1. Reduce this system to upper triangular form by two row operations:

$$2x + 3y + z = 8$$
$$4x + 7y + 5z = 20$$
$$-2y + 2z = 0$$

Circle the pivots. Solve by back substitution for z, y, x.

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 3 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right] \quad \therefore (x, y, z) = (2, 1, 1) \#$$

2. Suppose elimination takes A to U without row exchanges.
① Then row j of U is a combination of which rows of A?
② If $Ax = 0$, is $Ux = 0$?
③ If $Ax = b$, is $Ux = b$?
④ If A starts out lower triangular, what is the upper triangular U?

① row 1. 2. 3. 4 ⋯ j row of A # ($\because$ Guassian elimination starts form up.)

② Yes

$\because$ Ux is a combination of rows of A, it doesn't matter how 0 combines, it'll always remain 0 $\therefore$ If $Ax=0$. $Ux=0$ #

③ No

$\because$ Ux is a combination of rows of A, but we can't be sure that whether b still remains b after the combination. $\therefore$ we can't be sure that when $Ax=b$, $Ux=b$ #

④ $U=A^T$

(because Guassian elimination only has "row exchange", don't have to recombine the numbers)

3. $A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$

multiply A times BC. Then multiply AB times C.

$$BC = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A\cdot(BC) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \#$$

$$AB = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix}$$

$$(AB)\cdot C = \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \#$$

4. For which three numbers a will elimination fail to give three pivots?

$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}$ is singular for three values of a.

$$\begin{vmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{vmatrix} = a^3 + \cancel{3a^2} + 8a - \cancel{3a^2} - 2a^2 - 4a^2$$

$$= a^3 - 6a^2 + 8a = a(a^2 - 6a + 8)$$

$$= a(a-4)(a-2)$$

$a \quad -4$
$a \quad -2$

$a = 0.2.4$ #

5. Find A^{-1} and B^{-1} (*if they exist*) by elimination on $\begin{bmatrix} A & I \end{bmatrix}$ and $\begin{bmatrix} B & I \end{bmatrix}$:

$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 3 & 0 & -1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 3 & 0 & -1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 3 & 0 & -1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 0 & 0 & 4 & -1 & -1 & 3 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 3 & 0 & -1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \#$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 3 & -3 & 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 3 & -3 & 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 & -1 & 1 \\ -3 & -3 & 6 & 0 & 0 & 3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 3 & -3 & 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 & -1 & 1 \\ 0 & -6 & 6 & 1 & -1 & 3 \end{array}\right]$$

B^{-1} doesn't exist #

6. Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \text{ and } E_{32}E_{31}E_{21}A = U.$$

Multiply those E's to get one matrix M that does elimination: $MA = U$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \xrightarrow{①\times -4+②} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} \xrightarrow{①\times 2+③} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow{②\times(-2)+③} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \# \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \# \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \#$$

$$M = E_{32}E_{31}E_{21}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} \#$$

7. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

$$\begin{vmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{vmatrix} = 2c^2 + \cancel{8c^2} + 7c^2 - \cancel{8c^2} - c^3 - 14c$$

$$= 9c^2 - c^3 - 14c \qquad c \; -7$$

$$= -c(c^2 - 9c + 14) \qquad c \; -2$$

$$= -c(c-2)(c-7)$$

when A^{-1} doesnt exist $\Rightarrow -c(c-2)(c-7) = 0$

$\therefore c = 0 . 2 . 7 \#$

8. What three matrices E_{21} and E_{12} and D^{-1}reduce $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ to the identity matrix?

Multiply $D^{-1}E_{12}E_{21}$ to find A^{-1}.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \xrightarrow{②-①\times 2} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \xrightarrow[②/2]{①-②} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \#$$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad E_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \#$$

$$DE_{12}E_{21} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

9. If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $Ax = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?

(c) What happens to row 3 in elimination?

令 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ 又 $a_{11} + a_{21} = a_{31}$, $a_{12} + a_{22} = a_{32}$, $a_{13} + a_{23} = a_{33}$ $\Rightarrow (a_{11}+a_{12}+a_{13}+a_{21}+a_{22}+a_{23}) = (a_{31}+a_{32}+a_{33})$

(a) If $Ax = (1,0,0)$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$(a_{11}+a_{12}+a_{13})X = 1$

$(a_{21}+a_{22}+a_{23})X = 0$

$(a_{31}+a_{32}+a_{33})X = 0$

from $(a_{11}+a_{12}+a_{13})+(a_{21}+a_{22}+a_{23}) = (a_{31}+a_{32}+a_{33})$

$\xrightarrow{\cdot x} x(a_{11}+a_{12}+a_{13}) + x(a_{21}+a_{22}+a_{23}) = x(a_{31}+a_{32}+a_{33})$

$\Rightarrow 1 + 0 \neq 0$

so that we know $Ax = (1,0,0)$ doesn't have a solution #

(b)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} X = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\underbrace{(a_{11}+a_{12}+a_{13})X}_{b_1} + \underbrace{(a_{21}+a_{22}+a_{23})X}_{b_2} = \underbrace{(a_{31}+a_{32}+a_{33})X}_{b_3}$$

$\therefore$ when $b_1 + b_2 = b_3$, (b_1, b_2, b_3) might allow a solution to $Ax = b$ #

(c) row 3 = 0 #