AMATH 573, Problem Set 2

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1. Problem 1: The Benjamin-Ono equation

$$u_t + uu_x + \mathcal{H}u_{xx} = 0$$

is used to describe internal water waves in deep water. Here $\mathcal{H}f(x,t)$ is the spatial Hilbert transform of $f(x,t)$:

$$\mathcal{H}f(x,t) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{f(z,t)}{z-x} dz$$

and ⨍ denotes the Cauchy principle value integral. Write down the linear dispersion relationship for this equation linearized about the zero solution.

Solution:

We begin by linearizing u around the zero solution. We let $u(x,t) = \epsilon U(x,t)$, where $\epsilon << 1$. Then the Benjamin-Ono equation becomes

$$\epsilon U_t + \epsilon \mathcal{H} U_{xx} + \mathcal{O}(\epsilon^2) = 0$$

Dropping $\mathcal{O}(\epsilon^2)$ terms leaves us with the linear PDE

$$U_t = -\mathcal{H} U_{xx}$$

We will look for solutions of the form $U(x,t) = e^{ikx - i\omega t}$, which can be superimposed via the Fourier transform to construct a general solution. Plugging this into our equation and using the linearity of $\mathcal{H}$ gives us

$$-i\omega U = k^2 \mathcal{H} U$$

We will now work on evaluating $\mathcal{H}U(x,t)$. We have

$$\mathcal{H}U = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{e^{ikz - i\omega t}}{z-x} dz == \frac{e^{-i\omega t}}{\pi} ⨍_{-\infty}^{\infty} \frac{e^{ikz}}{z-x} dz$$

We make the substitution $s = k(z - x)$, $ds = kdz$, so that our integral becomes

$$\mathcal{H}U = \frac{U}{\pi} ⨍_{-\infty}^{\infty} \frac{e^{is}}{s} ds = \frac{U}{\pi} ⨍_{-\infty}^{\infty} \frac{\cos(s) + i\sin(s)}{s} ds$$

Note that the imaginary part of the integrand is even, and the real part is odd. Since we are integrating from $(-\infty, \infty)$, our integral can be rewritten as

$$\mathcal{H}U = \frac{iU}{\pi} ⨍_{-\infty}^{\infty} \frac{\sin(s)}{s} ds = \frac{iU}{\pi} \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} \frac{\sin(s)}{s} ds + \int_{\epsilon}^{\infty} \frac{\sin(s)}{s} ds \right] = \frac{2iU}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\sin(s)}{s} ds$$

We recognize that in the limit as $\epsilon \to 0$ the above integral approaches the Dirichlet integral $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$, and hence

$$\mathcal{H}U = iU$$

Therefore, returning to the linear dispersion relation derived above, we now have

$$-i\omega = ik^2 \implies \omega = -k^2$$

2. Problem 2: Derive the linear dispersion relationship for the one-dimensional surface water wave problem by linearizing around the trivial solution $\zeta(x,t) = 0$, $\phi(x,z,t) = 0$:

   (a) $\nabla^2\phi = 0$ at $-h < z < \zeta(x,t)$
   (b) $\phi_z = 0$ at $z = -h$
   (c) $\zeta_t + \phi_x\zeta_x = \phi_z$ at $z = \zeta(x,t)$
   (d) $\phi_t + g\zeta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = T\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}$ at $z = \zeta(x,t)$

   Here $z = \zeta(x,t)$ is the surface of the water wave, $\phi(x,z,t)$ is the velocity potential so that $v = \nabla\phi$ is the velocity of the water, g is the acceleration of gravity, and $T > 0$ is the coefficient of surface tension.

   Solution: We begin by linearizing around the trivial solution $\zeta(x,t) = 0$, $\phi(x,z,t) = 0$ by letting $\zeta(x,t) = 0 + \epsilon\xi(x,t)$ and $\phi(x,z,t) = 0 + \epsilon\varphi(x,z,t)$. Where $\epsilon << 1$ and $\xi, \varphi \neq 0$.

   Plugging these in to our above equations and considering only terms $\mathcal{O}(\epsilon)$ gives us the following linearized equations:

   (a) $\nabla^2\varphi = 0$ at $-h < z < \zeta(x,t)$
   (b) $\varphi_z = 0$ at $z = -h$
   (c) $\xi_t = \varphi_z$ at $z = \zeta(x,t)$
   (d) $\varphi_t + g\xi = T\xi_{xx}$ at $z = \zeta(x,t)$

   We will start by considering horizontal wave solutions of $\varphi(x,z,t)$ which have the form $\varphi(x,z,t) = f(z)e^{i\hat{k}x - i\hat{\omega}t} = f(z)\Phi(x,t)$.

   Equation (a) then becomes $f(z)\Phi_{xx} = -f''(z)\Phi$, and so

$$-\hat{k}^2 f\Phi = -f''(z)\Phi$$

   For $\Phi(x,t) \neq 0$ we have

$$f''(z) = -\hat{k}^2 f$$

   Which has solutions of the form $f(z) = Ae^{\hat{k}x} + Be^{-\hat{k}x}$. We can now apply boundary condition (b) to solve for one of our two unknown coefficients. We have

$$kAe^{\hat{k}h} - kBe^{-\hat{k}h} = 0$$

   Rearranging gives $B = Ae^{2\hat{k}h}$, and plugging this back in to $f(z)$ results in solutions of the form

$$f(z) = A\cosh(\hat{k}(z+h))$$

We turn now to equation (c). We will consider $\xi(x,t)$ of the form $\xi(x,t) = e^{ikx-i\omega t}$. Plugging this in to (3) along with our expression for φ results in

$$-i\omega e^{ikx-i\omega t} = f'(\epsilon\xi)e^{i\hat{k}x-i\hat{\omega}t}$$

Note that both the right and left-hand side are separable functions of x and y. By inspection we deduce that $\hat{k} = k$ and $\hat{\omega} = \omega$, and hence that $\varphi(x,z,t) = A\cosh(k(z-h))\xi(x,t)$. Hence we have, for $\xi(x,t) \neq 0$,

$$-i\omega = f'(\epsilon\xi)$$

Since we have assumed small ϵ, we may Taylor expand the right-hand side around the equilibrium solution $\zeta = 0$. We have $f(z) = f(0) + f'(0)z + \mathcal{O}(z^2)$, and therefore $f'(z) = f'(0) + \mathcal{O}(z)$. And so for for small z, $f'(z) \approx f'(0)$. Since we already derived an expression for $f(z)$, we can differentiate this and plug it in to get

$$-i\omega = kA\sinh(kh)$$

Solving for A, we find that

$$A = \frac{-i\omega}{k\sinh(kh)}$$

Lastly, we can use our above results to analyze equation (d). Plugging in our solutions for φ and ξ gives us

$$-i\omega\cosh(k(z+h))\xi + g\xi = -k^2T\xi$$

And since this occurs on the surface and we are assuming small z values, we may Taylor expand the cosh term and keep only the $\mathcal{O}(1)$ terms so that $\cosh(k(z+h)) \approx \cosh(kh)$. Making this substitution, canceling ξ terms, substituting the above expression for A, and rearranging gives us

$$i\omega\left(\frac{-i\omega}{k\sinh(kh)}\right)\cosh(kh) = g + k^2T$$

Therefore, solving for ω, we get our final answer:

$$\omega^2 = (gk + k^3T)\tanh kh$$

3. Problem 3: Having found that for the surface water wave problem without surface tension the linear dispersion relationship is $\omega^2 = gk \tanh kh$, find the group velocities for the case of long waves in shallow water (kh small), and for the case of deep water (kh big).

   Solution: For long shallow waves (kh small), we can Taylor expand $\tanh kh$ approximate the dispersion relation as $\omega^2 = gk(kh) = gk^2h$, which gives us $\omega = k\sqrt{gh}$. Therefore for long waves in shallow water the group velocity $v_g = \frac{d\omega}{dk} = \sqrt{gh}$.

   For short waves in deep water (kh big) we have $\lim_{k\to\infty} \tanh kh = 1$. Therefore as kh gets very large, our dispersion relation begins to approach $\omega^2 = gk$ and hence $\omega = \sqrt{gk}$. Therefore, for short waves in deep water the group velocity $v_g = \frac{d\omega}{dk} = \frac{g}{2\sqrt{gk}} = \frac{1}{2}\sqrt{\frac{g}{k}}$

4. Problem 4: Whitham wrote down what is now known as **the Whitham equation** to incorporate the full effect of water-wave dispersion for waves in shallow water by modifying the KdV equation $u_t + vu_x + uu_x + \gamma u_{xxx} = 0$ (where we have included the transport term) to

$$u_t + uu_x + \int_{-\infty}^{\infty} K(x-y)u_y(y,t)dy = 0$$

where

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk$$

and $c(k)$ is the postive phase speed for the water-wave problem: $c(k) = \sqrt{g\tanh(kh)/k}$.

   (a) What is the linear dispersion relation of the Whitham equation?

   (b) Show that the dispersion relation of the KdV equation is an approximation to that of the Whitham equation for long waves, i.e., for $k \to 0$. What are v and γ?

Note that using this process of "Whithamization", one could construct a KdV-like equation (i.e., an equation with the KdV nonlinearity) that has any desired dispersion relation. Similar procedures can be followed for other equations, like the NLS equation, etc.

Solution:

   (a) We begin by linearizing $u(x,t)$ around the zero soution. That is, we let $u(x,t) = \epsilon v(x,t)$. Plugging this solution into our equation and keeping only $\mathcal{O}(\epsilon)$ terms leaves us with

$$v_t + \int_{-\infty}^{\infty} K(x-y)v_y(y,t)dy = 0$$

   We will look for solutions of the form $v(x,t) = e^{ikx-i\omega t}$, which can be linearly superimposed to construct a general solution to this linearized PDE. Plugging this expression into our equation gets us

$$-i\omega v + ik\int_{-\infty}^{\infty} K(x-y)v(y,t)dy = 0$$

   Let us now focus on evaluating the integral $\int_{-\infty}^{\infty} K(x-y)v(y,t)dy$. Explicitly plugging our wave solution for v into this equation gives us

$$\int_{-\infty}^{\infty} K(x-y)e^{iky-i\omega t}dy = 0$$

We now make the variable substitution $s = x - y$, $ds = -dy$, so that our equation becomes

$$-\int_{\infty}^{-\infty} K(s)e^{ik(x-s)-i\omega t}ds = v\int_{-\infty}^{\infty} K(s)e^{-iks}ds = 0$$

Where we have pulled $v(x,t) = e^{ikx-i\omega t}$ out of the integral.

Looking at the definition of $K(x)$, we see that this function is nothing more than the Fourier transform of $c(k)$, $K(s) = \mathcal{F}\left\{c(k)\right\}(s)$. Furthermore, our above integral is in fact v times the inverse Fourier transform of $K(s)$, $\mathcal{F}^{-1}\left\{K(s)\right\}(k)$.

We note briefly that $c(k)$ is undefined at $k = 0$, however this is a removable singularity since the series expansion of $\tanh(k) = k - \frac{k^2}{3} + \frac{2k^5}{15} + \mathcal{O}(k^7)$. Therefore, we can redefine $c(k)$ to be an entire function by fixing $c(k) := 1$, which would enable us to integrate as usual. Alternatively, we could use the Cauchy principle value, as in the definition of $K(x)$. Both will give the same result, because $\lim_{k\to 0^+} c(k) = \lim_{k\to 0^-} c(k) = 1$.

Returning to our problem, since our equation involves the inverse Fourier transform applied to a Fourier transform, the outcome is simply the original function, $c(k)$. We therefore have

$$\int_{-\infty}^{\infty} K(x-y)v(y,t)dy = vc(k)$$

And so our equation has become

$$-i\omega v + ikvc(k) = -i\omega v + ikv\sqrt{g\tanh(kh)/k} = 0$$

Solving this equation for ω gives the result

$$\omega = \sqrt{gk\tanh(kh)}$$

(b) For small x, $\tanh(x)$ can be Taylor expanded to $\tanh(x) = x - \frac{x^3}{3} + \mathcal{O}(x^5)$. Using this fact, we can rewrite the Whitham equation dispersion relation for small k as

$$\omega = \sqrt{gk(kh - k^3h^3/3)} = k\sqrt{gh - gk^2h^3/3}$$

We now employ the Taylor expansion $\sqrt{a-x} = \sqrt{a} - \frac{x}{2\sqrt{a}} + \mathcal{O}(x^2)$ to expand the square root term. Then, again in the limit of small k, we have

$$\omega = \sqrt{gh}\left(k - \frac{k^3h^2}{6}\right)$$

For the KdV equation $u_t + vu_x + uu_x + \gamma u_{xxx} = 0$ we linearize around the zero solution to get the linearized KdV equation $u_t + vu_x + \gamma u_{xxx}$, whose disperion relation is

$$-i\omega + ivk - i\gamma k^3 = 0 \implies \omega = vk - \gamma k^3$$

Comparing with our above approximation of the Whitham equation for small k, we see that the KdV dispersion relation is indeed an approxition to that of the Whitham equation for long waves, with $v = \sqrt{gh}$ and $\gamma = \frac{\sqrt{gh}h^2}{6}$.

5. Problem 5: Consider the linear free Schrödinger ("free", because there's no potential) equation

$$i\psi_t + \psi_{xx} = 0$$

where $-\infty < x < \infty$, $t > 0$, $\psi \to 0$ as $|x| \to \infty$. With $\psi(x,0) = \psi_0(x)$ such that $\int_{-\infty}^{\infty} |\psi_0|^2 dx < \infty$.

   (a) Using the Fourier transform, write down the solution of this problem.
   (b) Using the Method of Stationary Phase, find the dominant behavior as $t \to \infty$ of the solution, along lines of constant x/t.
   (c) With $\psi_0(x) = e^{-x^2}$, the integral can be worked out exactly. Compare (graphically or otherwise) this exact answer with the answer you get from the Method of Stationary Phase. Use the lines $x/t = 1$ and $x/t = 2$ to compare.
   (d) Use your favorite numerical integrator (write your own, or use maple, mathematica or matlab) to compare (graphically or other) with the exact answer and the answer you get from the Method of Stationary Phase.

Solution:

   (a) We will now solve the linear free Schrödinger equation using the Fourier transform. We will consider solutions of the form $\psi(x,t) = e^{ikx-i\omega t}$. Plugging this in gives us the dispersion relation $\omega - k^2 = 0$ and therefore we have

$$\omega = k^2$$

Our solutions are functions of the form $\psi_k(x,t) = e^{ikx-ik^2t}$. We may write the general solution of the equation as

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{ikx-ik^2t}dk$$

To find the values of the coefficients $a(k)$ we use our initial condition $\psi(x,0) = \psi_0(x)$. We have

$$a(k) = \langle\psi_k|\psi_0\rangle = \int_{-\infty}^{\infty} \psi_k^*(x,0)\psi_0(x)dx = \int_{-\infty}^{\infty} e^{-ikx}\psi_0(x)dx$$

And so our final solution is

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{ikx-ik^2t}dk$$
$$a(k) = \int_{-\infty}^{\infty} e^{-ikx}\psi_0(x)dx$$

(b) We will now use the Method of Stationary Phase to find the dominant behaviour as $t \to \infty$ of the solution along lines of constant x/t. Looking again at our general solution

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{ikx-ik^2t}dk$$

we see that

$$\phi(k) = k\frac{x}{t} - k^2 \implies \phi'(k) = \frac{x}{t} - 2k \implies \phi''(k) = -2$$

Stationary points occur when $k = k_0 = \frac{x}{2t}$. The dominant contribution from waves with wave numbers near k_0 to $\Psi(x,t)$ is

$$\frac{1}{2\pi}\int_{k_0-\delta}^{k_0+\delta} a(k)e^{i\phi(k)t}dk$$

Since we are assuming $\delta << 1$, we Taylor expand $\phi(k)$ around k_0 and approximate this integral with

$$\frac{1}{2\pi}a(k_0)\int_{k_0-\delta}^{k_0+\delta} e^{i\left[\phi(k_0)+\frac{1}{2}(k-k_0)^2\phi''(k_0)\right]t}dk = \frac{1}{\pi}a(k_0)e^{i\phi(k_0)t}\int_{k_0}^{k_0+\delta} e^{i\frac{1}{2}(k-k_0)^2\phi''(k_0)t}dk$$

Where we have used the stationarity condition $\phi'(k_0) = 0$ and the evenness of the integrand around k_0. This integral can be evaluated by using the substitution $\kappa^2 = \frac{1}{2}(k-k_0)^2|\phi''(k_0)|t$, taking the limit as $t \to \infty$, and then using the Fresnel integral to evaluate the remaining term, as done in chapter 2.4 of *Deconinck - Nonlinear Waves*. Doing so results in the expression

$$\frac{a(k_0)}{\sqrt{2\pi t|\phi''(k_0)|}}e^{i\phi(k_0)t+i\pi\mathrm{sgn}(\phi''(k_0))/4}$$

which corresponds to the contribution of the stationary point k_0 to $\Psi(x,t)$. Applying this result with the $\phi(k)$ we derived above, we have

$$\Psi(x,t) \approx \frac{a(k_0)}{2\sqrt{\pi t}}e^{ik_0^2t-i\pi/4}$$

Lastly, we can evaluate $a(k_0)$ as

$$a(k_0) = \int_{-\infty}^{\infty} e^{-x^2(\frac{i}{2t}+1)}dx = \sqrt{\frac{\pi}{\frac{i}{2t}+1}} = \sqrt{\frac{2t\pi}{2t+i}}$$

Plugging this in to our approximate solution for $\Psi(x,t)$ gives us

$$\Psi(x,t) \approx \frac{e^{ik_0^2t-i\pi/4}}{\sqrt{4t+2i}}$$

(c) For $\psi_0(x) = e^{-x^2}$ we have

$$a(k) = \int_{-\infty}^{\infty} e^{-ikx-x^2} dx$$

Completing the square, we have

$$a(k) = e^{-k^2/4} \int_{-\infty}^{\infty} e^{(i\frac{k}{2}+x)^2} dx$$

Substituting $z = i\frac{k}{2} + x$, we have

$$a(k) = e^{-k^2/4} \int_{\frac{ik}{2}-\infty}^{\frac{ik}{2}+\infty} e^{-z^2} dz$$

We can evaluate this integral by considering the rectangular contour with corners at $\pm R$ and $\frac{ik}{2} \pm R$ as $R \to \infty$. The contribution of the vertical components to the integral goes to zero as R goes to infinity. The upper horizontal contour is the one we are interested in evaluating, while the lower contour is the familiar Gaussian integral on the real line, which we know evaluates to $-\sqrt{\pi}$, (the negative reflects that this is a leftward directed contour). Since this function is analytic, by Cauchy's Integral Theorem we know that $I_{\text{Top}} + I_{\text{Bottom}} = I_{\text{Top}} - \sqrt{\pi} = 0$ and therefore we conclude that the integral evaluates to $\sqrt{\pi}$. And so our final result is

$$a(k) = \sqrt{\pi} e^{-k^2/4}$$

and our full solution is

$$\Psi(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2/4} e^{ikx-ik^2 t} dk$$

Using Mathematica, we find that this becomes

$$\Psi(x,t) = \frac{e^{\frac{-x^2}{1+4it}}}{\sqrt{1+4it}}$$

For $x/t = 1$, this is

$$\Psi(t) = \frac{e^{\frac{-t^2}{1+4it}}}{\sqrt{1+4it}}$$

and for $x/t = 2$ it is

$$\Psi(t) = \frac{e^{\frac{-4t^2}{1+4it}}}{\sqrt{1+4it}}$$

Let us compare these to the approximate solutions we get from the Method of Stationary Phase.

For $x/t = 1$ we have $k_0 = 1/2$, and so

$$\Psi(x,t) \approx \frac{e^{it/4 - i\pi/4}}{\sqrt{4t + 2i}}$$

While for $x/t = 2$ we have $k_0 = 1$, which gives us

$$\Psi(x,t) \approx \frac{e^{it - i\pi/4}}{\sqrt{4t + 2i}}$$

Graphing these solutions in Mathematica, we find that the stationary phase approximation holds up remarkably well! We see that for $\frac{x}{t} = 1$ the solution oscillates more slowly, and the approximate solution is much more accurate, while for $\frac{x}{t} = 2$ the solution oscillates much more rapidly, and the approximate solution is less accurate.

Figure 3 is the most interesting to me. We see that while the error decreases precipitously initially, it does not approach zero but instead levels off rather quickly at some asymptotic value. The speed at which the error approaches this asymptote is a bit surprising to me, given that we derived this approximation in the limit of $t \to \infty$.

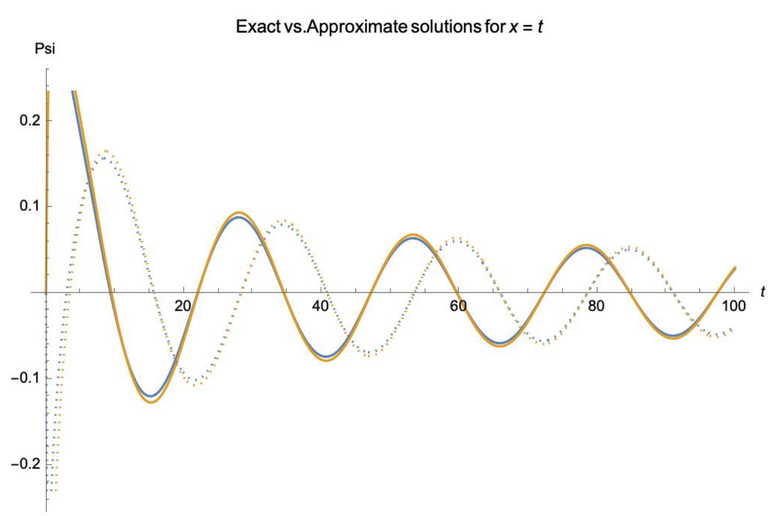

Figure 1: Exact and approximate solutions for $\frac{x}{t} = 1$

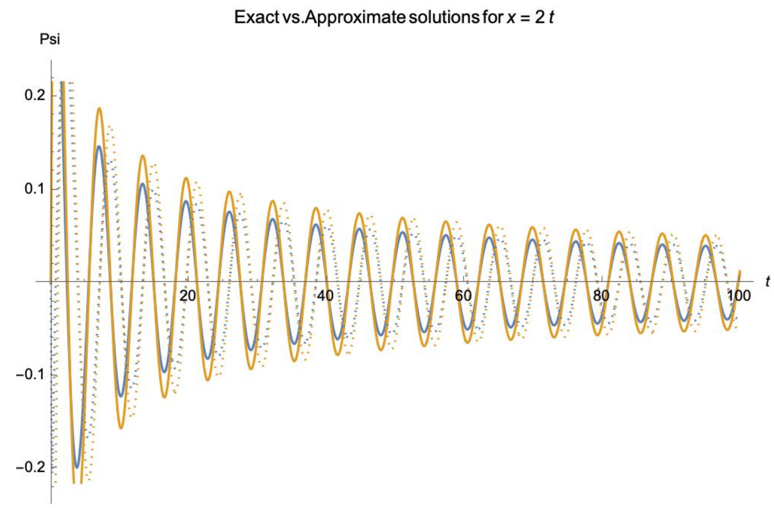

Figure 2: Exact and approximate solutions for $\frac{x}{t} = 2$

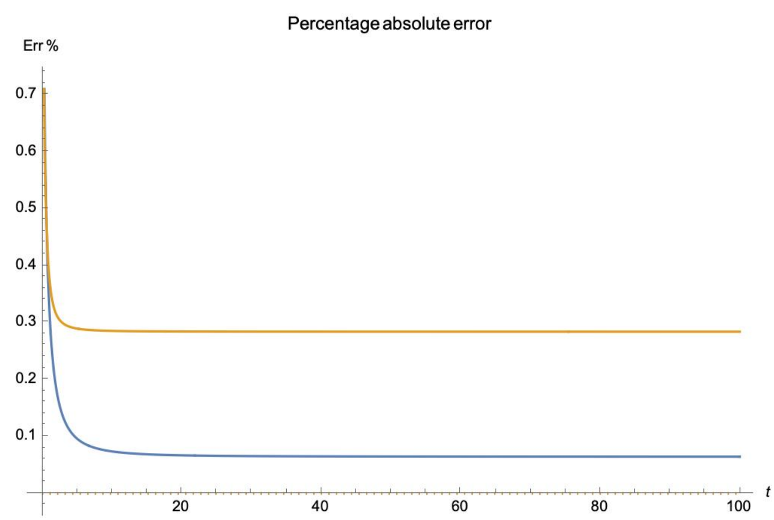

Figure 3: Error as percentage of exact solution.

(d) See Mathematica notebook.

6. Problem 6: Everything that we have done for continuous space equations also works for equations with a discrete space variable. Consider the discrete linear Schrödinger equation:

$$i\frac{d\psi_n}{dt} + \frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) = 0$$

where h is a real constant, n is any integer, $t > 0$, $\psi_n 0$ as $|n| \to \infty$, and $\psi_n(0) = \psi_{n,0}$ is given.

(a) The discrete analogue of the Fourier transform is given by

$$\psi_n(t) = \frac{1}{2\pi i}\oint_{|z|=1} \hat{\psi}(z,t) z^{n-1} dz$$

and its inverse

$$\hat{\psi}(z,t) = \sum_{m=-\infty}^{\infty} \psi_m(t) z^{-m}$$

Show that these two transformations are indeed inverses of each other.

(b) The dispersion relation of a semi-discrete problem is obtained by looking for solutions of the form $\psi_n = z^n e^{-i\omega t}$. Show that for the semi-discrete Schrödinger eqaution

$$\omega(z) = -\frac{(z-1)^2}{zh^2}$$

How does this compare to the dispersion relation of the continuous space problem? Specifically, demonstrate that you recover the dispersion relationship for the continuous problem as $h \to 0$.

Solution:

(a) We will first show that the discrete Fourier transform of its inverse recovers the original function.

Plugging the definition of $\hat{\psi}(z,t)$ into the definition of $\psi_n(t)$ gives us

$$\psi_n(t) = \frac{1}{2\pi i}\oint_{|z|=1} \left(\sum_{m=-\infty}^{\infty} \psi_m(t) z^{-m} \right) z^{n-1} dz$$

We exchange the integral and the summation to get

$$\psi_n(t) = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} \psi_m(t) \oint_{|z|=1} z^{n-m-1} dz$$

We now recall a useful result from complex analysis, which is that $\oint_{|z|=1} z^n dz = 2\pi i$ when $n = -1$ and 0 otherwise. Using this fact, we see that the above expression is

$$\psi_n(t) = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} \psi_m(t) 2\pi i \delta_{mn} = \psi_n(t)$$

So taking the discrete Fourier transform of an inverse discrete Fourier transform as defined above does indeed recover the original function.

We will now show the reverse direction is also valid. Plugging the definition of $\psi_n(t)$ into $\hat{\psi}(z,t)$ gives us

$$\hat{\psi}(z,t) = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2\pi i} \oint_{|z|=1} \hat{\psi}(z,t) z^{m-1} dz \right) z^{-m}$$

Note that since m ranges over all integers, positive and negative, we are free to change $m \to -m$ in the summand. Doing so gives us

$$\hat{\psi}(z,t) = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2\pi i} \oint_{|z|=1} \hat{\psi}(z,t) z^{-(m+1)} dz \right) z^{m}$$

We now recall the definition of the Laurent series, and recognize that the expression inside the parentheses is by definition the coefficient a_m of the Laurent series about the origin of $\hat{\psi}(z,t)$. Therefore, this equation is expressing that $\hat{\psi}(z,t)$ is equal to its Laurent series, which is clearly true.

We have shown that the two transformations as defined are indeed inverses of each other.

(b) Plugging $\psi_n = z^n e^{-i\omega t}$ into the discrete linear Schrödinger equation above yields the dispersion relation

$$\omega + \frac{1}{h^2}(z - 2 + z^{-1}) = 0$$

Rearranging gives

$$\omega = -\frac{z^2 - 2z + 1}{zh^2} = -\frac{(z-1)^2}{zh^2}$$

Solutions $\psi_n(t)$ of the discrete linear Schrödinger equation are functions defined on a one dimensional lattice with spacing h. We can move back to a continuous space picture by considering functions $\psi(x,t)$ such that $\psi(x,t) = \psi_n(t)$ and $\psi(x+h,t) = \psi_{n+1}(t)$. If we replace our discrete solutions with these continuous functions in the discrete linear Schrödinger equation, we get

$$i\dot{\psi}(x,t) + \frac{1}{h^2}(\psi(x+h,t) - 2\psi(x,t) + \psi(x-h,t))$$

In the limit of $h \to 0$ we can Taylor expand the $\psi(x \pm h)$ terms. We have $\psi(x+h,t) = \psi(x,t) + h\psi'(x,t) + \frac{h^2}{2}\psi''(x,t) + \mathcal{O}(h^3)$. When we plug this in to the above equation, we find that all $\psi(x,t)$ and $\psi'(x,t)$ terms cancel, and we are left with

$$i\dot{\psi}(x,t) + \frac{1}{h^2}(h^2\psi''(x,t)) = i\dot{\psi}(x,t) + \psi''(x,t) = 0$$

This is linear free Schrödinger equation we saw in question (5), with dispersion relation

$$\omega = k^2$$

Hence in the limit as $h \to 0$ the discrete problem approaches the continuous one, as expected.