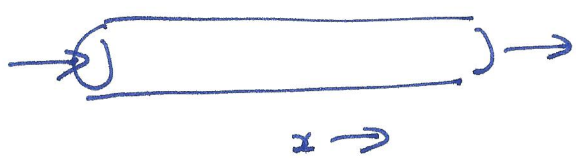
Lecture 4 (optional)

Method of characteristics for higher order PDE.

Example from 1-D gas flow in a tube



ρ density, u velocity in x-direction

ρu momentum per unit volume

Conservation of mass

$$\frac{\partial}{\partial t} A \Delta x \, \rho = -(\text{flux of mass out} - \text{flux in})$$

$$= -\left[A\rho u \Big|_{x+\Delta x} - A\rho u \Big|_{x} \right]$$

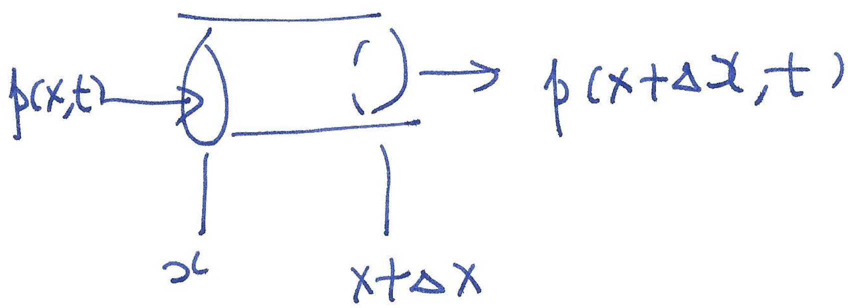
$$\Delta x \to 0, \quad \boxed{\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u)}$$

Conservation of momentum

$$\frac{\partial}{\partial t} A \Delta x \, \rho u = -(\text{flux of momentum out} - \text{flux in}) + \text{pressure force}$$

$$\boxed{\frac{\partial}{\partial t}\rho u = -\frac{\partial}{\partial x}(\rho u u) - \frac{\partial}{\partial x} p}$$

Pressure force



Net pressure force on the volume of fluid

$$A\,p(x,t) - A\,p(x+\Delta x,t)$$

$$= A\,[\,p(x,t) - p(x+\Delta x,t)\,]$$

$$= A\Delta x\;\frac{[\,p(x,t) - p(x+\Delta x,t)\,]}{\Delta x}$$

$$= -\Delta x A\,\frac{\partial}{\partial x}p \qquad \text{as } \Delta x \to 0$$

Equation of state

$$p = p(\rho)$$

[For isentropic flow, $p = p(\rho) = A\rho^{\gamma}$]

Let $c^2 \equiv \frac{dp}{d\rho}$

$$\boxed{\begin{aligned} &\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + \frac{c^2}{\rho}\frac{\partial}{\partial x}\rho = 0 \\ &\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\rho + \rho\frac{\partial}{\partial x}u = 0 \end{aligned}}$$

A set of two coupled nonlinear PDEs

Riemann Invariants

Chapter 7 Kevorkian

Let $\xi =$ constant be curves along which

$$\frac{dx}{dt} = u - c$$

and $\eta =$ const be curves along which

$$\frac{dx}{dt} = u + c$$

Now

$$du = dx \frac{\partial y}{\partial x} + dt \frac{\partial y}{\partial t}$$

$$d\rho = dx \frac{\partial \rho}{\partial x} + dt \frac{\partial \rho}{\partial t}$$

Along $\xi =$ const, substitute $\frac{dx}{dt} = u - c$

$$du = dt \left[\frac{\partial}{\partial t} + (u-c) \frac{\partial}{\partial x} \right] u$$

$$d\rho = dt \left[\frac{\partial}{\partial t} + (u-c) \frac{\partial}{\partial x} \right] \rho$$

So

$$du - \frac{c}{\rho} d\rho = dt \left\{ \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) u + \frac{c^2}{\rho} \frac{\partial}{\partial x} \rho - c \frac{\partial}{\partial x} u - \frac{c}{\rho} \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) \rho \right\}$$

$$= 0$$

More details:

$$du = dt\left[\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right]u \quad \text{along } \xi = \text{const}$$

but the momentum equation is

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u = -\frac{c^2}{\rho}\frac{\partial}{\partial x}\rho$$

So $$du = dt\left[-\frac{c^2}{\rho}\frac{\partial}{\partial x}\rho - c\frac{\partial}{\partial x}u\right]$$

$$d\rho = dt\left[\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right]\rho$$

but the mass equation is

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\rho = -\rho\frac{\partial}{\partial x}u$$

So $$d\rho = dt\left[-\rho\frac{\partial}{\partial x}u - c\frac{\partial}{\partial x}\rho\right]$$

$$du - \frac{c}{\rho}d\rho = dt\left[-\frac{c^2}{\rho}\frac{\partial}{\partial x}\rho - c\frac{\partial}{\partial x}u + \frac{c^2}{\rho}\frac{\partial}{\partial x}\rho + c\frac{\partial}{\partial x}u\right] = 0$$

$$\boxed{du - \frac{c}{\rho}d\rho = 0} \quad \text{along } \xi : \frac{dx}{dt} = u - c$$

similarly

$$\boxed{du + \frac{c}{\rho}d\rho = 0} \quad \text{along } \eta : \frac{dx}{dt} = u + c$$

Along $\xi = \text{const}$

$$\frac{du}{d\rho} = \frac{c}{\rho} \quad \text{or} \quad u - \int^{\rho} \frac{c}{\rho} d\rho = \text{const.}$$

For isentropic fluid, $c^2 \equiv \frac{dp}{d\rho} = A\gamma\rho^{\gamma-1}$

$$\frac{c}{\rho} = (A\gamma)^{1/2} \rho^{\frac{\gamma}{2} - \frac{3}{2}}$$

$$\int^{\rho} \frac{c}{\rho} d\rho = \int^{\rho} (A\gamma)^{1/2} \rho^{\frac{\gamma}{2} - \frac{3}{2}} d\rho$$

$$= (A\gamma)^{1/2} \rho^{\frac{\gamma}{2} - \frac{1}{2}} / \left(\frac{\gamma}{2} - \frac{1}{2}\right)$$

$$= \frac{2c}{\gamma - 1}$$

Along $\xi = \text{const}$

$$\mathcal{R}_1 \equiv \frac{u}{2} - \frac{c}{\gamma - 1} = \text{const}$$

Along $\eta = \text{const}$

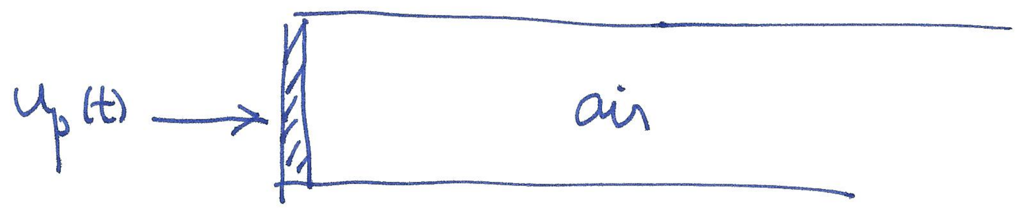
$$\mathcal{R}_2 \equiv \frac{u}{2} + \frac{c}{\gamma - 1} = \text{const.}$$

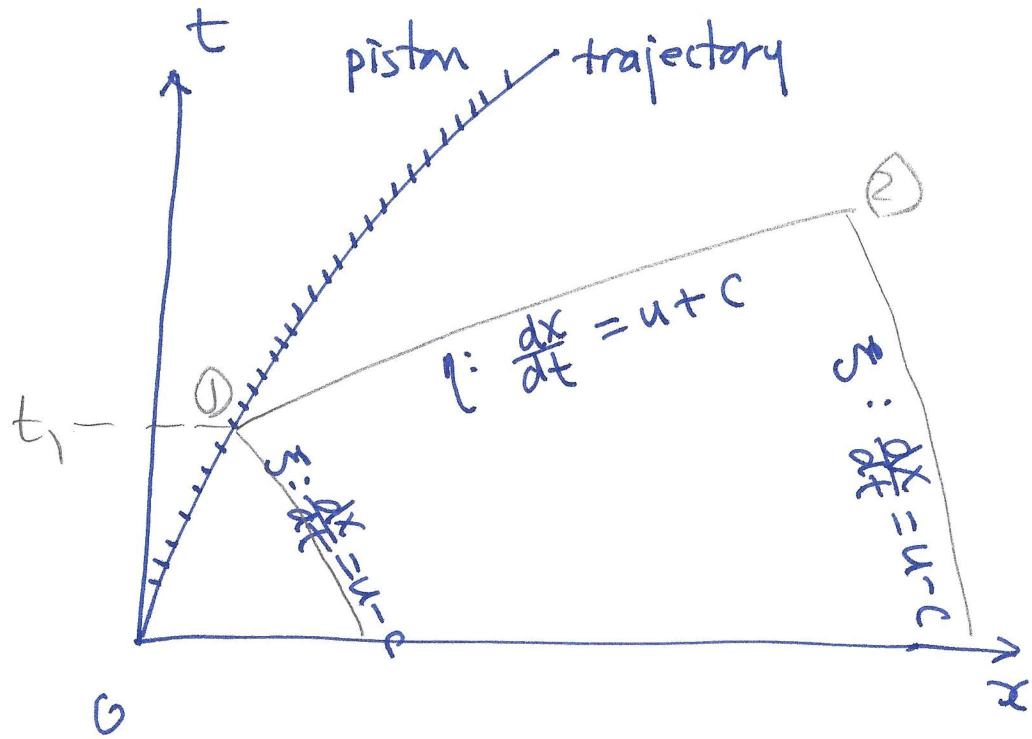
These are called the Riemann invariants.

u and c are the two unknowns

(ρ can be calculated once c is known)

Example shock tube driven by a piston.





Consider the characteristic $\xi = \text{const}$ intersecting ①

$$r_1 = \frac{u}{2} - \frac{c}{\gamma - 1} = \frac{u_p(t_1)}{2} - \frac{c_p(t_1)}{\gamma - 1}$$

also intersecting $t=0$: $\quad = -\frac{c_0}{\gamma - 1}$

So. $\quad c_p(t_1) - \frac{\gamma - 1}{2} u_p(t_1) = c_0$

At ②, along $\xi = \text{const}$

$$R_1 = \frac{u}{2} - \frac{c}{\gamma - 1} = -\frac{c_0}{\gamma - 1}$$

Also at ②, but along ①–② where $\eta = \text{const}$

$$R_2 = \frac{u}{2} + \frac{c}{\gamma - 1} = \frac{u_p(t_1)}{2} + \frac{c_p(t_1)}{\gamma - 1}$$

subtract to get rid of u

$$\frac{2c}{\gamma - 1} = \frac{c_0}{\gamma - 1} + \frac{c_p(t_1)}{\gamma - 1} + \cancel{\frac{\gamma-1}{}}\frac{u_p(t)}{2}$$

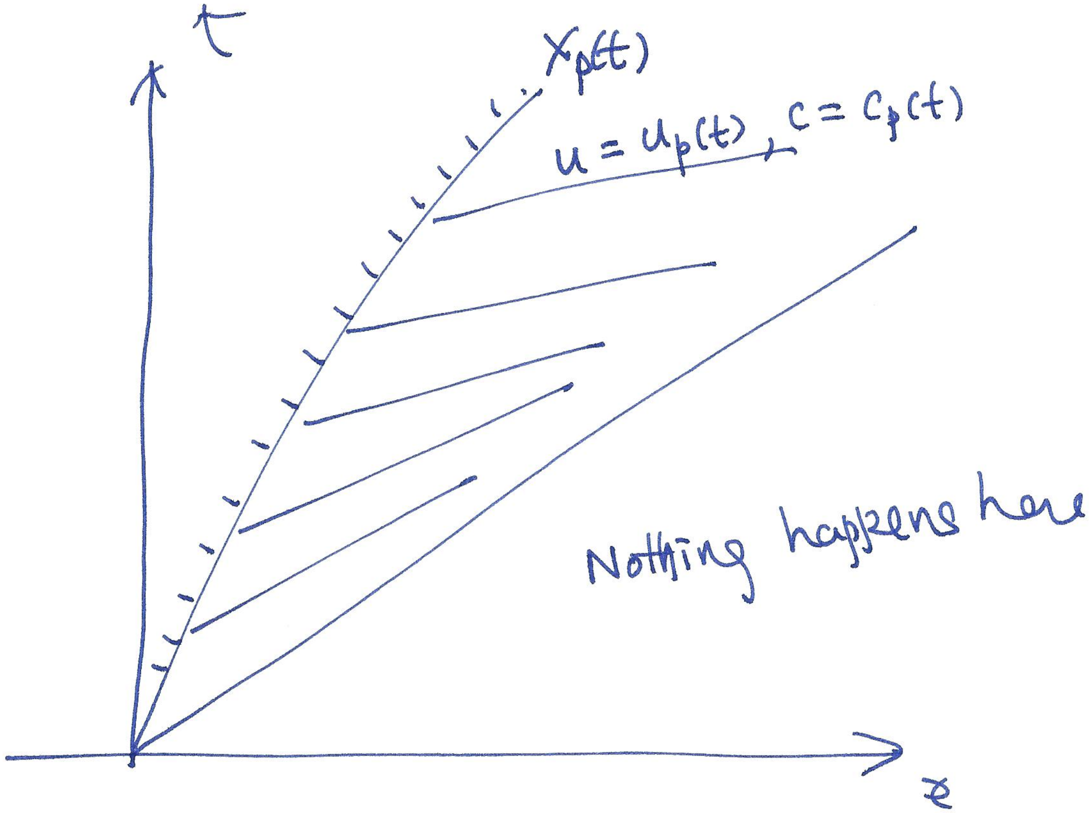
since $c_p(t_1) = c_0 + \frac{\gamma - 1}{2} u_p(t_1)$ from previous

$$\frac{2c}{\gamma - 1} = \frac{2\, c_p(t_1)}{\gamma - 1}$$

or $\boxed{\begin{aligned} c &= c_p(t_1) \\ u &= u_p(t_1) \end{aligned}}$ along ①–②

Therefore ①–② : $\frac{dx}{dt} = u + c = u_p(t_1) + c_p(t_1)$
$= \text{const}.$

This characteristic is a straight line



Characteristics are straight lines but the slope may depend on $u_p(t)$.

Example:

$$u_p(t) = U \sin\omega t$$

$$x_p = \int_0^t u_p(t)\,dt$$

$$= \frac{U}{\omega}(1-\cos\omega t)$$