Quiz 9

1. $W(5,\cos^2 t,\cos 2t) = \begin{vmatrix} 5 & \cos^2 t & \cos 2t \\ 0 & -2\sin t\cos t & -2\sin 2t \\ 0 & -2\cos 2t & -4\cos 2t \end{vmatrix}$

$W = 5\begin{vmatrix} -\sin 2t & -2\sin 2t \\ -2\cos 2t & -4\cos 2t \end{vmatrix} = 0$

$\begin{cases} y_1(t)=5 \\ y_2(t)=\cos^2 t \\ y_3(t)=\cos 2t \\ W(y_1(t).y_2(t).y_3(t))(t)=0 \end{cases}$ $, t\in I$

assume the contrary or $y_1.y_2.y_3$ form a fundamental set of solutions on I

$\Rightarrow y_3(t) = 2\cos^2 t - 1$

$= 2y_2(t) - \frac{1}{5}y_1(t)$ $\Rightarrow W(y_1(t),y_2(t).y_3(t))(t)=0$ $, \forall t\in I$ #

$\therefore y_1.y_2.y_3$ are linear dependent on I.

2. $y''' - 3y'' + y' + y = 0$

$\Rightarrow r^3 - 3r^2 + r + 1 = 0$

$\Rightarrow (r-1)(r^2-2r-1) = 0$

$\Rightarrow r = 1$ or $1\pm\sqrt{2}$

$y = C_1e^{t} + C_2e^{(1+\sqrt{2})t} + C_3e^{(1-\sqrt{2})t}$ #

3. $(A-rI)\xi = 0$

$\begin{bmatrix} 1-r & 2 \\ 3 & -4-r \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow (1-r)(-4-r)+6=0$

$\Rightarrow r^2+3r+2=0$

$\Rightarrow (r+1)(r+2)=0$

$\Rightarrow r=-1$ or -2

① $r=-1$

$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow x^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}$

$\Rightarrow \xi_2 = \xi_1 \Rightarrow \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② $r=-2$

$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow x^{(2)}(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}$

$\Rightarrow 2\xi_2 = 3\xi_1 \Rightarrow \xi_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

By ①.② $x = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}$ #

$\Rightarrow \begin{cases} x_1(t) = C_1e^{-t} + 2C_2e^{-2t} \\ x_2(t) = C_1e^{-t} + 3C_2e^{-2t} \end{cases}$

$\xRightarrow{t\to\infty} \begin{cases} \lim_{t\to\infty} x_1(t) = 0 \\ \lim_{t\to\infty} x_2(t) = 0 \end{cases}$ #

4. $x' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} x$, $x_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow (5-r)(1-r)+3=0$

$\Rightarrow r^2-6r+8=0$

$\Rightarrow (r-2)(r-4)=0$

$\Rightarrow r=2 \text{ or } 4$

① $r=2$

$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow x^{(2)}(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$

$\Rightarrow 3\xi_1 = \xi_2 \Rightarrow \xi^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

② $r=4$

$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow x^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$\Rightarrow \xi_1 = \xi_2 \Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

By ①.② $x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} c_1 = \frac{7}{2} \\ c_2 = -\frac{3}{2} \end{cases}$

$\therefore x = \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$ #

(as $t \to \infty$. unbounded)

5. $x' = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} x$

$\begin{bmatrix} 1-r & i \\ -i & 1-r \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow (1-r)^2 + i^2 = 0$

$\Rightarrow r^2 - 2r = 0$

$\Rightarrow r = 0 \text{ or } 2$

① $r=0$

$\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\Rightarrow \xi_1 + \xi_2 i = 0$

② $r=2$

$\begin{bmatrix} -1 & i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\Rightarrow -\xi_1 + i\xi_2 = 0$

By ①.② $x = c_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{2t}$ #