AMATH 567, Problem Set 1

Lucas Cassin Cruz Burke

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1. Problem 1: Evaluate $\oint_C f(z)dz$ where C is the unit circle centered at the origin and $f(z)$ is given by the following:

   (a) e^{iz}

   (b) e^{z^2}

   (c) $\frac{1}{z-1/2}$

   (d) $\frac{1}{z^2-4}$

   (e) $\frac{1}{2z^2+1}$

   (f) $\sqrt{z-4}$, with $0 \leq z-4 < 2\pi$

   Solution:

   (a) Since e^{iz} is analytic at all points in the complex plane, by Cauchy's Integral Theorem $\oint_C e^{iz}dz = 0$.

   (b) Since e^{z^2} is analytic at all points in the complex plane, by Cauchy's Integral theorem $\oint_C e^{z^2}dz = 0$.

   (c) $\frac{1}{z-1/2}$ has a simple pole at the point $z = 1/2$, which lies inside the unit circle. We can calculate the residue of f at $z = 1/2$ and use the Residue Theorem to calculate the integral. We have $Res(f, 1/2) = \lim_{z\to 1/2}(z - 1/2)\frac{1}{z-1/2} = 1$. Therefore by the residue theorem, $\oint_C \frac{1}{z-1/2}dz = 2\pi i$

   (d) $\frac{1}{z^2-4}$ has two simple poles at $z = \pm 2$. However, since these lie outside the unit circle C, by the Cauchy Integral Theorem $\frac{1}{z^2-4} = 0$

   (e) $\frac{1}{2z^2+1}$ has two simple poles at $z = \pm i/\sqrt{2}$. Since these both lie within C, we must calculate their residues and use the Residue Theorem as in (c). We find the residues to be ± 1 and their sums cancel, so that $\oint_C \frac{1}{2z^2+1} = 0$.

   (f) $\sqrt{z-4}$ is analytic at all points within $0 \leq z-4 < 2\pi$, therefore by Cauchy's Integral Theorem $\oint_C \sqrt{z-4} = 0$.

2. Problem 2: We wish to evaluate the integral

$$\int_0^\infty e^{ix^2}dx$$

Consider the contour

$$I_R = \oint_{C_{(R)}} e^{iz^2}dz$$

where $C_{(R)}$ is the closed circular sector in the upper half plane with boundary points 0, R, and $Re^{i\pi/4}$. Show that $I_R = 0$ and that

$$\lim_{R\to\infty}\int_{C_{1(R)}} e^{iz^2}dz = 0$$

where $C_{1(R)}$ is the line integral along the circular sector from R to $Re^{i\pi/4}$.
Hint: Use $\sin(x) \geq \frac{2x}{\pi}$ on $0 \leq x \leq \pi/2$.
Then, breaking up the contour $C_{(R)}$ into three component parts, deduce

$$\lim_{R\to\infty}\left(\int_0^R e^{ix^2}dx - e^{i\pi/4}\int_0^R e^{-r^2}dr\right) = 0$$

and from the well-known result of real integration:

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

deduce that $I = e^{i\pi/4}\sqrt{\pi}/2$.

Solution:

We begin by showing that $I_R = \oint_{C_{(R)}} e^{iz^2}dz = 0$, which can be done by observing that $f(z) = e^{iz^2}$ is analytic at all points in the complex plane, and therefore by Cauchy's Integral Theorem the integral of $f(z)$ around any closed loop is zero. Hence $I_R = 0$.
Next we need to show that $\lim_{R\to\infty}\int_{C_{1(R)}} e^{iz^2}dz = 0$. We can write z in polar coordinates as $z = Re^{i\theta}$. Since we are integrating along a circular sector centered at the origin, $r = R$ is constant for all points on $C_{1(R)}$. Our angle θ changes from 0 to $\pi/4$, and we have $dz = iRe^{i\theta}d\theta$. So changing our integration variable results in

$$\int_{C_{1(R)}} e^{iz^2}dz = \int_0^{\pi/4}(e^{i(Re^{i\theta})^2})(iRe^{i\theta})d\theta = iR\int_0^{\pi/4} e^{iR^2e^{2i\theta}+i\theta}d\theta$$

We can now use Euler's identity and split the exponential into real and imaginary terms to rewrite this integral as

$$iR\int_0^{\pi/4} e^{-R^2\sin 2\theta}e^{i(R^2\cos 2\theta+\theta)}d\theta$$

We can now use $\sin x \geq \frac{2x}{\pi}$ on $0 \leq x \leq \pi/2$ and see that

$$\left| \int_{C_{1(R)}} e^{iz^2} dz \right| \leq R \int_0^{\pi/4} e^{-R^2 \cdot 4\theta/\pi} d\theta = R \left[\frac{-\pi}{4R^2} e^{-4R^2\theta/\pi} \right]_0^{\pi/4} = \frac{\pi}{4R}(1-e^{-R^2})$$

The limit of the right-hand side as $R \to \infty$ is clearly zero, hence

$$\lim_{R\to\infty} \left| \int_{C_{1(R)}} e^{iz^2} dz \right| \leq 0$$

and it follows from this that

$$\lim_{R\to\infty} \int_{C_{1(R)}} e^{iz^2} dz = 0$$

We can break the contour $C_{(R)}$ into three component parts: the real part from 0 to R which we would like to evaluate (and which we will refer to as $C_{0(R)}$), the circular arc sector from R to $Re^{i\pi/4}$ whose limit we just analyzed above (called $C_{1(R)}$), and the line from $Re^{i\pi/4}$ to 0, which we will analyze now (called $C_{2(R)}$).

The integral over $C_{2(R)}$ is $\int_{Re^{i\pi/4}}^0 e^{iz^2} dz$. Since $\theta = \pi/4$ is constant along this curve, we can again write z in polar form as $z = re^{i\theta}$ and rewrite this integral in terms of r as it goes from 0 to R. We have $dz = e^{i\theta} dr$, and so

$$\int_{Re^{i\pi/4}}^0 e^{iz^2} dz = -\int_0^R e^{i(re^{i\pi/4})^2} (e^{i\pi/4}) dr = -e^{i\pi/4} \int_0^R e^{ir^2 e^{i\pi/2}} dr = -e^{i\pi/4} \int_0^R e^{-r^2} dr$$

We are now ready to return to the original integral we would like to evaluate. We know that $I_R = 0$, and we have broken this integral up into three directed line integrals described above. We now take the limit $\lim_{R\to\infty} I_R$ and, applying the result we found above that the contribution of the $C_{1(R)}$ contour goes to zero as $R \to \infty$, we find

$$\lim_{R\to\infty} I_R = \lim_{R\to\infty} \left(\int_0^R e^{ix^2} dx - e^{i\pi/4} \int_0^R e^{-r^2} dr \right) = 0$$

We can apply the well-known result of real integration $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ to evaluate the second term in parentheses. Doing so gives us

$$\lim_{R\to\infty} I_R = \int_0^\infty e^{ix^2} dx - e^{i\pi/4} \int_0^\infty e^{-r^2} dr = \int_0^\infty e^{ix^2} dx - e^{i\pi/4} \frac{\sqrt{\pi}}{2} = 0$$

And lastly we can solve for our original integral to find

$$\int_0^\infty e^{ix^2} dx = e^{i\pi/4} \frac{\sqrt{\pi}}{2}$$

3. Problem 3: Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

Show how to evaluate this integral by considering

$$\oint_{C_{(R)}} \frac{dz}{z^2+1}$$

where $C_{(R)}$ is the closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x axis. Hint: use

$$\frac{1}{z^2+1} = \frac{-1}{2i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right)$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \to \infty$. Verify your answer by usual integration in real variables.

Solution:

We would like to evaluate I by considering the contour integral

$$\oint_{C_{(R)}} \frac{dz}{z^2+1}$$

where $C_{(R)}$ is the closed semicircle in the upper half plane with radius R.

To evaluate I, we will split our contour into two parts, the first being the real line interval $[-R, R]$ (which we will call $C_{1(R)}$) and the second being the semicircular arc in the upper half complex plane (which we will call $C_{2(R)}$). Next we will show that as $R \to \infty$ the contribution of the semicircular arc vanishes as R goes to infinity. Lastly, we will evaluate the contour integral in the limit of $R \to \infty$ using the Residue Theorem to achieve our final result.

Let us consider the contribution of the upper semicircular arc component $C_{2(R)}$ of our contour to our contour integral. Since the radius is fixed for this section, we can use polar coordinates to write this contribution as the integral

$$I_{C_{2(R)}} = \int_0^{\pi} \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1} d\theta$$

Applying the integral inequality $\left|\int_C f(z)dz\right| \leq \int_C |f(z)|\, dz$, we have

$$|I_{C_{2(R)}}| \leq \int_0^{\pi} \frac{R}{|R^2e^{2i\theta}+1|} d\theta$$

We now note that $|a| \leq |a+b| + |b|$ and hence $|a| - |b| \leq |a+b|$. Using this fact, we see that $|R^2e^{2i\theta}+1| \geq R^2 - 1$, and therefore

$$\frac{R}{|R^2e^{2i\theta}+1|} \leq \frac{R}{R^2-1}$$

and so by transitivity of inequality

$$|I_{C_{2(R)}}| \leq \int_0^\pi \frac{R}{R^2 - 1} d\theta$$

In the limit as $R \to \infty$ we see by L'Hôpital's rule that the integrand goes to zero and hence,

$$\lim_{R \to \infty} |I_{C_{2(R)}}| = 0$$

So in the limit as R approaches infinity the contribution of the upper hemisphere component of the contour vanishes.

We now evaluate the contour integral using the Residue Theorem. We can factorize the denominator of the integrand and use partial fraction decomposition to rewrite our integral as

$$\oint_{C_{(R)}} \frac{-1}{2i} \left(\frac{1}{z+i} - \frac{1}{z-i} \right) dz$$

We see that the integral has simple poles at $z = \pm i$, and since $C_{(R)}$ covers only the upper half of the complex plane, only the $z = i$ residue contributes to the integral.

The residue at $z = i$ is

$$\lim_{z \to i} \frac{-1}{2i} \left(\frac{z-i}{z+i} - 1 \right) = \frac{1}{2i}$$

And so by the residue theorem

$$\oint_{C_{(R)}} \frac{dz}{z^2+1} = 2\pi i \cdot \frac{1}{2i} = \pi$$

We can verify this result through usual integration with real variables. We have

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = [\tan^{-1} x]_{-\infty}^{\infty} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$$

4. Problem 4: Let

$$f(z) = e^{\frac{t}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n$$

Show from the definition of Laurent series and using properties of integration that

$$J_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} d\theta = \frac{1}{\pi}\int_0^{\pi} \cos{(n\theta - t\sin\theta)} d\theta$$

The functions $J_n(t)$ are called Bessel functions, which are well-known special functions in mathematics and physics.

Solution:

The Laurent series of a function $f(z)$ which is analytic in an annulus $R_1 \le |z - z_0| \le R_2$ is defined as the infinite series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ in $R_1 < R_a \le |z - z_0| \le R_b < R_2$ where

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and C is a closed contour enclosing z_0 in the annulus.

Since $f(z)$ has one singularity, at $z = 0$, we will select our annulus to be centered at the origin and allow its inner radius R_1 to approach zero, and its outer radius R_2 to approach infinity.

Comparing with the above equation, we see that

$$J_n(t) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}} dz == \frac{1}{2\pi i}\oint_C \frac{e^{\frac{t}{2}(z-1/z)}}{z^{n+1}} dz$$

Where C is a closed contour enclosing $z_0 = 0$, which we can freely choose to be the unit circle for ease of computation. Changing our integration variable, we have

$$J_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta} - e^{-i\theta})}}{e^{in\theta}} d\theta$$

Using $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ we have

$$J_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it\sin\theta}}{e^{in\theta}} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} d\theta$$

And using Euler's identity we can write this as

$$J_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(n\theta - t\sin\theta) - i\sin(n\theta - t\sin\theta)]$$

Since our integration bounds are even, we can use the evenness of cos and the oddness of sin to simplify this expression to

$$\frac{1}{\pi}\int_0^{\pi} \cos(n\theta - t\sin\theta) d\theta$$