Lecture 3. Hilbert Spaces

So far this week, we introduced the defⁿ of Banach spaces & their duals. For most applications in Amath & comp. sci. Banach spaces are as general as we need to get. However, in many settings, specially when it comes to alg. we would like to have more structure / geometry. This is why Hilbert spaces are so useful.

3.1 Inner products

Defⁿ. Let X be a vector space. We say that a function/map $\langle \cdot , \cdot \rangle : X \times X \to \mathbb{R}$ is an inner product on X if it satisfies the following properties:

(i) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(ii) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$ $\qquad x,y,z \in X$

(iii) $\langle x, y\rangle = \langle y, x\rangle$ $\qquad \alpha \in \mathbb{R}$

(iv) $\langle x, x\rangle \geq 0$

(v) $\langle x, x\rangle = 0 \iff x = 0$

A vector space that is equipped with an inner product is called an inner product space or a pre-Hilbert space.

Note that a inner product naturally induces a norm on X

$$\|x\| = \sqrt{\langle x, x \rangle}$$

& so inner-product spaces are also normed spaces.

In fact, inner products are generalizations of the euclidean dot product & so allow us to define angles, $\theta := \cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}\right)$ & in particular the notion of orthogonality.

x & y are orth. if $\langle x, y \rangle = 0$

This is precisely the sort of geometry that is useful in applications as it allows us to "decompose" elements of X into smaller components.

3.3 Hilbert Spaces

Def^n A Hilbert space is a complete inner product space.

Summary of the desirable properties of Hilbert Spaces:

- We can represent a Hilbert space H as a direct sum of a closed subspace & its orth. complement.
- We can define orthonormal sets & sequences/bases
- The Riesz representation theorem shows that H^* is isomorphic to H!
- We can define the Hilbert adjoint T^* of an operator T, analogous to the transpose of a matrix!

If $\|x\|$ is a norm induced by an inner product we can compute

$$\|x-y\|^2 = \langle x-y, x-y \rangle = \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle$$
$$\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$$

(*) $\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$ (Parallelogram ident)

3 (**) $\Rightarrow \langle x,y\rangle = \frac{1}{4}\left(\|x-y\|^2 - \|x-y\|^2\right)$

In fact, we infer if (*) Holds for a norm then it is induced by an inner product!

eg. of Hilbert spaces

- Euclidean space $\mathbb{R}^n$

$$\langle x,y\rangle := x^T y$$

- Sequence space ℓ^2

$$\langle x,y\rangle := \sum_{j=1}^{\infty} x_j y_j$$

- Space $L^2([a,b])$

$$\langle x,y\rangle := \int_a^b x(t)\,y(t)\,dt$$

eg. of Banach spaces that aren't Hilbert

- Seq. spaces ℓ^p for $p\geq 1$ & $p\neq 2$

let $x = (1,1,0,0,\cdots)$
$y = (1,-1,0,0,\cdots)$

then $\|x\|_{\ell^p} = \|y\|_{\ell^p} = 2^{1/p}$ but $\|x-y\|_{\ell^p} = \|x+y\|_{\ell^p} = 2$

so Parallelogram fails!

- The space $C[a,b]$ equipped with sup norm is not Hilbert.

(Scribe please complete) Take $x(t) = 1$ & $y(t) = \frac{(t-a)}{(b-a)}$ & show parallelogram fails.

An important idea for us is the ability to obtain a Hilbert space by completion of a pre-Hilb. space (we will introduce RKHS spaces this way). We will show a result in this direction in the rest of the lecture.

First we need a technical lemma

(See 3.2.2 of Krey. for proof) **Lem** (Continuity of inner product) if X is a Pre-Hilbert space & $x_n \to x$, $y_n \to y$ are convergent sequences in X. Then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

(Scribe add proof)

An important consequence of this result is that any pre-Hilbert space can be completed to obtain a Hilbert space. (Recall our result on completion of normed spaces to obtain a Banach space).

Thm (Completion) For any pre-Hilbert space X there exists a Hilbert space H & an isomorphism A from X to a dense subspace $W \subset H$. The space H is unique up to isomorphisms.

A quick note on isomorphisms

Recall that an isomorphism on a normed space is a linear bijection that preserves distances

$$\|Tx - Ty\| = \|x - y\|$$

We can extend this def^n to pre-Hilbert spaces & observe that

$$\langle Tx, Ty\rangle = \frac{1}{4}\left(\|Tx+Ty\|^2 - \|Tx-Ty\|^2\right)$$

$$= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = \langle x, y\rangle$$

In other words, isomorphisms on pre-Hilbert spaces preserve inner products (ie, angles!)

Proof: Since X is normed then by the completion result for normed spaces we know that there exists a Banach space H & an isomorphism A s.t.

$A: X \to W \subset H$ & W is dense in H.

(The main difficulty right now is to show that H is Hilbert, ie, has an inner product!)

We can define an inner product on W using A

$$\langle w, w'\rangle_W := \langle A^{-1}w, A^{-1}w'\rangle_X \quad \forall w, w' \in W$$

(6) & further using Lem above, define an inner product on H, as follows:

Given a pair of points $h, h' \in H$ take sequences $\{h_n\}_{n=1}^{\infty}, \{h_n'\}_{n=1}^{\infty} \subset W$ so that $h_n \to h$ & $h_n' \to h'$. The $\{h_n\}, \{h_n'\}$ are called the "representatives" of h & h' respectively. We then define

$$\langle h, h' \rangle_H := \lim_{n\to\infty} \langle h_n, h_n' \rangle_W$$

The completion result for Banach spaces says that H is unique upto isometries. That is, if $\tilde{H}$ is another completion of X then $\tilde{H}$ & H are isomorphic as normed spaces!

$T: H \to \tilde{H}$ $\quad \|h - h'\|_H = \|Th - Th'\|_{\tilde{H}}$

for isomorphism H. Since both H & $\tilde{H}$ are Hilbert spaces by the above argument. Then (**) implies they are also isomorphic as Hilbert spaces, ie, $\langle h, h' \rangle_H = \langle Th, Th' \rangle_{\tilde{H}}$

$\blacksquare$