AMATH 568
Advanced Differential Equations
Homework 4

Lucas Cassin Cruz Burke

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1. Consider the weakly nonlinear oscillator:

$$\frac{d^2y}{dt^2} + y + \epsilon y^5 = 0$$

with $y(0) = 0$ and $y'(0) = A > 0$ and with $0 < \epsilon \ll 1$.

(a) Use a regular perturbation expansion and calculate the first two terms.
Solution: Let $y = y_0 + \epsilon y_1 + \ldots$. Then our ODE becomes

$$y_0'' + \epsilon y_1'' + y_0 + \epsilon y_1 + \epsilon y_0^5 + \mathcal{O}(\epsilon^2) = 0$$

Looking at the $\mathcal{O}(1)$ terms, we have

$$y_0'' + y_0 = 0 \Rightarrow y_0 = A\sin t + B\cos t$$

Applying the boundary condition $y_0(0) = 0$ and $y_0'(0) = A$ we find our first order solution to be

$$y_0(t) = A\sin t$$

Next, we look at the $\mathcal{O}(\epsilon)$ terms. Plugging in our solution for y_0, we have

$$y_1'' + y_1 + A^5\sin^5 t = 0 \Rightarrow y_1'' + y_1 = -A^5\sin^5 t$$

For the homogeneous problem, we have the general solution $y = c_1\sin(t) + c_2\cos(t)$. For the particular solution we use Mathematica and find

$$y_1(t) = \frac{-5}{128}A^5\sin(3t) + \frac{1}{384}A^5\sin(5t) + \frac{5}{16}A^5 t\cos(t) + c_1\sin(t) + c_2\cos(t)$$

From the boundary condition $y_1(0) = 0$ we see that $c_2 = 0$, and from the boundary condition $y_1'(0) = 0$ we have

$$y_1'(0) = \frac{-15}{128}A^5 + \frac{5}{384}A^5 + \frac{5}{16}A^5 + c_1 = 0 \Rightarrow c_1 = \frac{-5}{24}A^5$$

Hence, we may write our second order correction term as

$$y_1(t) = \frac{-5}{128}A^5\sin(3t) + \frac{1}{384}A^5\sin(5t) + \frac{5}{16}A^5 t\cos(t) - \frac{5}{24}A^5\sin(t)$$

Note that, since $\sin^5 t$ is not orthogonal to the null-space, we have secular growth terms $\sim t\cos(t)$.

Combining our y_0 and y_1, we find the following approximate solution from the regular expansion method:

$$y(t) = A\sin t + \epsilon\left[\frac{-5}{128}A^5\sin(3t) + \frac{1}{384}A^5\sin(5t) + \frac{5}{16}A^5 t\cos(t) - \frac{5}{24}A^5\sin(t)\right] + \mathcal{O}(\epsilon^2)$$

(b) Determine at what time the approximation of part (a) fails to hold.

Solution: We note that there is a secular growth term in $y_1(t)$ given by

$$\frac{5}{16}A^5 t\cos(t)$$

This term will grow linearly with time, and so for large time-scales our assumption that $\epsilon y_1(t)$ is a small correction begins to break down. In particular, we see that our approximation fails when

$$t \sim \frac{1}{A^5}\frac{1}{\epsilon}$$

(c) Use a Poincare-Lindstedt expansion and determine the first two terms and frequency corrections.

Solution: We now expand $y(t)$ in terms of slow time τ.

$$y(t) = y_0(\tau) + \epsilon y_1(\tau) + \mathcal{O}(\epsilon^2) \qquad \tau = \omega t = (\omega_0 + \epsilon\omega_1 + \dots)t$$

Making the change of variables from $t \to \tau$ results in

$$y_{tt} + y + \epsilon y^5 = 0 \qquad \Rightarrow \qquad \omega^2 y_{\tau\tau} + y + \epsilon y^5 = 0$$

Plugging in our power series expansions for y and ω gives us, to $\mathcal{O}(\epsilon)$,

$$(\omega_0^2 + 2\epsilon\omega_0\omega_1)(y_{0\tau\tau} + \epsilon y_{1\tau\tau}) + y_0 + \epsilon y_1 + \epsilon y_0^5 = 0$$

To $\mathcal{O}(1)$ we have

$$\omega_0^2 y_{0\tau\tau} + y_0 = 0 \qquad \Rightarrow \qquad y_0(\tau) = c_1 \sin(\tau/\omega_0) + c_2 \cos(\tau/\omega_0)$$

The Dirichlet boundary condition at $t = 0$ means that $c_2 = 0$, while the Neumann boundary condition means that

$$\frac{c_1}{\omega_0} = A \qquad \Rightarrow \qquad c_1 = A\omega_0$$

Hence our first order solution comes out to

$$y_0(\tau) = A\omega_0 \sin(\tau/\omega_0)$$

We now continue to $\mathcal{O}(\epsilon)$. Here we have

$$\omega_0^2 y_{1\tau\tau} + y_1 = -2\omega_0\omega_1 y_{0\tau\tau} - y_0^5 = 2A\omega_1 \sin(\tau/\omega_0) - A^5\omega_0^5 \sin^5(\tau/\omega_0)$$

For the homogeneous problem we have the solution $y = c_1 \sin(\tau/\omega_0) + c_2 \cos(\tau/\omega_0)$ and for the particular solution, we turn to Mathematica.

$$\begin{aligned} y_1(\tau) = &-\frac{7}{96}A^5\omega_0^5 \sin(\tau/\omega_0)\cos(2\tau/\omega_0) + \frac{1}{192}A^5\omega_0^5 \sin(\tau/\omega_0)\cos(4\tau/\omega_0) \\ &+ \frac{5}{16}A^5\omega_0^4 \tau \cos(\tau/\omega_0) - A\omega_1\tau\cos(\tau/\omega_0)/\omega_0 + c_1\sin(\tau/\omega_0) + c_2\cos(\tau/\omega_0) \end{aligned}$$

In order to avoid secular growth terms, we see that the frequency correction ω_1 must satisfy

$$\frac{5}{16}A^5\omega_0^4 - A\omega_1 = 0 \qquad \Rightarrow \qquad \omega_1 = \frac{5}{16}A^4\omega_0^4$$

Lastly, the boundary conditions require that $c_2 = 0$, and that

$$y_1'(0) = -\frac{7}{96}A^5\omega_0^4 + \frac{1}{192}A^5\omega_0^4 + c_1/\omega_0 = 0 \qquad \Rightarrow \qquad c_1 = \frac{13}{192}A^5\omega_0^5$$

And so the complete $\mathcal{O}(\epsilon)$ correction can be written down as

$$y_1(\tau) = \frac{A^5\omega_0^5}{192}\sin(\tau/\omega_0)\left(-14\cos(2\tau/\omega_0) + \cos(4\tau/\omega_0) + 13\right)$$

We can now put together our full approximate accurate solution. We will let $\omega_0 = 1$ to look for 2π-periodic solutions. Then $\tau = (1 + 5\epsilon A^4/16)t$. We then have

$$
\begin{aligned}
y(t;\epsilon) &= A\sin[(1+5\epsilon A^4/16)t] \\
&\quad + \epsilon\frac{A^5}{192}\sin[(1+5\epsilon A^4/16)t]\left(-14\cos[2(1+5\epsilon A^4/16)t] + \cos[4(1+5\epsilon A^4/16)t] + 13\right) \\
&= A\sin[(1+5\epsilon A^4/16)t]\left(1 - \epsilon\frac{A^4}{48}\sin[(1+5\epsilon A^4/16)t]\left(\cos[2(1+5\epsilon A^4/16)t] - 6\right)\right)
\end{aligned}
$$

$$
\Rightarrow y(t;\epsilon) = A\sin[(1+5\epsilon A^4/16)t)] + \epsilon\frac{A^5}{48}\sin^3(t)(6-\cos(2t)) + \mathcal{O}(\epsilon^2)
$$

(d) For $\epsilon = 0.1$, plot the numerical solution (from MATLAB), the regular expansion solution, and the Poincare-Lindstedt solution for $0 \le t \le 20$.

Solution:

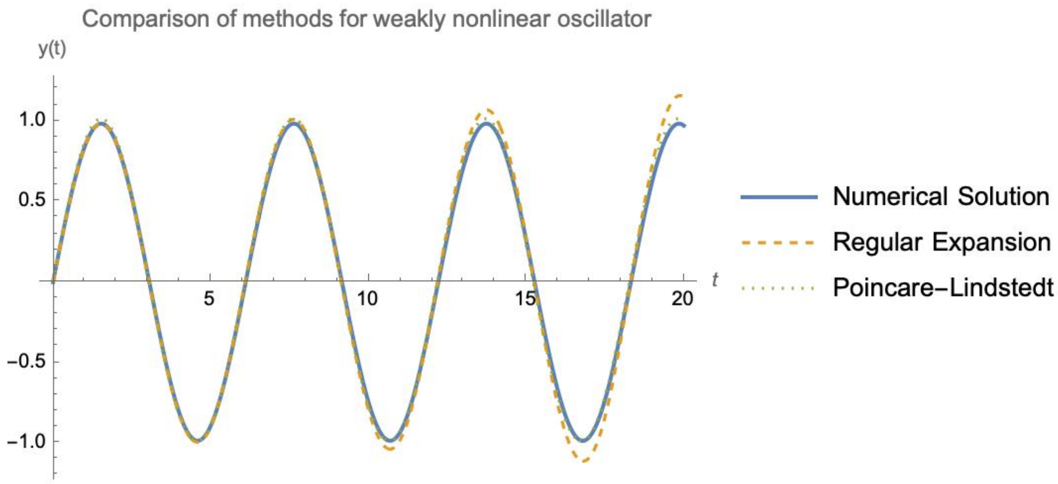

Figure 1: Solution comparison for different methods, with $\epsilon = 0.1$.

2. Consider Rayleigh's equation:

$$
\frac{d^2y}{dt^2} + y + \epsilon\left[-\frac{dy}{dt} + \frac{1}{3}\left(\frac{dy}{dt}\right)^3\right] = 0
$$

which has only one periodic solution called a "limit cycle" $(0 < \epsilon \ll 1)$. Given

$$
y(0) = 0
$$

and

$$
\frac{dy(0)}{dt} = A.
$$

(a) Use a multiple scale expansion to calculate the leading order behavior.

Solution: Let $y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \mathcal{O}(\epsilon^2)$, where $\tau = \epsilon t$. Then our ODE becomes

$$y_{0tt} + 2\epsilon y_{0t\tau} + \epsilon y_{1tt} + y_0 + \epsilon y_1 + \epsilon[-y_{0t} + \frac{1}{3}y_{0t}^3] = 0$$

To $\mathcal{O}(1)$ we have

$$y_{0tt} + y_0 = 0 \qquad \Rightarrow \qquad y_0 = A_0(\tau)\sin t + B_0(\tau)\cos t$$

By our Dirichlet boundary conditions, we have

$$y_0(0) = B_0(0) = 0$$

while from our Neumann boundary conditions we have

$$y_0'(0) = y_{0t}(0) + \epsilon y_{0\tau}(0) = A_0(0) + \epsilon B_0'(0) = A \qquad \Rightarrow \qquad \begin{cases} A_0(0) = A \\ B_0'(0) = 0 \end{cases}$$

To $\mathcal{O}(\epsilon)$ we have

$$\begin{aligned} y_{1tt} + y_1 &= y_{0t} - 2y_{0t\tau} - \frac{1}{3}y_{0t}^3 \\ &= (A_0 - 2A_0')\cos t - (B_0 - 2B_0')\sin t - \frac{1}{3}(A_0\cos t - B_0\sin t)^3 \end{aligned}$$

From Mathematica, we find the solution to this ODE to be

$$\begin{aligned} y_1(t,\tau) = \frac{1}{96}[&-2\cos(t)\left(48A_0' + 6B_0t\left(A_0^2 + B_0^2 - 4\right) + A_0\left(5A_0^2 + 9B_0^2 - 24\right) + 48(B_0't - B_1)\right) \\ &+ 2\sin(t)\left(-48A_0't - 6A_0^3t - 3A_0^2B_0 - 6A_0\left(B_0^2 - 4\right)t + B_0^3 + 48B_1\right) \\ &+ B_0\left(B_0^2 - 3A_0^2\right)\sin(3t) + A_0\left(A_0^2 - 3B_0^2\right)\cos(3t)] \end{aligned}$$

where $A_1 = A_1(\tau)$ and $B_2 = B_1(\tau)$. To ensure that there are no secular growth terms, we require that A_0 and B_0 satisfy

$$A_0' = \frac{1}{8}A_0(4 - A_0^2 - B_0^2) \qquad\qquad B_0' = \frac{1}{8}B_0(4 - A_0^2 - B_0^2)$$

Let $G(\tau) = \frac{1}{8}(4 - A_0^2 - B_0^2)$. Then $A_0' = A_0G$ and $B_0' = B_0G$, and

$$G' = -\frac{1}{4}(A_0A_0' + B_0B_0') = -\frac{1}{4}(A_0^2 + B_0^2)G$$
$$= 2G^2 - G$$

This is a separable equation. Dividing both sides by $2G^2 - G$, integrating, and rearranging yields the solution

$$G(\tau) = \frac{1}{Ce^\tau + 2}$$

where $C > 0$ is an integration constant. We can now use this expression to solve for $A_0(\tau)$ and $B_0(\tau)$. We have

$$\frac{dA_0}{d\tau} = A_0G = \frac{A_0}{Ce^\tau + 2}$$
$$\Rightarrow \int_A^{A_0} \frac{dA_0}{A_0} = \int_0^\tau \frac{d\tau}{Ce^\tau + 2}$$
$$\Rightarrow \log\left(\frac{A_0}{A}\right) = \frac{\tau}{2} - \frac{1}{2}\log\left(\frac{Ce^\tau + 2}{C + 2}\right)$$
$$\Rightarrow A_0(\tau) = Ae^{\tau/2}\sqrt{\frac{C+2}{Ce^\tau + 2}}$$

where we have used $A_0(0) = A$ from our boundary condition. Similarly, we find

$$B_0(\tau) = B(0)e^{\tau/2}\sqrt{\frac{C+2}{Ce^\tau + 2}} = 0$$

since $B_0(0) = 0$ from our boundary conditions.
Having solved for $A_0(\tau)$ and $B_0(\tau)$, we may write down our first order approximate solution as

$$y_0(t, \tau) = A\sin(t)e^{\tau/2}\sqrt{\frac{C+2}{Ce^\tau + 2}}$$

$$\Rightarrow y_0(t) = A\sin(t)e^{\epsilon t/2}\sqrt{\frac{C+2}{Ce^{\epsilon t} + 2}}$$

We note that in the limit $t \to \infty$ the right-hand side approaches $A\sin(t)$.

(b) Use a Poincare-Lindsted expansion and an expansion of $A = A_0 + \epsilon A_1 + \dots$ to calculate the leading-order solution and the first non-trivial frequency shift for the limit cycle.

Solution: Let

$$
\begin{aligned}
y &= y_0(\tau) + \epsilon y_1(\tau) + \dots \\
\tau &= \omega t = (\omega_0 + \epsilon\omega_1 + \dots)t \\
A &= A_0 + \epsilon A_1 + \dots
\end{aligned}
$$

Then our ODE becomes

$$
\begin{aligned}
\omega^2 y_{\tau\tau} + y + \epsilon\left[-\omega y_\tau + \frac{1}{3}\omega^3 y_\tau^3\right] &= 0 \\
(\omega_0^2 + 2\epsilon\omega_0\omega_1)(y_0'' + \epsilon y_1'') + y_0 + \epsilon y_1 + \epsilon\left[-\omega_0 y_0' + \frac{1}{3}\omega_0^3 {y_0'}^3\right] &= \mathcal{O}(\epsilon^2)
\end{aligned}
$$

Then to $\mathcal{O}(1)$ we have

$$
\omega_0^2 y_0'' + y_0 = 0 \qquad \Rightarrow \qquad y_0(\tau) = c_1 \sin(\tau/\omega_0) + c_2\cos(\tau/\omega_0)
$$

Applying our boundary condition $y(0) = 0$ means that $c_2 = 0$, while our second boundary condition results in

$$
\frac{c_1}{\omega_0} = A_0 \qquad \Rightarrow \qquad c_1 = \omega_0 A_0
$$

Hence our leading order solution becomes

$$
y_0(\tau) = A_0\omega_0 \sin(\tau/\omega_0)
$$

To find the first non-trivial frequency shift we must continue to $\mathcal{O}(\epsilon)$. Here we have

$$
\omega_0^2 y_1'' + 2\omega_0\omega_1 y_0'' + y_1 - \omega_0 y_0' + \frac{1}{3}\omega_0^3 {y_0'}^3 = 0
$$

$$
\begin{aligned}
\omega_0^2 y_1'' + y_1 &= \omega_0 y_0' - 2\omega_0\omega_1 y_0'' - \frac{1}{3}\omega_0^3 {y_0'}^3 \\
\omega_0^2 y_1'' + y_1 &= \omega_0 A_0 \cos(\tau/\omega_0) + 2\omega_1 A_0 \sin(\tau/\omega_0) - \frac{1}{3}\omega_0^3 A_0^3 \cos^3(\tau/\omega_0)
\end{aligned}
$$

We solve this in Mathematica to find

$$
\begin{aligned}
y_1(\tau) = \frac{A_0}{2}\left(1 - \frac{1}{4}A_0^2\omega_0^2\right)\tau\sin(\tau/\omega_0) - A_0\frac{\omega_1}{\omega_0}\tau\cos(\tau/\omega_0) \\
+ \frac{1}{96}A_0^3\omega_0^3\cos(3\tau/\omega_0) + c_1\sin(\tau/\omega_0) + c_2\cos(\tau/\omega_0)
\end{aligned}
$$

Where c_1 and c_2 are once again integration constants. To avoid secular growth terms, we need

$$1 - \frac{1}{4}A_0^2\omega_0^2 = 0 \qquad \Rightarrow \qquad \omega_0 = \frac{2}{A_0}$$

$$A_0\frac{\omega_1}{\omega_0} = 0 \qquad \Rightarrow \qquad \omega_1 = 0$$

Using these values of ω_0 and ω_1, our $\mathcal{O}(\epsilon)$ solution becomes

$$y_1(\tau) = \frac{1}{12}\cos(3A_0\tau/2) + c_1\sin(A_0\tau/2) + c_2\cos(A_0\tau/2)$$

We now apply our $\mathcal{O}(\epsilon)$ boundary conditions. We require that $y_1(0) = 0$, which means that $c_2 = -1/12$. We also require that $y_1'(0) = A_1$, which gives us

$$\frac{A_0c_1}{2} = A_1 \qquad \Rightarrow \qquad c_1 = \frac{2A_1}{A_0}$$

and so our $\mathcal{O}(\epsilon)$ correction is given by

$$y_1(\tau) = \frac{2A_1}{A_0}\sin(A_0\tau/2) + \frac{1}{12}(\cos(3A_0\tau/2) - \cos(A_0\tau/2))$$

Putting together our y_0 and y_1 solutions, we find our full solution up to $\mathcal{O}(\epsilon)$ is given by

$$y(t) = 2\left(1 + \epsilon\frac{A_1}{A_0}\right)\sin(t) + \frac{\epsilon}{12}(\cos(3t) - \cos(t)) + \mathcal{O}(\epsilon^2)$$

We see that the first non-trivial frequency shift is $3\omega_0$.

(c) For $\epsilon = 0.01, 0.1, 0.2$ and 0.3, plot the numerical solution and the multiple scale expansion for $0 \leq t \leq 40$ and for various values of A for your multiple scale solution. Also plot the limit cycle solution calculated from part (b).

Solution:

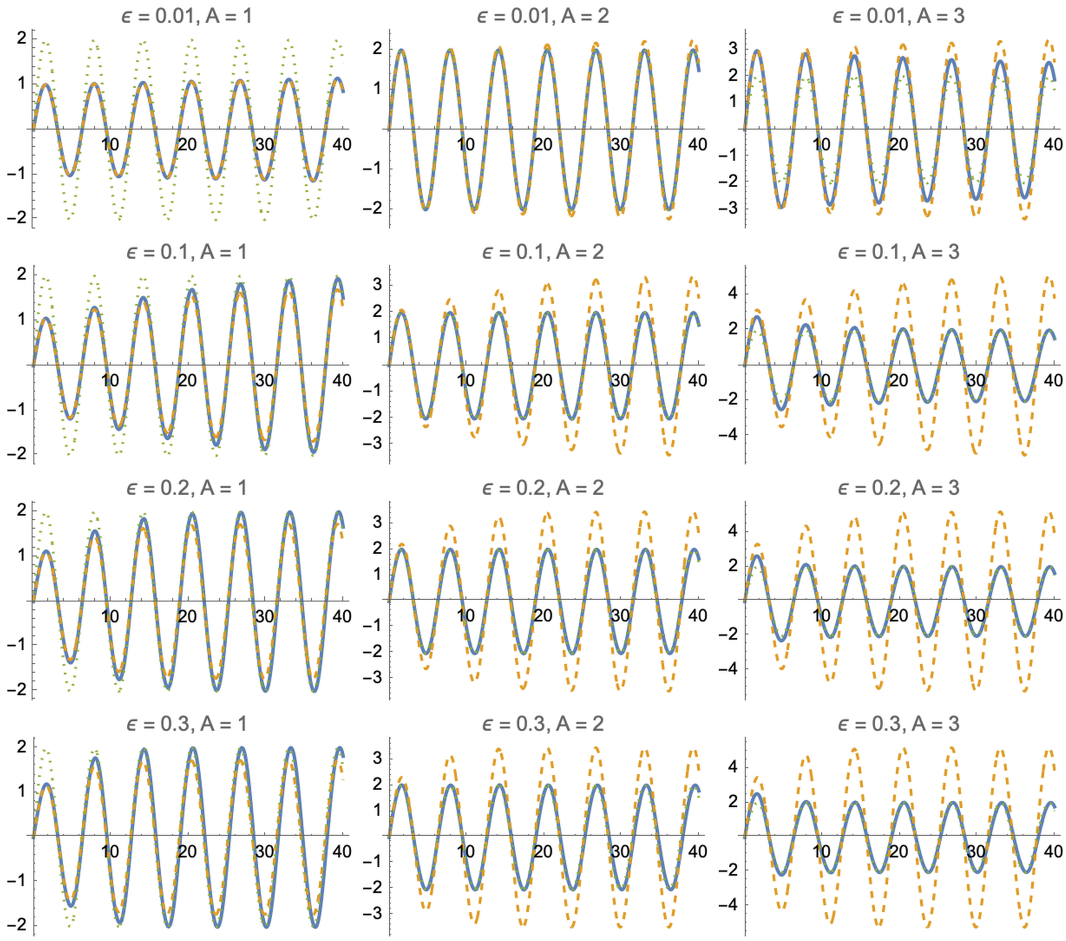

Figure 2: Multiple scale expansion for different ϵ and A values.

(d) Calculate the error

$$E(t) = |y_{numerical}(t) - y_{approximation}(t)|$$

as a function of time $(0 \leq t \leq 40)$ using $\epsilon = 0.01, 0.1, 0.2$ and 0.3.

Solution:

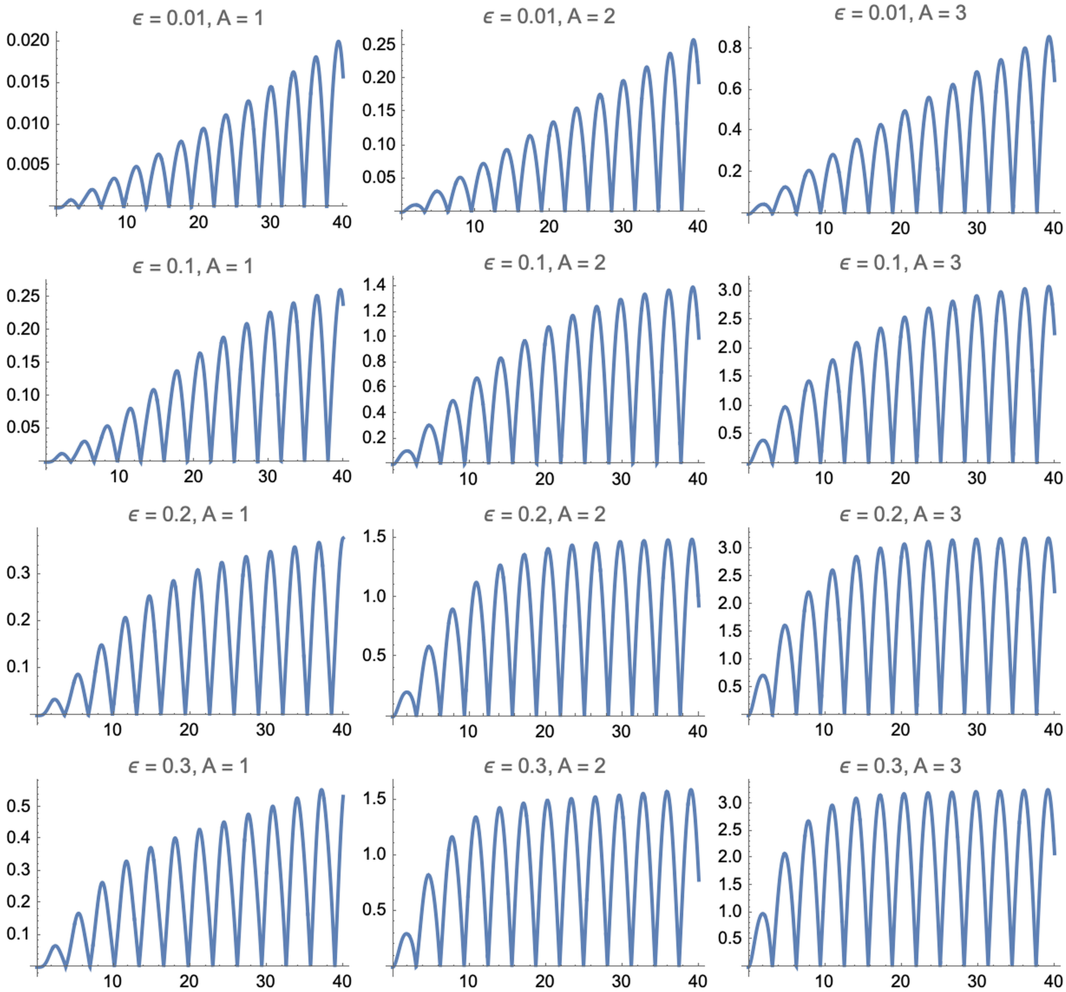

Figure 3: Plot of the error between the numerical solution and the multiple scale approximation.