AMATH 567, Homework 3

Lucas Cassin Cruz Burke

October 26, 2022

1. **Problem 1. From AF: 4.2.1 c d**

   (a) Evaluate
   $$I = \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$
   where $a^2, b^2 > 0$.

   Solution: We note that the integrand is even, therefore we can expand the bounds of our integral to
   $$I = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$
   We will evaluate this integral by considering the integral $I_{C(R)}$ over a positively oriented semicircular contour of radius R in the upper half of the complex plane.
   $$I_{C(R)} = \oint_{C(R)} \frac{dz}{(z^2+a^2)(z^2+b^2)}$$
   This closed semicircular contour can be divided into two sub-contours: a straight line $C_1(R)$ traveling from $-R$ to R along the x-axis, and a semicircular arc $C_2(R)$ in the upper complex plane beginning at R and ending at $-R$. Note that in the limit $R \to \infty$, $I_{C_1(R)} \to 2I$
   Since the radius R is fixed along C_2, we can evaluate the integral $I_{C_2(R)}$ by substituting $z = Re^i\theta$ and integrating with respect to θ as it goes from 0 to π:
   $$I_{C_2(R)} = \int_0^\pi \frac{iRe^{i\theta}}{(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2)}d\theta$$
   We recall that $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$, and therefore
   $$I_{C_2(R)} \leq \int_0^\pi \frac{R}{|R^2e^{2i\theta}+a^2||R^2e^{2i\theta}+b^2|}d\theta$$

We now take the limit $\lim_{R\to\infty} I_{C_2(R)}$. To evaluate the denominator we note that $|R^2e^{2i\theta} + a^2| = \sqrt{R^4 + a^4 + 2a^2R^2\cos 2\theta}$, which approaches R^2 as $R \to \infty$, therefore

$$\lim_{R\to\infty} I_{C_2} \leq \lim_{R\to\infty} \int_0^\pi \frac{1}{R^3} d\theta = 0$$

So in the limit of $R \to \infty$ the upper semicircular component of the contour C goes to zero.
Let us now use the Residue Theorem to evaluate the integral $I_{C(R)}$. We can expand the denominator of the integrand to be

$$I_{C(R)} = \oint_{C(R)} \frac{dz}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

We see that the integrand has poles at $z = \pm ia$ and $z = \pm ib$. Since our contour is in the upper half of the complex plane, only $z = ia$ and $z = ib$ will contribute to the integral. These are first order poles, and we can evaluate their residues as

$$Res(ia) = \lim_{z\to ia} \frac{1}{(z+ia)(z+ib)(z-ib)} = \frac{1}{(2ia)(ia+ib)(ia-ib)} = \frac{1}{2ia(b^2-a^2)}$$

And by symmetry, $Res(ib) = \frac{1}{2ib(a^2-b^2)}$.
Therefore by the Residue Theorem,

$$I_{C(R)} = 2\pi i \left(\frac{1}{2ib(a^2-b^2)} - \frac{1}{2ia(a^2-b^2)} \right) = \frac{\pi}{a+b}$$

Since we showed above that the contribution from the semicircular contour goes to zero as $R \to \infty$, this value must come from the horizontal contour along the x-axis, but as we noted above this is two times our original integral I that we wish to evaluate. So we conclude that

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}$$

(b) Evaluate

$$I = \int_0^\infty \frac{dx}{x^6+1}$$

Solution: We note that the integrand is even, and so we can use the same contour as used in part (a) to evaluate this integral. We again begin by showing that the semicircular component of the contour integral goes to zero as $R \to \infty$:

$$I_{C_2} = \lim_{R\to\infty} \int_0^\pi \frac{iRe^{i\theta}}{R^6 e^{6i\theta}+1} d\theta \leq \lim_{R\to\infty} \int_0^\pi \frac{R}{|R^6 e^{6i\theta}+1|} d\theta$$

And using the same argument in part (a) we can show that $|R^6 e^{6i\theta} + 1| \to R^6$ as $R \to \infty$, and therefore

$$I_{C_2} \leq \lim_{R\to\infty} \int_0^\pi \frac{1}{R^5} d\theta \Rightarrow I_{C_2} = 0$$

Let us now use the Residue Theorem to evaluate the contour integral

$$I_C = \oint_C \frac{dz}{z^6+1}$$

We note that the integral has poles wherever $z^6 = -1$. In terms of the argument θ of z, this means that $6\theta = \pi + 2\pi n$, and so we have poles when $\theta = \pm\frac{\pi}{2}, \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$. Since our contour envelopes only the upper half of the complex plane, this means we must calculate the residues at the simple poles $z = i, e^{i\pi/6}$ and $e^{i5\pi/6}$.

i. $Res(i) = \lim_{z\to i} \frac{(z-i)}{z^6+1} = \lim_{z\to i} \frac{1}{(z^4-z^2+1)(z+i)} = \frac{1}{6i}$

ii. $Res(e^{i\pi/6}) = \lim_{z\to e^{i\pi/6}} \frac{(z-e^{i\pi/6})}{z^6+1} = \lim_{z\to e^{i\pi/6}} \frac{1}{(z^2+1)(z-e^{i5\pi/6})(z-e^{-i5\pi/6})(z-e^{-i\pi/6})}$
$\Rightarrow Res(e^{i\pi/6}) = -\frac{i}{6}e^{-i\pi/3}$

iii. $Res(e^{i5\pi/6}) = \lim_{z\to e^{i5\pi/6}} \frac{(z-e^{i5\pi/6})}{z^6+1} = \lim_{z\to e^{i5\pi/6}} \frac{1}{(z^2+1)(z-e^{i\pi/6})(z-e^{-i\pi/6})(z-e^{-i5\pi/6})}$
$\Rightarrow Res(e^{i5\pi/6}) = \frac{i}{6}e^{-i2\pi/3}$

And so by the Residue theorem

$$I_C = \oint_C \frac{dz}{z^6+1} = 2\pi i \cdot \frac{-i}{6}\left(1 + e^{-i\pi/3} - e^{-i2\pi/3}\right) = \frac{2\pi}{3}$$

Since we determined that only the contour along the x axis contributes to this integral, and since the integral along this contour is twice our original integral we would like to evaluate, we conclude that

$$I = \int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$$

2. **Problem 2. From AF: 4.2.2 a, b h.** Evaluate the following integrals:

   (a) $\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2}dx$; $a^2 > 0$
   Solution: Since $\frac{x}{x^2+a^2} \to 0$ as $|x| \to \infty$, we can use Jordan's lemma and evaluate the integral $I_C = \int_C \frac{ze^{iz}}{z^2+a^2}$, where C is a semicircular contour in the upper half of the complex plane, and $\Im(I_C)$ is the integral we are interested in evaluating.

This integral has two simple poles at $z = \pm ia$, and since we are interested in the upper half of the complex plane we are only interested in the pole at $z = ia$. We can evaluate this residue with $Res(ia) = \lim_{z\to ia} \frac{ze^{iz}}{(z+ia)} = \frac{1}{2}e^{-a}$. Therefore, by the Residue Theorem, we have $I_C = 2\pi i \cdot \frac{1}{2}e^{-a} = i\pi e^{-a}$.

Since the integral we would like to evaluate is $\Im(I_C)$, our final answer is $I = \pi e^{-a}$.

(b) $\int_{-\infty}^{\infty} \frac{\cos(kx)dx}{(x^2+a^2)(x^2+b^2)}$; $a^2, b^2, k > 0$

Solution: Since $\frac{1}{(x^2+a^2)(x^2+b^2)} \to 0$ as $|x| \to \infty$, we may again use Jordan's Lemma, and we note that this integral is the real part of $I_C = \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)}$. We evaluate this integral using a semicircular contour of radius R in the upper half of the complex plane in the limit as $R \to \infty$. We see that integrand has poles at $x = \pm ia$ and $x = \pm ib$. Since we are only interested in the upper half of the complex plane, only the residues at ia and ib will contribute to the integral.

We have $Res(ia) = \lim_{z\to ia} \frac{e^{ikz}}{(z+ia)(z^2+b^2)} = \frac{e^{-ka}}{(2ia)(b^2-a^2)}$, and therefore by symmetry $Res(ib) = \frac{e^{-kb}}{(2ib)(a^2-b^2)}$. So by the Residue Theorem our integral is $I_C = 2\pi i \left(\frac{e^{-kb}}{(2ib)(a^2-b^2)} - \frac{e^{-ka}}{(2ia)(a^2-b^2)}\right) = \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-kb}}{b} - \frac{e^{-ka}}{a}\right)$.

Since the integral we would like to evaluate is $\Re I_C$, this means that our final answer is $I = \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-kb}}{b} - \frac{e^{-ka}}{a}\right)$.

(c) $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$

Solution: We can convert a trigonometric integral $\int_0^{2\pi} \mathcal{U}(\cos\theta, \sin\theta)d\theta$ into a contour integral $\oint_C \mathcal{U}(z)\frac{dz}{iz}$, where C is the unit circle.

We can use this fact, along with the identity $\sin\theta = \frac{1}{2i}(z - 1/z)$ (for z on the unit circle), to rewrite this real integral as the contour integral $I_C = \oint_C \frac{1}{(5-3(z-1/z)/(2i))^2}\frac{dz}{iz} = \oint_C \frac{4zi}{(-3z^2+10zi+3)^2}dz$.

Using the quadratic formula, we see that the denominator has roots at $z = 3i$ and $z = i/3$, and we can rewrite this integral as $I_C = \oint_C \frac{4zi}{(-3(z-3i)(z-i/3))^2}dz = \oint_C \frac{4zi}{9(z-3i)^2(z-i/3)^2}dz$.

The integrand has two second order poles, and since we are integrating around the unit circle only the pole at $z = i/3$ contributes to the integral. We can calculate the residue at this point using the general formula for calculating an mth order pole (see Lecture 10 notes, page 7). Applying this, we have $Res(i/3) = \lim_{z\to i/3} \frac{d}{dz}\frac{4zi}{9(z-3i)^2} = -\frac{5i}{64}$.

Therefore, by the Residue Theorem, $I_C = 2\pi i \cdot \left(-\frac{5i}{64}\right) = \frac{5\pi}{32}$, which is our final answer.

3. **Problem 3. From AF: 4.2.7.** Use a sector contour with radius R, as in Figure 4.2.6, centered at the origin with angle $0 \le \theta \le \frac{2\pi}{5}$ to find, for

$a > 0$,

$$I = \int_0^\infty \frac{dx}{x^5 + a^5} = \frac{\pi}{5a^4 \sin(\pi/5)}$$

Solution: We will call our sector contour C and divide it into three components, C_1 is the segment of the real line from 0 to R, C_2 is the circular arc segment from R to $Re^{i2\pi/5}$, and C_3 is the line segment from $Re^{i2\pi/5}$ to 0, so that $I_C = I_{C_1} + I_{C_2} + I_{C_3}$. Note that I_{C_1} corresponds to the real integral I which we would like to evaluate.

We begin by showing that the circular arc component of the integral goes to 0 as $R \to \infty$. Since the radius is fixed along this contour, we can write

$$I_{C_2} = \lim_{R\to\infty} \int_0^{2\pi/5} \frac{iRe^{i\theta}}{R^5 e^{i5\theta} + a^5} d\theta \leq \lim_{R\to\infty} \int_0^{2\pi/5} \frac{R}{|R^5 e^{i5\theta} + a^5|} d\theta$$

The denominator of the right side of the inequality can be calculated as $\sqrt{R^{10} + a^{10} + 2R^5 a^5 \cos(5\theta)}$, and it is clear that this expression approaches $\sqrt{R^{10}} = R^5$ in the limit as $R \to \infty$. Therefore,

$$I_{C_2} \leq \lim_{R\to\infty} \frac{1}{R^4} \int_0^{2\pi/5} d\theta \Rightarrow I_{C_2} = 0$$

Therefore, the circular arc component of our contour integral goes to zero as $R \to \infty$. We will now evaluate the integral I_{C_3} on the contour which is a straight line from $Re^{i2\pi/5}$ to 0. Since $\arg z = \frac{2\pi}{5}$ is fixed along this line, we can rewrite our integral in terms of r as it goes from $R \to \infty$ to 0. We have

$$I_{C_3} = -e^{i2\pi/5} \int_0^\infty \frac{dr}{r^5 + a^5}$$

Where we have swapped the limits of integration and picked up a minus sign, and used the fact that $z^5 = (re^{i2\pi/5})^5 = r^5$. When we compare this to our original integral I we note that $I_{C_3} = -e^{i2\pi/5} I$. Therefore our original contour integral becomes

$$I_C = (1 - e^{i2\pi/5}) I$$

We are now ready to evaluate the contour integral I_C using the residue theorem. The integrand of I_C has poles at $z = -a, ae^{\pm i\pi/5}$, and $ae^{\pm i3\pi/5}$. However, only $ae^{i\pi/5}$ lies in the interior of our contour, therefore only the residue at this pole will contribute to I_C. This is a simple pole, and we can calculate its residue by using the formula for finding the residue at a simple pole z_0 of a function $f(z) = P(z)/Q(z)$ where P, Q are analytic, which is $Res(z_0) = P(z_0)/Q'(z_0)$. Applying this, we find that

$$Res(ae^{i\pi/5}) = \left.\frac{1}{5z^4}\right|_{z=ae^{i\pi/5}} = \frac{1}{5a^4e^{i4\pi/5}}$$

Therefore, by the Residue Theorem,

$$I_C = \frac{2\pi i}{5a^4e^{i4\pi/5}}$$

Comparing this with our above result, we see that our original integral I can be expressed by

$$I = \frac{2\pi i}{5a^4}\frac{1}{e^{i4\pi/5}(1-e^{i2\pi/5})} = \frac{2\pi i}{5a^4}\frac{1}{(e^{i\pi/5}-e^{-i\pi/5})} = \frac{2\pi i}{5a^4}\frac{1}{2i\sin(\pi/5)}$$

Simplifying once more gives the result

$$I = \frac{\pi}{5a^4\sin(\pi/5)}$$