# LECTURE 9 (09/02/26)

UP TO NOW WE CONSIDERED:

- LINEAR PROBLEMS $\Rightarrow$ EXPONENTIAL QUADRATURE RULES
- SEMILINEAR PROBLEMS $\Rightarrow$ EXPONENTIAL RUNGE-KUTTA METHODS

(EXPONENTIAL MULTISTEP METHODS)

WHAT HAPPENS FOR FULLY NONLINEAR PROBLEMS

$$(\Delta) \quad \begin{cases} y'(t) = f(y(t)) & t \in [0, t^*] \\ y(0) = y_0 \end{cases}$$

EXPRK $\quad f(y(t)) = Ay(t) + g(y(t))$

$y(t) \in \mathbb{R}^N$. WE LINEARIZE $(\Delta)$ AROUND A "FIXED" STATE (LOCALLY)

$y_n$, WHICH IS THE NUMERICAL SOLUTION AT TIME $t_n$,

AND WE GET

$$\begin{cases} y(t) = J_n y(t) + g_n(y(t)) \\ y(0) = y_0 \end{cases} \qquad (\square)$$

WITH $\quad J_n = \dfrac{df}{dy}(\underline{\underline{y_n}}) \quad$ AND $\quad g_n(y(t)) = f(y(t)) - J_n y(t)$

$J_n$: JACOBIAN (ASSUME TO EXIST)

$g_n$: NONLINEAR REMAINDER

FROM $(\Delta)$ TO $(\square)$ WE DIDN'T DO ANY KIND OF APPROXIMATION, WE SIMPLY IDENTIFIED A "NATURAL" LINEAR PART OF THE RHS (THINK ABOUT TAYLOR)

IF WE APPLY TO (D) THE EXPONENTIAL EULER METHOD WE GET

$$y_{m+1} = e^{\tau J_m} y_m + \tau \varphi_1(\tau J_m)\, g_m(y_m) \qquad (o)$$

$$= \cancel{y_m} + \tau \varphi_1(\tau J_m)\, f(y_m)$$

THE OBTAINED SCHEME IS CALLED EXPONENTIAL ROSENBROCK EULER METHOD. IT IS A SECOND ORDER METHOD.

WE PROVE THIS IN THE SIMPLIFIED SETTING

$$f(y(t)) = Ay(t) + g(y(t))$$

$$J = \frac{df}{dy} = A + \frac{dg}{dy}$$

## THEOREM

CONSIDER PROBLEM (1) AND INTEGRATOR (o). THEN, ASSUMING $g$ SUFFICIENTLY OFTEN DIFFERENTIABLE (WITH BDD DERIVATIVES) THEN

$$\|y(t_m) - y_m\| \leq C\tau^2$$

## PROOF

FROM THE VOC WE GET

$$y(t_{m+1}) = e^{\tau J_m} y(t_m) + \int_0^\tau e^{(\tau-s)J_m} g_m(y(t_m+s))\, ds$$

WE CALL $h_m(t) \doteq g_m(y(t))$ SO THAT

$$y(t_{m+1}) = e^{\tau J_m} y(t_m) + \int_0^\tau e^{(\tau-s)J_m} h_m(t_m+s)\, ds$$

BY TAYLOR EXPANSION OF $h_n$ WE GET

$$h_n(t_n+s) = h_n(t_n) + h_n'(t_n)\,s + \int_0^s h''(t_n+\sigma)(\sigma-s)\,d\sigma$$

SO THAT

$$y(t_{n+1}) = e^{\tau J_n} y(t_n) + \tau\varphi_1(\tau J_n) h_n(t_n) + \tau^2 \varphi_2(\tau J_n) h_n'(t_n)$$
$$+ \int_0^\tau e^{(\tau-s)J_n} \int_0^s h''(t_n+\sigma)(\sigma-s)\,d\sigma\,ds$$

NOW WE COMPARE THIS EXPANSION WITH THE NUMERICAL SOLUTION

$$y_{n+1} = e^{\tau J_n} y_n + \tau\varphi_1(\tau J_n)\, g_n(y_n)$$

$$y(t_{n+1}) - y_{n+1} = e^{\tau J_n}(y(t_n) - y_n) + \underline{\tau\varphi_1(\tau J_n)(h_n(t_n) - g_n(y_n))}$$
$$+ \underline{\tau^2\varphi_2(\tau J_n) h_n'(t_n)} + \underline{\int_0^\tau e^{(\tau-s)J_n} \int_0^s h''(t_n+\sigma)(\sigma-s)\,d\sigma\,ds}$$

LET'S START WITH

$$h_n(t_n) - g_n(y_n) = g_n(y(t_n)) - g_n(y_n)$$

$$= (f(y(t_n)) - J_n y(t_n)) - (f(y_n) - J_n y_n)$$

$$= (A y(t_n) + g(y(t_n)) - J_n y(t_n)) - (A y_n + g(y_n) - J_n y_n)$$

HOWEVER $J_n = \frac{df}{dy}(y_n) = A + \frac{dg}{dy}(y_n)$

$$= \left(g(y(t_n)) - \frac{dg}{dy}(y_n)\, y(t_n)\right) - \left(g(y_n) - \frac{dg}{dy}(y_n)\, y_n\right)$$

$$= \left(g(y(t_n)) - g(y_n)\right) - \frac{dg}{dy}(y_n)\left(y(t_n) - y_n\right)$$

WE LIKE THIS FORM BECAUSE SINCE $g$ IS LIPSCHITZ

$$\Rightarrow \quad \|h_n(t_n) - g_n(t_n)\| \le C\,\|y(t_n) - y_n\|$$

FOR $\textcolor{orange}{\bullet}$ WE HAVE

$$h_n'(t_n) = \frac{\partial g_n}{\partial y}(y(t_n))\, y'(t_n) \overset{!}{=} \left(\frac{\partial g}{\partial y}(y(t_n)) - \frac{\partial g}{\partial y}(y_n)\right) y'(t_n)$$

BY LIPSCHITZIANITY WE GET

$$\|h_n'(t_n)\| \le C\,\|y(t_n) - y_n\|$$

FINALLY FOR $\textcolor{blue}{\bullet}$ WE HAVE $\|\textcolor{blue}{\bullet}\| \le C\tau^3$

THEREFORE IF WE CALL $y(t_n) - y_n \doteq \varepsilon_n$ AND

$\textcolor{red}{\bullet} + \textcolor{orange}{\bullet} + \textcolor{blue}{\bullet} \doteq \delta_{n+1}$ THEN

$$\varepsilon_{n+1} = e^{\tau J_n}\varepsilon_n + \delta_{n+1} \iff \varepsilon_n = \sum_{j=1}^{n}\left(\prod_{k=1}^{n-j} e^{\tau J_{n-k}}\right)\delta_j$$

HENCE USING THE PREVIOUS BOUNDS WE GET

$$\|\varepsilon_n\| \le C\sum_{j=1}^{n}\|\delta_j\| \le C\tau\sum_{j=0}^{n-1}\|\varepsilon_j\| + C\tau^2\sum_{j=0}^{n-1}\|\varepsilon_j\| + \underbrace{C\tau^3\sum_{j=0}^{n-1} 1}_{\le C\tau^2}$$

BY A DISCRETE GRONWALL LEMMA (SEE FIRST LECTURES) WE GET

$$\|\varepsilon_n\| \leq c\tau^2 \iff \|y(t_n) - y_n\| \leq c\tau^2$$

$\square$

HIGHER ORDER METHODS OF THIS CLASS DO EXIST (CREATE INTERMEDIATE STAGES, PLAY WITH ORDER CONDITIONS, ...) BUT WE DON'T CONSIDER THEM HERE.

## NON-AUTONOMOUS CASE

$$y'(t) = \underbrace{Ay(t) + g(y(t))}_{= f(y(t))} \quad \overset{\text{EXPEULER}}{\Longrightarrow} \quad y_{n+1} = y_n + \tau\varphi_1(\tau A) f(y_n) \checkmark$$

$$y'(t) = \underbrace{Ay(t) + g(t, y(t))}_{= f(t, y(t))} \quad \overset{\text{EXPEULER}}{\Longrightarrow} \quad y_{n+1} = y_n + \tau\varphi_1(\tau A) f(t_n, y_n) \checkmark$$

$$y'(t) = f(y(t)) \quad \overset{\text{EXPRBEULER}}{\Longrightarrow} \quad y_{n+1} = y_n + \tau\varphi_1(\tau J_n) f(y_n) \checkmark$$

$$y'(t) = f(t, y(t)) \quad \overset{\text{EXPRBEULER}}{\Longrightarrow} \quad \boxed{y_{n+1} = y_n + \tau\varphi_1(\tau J_n) f(t_n, y_n)}$$

⚠ NO!!!

THIS IS NOT CORRECT BECAUSE A LINEARIZATION IN THE VARIABLE $t$ IS MISSING. TO GET THE CORRECT INTEGRATOR WE FIRST REWRITE THE SYSTEM IN AUTONOMOUS FORM.

$$\mathbb{R}^{N+1} \ni Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} t \\ y(t) \end{bmatrix}, \quad Y_1(t) \in \mathbb{R}, \quad Y_2(t) \in \mathbb{R}^N$$

AND HENCE

$$Y'(t) = \begin{bmatrix} Y_1'(t) \\ Y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 \\ f(Y_1(t), Y_2(t)) \end{bmatrix} \doteq F(Y(t))$$

+ INITIAL CONDITION

NOW WE LINEARIZE

$$\frac{dF}{dY} = \begin{bmatrix} 0 & \mathbb{0} \\ \frac{\partial f}{\partial Y_1} & \frac{\partial f}{\partial Y_2} \end{bmatrix} = \begin{bmatrix} \overset{\in \mathbb{R}}{0} & \overset{\in \mathbb{R}^{1\times N}}{\mathbb{0}} \\ \underset{\in \mathbb{R}^{N\times 1}}{\frac{\partial f}{\partial t}} & \underset{\in \mathbb{R}^{N\times N}}{\frac{\partial f}{\partial y}} \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}$$

AND

$$\left.\frac{dF}{dY}\right|_{Y_n} = \begin{bmatrix} 0 & \mathbb{0} \\ \underbrace{\left.\frac{\partial f}{\partial t}\right|_{(t_n, y_n)}}_{\doteq V_n} & \underbrace{\left.\frac{\partial f}{\partial y}\right|_{(t_n, y_n)}}_{\doteq J_n} \end{bmatrix} \doteq \begin{bmatrix} 0 & \mathbb{0} \\ V_n & J_n \end{bmatrix} \doteq \tilde{J}_n$$

FOR THE EXP. RB. EULER METHOD WE HAVE TO COMPUTE

$$\varphi_1(\tau \tilde{J}_n) = \varphi_1\left(\tau \begin{bmatrix} 0 & \mathbb{0} \\ V_n & J_n \end{bmatrix}\right)$$

BLOCK LOWER TRIANG

$$\begin{matrix} * & \mathbb{0} \\ * & * \end{matrix}$$

$$= \begin{bmatrix} \overset{\leftarrow \varphi_1(0)}{1} & \mathbb{0} \\ \tau\varphi_2(\tau J_n)V_n & \varphi_1(\tau J_n) \end{bmatrix}$$

CAN BE PROVEN BY USING THE DEFINITION OF $\varphi_1$ + LOWER BLOCK DIAG. STRUCTURE

SO WE HAVE THE TIME MARCHING

$$Y_{n+1} = Y_n + \tau\, \varphi_1(\tau \tilde{J}_n)\, F(Y_n)$$

$$= \begin{bmatrix} Y_{1,n} \\ Y_{2,n} \end{bmatrix} + \tau \begin{bmatrix} 1 & 0 \\ \tau \varphi_2(\tau J_n) v_n & \varphi_1(\tau J_n) \end{bmatrix} \begin{bmatrix} 1 \\ f(Y_{1,n}, Y_{2,n}) \end{bmatrix}$$

$\approx y(t_n)$

$$= \begin{bmatrix} * \\ Y_{2,n} + \tau \varphi_1(\tau J_n) f(Y_{1,n}, Y_{2,n}) + \tau^2 \varphi_2(\tau J_n) v_n \end{bmatrix}$$

GOING BACK TO THE ORIGINAL VARIABLES WE GET THE TIME MARCHING

$$\boxed{y_{n+1} = y_n + \tau \varphi_1(\tau J_n) f(t_n, y_n) + \tau^2 \varphi_2(\tau J_n) v_n}$$

WHICH IS THE CORRECT FORM OF EXPONENTIAL ROSENBROCK EULER FOR NON-AUTONOMOUS SYSTEMS.

## (STANDARD) ROSENBROCK METHODS

ROSENBROCK METHODS ARE IMPLICIT METHODS WHICH EXPLOIT INFORMATION FROM THE JACOBIAN OF THE RIGHT HAND SIDE. THEY ARE IMEX SCHEMES WHICH ARE USEFUL IN PRESENCE OF STIFFNESS.

IDEA: FOR $y'(t) = A y(t) + g(y(t))$

WE HAVE IMEX-EULER $\Rightarrow$ $(I - \tau A) y_{n+1} = y_n + \tau g(y_n)$

FOR A NONLINEAR SYSTEM $y'(t) = f(y(t))$

A SECOND-ORDER A-STABLE ROSENBROCK METHOD IS

$$\boxed{\begin{aligned} \left(I - \frac{\tau}{2} J_n\right) k_1 &= \tau f(y_n) \\ y_{n+1} &= y_n + k_1 \end{aligned}}$$

WITH $J_n = \left.\dfrac{df}{dy}\right|_{y_n}$.