The Impossible Problem*

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1 Introduction

The Impossible Problem is a very beautiful problem about integer numbers. Its original form was given by Freudenthal [1] before being popularized by Martin Gardner [2]. The two problems are not exactly the same. Gardner's version is as follows:

> Two numbers (not necessarily different) are chosen from the range of positive integers greater than 1 and not greater than 20. Only the sum of the two numbers is given to mathematician S. Only the product of the two is given to mathematician P.
>
> On the telephone S says to P, "I see no way you can determine my sum." An hour later P calls him back to say, "I know your sum." Later S calls P again to report, "Now I know your product."
>
> What are the two numbers?

In the original problem Freudenthal gave an upper bound of 100 (not for the numbers but for their sum). To simplify the problem, Gardner gave the bound of 20. Doing that, "the impossible problem turned out to be literally impossible" as said by Gardner himself [3]. In this text we will explain this point.

It's surprising that the original problem is well posed since one can think that the little conversation has no relevant information about the numbers. But as we shall see, this conversation is plenty of mathematical information.

The Impossible Problem is a 3-in-1 problem. The problems for P and S differ and none of them is the same as ours. Both P and S have additional information (the product for P and the sum for S) that we don't. This makes a big difference.

Section 2 shows how the problem can be solved with a computer's help. Section 3 explains why although the numbers are within the range $[2, 20]$, if we set $N = 20$ then the solution no longer holds. Section 4 shows how P and S can solve their problems with no need of a computer. Finally, Section 5 shows that we can also find the solution without a computer.

2 Solving the problem

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Latest version available at github.com/cassioneri/Impossible.

To solve the problem we shall use a computer program since there are too many cases to consider. A code in C language, available on [7], gives the answer. Later, we shall see another solution that do not require computer assistance.

Now we explain the method of solution. Roughly speaking, we begin with two large sets: the set of all possible products and the set of all possible sums. From each sentence of the conversation we extract information that reduces the sizes of these sets. At the end, we have only one possible pair of product and sum. By solving a second degree polynomial equation, we determine the numbers.

We assume the following fundamental hypothesis, otherwise the problem has no mathematical meaning.

Fundamental Hypothesis: P and S are telling the truth.

This hypothesis deserves a philosophical comment. A reader might think that such hypothesis doesn't hold since S said that P couldn't find the sum and, an hour later, P annouced that she did. Is anyone lying? Not necessarily. The hypothesis is still plausible when time is considered. Initially, P couldn't solve the problem. For some reason, that we shall see below, S knew that and told P. After thinking, P figured out the reason why S was so sure. Only after that (hence, not before S has spoken) P could determine the sum.

Let p be the product and s the sum of the two numbers. Let N be the given upper bound. In this section $N = 100$. We know *a priori* that p is not prime because it's a product of two integers greater than one. In addition, p cannot have any prime divisor bigger than N (for instance, p cannot be $2 \cdot 101$). There are other restrictions, for instance, $p \neq 11 \cdot 11 \cdot 11$ and $p \neq 11 \cdot 13 \cdot 17$. Notice that we know a lot about p. On the other hand, we have no restriction on s.

To simplify the presentation, we make the following definitions.

Definition: Given an integer $m \geq 4$, we say that the pair (i, j) is a *factorization* of m if $m = ij$ with i and j integers in $[2, N]$.

Definition: Given an integer $m \geq 4$, we say that the pair (i, j) is a *decomposition* of m if $m = i + j$ with i and j integers in $[2, N]$.

Notice that these definitions depends on N. So $(2, 8)$ is a factorization of 16 if, and only if, $N \geq 8$. When refering to factorizations and decompositions we make the convention that $(i, j) = (j, i)$.

The restrictions on p seen above are consequences of the following fact: there exist at least one factorization of p. The sets of admissible products and sums are given, respectively, by

$$\begin{aligned} P_0 &:= \{m \in \mathbb{N};\ 4 \leq m \leq N^2 \text{ and } m \text{ has at least one factorization}\}, \\ S_0 &:= \{m \in \mathbb{N};\ 4 \leq m \leq 2N\}. \end{aligned}$$

The program tels us that P_0 has 2880 elements. It tels also that S_0 has 197 elements but this is trivially obtained with no need of a computer.

We now analyse the dialogue. First S says that P cannot determine the sum. This means that p has at least two distinct factorizations. Indeed, we already know that p has a factorization. If (i, j) was the only one, then P would know that $s = i + j$. Hence, the set of possible products reduces to

$$P_1 := \{m \in P_0\ ;\ m \text{ has at least two factorizations}\}.$$

We have $p \in P_1$ and, accordingly to the program, P_1 has 1087 elements. One can easily see that if i and j are primes and $i, j \leq N$, then $ij \in P_0 \setminus P_1$. However, numbers of this form constitute just a small part of $P_0 \setminus P_1$. Other examples of numbers in $P_0 \setminus P_1$ are N^2 and $N(N-1)$.

There is another consequence of S's first sentence: for every decomposition (i, j) of s we have $ij \in P_1$. Indeed, suppose that s has a decomposition (i, j) such that $ij \notin P_1$, that is, (i, j) is the unique factorization of ij. At this stage, S couldn't deny the possibility of p being ij and, if this was the case, then P would know that the numbers were i and j. Hence S wouldn't be sure that P was unable to find the sum. Therefore, the set of admissible sums reduces to

$$S_1 := \{m \in S_0 \ ; \text{ for each decomposition } (i,j) \text{ of } m \text{ we have } ij \in P_1\}.$$

We have $s \in S_1$ and, accordingly to the program,

$$S_1 = \{11, 17, 23, 27, 29, 35, 37, 41, 47, 53\}.$$

Next P says that she knows the sum. It follows that there exists a unique factorization (i,j) of p such that $i+j \in S_1$. Indeed, since $s \in S_1$, p has a factorization (the one we are trying to find) whose sum is in S_1. If there was another such factorization, then P could not know which one was correct. Now, the set of admissible products reduces to

$$P_2 := \{m \in P_1 \ ; \text{ there exists a unique factorization } (i,j) \text{ of } m \text{ with } i+j \in S_1\}.$$

We have $p \in P_2$. The computer program shows that this set has 86 elements.

Finally, S clais that she knows the product. By an argument analougous to that of the last paragraph, we conclude that there exists a unique decomposition (i,j) of s such that $ij \in P_2$. Therefore, the set of admissible sums shrinks to

$$S_2 := \{m \in S_1 \mid \text{there exists a unique decomposition } (i,j) \text{ of } m \text{ with } ij \in P_2\}.$$

We have $s \in S_2$ and the program tells us that $S_2 = \{17\}$, that is, $s = 17$.

At this point S has announced having found the product. Since S is telling the truth, we can find it too. Indeed, now we know that the sum is 17. So we can put ourselves in S's shoes. Another way to find the solution is reiterating the step which gives P_2 from S_1 to construct P_n from S_{n-1} and that which gives S_2 from P_2 to construct S_n from P_n. We repeat until we find $n \in \mathbb{N}$ such that the sets P_n and S_n are singletons. More precisely, for $n \geq 3$, we set

$$P_n := \{m \in P_{n-1} \mid \text{there exists a unique factorization } (i,j) \text{ of } m \text{ with } i+j \in S_{n-1}\}.$$
$$S_n := \{m \in S_{n-1} \mid \text{there exists a unique decomposition } (i,j) \text{ of } m \text{ with } ij \in P_n\}.$$

We obtain $P_3 = \{52\}$ and $S_3 = \{17\}$, hence $p = 52$ and $s = 17$ and then the numbers are 4 and 13.

3 Gardner's mistake

Knowing the answer 4 and 13, intuition would say that the same solution holds if we change, as Gardner did, the upper bound N from 100 to 20. Funilly enough, this is wrong! We shall see that for $N \leq 61$ the numbers 4 and 13 do not solve the problem!

What happens if $N \leq 61$? In this case $17 \notin S_2$. In fact, the computer program shows that $S_2 = \varnothing$ which means the problem has no solution. One can argue that this might be a default of the program and the solution 4 and 13 can still hold. However, let us look at the mathematical argument again. It's as follows:

1. Before the dialogue : $p \in P_0$ e $s \in S_0$;
2. After S's first sentence: $p \in P_1$ e $s \in S_1$;
3. After P's first sentence: $p \in P_2$;

4. After S's second sentence: $s \in S_2$.

Therefore, if the numbers are 4 and 13, then $p = 52 \in P_2 \subset P_1 \subset P_0$ and $s = 17 \in S_2 \subset S_1 \subset S_0$. Suppose now that $N \leq 61$. We shall prove that $52 \notin P_2$ or $17 \notin S_2$ and it will follow that the numbers cannot be 4 and 13. We proceed by contradiction assuming that $52 \in P_2$ and $17 \in S_2$.

We claim that $19, 37 \notin S_1$. Indeed, since 34 has at most one factorization (which is $(2, 17)$) we have $34 \notin P_1$ and thus $19 = 2 + 17 \notin S_1$. Notice that 186 can be written as a product of two natural numbers only in the following ways: $1 \cdot 186$, $2 \cdot 93$, $3 \cdot 62$ and $6 \cdot 31$. Since $N \leq 61$ at most the last form is a factorization of 186 and thus $186 \notin P_1$. It follows that $37 = 6 + 31 \notin S_1$.

Now, let us show that $70 \in P_2$. We can write 70 as $1 \cdot 70$, $2 \cdot 35$, $5 \cdot 14$ and $7 \cdot 10$. As we have seen $37 = 2 + 35$ and $19 = 5 + 14$ are not in S_1 but $17 \in S_1$. Hence $70 \in P_2$.

Finalmente, entre todas as decomposies de 17 temos $(4, 13)$ e $(7, 10)$. Como $52, 70 \in P_2$ temos que $17 \notin S_2$ o que uma contradio.

Finally, among all decompositions of 17 we have $(4, 13)$ and $(7, 10)$. Since $52,\ 70 \in P_2$ we have $17 \notin S_2$, which is a contradiction.

I emphasize the following facts:

1. We have shown only that $(4, 13)$ is no longer a solution. Pehaps another solution arises for $N \leq 61$. But the program in [7] shows that, in fact, $S_2 = \varnothing$ and thus the problem has no solution.

2. Even ther purists who don't accept computer assisted proofs should accept that $(4, 13)$ is not a solution for $N \leq 61$.

3. When I say the problem has no solution, I mean *our* problem has no solution. On the other hand, the problem given to P have solution provided she has good data. The situation for S is not very different from ours. We shall see more details in the next section.

4 Problem for P and S

In this section we consider the other problems, specially, the one given to P. In [7] we give a code in C language that could be used by P (or by you in his place) and also a code for S. For the time being, we go back to assuming that $N = 100$.

Despite the use of a computer to assist our solution, P and S don't need a great computational ability to find the answer. A great memory (specially for S) or at most a pencil and a piece of paper are enough for them.

P knows that $p = 52$ and its factorizations are $(2, 26)$ and $(4, 13)$. Thus she knows that s is either 28 or 17. After S talks for the first time, P find the decompositions of 28: $(2, 26)$, $(3, 25)$, $(4, 24)$, $(5, 23)$, ... She can stop at $(5, 23)$ and conclude that $s \neq 28$ because $(5, 23)$ is the unique factorization of 115 (5 and 23 are primes) and thus $115 \notin P_1$. She can now seek the decompositions of 17 (just to be sure that $s = 17$): $(2, 15)$, $(3, 14)$, $(4, 13)$, $(5, 12)$, $(6, 11)$, $(7, 10)$ and $(8, 9)$. These pairs are factorizations of, respectively, 30, 42, 52, 60, 66, 70 and, 72. All these numbers are in P_1 since each of them has another factorization which differ from the one shown here. Now P can conclude that the sum of numbers are 17 and thus she is able to find the two numbers.

S must work more than P. Nevertheless, she is also able to find the product after some minutes[1].

[1] Playing the role of S, I found the product in about 15 minutes. To simplify my task, in my computations I set $P_0 \setminus P_1$ to have only numbers which are product of two primes.

Now we consider other values of N. As we have seen, for smal values we cannot solve the problem. Analogously, for N large, *e.g.* $N = 866$, we are not able to find the numbers since because $P_n = \{52, 244\}$ and $S_n = \{17, 65\}$ for all $n \geq 3$. The effect of changing N for P is almost null: knowing that $p = 52$ she finds the solution $(4, 13)$ for all values of $N \geq 13$ (which agrees with intuiton). Even the ambiguity that we have for $N = 866$ disappears. In this case, had P been given $p = 244$ then she would find the solution $(4, 61)$.

When $N \leq 25$, $(2, 26)$ is no longer a factorization of 52. In this case, P would have no doubt that the $s = 17$. Hence S wouldn't say that P cannot determine the sum. P could find s but S wouldn't find p as we shall see below.

For $N \leq 61$, S's situation is similar to ours: she cannot find p. The reason is the same as before: $(4, 13)$ and $(7, 10)$ are two decompositions of 17 with products in P_2. Another way to see this fact is the following. Let $N = 61$ and put yourself in P's shoes, first, with $p = 52$ and, second, with $p = 70$. In both cases, after S's first sentence, you would find $s = 17$. Hence S couldn't decide whether $p = 52$ or $p = 70$. In fact there is a third possibility for $N = 61$, namely, $p = 66$. For $N = 866$, S find the $p = 52$, when $s = 17$, or $p = 244$, when $s = 65$.

5 Solving without a computer

Let us see now another way to solve the original problem, where $N = 100$, that doesn't need computer assistence.

Finding P_0 or P_1 manually would be very laborous (recall that, according to the program, these sets have, respectively, 2880 and 1087 elements). Let us find S_1 with no computer but a bit of patience.

To prove that $m \notin S_1$ it suffices to find a decomposition (i, j) of m which is the unique factorization of ij. Let's see some cases (whose verification is left to the reader). For $m = 200$ we have the decomposition $(100, 100)$. For $m = 199$ and $m = 198$ we have $(99, 100)$ and $(99, 99)$, respectively.

For $99 \leq m \leq 197$ we have $2 \leq m - 97 \leq 100$. Then $(i, j) = (97, m - 97)$ is a decomposition of m. Now, 97 is a prime factor of $ij = 97(m - 97)$ and then, either i or j is divisible by 97. Assume, without loss of generality, that i is divisible by 97. In this case, we must have $i = 97$ because, otherwise, i has another factor and then $i \geq 2 \cdot 97 > 100$ which is absurd. Hence $(97, m - 97)$ is the unique factorization of $97(m - 97)$. In the same way, one can show that if $55 \leq m \leq 153$, then $(53, m - 53)$ is the unique factorization of $53(m - 53)$.

In summary, we have shown that if $m \in S_1$, then $m < 55$.

If $m \leq 54$ is even, then one can verify, case-by-case, that m has a decomposition (i, j) with i and j primes[2]. Such decomposition is the unique factorization of ij. Analogously, for $m \in \{5, 7, 9, 13, 15, 19, 21, 25, 31, 33, 39, 43, 45\}$, the decomposition $(i, j) = (2, m - 2)$ has both i and j primes, meaning this is the only factorization of $2(m - 2)$. Finally, $(17, 34)$ is a decomposition of 51 which is the unique factorization of $17 \cdot 34 = 2 \cdot 17^2$ (since $17^2 > 100$).

It follows that $S_1 \subset \tilde{S}_1 := \{11, 17, 23, 27, 29, 35, 37, 41, 47, 53\}$. Now, we shall prove the opposite inclusion to conclude that $S_1 = \tilde{S}_1$. For this, we need to prove that for all $m \in \tilde{S}_1$ and every decomposition (i, j) of m there exists a distinct factorization of ij.

Let $m \in \tilde{S}_1$ and (i, j) be a decomposition of m. Since m is odd, either i of j is even. Without loss of generality, we assume i is even. We consider first the case $i \geq 4$ and then the case $i = 2$.

If $i \geq 4$, then $j = m - i \leq m - 4 \leq 53 - 4 < 50$ and we obtain $2j < 100$. Hence, $(i/2, 2j)$ is a factorization of ij which distinct from (i, j) unless $i = 2j$ and $m = 3j$. The unique multiple of 3 in $\tilde{S}_1$ is

[2] The general case, *i.e*, with no upper bound for m, is an open problem known as the Goldbach's conjecture [4].

27 for which the decomposition in question is $(18, 9)$ but its product also admits the factorization $(6, 27)$.

If $i = 2$, then we have to present a factorization of $2(m-2)$ distinct from $(2, m-2)$, which is done in Table 1.

m	$(2, m-2)$	$2(m-2)$	Distinct decomposition
11	$(2, 9)$	18	$(3, 6)$
17	$(2, 15)$	30	$(3, 10)$
23	$(2, 21)$	42	$(3, 14)$
27	$(2, 25)$	50	$(5, 10)$
29	$(2, 27)$	54	$(3, 18)$
35	$(2, 33)$	66	$(3, 22)$
37	$(2, 35)$	70	$(5, 14)$
41	$(2, 39)$	78	$(3, 26)$
47	$(2, 45)$	90	$(3, 30)$
53	$(2, 51)$	102	$(3, 34)$

Table 1: Completing the proof that $\tilde{S}_1 \subset S_1$.

Finding P_2 would also be very labourous and we go directly to S_2. By definition $m \in S_2$ if there exists a unique decomposition (i, j) of m which is also the unique factorization of ij whose sum belongs to S_1. Hence, to conclude that $m \in S_1 \setminus S_2$ it suffices to find two distinct decompositions of m that are the unique factorizations of their product having sum in S_1. This is done for all $m \in S_1 \setminus \{17\}$ in Table 2. It follows that $S_1 \setminus \{17\} \subset S_1 \setminus S_2$ or, in other words, $S_2 \subset \{17\}$. Since $s \in S_2$ we have $S_2 = \{17\}$ and $s = 17$.

m	1^{st} decomp.	Product	Other fact(s).	Sum(s)	2^{nd} decomp.	Product	Other fact(s).	Sum(s)
11	$(2, 9)$	18	$(3, 6)$	9	$(4, 7)$	28	$(2, 14)$	16
23	$(4, 19)$	76	$(2, 38)$	40	$(10, 13)$	130	$(2, 65)$, $(5, 26)$	67, 31
27	$(2, 25)$	50	$(5, 10)$	15	$(4, 23)$	92	$(2, 46)$	48
29	$(4, 25)$	100	$(2, 50)$, $(5, 20)$	52, 25	$(6, 23)$	138	$(2, 69)$, $(3, 46)$	71, 49
35	$(4, 31)$	124	$(2, 62)$	64	$(6, 29)$	174	$(2, 87)$, $(3, 58)$	89, 61
37	$(6, 31)$	186	$(2, 93)$, $(3, 62)$	95, 65	$(8, 29)$	232	$(4, 58)$	62
41	$(6, 35)$	210	$(3, 70)$	73	$(7, 34)$	238	$(14, 17)$	31
47	$(4, 43)$	172	$(2, 86)$	88	$(6, 41)$	246	$(3, 82)$	85
53	$(4, 49)$	196	$(2, 98)$	100	$(10, 43)$	430	$(5, 86)$	91

Table 2: The proof that $S_2 \subset \{17\}$. Notice that no number in either *Sum(s)*' columns are in S_1.

We know that $p = ij$ for some decomposition (i, j) of 17. In addition, $p \in P_2$, that is, (i, j) is the unique factorization of p whose sum is in S_1. We shall show now that $(4, 13)$ is the only decomposition of 17 with this property and it will follow that the numbers are 4 and 13.

Indeed 52 has only two factorizations $(2, 26)$, $(4, 13)$ but only the latter has the sum in S_1. It remains to prove that every other decomposition (i, j) of 17 has a product that admits a factorization distinct from (i, j) with sum in S_1. This is done in Table 3.

Decomposition	Product	Factorization	Sum
$(2, 15)$	30	$(5, 6)$	11
$(3, 14)$	42	$(2, 21)$	23
$(5, 12)$	60	$(3, 20)$	23
$(6, 11)$	66	$(2, 33)$	35
$(7, 10)$	70	$(2, 35)$	37
$(8, 9)$	72	$(3, 24)$	27

Table 3: Any decomposition of 17 but $(4, 13)$ have a product that doesn't belong to P_2.

References

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[7] URL: www.labma.ufrj.br/~cassio/f-impossible.html