The eigenvalues of the Sturm-Liouville system are discrete, and form an increasing sequence:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

with $\lambda_j \to \infty$ as $j \to \infty$

provided that the domain is finite and $p(x) > 0$, $r(x) > 0$ in $a < x < b$.

We show first that the eigenvalues are discrete:

Let λ be a value intermediate between λ_n and λ_{n+1}. λ is not an eigenvalue. Let $\phi(x)$ be the solution corresponding to λ. Since λ is not an eigenvalue, $\phi(x)$ is not an eigenfunction. We let $\phi(x)$ satisfy the boundary condition at $x = a$ but not $x = b$.

From a previous result:

$$(\lambda - \lambda_n) \int_a^b r \phi_n(x) \phi(x) dx$$

$$= p(b) [\phi(b) \phi_n'(b) - \phi_n(b) \phi'(b)]$$

$$\neq 0$$

Thus unless $\int_a^b r \phi_n(x) \phi(x) dx$ is infinite,

$\lambda - \lambda_n$ is finite, even as $n \to \infty$

Since $\lambda_{n+1} > \lambda$, we have

$(\lambda_{n+1} - \lambda_n)$ is finite.

It then follows that the sequence

$\lambda_1, \lambda_2, \lambda_3, \dots ; \lambda_n, \lambda_{n+1}, \dots$

can have no upper bound, but must continue to $+\infty$.

Possibility exists for a continuous distribution of λ only if $(b-a)$ is infinite (so that $\int_a^b r\phi_n(x)\phi(x)dx$ is infinite)

Nevertheless, if the solution decays sufficiently rapidly as $x\to\infty$ so that $\int_a^b r\phi_n\phi dx$ is finite, the eigenvalues are still discrete even if the domain is infinite.

Variational Principle

$$\Omega(\psi) \equiv \int_a^b (p\psi'^2 + q\psi^2) \Big/ \int_a^b r\psi^2 dx$$
$$\geq 0$$

Rayleigh quotient for any piecewise continuous function $\psi(x)$. $\psi(x)$ does not need to satisfy the S-L equation, but we assume that it satisfies the same boundary conditions.

Since $\Omega(\psi)$ is nonnegative, it must have a greatest lower bound. That minimum turns out to be λ_1, the lowest eigenvalue, i.e.

$$\lambda_1 = \Omega(\phi_1) = \min_{\psi} \Omega(\psi)$$

To find the minimum for $\Omega(\psi)$, we evaluate the functional derivative of $\Omega(\psi)$ as

$$\left.\frac{\partial \Omega(\psi+\alpha g)}{\partial \alpha}\right|_{\alpha=0} = 0$$

for any continuous differentiable $g(x)$ which vanishes at $x=a$ and $x=b$.

$$0 = \frac{2\int_a^b (p\psi' g' + q\psi g)\,dx}{\int_a^b r\psi^2\,dx} - \frac{2\int_a^b r\psi g\,dx \int_a^b (p\psi'^2 + q\psi^2)\,dx}{\left(\int_a^b r\psi^2\,dx\right)^2}$$

Let the minimum of $\Omega(\psi)$ be μ.

$$\int_a^b (p\psi' g' + q\psi g)\,dx$$

$$= \frac{\int_a^b (p\psi'^2 + q\psi^2)\,dx}{\int_a^b r\psi^2\,dx} \cdot \int_a^b r\psi g\,dx$$

$$= \mu \int_a^b r\psi g\,dx$$

$$\int_a^b [\, p\psi' g' + q\psi g - \mu r \psi g \,]\,dx = 0$$

Integrate by parts:

$$-\int_a^b g\,[(p\psi')' + (\mu r - q)\psi]\,dx = 0$$

Since it is true for every admissible function $g(x)$, $[\ \ ]$ in the integrand must vanish.

$$[(p\psi')' + (\mu r - q)\psi] = 0$$

i.e. ψ must be an ~~eigenvalue~~ eigenfunction, μ must be an eigenvalue

So :

$$\lambda_1 = \min \Omega(\psi) = \Omega(\phi_1).$$

The minimum of $\Omega(\psi)$ is the lowest eigenvalue λ_1, achieved when ψ is the lowest eigenfunction ϕ_1.

Next find the minimum of $\Omega(\psi)$ subject to the additional constraint that ψ must be orthogonal to the first eigenfunction, i.e. $\int_a^b r\psi\phi_1\,dx = 0$.

The minimum of $\Omega(\psi)$ is λ_2, and since there is one additional constraint,

$$\lambda_2 > \lambda_1.$$

The minimizing ψ is ϕ_2.

Continuing, we can show that

$$\lambda_k = \min \Omega(\psi)$$

subject to the constraint ψ be orthogonal to $\phi_1, \phi_2, \cdots, \phi_{k-1}$.