

AMATH 562
Advanced Stochastic Processes
Homework 5

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1. Please give an integer-valued stochastic Lévy process $(\eta_t)_{t \geq 0}$,

$$\eta_t \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\},$$

which, furthermore, is a martingale.

Solution: Let $(P_t)_{t \geq 0}$ be a Poisson process with intensity λ and let $(X_n)_{n \in \mathbb{N}}$ be a sequence of iid random variables with values in \mathbb{Z} with $\mathbb{E}X_i = 0$. Now let $\eta = (\eta_t)_{t \geq 0}$ be the compound Poisson process defined by

$$\eta_t = \sum_{n=1}^{P_t} X_n$$

This is a Lévy process, as are all compound Poisson processes. It is clearly integer valued, since the X_n summands are integer valued. Lastly, we have

$$\begin{aligned} \mathbb{E}[\eta_{t+s} | \mathcal{F}_t] &= \mathbb{E}[\eta_t + (\eta_{t+s} - \eta_t) | \mathcal{F}_t] \\ &= \eta_t + \mathbb{E}[\eta_{t+s} - \eta_t | \mathcal{F}_t] \\ &= \eta_t + \mathbb{E}\left[\sum_{n=P_t+1}^{P_{t+s}} X_n \middle| \mathcal{F}_t\right] \\ &= \eta_t + \sum_{n=P_t+1}^{P_{t+s}} \mathbb{E}[X_n] \\ &= \eta_t \end{aligned}$$

Hence η_t is also martingale.

2. Discuss why the Lévy process as defined in Definition 10.1.1. cannot be improved to a Gaussian distribution.

Solution: The Central Limit Theorem applies to infinite sequences of i.i.d. random variables with finite expectation and variance. Since Brownian motion is defined as the limit of a sequence of scaled random walks, which are themselves sums of i.i.d. random variables, it follows from CLM that the increments of a Brownian motion are Gaussian distributed.

The reason this does not hold for more general Lévy processes is that Lévy processes may have discontinuous jumps whose behaviour cannot be replicated by normally distributed i.i.d. increments. The jump behavior of a Lévy process is characterized by the Poisson random measure, which contains much more information than the mean and variance which characterise Gaussian distributed processes. Allowing for non-normally distributed increments enables us to work with a much larger class of processes.

3. A probability distribution is *infinitely divisible* if it can be expressed as the probability distribution of the sum of an arbitrary number of independent and identically distributed (i.i.d.) random variables. It turns out that every infinitely divisible probability distribution corresponds to a Lévy process.

- (a) Show that the normal distribution on \mathbb{R} is infinitely divisible.

Solution: Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then its characteristic function can be calculated to be

$$\phi_X(t) = \mathbb{E}e^{itX} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{itx} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right)$$

Now for some $n \in \mathbb{N}$ define S_n to be the sum of n iid random variables $(Z_i)_{1 \leq i \leq n}$.

$$S_n = \sum_{i=1}^n Z_i \qquad Z_i \sim \mathcal{N}(\mu/n, \sigma^2/n)$$

The characteristic function of S_n can be calculated as

$$\begin{aligned} \phi_{S_n}(t) &= (\phi_Z(t))^n = \exp\left(i\mu t/n - \frac{1}{2}\sigma^2 t^2/n\right)^n = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right) \\ &= \phi_X(t) \end{aligned}$$

Since we can express $\mathcal{N}(\mu, \sigma^2)$ as the distribution of the sum of an arbitrary number of iid random variables, the normal distribution is infinitely divisible.

- (b) Show that the Poisson distribution on \mathbb{N} is infinitely divisible.

Solution: Let $P \sim \text{Poi}(\lambda)$. Then we can calculate the characteristic function

$$\begin{aligned}
\phi_P(t) &= \mathbb{E}e^{itP} = \sum_{n=0}^{\infty} e^{itn} \frac{\lambda^n}{n!} e^{-\lambda} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^{it})^n}{n!} \\
&= \exp(-\lambda(1 - e^{it}))
\end{aligned}$$

Now for some $n \in \mathbb{N}$ define S_n to be the sum of n iid random variables $(Z_i)_{1 \leq i \leq n}$, with

$$S_n = \sum_{i=1}^n Z_i \qquad Z_i \sim \text{Poi}(\lambda/n)$$

Then the characteristic function of S_n can be calculated as

$$\begin{aligned}
\phi_{S_n}(t) &= (\phi_Z(t))^n = \exp(-\lambda(1 - e^{it})/n)^n = \exp(-\lambda(1 - e^{it})) \\
&= \phi_P(t)
\end{aligned}$$

Since we can express $\text{Poi}(\lambda)$ as the distribution of the sum of an arbitrary number of iid random variables, the Poisson distribution is infinitely divisible.

(c) The pdf of the Cauchy distribution is

$$f_X(x) = \frac{\gamma}{\pi(\gamma^2 + x^2)}$$

Show that its characteristic function $\phi_X(u) := \mathbb{E}[e^{iuX}]$ is $e^{-\gamma|u|}$. What are its expected value and its variance?

Solution: We have

$$\begin{aligned}
\phi_X(u) &= \mathbb{E}[e^{iuX}] \\
&= \int_{-\infty}^{\infty} e^{iux} f_X(x) dx \\
&= \frac{\gamma}{\pi} \int_{-\infty}^{\infty} \frac{e^{iux}}{\gamma^2 + x^2} dx \\
&= \frac{\gamma}{\pi} I
\end{aligned}$$

To evaluate the integral I , consider the contour integral

$$I_C = \int_C \frac{e^{iuz}}{\gamma^2 + z^2} dz$$

where C is the positively-oriented semicircular contour enclosing the upper half plane if $u > 0$, and the negatively-oriented semicircular contour enclosing the lower half plane if $u < 0$. Since $1/(\gamma^2 + z^2) \rightarrow 0$ as $|z| \rightarrow \infty$, by Jordan's lemma we have that $I_C = I$. We will now calculate I_C using the residue theorem. We note that the integrand has simple poles at $z = \pm i\gamma$, whose residues we calculate as

$$\begin{aligned}\text{Res}(i\gamma) &= \lim_{z \rightarrow i\gamma} \frac{e^{iuz}}{z + i\gamma} = \frac{e^{-\gamma u}}{2i\gamma} \\ \text{Res}(-i\gamma) &= \lim_{z \rightarrow -i\gamma} \frac{e^{iuz}}{z - i\gamma} = \frac{-e^{\gamma u}}{2i\gamma}\end{aligned}$$

Hence, by the residue theorem we find

$$\begin{aligned}I_C &= \begin{cases} 2\pi i \text{Res}(i\gamma) = \frac{\pi}{\gamma} e^{-\gamma u} & u > 0 \\ -2\pi i \text{Res}(-i\gamma) = \frac{\pi}{\gamma} e^{\gamma u} & u < 0 \end{cases} \\ &= \frac{\pi}{\gamma} e^{-\gamma|u|}\end{aligned}$$

Returning to our original calculation, we have

$$\begin{aligned}\phi_X(u) &= \frac{\gamma}{\pi} I = \frac{\gamma}{\pi} I_C \\ &= e^{-\gamma|u|}\end{aligned}$$

Note that $\phi_X(u)$ is not differentiable at $u = 0$, and hence the mean, variance, and all $n > 0$ moments of X are undefined.

- (d) A real-valued random variable X is infinitely divisible if and only if its characteristic function $\phi_X(u)$ is of the form $e^{\psi(u)}$, with

$$\psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iuz} - 1 - iuz\mathbb{1}_{|z|<1}) \nu(dz). \quad (1)$$

We see that when the $\nu(dz) = 0$, the $\phi_X(u)$ corresponds to the normal distribution $\mathcal{N}(\mu, \sigma^2)$. Taking Eq. 1 as given, with $\mu = \sigma = 0$, find the $\nu(dz)$ such that X has a Cauchy distribution.

Solution: If X is Cauchy-distributed, then we know from part (c) that $\phi_X(u) = e^{-\gamma|u|}$, which means that $\psi(u) = -\gamma|u|$. Hence, letting $\mu = \sigma = 0$, we have

$$-\gamma|u| = \int_{\mathbb{R}} (e^{iuz} - 1 - iuz\mathbb{1}_{|z|<1}) \nu(dz) = \int_{\mathbb{R}} (e^{iuz} - 1) \nu(dz) - iu \int_{-1}^1 z \nu(dz)$$

For the right hand side to converge we need $\nu(dz) \sim z^{-k}dz$ for some $k \geq 1$. We also note that for even k the second integral goes away, as the integrand is odd. Lastly, we note that $\mathcal{F}\{|x|\}(k) \sim \frac{1}{k^2}$. For these reasons, let $k = 2$ and let $\nu(dz) = A \frac{dz}{z^2}$. With this, we have

$$-\gamma|u| = A \int_{\mathbb{R}} (e^{iuz} - 1) \frac{dz}{z^2} = AI$$

Now let C be a positively oriented semicircular contour enclosing the upper half of the complex plane if $u > 0$, and a negatively oriented semicircular contour enclosing the lower half plane if $u < 0$. Then we have

$$\mathcal{P}I = I_C = \int_C \frac{e^{iuz} - 1}{z^2} dz$$

where $\mathcal{P}I$ denotes the Cauchy principle value of the integral I over $\mathbb{R} \setminus \{0\}$. I_C has a simple pole on the contour at $z = 0$. We evaluate the residue to be

$$\text{Res}(0) = \lim_{z \rightarrow 0} \frac{e^{iuz} - 1}{z} = \lim_{z \rightarrow 0} \frac{iue^{iuz}}{1} = iu$$

Hence by the residue theorem, (and taking care to change the direction of integration for $u < 0$), we have

$$\mathcal{P}I = I_C = -\pi|u|$$

Applying this result above, we have

$$-\gamma|u| = -A\pi|u| \Rightarrow A = \frac{\gamma}{\pi}$$

Hence, we conclude that

$$\nu(dz) = -\frac{\gamma}{\pi} \frac{dz}{z^2}$$

corresponds to the Cauchy distribution.

4. Let P_t be a Poisson process with rate λ . Consider the compensated Poisson process defined by

$$\tilde{P}_t = P_t - \mathbb{E}[P_t]$$

Show that its characteristic function has the form

$$\mathbb{E} \left[e^{iu\tilde{P}_t} \right] = e^{t\psi(u)}.$$

and give an expression for $\psi(u)$.

Solution: We have

$$\begin{aligned}
\phi_{\tilde{P}_t}(u) &= \mathbb{E} \left[e^{iu\tilde{P}_t} \right] \\
&= \mathbb{E} \left[e^{iu(P_t - \mathbb{E}[P_t])} \right] \\
&= e^{-iu\lambda t} \mathbb{E} \left[e^{iuP_t} \right] \\
&= e^{-iu\lambda t} \sum_{k=0}^{\infty} e^{iuk} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= e^{-\lambda t(1+iu)} \sum_{k=0}^{\infty} \frac{(\lambda t e^{iu})^k}{k!} \\
&= e^{-\lambda t(1+iu-e^{iu})}
\end{aligned}$$

Hence, we see that the characteristic function of \tilde{P}_t has the form $\phi_{\tilde{P}_t}(u) = e^{t\psi(u)}$ with

$$\psi(u) = -\lambda(1 + iu - e^{iu})$$

5. **Exercise 10.1:** Let $P = (P_t)_{t \geq 0}$ be a Poisson process with intensity λ .

(a) What is the Lévy measure ν of P ?

Solution: By definition of the Lévy measure we have

$$\nu(U) := \mathbb{E}N(1, U) = \mathbb{E} \int_0^1 \mathbb{1}_{\{\Delta P_s \in U\}} ds \quad \forall U \in \mathcal{B}_0 := \mathcal{B}(\mathbb{R}) \setminus \{0\}$$

where $N(t, U)$ is the Poisson random measure of P , and ΔP_s is the jump of P at time s . Since Poisson processes are pure jump processes we have $P_t = \int_0^t \Delta P_s ds$. Furthermore, since Poisson processes are counting processes, we have $\forall s \in (0, t] : \Delta P_s \in \{0, 1\}$ almost surely. Now let $U \in \mathcal{B}_0$ such that $1 \in U$. Then from the above properties and the definition of the Poisson random measure we have

$$N(t, U) = \int_0^t \mathbb{1}_{\{\Delta P_s \in U\}} ds = \int_0^t \mathbb{1}_{\{\Delta P_s = 1\}} ds = \int_0^t \Delta P_s ds = P_t$$

For a general $U \in \mathcal{B}_0$ we have

$$N(t, U) = \begin{cases} P_t & 1 \in U \\ 0 & 1 \notin U \end{cases} = P_t \mathbb{1}_{\{1 \in U\}}$$

With this we can calculate the Lévy measure ν of P to be

$$\begin{aligned}\nu(U) &= \mathbb{E}N(1, U) = \mathbb{E}[P_1 \mathbf{1}_{\{1 \in U\}}] = \mathbf{1}_{\{1 \in U\}} \mathbb{E}[P_1] \\ &= \lambda \mathbf{1}_{\{1 \in U\}}\end{aligned}$$

- (b) Let $dX_t = dP_t$. Define $u(t, x) := \mathbb{E}[\varphi(X_T) | X_t = x]$. Find $u(t, x)$ and verify that it solves the Kolmogorov backward equation.

Solution: We have

$$\begin{aligned}u(t, x) &= \mathbb{E}[\varphi(X_T) | X_t = x] \\ &= \sum_{n=0}^{\infty} \varphi(n) \mathbb{P}(X_T = n | X_t = x) \\ &= \sum_{n=x}^{\infty} \varphi(n) \frac{(\lambda(T-t))^{n-x}}{(n-x)!} e^{-\lambda(T-t)} \\ &= e^{-\lambda(T-t)} \sum_{k=0}^{\infty} \varphi(k+x) \frac{(\lambda(T-t))^k}{k!}\end{aligned}$$

To show that $u(t, x)$ solves the Kolmogorov backward equation, we will first show that $u(t, X_t)$ is martingale. We have

$$\mathbb{E}[u(t+s, X_{t+s}) | \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[\varphi(X_T) | \mathcal{F}_{t+s}] | \mathcal{F}_t] = \mathbb{E}[\varphi(X_T) | \mathcal{F}_t] = u(t, X_t)$$

Hence $u(t, X_t)$ is martingale. Next, we use the Itô formula (Lorig 10.2.1) to calculate the differential of $u(t, X_t)$. Using $dX_t = dP_t = \int_{\mathbb{R}} \tilde{N}(dt, dz)$, and the expressions for $N(t, U)$ and $\nu(U)$ derived in (a), we have

$$\begin{aligned}du(t, X_t) &= \frac{\partial u}{\partial t}(t, X_t)dt + \int_{\mathbb{R}} [u(t, X_{t-} + 1) - u(t, X_{t-})] N(dt, dz) - \int_{\mathbb{R}} \frac{\partial u}{\partial x}(t, X_t) \nu(dz)dt \\ &= \frac{\partial u}{\partial t}(t, X_t)dt + \lambda \left[u(t, X_{t-} + 1) - u(t, X_{t-}) - \frac{\partial u}{\partial x}(t, X_t) \right] dt \\ &= [\partial_t + \lambda(\theta_1 - 1 - \partial_x)] u(t, X_t)dt\end{aligned}$$

where we have utilized the shift operator θ_1 defined by $\theta_1 u(t, x) = u(t, x+1)$. Since $u(t, X_t)$ is martingale the dt -term must equal to zero, which means that

$$[\partial_t + \lambda(\theta_1 - 1 - \partial_x)] u(t, X_t) = 0 \tag{2}$$

We can use Lorig 10.3.3. to calculate the generator $\mathcal{A}(t)$ of the SDE solution X as

$$\begin{aligned}\mathcal{A}(t) &= \int_{\mathbb{R}} \nu(dz) (\theta_1 - 1 - \partial_x) \\ &= \lambda (\theta_1 - 1 - \partial_x)\end{aligned}$$

Hence we conclude that $u(t, x)$ must satisfy

$$(\partial_t + \mathcal{A}(t))u(t, X_t) = 0$$

along with the terminal condition $u(T, X_T) = \varphi(X_T)$. This shows that $u(t, x)$ solves the Kolmogorov backward equation.