Greens function for the heat equation

$$\left(\frac{\partial}{\partial t}u - D\nabla^2 u\right) = Q(\vec{x}, t)$$

where Q is a known (heat) source subject to homogeneous boundary conditions and zero initial condition.

The Greens function $G(\vec{x}, t; \vec{\xi}, \tau)$ is defined by

$$\left(\frac{\partial}{\partial t}G - D\nabla^2 G\right) = \delta_3(\vec{x}-\vec{\xi})\,\delta(t-\tau)$$

subject to the same homogeneous boundary conditions and zero initial condition.

The solution $u(\vec{x}, t)$ can be constructed from G:

$$\boxed{u(\vec{x}, t) = \int_0^\infty d\tau \iiint_V G(\vec{x}, t; \vec{\xi}, \tau)\, Q(\vec{\xi}, \tau)\, d^3\vec{\xi}}$$

$$d^3\vec{\xi} = d\xi\, d\eta\, d\zeta$$

Basically a superposition.

check:

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) u(\vec{x}, t)$$

$$= \int_0^\infty d\tau \iiint_V \left(\frac{\partial}{\partial t} - D\nabla^2\right) G(\vec{x}, t; \vec{\xi}, \tau) \, Q(\vec{\xi}, \tau) \, d^3\vec{\xi}$$

$$= \int_0^\infty d\tau \iiint_V \delta(\vec{x} - \vec{\xi}) \, \delta(t - \tau) \, Q(\vec{\xi}, \tau) \, d^3\vec{\xi}$$

$$= Q(\vec{x}, t)$$

BC and IC are also satisfied.

Fundamental solution

PDE: $\frac{\partial}{\partial t} G - D\nabla^2 G = \delta_3(\vec{x}-\vec{\xi})\delta(t-\tau)$

IC: $G = 0$ at $t=0$

BC: homogeneous

Solve a simpler problem for $t > \tau > 0$

and $0 < t < \tau$.

In both cases, the source term vanishes:

For $0 < t < \tau$, the initial value problem is

$$\frac{\partial}{\partial t} G - D\nabla^2 G = 0, \qquad t < \tau$$

$$G = 0 \text{ at } t = 0$$

yielding the trivial solution

$$\boxed{G(\vec{x}, t; \vec{\xi}, \tau) \equiv 0 \quad \text{for } 0 \le t < \tau}$$

For $t > \tau > 0$:

$$\frac{\partial}{\partial t} G - D \nabla^2 G = 0 .$$

We need to specify an "initial" condition at $t = \tau$. If this "initial" condition is $G \equiv 0$ and $t = \tau$,

then again $G \equiv 0$ for $t > \tau$.

This is inconsistent with the behavior of the PDE at $t = \tau$.

So G cannot be $\equiv 0$ as $t \to \tau^+$.

If G is nonzero as $t \to \tau^+$, but $G \equiv 0$ as $t \to \tau^-$, there must be a discontinuity in t at $t = \tau$.

$$\int_{\tau^-}^{\tau^+} dt \left[\frac{\partial}{\partial t} G - D \nabla^2 G \right] = \delta_3(\vec{x} - \vec{\xi}) \int_{\tau^-}^{\tau^+} \delta(t-\tau) dt$$

$$= \delta_3(\vec{x} - \vec{\xi})$$

$$\int_{\tau^-}^{\tau^+} \frac{\partial}{\partial t} G \, dt = G \Big|_{t=\tau^-}^{t=\tau^+}$$

$$D \int_{\tau^-}^{\tau^+} \nabla^2 G \, dt \to 0 \quad \text{as } \tau^+ \to \tau^-$$

$$G \Big|_{t=\tau^-}^{t=\tau^+} = \delta_3 (\vec{x} - \vec{\xi})$$

$$G \Big|_{t=\tau^+} = \delta_3 (\vec{x} - \vec{\xi})$$

because $G \equiv 0$ for $t < \tau$

$$\begin{cases} \text{PDE}: & \frac{\partial}{\partial t} G - D\nabla^2 G = \delta_3(\vec{x}-\vec{\xi}\,)\,\delta(t-\tau) \\ & t > 0 \\ \text{IC}: & G = 0 \text{ at } t = 0 \\ \text{BC}: & \text{homogeneous} \end{cases}$$

is equivalent to

$$\begin{cases} \text{PDE}: & \frac{\partial}{\partial t} G - D\nabla^2 G = 0, \quad t > \tau \\ \text{IC}: & G = \delta_3(\vec{x}-\vec{\xi}\,), \quad \text{at } t = \tau \\ & G \equiv 0 \quad \text{for } 0 < t < \tau \end{cases}$$

BC: homogeneous

This is the Fundamental problem of the heat equation.

The Fundamental problem for the heat equation is the same as the Drunken sailor problem. Solution has been obtained previously:

1-D:

$$G(x,t;\xi,\tau) = \begin{cases} \dfrac{1}{\sqrt{4\pi D(t-\tau)}} \exp\left\{-\dfrac{(x-\xi)^2}{4D(t-\tau)}\right\} & , t>0 \\ 0 & , t<0 \end{cases}$$

n-D:

$$G(\vec{x},t;\vec{\xi},\tau) = \begin{cases} \left[\dfrac{1}{4\pi D(t-\tau)}\right]^{\frac{n}{2}} \exp\left\{-\dfrac{|\vec{x}-\vec{\xi}|^2}{4D(t-\tau)}\right\} & , t>0 \\ 0 & , t<0 \end{cases}$$

Solution to the original problem in 3-D:

$$u(\vec{x},t) = \int_0^\infty d\tau \iiint_V G(\vec{x},t;\vec{\xi},\tau)\, Q(\vec{\xi},\tau)\, d^3\vec{\xi}$$

$$= \int_0^t d\tau \iiint_V \left[\frac{1}{4\pi D(t-\tau)}\right]^{\frac{3}{2}} \exp\left\{-\frac{|\vec{x}-\vec{\xi}|^2}{4D(t-\tau)}\right\} Q(\vec{\xi},\tau)\, d^3\vec{\xi}\,.$$

What about nonzero initial condition for u?

IC: $u(\vec{x},0)=f(\vec{x})$

The solution is, by superposition

$$u(\vec{x},t)=\int_0^\infty d\tau \iiint_V G(\vec{x},t;\vec{\xi},\tau)\,Q(\vec{\xi},\tau)\,d^3\vec{\xi}$$

$$+\,v(\vec{x},t)$$

where $v(\vec{x},t)$ satisfies homogeneous PDE but nonzero IC.

$$\left(\frac{\partial}{\partial t}v - D\nabla^2 v\right)=0 \quad , t>0$$

$$v(\vec{x},0)=f(\vec{x})$$

We claim that $v(\vec{x},t)$ can also be constructed using the same G as

$$\boxed{v(\vec{x},t)=\iiint_V G(\vec{x},t;\vec{\xi},0)\,f(\vec{\xi})\,d^3\vec{\xi}\,.}$$

$t>0$

Show this :

PDE: $\left(\frac{\partial}{\partial t} - D\nabla^2\right) v(\vec{x}, t)$

$$= \iiint_V \left(\frac{\partial}{\partial t} - D\nabla^2\right) G(\vec{x}, t ; \vec{\xi}, 0) f(\vec{\xi}) d^3\vec{\xi}$$

$$= \iiint_V \delta(\vec{x} - \vec{\xi}) \delta(t) f(\vec{\xi}) d^3\vec{\xi}$$

$= 0$ for $t > 0$. satisfies homogeneous PDE

IC: since $G(\vec{x}, t ; \vec{\xi}, 0) = \delta_3(\vec{x} - \vec{\xi})$ at $t = \tau$

For $\tau = 0$, $t \to 0$, then $G(\vec{x}, t ; \vec{\xi}, 0)$
$= \delta_3(\vec{x} - \vec{\xi})$

$$v(\vec{x}, t) = \iiint_V G(\vec{x}, t ; \vec{\xi}, 0) f(\vec{\xi}) d^3\vec{\xi}$$

$$t \to 0 \Rightarrow v(\vec{x}, 0) = \iiint_V \delta_3(\vec{x} - \vec{\xi}) f(\vec{\xi}) d^3\vec{\xi}$$

$= f(\vec{x})$

satisfies nonzero IC.