Performance Criteria in `interpTools`

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The following are the function definitions for the criteria used in the evaluation of statistical performance of interpolation. Here, we define $\{X_i\}_{i=1}^{I}$ as the **interpolated points** in the repaired series, and $\{x_i\}_{i=1}^{I}$ as the corresponding points in the **original** series, where I is the total number of interpolated data points. The collection of metrics was gathered from the 2017 review by Lepot *et al.* [1] and summary provided by Van Bussel [2].

1 Correlation Coefficient

Sometimes referred to as *Pearson's* r, the correlation coefficient has range $-1 \leq r \leq 1$ and measures the linear correlation between the interpolated and original values:

$$r = \frac{\sum_{i=1}^{I}(X_i - \bar{X})(x_i - \bar{x})}{\sqrt{\sum_{i=1}^{I}(X_i - \bar{X})^2}\sqrt{\sum_{i=1}^{I}(x_i - \bar{x})^2}} \tag{1}$$

2 Coefficent of Determination

Simply the square of the correlation coefficient, this criterion represents how much of the variation in the interpolated values is attributable to the relationship between the interpolated and original values:

$$r^2 = \left(\frac{\sum_{i=1}^{I}(X_i - \bar{X})(x_i - \bar{x})}{\sqrt{\sum_{i=1}^{I}(X_i - \bar{X})^2}\sqrt{\sum_{i=1}^{I}(x_i - \bar{x})^2}}\right)^2 \tag{2}$$

3 Absolute Differences

The absolute differences criterion is the sum of the absolute differences between each pair of original and interpolated values:

$$\mathbf{AD} = \sum_{i=1}^{I} |X_i - x_i| \tag{3}$$

This criterion can be difficult to interpret since there is no scale factor.

4 Mean Bias Error

The Mean Bias Error (MBE) is the sum of the differences between the interpolated and original values, scaled by n:

$$\mathbf{MBE} = \frac{\sum_{i=1}^{I} X_i - x_i}{I} \tag{4}$$

5 Mean Error

The Mean Error (ME) has identical magnitude to the MBE, except with opposite sign:

$$\mathbf{ME} = \frac{\sum_{i=1}^{I} x_i - X_i}{I} \tag{5}$$

6 Mean Absolute Error

The Mean Absolute Error (MAE) is very similar to both the MBE and the ME, but instead uses the absolute value of the sum of the differences:

$$\mathbf{MAE} = \frac{\sum_{i=1}^{I} |X_i - x_i|}{I} \tag{6}$$

7 Mean Relative Error

The Mean Relative Error (MRE) is the relative differences between the original and interpolated values:

$$\mathbf{MRE} = \sum_{i=1}^{I} \frac{x_i - X_i}{x_i} \tag{7}$$

Note that this cannot be used if $x_i = 0$ for some i.

8 Mean Absolute Relative Error

The Mean Absolute Relative Error (MARE) is similar to the MRE, except that the absolute value of each term in the summation is used:

$$\mathbf{MARE} = \frac{1}{I} \sum_{i=1}^{I} \left| \frac{x_i - X_i}{x_i} \right| \tag{8}$$

Again, note that $x_i \neq 0$ for any i.

9 Mean Absolute Percentage Error

The Mean Absolute Percentage Error (MAPE) is essentially the MARE converted into a percentage:

$$\mathbf{MAPE} = \frac{100}{I} \sum_{i=1}^{I} \left| \frac{x_i - X_i}{x_i} \right| \tag{9}$$

This percentage-metric is easier to interpret than the MARE. Note that $x_i \neq 0$ for any i.

10 Sum of Squared Errors

The Sum of Squared Errors (SSE) is the sum of the squared differences:

$$\mathbf{SSE} = \sum_{i=1}^{I} (X_i - x_i)^2 \tag{10}$$

11 Mean Square Error

The Mean Square Error (MSE), sometimes referred to as the "standard error" in the literature, is the SSE scaled by n:

$$\mathbf{MSE} = \frac{1}{I} \sum_{i=1}^{I} (X_i - x_i)^2 \tag{11}$$

12 Root Mean Squares

The Root Mean Squares (RMS) is sometimes referred to as the Root Mean Square Errors of Prediction (RMSEP). It is essentially the square root of the MSE, except that the squared differences are scaled by the original observations:

$$\mathbf{RMS} = \sqrt{\frac{1}{I} \sum_{i=1}^{I} \left(\frac{X_i - x_i}{x_i} \right)^2} \tag{12}$$

Again, note the restriction $x_i \neq 0 \quad \forall t$.

13 Normalized Mean Square Error

According to [1], the Normalized Mean Square Error criterion (denoted NMSE) is rarely present in the literature. Its definition is:

$$\mathbf{NMSE} = \frac{\sum_{i=1}^{I} (x_i - X_i)^2}{\sum_{i=1}^{I} (x_i - \bar{x})^2} \tag{13}$$

14 Reduction of Error

The Reduction of Error (RE) is sometimes referred to as the Nash-Sutcliffe coefficient (NS). It is simply 1 - NMSE:

$$\mathbf{RE} = 1 - \frac{\sum_{i=1}^{I} (x_i - X_i)^2}{\sum_{i=1}^{I} (x_i - \bar{x})^2} \tag{14}$$

15 Root Mean Square Error

Sometimes referred to as the Root Mean Squares Deviation (RMSD), the Root Mean Square Error (RMSE) is given by:

$$\mathbf{RMSE} = \sqrt{\frac{1}{I} \sum_{i=1}^{I} (X_i - x_i)^2} \tag{15}$$

16 Normalized Root Mean Square Deviation

The Normalized Root Mean Square Deviation (NRMSD) is a normalized version of the RMSE criterion, where the range of the original values is used as a scale factor. For easier interpretation, the metric is multiplied by 100:

$$\mathbf{NRMSD} = 100 \times \frac{\sqrt{\frac{1}{I}\sum_{i=1}^{I}(X_i - x_i)^2}}{\max(x_i) - \min(x_i)} \tag{16}$$

17 Root Mean Square Standardized Error

Similar to the Root Mean Square (RMS), the Root Mean Square Standardized Error (RMSS) has a denominator which is the standard deviation of the original values:

$$\mathbf{RMSS} = \sqrt{\frac{1}{I}\sum_{i=1}^{I}\left(\frac{X_i - x_i}{\sigma_x}\right)^2} \tag{17}$$

Note that this metric will be undefined if the $\sigma_x = 0$, which is unlikely to happen in practice.

18 Median Absolute Percentage Error

Similar to the Mean Absolute Percentage Error (MARE), the Median Absolute Percentage Error (MdAPE) considers the median of the absolute differences, converted into a percentage:

$$\mathbf{MdAPE} = \text{median}\left|\frac{x_i - X_i}{x_i}\right| \times 100\% \tag{18}$$

This metric is less sensitive to skewness than the MAPE. Note that $x_i \neq 0$ for any i.

References

[1] Mathieu Lepot, Jean-Baptiste Aubin, and François Clemens. "Interpolation in time series: An introductive overview of existing methods, their performance criteria and uncertainty assessment". In: *Water* 9.10 (2017), p. 796.

[2] Melissa Van Bussel. "Time Series Interpolation Algorithms: Honours Thesis, Trent University". In: (2019).