1. Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution$\tilde{x} = (C, D)$ and draw the closest line.

t	-1	1	2
b	7	7	21

$\bar{t} = 2/3$

$\bar{b} = 35/3$

$\rightarrow$

$t-\bar{t}$	$-5/3$	$1/3$	$4/3$
$b-\bar{b}$	$-14/3$	$-14/3$	$28/3$

$$D = m = \frac{\Sigma(b-\bar{b})(t-\bar{t})}{\Sigma(t-\bar{t})^2}$$

$$= \frac{\frac{1}{\cancel{9}}(70 - 14 + 112)}{\frac{1}{\cancel{9}}(25 + 1 + 16)} = \frac{168}{42} = 4$$

$(b - \frac{35}{3}) = 4(t - \frac{2}{3})$

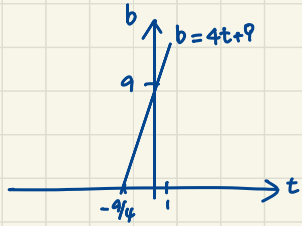
$\Rightarrow 3b - 35 = 4(3t - 2)$

$\Rightarrow 3b = 12t - 8 + 35$

$\Rightarrow 3b = 12t + 27$

$\Rightarrow b = 4t + 9$

$\Rightarrow (C, D) = \underline{(9, 4)}$ #



2. Find the projection $p = A\tilde{x}$ in problem 1. This gives the three heights of the closest line. Show that the error vector is $e = (2, -6, 4)$. Why is $Pe = 0$?

$b = C + Dt$

$C - D = 7$

$C + D = 7$

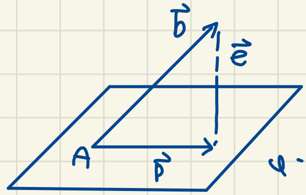
$C + 2D = 21$

$$\underset{A}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}}$$

$$p = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$$ #

$\vec{e} = \vec{b} - \vec{p}$

$$= \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} - \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$$ #



$\because \vec{p} \perp \vec{e}$

$\therefore Pe = 0$ #

3. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ with $\vec{a_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{a_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\vec{a_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(a) perform Gram-schmidt to get orthonormal $\vec{q_1}, \vec{q_2}, \vec{q_3}$ from $\vec{a_1}, \vec{a_2}, \vec{a_3}$

(b) QR decompose A

(a) $\|\vec{a_1}\| = \sqrt{\vec{a_1}^T \cdot \vec{a_1}} = \sqrt{2}$ $\quad \vec{q_1} = \frac{\vec{a_1}}{\|\vec{a_1}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ #

$[1\ 1\ 0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2$

$\vec{e_2} = \vec{a_2} - (\vec{q_1}^T \vec{a_2})\vec{q_1} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$ $\quad \vec{q_2} = \frac{\vec{e_2}}{\|\vec{e_2}\|} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$ #

$\vec{e_3} = \vec{a_3} - (\vec{q_1}^T \vec{a_3})\vec{q_1} - (\vec{q_2}^T \vec{a_3})\vec{q_2} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ $\quad \vec{q_3} = \frac{\vec{e_3}}{\|\vec{e_3}\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ #

(b) $Q = [q_1\ q_2\ q_3] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ $\quad R = \begin{bmatrix} \vec{v_1}^T\vec{q_1} & \vec{v_2}^T\vec{q_1} & \vec{v_3}^T\vec{q_1} \\ 0 & \vec{v_2}^T\vec{q_2} & \vec{v_3}^T\vec{q_2} \\ 0 & 0 & \vec{v_3}^T\vec{q_3} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$ #

4. If Q has orthonormal columns, what is the least squares solution x to $Qx = b$?

$\therefore Q^T Q = I$

$Q\vec{x} = \vec{b}$

$\Rightarrow Q^T Q \vec{x} = Q^T \vec{b}$

$\Rightarrow I \vec{x} = Q^T \vec{b}$

$\Rightarrow \vec{x} = \underline{Q^T \vec{b}}$ #

5. Find the determinants of rotations and reflections:

$$①\ Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } ②\ Q = \begin{bmatrix} 1-2\cos\theta^2 & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1-2\sin\theta^2 \end{bmatrix}$$

① $|Q| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta) = \underline{1}_{\#}$

② $|Q| = \begin{vmatrix} 1-2\cos\theta^2 & -2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 1-2\sin\theta^2 \end{vmatrix} = \begin{vmatrix} -\cos2\theta & -\sin2\theta \\ -\sin2\theta & \cos2\theta \end{vmatrix} = -\cos^2 2\theta - \sin^2 2\theta = \underline{-1}_{\#}$

6. By applying row operations to produce an upper triangular U, compute

$$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \text{ and } \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix} = \underline{36}_{\#}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \rightarrow \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{vmatrix} = \underline{5}\ \#$$

7. Find the cofactor matrix C and multiply A times C^T. Compare AC^T with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$M_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1=3$ $\quad M_{21} = \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = -2$ $\quad M_{31} = \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1$ $\quad \Rightarrow C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$M_{12} = \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = -2$ $\quad M_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$ $\quad M_{32} = \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = -2$

$M_{13} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$ $\quad M_{23} = \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = -2$ $\quad M_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$$AC^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

8. Solve the linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

$$2x_1 + 5x_2 = 1$$
$$x_1 + 4x_2 = 2$$

$$x_1 = \frac{D_{x_1}}{D} = \frac{\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}} = \frac{-6}{3} = -2$$

$(x_1, x_2) = (-2, 1)$ #

$$x_2 = \frac{D_{x_2}}{D} = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}} = \frac{3}{3} = 1$$

9. Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

① $AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

$|AB - \lambda_1 I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda_1 & 2 \\ 1 & 3-\lambda_1 \end{vmatrix} = (1-\lambda_1)(3-\lambda_1) - 2 = 0$

$\Rightarrow \lambda_1^2 - 4\lambda_1 + 1 = 0$

② $BA = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$|BA - \lambda_2 I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda_2 & 2 \\ 1 & 1-\lambda_2 \end{vmatrix} = (3-\lambda_2)(1-\lambda_2) - 2$

$\Rightarrow \lambda_2^2 - 4\lambda_2 + 1 = 0$

③ $|A - \lambda_3 I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda_3 & 0 \\ 1 & 1-\lambda_3 \end{vmatrix} = (1-\lambda_3)^2 = 0$

$\Rightarrow \lambda_3 = 1$

④ $|B - \lambda_4 I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda_4 & 2 \\ 0 & 1-\lambda_4 \end{vmatrix} = (1-\lambda_4)^2 = 0$

$\Rightarrow \lambda_4 = 1$

(a) No #

(b) Yes #