Lecture 2

Time when shock first forms

Define shock to be where a discontinuity in u occurs. If the discontinuity is in the initial condition, we say the shock forms immediately.

$$u(x,t) = u_0(\xi)$$

$$u_t = u_0'(\xi)\,\xi_t$$

$$u_x = u_0'(\xi)\,\xi_x$$

$$x = \xi + t\,c(u_0(\xi))$$

$$\frac{\partial}{\partial x} x = \xi_x + t\,c'(u_0)\,u_{0\xi}\,\xi_x$$

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$$\xi_x = \frac{1}{1 + t\,c'(u_0(\xi)) \cdot u_{0\xi}}$$

Similarly

$$\xi_t = \frac{-c(u_0(\xi))}{1 + t\,c'(u_0(\xi))\,u_{0\xi}}$$

Breaking time determined by $|u_x| \to \infty$

or $|u_t| \to \infty$.

$$u_x = u_0'(\xi)\,\xi_x$$

$|\xi_x| \to \infty$ when

$$1 + t\,c'(u_0(\xi))\,u_{0\xi} = 0$$

$$t^* = -\frac{1}{c'(u_0(\xi))\,u_{0\xi}}$$

[Same for $|\xi_t| \to \infty$]

We want the minimum.

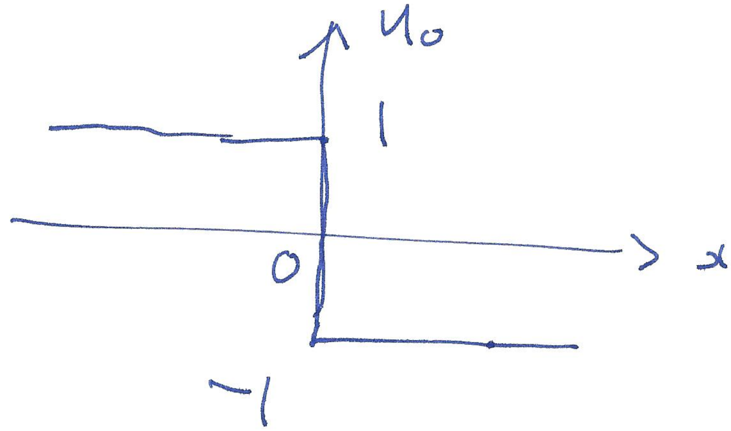
Solution valid for $t < t^*$

After t^* solution becomes multi-valued unless fixes can be applied, usually physics based.

Example

$$u_t + u u_x = 0$$

$$u(x,0) = u_0(x) = \begin{cases} -1, & x > 0 \\ +1, & x < 0 \end{cases}$$



$$u(x,t) = u_0(\xi), \quad x = \xi + t \cdot u_0(\xi)$$

shocks first form when

$$t^* = \min\left\{ -\frac{1}{c' \cdot u_{0\xi}} \right\} = 0$$

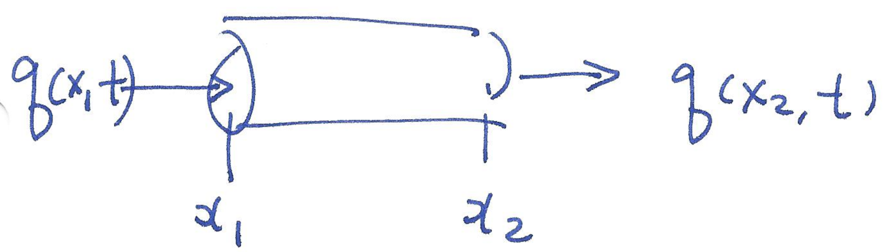
$u_{0\xi}$ is $-\infty$ at $\xi = 0$, which is $x = 0$ at $t = 0$. shock forms right away when there is a negative discontinuity in the initial condition.

What happens at the shock:

Rankine - Hugoniot condition, just one of many fixes.

Use integral form of the conservation law:

$$\frac{d}{dt}\int_{x_1}^{x_2} u\,dx = -\left[q(x_2,t) - q(x_1,t) \right]$$



Let $X(t)$ be the location of the shock

$$x_1 < X(t) < x_2$$

$$\frac{d}{dt}\int_{x_1}^{X(t)} u\,dx + \frac{d}{dt}\int_{X(t)}^{x_2} u\,dx = -(q_2 - q_1)$$

$$\dot{X}(t)\,u_1 - \dot{X}(t)\,u_2 = -(q_2 - q_1)$$

$$\boxed{\dot{X}(t) = \frac{q_2 - q_1}{u_2 - u_1}}$$ shock speed.

The Rankine - Hugoniot condition may not necessarily be the right condition.

For example, if we multiply u to

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2} u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{3} u^3\right) = 0$$

Integrating, we would have gotten

$$\dot{X}(t) = \frac{2}{3} \frac{u_2^3 - u_1^3}{u_2^2 - u_1^2} = \frac{2}{3} \frac{u_2^2 + u_1 u_2 + u_1^2}{u_2 + u_1}$$

So the original equation itself $\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = 0$ does not contain the information to uniquely determine the jump condition.

This condition happens to work for equations such as the Burger's equation:

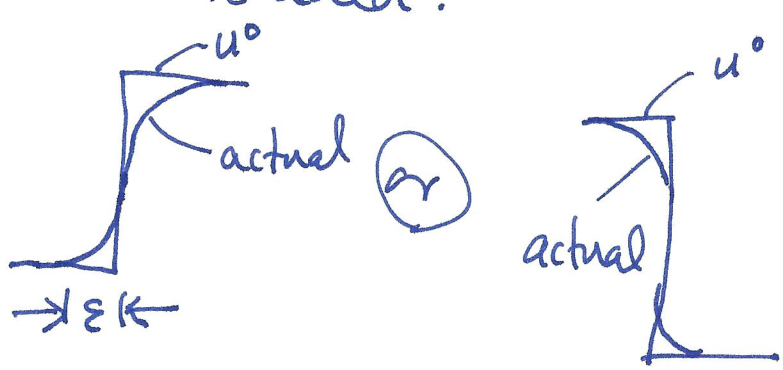
$$u_t + u u_x = \nu u_{xx}$$

in the limit $\nu \to 0^+$.

Away from the shock

$$\frac{\partial}{\partial t} u^0 + u \frac{\partial}{\partial x} u^0 = 0$$

Within the shock, u_{xx} large, so νu_{xx} is not small. ν-term needs to be included.



Let $X(t)$ be the location of the shock, the shock layer is from $X(t) - \varepsilon$ to $X(t) + \varepsilon$. Away from this layer, the ν-term can be ignored.

Integrate Burger's equation:

$$\int_{X-\varepsilon}^{X+\varepsilon} u_t \, dx + \int_{X-\varepsilon}^{X+\varepsilon} u u_x \, dx$$

$$= \cancel{\nu [u_x]_{X-\varepsilon}^{X+\varepsilon}} \quad 0$$

$$\int_{X-\varepsilon}^{X+\varepsilon} u_t \, dx + \frac{1}{2} u^2 \Big|_{X-\varepsilon}^{X+\varepsilon} = 0$$

The first term is

$$\dot{X}(t) (u^- - u^+)$$

The second term is $\frac{1}{2}(u^{+2} - u^{-2})$

So $\dot{X}(t) = -\frac{1}{2}(u^{+2} - u^{-2}) / (u^- - u^+)$

$$\boxed{\dot{X}(t) = \frac{1}{2}(u^+ + u^-)}$$

shock travels at the average speeds

Details :

Consider $$\int_{x_1(t)}^{x_2(t)} u_t \, dt = \frac{d}{dt}\int_{x_1(t)}^{x_2(t)} u \, dx$$

$$- \frac{dx_2}{dt} \cdot u(x_2,t) + \frac{dx_1}{dt} u(x_1,t)$$

but $$\frac{d}{dt}\int_{x_1}^{x_2} = \frac{d}{dt}\int_{x_1}^{X(t)} + \frac{d}{dt}\int_{X(t)}^{x_2}$$

$$= \dot{X} u^- - \frac{d}{dt}x_1 \cdot u(x_1,t)$$

$$+ \frac{d}{dt}x_2 \cdot u(x_2,t) - \dot{X} u^+$$

So $$\int_{x_1(t)}^{x_2(t)} u_t \, dt = \dot{X}(u^- - u^+)$$

Equal Area Rule

For continuous initial conditions, it is more difficult to find the shock location. Note that, for the Burger's equation, and many other equations, the area under the curve is conserved:

$$\frac{d}{dt}\int_{\infty}^{\infty} u(x,t)\,dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)\,dx = 0$$

so $\int_{-\infty}^{\infty} u\,dx$ is conserved.

This is true for the multi-valued solution (if we do not fit a shock) and the solution fitted with a shock to eliminate multivaluedness.