Lecture 3

(Chapter 5 of Kevorkian
Chapter 2 of Whitham)

Burger's equation

is a prototype equation for the Navier-Stokes equation governing fluid motion.

Navier-Stokes in 2-D

$$\frac{\partial}{\partial t}u + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)u = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u$$

In 1-D

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\frac{\partial^2}{\partial x^2}u$$

Two unknowns, so need more equations.

Burger's equation:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = \nu\frac{\partial^2}{\partial x^2}u.$$

It has nonlinear advection which tends to steepen the wave, and viscous diffusion which tends to smooth out high gradients

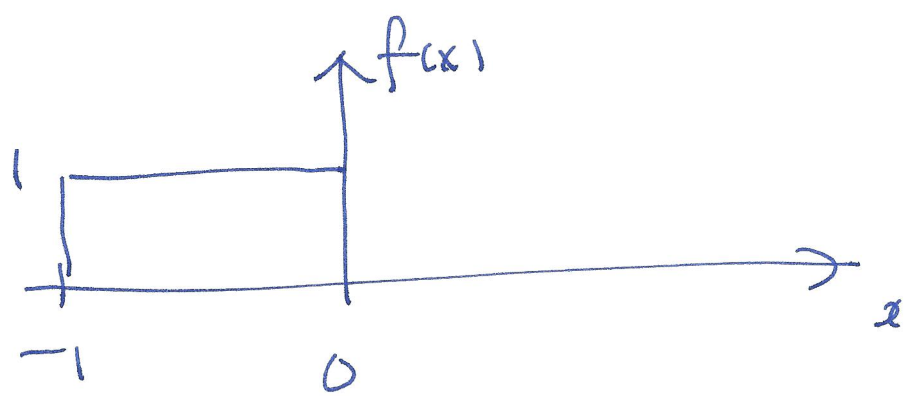
Example :

PDE: $u_t + u u_x = \nu u_{xx}$, $-\infty < x < \infty$

ν small.

IC: $u(x,0) = f(x)$

$$f(x) = \begin{cases} 1 & , -1 < x < 0 \\ 0 & , \text{otherwise} \end{cases}$$



As $\nu \to 0^+$, "outer" equation

$$u^0_t + u^0 u^0_x = 0$$

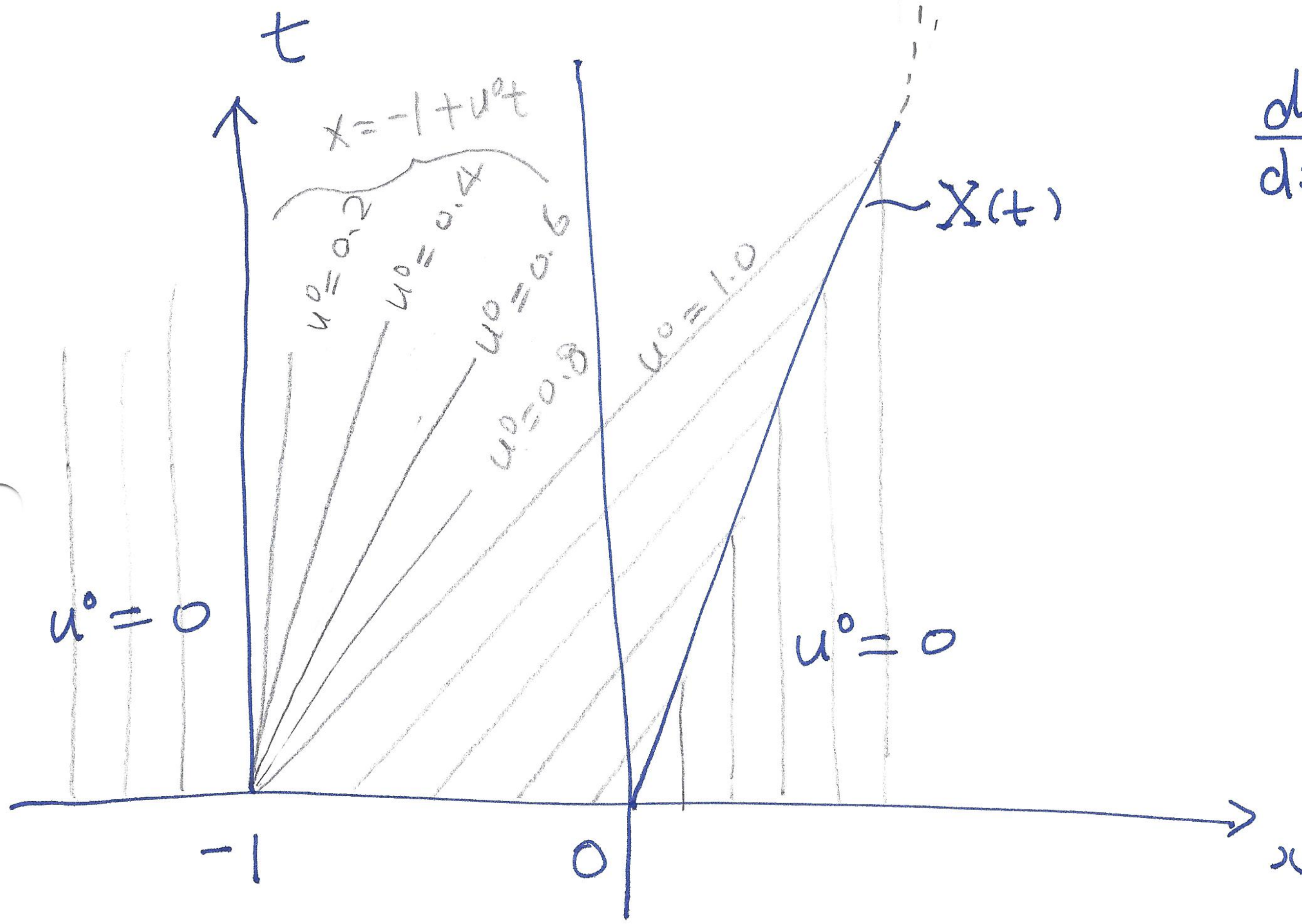
Solution: $u^0 = f(\xi)$, $\xi = x - \int_0^t u^0 dt$

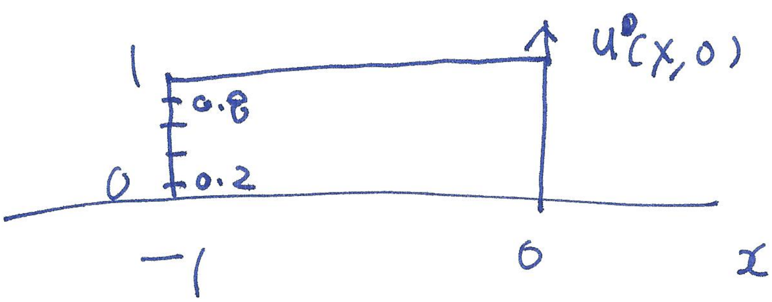
So $$u^0(x,t) = \begin{cases} 1 & , -1 < \xi < 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\frac{du^\circ}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = u^\circ, \text{ a constant}$$

determined from IC

$$\frac{dt}{dx} = \frac{1}{u^\circ}$$





shock fitting

$$\frac{dX}{dt} = \frac{1}{2}(u^+ + u^-), \quad \begin{array}{l} u^+ \text{ ahead of the shock} \\ u^- \text{ behind of the shock} \end{array}$$

On the lower part of the shock originating at $x=0$ at $t=0$, $u^+ = 0$, $u^- = 1$

$$\frac{dX}{dt} = \frac{1}{2}, \quad X(t) = \frac{1}{2}t, \text{ a straight line}$$

This straight part of the shock terminates at where the rightmost characteristic (with slope 1) intersects $X(t)$.

The rightmost characteristic originating at $x=-1$ is given by $x = -1 + 1 \cdot t$.

It intersects $X(t)$ at a time

$$\frac{1}{2}t = t - 1 \quad \sim \quad t = 2, \; X(2) = 1$$

For $t>2$, the shock curves up in the x-t plane as the fan wave hits the shock.

$$\frac{d}{dt}X = \frac{1}{2}u^- .$$

u^- varies from 0 to 1.

The characteristics are described by

$$x = -1 + u^- \cdot t$$

Rewriting it: $u^- = \frac{x+1}{t}$

At where it intersects $X(t)$, it is

$$u^- = \frac{X+1}{t}$$

$$\frac{dX}{dt} = \frac{1}{2}u^- = \frac{1}{2}\,\frac{x+1}{t} = \frac{1}{2}\frac{(X+1)}{t}$$

$$\frac{d}{dt}(X+1) - \frac{1}{2t}(X+1) = 0, \quad X(2)=1$$

Solving

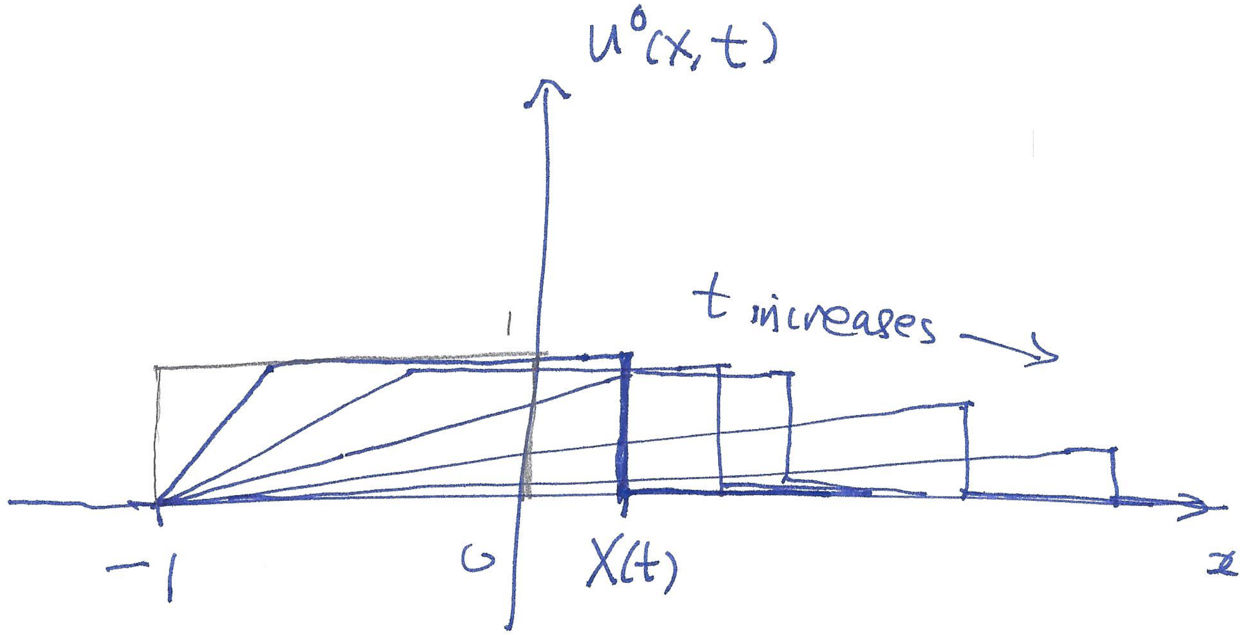
$$\boxed{X(t)+1 = \sqrt{2t}, \quad t \geq 2}$$

$$u^- = \frac{X+1}{t} = \frac{\sqrt{2t}}{t} = \sqrt{\frac{2}{t}}$$

The jump (or drop) across the shock

$$u^- - u^+ = u^- - 0 = \sqrt{\frac{2}{t}}$$

The jump eventually $\to 0$ as $t \to \infty$



Asymptotic solution of the Burger's equation

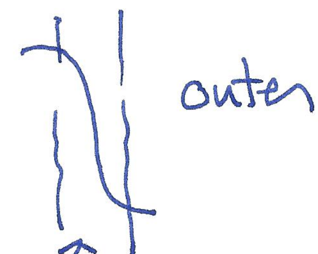
(optional) p101 of Whitham.

$$u_t + u u_x = \nu u_{xx}, \quad -\infty < x < \infty$$

ν small.

Outer solution as $\nu \to 0^+$

$$u^0_t + u^0 u^0_x = 0 .$$



Inner solution: First move with the (unknown) shock speed $U \equiv \dot{X}(t)$.

Let $\hat{x} = x - \int^t U\,dt$

$$u_t = -U u_{\hat{x}}$$
$$= -U u_{\hat{x}}$$

$$(u - U) u_{\hat{x}} = \nu u_{\hat{x}\hat{x}}$$

Scale $\hat{x}$: $\bar{x} = \hat{x}/\varepsilon$

$$(u - U) u_{\bar{x}} \frac{1}{\varepsilon} = \frac{\nu}{\varepsilon^2} u_{\bar{x}\bar{x}}$$

$$(u - U) u_{\bar{x}} = \frac{\nu}{\varepsilon} u_{\bar{x}\bar{x}}$$

Choose $\varepsilon = \nu$

$$\boxed{(u - U) u_{\bar{x}} = u_{\bar{x}\bar{x}}}$$

Integrate w.r.t. $\bar{x}$:

$$\frac{1}{2}u^2 - Uu + C = \frac{\nu}{\varepsilon}u_{\bar{x}}$$

where C is a constant of integration.

Match to the outer solution as $\bar{x} \to \pm\infty$

$\bar{x} \to -\infty, \quad u \to u^{0-}$

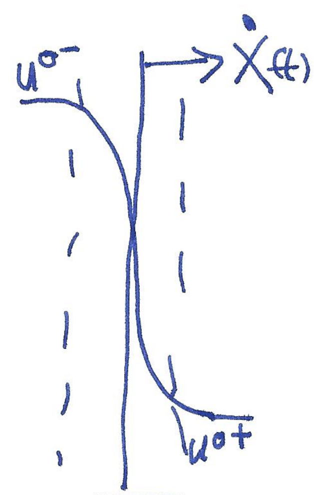
$\bar{x} \to +\infty, \quad u \to u^{0+}$

Noting that $u_{\bar{x}} \to 0$ as $\bar{x} \to \pm\infty$ (i.e. as $\varepsilon \to 0$ for $\hat{x} > 0$, $\hat{x} < 0$)

we get $C = \frac{1}{2}u^{0-}u_0^{+}$

$$\boxed{U = \frac{1}{2}(u^{0-} + u^{0+})}$$

$$(u - u^{0+})(u^{0-} - u) = -2u_{\bar{x}}$$



$$\frac{du}{(u-u^{0+})(u^{0-}-u)} = -2d\bar{x}$$

$$\frac{\hat{x}}{\nu} = \bar{x} = \frac{2}{(u^{0-}-u^{0+})}\ln\left(\frac{u^{0-}-u}{u-u^{0+}}\right)$$

$$\boxed{u = u^{0+} + \frac{u^{0-}-u^{0+}}{1+\exp\left\{\frac{u_0^{-}-u^{0+}}{2\nu}(x-Ut)\right\}}}$$