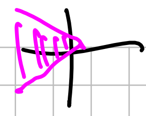
# LECTURE 2 (14/01/25)

L-STABLE

A($\alpha$)-STABLE

I-STABLE



WE SAW THAT SOME METHODS ARE A-STABLE WHILE SOME OTHERS ARE NOT

→ FORWARD EULER

- EXPLICIT EULER IS NOT A-STABLE
- IMPLICIT EULER IS A-STABLE

↳ BACKWARD EULER

DO EXPLICIT A-STABLE METHODS EXIST?

YES ⟹ EXPONENTIAL INTEGRATORS

WE CONSIDER THE FOLLOWING SYSTEM OF ODES

$$(1)\quad \begin{cases} y'(t) = Ay(t) + g(y(t)) = f(y(t)), & t \in (0, t^*] \\ y(0) = y_0 \end{cases}$$

- $y(t) \in \mathbb{C}^N$ UNKNOWN
- $A \in \mathbb{C}^{N\times N}$ WHICH ACCOUNTS FOR THE STIFFNESS
- $g(y(t)) \in \mathbb{C}^N$ NON STIFF

WE ASSUME EXISTENCE AND UNIQUENESS OF SOLUTION

→ DUHAMEL PRINCIPLE

THE VARIATION-OF-CONSTANTS FORMULA GIVES US THE EXACT SOLUTION OF SYSTEM (1)

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} g(y(s))\, ds \qquad (2)$$

$e^X$ IS THE MATRIX EXPONENTIAL $\left( e^X = \sum_{k=0}^{+\infty} \frac{X^k}{k!} \right.$

$X \in \mathbb{C}^{N\times N}$

INDEED

$$y(t) = e^{tA} y_0 + e^{tA}\int_0^t e^{-sA} g(y(s))\,ds$$

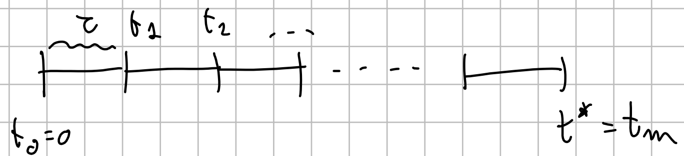
$$y'(t) = \underline{\underline{A}}\, e^{tA} y_0 + \underline{\underline{A}}\, e^{tA}\int_0^t e^{-sA} g(y(s))\,ds + e^{tA}\,\boxed{\frac{d}{dt}\int_0^t e^{-sA} g(y(s))\,ds} \to e^{-tA} g(y(t))$$

$$= A y(t) + \cancel{e^{tA}}\,\cancel{e^{-tA}}\, g(y(t)) = A y(t) + g(y(t)) \quad \checkmark$$

$$y(0) = y_0 \quad \checkmark$$

WE INTRODUCE A TIME DISCRETIZATION ($\tau$ CONSTANT)

$$t_0 = 0 < t_1 = \tau < t_2 = 2\tau < \dots < t_n = n\tau < \dots < t_m = t^*$$




WE WRITE THE $\underline{\text{EXACT}}$ SOLUTION UP TO $t_{n+1}$ STARTING FROM $t_n$

$$y(t_{n+1}) = e^{\tau A}\, \underline{\underline{y(t_n)}} + \int_0^\tau e^{(\tau-s)A}\, \underbrace{g(\underline{\underline{y(t_n+s)}})}\, ds$$

$$g(y(t_n+s)) \approx g(y(t_n))$$

IF WE CALL $y_n \approx y(t_n)$ $\quad y_{n+1} \approx y(t_{n+1})$ THEN WE GET

$$\boxed{y_{n+1} = e^{\tau A} y_n + \left(\int_0^\tau e^{(\tau-s)A}\, ds\right) g(y_n)} \qquad (*)$$

$$\int_0^\tau e^{(\tau-s)A}\,ds \underset{\vartheta=\frac{s}{\tau}}{=} \tau\int_0^1 e^{(1-\vartheta)\tau A}\,d\vartheta = \tau\left[-e^{(1-\vartheta)\tau A}(\tau A)^{-1}\right]_0^1$$

$$= \tau\left[(I-e^{\tau A})(\tau A)^{-1}\right]$$

$$X\varphi_1(X)=e^X-I$$

$$= \tau\varphi_1(\tau A)$$

$$(*) \Leftrightarrow \boxed{y_{n+1}=e^{\tau A}y_n+\tau\varphi_1(\tau A)\,g(y_n)} \qquad (EE)$$

EXPONENTIAL EULER

THE $\varphi_1$ FUNCTION IS JUST A SPECIAL CASE OF THE SO-CALLED $\varphi$-FUNCTIONS WHICH MAY BE DEFINED AS

$l\in\mathbb{N}$

$$\varphi_l(X)=\begin{cases} e^X, & l=0\\ \dfrac{1}{(l-1)!}\displaystyle\int_0^1 e^{(1-\vartheta)X}\vartheta^{l-1}\,d\vartheta, & l>0\end{cases}$$

$$\varphi_l(X)=\sum_{k=0}^{+\infty}\frac{X^k}{(k+l)!} \qquad \varphi_1(X)=\sum_{k=0}^{+\infty}\frac{X^k}{(k+1)!}$$

A COUPLE OF FEATURES

$$\boxed{y_{n+1}=e^{\tau A}y_n+\tau\varphi_1(\tau A)\,g(y_n)}$$

- IF $A\equiv \mathbb{O}$ $\Rightarrow$ $y_{n+1}=y_n+\tau g(y_n)$ $\rightarrow$ EXPLICIT EULER
- IF WE CONSIDER $g(y(t))=\mathbf{b}$

$$(\Delta)\ \begin{cases} y'(t)=Ay(t)+b\\ y(0)=y_0\end{cases} \Rightarrow y(t)=e^{tA}y_0+t\varphi_1(tA)\,\mathbf{b}$$

SO EXPONENTIAL EULER IS EXACT ON $(\Delta)$

IF THEN $b \equiv 0$ THEN WE ARE INTEGRATING EXACTLY

$$\begin{cases} y'(t) = Ay(t) \\ y(0) = y_0 \end{cases}$$

$\Rightarrow$ BY CONSTRUCTION EXPONENTIAL EULER IS A-STABLE

$\Rightarrow$ WELL SUITED FOR STIFF SYSTEMS

- $$y_{n+1} = e^{\tau A} y_n + \tau \varphi_1(\tau A) g(y_n) \leftarrow$$

$$\Leftrightarrow$$

$$y_{n+1} = y_n + \tau \varphi_1(\tau A)(A y_n + g(y_n))$$
$$\overset{!}{=} y_n + \tau \varphi_1(\tau A) f(y_n) \leftarrow$$

INDEED:

$$X \varphi_1(X) = e^X - I$$

$$y_{n+1} = y_n + \tau \varphi_1(\tau A) A \, y_n + \tau \varphi_1(\tau A) g(y_n)$$
$$\overset{!}{=} y_n + (e^{\tau A} - I) y_n + \tau \varphi_1(\tau A) g(y_n)$$
$$\overset{!}{=} e^{\tau A} y_n + \tau \varphi_1(\tau A) g(y_n)$$

NOW WE PROVE THAT (EE) IS INDEED A FIRST ORDER METHOD. TO DO THIS WE WILL AVAIL OF THE FOLLOWING RESULT

**LEMMA 1 (DISCRETE GRONWALL LEMMA)**

LET $\tau > 0$, $t^* > 0$, $0 \le t_n = n\tau \le t^*$. ASSUME THAT THE SEQUENCE OF NON NEGATIVE NUMBERS $\gamma_n$ SATISFIES

$$\gamma_n \le a \tau \sum_{j=1}^{n-1} t_{n-j}^{-\rho} \gamma_j + b t_n^{-\sigma}$$

FOR $\rho \ge 0$, $\sigma < 1$, $a \ge 0$, $b \ge 0$. THEN THE ESTIMATE

$$\gamma_n \le c b t_n^{-\sigma}$$

HOLDS FOR C POSSIBLY DEPENDENT ON $\rho, \sigma, a$ AND $t^*$.

FRÉCHET

## THEOREM 1 ( EXPONENTIAL EULER)

LET $g$ IS DIFFERENTIABLE, WITH UNIFORMLY BDD DERIVATIVE. THEN, THE EXPONENTIAL EULER METHOD (EE) IS FIRST-ORDER ACCURATE. THAT IS

$$\|y(t_n) - y_n\| \leq C\tau$$

WHERE $C$ MAY DEPEND ON $t^*$ BUT NOT ON $n$ $(0 \leq t_n \leq t^*)$

PROOF

RECALL THE VARIATION-OF-CONSTANTS FORMULA

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau-s)A} g(y(t_n+s))\,ds$$

INTRODUCE THE NOTATION $h(t) \doteq g(y(t))$ SO THAT WE GET

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau-s)A} h(t_n+s)\,ds$$

NOW WE TAYLOR EXPAND $h$

$$h(t_n+s) = h(t_n) + \int_0^s h'(t_n+\sigma)\,d\sigma$$

THEN

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \tau\varphi_1(\tau A) h(t_n) + \int_0^\tau e^{(\tau-s)A} \int_0^s h'(t_n+\sigma)\,d\sigma\,ds$$

NOW WE COMPARE WITH THE NUMERICAL SOLUTION $y_{n+1}$

$$y(t_{n+1}) - y_{n+1} = e^{\tau A}(y(t_n) - y_n) + \tau\varphi_1(\tau A)(h(t_n) - g(y_n)) + \int_0^\tau e^{(\tau-s)A} \int_0^s h'(t_n+\sigma)\,d\sigma\,ds$$

LET US CALL
$$\varepsilon_{n+1} \doteq y(t_{n+1}) - y_{n+1}$$
$$\delta_{n+1} \doteq \int_0^{\tau} e^{(\tau-s)A} \int_0^{s} h'(t_n+\sigma)\,d\sigma\,ds$$

SO THAT WE HAVE

$$\varepsilon_{n+1} = e^{\tau A}\varepsilon_n + \tau\varphi_1(\tau A)\left(h(t_n) - g(y_n)\right) + \delta_{n+1} \qquad (\star)$$

$$\varepsilon_n = \dots\ \varepsilon_0 + \dots \qquad\qquad \varepsilon_0 \doteq y(t_0) - y_0 = 0$$

$n=0$
$$\varepsilon_1 = \cancel{e^{\tau A}\varepsilon_0} + \tau\varphi_1(\tau A)\cancel{\left(h(t_0) - g(y_0)\right)} + \delta_1$$
$$h(t_0) = g(y(t_0)), \quad y(t_0) = y_0$$
$$\varepsilon_1 = \delta_1$$

$n=1$
$$\varepsilon_2 = e^{\tau A}\varepsilon_1 + \tau\varphi_1(\tau A)\left(h(t_1) - g(y_1)\right) + \delta_2$$
$$= e^{\tau A}\delta_1 + \tau\varphi_1(\tau A)\left(h(t_1) - g(y_1)\right) + \delta_2$$

$n=2$
$$\varepsilon_3 = e^{\tau A}\varepsilon_2 + \tau\varphi_1(\tau A)\left(h(t_2) - g(y_2)\right) + \delta_3$$
$$= e^{2\tau A}\delta_1 + e^{\tau A}\tau\varphi_1(\tau A)\left(h(t_1) - g(y_1)\right) + e^{\tau A}\delta_2 + \tau\varphi_1(\tau A)\left(h(t_2) - g(y_2)\right) + \delta_3$$

BY INDUCTION YOU CAN PROVE THAT THE RECURSION $(\star)$ IS SOLVED AS

$$\varepsilon_n = \tau\varphi_1(\tau A)\sum_{j=1}^{n-1} e^{(n-j-1)\tau A}\left(h(t_j) - g(y_j)\right) + \sum_{j=1}^{n} e^{(n-j)\tau A}\delta_j$$

NOW WE INTRODUCE SOME BOUNDS

$$\|\varepsilon_n\| \le \tau\|\varphi_1(\tau A)\|\sum_{j=1}^{n-1}\|e^{(n-j-1)\tau A}\|\,\|h(t_j) - g(y_j)\| + \sum_{j=1}^{n}\|e^{(n-j)\tau A}\|\,\|\delta_j\|$$

BY POWER SERIES DEF

- $\|\varphi_1(\tau A)\| \leq \varphi_1(\tau\|A\|) \leq \varphi_1(t^*\|A\|) \leq C$
- $\|e^{J\tau A}\| \leq e^{J\tau\|A\|} \leq e^{t^*\|A\|} \leq C$
- $\|h(t_J) - g(y_J)\| = \|g(y(t_J)) - g(y_J)\|$

$$\underset{g \text{ LIPSCHITZ}}{\leq} C_L \|y(t_J) - y_J\| = C_L \varepsilon_J$$

- $\|\delta_J\| = \left\| \int_0^\tau e^{(\tau-s)A} \int_0^s h'(t_{J-1}+\sigma)\, d\sigma\, ds \right\| \leq \int_0^\tau \underbrace{\|e^{(\tau-s)A}\|}_{\leq C} \int_0^s \underbrace{\|h'(t_{J-1}+\sigma)\|}_{\leq \tilde{C}}\, d\sigma\, ds$

$$\leq C\tau^2$$

! $\|e^{tA}\| \leq C$ WITH $C$ INDEPENDENT OF $\|A\|$ $\Rightarrow$ !!

PUTTING ALL TOGETHER WE HAVE

$$\|\varepsilon_m\| \leq C\tau \sum_{J=1}^{m-1} \|\varepsilon_J\| + \sum_{J=1}^{m} C\tau^2$$

$$\leq C\tau \sum_{J=1}^{m-1} \|\varepsilon_J\| + C\tau^2 m \qquad m \leq \frac{t^*}{\tau}$$

$$\leq C\tau \sum_{J=1}^{m-1} \|\varepsilon_J\| + C\tau$$

BY USING DISCRETE GRONWALL LEMMA WE GET

( SETTING $\gamma_m = \|\varepsilon_m\|$, $a = C$, $p = 0$, $\gamma_J = \|\varepsilon_J\|$

$b = C\tau$ $\quad \sigma = 0$ )

$$\|\varepsilon_m\| \leq C\tau \iff \|y(t_n) - y_n\| \leq C\tau$$

$\square$

FINAL REMARK $\Rightarrow$ IMEX METHODS

$$\begin{cases} y'(t) = Ay(t) + g(y(t)) = f(y(t)) \\ y(0) = y_0 \end{cases}$$

THE IDEA OF $\underbrace{\text{IM}}_{\text{IMPLICIT}}\underbrace{\text{EX}}_{\text{EXPLICIT}}$ METHODS IS TO "SEPARATE" LINEAR STIFF FROM NONLINEAR NONSTIFF.

THAT IS

$$\frac{y_{n+1} - y_n}{\tau} = A y_{n+1} + g(y_n)$$

$$\Leftrightarrow$$

$$y_{n+1} = y_n + \tau A y_{n+1} + \tau g(y_n)$$

$$\Leftrightarrow$$

$$(I - \tau A) y_{n+1} = y_n + \tau g(y_n)$$

THIS IS CALLED BACKWARD - FORWARD EULER

IT IS A FIRST ORDER METHOD A - STABLE

$$y_{n+1} = (I - \tau A)^{-1} y_n + \tau (I - \tau A)^{-1} g(y_n)$$

$\frac{1}{1-x} \approx e^x$ $\qquad$ $\frac{1}{1-x} \approx \varphi_1(x)$ $\qquad$ PADÉ APPROXIMATION $[0,1]$

$x$ SMALL