform AB tests to determine the existence of branch continuities.

(a) $-3\pi/2 < \theta_1 \leq \pi/2$, $-3\pi/2 < \theta_2 \leq \pi/2 \Rightarrow$ We will first consider $A = -1 - \epsilon + i\delta$ and $B = -1 + \epsilon + i\delta$ as $\epsilon \to 0^+$, and then $\delta \to 0^+$. Geometrically, these points both approach -1 from above in the complex plane. For A we have $\theta_1 = -\pi$ and $\theta_2 \to -3\pi/2$, so $f(A) = \ln(r_1 r_2) - i\frac{5\pi}{2}$. Meanwhile at B we have $\theta_1 = -\pi$ and $\theta_2 = \pi/2$, so that $f(B) = \ln(r_1 r_2) + -i\pi/2$. We have $f(B) - f(A) = i\pi/2$, and so $f(z)$ is discontinuous along $z \in [-1, i\infty)$.

Next we will consider $C = 1 - \epsilon + i\delta$ and $B = 1 + \epsilon + i\delta$ as $\epsilon \to 0^+$, and then $\delta \to 0^+$. These points both approach 1 from above in the complex plane. For C we have $\theta_1 \to -3\pi/2$ and $\theta_2 = 0$, so $f(C) = \ln(r_1 r_2) - i3\pi/2$. Meanwhile at D we have $\theta_1 = \pi/2$ and $\theta_2 = 0$, so that $f(B) = \ln(r_1 r_2) + i\pi/2$. We have $f(B) - f(A) = -i\pi$, and so $f(z)$ is discontinuous along $z \in [1, i\infty)$.

(b) $0 < \theta_1 \leq 2\pi$, $0 < \theta_2 \leq 2\pi \Rightarrow$ We will start by considering $A = -1 + \delta + i\epsilon$ and $B = -1 + \delta - i\epsilon$ where $0 < \delta < 2$ and $\epsilon \to 0$. For A we have $\theta_1 = \pi$ and $\theta_2 \to 0$, so $f(A) = \ln(r_1 r_2) + i\pi$. For B we have $\theta_1 = \pi$ and $\theta_2 = 2\pi$, so that $f(B) = \ln(r_1 r_2) + i3\pi$. $f(B) - f(A) = i2\pi$, so there is a discontinuity across $[-1, 1)$.

We next consider $C = 1 + i\epsilon$ and $D = 1 - i\epsilon$ as $\epsilon \to 0$. For C we have $\theta_1 = \pi/2$ and $\theta_2 \to 0$, so $f(C) = \ln(r_1 r_2) + i\pi/2$. For D we have $\theta_1 = 3\pi/2$ and $\theta_2 = 2\pi$, so that $f(D) = \ln(r_1 r_2) + i7\pi/2$. We have $f(D) - f(C) = i2\pi$, so there is also a discontinuity at $z = 1$.

Lastly we consider $E = 1 + \delta i\epsilon$ and $F = 1 + \delta - i\epsilon$ for $\delta > 0$ as $\epsilon \to 0$. For E we have $\theta_1 \to 0$ and $\theta_2 \to 0$, so $f(C) = \ln(r_1 r_2)$. For F we have $\theta_1 = 2\pi$ and $\theta_2 = 2\pi$, so that $f(F) = \ln(r_1 r_2) + i4\pi$. We have $f(F) - f(E) = i4\pi$, so there is also a discontinuity across $z \in (0, \infty)$.

In sum, we have a branch cut from $[-1, \infty)$.

(c) $-\pi < \theta_1 \leq \pi$, $0 < \theta_2 \leq 2\pi \Rightarrow$ We first consider $A = -1 - \delta + i\epsilon$ and $B = -1 - \delta - i\epsilon$ for $\delta > 0$ and $\epsilon \to 0$. For A we have $\theta_1 = \pi$ and $\theta_2 = \pi$, so that $f(A) = \ln(r_1 r_2) + i2\pi$. For B we have $\theta_1 = -\pi$ and $\theta_2 = \pi$, so that $f(B) = \ln(r_1 r_2)$. We have $f(B) - f(A) = -i2\pi$, and so there exists a discontinuity across $z \in (-\infty, -1)$.

Next we consider $C = \delta + i\epsilon$ and $D = \delta - i\epsilon$ for $\delta \in (-1, 1)$ and $\epsilon \to 0$. For C we have $\theta_1 = \pi$ and $\theta_2 \to 0$, so that $f(C) = \ln(r_1 r_2) + i\pi$. For D we have $\theta_1 = -\pi$ and $\theta_2 = 2\pi$, so that $f(D) = \ln(r_1 r_2) + i\pi$. We have $f(C) - f(D) = 0$, and so there is no discontinuity across $z \in (-\infty, -1)$. The branch cuts at z_1 and

z_2 effectively cancel out in the region between -1 and 1.

Next we consider $E = 1 + \delta + i\epsilon$ and $F = 1 + \delta - i\epsilon$ for $\delta > 0$ and $\epsilon \to 0$. For E we have $\theta_1 = 0$ and $\theta_2 \to 0$, so that $f(E) = \ln(r_1 r_2)$. For F we have $\theta_1 = 0$ and $\theta_2 = 2\pi$, so that $f(F) = \ln(r_1 r_2) + i2\pi$. We have $f(F) - f(E) = i2\pi$, and so there exists a discontinuity across $z \in (1, \infty)$.

Lastly, we consider the points $G = -1 + i\epsilon$, $H = -1 - i\epsilon$, $I = 1 + i\epsilon$, and $J = 1 - i\epsilon$. For G we have $\theta_1 = \pi$ and $\theta_2 = \pi/2$, and so $f(G) = \ln(r_1 r_2) + i3\pi/2$. For H we have $\theta_1 = -\pi$ and $\theta_2 = 3\pi/2$, so that $f(H) = \ln(r_1 r_2) + i\pi/2$. We have $f(H) - f(G) = -i\pi$, so we have a discontinuity across $z = -1$. Likewise, for I we have $\theta_1 = \pi/2$ and $\theta_2 \to 0$, so that $f(I) = \ln(r_1 r_2) + i\pi/2$. For J we have $\theta_1 = -\pi/2$ and $\theta_2 = 2\pi$, so that $f(J) = \ln(r_1 r_2) + i3\pi/2$. We have $f(J) - f(I) = i\pi$, so there is also a discontinuity across $z = 1$.

In sum, we have branch cut discontinuities across $z \in (-\infty, -1] \cup [1, \infty)$.