獨立

1. Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} V_1 &= V_2 - V_4 & \quad V_4 &= V_2 - V_1 \\ V_2 &= V_3 - V_6 & \Rightarrow \quad V_5 &= V_3 - V_1 \\ V_3 &= V_1 + V_5 & \quad V_6 &= V_3 - V_2 \end{aligned}$$

$$[V_1\ V_2\ V_3] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \#$$

2. Find a basis for each of these subspaces of R^4:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero. 垂直

(c) All vectors that are perpendicular to (1, 1, 0, 0) and (1, 0, 1, 1).

(d) The column space and the nullspace of I (4 by 4).

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \#$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\} \#$

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\} \#$ $\quad \begin{bmatrix} 1 \\ -1 \\ \\ \end{bmatrix} [1\ 1\ 0\ 0] = [0], \quad \begin{bmatrix} 1 \\ -1 \\ \\ \end{bmatrix} [1\ 0\ 1\ 1] = [0]$

$\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$

(d) column space $\rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \#$

null space $\rightarrow \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \#$

3. Choose $x = (x_1, x_2, x_3, x_4)$ in R^4. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including x itself, span a subspace S. Find specific vectors x so that the dimension of S is: (a) zero, (b) one, (c) three, (d) four.

(a) $x = 0$

(b) $x = (1, 1, 1, 1)$

(c) $x = (1, 1, -1, -1)$

(d) $x = (0, 1, 2, 3)$ #

4. Find bases and dimensions for the four subspaces associated with A and B:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \text{and} B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

Ⓐ $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A) = \left\{ x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \middle| x_1 \in \mathbb{R} \right\}$

bases → $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ dimension = 1 #

$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix}$ $C(A^T) = \left\{ x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \middle| x_1 \in \mathbb{R} \right\}$

bases → $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$ dimension = 1 #

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$N(A) \Rightarrow$ bases $\rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ dimension $= 2$ #

$N(A^T) \Rightarrow$ bases $\rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ dimension $= 1$ #

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

$C(B) \Rightarrow$ bases $\rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ dimension $= 2$ #

$$B^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 0 \end{bmatrix}$$

$C(B^T) \Rightarrow$ bases $\rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ dimension $= 2$ #

$N(B) \Rightarrow$ bases $\rightarrow \left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ dimension $= 1$ #

$N(B^T) \Rightarrow$ bases $\rightarrow \{ [0] \}$ dimension $= 0$ #

5. Suppose the m by n matrices A and B have the same four subspaces. If they are both in row reduced echelon form, prove that F must equal G:

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}$$

$m \times n$

$$A = \begin{array}{c} \\ m-1 \left\{ \right. \\ \\ \\ \end{array} \left[\begin{array}{c|c} I & F \\ \hline 0 & 0 \end{array} \right]$$
$$\qquad \underbrace{\quad}_{m-1} \; \underbrace{\quad}_{n+1-m}$$

Each row of A is combined by the rows of B.

Besides, because of the matrix I, the only combination is the same row of B (i.e. row t in A is same with rows in B)

Therefore, $F = G$ #

6. If a subspace S is contained in a subspace V, prove that $S^{\perp}$ contains $V^{\perp}$.

Suppose x in $V^{\perp}$ is perpendicular to any vector in V

$\because S \subset V$, $x \perp S$ $\quad \therefore$ every x in $V^{\perp}$ is also in $S^{\perp}$ #

7. Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

$$\begin{bmatrix} \text{column 1} & \text{column 2} & \text{column 3} \end{bmatrix} = A \qquad \begin{cases} (\text{column 1}) \cdot (\text{column 2}) = 0 \\ (\text{column 2}) \cdot (\text{column 3}) = 0 \\ (\text{column 3}) \cdot (\text{column 1}) = 0 \end{cases} \quad (\text{内積為 } 0)$$

$$A^T \cdot A = \begin{bmatrix} \text{column 1} \\ \text{column 2} \\ \text{column 3} \end{bmatrix} \begin{bmatrix} \text{column 1} & \text{column 2} & \text{column 3} \end{bmatrix}$$

$$= \begin{bmatrix} (\text{column 1})^2 & (\text{column 1}) \cdot (\text{column 2}) & (\text{column 1}) \cdot (\text{column 3}) \\ (\text{column 2}) \cdot (\text{column 1}) & (\text{column 2})^2 & (\text{column 2}) \cdot (\text{column 3}) \\ (\text{column 3}) \cdot (\text{column 1}) & (\text{column 3}) \cdot (\text{column 2}) & (\text{column 3})^2 \end{bmatrix}$$

$$= \begin{bmatrix} (\text{column 1})^2 & 0 & 0 \\ 0 & (\text{column 2})^2 & 0 \\ 0 & 0 & (\text{column 3})^2 \end{bmatrix} \rightarrow \text{a diagonal matrix} \ \#$$

8. Suppose S is spanned by the vectors (1, 2, 2, 3) and (1, 3, 3, 2). ① Find two vectors that span $S_\perp$. ② This is the same as solving $Ax = 0$ for which A?

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix} = 0 \qquad \begin{aligned} \cancel{x_1} + 2x_2 + 2x_3 + 3x_4 &= 0 \\ -)\ \cancel{x_1} + 3x_2 + 3x_3 + 2x_4 &= 0 \\ \hline x_2 + x_3 - x_4 &= 0 \end{aligned} \qquad \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \begin{aligned} x_1 &= -5x_2 - 5x_3 \\ x_4 &= x_2 + x_3 \end{aligned} , x_2, x_3 \in \mathbb{R} \right\}$$

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix} = 0 \qquad \begin{cases} x_4 = x_2 + x_3 \\ x_1 = -5x_2 - 5x_3 \end{cases} \Rightarrow \left\{ x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \end{bmatrix} \middle| x_2, x_3 \in \mathbb{R} \right\}$$

①

$$\text{vector I} \rightarrow \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{II} \rightarrow \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \end{bmatrix} \#$$

②

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \#$$

9. Project b onto the column space of A by solving $A^T A\hat{x} = A^T b$ and $p = A\hat{x}$:

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$

Find $e = b - p$. It should be perpendicular to the columns of A.

(a) $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} a \\ b \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}$ $a+b=2$, $a+2b=5$ $b=3,\ a=-1$

$\hat{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ #

$p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ #

$e = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$e \perp$ columns of A #

(b) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$ $2a+2b=8$, $2a+3b=14$ (−) $b=6,\ a=-2$

$\hat{x} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ #

$p = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$ #

$e = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$e \perp$ columns of A #