AMATH 568
Advanced Differential Equations
Homework 2

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1. Consider the nonhomogeneous problem $\vec{x}' = \mathbf{A}\vec{x} + \vec{g}(t)$.

   Let $\vec{x} = \mathbf{M}\vec{y}$ where the columns of $\mathbf{M}$ are the eigenvectors of $\mathbf{A}$.

   Writing the nonhomogeneous equation in terms of $\vec{y}$ gives us

$$\begin{aligned} (\mathbf{M}\vec{y})' &= \mathbf{A}(\mathbf{M}\vec{y}) + \vec{g}(t) \\ \Rightarrow \mathbf{M}\vec{y}' &= \mathbf{A}\mathbf{M}\vec{y} + \vec{g}(t) \end{aligned}$$

   Then, multiplying from the left by $\mathbf{M}^{-1}$, we have

$$\begin{aligned} \vec{y}' &= \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\vec{y} + \mathbf{M}^{-1}\vec{g}(t) \\ &= \mathbf{D}\vec{y} + \vec{h}(t) \end{aligned}$$

   where $\mathbf{D} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ is a diagonal matrix of eigenvalues $\lambda_i = D_{ii}$, and $\vec{h}(t) = \mathbf{M}^{-1}\vec{g}(t)$. Since $\mathbf{D}$ is diagonal, we have *decoupled* the system so that each component of $\vec{y}$ can be solved independently of the other components.

$$y_i' = \lambda_i y_i + h_i(t)$$

2. Given $L = -d^2/dx^2$ find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -10\exp(x)$$

   where $y(0) = 0$ and $y'(1) = 0$.

   Solution: We begin by writing the problem in Sturm-Liouville form as

$$-y_{xx} = 2y + 10\exp(x)$$

   where $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $\mu = 2$, and $f(x) = 10\exp(x)$. We now consider the associated eigenvalue equation

$$-u_{xx} = \lambda^2 u$$

This equation has the general solution

$$u = A\sin(\lambda x) + B\cos(\lambda x)$$

The first boundary condition $u(0) = 0$ requires that $B = 0$. The second boundary condition gives us

$$u'(1) = \lambda A\cos(\lambda) = 0 \Rightarrow \lambda_n = (2n+1)\frac{\pi}{2}$$

for $n \in \{0, 1, \dots\}$. We can normalize the u_n by solving for the coefficients A_n such that

$$\langle u_n, u_n\rangle = \int_0^1 A_n^2 \sin^2(\lambda_n x)dx = 1$$
$$\Rightarrow A_n = 2\sqrt{\frac{\lambda_n}{2\lambda_n - \sin(2\lambda_n)}} = \sqrt{2}$$

where we have used $\sin(2\lambda_n) = \sin((2n+1)\pi) = 0$. Putting this together, we find the normalized eigenfunctions of the operator L to be

$$u_n(x) = \sqrt{2}\sin(\lambda_n x) \qquad\qquad \lambda_n = (2n+1)\frac{\pi}{2}$$

We can then write our solution as an eigenfunction expansion $y = \sum_i c_i u_i$. Doing so, we can write our problem as

$$L\left(\sum_{i=1}^{\infty} c_i u_i\right) = 2\left(\sum_{i=1}^{\infty} c_i u_i\right) + 10e^x$$
$$\Rightarrow \sum_{i=1}^{\infty}(\lambda_i^2 - 2)c_i u_i = 10e^x$$

Lastly, we can use the orthogonality of $\{u_i\}$ to compute the expansion coefficients c_i. Taking an inner product with u_n, we find

$$(\lambda_n^2 - 2)c_n = 10\langle e^x, u_n\rangle = 10\int_0^1 \sqrt{2}e^x \sin(\lambda_n x) = 10\sqrt{2}\frac{\lambda_n + e(\sin\lambda_n - \lambda_n\cos\lambda_n)}{\lambda_n^2 + 1}$$

Replacing λ_n with $(2n+1)\frac{\pi}{2}$ results in the cos term going to zero and the sin term becoming $(-1)^n$, and so we find

$$c_n = 10\sqrt{2}\frac{\lambda_n + (-1)^n e}{(\lambda_n^2 + 1)(\lambda_n^2 - 2)}$$

To conclude, we have found the eigenfunction expansion solution of the given Sturm-Liouville problem to be

$$y = 20\sum_{n=0}^{\infty} \frac{\lambda_n + (-1)^n e}{(\lambda_n^2 + 1)(\lambda_n^2 - 2)}\sin(\lambda_n x) \qquad\qquad \lambda_n = (2n+1)\frac{\pi}{2}$$

3. Given $L = -d^2/dx^2$ find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -x$$

where $y(0) = 0$ and $y(1) + y'(1) = 0$.

Solution: We begin by writing the problem in Sturm-Liouville form as

$$-y_{xx} = 2y + x$$

where $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $\mu = 2$, and $f(x) = x$. We now consider the associated eigenvalue problem

$$-y_{xx} = \lambda^2 y$$

This equation has the general solution

$$y = A\sin(\lambda x) + B\cos(\lambda x)$$

From the boundary condition $y(0) = 0$ we see that $B = 0$, while the second boundary condition $y(1) + y'(1) = 0$ gives us a transcendental equation which the eigenvalues must satisfy.

$$A\sin(\lambda) + \lambda A\cos(\lambda) = 0 \Rightarrow \lambda + \tan\lambda = 0$$

There are countably infinite λ_i which satisfy this relation. Next, we can normalize our eigenfunctions u_n by solving for the coefficients A_n such that

$$\langle u_n, u_n\rangle = \int_0^1 A_n^2 \sin^2(\lambda_n x)dx = 1$$
$$\Rightarrow A_n = 2\sqrt{\frac{\lambda_n}{2\lambda_n - \sin(2\lambda_n)}} = \frac{2}{\sqrt{\cos(2\lambda_n) + 3}}$$

where we have used $\lambda_n = -\tan(\lambda_n)$ in the last step. Hence, our normalized eigenfunctions are given by

$$u_n = \frac{2\sin(\lambda_n x)}{\sqrt{\cos(2\lambda_n) + 3}}$$

We can now write our solution as an eigenfunction expansion $y = \sum_i c_i u_i$ and write our problem as

$$L\left(\sum_{i=1}^{\infty} c_i u_i\right) = 2\left(\sum_{i=1}^{\infty} c_i u_i\right) + x$$
$$\Rightarrow \sum_{i=1}^{\infty}(\lambda_i^2 - 2)c_i u_i = x$$

Lastly, we can exploit the orthogonality of $\{u_i\}$ to solve for c_n by taking an inner product of both sides with u_n. We have

$$
\begin{aligned}
(\lambda_n^2 - 2)c_n = \langle x, u_n \rangle &= \frac{2}{\sqrt{\cos(2\lambda_n) + 3}} \int_0^1 x \sin(\lambda_n x) dx \\
&= \frac{2}{\sqrt{\cos(2\lambda_n) + 3}} \frac{\sin(\lambda_n) - \lambda_n \cos(\lambda_n)}{\lambda_n^2} \\
&= \frac{2}{\sqrt{\cos(2\lambda_n) + 3}} \frac{-2\cos(\lambda_n)}{\lambda_n}
\end{aligned}
$$

$$
\Rightarrow c_n = \frac{4\cos(\lambda_n)}{\lambda_n(2 - \lambda_n^2)\sqrt{\cos(2\lambda_n) + 3}}
$$

Hence we can write our final solution as

$$
y = \sum_{n=1}^{\infty} \frac{8\cos(\lambda_n)}{\lambda_n(2 - \lambda_n^2)(\cos(2\lambda_n) + 3)} \sin(\lambda_n x)
$$

where $\lambda_n + \tan \lambda_n = 0$ and $\lambda_n > 0$.

4. Consider the Sturm-Liouville eigenvalue problem

$$
Lu = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda \rho(x) u
$$

for $0 < x < l$ with boundary conditions

$$
\begin{aligned}
\alpha_1 u(0) - \beta_1 u'(0) = 0 \\
\alpha_2 u(l) - \beta_2 u'(l) = 0
\end{aligned}
$$

and with $p(x) > 0$, $\rho(x) > 0$, and $q(x) \geq 0$ and with $p(x), \rho(x), q(x)$, and $p'(x)$ continuous over $0 < x < l$, and the weighted inner product $\langle \phi, \psi \rangle_\rho = \int_0^l \rho(x)\phi(x)\psi^*(x)dx$. Show the following:

(a) L is a self-adjoint operator.

Solution: Consider the ℓ_2 inner product $\langle Lu, v \rangle$. Note that this is a standard inner product, or equivalently, the weighted inner product $\langle Lu, v \rangle_\rho$ for $\rho(x) =$

1. From the above definition for L we have

$$\begin{aligned}
\langle Lu, v\rangle &= \int_0^l \left[-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u\right] v^* dx \\
&= -\int_0^l (p(x)u')'\, v^* dx + \int_0^l q(x)uv^* dx \\
&= -p(x)u'v^*\Big|_0^l + \int_0^l p(x)u'v^{*\prime} dx + \int_0^l q(x)uv^* dx \\
&= \left[-p(x)u'v^* + p(x)uv^{*\prime}\right]\Big|_0^l + \int_0^l u\left[-(p(x)v^{*\prime})' + q(x)v^*\right] dx \\
&= J(u,v) + \langle u, Lv\rangle
\end{aligned}$$

Hence, using integrating by parts, we see that L is formally self-adjoint with respect to the standard unweighted ℓ_2 norm, with the conjunct

$$J(u,v) = p(x)\left[uv^{*'} - u'v^*\right]\Bigg|_0^l$$

However, if both u and v satisfy the given boundary conditions, then at $x = 0$ and $x = l$, $u' = \gamma_i u$ and $v' = \gamma_i v$, where $\gamma_i = \frac{\alpha_i}{\beta_i}$. We can use this to get rid of the derivative terms in the conjuct $J(u,v)$, which results in

$$\begin{aligned}
J(u,v) &= p(x)\left[uv^{*'} - u'v^*\right]\Bigg|_0^l \\
&= p(x)\gamma_i\left[uv^* - uv^*\right]\Big|_0^l \\
&= 0
\end{aligned}$$

and so we see that $\langle Lu, v\rangle = \langle u, Lv\rangle$, and L is self-adjoint with respect to the ℓ_2 inner product. Furthermore, we note that since $\rho(x) > 0$ we are always able to define a new differential operator $\hat{L} = \frac{1}{\rho(x)}L$. Since $\langle \hat{L}\phi, \psi\rangle_\rho = \langle L\phi, \psi\rangle$, it follows from the above that the operator $\hat{L}$ is self-adjoint with respect to the weighted inner product $\langle\cdot,\cdot\rangle_\rho$.

(b) Eigenfunctions corresponding to different eigenvalues are orthogonal, i.e. $\forall n \neq m : \langle u_n, u_m\rangle_\rho = 0$.

Solution: Let u_n, u_m be eigenfunctions of $\hat{L} = \frac{1}{\rho(x)}L$ with eigenvalues λ_n, λ_m respectively and $\lambda_n \neq \lambda_m$. Then by the self-adjointness of $\hat{L}$ with respect to the ρ-weighted norm $\langle\cdot,\cdot\rangle_\rho$ we have

$$\langle \hat{L}u_n, u_m\rangle_\rho = \langle u_n, \hat{L}u_m\rangle_\rho$$

Additionally, since u_n and u_m are eigenfunctions of $\hat{L}$, we have

$$\langle \hat{L}u_n, u_m\rangle_\rho = \lambda_n\langle u_n, u_m\rangle_\rho$$

and

$$\langle u_n, \hat{L}u_m \rangle_\rho = \lambda_m \langle u_n, u_m \rangle_\rho$$

Together with the first equation, these imply that

$$\lambda_n \langle u_n, u_m \rangle_\rho = \lambda_m \langle u_n, u_m \rangle_\rho$$

and since $\lambda_n \neq \lambda_m$, we conclude that

$$\langle u_n, u_m \rangle_\rho = 0$$

This proves that eigenfunctions with different eigenvalues are orthogonal.

(c) Eigenvalues are real, non-negative, and eigenfunctions may be chosen to be real valued.

Solution:

i. *Eigenvalues are real.*
Consider the inner product $\langle \hat{L}u, u \rangle$ for some eigenfunction $u \neq 0$. By the self-adjointness of $\hat{L}$ with respect to the weighted inner product $\langle \cdot, \cdot \rangle_\rho$ we have

$$\langle \hat{L}u, u \rangle = \langle u, \hat{L}u \rangle_\rho$$

and, since u is an eigenfunction of $\hat{L}$, we also have

$$\langle \hat{L}u, u \rangle_\rho = \lambda \langle u, u \rangle_\rho$$

and

$$\langle u, \hat{L}u \rangle_\rho = \lambda^* \langle u, u \rangle_\rho$$

Together this means that

$$\lambda \langle u, u \rangle_\rho = \lambda^* \langle u, u \rangle_\rho$$

It therefore follows from $u \neq 0$ and $\rho(x) > 0$ that $\langle u, u \rangle_\rho > 0$, and so it must be that $\lambda = \lambda^*$, meaning λ is real valued.

ii. *Eigenvalues are (mostly) non-negative.*
Consider the inner product $\langle \hat{L}u, u \rangle_\rho$ for some eigenfunction u. Since u is an eigenfunction of $\hat{L}$ we have

$$\langle \hat{L}u, u \rangle_\rho = \lambda \langle u, u \rangle_\rho$$

We also have, by the definition of $\hat{L} = \frac{1}{\rho(x)} L$, and letting $\gamma_i = \frac{\alpha_i}{\beta_i}$,

$$\begin{aligned}
\langle \hat{L}u, u \rangle_\rho &= \int_0^l -(p(x)u')' u^* dx + \int_0^l q(x) u u^* dx \\
&= -p(x) u' u^* \Big|_0^l + \int_0^l p(x) u' u'^* dx + \int_0^l q(x) u u^* dx \\
&= p(0)\gamma_1 \left\| u(0) \right\|^2 - p(l)\gamma_2 \left\| u(l) \right\|^2 + \langle u', u' \rangle_p + \langle u, u \rangle_q
\end{aligned}$$

Equating these two expressions we see that

$$\lambda = \frac{p(0)\gamma_1 \|u(0)\|^2 - p(l)\gamma_2 \|u(l)\|^2 + \langle u', u'\rangle_p + \langle u, u\rangle_q}{\langle u, u\rangle_\rho}$$

and so we find the lower bound for λ:

$$\frac{\gamma_1 p(0) \|u(0)\|^2 - \gamma_2 p(L) \|u(L)\|^2}{\langle u, u\rangle_\rho} \leq \lambda$$

From this we see that if $\gamma_1 p(0) \|u(0)\|^2 \geq \gamma_2 p(L) \|u(L)\|^2$, then λ is always non-negative.

iii. *Eigenfunctions may be chosen to be real valued.*
Let u be an eigenfunction of $\hat{L}$ with eigenvalue λ. Then u^* is also an eigenfunction of $\hat{L}$ with the same eigenvalue, since

$$\hat{L}u = \lambda u \Rightarrow (\hat{L}u)^* = (\lambda u)^* \Rightarrow \hat{L}u^* = \lambda u^*$$

where we have used the self-adjointness of $\hat{L}$ and the result from (i) that λ is real.
Since both u and u^* are eigenfunctions with eigenvalue λ, it follows from linearity that $u + u^*$ is also an eigenfunction with the same eigenvalue. But $u + u^* = 2\Re\{u\}$, so we conclude that given a general eigenfunction u we can always get a real valued eigenfunction by taking $\Re\{u\}$.

(d) Each eigenvalue is simple, i.e. it only has one eigenfunction.
Solution: Let u and v be real eigenfunctions of $\hat{L}$ with a shared eigenvalue λ and consider the quantity $v(Lu) - u(Lv)$. By the definition of L we have

$$\begin{aligned}
v(Lu) - u(Lv) &= -(p(x)u')'v + q(x)uv + (p(x)v')'u - q(x)uv \\
&= -p'(x)u'v - pu''v + p'(x)v'u + p(x)v''u \\
&= p(x)(v''u - u''p) + p'(x)(v'u - u'v) \\
&= [p(x)(v'u - u'v)]' = [p(x)W(v, u)]'
\end{aligned}$$

where $W(v, u) = v'u - u'v$ is the Wronskian of v and u.
Additionally, since u and v are both eigenfunctions of $\hat{L}$, we have

$$v(Lu) - u(Lv) = \lambda\rho(x)(vu - uv) = 0$$

Together with the above, this means that

$$[p(x)W(v, u)]' = 0 \Rightarrow p(x)W(v, u) = \text{Const.}$$

But from the boundary conditions we have

$$\begin{aligned}
W(v, u)(0) &= v'(0)u(0) - v(0)u'(0) = \gamma_1(v(0)u(0) - v(0)u(0)) \\
&= 0
\end{aligned}$$

Which means that $p(x)W(v, u) = 0$ for all x. And since $p(x) > 0$, it follows that $W(v, u) = 0$ for all x, and we conclude that u and v are linearly dependent.