(My book) 1
chapter 11

Boundary - Value Problems

So far we have discussed problems in infinite domains, unaffected by the presence of boundaries.

Consider the solution to the wave-equation in an infinite domain

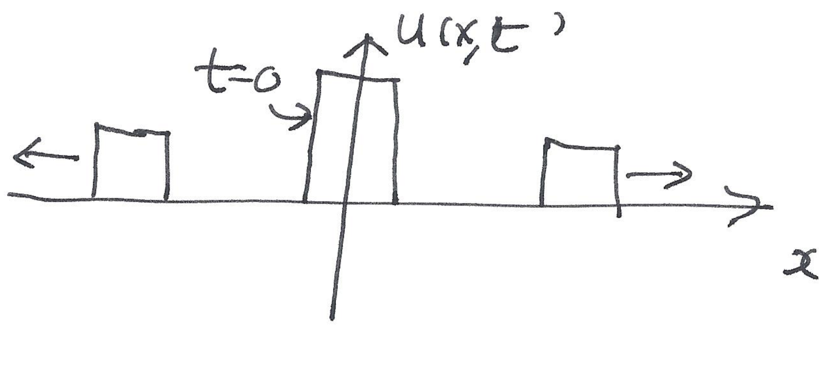
$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0, \quad -\infty < x < \infty$$
$$t > 0$$

ICs: $u(x,0) = f(x)$

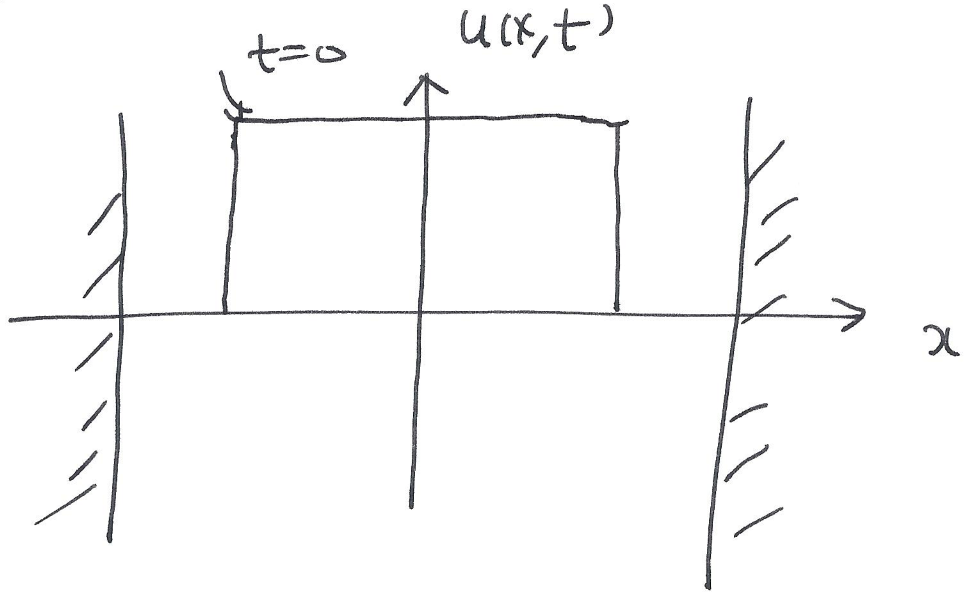
$u_t(x,0) = 0$

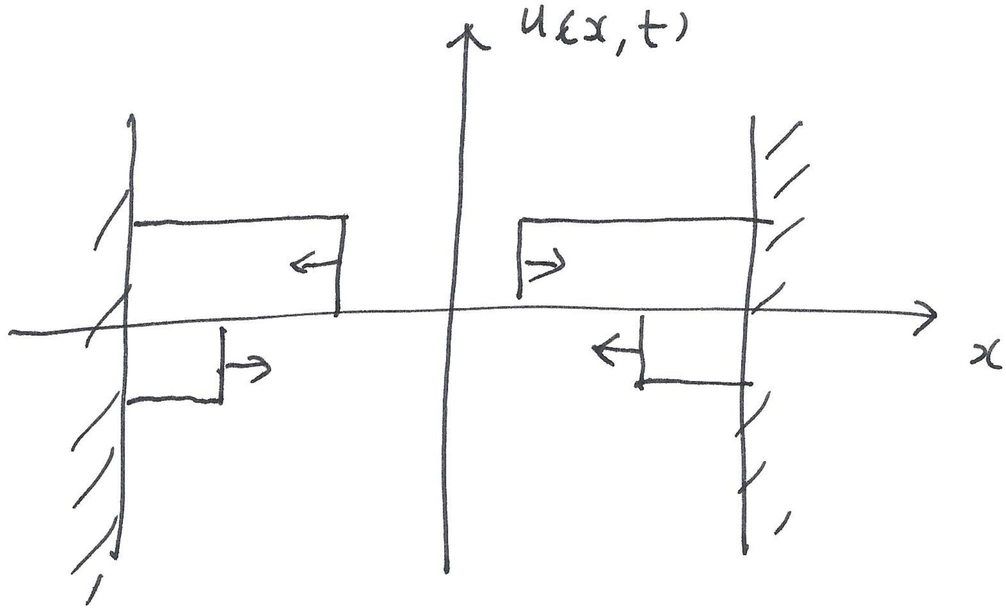
d'Alembert's solution

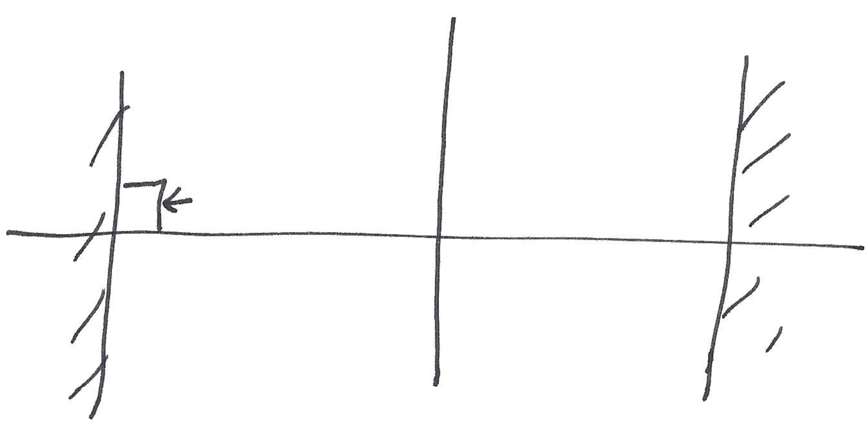
$$u(x,t) = \tfrac{1}{2}f(x-ct) + \tfrac{1}{2}f(x+ct).$$



In the presence of boundaries, where say $u = 0$:







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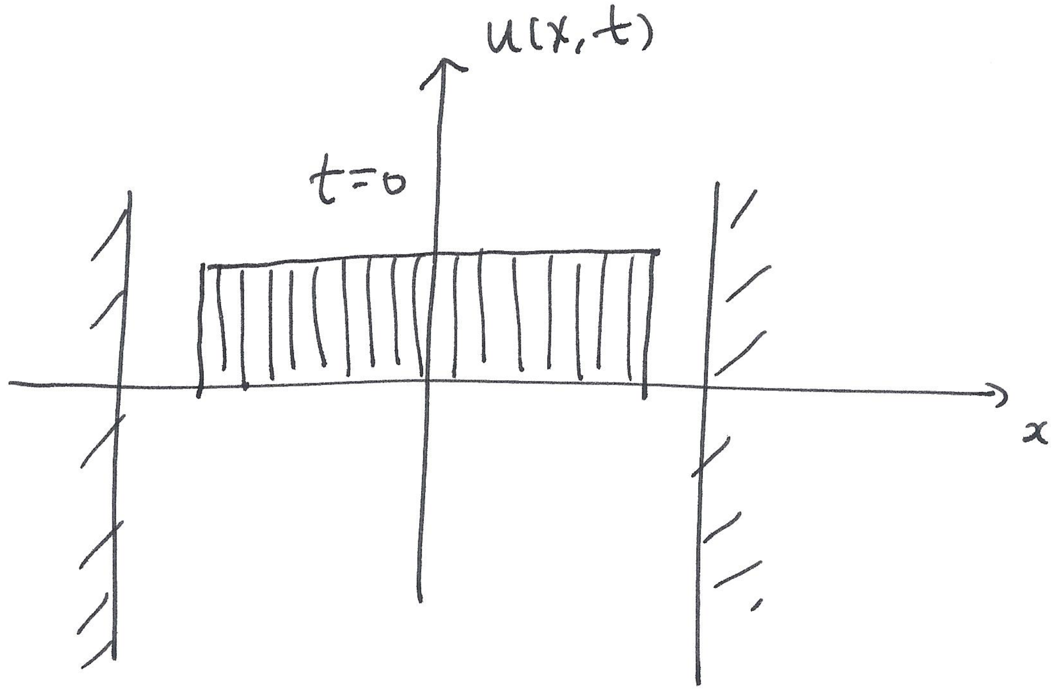
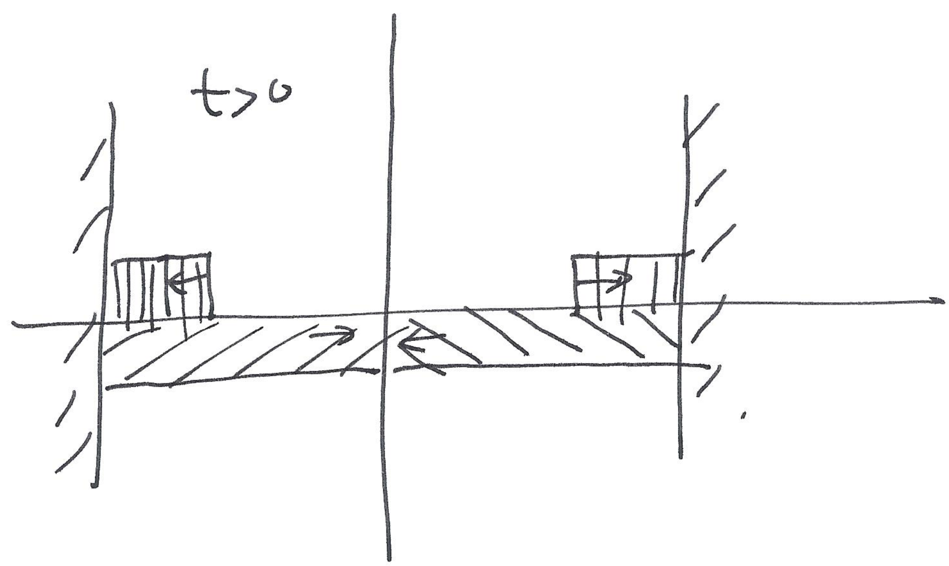
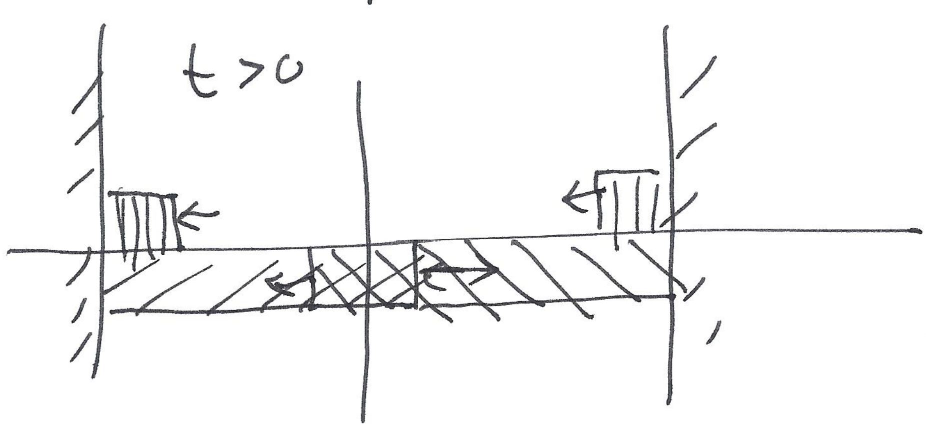
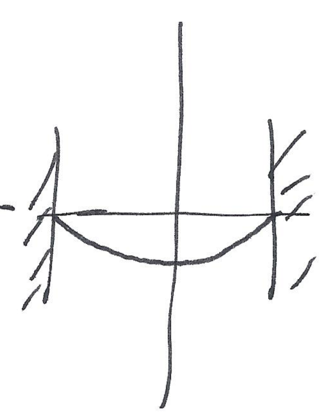
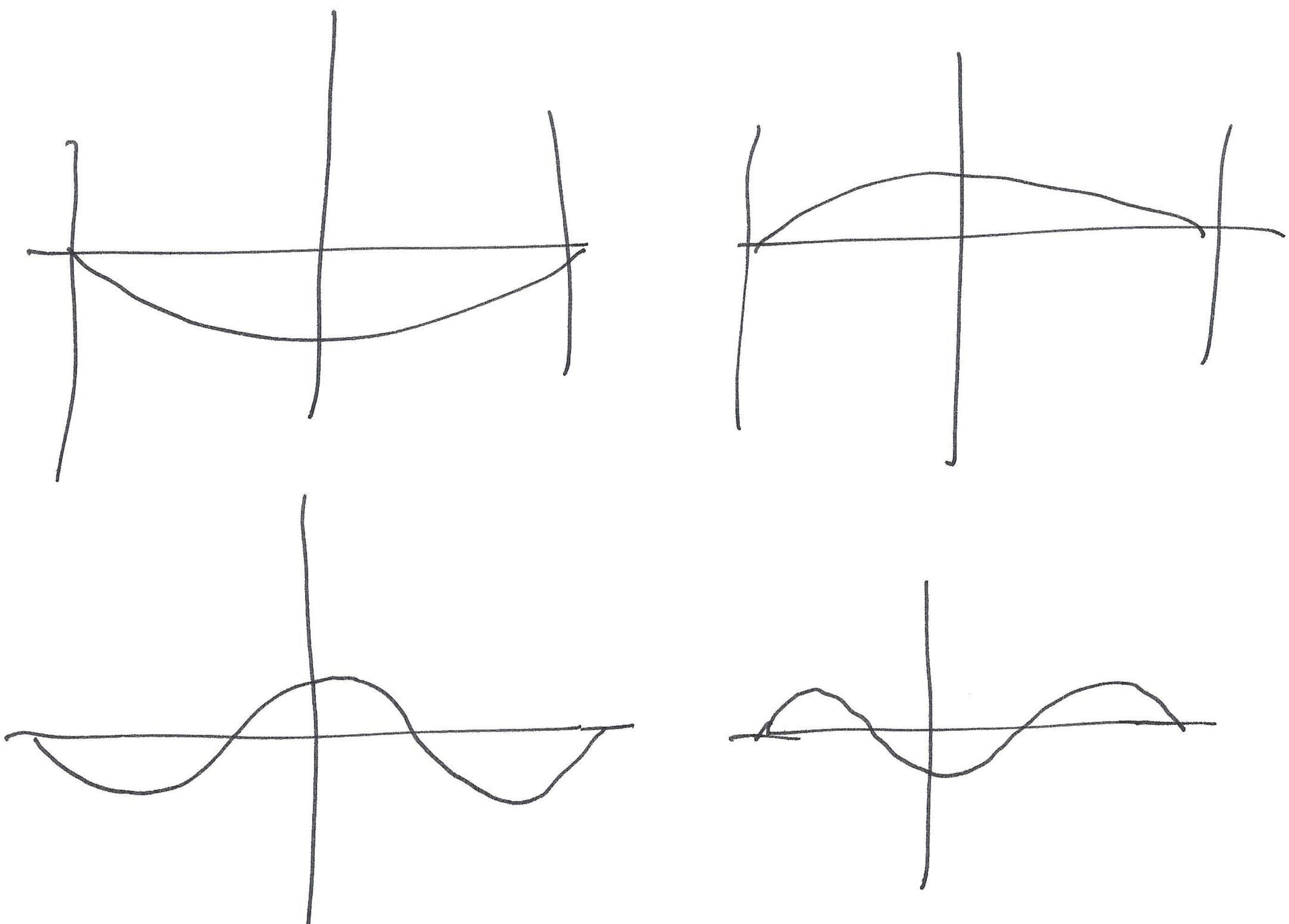








Eventually, the spatial structure will be shaped by the boundaries to become of the shape, or a combination of the shapes :



etc.

These are "normal modes".

In the presence of finite boundaries, the spatial shape of the solution is influenced more by the boundaries than the dynamics of different PDEs.

Use separation of variables to separate the time dependence from the spatial dependence.

Helmholtz equation in 3-D

Wave-equation:

$$\frac{\partial^2}{\partial t^2} u = c^2 \nabla^2 u$$

Heat equation:

$$\frac{\partial}{\partial t} u = \alpha^2 \nabla^2 u$$

Laplace equation:

$$\nabla^2 u$$

Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2\mu}\nabla^2\psi$$

They all involve the spatial derivatives in the form of a Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Separation of variables

As an intermediate step, assume

$$u(\vec{x},t) = \phi(\vec{x})\,T(t)$$

substitute into the PDE:

wave eq: $\quad \frac{\partial^2}{\partial t^2} u = c^2 \nabla^2 u$

$$T''\phi = c^2 \nabla^2\phi\, T$$

$$\frac{T''(t)}{T(t)} = \frac{\nabla^2\phi(\vec{x})}{\phi(\vec{x})} = \text{const} \equiv -\lambda^2$$

The Left-Hand side is a function of t only, while the Right-Hand side is a function of $\vec{x}$ only. The only way they can be equal to each other is for each to be equal to a constant. $\boxed{\nabla^2\phi = -\lambda^2\phi}$ Helmholtz eq.

The heat equation:

$$\frac{\partial}{\partial t} u = \alpha^2 \nabla^2 u$$

$$u(\vec{x}, t) = \phi(\vec{x})\, T(t)$$

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{\nabla^2 \phi(\vec{x})}{\phi(\vec{x})} \equiv -\lambda^2$$

$$\boxed{\nabla^2 \phi = -\lambda^2 \phi}$$ Helmholtz equation.

$$T(t) = T(0)\, e^{-\alpha^2 \lambda^2 t}$$

But λ^2 is not yet known.
Needs to be solved as an eigenvalue of the Helmholtz equation subject to appropriate boundary conditions.

Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} u = -\frac{\hbar^2}{2\mu} \nabla^2 u$$

$$u(\vec{x},t) = \phi(\vec{x})\, T(t)$$

$$\frac{i\hbar T'(t)}{T(t)} = -\frac{\hbar^2}{2\mu} \frac{\nabla^2 \phi(\vec{x})}{\phi(\vec{x})}$$

$$= \text{Const} \equiv E \quad , \qquad \lambda^2 = 2\mu E/\hbar^2$$

$$\frac{i\hbar T'(t)}{T(t)} = E \qquad \Bigg| \qquad \boxed{\nabla^2 \phi = -\lambda^2 \phi}$$

Helmholtz eq.

$$T(t) = T(0)\, e^{-i(E/\hbar)t},$$

where the separation constant E is interpreted as the energy of the electron since $E/\hbar$ is the frequency, and energy in quantum mechanics is $\hbar$ times frequency.