Uniqueness Th^m for the wave equation

The solution to

$$\frac{\partial^2}{\partial t^2} u - c^2 \nabla^2 u = F(\vec{x}, t)$$

satisfying $u(\vec{x}, 0) = f(\vec{x})$

$u_t(\vec{x}, 0) = g(\vec{x})$

is unique.

Proof: Let $u_1(\vec{x}, t)$ and $u_2(\vec{x}, t)$ be two different solutions satisfying the same IC and BC. Let

$$w(\vec{x}, t) \equiv u_2(\vec{x}, t) - u_1(\vec{x}, t)$$

w satisfies homogeneous PDE, IC and BC. From the ~~previous~~ next result, $w \equiv 0$.

Therefore u_1 and u_2 must be the same solution.

Wave equation:

Let $w(\vec{x},t)$ satisfy

$$\frac{\partial^2}{\partial t^2} w = c^2 \nabla^2 w$$

$$w(\vec{x},0) = 0$$

$$w_t(\vec{x},0) = 0$$

and homogeneous boundary conditions.

$\times\, w_t$:

$$w_t \frac{\partial^2}{\partial t^2} w = \frac{\partial}{\partial t} \frac{1}{2} w_t^2$$

$$w_t \nabla^2 w = -\frac{1}{2} \frac{\partial}{\partial t} |\vec{\nabla} w|^2 + \vec{\nabla} \cdot (w_t \vec{\nabla} w)$$

So the wave equation becomes

$$\frac{\partial}{\partial t} \left(\frac{1}{2} w_t^2 + \frac{1}{2} c^2 |\nabla w|^2 \right) = \vec{\nabla} \cdot (w_t \vec{\nabla} w)$$

$$\frac{\partial}{\partial t} \iiint_V \left(\frac{1}{2} w_t^2 + \frac{1}{2} c^2 |\nabla w|^2 \right) dV$$

$$= \iint_{\partial V} w_t \vec{\nabla} w \cdot \vec{n}\, dS = 0$$

$$\frac{\partial}{\partial t} \underbrace{\iiint_V \left(\frac{1}{2} w_t^2 + \frac{1}{2} c^2 |\nabla w|^2 \right) dV}_{I} = 0$$

Let $I = \iiint_V \left(\frac{1}{2} w_t^2 + \frac{1}{2} c^2 |\nabla w|^2\right) dV$

Since $I = 0$ at $t = 0$

and $\frac{\partial}{\partial t} I = 0$ for $t > 0$,

we must have $I \equiv 0$ for all $t > 0$

But the integrand is the sum of squares, and consequently I can vanish only if $|\nabla w| \equiv 0$ and $w_t \equiv 0$.

Therefore w can at most be a constant, but since $w = 0$ at $t = 0$, we have

$w \equiv 0$.

The heat equation

Let w satisfy

$$\frac{\partial}{\partial t} w = D \nabla^2 w$$

$w = 0$ at $t = 0$, $w = 0$ on the boundary

$$w \frac{\partial}{\partial t} w = D\, w \nabla^2 w$$

$$\frac{\partial}{\partial t} \frac{1}{2} w^2 = D \vec{\nabla} \cdot (w \vec{\nabla} w) - D \vec{\nabla} w \cdot \vec{\nabla} w$$

$$\frac{\partial}{\partial t} \iiint_V \frac{1}{2} w^2 dV + D \iiint_V |\vec{\nabla} w|^2 dV = D \iiint_V \vec{\nabla} \cdot (w \vec{\nabla} w)\, dV$$

$$= D \iint_{\partial V} w \vec{\nabla} w \cdot \vec{n}\, dS = 0$$

because of zero BC.

So $$\frac{\partial}{\partial t} \iiint_V \frac{1}{2} w^2 dV = - \iiint_V D |\vec{\nabla} w|^2 dV$$

$$\frac{\partial}{\partial t} \iiint_V \frac{1}{2} w^2 dV \le 0$$

$w^2 = 0$ at $t = 0$. It cannot go negative

$$w \equiv 0.$$

Uniqueness Theorem for the heat equation

The solution of

$$\frac{\partial}{\partial t} u = D \nabla^2 u + F(\vec{x}, t)$$

satisfying $u(\vec{x}, 0) = f(\vec{x})$

is unique.

Proof: Let $u_1(\vec{x}, t)$ and $u_2(\vec{x}, t)$ be two solutions satisfying the same IC and BC. Let

$$w(\vec{x}, t) \equiv u_2 - u_1$$

Then w satisfies the homogeneous PDE, IC and BC.

From the previous page,

$w \equiv 0$. Therefore u_1 and u_2 must be the same solution.