LECTURE 6 (28/01/26)

THE SETTING IS NOW

$$\begin{cases} y'(t) = Ay(t) + g(y(t)) = f(y(t)) & t \in (0, t^*] \\ y(0) = y_0 \end{cases}$$

WE WRITE THE V.O.C. FORMULA

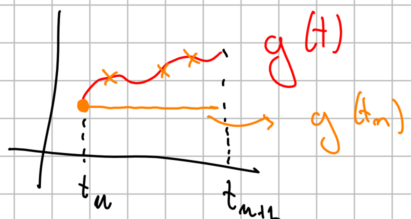
$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^{\tau} e^{(\tau - s)A} g(y(t_n + s))\, ds$$

⚠ IMPLICIT FORMULA

$$\left[y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^{\tau} e^{(\tau - s)A} g(t_n + s)\, ds \right]$$

EXPLICIT FORMULA

IN FACT TO GO HIGHER ORDER IF WE DO THE SAME PROCEDURE AS DID IN THE LINEAR CASE WE HAVE



$$g(y(t_n + s)) \approx g(y(t_n + c_2 \tau)) \qquad c_2 \in [0, 1]$$

SINCE WE DON'T HAVE $y(t_n + c_2\tau)$ WE APPROXIMATE IT (SUITABLY) SO THAT WE CAN END UP WITH A TIME MARCHING SCHEME. A REASONABLE CHOICE TO APPROXIMATE $y(t_n + c_2\tau)$ IS TO COMPUTE IT BY MEANS OF EXPONENTIAL EULER, WHICH USES INFORMATION THAT WE HAVE (AT TIME t_n)

$y_n \approx y(t_n)$

$$y(t_n + c_2\tau) \approx Y_{n2} = e^{c_2 \tau A} + c_2 \tau \varphi_1(c_2 \tau A)\, g(y_n)$$

THEN IN THE V.O.C. FORMULA WE USE THIS INFORMATION, AND WE GET

$$y_{n+1} = e^{\tau A} y_n + \left(\int_0^{\tau} e^{(\tau - s)A}\, ds \right) g(Y_{n2})$$

$$= e^{\tau A} y_n + \tau \varphi_1(\tau A)\, g(Y_{n2})$$

THERE OVERALL WE HAVE THE SCHEME

$$Y_{n2} = e^{c_2 \tau A} y_n + c_2 \tau \varphi_1 (c_2 \tau A) g(y_n)$$ INTERMEDIATE STAGE

$$y_{n+1} = e^{\tau A} y_n + \tau \varphi_1 (\tau A) g(Y_{n2})$$ → FINAL APPROXIMATION

THE CONSTRUCTION RESEMBLE THE ONE OF "STANDARD" EXPLICIT RUNGE-KUTTA METHODS. FOR THIS REASON THIS INTEGRATOR BELONGS TO THE CLASS OF THE SO-CALLED (EXPLICIT) EXPONENTIAL RUNGE-KUTTA METHODS (OF COLLOCATION TYPE). IN BUTCHER TABLEAU WAY YOU HAVE

$$c \left\{ \begin{array}{c|cc} 0 & & \\ c_2 & c_2\varphi_1(c_2\,\cdot) & \\ \hline & \underbrace{0 \quad \varphi_1(\cdot)}_{b} & \end{array} \right. \rightarrow \mathbf{A}$$

NOT SURPRISINGLY WE CAN GET <u>SECOND ORDER</u> OF CONVERGENCE IF $c_2 = \frac{1}{2}$ (AS FOR THE LINEAR CASE).

WHAT YOU CAN PROVE IS THAT:

- IF $c_2 \neq \frac{1}{2}$ WE HAVE FIRST ORDER
- IF $c_2 = \frac{1}{2}$ WE (MAY GET) SECOND ORDER
  ↳ WE MAY ENCOUNTER ORDER REDUCTION IN SPECIFIC STIFF PROBLEMS.

NOTICE THAT WE ARE NOT USING IN y_{n+1} AN INFORMATION THAT WE HAVE, NAMELY $g(y_n)$.

CAN WE USE IT?

⇒ YES!

$$Y_{n2} = e^{c_2 \tau A} y_n + c_2 \tau \varphi_1(\tau A)\, g(y_n) \qquad (*)$$

$$y_{n+1} = e^{\tau A} y_n + \tau \left(b_1(\cdot) g(y_n) + b_2(\cdot) g(Y_{n2}) \right)$$

WHERE b_1 AND b_2 ARE (MATRIX) COEFFICIENTS TO BE DETERMINED. TO DETERMINE b_1 AN b_2 SO THAT WE GET SECOND-ORDER WE LOOK AT THE LOCAL ERROR OF APPROXIMATON $(*)$ [IF $\mathcal{O}(\tau^3)$ LOCALLY $\Rightarrow$ $\mathcal{O}(\tau^2)$ GLOBALLY]

TO DO THIS WE SET FOR SIMPLICITY OF NOTATION

$h(t) \doteq g(y(t))$ AND SO THE V.O.C. FORMULA GIVES

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau - s)A} h(t_n + s)\, ds$$

AND BY TAYLOR EXPANSION

$$h(t_n + s) = h(t_n) + s\, h'(t_n) + \int_0^s h''(t_n + \sigma)(s - \sigma)\, d\sigma$$

THEREFORE

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \underbrace{\boxed{\left(\int_0^\tau e^{(\tau - s)A}\, ds \right)}}_{\tau \varphi_1(\tau A)} h(t_n)$$

$$+ \underbrace{\boxed{\left(\int_0^\tau e^{(\tau - s)A}\, s\, ds \right)}}_{\tau^2 \varphi_2(\tau A)} h'(t_n)$$

$$+ \int_0^\tau e^{(\tau - s)A} \int_0^s h''(t_n + \sigma)(s - \sigma)\, d\sigma\, ds$$

SO

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \tau \varphi_1(\tau A) h(t_n) + \tau^2 \varphi_2(\tau A) h'(t_n) + \int_0^{\tau} e^{(\tau-s)A} \int_0^{s} h''(t_n+\sigma)(s-\sigma)\, d\sigma\, ds$$

WE NOW DENOTE THE NUMERICAL SOLUTION ASSUMING WE ARE STARTING FROM AN EXACT QUANTITY AS y^*_{n+1}, NAMELY

$$y^*_{n+1} = e^{\tau A} y(t_n) + \tau\left(b_1 g(y(t_n)) + b_2 g(y(t_n + c_2\tau))\right)$$
$$= e^{\tau A} y(t_n) + \tau\left(b_1 h(t_n) + b_2 h(t_n + c_2\tau)\right)$$

BY TAYLOR EXPANSION

$$h(t_n + c_2\tau) = h(t_n) + c_2\tau h'(t_n) + \int_0^{c_2\tau} h''(t_n+\sigma)(s-\sigma)\, d\sigma$$

THEN WE GET

$$y^*_{n+1} = e^{\tau A} y(t_n) + \tau(b_1 + b_2) h(t_n) + b_2 c_2 \tau^2 h'(t_n) + \tau b_2 \int_0^{c_2\tau} h''(t_n+\sigma)(s-\sigma)\, d\sigma$$

NOW WE CAN COMPARE $y(t_{n+1})$ WITH y^*_{n+1}

$$y^*_{n+1} - y(t_{n+1}) = \tau\left(b_1 + b_2 - \varphi_1(\tau A)\right) h(t_n) + \tau^2\left(b_2 c_2 - \varphi_2(\tau A)\right) h'(t_n) + \tau b_2 \int_0^{c_2\tau} h''(t_n+\sigma)\underline{(s-\sigma)}\, d\sigma - \int_0^{\tau} e^{(\tau-s)A} \int_0^{s} h''(t_n+\sigma)\underline{(s-\sigma)}\, d\sigma\, ds$$

$\hookrightarrow \mathcal{O}(\tau^3)$ $\hookrightarrow \mathcal{O}(\tau^3)$

IF WE PASS TO $\|\cdot\|$ LEFT AND RIGHT WE WANT TO GET $\mathcal{O}(\tau^3)$. THEREFORE WE REQUIRE

$$\begin{cases} b_1 + b_2 - \varphi_1(\tau A) = 0 \\ b_2 c_2 - \varphi_2(\tau A) = 0 \end{cases}$$

$$\Rightarrow \quad b_2 = \frac{1}{c_2}\varphi_2(\tau A), \qquad b_1 = \varphi_1(\tau A) - \frac{1}{c_2}\varphi_2(\tau A)$$

$c_2 \neq 0$

IF WE SUBSITUTE BACK IN OUR SCHEME WE GET THE METHOD

$$\begin{aligned} Y_{n2} &= e^{c_2 \tau A} y_n + c_2 \tau \varphi_1(c_2 \tau A)\, g(y_n) \\ y_{n+1} &= e^{\tau A} y_n + \tau\left(\varphi_1(\tau A) - \frac{1}{c_2}\varphi_2(\tau A)\right) g(y_n) \\ &\quad + \frac{\tau}{c_2}\varphi_2(\tau A)\, g(Y_{n2}) \end{aligned} \qquad (***)$$

$c_2 = 1 \Rightarrow$ ETD2RK

OR IN BUTCHER TABLEAU WAY

$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2\varphi_{1,2} & \\ \hline & \varphi_1 - \frac{1}{c_2}\varphi_2 & \frac{1}{c_2}\varphi_2 \end{array}$$

$\varphi_1(c_2 \cdot)$

WE CAN PROVE THAT INTEGRATOR $(***)$ IS SECOND ORDER ACCURATE (IF g SUFFICIENTLY OFTEN DIFFERENTIABLE WITH BOUNDED DERIVATIVES). MOREOVER THIS IS TRULY STIFF RESISTANT.

LAWSON METHODS

V.O.C. FORMULA

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau \underbrace{e^{(\tau-s)A} g(y(t_n+s))}_{\text{WE APPROXIMATE THE WHOLE INTEGRAND, NOT JUST } g(\dots)} ds$$

IF WE DO THIS AS FOR EXPONENTIAL EULER (APPROXIMATING AT LEFT POINT)

WE GET

$$y_{n+1} = e^{\tau A} y_n + e^{\tau A} g(y_n) \left(\int_0^\tau 1 \, ds \right)$$
$$= e^{\tau A} \left(y_n + \tau g(y_n) \right)$$

THIS IS CALLED LAWSON-EULER METHOD

- FIRST-ORDER ACCURATE
- A STABLE BY CONSTRUCTION (IT INTEGRATES EXACTLY WHEN $g \equiv 0$)
- COMPARED TO EXPONENTIAL EULER IT <u>NOT</u> EXACT IF $g(y(t)) \equiv b$
- THEY JUST REQUIRE MATRIX EXPONENTIALS, NOT φ FUNCTIONS

HISTORICALLY THEY ARE DERIVED BY DOING THE CHANGE OF VARIABLES $z(t) := e^{-tA} y(t)$. THEN ON THE OBTAINED SYSTEM YOU APPLY A <u>STANDARD</u> RUNGE-KUTTA METHOD AND THEN YOU GO BACK THE ORIGINAL VARIABLE $y(t)$.