A-stability linear stability

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases} \qquad y : \mathbb{R} \to \mathbb{C}^N$$

$$y_{N+1}(t) := t$$

$$\begin{cases} y'(t) = \lambda y(t) \\ y(t_0) = y_0 \end{cases} \qquad \lambda \in \mathbb{C} \qquad y : \mathbb{R} \to \mathbb{R}$$

$$y(t) = e^{(t-t_0)\lambda} y_0$$

if $Re(\lambda) < 0$ $\boxed{\lim_{t \to \infty} y(t) = 0}$

Fix a time step τ constant τ

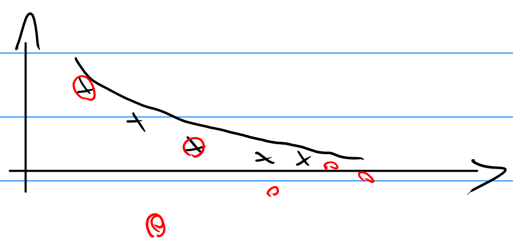
FORWARD EULER

$$y_{n+1} = y_n + \tau \lambda y_n \qquad y_n \approx y(t_n)$$

$$= (1 + \tau\lambda) y_n$$

$$y_n = (1 + \tau\lambda)^n y_0$$

We would like $\lim_{n \to \infty} y_n = 0$



$$|1 + \tau\lambda| < 1$$

$$|1+\tau\lambda|^2 < 1^2$$

$$(1+\tau \mathrm{Re}(\lambda))^2 + (\tau \mathrm{Im}(\lambda))^2 < 1$$

$$\cancel{1} + \tau^{\cancel{2}} \mathrm{Re}(\lambda)^2 + 2\cancel{\tau} \mathrm{Re}(\lambda) + \tau^{\cancel{2}} \mathrm{Im}(\lambda)^2 < \cancel{1}$$

$$\tau|\lambda|^2 < -2\mathrm{Re}(\lambda)$$

$$\tau < \frac{-2\mathrm{Re}(\lambda)}{|\lambda|^2}$$

$$\lim_{n\to\infty} y_n = 0 \iff \tau < -\frac{\mathrm{Re}(\lambda)}{|\lambda|^2}$$

BACKWARD EULER

$$y_{n+1} = y_n + \tau\lambda y_{n+1}$$

$$y_n = \left(\frac{1}{1-\tau\lambda}\right)^n y_0$$

$$\left|\frac{1}{1-\tau\lambda}\right| < 1 \qquad |1-\tau\lambda| > 1$$

$$|\underbrace{1-\tau \mathrm{Re}(\lambda)}_{>1} - i\tau \mathrm{Im}(\lambda)| > 1$$

$$\lim_{n\to\infty} y_n = 0 \quad \text{ALWAYS}$$

TRAPEZOIDAL RULE

$$y_{m+1} = y_m + \frac{\tau}{2}\lambda y_m + \frac{\tau}{2}\lambda y_{m+1}$$

$$y_m = \left(\frac{1+\frac{\tau}{2}\lambda}{1-\frac{\tau}{2}\lambda}\right)^m y_0$$

$$\left|\frac{1+\frac{\tau}{2}\lambda}{1-\frac{\tau}{2}\lambda}\right| < 1 \qquad \left|1+\frac{\tau}{2}\lambda\right| < \left|1-\frac{\tau}{2}\lambda\right|$$

$$\lim_{m\to\infty} y_m = 0 \qquad \text{ALWAYS}$$

REGION OF ABSOLUTE STABILITY

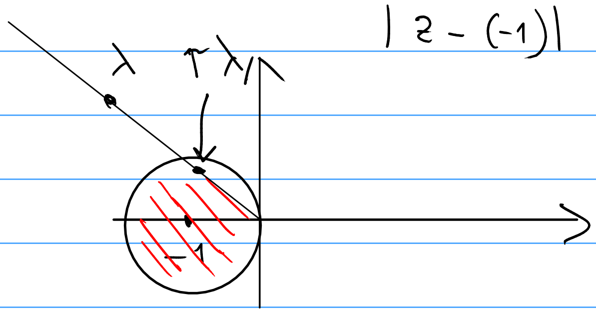
set of $z = \tau\lambda$ such that

$$\lim_{m\to\infty} y_m = 0$$

FORWARD EULER

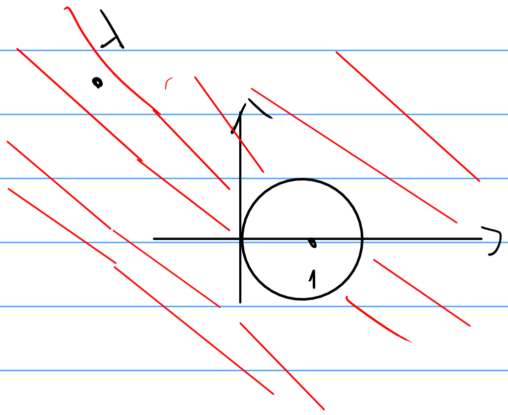
$$|1+\tau\lambda| < 1 \qquad |1+z| < 1$$

$$|z-(-1)| < 1$$



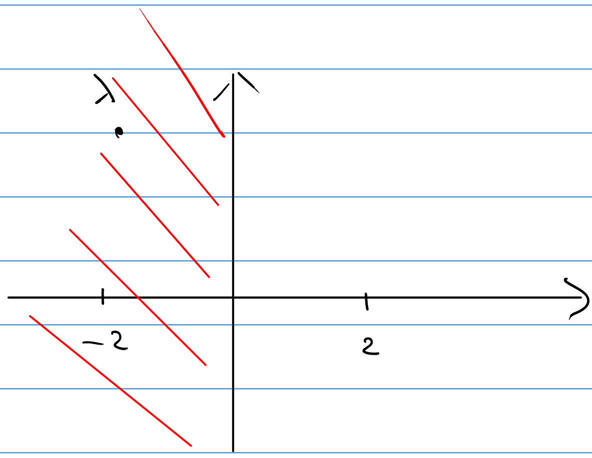
BACKWARD EULER

$|1 - \tau\lambda| > 1 \qquad |1 - z| > 1$



TRAPEZOIDAL RULE

$\left|1 - \frac{\tau}{2}\lambda\right| > \left|1 + \frac{\tau}{2}\lambda\right| \qquad |z - 2| > |z - (-2)|$



A method is A-stable if its absolute stability region contains $\mathbb{C}^- = \{ z \in \mathbb{C} \mid Re(z) < 0 \}$

$$\begin{cases} y'(t) = \lambda y(t) + b \quad \checkmark \\ y(t_0) = y_0 \quad \checkmark \end{cases}$$

$$y(t) = e^{(t-t_0)\lambda} y_0 + e^{(t-t_0)\lambda} \frac{b}{\lambda} - \frac{b}{\lambda}$$

CHECK

$$y(t_0) = y_0 + \frac{b}{\lambda} - \frac{b}{\lambda} = y_0 \quad \checkmark$$

$$y'(t) = \lambda e^{(t-t_0)\lambda} y_0 + \lambda e^{(t-t_0)\lambda} \frac{b}{\lambda} =$$

$$\lambda y(t) + b \quad \checkmark$$

$Re(\lambda) < 0$ $$\lim_{t \to \infty} y(t) = -\frac{b}{\lambda}$$

FORWARD EULER

$$y_{n+1} = y_n + \tau(\lambda y_n + b) = (1 + \tau\lambda) y_n + \tau b$$

$$y_n = (1 + \tau\lambda) y_{n-1} + \tau b$$

$$y_{n+1} = (1 + \tau\lambda)\left[(1 + \tau\lambda) y_{n-1} + \tau b\right] + \tau b$$

$$= (1 + \tau\lambda)^2 y_{n-1} + \left[(1 + \tau\lambda) + 1\right] \tau b$$

$$y_n = (1 + \tau\lambda)^n y_0 + \frac{(1 + \tau\lambda)^n - 1}{\cancel{1} + \cancel{\tau}\lambda - \cancel{1}} \cancel{\tau} b$$

$$= (1 + \tau\lambda)^n y_0 + \frac{(1 + \tau\lambda)^n - 1}{\lambda} b$$

$$\lim_{n \to \infty} y_n = -\frac{b}{\lambda} \iff |1 + \tau\lambda| < 1$$

$$y'(t) = \lambda y(t) + b$$

$$y(t) = e^{(t-t_0)\lambda} y_0 + e^{(t-t_0)\lambda} \frac{b}{\lambda} - \frac{b}{\lambda}$$

$$= e^{(t-t_0)\lambda} y_0 + (t-t_0) \boxed{\frac{e^{(t-t_0)\lambda} - 1}{(t-t_0)\lambda}} b$$

$$e^{(t-t_0)\lambda} y_0 + (t-t_0)\, \varphi_1((t-t_0)\lambda)\, b$$

$z \in \mathbb{C}$

$$\varphi_1(z) = \begin{cases} \frac{e^z - 1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases} \qquad (e^A - I)A^{-1} \quad ?$$

$$\boxed{\frac{d}{dt} t\, \varphi_1(t\lambda) = e^{t\lambda}}$$

NO EXPLICIT RUNGE-KUTTA METHOD IS A-STABLE
NO EXPLICIT MULTISTEP METHOD IS A-STABLE
SOME IMPLICIT METHODS ARE A-STABLE

EXAMPLE

$$\begin{cases} y'(t) = -100\, y(t) \\ y(0) = 1 \end{cases} \qquad \lambda = -100$$

FORWARD EULER $\qquad \tau < -\frac{2}{-100} = \frac{1}{50} = 0.02$

$$y(t) = e^{-100t}$$

$$y(0.4) < 10^{-17} \qquad \varepsilon \approx 10^{-16}$$

double precision

$$y(0) - y(0.4) = 1 - 10^{-17} = 1$$

we need 21 time steps

$$y'(t) = \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} y(t)$$

$$y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_1(t) = e^{-100t} \leftarrow$$

$$y_2(t) = e^{-t} \leftarrow$$

$$\| y(40) \|_\infty < 10^{-17}$$

FORWARD EULER $\quad \tau < \frac{1}{50} = 0.02$

we need 2001 time steps

$$y'(t) = Ay(t) \qquad A \in \mathbb{C}^{N \times N}$$

if $AV = V\Lambda \qquad \Lambda$ diagonal

$$\underbrace{V^{-1} y'(t)}_{z'(t)} = V^{-1} A V V^{-1} y(t)$$

$$z'(t) = \Lambda z(t) \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}$$

$$\exp(tA) = \sum_{i=0}^{\infty} \frac{(tA)^i}{i!}$$

$$y'(t) = Ay(t) + b \qquad y(t) = \exp(tA)\,y_0 + t\,\varphi_1(tA)\,b$$

if $AV = V\Lambda$

$$z'(t) = \Lambda z(t) + V^{-1} b$$

$$y'(t) = f(y(t)) \qquad \text{AUTONOMOUS}$$

LINEARIZATION $= f(y(t_m)) + J_{f_m}(y(t) - y(t_m)) + \dots$

$$= J_{f_m}\, y(t) + \left(f(y(t_m)) - J_{f_m}\, y(t_m) \right) + \dots$$

DEFINITION

$y'(t) = f(y(t))$ is $\boxed{\text{STIFF}}$ around t_m if

1) J_{f_m} has at least two eigenvalues λ_1, λ_2, $Re(\lambda_1) < 0$ $Re(\lambda_2) < 0$

2) $Re(\lambda_1) \ll Re(\lambda_2)$

ANOTHER DEFINITION

A problem is stiff if implicit methods perform much better then explicit ones.

$$\boxed{\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}} \qquad \lim_{m \to \infty} \left(I + \frac{A}{m} \right)^m$$

$$AV = V\Lambda$$
$$A = V \Lambda V^{-1}$$
$$A^2 = (V \Lambda V^{-1})(V \Lambda V^{-1}) = V \Lambda^2 V^{-1}$$

$$\exp(A) = V \exp(\Lambda) V^{-1}$$

$$\exp(\Lambda) = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_N} \end{bmatrix}$$

$$\varphi_1(z) = \begin{cases} \frac{e^z - 1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

$$\cancel{1} + \cancel{z} + \frac{z^{\cancel{2}\,1}}{2} + \frac{z^{\cancel{3}\,2}}{6} + \frac{z^{\cancel{4}\,3}}{24} + \dots = \sum_{i=0}^{\infty} \frac{z^i}{(i+1)!}$$

ENTIRE FUNCTION

$$\varphi_1(A) = \sum_{i=0}^{\infty} \frac{A^i}{(i+1)!} \qquad \boxed{A\varphi_1(A) = e^A - I}$$

$$AV = V\Lambda$$

$$\varphi_1(A) = V \varphi_1(\Lambda) V^{-1}$$

DO NOT TRY TO COMPUTE $\frac{e^\lambda - 1}{\lambda}$ for $\lambda \approx 0$

in finite arithmetic

suppose $AV = V\Lambda$ exactly

$$\|e^A - V e^{\tilde{\Lambda}} V^{-1}\| =$$

$$\| V e^{\Lambda} V^{-1} - V e^{\tilde{\Lambda}} V^{-1} \| \leq$$

$$\|V\| \|V^{-1}\| \|e^{\Lambda} - e^{\tilde{\Lambda}}\| =$$

$$\mathrm{cond}(V) \cdot \boxed{\text{something small}}$$

if A is normal (symm. skew symm. orthog.)

$$A = V \Lambda V^H \qquad V^H = \overline{V^T}$$

$$\mathrm{cond}_2(V) = 1$$

small normal matrices : diagonalization!