Testing for homogeneity of a Poisson process

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5ModIA

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The situation is getting worse

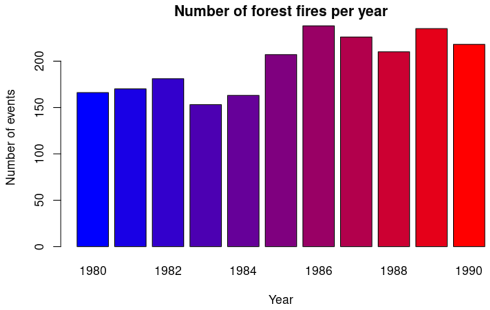


Figure 1 – Slight increase over the last few years

An increasing trend : costs are becoming higher

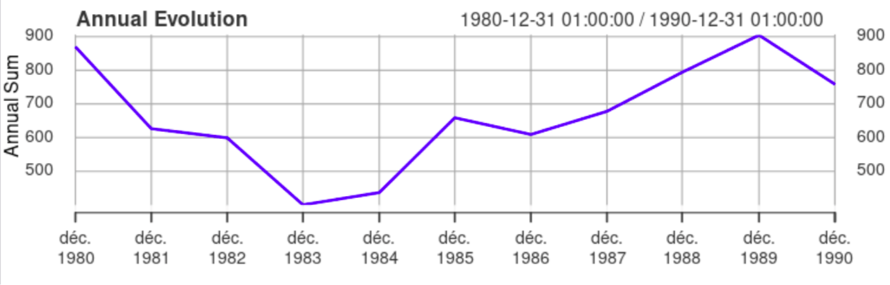


Figure 2 – A rising trend : costs over time

Changing climate [1]

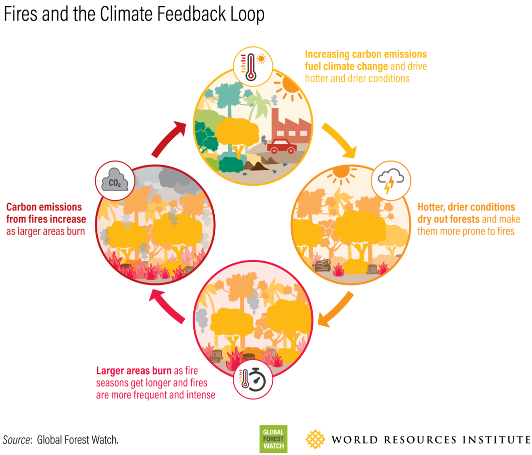


Figure 3 – How global warming impacts forest fires occurencies

Paving the way

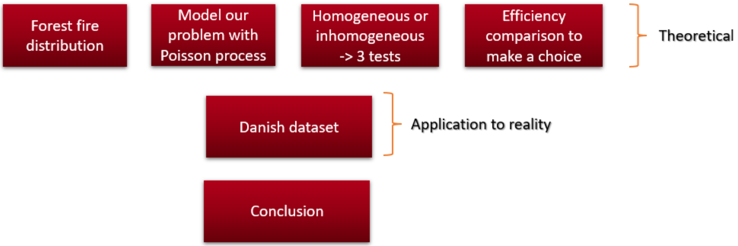


Figure 4 – Telling a clear story

Diving into mathematics

Objective : Are forest fires constant over time ?

Characteristics of a forest fire

- Random, infrequent
- Temporal independence of fires
- Constant or varies over time

$\rightsquigarrow$ **Poisson process**

Linear and non linear intensity function

2 types of Poisson process

- Homogeneous (λ constant over time)
- Inhomogeneous (λ varies over time)

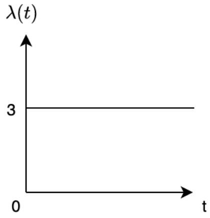


Figure 5 – Linear intensity function of a homogeneous Poisson process

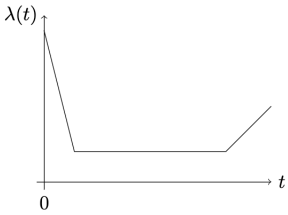


Figure 6 – Non linear intensity function of a inhomogeneous Poisson process

Modeling the occurrence of forest fires

Objective of the project

$$H_0 : \lambda \quad \text{constant} \quad \text{against} \quad H_1 : \lambda \quad \text{varies over time}$$

Laplace test [2]

Central limit theorem

This theorem establishes that for i.i.d. random variables, the sampling distribution tends towards the normal distribution.

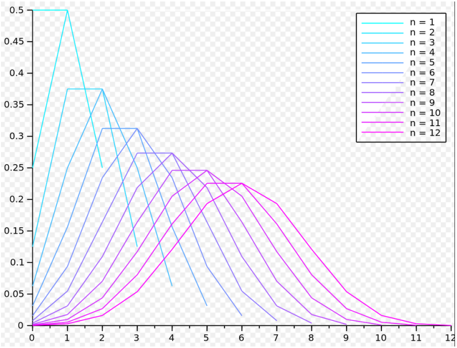


Figure 7 – Illustration of the central limit theorem

Laplace test [3]

Intensity function and hypothesis

Intensity function : $\lambda(t) = \exp(\beta t)$ for $\beta \geq 0$

Test :

$$H_0 : \beta = 0 \quad \text{against} \quad H_1 : \beta > 0$$

Test statistics and rejection zone

- Test statistics :

$$L := \sum_{i=1}^{n} \left(\frac{T_i}{T^*} \right) \tag{1}$$

with T_is the first arrival times, T^* the time period over which we observe the Poisson process.

- Rejection zone :

$$R_\alpha = \left\{ L \geq \sqrt{\frac{n}{12}} \cdot z_{1-\alpha} + \frac{n}{2} \right\} \tag{2}$$

Weibull test [4]

Intensity function and hypothesis

Intensity function : $\lambda(t) = \beta \cdot t^{\beta-1}$ for $\beta > 0$

Test :

$$H_0 : \beta = 1 \quad \text{against} \quad H_1 : \beta > 1$$

Test statistics and rejection zone

- Test statistics :

$$Z = 2\sum_{i=1}^{n} \log\left(\frac{T^*}{T_i}\right) \sim \chi^2(2n) \tag{3}$$

- Rejection zone :

$$R_\alpha = \{Z \leq s_\alpha\} = \alpha \tag{4}$$

with s_α the α-quantile of a $\chi^2(2n)$

Barlow test [5]

Barlow test

- The test stastics is :

$$F = \frac{(n-d)T_d}{d(T_n - T_d)} \tag{5}$$

Theoretical simulation [6]

Use of the exponential intensity : $\lambda(t) = \exp(\beta t)$ for $\beta \geq 0$

Table 1 – Theoretical simulation on Laplace and Weibull test for different values of T^* and n with the exponential intensity

Test with exponential intensity			
T^*	1	2	4
n	1.71	6.38	53.59
L (Laplace)	0.2467	0.3877	1
Z (Weibull)	0.2643	0.3703	1

The Laplace test is better with the exponential intensity.

Theoretical simulation [6]

Use of the logarithmic intensity : $\theta * log(t+1)$

Table 2 – Theoretical simulation on Laplace and Weibull test for different values of T^* and n with the logarithmic intensity

Test with logarithmic intensity			
T^*	10	15	25
n	15.37	28.36	58.71
L (Laplace)	0.4061	0.5478	0.7543
Z (Weibull)	0.4791	0.674	0.8812

The Weibull test is better with the logarithmic intensity

Theoretical simulation [6]

Use of the weibull intensity : $\theta * t^{(\theta-1)}$

Table 3 – Theoretical simulation on Laplace and Weibull test for different values of θ and n with the Weibull intensity

Test with Weibull intensity			
θ	1	2	4
n	2	4	16
L (Laplace)	0.175	0.2912	0.9991
Z (Weibull)	0.177	0.3048	0.9992

The Weibull test is better with the weibull intensity.

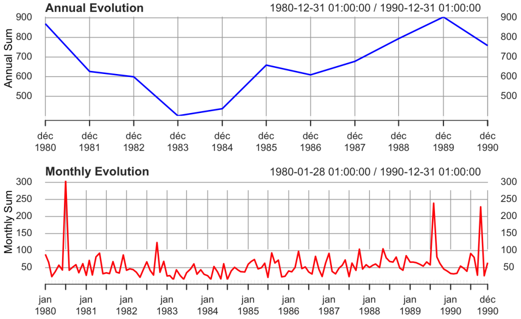
1. Theoretical part

2. Simulation using the Danish dataset
   - Results

Danish dataset

- Dataset : Costs of forest fires between 1980 and 1990.
- Goal : Determine if the frequency of forest fire remains the same over time or not.

Figure 8 – Variation of the cost of forest fires annually and monthly



Linear variation of the number of forest fire

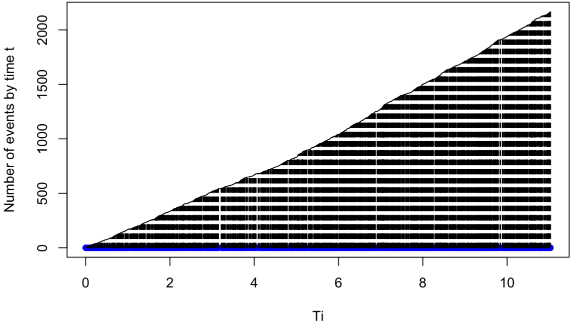


Figure 9 – Cumulative number of forest fire from 1980-01-03 to 1990-12-31

Rejection of H_0

Table 4 – Estimated p-value of Laplace and Weibull tests on the Danish dataset with $T^* = 11 + \frac{3}{365}$, $n = 2197$

	Laplace test	Weibull test	Barlow test
p-value	$9.35 * 10^{-9}$	$4.30 * 10^{-7}$	$3.97 * 10^{-11}$

p-value < 0.05 => **we reject** H_0 (homogeneous Poisson process) at a 5% risk level.

Limitations and future work

- Consider a longer **timeframe**, and explore forest fire occurrences **across Europe**.
- When interpreting graphs, it is crucial to pay attention to **details** and consider the **context** for accurate understanding.

References

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[2] Théorème central limite. (January 19, 2023) In Wikipedia. https ://fr.wikipedia.org/wiki/Th

[3] Ascher, H. E., and Feingold, H. (1978), "Application of Laplace's Test to Repairable System Reliability", in *Proceedings of the International Conference on Reliability and Maintainability* (Societe Pour la Diffusion des Sciences et des Arts, France), 219-225.

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[6] Bain, L. J., Engelhardt, M. and Wright F. T. (1985), "Tests for an Increasing Trend in the Intensity of a Poisson Process : A Power Study," in *Journal of the American Statistical Association, Vol. 80, No. 390,* pp. 419-422

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Diving into mathematics

Homogeneous Poisson process

A Poisson process is homogeneous if the intensity $\lambda > 0$ (average number of events) is constant and satisfies :

- $N_0 = 0$
- Independant increments
- Stationary increments
- $\forall t > 0, \forall s > 0,$
$$N_{s+t} - N_t \sim \mathcal{P}(\lambda t)$$

Inhomogeneous Poisson process

Inhomogeneous Poisson Process

A Poisson process is inhomogeneous if the intensity $\lambda \in \mathbf{R}_+$ varies over time :

- $N_0 = 0$
- Independant increments
- Stationary increments
- $\forall t > 0$,

$$N_t \sim \mathcal{P}(\int_0^t \lambda(u)\, du)$$