# LECTURE 4 (21/01/25)

UP TO NOW WE STUDIED FIRST ORDER METHODS (EXPONENTIAL EULER). THE IDEA IS THAT WE WANT TO GO HIGHER ORDER, BUT TO DO THAT WE START WITH A SIMPLIFIED SETTING.

$\Rightarrow$ LINEAR TIME-DEPENDENT SYSTEMS

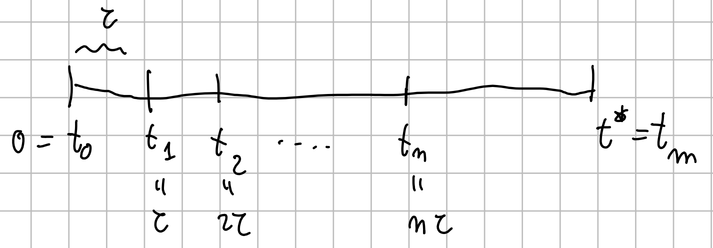
$$\begin{cases} y'(t) = A\,y(t) + g(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases} \qquad (*)$$

$y(t) \in \mathbb{C}^N$, $A \in \mathbb{C}^{N\times N}$ (STIFF PART), $g(t) \in \mathbb{C}^N$ SOURCE TERM.

WE START WITH THE VARIATION OF CONSTANTS FORMULA FOR $(*)$

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} g(s)\, ds$$

AND WE INTRODUCE A TIME DISCRETIZATION (CONSTANT)




$$\tau = \frac{t^*}{m}$$

SO THE V.O.C. TELLS US AT TIME $t_{n+1}$

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau - s)A} g(t_n + s)\, ds \qquad (**)$$

WE DO EXACTLY AS WE DID FOR EXPONENTIAL EULER, THAT IS $g(t_n + s) \approx g(t_n)$, THEN BY CALLING $y_{n+1} \approx y(t_{n+1})$ $y_n \approx y(t_n)$ WE GET THE TIME-MARCHING SCHEME

$$y_{n+1} = e^{\tau A} y_n + \left(\int_0^\tau e^{(\tau-s)A}\, ds\right) g(t_n) \qquad \to \tau\varphi_1(\tau A)$$

$$= e^{\tau A} y_n + \tau\varphi_1(\tau A)\, g(t_n) \qquad \nearrow \; x\varphi_1(x) = e^x - I$$

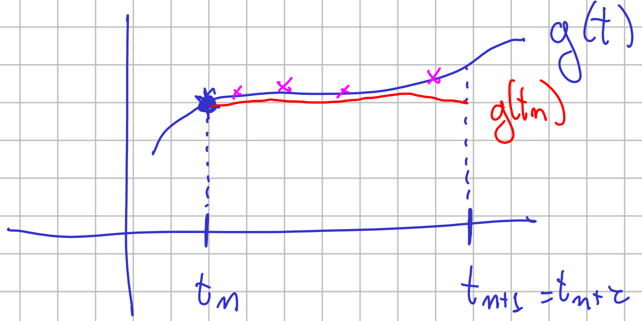
$$= y_n + \tau\varphi_1(\tau A)\left(A y_n + g(t_n)\right)$$

$$= y_n + \tau\varphi_1(\tau A)\, f(t_n, y_n)$$

THIS IS EXACTLY EXPONENTIAL EULER APPLIED TO (*), WHICH IN THIS CASE GOES UNDER THE NAME OF AN EXPONENTIAL QUADRATURE RULE (OF COLLOCATION TYPE)

↳ BECAUSE IT'S ESSENTIALLY RECTANGLE LEFT POINT QUADRATURE RULE APPLIED TO (**) ON $g(t_n+s)$

OBVIOUSLY THIS SCHEME IS FIRST-ORDER ACCURATE AND A-STABLE.




IN OTHER WORDS IF INSTEAD OF DOING $g(t_n+s) \approx g(t_n)$

WE DO $g(t_n+s) \approx g(t_n + c_1\tau)$ $\quad (\Delta) \quad c_1 \in [0,1]$

CAN WE GET HIGHER ORDER? → SPOILER YES!

IF WE INSERT $(\Delta)$ IN $(**)$ WE GET

$$y_{n+1} = e^{\tau A} y_n + \tau \varphi_1(\tau A)\, g(t_n + c_1 \tau) \qquad (\square)$$

HOW DO WE CHOOSE $c_1$ S.T. WE GAIN ORDER?

WE PROCEED SIMILARLY AS DONE FOR EXPONENTIAL EULER

$$g(t_n + s) = g(t_n) + g'(t_n)\, s + \int_0^s g''(t_n+\sigma)(s-\sigma)\, d\sigma$$

INSERTING IN V.O.C. WE GET

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau - s)A} g(t_n + s)\, ds$$

$$\varphi_\ell(x) = \int_0^1 e^{(1-\vartheta)x} \frac{\vartheta^{\ell-1}}{(\ell-1)!}\, d\vartheta$$

$$= e^{\tau A} y(t_n) + \tau \varphi_1(\tau A)\, g(t_n)$$

$$+ \left(\int_0^\tau e^{(\tau-s)A} s\, ds\right) g'(t_n) \qquad \tau^2 \varphi_2(\tau A)\, g'(t_n)$$

$$\vartheta = \frac{s}{\tau}$$

$$\tau^2 \int_0^1 e^{(1-\vartheta)\tau A} \vartheta\, d\vartheta = \tau^2 \varphi_2(\tau A) \quad (\ell = 2)$$

$$+ \int_0^\tau e^{(\tau-s)A} \int_0^s g''(t_n + \sigma)(s-\sigma)\, d\sigma\, ds$$

$$= e^{\tau A} y(t_n) + \tau \varphi_1(\tau A)\, g(t_n) + \tau^2 \varphi_2(\tau A)\, g'(t_n)$$
$$+ \int_0^\tau e^{(\tau-s)A} \int_0^s g''(t_n+\sigma)(s-\sigma)\, d\sigma\, ds$$

WE NOW COMPARE THIS WITH OUR NUMERICAL SOLUTION

$$y_{n+1} - y(t_{n+1}) = e^{\tau A}(y_n - y(t_n)) + \boxed{\tau \varphi_1(\tau A)\left(g(t_n + c_1\tau) - g(t_n)\right)}$$
$$\boxed{-\tau^2 \varphi_2(\tau A)\, g'(t_n)} - \underbrace{\int_0^\tau e^{(\tau-s)A} \int_0^s g''(t_n+\sigma)(s-\sigma)\, d\sigma\, ds}_{\mathcal{O}(\tau^3)}$$

THE IDEA IS TO OBTAIN $\mathcal{O}(\tau^3)$ (LOCALLY) FROM THE BOXED QUANTITIES

$$g(t_n + c_1\tau) = g(t_n) + g'(t_n) c_1 \tau + \int_0^{c_1\tau} g''(t_n+\sigma)(c_1\tau - \sigma)\,d\sigma$$

$$\Leftrightarrow$$

$$\underline{g(t_n + c_1\tau) - g(t_n)} = g'(t_n) c_1 \tau + \int_0^{c_1\tau} g''(t_n+\sigma)(c_1\tau-\sigma)\,d\sigma$$

SO BY SUBSTITUTING WE GET

$$y_{n+1} - y(t_{n+1}) = e^{\tau A}(y_n - y(t_{n+1})) +$$
$$+\tau^2 \left( c_1 \varphi_1(\tau A) - \varphi_2(\tau A)\right) g'(t_n)$$
$$\left. + \tau \varphi_1(\tau A) \int_0^{c_1\tau} g''(t_n+\sigma)(c_1\tau - \sigma)\,d\sigma \right) \to \mathcal{O}(\tau^3)$$
$$\left. - \int_0^{\tau} e^{(\tau - s)A} \int_0^{s} g''(t_n+\sigma)(s-\sigma)\,d\sigma \right) \to \mathcal{O}(\tau^3)$$

SO WE ARE HAPPY IF

$$c_1\varphi_1(\tau A) - \varphi_2(\tau A) = \tau\, \eta(\tau A) \qquad (0)$$

HOW DO WE GUESS $c_1$ AND $\eta$ ?

$$\varphi_1(\tau A) = \sum_{k=0}^{+\infty} \frac{(\tau A)^k}{(k+1)!} = I + \frac{\tau A}{2} + \ldots$$

$$\varphi_2(\tau A) = \sum_{k=0}^{+\infty} \frac{(\tau A)^k}{(k+2)!} = \frac{I}{2} + \frac{\tau A}{6} + \ldots$$

$$\Rightarrow c_1\varphi_1(\tau A) - \varphi_2(\tau A) = \left(c_1 - \frac{1}{2}\right) I + \underbrace{\tau\left(\frac{c_1}{2} - \frac{1}{6}\right) A + \ldots}$$

$$\boxed{c_1 = \frac{1}{2}}$$

THEREFORE OUR GUESS FOR $c_1 = \frac{1}{2}$, THEN WE JUST EXPLOIT

THE RELATIONS OF THE $\varphi$ FUNCTIONS. IN PARTICULAR

$$X \varphi_{\ell+1}(x) = \varphi_\ell(x) - \varphi_\ell(0)$$

FROM THIS WE GET

$$\tau A \varphi_3(\tau A) = \varphi_2(\tau A) - \frac{1}{2} I \qquad \tau A \varphi_2(\tau A) = \varphi_1(\tau A) - I$$

$$\frac{1}{2} \tau A \varphi_2(\tau A) = \frac{1}{2}\varphi_1(\tau A) - \frac{1}{2} I$$

$\ominus$

$$\tau A \left( \frac{1}{2} \varphi_2(\tau A) - \varphi_3(\tau A) \right) = \frac{1}{2} \varphi_1(\tau A) - \varphi_2(\tau A)$$

$$\boxed{\frac{1}{2} \varphi_1(\tau A) - \varphi_2(\tau A) = \tau A \left( \underline{\frac{1}{2} \varphi_2(\tau A) - \varphi_3(\tau A)} \right)}$$

SO $(\sigma)$ IS VALID FOR $c_1 = \frac{1}{2}$ AND $\eta =$ .

GOING BACK TO OUR EXPANSION WE GET $\left(c_1 = \frac{1}{2}\right)$

$$y_{n+1} - y(t_{n+1}) = e^{\tau A}(y_n - y(t_n)) + \underline{\tau^3 \left( \frac{1}{2}\varphi_2(\tau A) - \varphi_3(\tau A) \right) A g'(t_n)}$$

$$\underline{+ \tau \varphi_1(\tau A) \int_0^{c_1 \tau} g''(t_n + \sigma)(c_1 \tau - \sigma) d\sigma - \int_0^\tau e^{(\tau - s)A} \int_0^s g''(t_n + s)(s - \sigma) d\sigma \, ds}$$

SIMILARLY TO THE PROOF OF EXP. EULER WE DEFINE

$$\varepsilon_n \doteq y_n - y(t_n) \qquad \varepsilon_{n+1} = y_{n+1} - y(t_{n+1})$$

$$\delta_{n+1} \doteq$$

SO THAT $\varepsilon_{n+1} = e^{\tau A} \varepsilon_n + \delta_{n+1}$. NOW LOOK AT LECTURE 2 AND YOU SOLVE THIS RECURSION AS

$$\boxed{\varepsilon_m = \sum_{j=0}^{m-1} e^{j\tau A}\, \delta_{m-j}}$$

THEN
$$\|\varepsilon_m\| \le \sum_{j=0}^{m-1} \|e^{j\tau A}\| \, \|\delta_{m-j}\| \qquad \le C$$

$$\le C \sum_{j=0}^{m-1} \|\delta_{m-j}\|$$

$\le C$

$\|e^{tA}\| \le e^{t^*\|A\|}$

$\le \|A\| \, \|g'(t_-)\|$

$A \approx \partial_{xx}$ ⊕ H.DIR

$A \approx i\partial_{xx}$ ⊕ H.DIR

$\le C$ $\quad \le C$

$$\|\delta_{m-j}\| \le \tau^3 \left\| \frac{1}{2}\varphi_2(\tau A) - \varphi_3(\tau A)\right\| \|A g'(t_\otimes)\| \qquad \| \; c\tau^3$$

$$+ \tau \|\varphi_1(\tau A)\| \int_0^{c_1\tau} \|g''(t_\otimes + \sigma)\| \, |(c_1\tau - \sigma)| \, d\sigma \qquad \| \; c\tau^3$$

$$+ \left\| \int_0^{\tau} e^{(\tau - s)A} \int_0^{s} g''(t_\otimes + \sigma)(s-\sigma)\, d\sigma \right\| \qquad \| \; c\tau^3$$

$$\le c\tau^3$$

$\le m$

$\le m = \frac{t^*}{\tau}$

SO THAT
$$\|\varepsilon_m\| \le \sum_{j=0}^{m-1} c\tau^3 \le c\tau^3 \sum_{j=0}^{m-1} 1 \le c\tau^2$$

SO WE GET A SECOND ORDER METHOD.

## THEOREM 2 (CONVERGENCE OF EXPONENTIAL QUADRATURE RULE 1 POINT)

ASSUME $g$ SUFFICIENTLY OFTEN DIFFERENTIABLE WITH BOUNDED DERIVATIVES. THEN FOR INTEGRATOR $(\square)$

- IF $c_1 \neq \frac{1}{2}$ $\quad \|y(t_n) - y_n\| \le c\tau$ $\qquad (t_n \in [0, t^*])$
- IF $c_1 = \frac{1}{2}$ $\quad \|y(t_n) - y_n\| \le c\tau^2$

$C$ CONSTANT INDEPENDENT OF $m$, BUT MAY BE DEPENDENT ON $t^*$