Signals and Systems I (2016506)
Faculty of Engineering
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Lecture 5 - Fourier Series

Prelude: Fourier Series (FS) allows us to represent a periodic signal of interest as an infinite linear combination of pure sinusoids. The frequencies of such sinusoids are integer multiples of a fundamental frequency ω_0.

There are multiple algorithms to evaluate numerically the FS of a periodic signal in Matlab (or other languages). We will introduce a simple one (perhaps, the simplest). Assume a time interval Δt. An approximation of the DC component can be written as

$$c_0 = \frac{1}{T}\int_{<T>} x(t)\,dt \approx \frac{1}{T}\Delta t \sum_{<T>} x(t) \tag{1}$$

In other words, we "extract" the differential dt as a constant from the integral, divide it by the fundamental period, and multiply this result by the sum of all values within the vector x[a], containing the information of a period of the signal $x(t)$. A similar approximation can be made for the Exponential Fourier Series coefficients. Recall that these coefficients can be computed via the analysis equation by

$$\bar{c}_k = \frac{1}{T}\int_{<T>} x(t)\, e^{-jk\omega_0 t} dt \approx \frac{1}{T}\Delta t \sum_{<T>} x(t)\, e^{-jk\omega_0 t} \tag{2}$$

In this case, the inner summation $\sum_{<T>} x(t)\, e^{-jk\omega_0 t}$ must be evaluated for each value of k of interest. For example, if we are interested in computing the coefficient $\bar{c}_3$ $(k = 3)$, then we have to calculate the sum

$$\frac{1}{T}\Delta t \sum_{<T>} x(t)\, e^{-j3\omega_0 t}$$

`for` loops can be useful when multiple harmonics are requested. The following piece of code illustrates this procedure for computing 3 harmonics. Notice that we assume that the signal $x(t)$ has already been computed within the program, and its fundamental period `T0` has been declared.

```
%Assume x is a vector containing exactly one period of the signal
% coefficients will be computed up to the third harmonic
index=1;
for k=-3:3
% Notice that this expression is also valid for k=0
    c(index)=dt*sum(x.*exp(-1i*k*2*pi*t/T0))/T0;
% Or equivalently: c(index)=mean(x.*exp(-1i*k*2*pi*t/T0));
    index=index+1;
end
%      c=[c_m3 c_m2 c_m1 c_0 c_1 c_2 c_3]
% index=  1     2     3     4    5    6    7
```

[a] Notice that the left-hand part of equation 1 is the exact definition of the mean value, while the right-hand part is nothing but a Riemann sum divided by T.

1. **Consider the following periodic signal:**

$$x(t) = \begin{cases} \sin(\omega_0 t) & 0 \le t < \frac{T_0}{2} \\ 0 & \frac{T_0}{2} \le t \end{cases}$$

with $\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$**. Assume** $f_0 = 60$ **Hz**

a) **Plot the signal $x(t)$ for 10 periods. Use a time interval of $\Delta t = \frac{1}{100 f_0}$ (Hint:** construct a plot with three subplots. The two remaining spaces will be used in part e) of this exercise**)**

b) **Compute the DC level of the signal (Hint:** since the signal is constructed in ten periods, divide the approximation of the integral by ten to obtain the exact result**)**

c) **Determine the value of the following Exponential Fourier Series (EFS) coefficients and fill the table:**

k	$\bar{c}_k$
-3	
-2	
-1	
0	
1	
2	
3	

d) **Repeat the previous exercise for $f_0 = 120$ Hz. Compare your results and conclude.**

k	$\bar{c}_k$
-3	
-2	
-1	
0	
1	
2	
3	

e) **In the second and third subplots, graph the magnitude and the phase of the first 5 EFS harmonics (both positive and negative: $-5 \leq n \leq 5$), respectively. How can you describe the magnitude and phase spectra?**

2. **Consider the same signal as in Exercise 1. Assume $f_0 = 60$ Hz. It is recommended to develop this second exercise in a new .m file**

   a) **Complete the following code. Which contains a function receiving one period of the signal (x), the period (T) in seconds and the desired number of harmonics for which the Fourier coefficients will be computed (n) . Using the Fourier coefficients, the function returns the reconstructed approximation of the signal x.**

```
% Complete the code where necessary
function [f] = FourierSeriesSynthesis(x,T,n)
    % delta t and t vector
    dt=T/length(x);
    t=0:dt:T-dt;
    % Analysis equation for the coefficients of Fourier Series
    for k=-n:n
        % Complete the following lines
    end
    % Synthesis equation
    f=0;
    for k=-n:n
        % Complete the following line
        % f=f+ (...) ;
    end
end
```

   b) **Using the `FourierSeriesSynthesis` function, compute a Fourier Series approximation for the signal $x(t)$ using 1, 2, 5 and 15 harmonics**

c) **Compute the sumation of the absolute approximation error for each Fourier Series approximation. Let the sumation of the absolute approximation error be**

$$e(i) = \sum_{i=0}^{n} |x(i) - \tilde{x}(i)| \tag{3}$$

d) **Plot the original signal and each of its Fourier Series approximations, place in the title of each plot the corresponding sumation of the absolute approximation error alongside the number of computed coefficients. Superimpose the plots of the original signal and its approximation. (Hint:** generate a 2×2 subplot –e.g., `subplot(2,2,1)`- and graph the superimposed signals $x(t)$ and its approximation $\tilde{x}(t)$ in each case. **)**