1. Let A (m×n) , B (n×k) be two matrices. ① Show that the columns of AB are linear combinations of the columns of A. ② Then show that C(AB) ⊆ C(A).

①
$$A = \begin{bmatrix} C_1 & C_2 & C_3 & \cdots & C_n \end{bmatrix}_{m\times n} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & & & & \\ \vdots & & \ddots & & \\ b_{n1} & & & & b_{nm} \end{bmatrix}$$

$$AB = \begin{bmatrix} C_1' & C_2' & C_3' & \cdots & C_m' \end{bmatrix}_{m\times m}$$

$$\rightarrow C_1' = b_{11}[C_1] + b_{21}[C_2] + b_{31}[C_3] + \cdots + b_{n1}[C_n]$$
$$C_2' = b_{12}[C_1] + b_{22}[C_2] + b_{32}[C_3] + \cdots + b_{n2}[C_n]$$
$$\vdots$$
$$C_m' = b_{1m}[C_1] + b_{2m}[C_2] + b_{3m}[C_3] + \cdots + b_{nm}[C_n]$$

⇒ the columns of AB are linear combinations of the colums of A #

②
$B = [B_1\ B_2\ \cdots\ B_p]$

$\Rightarrow AB = [AB_1\ AB_2\ AB_3\ \cdots\ AB_m]$

$\Rightarrow C(AB)$ = the set of all linear combination of $AB_1\ AB_2 \cdots AB_m$

Let $V \in C(AB)$ (V is a linear combination of $\underbrace{AB_1\ AB_2 \cdots AB_m}_{\text{linear combinations of } A}$)

$\Rightarrow V$ is a linear combination of columns of A

$\therefore V \in C(A)$

$\therefore C(AB) \subseteq C(A)$ #

2. (a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columns of AB. Then AB cannot have new pivot columns, so $rank(AB) \leq rank(B)$.

(b) Find A_1 and A_2 so that rank(A_1B) = 1 and rank(A_2B)=0 for B $= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) matrix AB can be seen as A times B

(A's rows time B's columns)

so if column j of B is a combination of the previous B, the column j of AB will also be the same combination of previous columns of AB. #

Then rank(AB) ≤ rank(B). no new pivot columns #

(b) $(A_1B) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ #

$(A_2B) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ #

3. Consider A $= \begin{bmatrix} 2 & 8 & 1 & 0 & 7 \\ -3 & -12 & 0 & 2 & 2 \\ 5 & 20 & -2 & -1 & 0 \end{bmatrix}$ Find $N(A)$ the null space of A. Please also identify pivot variables and free variables.

free variables

$A = \begin{bmatrix} 2 & 8 & 1 & 0 & 7 \\ -3 & -12 & 0 & 2 & 2 \\ 5 & 20 & -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 1 & 0 & 7 \\ 0 & 0 & 3 & 4 & 25 \\ 0 & 0 & -9 & -2 & -35 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 1 & 0 & 7 \\ 0 & 0 & 3 & 4 & 25 \\ 0 & 0 & 0 & 10 & 40 \end{bmatrix}$

pivot variables

$\rightarrow \begin{cases} 2x_1 + 8x_2 + x_3 + 7x_5 = 0 \\ 3x_3 + 4x_4 + 25x_5 = 0 \\ 10x_4 + 40x_5 = 0 \end{cases} \rightarrow \begin{cases} x_1 + 4x_2 + 2x_5 = 0 \\ x_3 + 3x_5 = 0 \\ x_4 + 4x_5 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -4x_2 - 2x_5 \\ x_3 = -3x_5 \\ x_4 = -4x_5 \end{cases}$

$N(A) = \left\{ \begin{bmatrix} -4x_2 - 2x_5 \\ x_2 \\ -3x_5 \\ -4x_5 \\ x_5 \end{bmatrix} \middle| \ x_2, x_5 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -3 \\ -4 \\ 1 \end{bmatrix} \middle| \ x_2, x_5 \in \mathbb{R} \right\}$ #

4. Which of the following subsets of R^3 are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $v = (1,4,0)$ and $w = (2,2,2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \leq b_2 \leq b_3$

(b) $b_1 = 1$

$(1,1,1) + (1,1,1) = (2,2,2),\ b_1 \neq 1$

(c) $b_1 \cdot b_2 \cdot b_3 = 0$

$(0,0,1) + (1,1,1) = (1,1,2)$

(f) $(1,2,3) - (4,8,10) = (-3,-6,-7)$ $b_1 > b_2 > b_3$

5. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space M of all 2 by 2 matrices. Write down the zero vector in this space, the vector $\frac{1}{2}$ A, and the vector -A. What matrices are in the smallest subspace containing A?

zero vector $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ #

$-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$ #

$\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ #

smallest subspace contain $A = k \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ $(k \in R)$ #

6. Suppose you know that the 3 by 4 matrix A has the vector $s = (2, 3, 1, 0)$ as the only special solution to $Ax = 0$.

(a) What is the *rank* of A and the complete solution to $Ax = 0$?

(b) What is the exact row reduced echelon form R of A?

(c) How do you know that $Ax = b$ can be solved for all b?

$A_{3\times4}$ $\quad\quad \vec{x_p} = (2,3,1,0) \quad\quad A = \vec{x_p} + \vec{x_n}$

$$A = \begin{bmatrix} \rule{3em}{0.4pt} \\ \rule{3em}{0.4pt} \\ \rule{3em}{0.4pt} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} = 0$$

(a) $\text{rank}(A) = 3$

$x = ks$ (k is any scalar) #

(b)
$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \#$$

(c) because the change of b only effect on the special solution, so $\vec{x_n}$ won't change, $Ax=b$ can still be solved #

(Also, A and R have full rank $r=3$)