Lecture 13 (chapter 16)

Greens function for the wave equation

To solve the wave equation with forcing

$$\frac{\partial^2}{\partial t^2} u - c^2 \nabla^2 u = Q(\vec{x}, t)$$

subject to two ICs:

$$u(\vec{x}, 0) = f(\vec{x})$$

$$\frac{\partial}{\partial t} u(\vec{x}, 0) = g(\vec{x}),$$

we define the Greens function as solution due to a concentrated source at $\vec{x} = \vec{\xi}$ acting only at $t = \tau$:

$G(\vec{x}, t; \vec{\xi}, \tau)$ satisfies

$$\frac{\partial^2}{\partial t^2} G - c^2 \nabla^2 G = \delta_3(\vec{x} - \vec{\xi})\,\delta(t - \tau)$$

subject to the same homogeneous BC but zero ICs.

$$G\big|_{t=0} = 0$$

$$\frac{\partial}{\partial t} G\big|_{t=0} = 0$$

If $0 < t < \tau$, it is physically obvious that with zero ICs, and before the source acts at $t = \tau$, the response has to be identically zero:

$$G(\vec{x}, t; \vec{\xi}, \tau) \equiv 0 \quad \text{for } 0 < t < \tau.$$

(causality condition).

For $t > \tau$, the source term is also zero:

$$\frac{\partial^2}{\partial t^2} G - c^2 \nabla^2 G = 0, \quad t > \tau$$

subject to homogeneous BCs.

Find the "initial" conditions at $t = \tau$

Claim that they are

$$G = 0 \text{ at } t = \tau$$

$$\frac{\partial}{\partial t} G = \delta_3(\vec{x} - \vec{\xi}) \text{ at } t = \tau$$

$$\int_{\tau^-}^{\tau^+} dt \left[\frac{\partial^2}{\partial t^2} G - c^2 \nabla^2 G\right]$$

$$= \int_{\tau^-}^{\tau^+} dt \; \delta_3(\vec{x} - \vec{\xi}) \, \delta(t - \tau)$$

$$= \delta_3(\vec{x} - \vec{\xi}) .$$

$$\int_{\tau^-}^{\tau^+} dt \frac{\partial^2}{\partial t^2} G = \frac{\partial}{\partial t} G \Big|_{t=\tau^-}^{t=\tau^+}$$

$$c^2 \int_{\tau^-}^{\tau^+} dt \; \nabla^2 G = 0 \quad \text{as } \tau^+ - \tau^- \to 0$$

$$\frac{\partial}{\partial t} G \Big|_{t=\tau^-}^{t=\tau^+} = \delta_3(\vec{x} - \vec{\xi})$$

$$\frac{\partial}{\partial t} G \Big|_{t=\tau^+} = \delta_3(\vec{x} - \vec{\xi})$$

G has to be continuous in t across $t = \tau$ otherwise $\frac{\partial}{\partial t} G$ will have $\sim \delta(t - \tau)$

Thus $G\big|_{t=\tau^+} = G\big|_{t=\tau^-} = 0$

Fundamental Problem for the wave equation

PDE: $\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2\right) G = 0, \quad t > \tau$

BC: homogeneous

ICs: $G(\vec{x}, t; \vec{\xi}, \tau) = 0$ at $t = \tau$

$\frac{\partial}{\partial t} G(\vec{x}, t; \vec{\xi}, \tau) = \delta_3(\vec{x} - \vec{\xi})$

at $t = \tau$

$G(\vec{x}, t; \vec{\xi}, \tau) \equiv 0$ for $t < \tau$

Construct the solution $u(\vec{x},t)$ to the original problem with nonzero ICs:

$$\boxed{u(\vec{x},t) = \int_0^\infty d\tau \iiint_V G(\vec{x},t;\vec{\xi},\tau)\,Q(\vec{\xi},\tau)\,d^3\vec{\xi} - \iiint_V \left[f(\vec{\xi}) \frac{\partial}{\partial\tau} G\Big|_{\tau=0} - g(\vec{\xi})\, G\Big|_{\tau=0} \right] d^3\vec{\xi}}$$

See my book, section 16.4.4, for a derivation.

For the wave equation I suggest using the approach:

$$u = u_p + u_H$$

$$u_p = \int_0^\infty d\tau \iint_V G(\vec{x},t;\vec{\xi},\tau)\,Q(\vec{\xi},\tau)\,d^3\vec{\xi}$$

u_H satisfies the homogeneous PDE but nonzero ICs.

Greens function for wave equation in 1-D

$$\frac{\partial^2}{\partial t^2} G - c^2 \frac{\partial^2}{\partial x^2} G = \delta(x-\xi)\,\delta(t-\tau)$$

$t > 0, \tau > 0$

subject to zero initial conditions.

The above problem is equivalent to

$$\frac{\partial^2}{\partial t^2} G - c^2 \frac{\partial^2}{\partial x^2} G = 0, \quad t > \tau$$

$G = 0$ at $t = \tau$

$\frac{\partial}{\partial t} G = \delta(x-\xi)$ at $t = \tau$

Since the PDE is homogeneous, the solution can be found either using d'Alembert's method or the Fourier transform

$$G(\vec{x}, t; \xi, \tau) = \frac{1}{2c} \int_{x-c(t-\tau}^{x+c(t-\tau)} \delta(x'-\xi)\,dx'$$

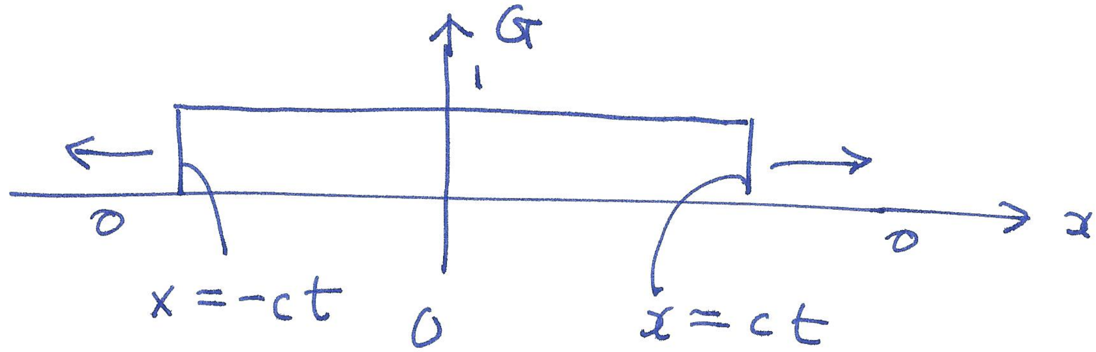
$t > \tau$

$$= \frac{1}{2c} \{ H((x-\xi) + c(t-\tau)) - H(x-\xi) - c(t-\tau)) \}, \quad t > \tau$$

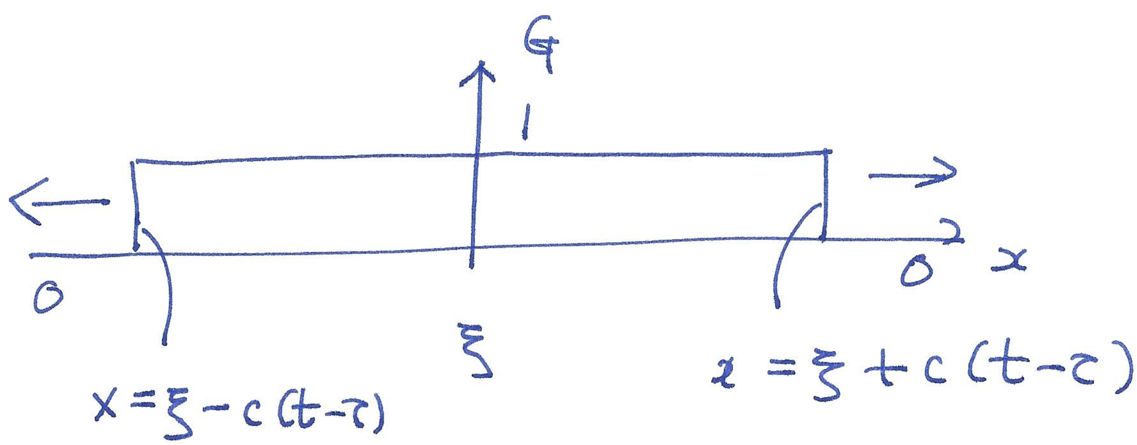
$G \equiv 0 \quad , t < \tau.$

The Greens function represents a rectangular pulse expanding from $x = \xi$ to the left and to the right, with speed c. Ahead of the expanding fronts, $G = 0$.

For $\xi = 0$, $\tau = 0$ the solution looks like



For $\xi \neq 0$, $\tau \neq 0$



So the Fundamental solution for the 1-D wave equation is

$$G(x,t;\xi,\tau) = \begin{cases} \frac{1}{2c}, & x-c(t-\tau)<\xi < x+c(t-\tau) \\ & t>\tau. \\ 0, & \text{otherwise} \end{cases}$$

Very simple!

To solve the problem:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = Q(x,t) \quad \text{subject to zero IC,}$$

we use

$$u(x,t) = \int_0^t d\tau \int_{-\infty}^{\infty} G(x,t;\xi,\tau)\,Q(\xi,\tau)\,d\xi$$

$$= \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} Q(\xi,\tau)\,d\xi$$

[Since $G \equiv 0$ for $t<\tau$, $\int_0^\infty d\tau\, G = \int_0^t d\tau\, G$]

Greens function for the wave equation in 3D

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2\right) G = \delta_3(\vec{x} - \vec{\xi})\,\delta(t - \tau)$$

subject to zero ICs at $t = 0$

Fundamental problem:

PDE: $\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2\right) G = 0 \quad , t > \tau$

BC: homogeneous

ICs: $G(\vec{x}, t; \vec{\xi}, \tau) = 0 \quad$ at $t = \tau$

$\frac{\partial}{\partial t} G(\vec{x}, t; \vec{\xi}, \tau) = \delta_3(\vec{x} - \vec{\xi})$ at $t = \tau$

$G \equiv 0$ for $t < \tau$.

Spherical symmetry: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$.

$$\nabla^2 G = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) G = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rG)$$

For $r > 0$: $\quad r = |\vec{x} - \vec{\xi}|$

$$\frac{\partial^2}{\partial r^2}(rG) - c^2 \frac{\partial^2}{\partial r^2}(rG) = 0$$

which is the same as 1-D wave equation.

d'Alembert's method:

$$rG = R(r - c(t-\tau)) + L(r + c(t-\tau))$$

IC: $G = 0$ at $t = \tau \Rightarrow L(r) = -R(r)$

So $rG = R(r - c(t-\tau)) - R(r + c(t-\tau))$.

IC: $\frac{\partial}{\partial t} G = \delta_3(\vec{r})$ at $t = \tau$

$$r \frac{\partial}{\partial t} G = -cR'(r - c(t-\tau)) - cR'(r + c(t-\tau))$$

$$r \frac{\partial}{\partial t} G \Big|_{t=\tau} = -2cR'(r)$$

$$-\frac{2c}{r} R'(r) = \delta_3(\vec{r})$$

$r \neq 0$, $R'(r) = 0$ and so $R(r) = B$, which we set to zero.

For $r = 0$, we have a singularity

Integrate over a sphere of radius ε

$$\iiint \delta_3(\vec{r}) dV = 1.$$

$$1 = -2c \int_0^{\varepsilon} \frac{1}{r} R'(r) 4\pi r^2 dr$$

$$1 = -8\pi c \int_0^{\varepsilon} r R'(r)\,dr$$

$$= -8\pi c \left[r R(r) \Big|_0^{\varepsilon} - \int_0^{\varepsilon} R(r)\,dr \right]$$

$$= 8\pi c \int_0^{\varepsilon} R(r)\,dr \, .$$

From this we see that $R(r)$ must be a delta function

$$R(r) = \frac{1}{4\pi c}\,\delta(r)$$

since $\int_0^{\varepsilon} \delta(r)\,dr = \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} \delta(r)\,dr = \frac{1}{2}$

Finally

$$G(\vec{x}, t\,;\, \vec{\xi}, \tau) = \frac{1}{4\pi c}\,\frac{\left[\delta(r - c(t-\tau)) - \delta(r + c(t-\tau))\right]}{r}$$

$$= \begin{cases} \dfrac{1}{4\pi c}\,\dfrac{\delta(r - c(t-\tau))}{r} & \text{for } r > 0,\ t > \tau \\ 0 & \text{otherwise.} \end{cases}$$

This represents a concentrated impulse spreading out from the source in a spherical shell with velocity c.

Arbitrary source in 3D

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u = Q(\vec{x}, t)$$

subject to zero ICs and BCs

$$u(\vec{x}, t) = \int_0^\infty d\tau \iiint_V Q(\vec{\xi}, \tau) G(\vec{x}, t; \vec{\xi}, \tau) d^3\vec{\xi}$$

$$= \int_0^t d\tau \iiint_V Q(\vec{\xi}, \tau) \frac{\delta(r - c(t-\tau))}{4\pi c r} d^3\vec{\xi}$$

Do the τ-integral first:

$$\delta(r - c(t-\tau)) = \delta(c\tau + (r - ct))$$

$$= \frac{1}{c}\delta(\tau - (t - r/c))$$

(which is zero for $r > ct$)

$$\boxed{u(\vec{x}, t) = \iiint_{0<r<ct} \frac{Q(\vec{\xi}, t - r/c)}{4\pi c^2 r} d^3\vec{\xi}}$$

The signal is received a distance r away from the source located at $\vec{\xi}$ that was sent out r/c earlier. The form of the source is unchanged in shape except attenuated by a factor $1/r$. There is no signal for $r > ct$.

Greens function for the wave equation

in 2-D (More difficult than in 3-D)

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) G_{2D} = \delta(x-\xi_0)\delta(y-\eta_0)\delta(t-\tau_0)$$

can be obtain from the 3-D nonhomogeneous problem with a 2-D source:

$$Q(\vec{x},t) = \delta(x-x_0)\delta(y-\eta_0)\delta(t-\tau_0)$$

Let

$$u(\vec{x},t) = G_{2D}(\vec{x},t;\vec{\xi}_0,\tau_0)$$

$$= \frac{1}{4\pi c^2}\iiint\limits_{0<r<c(t-\tau_0)} Q(\vec{\xi}, t - r/c)/r \, d^3\vec{\xi}$$

$$= \frac{1}{4\pi c^2}\iiint\limits_{0<r<c(t-\tau_0)} \frac{\delta(\xi-\xi_0)\delta(\eta-\eta_0)\delta((t-t_0)-\frac{r}{c})}{r} \, d^3\vec{\xi}$$

$$r = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}$$

$$G_{2D} = \frac{1}{4\pi c^2}\int_{-\infty}^{\infty} \frac{\delta((t-\tau_0)-r/c)}{r}\, d\zeta$$

$$0 < r < c(t-\tau_0)$$

$$r = [(x-\xi_0)^2 + (y-\eta_0)^2 + (z-\zeta)^2]^{1/2}$$

Let $\rho^2 = (x-\xi_0)^2 + (y-\eta_0)^2$

$$(\zeta - z)^2 = r^2 - \rho^2$$

Let $\zeta' = \zeta - z$

then $\zeta'^2 = r^2 - \rho^2 \leq c^2(t-\tau_0)^2 - \rho^2$

since $r < c(t-\tau_0)$

$$G_{2D} = \frac{1}{4\pi c^2}\int_{-\sqrt{c^2(t-\tau_0)^2-\rho^2}}^{\sqrt{c^2(t-\tau_0)^2-\rho^2}} d\zeta' \, \frac{\delta((t-\tau_0)-r/c)}{r}$$

$$= \frac{1}{2\pi c^2}\int_{0}^{\sqrt{c^2(t-\tau_0)^2-\rho^2}} d\zeta' \, \frac{\delta((t-\tau_0)-\frac{r}{c})}{r}$$

since delta function is zero for positive arguments.

$$\zeta'^2 = r^2 - \rho^2 \quad , \quad \zeta' \geq 0$$

$$\zeta' d\zeta' = r dr \quad , \quad 0 < r < c(t - \tau_0)$$

$$\frac{d\zeta'}{r} = \frac{dr}{\zeta'} = \frac{dr}{r^2 - \rho^2}$$

$$G_{2D} = \frac{1}{2\pi c^2} \int_0^{c(t-\tau_0)} \frac{\delta((t-\tau_0) - \frac{r}{c})\, dr}{\sqrt{r^2 - \rho^2}}$$

$$= \frac{c}{2\pi c^2} \int_0^{(t-\tau_0)} \frac{\delta((t-\tau_0) - r/c)\, d(\frac{r}{c})}{\sqrt{r^2 - \rho^2}}$$

$$= \frac{1}{2\pi c} \frac{1}{\sqrt{c^2(t-\tau_0)^2 - \rho^2}}$$

if $\rho^2 \leq r^2 < c^2(t-\tau_0)^2$

$$\boxed{G_{2D} = \frac{1}{2\pi c} \frac{H(c(t-\tau_0) - \rho)}{\sqrt{c^2(t-\tau_0)^2 - \rho^2}}}$$

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