LECTURE 5(26/01/26)

WE ARE CONSIDERING

$$\begin{cases} \partial_t y(t,x) = \delta \partial_{xx} y(t,x) + \underbrace{g(t,x)}_{= f(t,y)} & \mathcal{U} = (0,1) \\ y(0,x) = y_0(x) & t \in [0,t^*] \\ y\big|_{\partial\mathcal{U}} = 0 & \rightsquigarrow \text{HOM DIR} \end{cases}$$

WHAT WE CONSIDERED IS $\delta = 1$; AND TOOK

(a) $g(t,x) = e^t \left(1 + \delta(2\pi)^2 \right) \sin(2\pi x)$

(b) $g(t,x) = 4e^t \left(x(1-x) + 2\delta \right)$

AND WE OBSERVE SECOND ORDER FOR (a) BUT A "NONSMOOTH"
BEHAVIOUR FOR (b)

WHAT'S GOING ON ?

RECALL THAT WE ARE CONSIDERING THE INTEGRATOR

$$y_{m+1} = y_m + \tau \varphi_1(\tau A) f\left(t_m + \frac{\tau}{2}, y_m \right)$$

$$= e^{\tau A} y_m + \tau \varphi_1(\tau A) g\left(t_m + \frac{\tau}{2} \right)$$

IN OUR PROOF WE WROTE THE ERROR RECURRENCE AS

$$\mathcal{E}_m = \sum_{J=0}^{m-1} e^{J\tau A} \delta_{m-J}$$

WHERE

$$\delta_{m-J} = \tau^3 \left(\frac{1}{2} \varphi_2(\tau A) - \varphi_3(\tau A) \right) A g'(t_{m-J-1}) +$$

$$+ \tau \varphi_1(\tau A) \int_0^{c_1 \tau} g''(t_{m-J-1} + \sigma)(c_1 \tau - \sigma) d\sigma - \int_0^{\tau} e^{(\tau-s)A} \int_0^s g''(t_{m-J-1} + \sigma)(s-\sigma) d\sigma ds$$

WE PROVED THAT $\|\varepsilon_m\| \leq C\tau^2 \|\exists_m\|$. IN MORE DETAIL

$$\|\varepsilon_m\| \leq \sum_{J=0}^{m-1} \underbrace{\|e^{J\tau A}\|}_{\leq C} \|\delta_{m-J}\| \leq C \sum_{J=0}^{m-1} \|\textcolor{green}{\bullet}\| + C \sum_{J=0}^{m-1} \underbrace{\|\textcolor{blue}{\bullet}\|}_{\leq C\tau^2}$$

THE BLUE PART HAS NO ISSUE. BUT THE GREEN PART HAS A "FREE" A

$$\|\textcolor{green}{\bullet}\| \leq C\tau^3 \|A g'(t_{m-J-1})\| \qquad \left(\|\varphi_e(\tau A)\| \leq C \text{ INDEPENDENTLY OF } \|A\|\right)$$

RECALL THAT $A \simeq \delta \partial_{xx} \oplus$ HOMDIR. TO WORK "PROPERLY"

$$A_N$$

(1) N SHOULD BE SUFFICIENTLY SMOOTH (AT LEAST TWICE DIFFERENTIABLE

(2) N MUST SATISFY HOMDIR B.C.

BUT IN OUR CASE (1) IS ALWAYS OK BUT (2) IS NOT REQUIRED ANYWHERE! SO

- IN CASE (a) $g'(t)$ SATISFIES HOMDIR

   $\Rightarrow \|A g'(t_{m-J-1})\| \leq C \Rightarrow \textcolor{green}{\checkmark}$

- IN CASE (b) $g'(t)$ DOES $\underline{\underline{NOT}}$ SATISFY HOMDIR

 $\Rightarrow \|A g'(t_{m-J-1})\|$ IS NOT BOUNDED INDEPENDENTLY OF THE SIZE OF A

THIS IS PECULIAR TO DISCRETIZED PDES AND YOU CAN TRANSLATE THIS IN "CONTINUOUS SETTING" $\left(A = \partial_{xx} \oplus \text{HOMDIR}\right)$
$\hookrightarrow \text{OPERATOR}$

AS $g'(t) \notin \underline{\underline{\mathcal{D}}}(A)$ [SEMIGROUP LANGUAGE]
$\hookrightarrow \text{DOMAIN}$

DOES THIS APPLY TO ALL KINDS OF MATRICES A $\to$ NO!

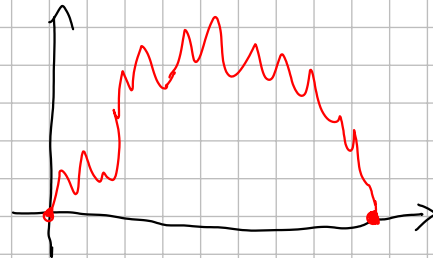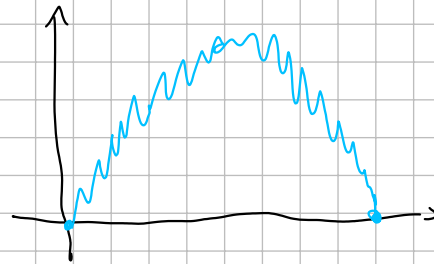IF $S = 1$, WE DON'T SEE PROBLEMS FOR NEITHER OF THE TWO CASES. THE EXPLANATIONS OF THIS LIES IN SOMETHING CALLED PARABOLIC SMOOTHING.

$A = \partial_{xx}$

$A = i \partial_{xx}$

t

$u(0)$

$u(t)$
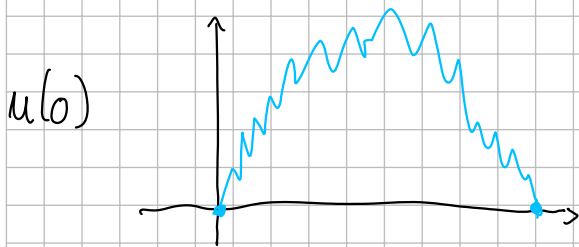
SMOOTHING EFFECT

NO SMOOTHING EFFECT

IN OUR SETTING THIS TRANSLATES INTO AN ESTIMATE OF THE FORM

$$\| t A e^{tA} \| \le C \qquad \forall t \in [0, t^*]$$

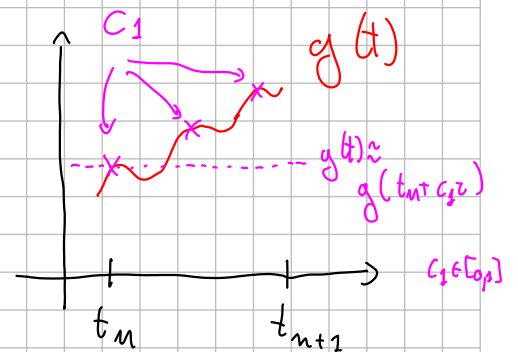FOR $A \not= \partial_{xx}$, WHICH IS NOT TRUE FOR $A = i \partial_{xx}$.

WITH THIS ESTIMATE AT HAND WE STOP THE PROOF BEFORE AND WE GO FROM

$$\varepsilon_M = \sum_{J=0}^{M-1} e^{JZA} \delta_{M-J} = \sum_{J=0}^{M-1} e^{JZA} \left(\bullet + \bullet \right)$$
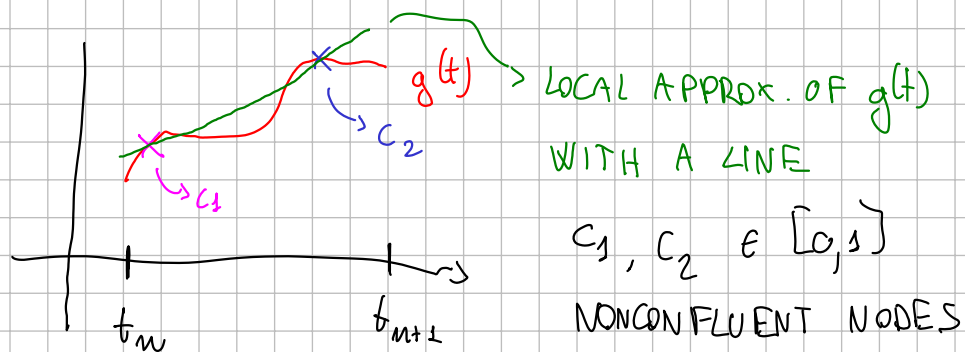
$\bullet \quad \bullet \quad \bullet \quad \bullet$

$$\begin{cases} y'(t) = A y(t) + g(t) \\ y(0) = y_0 \end{cases}$$

IF WE USE TWO COLLOCATION POINTS
WE GET GEOMETRICALLY

LOCAL APPROX. OF $g(t)$
WITH A LINE

$c_1, c_2 \in [0,1]$
NONCONFLUENT NODES

THEN IN OUR VOC FORMULA

$$y(t_{m+1}) = e^{\tau A} y(t_m) + \int_0^\tau e^{(\tau - s)A} g(t_m + s)\, ds \qquad (\ast\ast)$$

WE INSERT OUR APPROXIMATION

$$g(t_{m+s}) \approx \alpha_s\, g(t_m + c_1 \tau) + \beta_s\, g(t_m + c_2 \tau)$$

OF DEGREE ONE

α_s, β_s POLYNOMIALS $\vee$ STEMMING FROM INTERPOLATION CONDITIONS.

THEN WE END UP WITH THE CLASS OF TIME-MARCHING
SCHEMES

$$y_{m+1} = e^{\tau A} y_m + \tau \left(\frac{c_2}{c_2 - c_1} \varphi_1(\tau A) - \frac{1}{c_2 - c_1} \varphi_2(\tau A) \right) g(t_m + c_1 \tau)$$

$$+ \tau \left(-\frac{c_1}{c_2 - c_1} \varphi_1(\tau A) + \frac{1}{c_2 - c_1} \varphi_2(\tau A) \right) g(t_m + c_2 \tau)$$

IF WE ASSUME g SUFFICIENTLY OFTEN DIFFER. THE INTEGRATOR IS SECOND ORDER ACCURATE (FOR $c_2 \neq c_1$) BUT IF SOME CONDITIONS ON THE NODES ARE VERIFIED (⊕ ADDITIONAL SMOOTHNESS) THEN WE MAY GET HIGHER ORDER. IN PARTICULAR IF

$$\frac{1}{3} - \frac{1}{2}\left(c_1 + c_2\right) + c_1 c_2 = 0$$

⤳ GAUSS - RADAU
⤳ GAUSS - LEGENDRE

THEN WE GET THIRD ORDER

REMARK

IF $g(t)$ IS A POLYNOMIAL YOU CAN ALWAYS FIND AN EXPONENTIAL INTEGRATOR WHICH IS EXACT ON YOUR SYSTEM. IN OTHER WORDS YOU ARE WRITING THE EXACT SOLUTION IN TERMS OF φ- FUNCTIONS.

SEMILINEAR CASE

WE NOW GO BACK TO THE SYSTEM

$$\begin{cases} y'(t) = A y(t) + g\left(y(t)\right) \\ y(0) = y_0 \end{cases}$$

THE VARIATION OF CONSTANTS FORMULA TELLS US

IMPLICIT FORMULA !

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau - s)A} g\left(y(t_n + s)\right) ds$$

(*)

FOR EXPONENTIAL EULER WE DID THE APPROXIMATION

$$g\left(y(t_n + s)\right) \approx g\left(y(t_n)\right)$$

$$\Rightarrow y_{n+1} = e^{\tau A} y_n + \tau \varphi_1 (\tau A) g\left(y_n\right)$$

WE WANT TO GO HIGHER ORDER. SIMILARLY TO EXPONENTIAL QUADRATURE RULES WE TRY TO CHANGE COLLOCATION POINT, THAT IS

$$g\left(y(t_{n+s})\right) \approx g\left(y(t_n + c_1 \tau)\right) \qquad (c_1 \neq 0)$$

BUT THE PROBLEM IS THAT WE DON'T HAVE INFORMATION ON $y(t_n + c_1 \tau)$ $\Rightarrow$ WE "CREATE" THIS INFORMATION FROM WHAT WE HAVE $\left(y(t_n)\right)_{\approx y_n}$