LECTURE 9 (09/02/26)

UP TO NOW WE CONSIDERED:

- LINEAR PROBLEMS $\Rightarrow$ EXPONENTIAL QUADRATURE RULES
- SEMILINEAR PROBLEMS $\Rightarrow$ EXPONENTIAL RUNGE-KUTTA METHODS (EXPONENTIAL MULTISTEP METHODS)

WHAT HAPPENS FOR FULLY NONLINEAR PROBLEMS

$$(\Delta) \quad \begin{cases} y'(t) = f(y(t)) & t \in [0, t^*] \\ y(0) = y_0 \end{cases}$$

EXPRK $\quad f(y(t)) = Ay(t) + g(y(t))$

$y(t) \in \mathbb{R}^N$. WE LINEARIZE (Δ) AROUND A "FIXED" (LOCALLY) STATE y_n, WHICH IS THE NUMERICAL SOLUTION AT TIME t_n, AND WE GET

$$\begin{cases} y(t) = J_n y(t) + g_n(y(t)) \\ y(0) = y_0 \end{cases} \quad (\square)$$

WITH $J_n = \dfrac{df}{dy}(\underline{\underline{y_n}})$ AND $g_n(y(t)) = f(y(t)) - J_n y(t)$

J_n: JACOBIAN (ASSUME TO EXIST)

g_n: NONLINEAR REMAINDER

FROM (Δ) TO $(\square)$ WE DIDN'T DO ANY KIND OF APPROXIMATION, WE SIMPLY IDENTIFIED A "NATURAL" LINEAR PART OF THE RHS (THINK ABOUT TAYLOR)

IF WE APPLY TO (D) THE EXPONENTIAL EULER METHOD WE GET

$$y_{n+1} = e^{\tau J_n} y_n + \tau \varphi_1(\tau J_n) g_n(y_n) \qquad (o)$$

$$= \cancel{y_n} + \tau \varphi_1(\tau J_n) f(y_n)$$

THE OBTAINED SCHEME IS CALLED EXPONENTIAL ROSENBROCK EULER METHOD. IT IS A SECOND ORDER METHOD.

WE PROVE THIS IN THE SIMPLIFIED SETTING

$$f(y(t)) = Ay(t) + g(y(t))$$

$$J = \frac{df}{dy} = A + \frac{dg}{dy}$$

THEOREM

CONSIDER PROBLEM (1) AND INTEGRATOR (o). THEN, ASSUMING g SUFFICIENTLY OFTEN DIFFERENTIABLE (WITH BDD DERIVATIVES) THEN

$$\|y(t_n) - y_n\| \le C\tau^2$$

PROOF

FROM THE VOC WE GET

$$y(t_{n+1}) = e^{\tau J_n} y(t_n) + \int_0^\tau e^{(\tau - s)J_n} g_n(y(t_n + s))\, ds$$

WE CALL $h_n(t) \doteq g_n(y(t))$ SO THAT

$$y(t_{n+1}) = e^{\tau J_n} y(t_n) + \int_0^\tau e^{(\tau-s)J_n} h_n(t_n + s)\, ds$$

BY TAYLOR EXPANSION OF h_n WE GET

$$h_n(t_n+s) = h_n(t_n) + h_n'(t_n)s + \int_0^s h''(t_n+\sigma)(\sigma-s)d\sigma$$

SO THAT

$$y(t_{n+1}) = e^{\tau J_n} y(t_n) + \tau\varphi_1(\tau J_n) h_n(t_n) + \tau^2\varphi_2(\tau J_n) h_n'(t_n) + \int_0^\tau e^{(\tau-s)J_n} \int_0^s h''(t_n+\sigma)(\sigma-s)\,d\sigma\,ds$$

NOW WE COMPARE THIS EXPANSION WITH THE NUMERICAL SOLUTION

$$y_{n+1} = e^{\tau J_n} y_n + \tau\varphi_1(\tau J_n) g_n(y_n)$$

$$y(t_{n+1}) - y_{n+1} = e^{\tau J_n}(y(t_n) - y_n) + \tau\varphi_1(\tau J_n)(h_n(t_n) - g_n(y_n)) + \tau^2\varphi_2(\tau J_n) h_n'(t_n) + \int_0^\tau e^{(\tau-s)J_n} \int_0^s h''(t_n+\sigma)(\sigma-s)\,d\sigma\,ds$$

LET'S START WITH •

$$h_n(t_n) - g_n(y_n) = g_n(y(t_n)) - g_n(y_n)$$

$$= (f(y(t_n)) - J_n y(t_n)) - (f(y_n) - J_n y_n)$$

$$= (Ay(t_n) + g(y(t_n)) - J_n y(t_n)) - (Ay_n + g(y_n) - J_n y_n)$$

HOWEVER $J_n = \frac{df}{dy}(y_n) = A + \frac{dg}{dy}(y_n)$

$$= \left(g(y(t_n)) - \frac{dg}{dy}(y_n)\, y(t_n)\right) - \left(g(y_n) - \frac{dg}{dy}(y_n)\, y_n\right)$$

$$= \left(g(y(t_n)) - g(y_n)\right) - \frac{dg}{dy}(y_n)\left(y(t_n) - y_n\right)$$

WE LIKE THIS FORM BECAUSE SINCE g IS LIPSCHITZ

$$\Rightarrow \|h_n(t_n) - g_n(t_n)\| \leq C \|y(t_n) - y_n\|$$

FOR $\bullet$ WE HAVE

$$h_n'(t_n) = \frac{\partial g_n(y(t_n))}{\partial y} y'(t_n)$$
$$= \left(\frac{\partial g}{\partial y}(y(t_n)) - \frac{\partial g}{\partial y}(y_n)\right) y'(t_n)$$

BY LIPSCHITZIANITY WE GET

$$\|h_n'(t_n)\| \leq C \|y(t_n) - y_n\|$$

FINALLY FOR $\bullet$ WE HAVE $\|\bullet\| \leq C\tau^3$

THEREFORE IF WE CALL $y(t_n) - y_n \doteq \varepsilon_n$ AND

$\bullet + \bullet + \bullet \doteq \delta_{n+1}$ THEN

$$\varepsilon_{n+1} = e^{\tau J_n}\varepsilon_n + \delta_{n+1} \iff \varepsilon_n = \sum_{j=1}^{n}\left(\prod_{k=1}^{n-j} e^{\tau J_{n-k}}\right)\delta_j$$

HENCE USING THE PREVIOUS BOUNDS WE GET

$$\|\varepsilon_n\| \leq C \sum_{j=1}^{n} \|\delta_j\| \leq C\tau \sum_{j=0}^{n-1} \|\varepsilon_j\|$$
$$+ C\tau^2 \sum_{j=0}^{n-1} \|\varepsilon_j\|$$
$$+ C\tau^3 \sum_{j=0}^{n-1} 1 \longrightarrow \leq C\tau^2$$

BY A DISCRETE GRONWALL LEMMA (SEE FIRST LECTURES) WE GET

$$\|\varepsilon_n\| \le c\tau^2 \iff \|y(t_n) - y_n\| \le c\tau^2$$

$\square$

HIGHER ORDER METHODS OF THIS CLASS DO EXIST (CREATE INTERMEDIATE STAGES, PLAY WITH ORDER CONDITIONS, ...) BUT WE DON'T CONSIDER THEM HERE.

NON-AUTONOMOUS CASE

$$\underbrace{y'(t) = Ay(t) + g(y(t))}_{\overset{!}{=} f(y(t))} \overset{\text{EXPEULER}}{\Longrightarrow} y_{n+1} = y_n + \tau\varphi_1(\tau A) f(y_n) \ \checkmark$$

$$\underbrace{y'(t) = Ay(t) + g(t, y(t))}_{\overset{!}{=} f(t, y(t))} \overset{\text{EXPEULER}}{\Longrightarrow} y_{n+1} = y_n + \tau\varphi_1(\tau A) f(t_n, y_n) \ \checkmark$$

$$y'(t) = f(y(t)) \overset{\text{EXPRBEULER}}{\Longrightarrow} y_{n+1} = y_n + \tau\varphi_1(\tau J_n) f(y_n) \ \checkmark$$

$$y'(t) = f(t, y(t)) \overset{\text{EXPRBEULER}}{\Longrightarrow} \boxed{y_{n+1} = y_n + \tau\varphi_1(\tau J_n) f(t_n, y_n) \quad \text{NO!!!}}$$

THIS IS NOT CORRECT BECAUSE A LINEARIZATION IN THE VARIABLE t IS MISSING. TO GET THE CORRECT INTEGRATOR WE FIRST REWRITE THE SYSTEM IN AUTONOMOUS FORM.

$$\mathbb{R}^{N+1} \ni Y(t) = \begin{bmatrix} \overset{\in \mathbb{R}}{Y_1(t)} \\ \underset{\in \mathbb{R}^N}{Y_2(t)} \end{bmatrix} = \begin{bmatrix} t \\ y(t) \end{bmatrix}$$

AND HENCE

$$Y'(t) = \begin{bmatrix} Y_1'(t) \\ Y_2'(t) \end{bmatrix} = \begin{bmatrix} t \\ y'(t) \end{bmatrix} = \begin{bmatrix} Y_1(t) \\ f(Y_1(t), Y_2(t)) \end{bmatrix} \doteq F(Y(t))$$

+ INITIAL CONDITION

NOW WE LINEARIZE

$$\frac{dF}{dY} = \begin{bmatrix} 0 & \mathbb{0} \\ \frac{\partial f}{\partial Y_1} & \frac{\partial f}{\partial Y_2} \end{bmatrix} = \begin{bmatrix} \overset{\nearrow \in \mathbb{R}}{0} & \overset{\nearrow \in \mathbb{R}^{1\times N}}{\mathbb{0}} \\ \underset{\downarrow \in \mathbb{R}^{N\times 1}}{\frac{\partial f}{\partial t}} & \underset{\downarrow \in \mathbb{R}^{N\times N}}{\frac{\partial f}{\partial y}} \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}$$

AND

$$\left.\frac{dF}{dY}\right|_{Y_n} = \begin{bmatrix} 0 & \mathbb{0} \\ \underbrace{\left.\frac{\partial f}{\partial t}\right|_{(t_n, y_n)}}_{\doteq V_n} & \underbrace{\left.\frac{\partial f}{\partial y}\right|_{(t_n, y_n)}}_{\doteq J_n} \end{bmatrix} \doteq \begin{bmatrix} 0 & \mathbb{0} \\ V_n & J_n \end{bmatrix} \doteq \tilde{J}_n$$

FOR THE EXP. RB. EULER METHOD WE HAVE TO COMPUTE

$$\varphi_1(\tau \tilde{J}_n) = \varphi_1\left(\tau \begin{bmatrix} 0 & \mathbb{0} \\ V_n & J_n \end{bmatrix}\right)$$

BLOCK LOWER TRIANG

$$\begin{matrix} * & \mathbb{0} \\ * & * \end{matrix}$$

$$= \begin{bmatrix} \overset{\leftarrow \varphi_1(0)}{1} & \mathbb{0} \\ \tau\varphi_2(\tau J_n) V_n & \varphi_1(\tau J_n) \end{bmatrix}$$

CAN BE PROVEN BY USING THE DEFINITION OF φ_1 + LOWER BLOCK DIAG. STRUCTURE

SO WE HAVE THE TIME MARCHING

$$Y_{n+1} = Y_n + \tau \varphi_1(\tau \tilde{J}_n) F(Y_n)$$

$$= \begin{bmatrix} Y_{1,n} \\ Y_{2,n} \end{bmatrix} + \tau \begin{bmatrix} 1 & 0 \\ \tau\varphi_2(\tau J_n) v_n & \varphi_1(\tau J_n) \end{bmatrix} \begin{bmatrix} 1 \\ f(Y_{1,n}, Y_{2,n}) \end{bmatrix}$$

$\approx y(t_n)$

$$= \begin{bmatrix} * \\ Y_{2,n} + \tau\varphi_1(\tau J_n) f(Y_{1,n}, Y_{2,n}) + \tau^2 \varphi_2(\tau J_n) v_n \end{bmatrix}$$

GOING BACK TO THE ORIGINAL VARIABLES WE GET THE TIME MARCHING

$$\boxed{y_{n+1} = y_n + \tau\varphi_1(\tau J_n) f(t_n, y_n) + \tau^2 \varphi_2(\tau J_n) v_n}$$

WHICH IS THE CORRECT FORM OF EXPONENTIAL ROSENBROCK EULER FOR NON-AUTONOMOUS SYSTEMS.

(STANDARD) ROSENBROCK METHODS

ROSENBROCK METHODS ARE IMPLICIT METHODS WHICH EXPLOIT INFORMATION FROM THE JACOBIAN OF THE RIGHT HAND SIDE. THEY ARE IMEX SCHEMES WHICH ARE USEFUL IN PRESENCE OF STIFFNESS.

IDEA : FOR $y'(t) = A y(t) + g(y(t))$

WE HAVE IMEX - EULER $\Rightarrow$ $(I - \tau A) y_{n+1} = y_n + \tau g(y_n)$

FOR A NONLINEAR SYSTEM $y'(t) = f(y(t))$

A SECOND-ORDER A-STABLE ROSENBROCK METHOD IS

$$\boxed{\begin{aligned} \left(I - \frac{\tau}{2} J_n\right) k_1 &= \tau f(y_n) \\ y_{n+1} &= y_n + k_1 \end{aligned}}$$

WITH $J_n = \left.\frac{df}{dy}\right|_{y_n}$.