Linearization

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\rho + \rho\frac{\partial}{\partial x}u = 0$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + \frac{c^2}{\rho}\frac{\partial}{\partial x}\rho = 0$$

Linearize about a basic state $(\ )_0$

by assuming

$$\left|\frac{u-u_0}{u_0}\right| << 1, \quad \left|\frac{\rho-\rho_0}{\rho_0}\right| << 1$$

Write $u = u_0 + u'$, $\rho = \rho_0 + \rho'$

u_0, ρ_0 assumed here to be constants (not necessary)

$$\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)\rho' + \rho_0\frac{\partial}{\partial x}u' = 0$$

$$\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)u' + \frac{c_0^2}{\rho_0}\frac{\partial}{\partial x}\rho' = 0$$

$\frac{c_0^2}{\rho_0}\frac{\partial}{\partial x}$ the first equation:

$$\frac{c_0^2}{\rho_0}\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)\rho_x' + \frac{c_0^2}{\rho_0}\rho_0\frac{\partial^2}{\partial x^2}u' = 0$$

$\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)$ of the second equation:

$$\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)^2 u' + \frac{c_0^2}{\rho_0}\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)\rho_x'$$

Subtract:

$$c_0^2\frac{\partial^2}{\partial x^2}u' - \left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)^2 u'$$

Or: $\boxed{\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)^2 u' = c_0^2\frac{\partial^2}{\partial x^2}u'}$

Similarly

$$\boxed{\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)^2 \rho' = c_0^2\frac{\partial^2}{\partial x^2}\rho'}$$

Wave equation; Normally $\frac{\partial^2}{\partial t^2}u = c_0^2\frac{\partial^2}{\partial x^2}u$

but we have added advection by u_0.

Solution:

$$\left[\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right) - c_0 \frac{\partial}{\partial x}\right]\left[\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right) + c_0 \frac{\partial}{\partial x}\right]$$

$$\cdot u' = 0$$

Or $$\left[\left(\frac{\partial}{\partial t} + (u_0 - c_0)\frac{\partial}{\partial x}\right]\right]\left[\left(\frac{\partial}{\partial t} + (u_0 + c_0)\frac{\partial}{\partial x}\right]\right]$$

$$\cdot u' = 0$$

D'Alembert's solution

$$u'(x,t) = f(\xi) + g(\eta)$$

where $$\left.\begin{aligned} \xi &= x - (u_0 - c_0)t \\ \eta &= x - (u_0 + c_0)t \end{aligned}\right\} \text{characteristics.}$$

Alternatively, the equation can be transformed into:

$$\boxed{u'_{\xi\eta} = 0}$$

The simplest form of second order PDE

Second-order wave equation has two families of characteristics :

$$u_{tt} = c^2 u_{xx}$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

$$\left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x}\right) u = 0$$

So $\left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x}\right) u = 0$

and $\left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x}\right) u = 0$

The first equation is satisfied if

$$u(x,t) = g(x+ct)$$

and the second equation is satisfied if

$$u(x,t) = f(x-ct)$$

So in general

$$u = f(\xi) + g(\eta)$$

where $\xi = x - ct, \quad \eta = x + ct$

Transform x, t to ξ, η

In terms of the characteristic coordinates the wave equation becomes

$$u_{\xi\eta} = 0$$

The solution can be obtained simply by integration.

A more complicated wave equation

$$\frac{\partial^2}{\partial t^2} u - c^2 \frac{\partial^2}{\partial x^2} u = F(x, t, u, u_t, u_x)$$

(could be nonlinear, called quasi-linear because the nonlinearity does not appear at the highest order derivatives)

Can be transformed into

$$u_{\xi\eta} = F$$

Can all second-order quasi-linear PDE be transformed into this form?