AMATH 581, Homework 3 Presentation mastery

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1 Discussing problems from a physical perspective

1.1 Introduction

In this paper we will examine the 1D advection equation, which describes the flow of a quantity being advected by a velocity field, and numerically solve it with periodic boundary conditions and a Gaussian initial value.

The 1D advection equation is given by

$$u_t + c(x,t)u_x = 0$$

This equation describes the flow of a quantity $u(x,t)$, such as water waves on a beach, heat or humidity in the atmosphere, as it is advected by a velocity field, c.

Before we evaluate this equation numerically, we might compare it to other similar equations to get an intuition for its physical effects. We note that for constant $c(x,t) = C$ the equation is equivalent to the **transport equation**

$$u_t = -Cu_x$$

Like the advection equation, this equation has many physical applications and arises when an initial condition $f(x,0) = f_0(x)$ is being transported to the left or right at a constant velocity, so that $f(x,t) = f_0(x - Ct)$. We would expect the advection equation to have a similar effect of transporting an initial condition, except for the in advection equation the velocity field $c(x,t)$ is not constant, but may vary across time and space.

A standard example of a physical phenomena whose behaviour can be modeled by the advection equation, and in certain circumstances by the transport equation, is water waves. Under certain circumstances water waves can maintain a fairly constant shape, which moves at a velocity proportional to the square root of the water depth. Hence, for areas of roughly constant depth, water waves travel at a constant speed and maintain their original shape as they are transported forward. On the other hand if the water depth is changing or irregular it can be modeled by the advection equation, where $c(x,t) \propto \sqrt{h(x,t)}$. There are many physical situations where this is the case, such as beaches, whose water depth decreases as it gets nearer to the shore, or which may have reefs or discontinuous dropoffs near islands which affect the surface wave velocity.

To better understand the effect of the velocity c we will numerically solve this equation with periodic boundary conditions and initial condition $u(x,0) = e^{-(x-5)^2}$, for different velocity flows. First, we will consider the constant velocity flow $c(x,t) = -0.5$. This flow can be thought of as modeling a water wave traveling in a region of constant depth. We have already discussed that we expect this velocity flow to result in a transport of the initial condition to the left at a constant velocity of 0.5. We will next look at the velocity flow $c(x,t) = -(1+2\sin(5t) - H(x-4))$, where $H(x-4)$ is the Heaviside function which is 0 for $x < 4$ and 1 for $x > 4$. This can be thought of as modeling a water wave traveling across a region whose depth is varying with time, and which has a discontinuous drop-off at $x = 4$.

1.2 Results

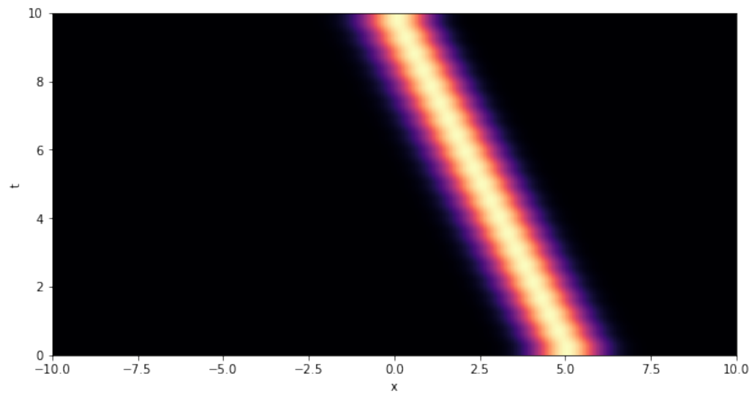

Figure 1: Solution for $c(x,t) = -0.5$

For the $c(x,t) = -0.5$ case, our results agree with our expectation. In this case, the initial condition is simply transported to the left at constant velocity. We can see this in Figure 1. At $t = 0$, $u(x,0)$ is a Gaussian function centered at $x = 5$. Over time, it moves to the left at a speed of 0.5, so that by time $t = 10$ it is centered over the origin.

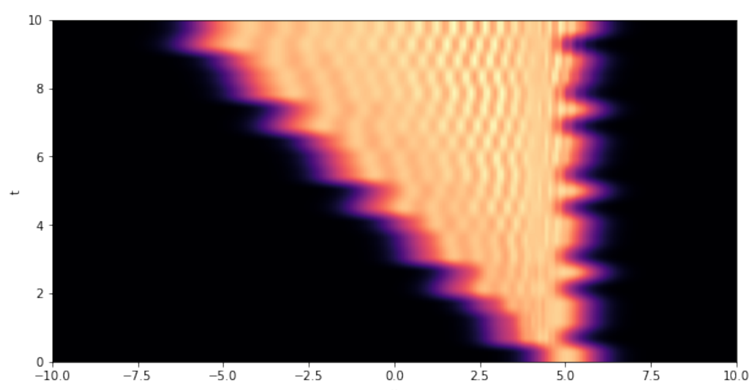

Figure 2: Solution for $c(x,t) = -0.5$

Our results are much different for $c(x,t) = -(1+2\sin(5t) - H(x-4))$. We note that for $x > 4$ the velocity flow is simply $\sin(5t)$, and hence we would expect that the righthand tail of our Gaussian initial condition would oscillate to the left and right with time. Meanwhile, for $x < 4$ the oscillatory term remains, but it is shifted downwards by a constant term of -1. Hence, we would expect the portions of our initial condition to the left of $x = 5$ to drift to the left at an average speed which is twice that of our original $c(x, t)$, but which oscillates between -3 and 1. Indeed, this is the behaviour we find in our numerical results.

1.3 Conclusion

In this paper we examined the advection equation and discussed its physical applications and relation to the the transport equation. Using our understanding of the physical effects transport equation, we discussed the expected effect of the velocity flow function $c(x, t)$ on our solutions. We solved the initial value problem with different velocity flow functions and discussed our results.