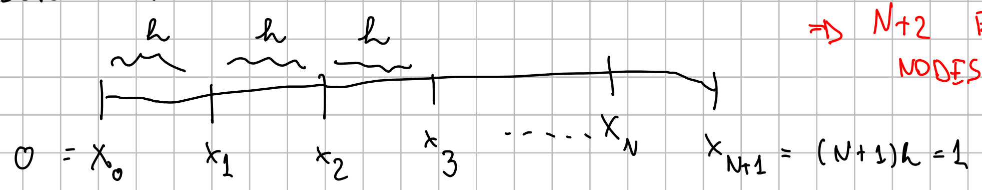
LECTURE 3 (19/01/26)

THE STARTING POINT IS THIS DIFFUSION REACTION EQUATION

$$\begin{cases} \partial_t y(t,x) = \delta\, \partial_{xx} y(t,x) + \dfrac{1}{1+y(t,x)^2}, & x \in (0,1) \quad t \in (0, t^*] \\ y(0,x) = y_0(x) & \\ y(t,0) = y(t,1) = 0 & \Rightarrow \text{HOM DIR B.C.} \end{cases}$$

WE SEMIDISCRETIZE IN SPACE (METHOD OF LINES) USING SECOND ORDER CENTERED FINITE DIFFERENCES.

OUR DISCRETIZATION IS:



$\Rightarrow$ N+2 EQUISPACED NODES

FOR THE INNER NODES x_i FOR $i = 1, \dots, N$. THEN IF WE DENOTE $y(t, x_i) \approx y_i(t)$ WE HAVE

$$\partial_{xx} y(t, x_i) \approx \frac{y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)}{h^2}$$

FOR THE NODES NEXT TO THE BOUNDARY WE EXPLOIT BDY CONDITIONS. FOR THE NODE x_1 WE USE $y(t, x_0) = 0$

$$\partial_{xx} y(t, x_1) \approx \frac{\overset{0}{\overset{\shortparallel}{y_0}}(t) - 2y_1(t) + y_2(t)}{h^2} = \frac{-2y_1(t) + y_2(t)}{h^2}$$

SIMILARLY FOR x_N $\Rightarrow$ $\partial_{xx} y(t, x_N) \approx \dfrac{y_{N-1}(t) - 2y_N(t) + \overset{0}{\overset{\shortparallel}{y_{N+1}}}(t)}{h^2}$

ALL IN ALL WE GET BY DEFINING

$$y(t) \doteq \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{bmatrix} \in \mathbb{R}^N \quad \text{AND} \quad A = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & -2 \end{bmatrix} \in \mathbb{R}^{N \times N}$$

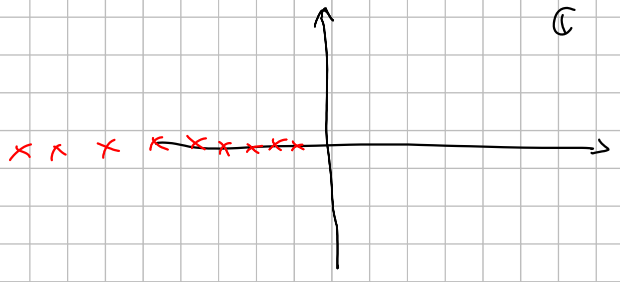
WE GET THE SYSTEM OF ODES

$$\begin{cases} y'(t) = \delta A y(t) + \overbrace{\dfrac{1}{1 + y(t)^2}}^{g(y(t))} = f(y(t)) \\ y(0) = y_0 \end{cases}$$

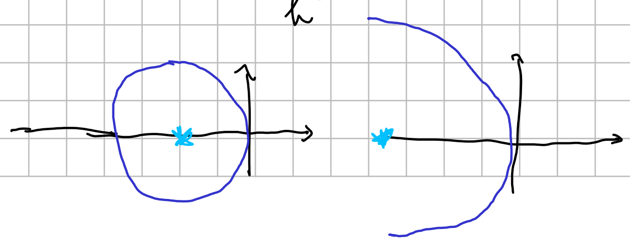
WHICH HAS N DEGREES OF FREEDOM, ONE FOR EACH INNER NODE.

REMARKS ABOUT THE MATRIX A

① THE SYSTEM IS STIFF BECAUSE A HAS NEGATIVE EIGENVALUES WITH LARGE MAGNITUDE



- A IS SYMMETRIC $\Rightarrow$ JUST REAL EIGENVALUES
- BY GERSHGORIN'S CIRCLE THEOREM THE CENTRE OF THE DISKS ARE $-\frac{2}{h^2}$ AND RADIUS $\frac{2}{h^2}$



THEREFORE THE EIGENVALUES ARE NEGATIVE

* SINCE A IS SYMMETRIC IT IS NORMAL $\Rightarrow$ "WELL" DIAGONALIZABLE $(\mathrm{cond}_2(V) = 1)$

$$A V = V D \iff A = V D V^{-1} \overset{!}{=} V D V^T$$

MATRIX WHOSE COLUMNS ARE EIGENVECTORS

EIGENVALUES (DIAG)

* THE "GOOD" PROOF OF EXPONENTIAL EULER REQUIRES $\|e^{tA}\| \leq C$ FOR EACH $t \in [0, t^*]$

* 2-NORM

$$e^{tA} = e^{t V D V^T} = V e^{tD} V^T \Rightarrow \|e^{tA}\|_2 \leq \underbrace{\|V\|_2 \|V^T\|_2}_{\mathrm{cond}_2(V) = 1} \|e^{tD}\|_2$$

$$\|e^{tA}\|_2 \leq \|e^{tD}\|_2$$

BUT D IS DIAGONAL, HENCE $e^{tD} = \begin{pmatrix} e^{-t\lambda_1} & & & \\ & e^{-t\lambda_2} & & \\ & & e^{-t\lambda_3} & \\ & & & \ddots \end{pmatrix}$

$$D \doteq \begin{pmatrix} -\lambda_1 & & \\ & -\lambda_2 & \\ & & \ddots \end{pmatrix}$$ WITH $\lambda_k \geq 0 \quad k = 1, \dots, N$

$\hookrightarrow$ EIGENVALUES

THEN $|e^{-t\lambda_k}| \leq 1 \Rightarrow \|e^{tD}\|_2 \leq 1 \quad \forall t \in [0, t^*]$

INDEPENDENTLY OF N

$\Rightarrow \|e^{tA}\|_2 \leq 1$ INDEPENDENTLY OF N AND OF $\|A\|_2$

* ∞ - NORM

IF YOU DO THE SAME THEN

$$\|e^{tA}\|_\infty \le \mathrm{cond}_\infty(V)\|e^{tD}\|_\infty \le C N$$

$\mathrm{cond}_\infty(V)$ GOES AS N $\mathcal{O}(N)$; $\|e^{tD}\|_\infty \le 1$

THEOREM (11.1 IN HNW08, SIMPLIFIED)

$$\|e^{tA}\|_\infty \le e^{\mu_\infty(A)} \qquad \forall t \in [0, t^*]$$

WHERE $\mu_\infty(A)$ DENOTES THE LOGARITHMIC NORM (IN THE ∞ SENSE) OF A. FOR A GENERIC MATRIX M

$$\mu_\infty(M) \doteq \max_{k=1,\dots,N} \Big(m_{kk} + \sum_{i \ne k} |m_{ki}| \Big).$$

IN OUR CASE

$$\mu_\infty(A) = \frac{1}{R^2} \max_{k=1,\dots,N} \big(\{-1, 0, 0, \dots, 0, 0, -1\} \big) = 0$$

$\Rightarrow \mu_\infty(A) = 0$ AND THEN $\|e^{tA}\|_\infty \le e^0 = 1$

$\square$

IMEX $\quad (I - \tau A) y_{n+1} = y_n + \tau g(u_n) \Rightarrow y_{n+1} = (I-\tau A)^{-1} (y_n + \tau g(u_n))$

$$= (V(I-\tau D)V^T)^{-1}(\dots) = V(I-\tau D)^{-1}V^T(\dots)$$

EXPEULER $\quad y_{n+1} = y_n + \tau \varphi_1(\tau A) f(u_n)$

$$\downarrow$$

$$V \varphi_1(\tau D) V^T$$

$\hookrightarrow$ SCALAR COMPUTATION $\qquad \varphi_1(z) = \dfrac{e^z - 1}{z}$

ADDITIONAL REMARKS

* STANDARD DISCRETIZATION OF HOMNEU $\frac{1}{h^2}e[-2,2] \Rightarrow \checkmark$
* NON HOMOGENEOUS DIRICHLET $\Rightarrow$ LIFTING $\Rightarrow \checkmark$
* ADVECTION - DIFFUSION REACTION EQUATION MAY BE ILL CONDITIONED
* SEMIGROUP THEORY $\mathcal{A} \doteq \partial_{xx} \oplus$ HOMDIR B.C.

$$\| e^{t\mathcal{A}} \| \leq 1 \qquad t \in [0, t^*]$$